Introduction to Process Control

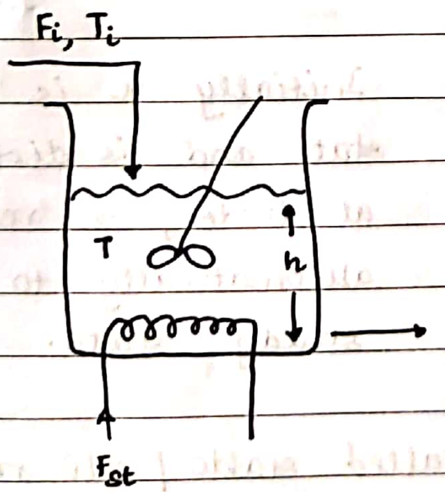
Raw Material → Chemical Plant → Product

The plant recieves raw materials, using various sources of energy, it converts these into product in the most economical way

Requirements :
(i) Safety
(ii) Production Specification
(iii) Environmental Requirements
(iv) Operational Constraints
(v) Economics (Min op cost, Max profit)

Issues :
(i) Influence of external disturbances
(ii) Stability of a chemical process
(iii) Performance of a chemical process

External Disturbance

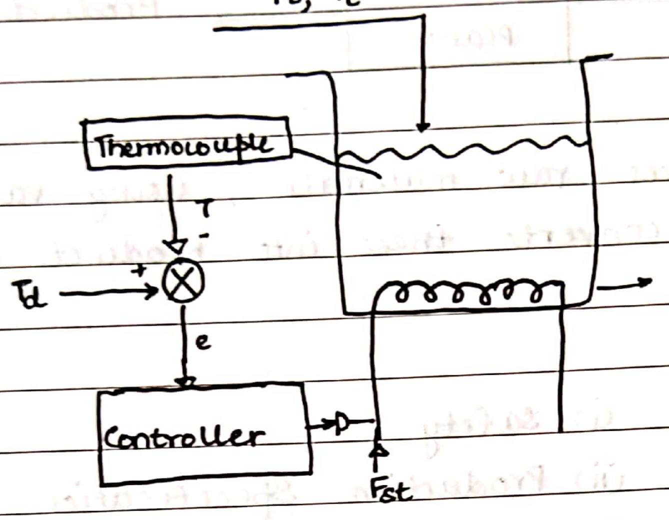


Objective :
$T = T_d$
$h = h_d$

Start → Steady state

The process remains at steady state if the above variables don't change. But this isn't the case, as F_i, T_i may change and hence we require a controller.

To avoid (stay at steady state) we use a controller,



e → error signal

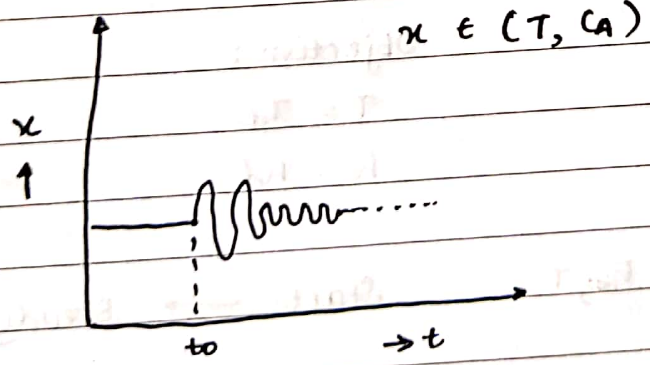
$e = T_d - T$

If error > 0, $T_d > T$ (more steam flow is req)

If error < 0, $T_d < T$ (reduce steam flow is req)

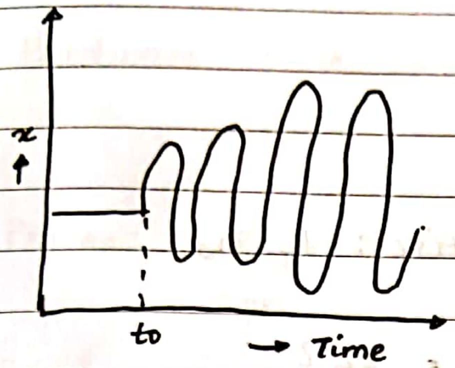
Consider the situation where F_i increases, in such case T↓, thus controller will increase the steam flow rate (F_{st}↑)

Stability of a chemical process



Initially x is at steady state and is disturbed at $t = t_0$, x returns automatically to the steady state.

This type of process is called stable / self regulating process. No need of external intervension



when x is disturbed at $t = t_0$, it doesn't return to the steady state.

This type of process is called unstable process. There is a need of external intervension

Optimize the performance of a process

Main Objectives : (i) Safety (ii) Production specific.

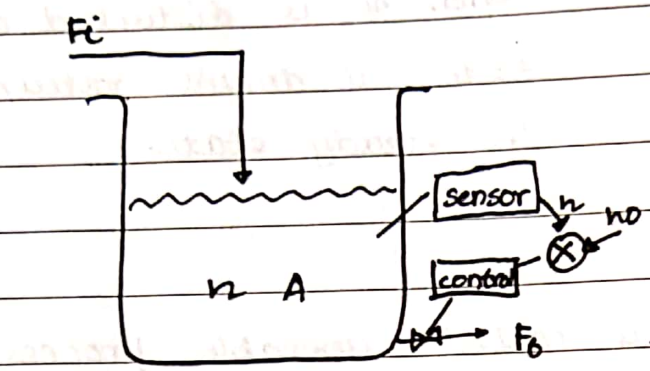
Economic Objective :

$$\text{Maximize profit } \phi = \int_0^t f(\underbrace{B}_{\text{pdt}}, \underbrace{A}_{\text{reactant}}, \text{coolant})\, dt$$

Classifacation of Variables

- Variable
  - Input Variable
    - → Disturbance / Load Variable (LV)
    - → Manipulated / Control Variable (CV)
  - Output Variable
    - → Measured output
    - → unmeasured output

Liquid Tank System



Objective : $h = h_d$

CV : $\{h\}$

Input variable : F_i (F_o)

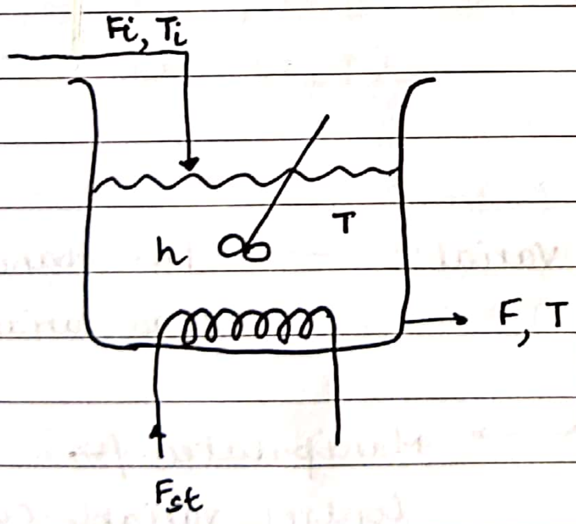
Output variable : F_o, h

A possible pair of CV, MV is (h, F_o) or (h, F_i)

We can control F_i or F_o to maintain h

If F_o is Manipulated variable, then F_o is input variable

Heating Tank System



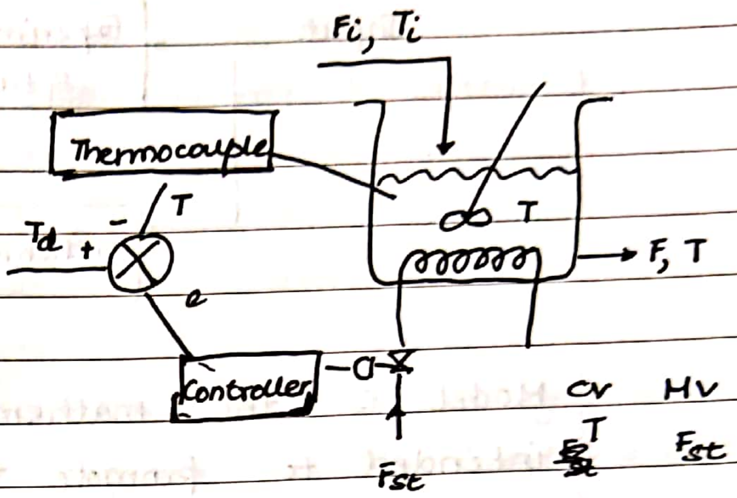
Input : $\underbrace{F_i, F_{st}}_{MV}$, $\underbrace{T_i}_{LV}$ (F)

Output : F, $\underbrace{T, h}_{CV}$

Objectives : $T = T_d$, $h = h_d$

MV	CV
F_{st}	T
F, F_i	h

Hardware of a control system



Process : Physical and chemical ops.

Measuring instruments or sensors :

Variable	Sensor
Temperature	Thermocouple, Resistance Thermometer
Pressure	Manometer, Digaphram Element
Flow rate	Orifice meter, Venturi meter
Liquid Level	DP cell
Comp.	Chromatographic Analyser

Transducer : Measurements to physical quantities

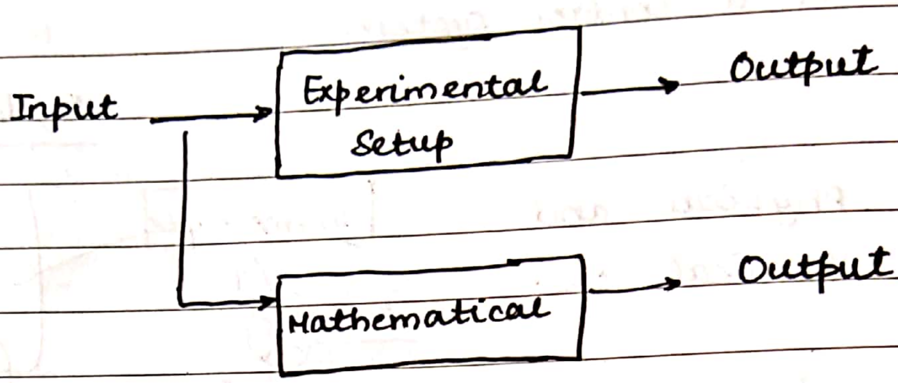
Transmission lines : Used to carry the measurement signals

Controller : Based on measurement signal from sensor, it decides what action is to be taken.

Final Control Element (FCE) : Controller decision is implemented through FCE

Recording device : To visualize the behaviour

Mathematical Modeling



Model is the mathematical rep of a process intended to promote the understanding of a real system

Use of process model
- → To understand the process behaviour
- → To train the operating personnel
- → Selection of control pairs (CV - MV)
- → Optimizing the process operating conditions.

Process Model →
- Theoretical Model (Principle of Conservation)
- Emperical Model (By Fitting exp. data)
- Semi-Emperical Model (Combination of EM, TM)

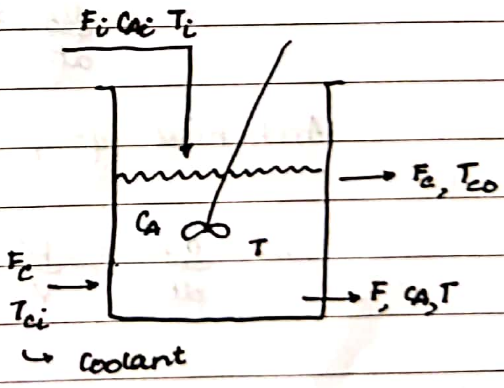
State Variable

- Describes the natural state of a process
- Fundamental quantities - (Mass, Energy, Momentum), they're not directly measure
- They are characterized by T, P, x, F
- They arise in the accumulation term

State Equations : The eqn's which are derived by the application of conservation principle on the fundamental quantities to relate the state variables with other variables are called state eqn's

Conservation Principle

$$\begin{pmatrix}\text{Rate of}\\\text{Accumulate}\end{pmatrix} = \begin{pmatrix}\text{Rate of}\\\text{Input}\end{pmatrix} - \begin{pmatrix}\text{Rate of}\\\text{Output}\end{pmatrix} + \begin{pmatrix}\text{Rate of}\\\text{Generation}\end{pmatrix} - \begin{pmatrix}\text{Rate of}\\\text{Depletion}\end{pmatrix}$$



F_i, F, F_c → Volumetric flow rates

C_A, C_{Ai} → mol / vol.

Assumptions

① Perfect mixing

② ρ, C_p → const.

③ Exothermic, first order rxn:

④ Perfectly insulated

⑤ Coolant is perfectly mixed in jacket

⑥ No energy balance for the jacket

Overall Mass Balance

Rate of Acc. = Rate of mass in − Rate of mass out

$$\frac{d}{dt}(V\rho) = F_i\rho - F\rho$$

since ρ is const., $\rho\frac{dV}{dt} = (F_i - F)\rho$

$$\therefore \frac{dV}{dt} = F_i - F$$

Mass Balance of comp A

Rate of Acc. of A = Rate of in of comp A + Rate of gen. of comp A − Rate of out of comp A

$$\frac{d}{dt}(VC_A) = F_iC_{Ai} - (-r_A)V - FC_A$$

On solving,

$$\frac{dC_A}{dt} = \frac{F_i}{V}(C_{Ai} - C_A) - (-r_A)$$

Arrhenius eqn., $-r_A = k_0 e^{-E_A/RT} C_A$

$$\frac{dC_A}{dt} = \frac{F_i}{V}(C_{Ai} - C_A) - k_0 C_A e^{-E_A/RT}$$

Using energy balance eqn:

$$\frac{dT}{dt} = \frac{F_i}{V}(T_i - T) - \frac{Q}{V\rho C_p} + \frac{(-\Delta H)k_0 C_A e^{-E/RT}}{\rho C_p}$$

Input Variables : C_{Ai}, F_i, T_i, Q, (F)

Output Variables : V, C_A, T (state variables)

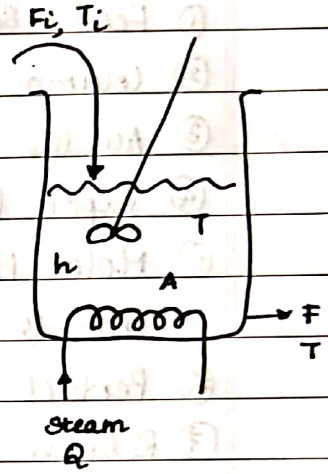
MV	CV
F	V
Q	T

Degrees of freedom (F) = V − E

V → No. of independent process variables

E → No. of Independent eqn's

Stirred Tank Heater



Assumptions

① Perfectly mixed

② ρ, c_p → constant

③ Perfectly insulated

Total mass balance

$$\frac{d}{dt}(hA\rho) = F_i\rho - F\rho \Rightarrow A\frac{dh}{dt} = F_i - F \quad -①$$

Energy Balance

$$\frac{d}{dt}\left(hA\rho C_p(T - T_{ref})\right) = F_i\rho C_p(T_i - T_{ref}) - F\rho C_p(T - T_{ref}) + Q$$

Considering $T_{ref} = 0$

$$Ah\frac{dT}{dt} = F_i(T_i - T) + \frac{Q}{\rho C_p} \quad -②$$

V → {h, F_i, F, T, T_i, Q}   F = 6 − 2

E → {2}   = 4

① Load Variables → {F_i, T_i}   F = 4 − 2 = 2

Steady state value of F → desired value

② $F = F_s + K_{cf}(h_d - h)$ → liquid height

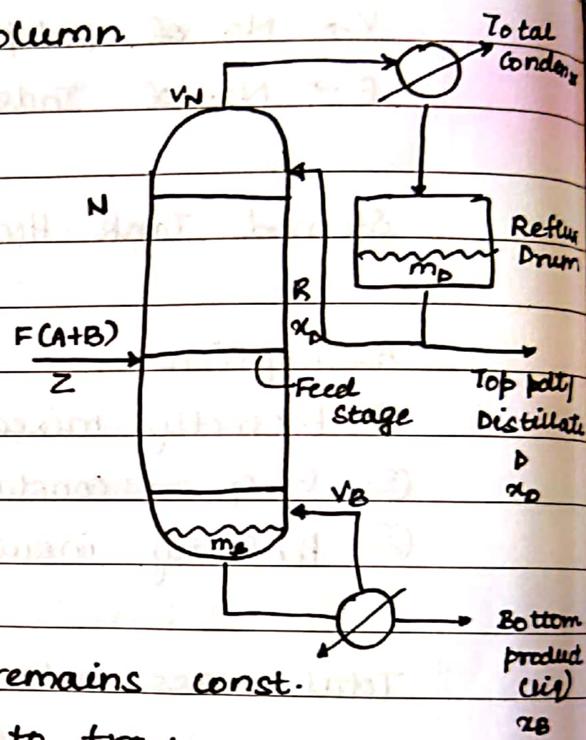
→ Tuning parameter

$F = 2 - 2$

$= 0$

$Q = Q_s + K_{cQ}(T_d - T)$

Modelling of a Distillation Column

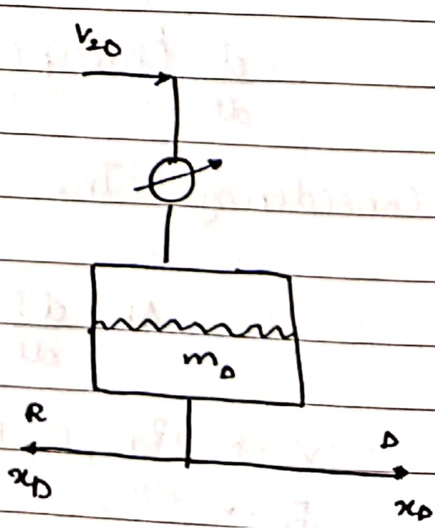


Assumptions

1. Feed - Saturated liquid (BP temp)
2. Column is perfectly insulated
3. All the trays are ideal
4. Vapor holdup is neglected
5. Molar heats of vaporization of A and B are approx. equal
6. Perfect mixing
7. Relative volatility of A and B remains const.
8. Liquid hold up varies from tray to tray
9. Condensor and reboiler dynamics are neglected.

From assumptions 2, 4, 5

$$V_1 = V_2 \cdots = V_N = V_B$$



Total Mass Bal.

$$\frac{d}{dt}(m_D) = V_{20} - D - R$$

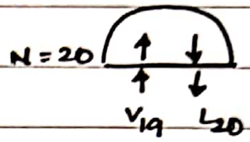
Comp. Mass Bal.

$$\frac{dx_D}{dt} = \frac{V_{20}}{m_D}(y_{20} - x_D)$$

Top stage (20th stage)

Total : $\frac{d}{dt}(m_{20}) = R + V_{19} - L_{20} - V_{20}$

$= R - L_{20}$



$V_1 = V_2 = \cdots = V_{20}$

Comp. mass balance

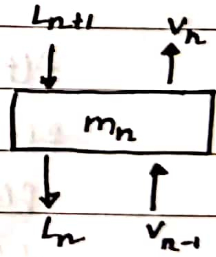
$$\frac{d(m_{20}x_{20})}{dt} = Rx_D + V_{19}y_{19} - L_{20}x_{20} - V_{20}y_{20}$$

$$\Rightarrow m_{20}\frac{dx_{20}}{dt} + x_{20}\frac{dm_{20}}{dt} = Rx_D + V_{19}y_{19} - L_{20}x_{20} - V_{20}y_{20}$$

$$V_{19} = V_{20} = V_B$$

$$\frac{dx_{20}}{dt} = \frac{1}{m_{20}}\left[R(x_D - x_{20}) + V_B(y_{19} - y_{20})\right]$$

n^{th} stage



Total : $\frac{dm_n}{dt} = L_{n+1} - L_n$

Comp. : $\frac{d(m_n x_n)}{dt} = L_{n+1}x_{n+1} + V_{n-1}y_{n-1} - L_n x_n - V_n y_n$

Feed stage

Total : $\frac{dm_{10}}{dt} = (L_{11} + F + V_9) - L_{10} - V_{10}$

Comp : $\frac{d(m_{10}x_{10})}{dt} = L_{11}x_{11} + Fz + V_9 y_9 - L_{10}x_{10} - V_{10}y_{10}$

Relative Volatility (α)

$$\alpha_{ij} = \frac{K_i}{K_j}$$

$\alpha_{ij} \rightarrow$ Relative Volatility of comp i ~~and~~ w.r.t j

$K \rightarrow$ Vap-liq eq. co-eff

$K_i = \frac{y_i}{x_i}$; $K_j = \frac{y_j}{x_j}$

$$\alpha_{ij} = \frac{K_i}{K_j} = \frac{y_i / x_i}{(1-y_i)/(1-x_i)}$$ (For binary soln.)

Laplace Transform

$$L[f(t)] = \bar{F}(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

Time fn.	Laplace Transform
$f(t) = 1$	$1/s$
$f(t) = t$	$1/s^2$
$f(t) = t^2$	$2!/s^3$
$f(t) = t^n$	$n!/s^{n+1}$
$f(t) = e^{-at}$	$1/(s+a)$
$f(t) = t^n e^{-at}$	$n!/(s+a)^{n+1}$
$f(t) = \sin \omega t$	$\omega / s^2+\omega^2$
$f(t) = \cos \omega t$	$s / s^2+\omega^2$
$f(t) = \sinh \omega t$	$\omega / s^2-\omega^2$
$f(t) = \cosh \omega t$	$s / s^2-\omega^2$
$f(t) = e^{-at} \sin \omega t$	$\omega / [(s+a)^2+\omega^2]$
$f(t) = e^{-at} \cos \omega t$	$(s+a) / [(s+a)^2+\omega^2]$

$$L\left[\frac{df(t)}{dt}\right] = s\bar{f}(s) - f(0)$$

$$L\left[\frac{d^2f(t)}{dt}\right] = s^2\bar{f}(s) - sf(0) - f'(0)$$

$$L\left[\frac{d^nf(t)}{dt^n}\right] = s^n\bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) \cdots - f^{n-1}$$

$f \rightarrow$ Derivation variable / fn.

$= f_{at\ time\ t} - f_{at\ ss}$

$= f_{as\ t=0} - f_{at\ ss} = 0$

$$L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}\bar{f}(s)$$

Final Value Theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0}[s\bar{f}(s)]$$

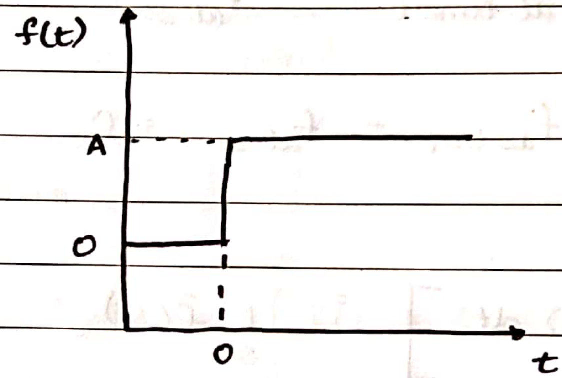
Initial Value Theorem

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty}[s\bar{f}(s)]$$

Dynamic Behavior

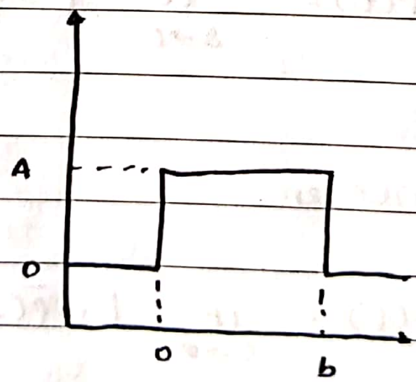
Forcing fn:

Step fn:

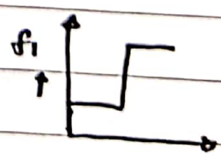


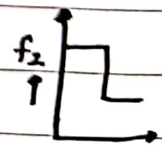
$$L[f(t)] = A/s$$

Rectangular Pulse fn:



$$f(t) = \begin{cases} 0 & t < 0 \\ A & 0 < t < b \\ 0 & t > b \end{cases}$$



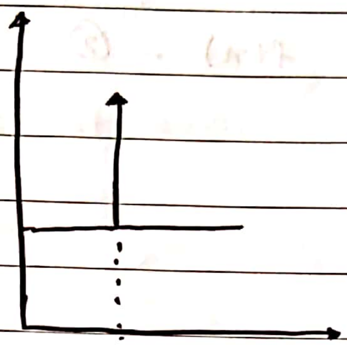


$$f(t) = f_1(t) - f_2(t-b)$$

Scanned with CamScanner

$\therefore \; L[f(t)] = \frac{A}{s}(1 - e^{-bs})$ ← For pulse fn.

Unit impulse fn:



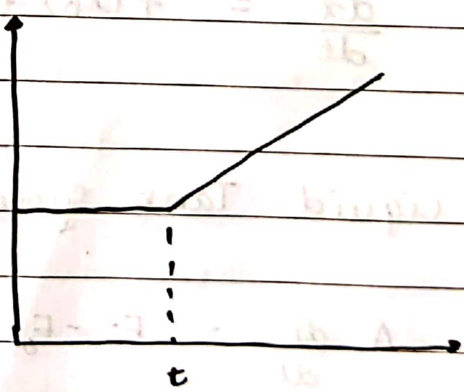
dirac fn, $\delta(t)$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

$L[\delta(t)] = 1$

Ramp fn:

$$f(t) = \begin{cases} 0 & t < 0 \\ At & t > 0 \end{cases}$$



$L[f(t)] = A/s^2$

Linearization of single variable systems

$$\frac{dx}{dt} = f(x, t) \quad - \text{①}$$

$$\frac{dx}{dt} = f(x) \quad - \text{②}$$

Taylor Series

$$f(x) = f(x_0) + \left(\frac{df}{dx}\right)_{x=x_0} (x-x_0) + \left(\frac{d^2f}{dx^2}\right)_{x=x_0} \frac{(x-x_0)^2}{2!} + \ldots$$

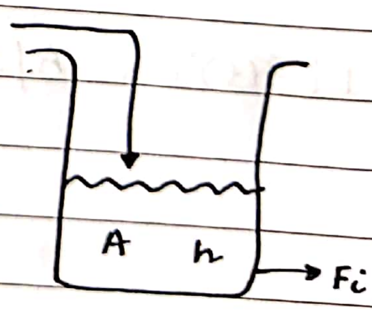
Neglecting order 2 and higher

$$f(x) \approx f(x_0) + \left(\frac{df}{dx}\right)_{x=x_0} (x-x_0)$$

$$\frac{dx}{dt} = f(x_0) + \left(\frac{df}{dx}\right)(x-x_0)$$

Liquid Tank System

$$A\frac{dh}{dt} = F_i - F_o \quad \text{(Mass Bal)}$$



Case 1 : $F \propto h \Rightarrow F = \beta h$

$$A\frac{dh}{dt} + \beta h = F_i \quad \text{(linear)}$$

Case 2 : $F_o \propto \sqrt{h} \rightarrow F = \alpha\sqrt{h}$

Taylor series

$$\alpha\sqrt{h} = \alpha\sqrt{h_0} + \left[\frac{d}{dh}(\alpha\sqrt{h})\right]_{h=h_0}(h-h_0) + \left[\frac{d^2}{dh^2}(\alpha\sqrt{h})\right]_{h=h_0}\frac{(h-h_0)^2}{2!}$$

Neglecting terms of order 2 and higher

$$\alpha\sqrt{h} = \alpha\sqrt{h_0} + \frac{\alpha}{2\sqrt{h_0}}(h-h_0)$$

$$A\frac{dh}{dt} + \frac{\alpha}{2\sqrt{h_0}}h = F_i - \frac{\alpha}{2}\sqrt{h_0} \rightarrow \text{Linearized Form}$$

Deviation variables

$$\frac{dx_s}{dt} = 0 = f(x_s)$$

$$f(x) = \frac{dx}{dt} = f(x_s) + \left(\frac{df}{dx}\right)_{x=x_s}(x-x_s)$$

$$\frac{d(x-x_s)}{dt} = \left(\frac{df}{dx}\right)_{x_s}(x-x_s)$$

Let $x' = x - x_s$, $\dfrac{dx'}{dt} = \left(\dfrac{df}{dx}\right)_{x_s} x'$

Linearized Form (Liquid Tank)

$$A\frac{dh}{dt} + \frac{\alpha}{2\sqrt{h_0}}h = F_i - \frac{\alpha}{2}\sqrt{h_s}$$

At SS : $A\dfrac{dh_s}{dt} + \alpha\sqrt{h_s} = F_{is}$

$$A \frac{d(h-h_s)}{dt} + \frac{\alpha}{2\sqrt{h_s}}(h-h_s) = F_i - F_{is}$$

Deviation variable, $F_i' = F_i - F_{is}$, $h' = h - h_s$

$$A \cdot \frac{dh'}{dt} + \frac{\alpha}{2\sqrt{h_s}} h' = F_i'$$

(Linearize form of a liquid Tank system)

System having 2 SV's

Linearization of multivariable systems

$$\frac{dx_1}{dt} = f_1(x_1, x_2)$$

$$\frac{dx_2}{dt} = f_2(x_1, x_2)$$

$x_1, x_2 \rightarrow$ State variables
$f_1, f_2 \rightarrow$ Nonlinear fn.

Taylor series

$$\frac{dx_1}{dt} = f_1(x_1, x_2) = f_1(x_{10}, x_{20}) + \left(\frac{\partial f_1}{\partial x_1}\right)_{(x_{10}, x_{20})}(x_1 - x_{10}) + \left(\frac{\partial f_1}{\partial x_2}\right)_{(x_{10}, x_{20})}(x_2 - x_{20})$$

$$\frac{dx_2}{dt} = f_2(x_1, x_2) = f_2(x_{10}, x_{20}) + \left(\frac{\partial f_2}{\partial x_1}\right)(x_2 - x_{20}) + \left(\frac{\partial f_2}{\partial x_2}\right)_{(x_{10}, x_{20})}(x_2 - x_{20})$$

At SS, $x_{10} = x_{1s}$, $x_{20} = x_{2s}$

$$\frac{dx_{1s}}{dt} = 0 = f_1(x_{1s}, x_{2s}) \quad - (1)$$

$$\frac{dx_{2s}}{dt} = 0 = f_2(x_{1s}, x_{2s}) \quad - (2)$$

$$\frac{d(x_1 - x_{1s})}{dt} = \left(\frac{\partial f_1}{\partial x_1}\right)_{(x_{1s}, x_{2s})}(x_1 - x_{1s}) + \left(\frac{\partial f_1}{\partial x_2}\right)_{(x_{1s}, x_{2s})}(x_2 - x_{2s})$$

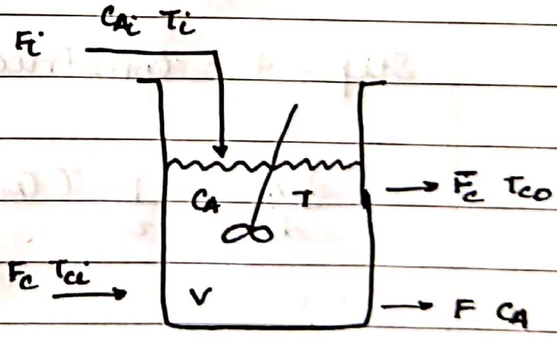
$$\frac{d}{dt}(x_2 - x_{2s}) = \left(\frac{\partial f_2}{\partial x_1}\right)_{(x_{1s}, x_{2s})}(x_1 - x_{1s}) + \left(\frac{\partial f_2}{\partial x_2}\right)_{(x_{1s}, x_{2s})}(x_2 - x_{2s})$$

Deviation variables, $x_1' = x_1 - x_{1s}$
$x_2' = x_2 - x_{2s}$

$$\left.\begin{aligned} \frac{dx_1'}{dt} &= a_{11} x_1' + a_{12} x_2' \\ \frac{dx_2'}{dt} &= a_{21} x_1' + a_{22} x_2' \end{aligned}\right\} \text{linearized form}$$

Irrev. first order rn:,

$A \rightarrow B$



$$\frac{dV}{dt} = F_i - F \quad \text{(Total mass)}$$

$$\frac{dC_A}{dt} = \frac{F_i}{V}(C_{Ai} - C_A) - (k_0 e^{-E_A/RT}) C_A \quad \text{(Comp. bal.)}$$

$$\frac{dT}{dt} = \frac{F_i}{V}(T_i - T) - \frac{Q}{V\rho C_p} + \frac{(-\Delta H) k_0 e^{-E_A/RT} C_A}{\rho C_p} \quad \text{(Energy bal.)}$$

Step - 1 : Develop the dynamic model

Assume v is const.

$$\frac{dC_A}{dt} = \frac{1}{\tau}(C_{Ai} - C_A) - K_0 e^{-E/RT} C_A \quad , \quad \tau = \frac{v}{F_i}$$

$$\frac{dT}{dt} = \frac{1}{\tau}(T_i - T) - \frac{Q}{v\rho C_p} + s K_0 e^{-E/RT} C_A \quad , \quad s = -\frac{\Delta H}{\rho C_p}$$

Step 2 : Identify the non-linear term

In eqn ①, $K_0 e^{-E/RT} C_A$ is non-linear term

In eqn ②, $s K_0 e^{-E/RT} C_A$ is non-linear term

Step - 3 : Linearize the non-linear terms

$$e^{-E/RT} C_A = e^{-E/RT_0} C_{A0} + \left(\frac{E}{RT_0^2} e^{-E/RT} C_{A0}\right)(T - T_0) + e^{-E/RT_0}(C_A - C_{A0})$$

Step - 4 : Construct the linearized model

$$\frac{dC_A}{dt} = \frac{1}{\tau}(C_{Ai} - C_A) - K_0 \left[e^{-E/RT_0} C_{A0} + \frac{E}{RT_0^2} e^{-E/RT_0} C_{A0} (T - T_0) + e^{-E/RT_0}(C_A - C_{A0}) \right]$$

$$\frac{dT}{dt} = \frac{1}{\tau}(T_i - T) - \frac{Q}{v\rho C_p} + s K_0 \left[e^{-E/RT_0} C_{A0} + \frac{E}{RT_0^2} e^{-E/RT_0} C_{A0} + e^{-E/RT_0}(C_A - C_{A0}) \right]$$

Scanned with CamScanner

Step 5: Represent the model at SS cond.

$$\frac{dC_A}{dt} = 0 = \frac{1}{\tau}(C_{Ai0} - C_{A0}) - k_0 e^{-E/RT} C_{A0}$$

$$\frac{dT}{dt} = 0 = \frac{1}{\tau}(T_{i0} - T_0) - \frac{Q_0}{V\rho C_p} + s k_0 e^{-E/RT} C_{A0}$$

Step 6: Derive the linearized model in terms of deviation variables

$$\frac{d(C_A - C_A)}{dt} = \frac{1}{\tau}\left((C_{Ai} - C_{A0}) - (C_A - C_{A0})\right)$$

$$- \frac{k_0 E}{RT_0} e^{-E/RT} C_{A0}(T - T_0) - k_0 e^{-E/RT_0}(C_A - C_{A0})$$

$$\frac{dC_A'}{dt} = \frac{1}{\tau}(C_{Ai}' - C_A') - \frac{k_0 E}{RT_0^2} e^{-E/RT_0} C_{A0} T'$$

$$- k_0 \frac{\cancel{E}}{\cancel{R}T_0^2} e^{-E/RT_0} C_A'$$

The deviation variables, $C_{Ai}' = C_{Ai} - C_{Ai0}$

$C_A' = C_A - C_{A0}$

$T' = T - T_0$

$$\frac{d(T - T_0)}{dt} = \frac{1}{\tau}\left[(T_i - T_{i0}) - (T - T_0)\right] - \left(\frac{Q - Q_0}{V\rho C_p}\right)$$

$$+ s k_0 \left[\frac{E}{RT_0^2} e^{-E/RT_0} C_{A0}(T - T_0) + e^{-E/RT_0}(C_A - C_{A0})\right]$$

$$\frac{dT'}{dt} = \frac{1}{\tau}(T_i' - T') - \frac{Q'}{V\rho C_p} + s k_0 \left(\frac{E}{RT_0^2} e^{-E/RT_0} C_{A0} T'\right.$$

$$\left. + e^{-E/RT_0} C_A'\right)$$

$C_A' = C_A - C_{A0}$ $\quad T_i' = T_i - T_{i0}$ $\quad Q' = Q - Q_0$

Transfer Functions

A single input single output (SISO) process,

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b\, f(t).$$

$$L\left[\frac{d^n y}{dt^n}\right] = s^n \bar{y}(s) - s^{n-1} y(0) - s^{n-2} y'(0) \cdots - y^{n-1}(0)$$

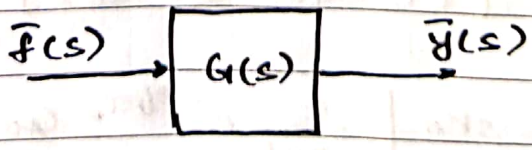
y, f are deviation variables thus, $y(0) = y'(0) \cdots = 0$

$$\Rightarrow \quad L\left[\frac{d^n y}{dt^n}\right] = s^n \bar{y}(s)$$

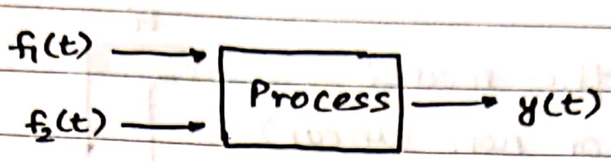
$$a_n s^n \bar{y}(s) + a_{n-1} s^{n-1} \bar{y}(s) + \cdots + a_0 \bar{y}(s) = b \bar{f}(s)$$

$$\frac{\bar{y}(s)}{\bar{f}(s)} = \frac{b}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0} = G(s)$$

Transfer fn:



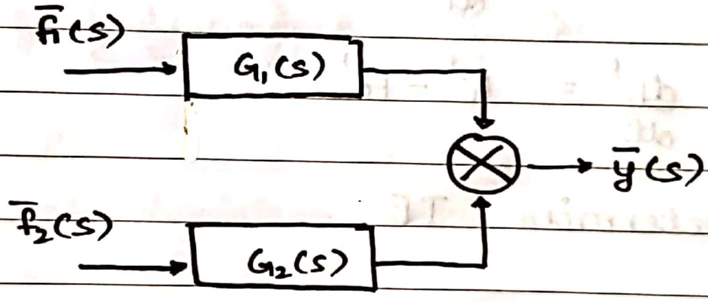
For a MISO system,



Process : $a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_0 y = b_1 f_1 + b_2 f_2$

$$\bar{y}(s) = \frac{b_1}{a_n s^n + a_{n-1} s^{n-1} + \dots a_0} \bar{f}_1(s) + \frac{b_2}{a_n s^n + a_{n-1} s^{n-1} + \dots a_0} \bar{f}_2(s)$$

$$\bar{y}(s) = G_1(s)\, \bar{f}_1(s) + G_2(s)\, \bar{f}_2(s)$$



For a SISO

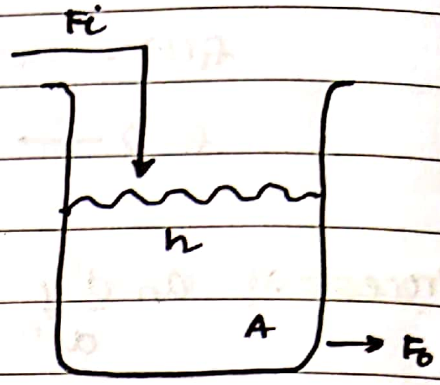
Change in input variable $f(t)$ ⟶ $\bar{f}(s)$ ⟶ $\bar{y}(s) = [G(s)\bar{f}(s)]$

Inverse Laplace

$y(t)$

A $\qquad$ A/s

Transfer fn: of a liquid tank system,



Step 1 : Develop the model
(linear or non-linear)

$$A \frac{dh}{dt} = F_i - F_o$$

Step 2 : Construct the model in terms of deviation variable

At SS : $$A \frac{dh_s}{dt} = F_{is} - F_{os}$$

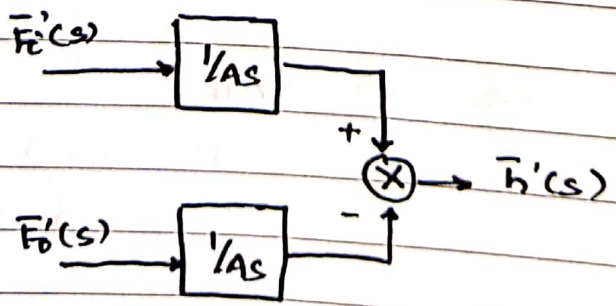
$$A \frac{dh'}{dt} = F_i' - F_o'$$

Step 3 : Determine TF

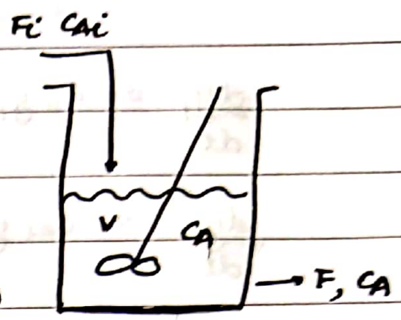
$$As\, \bar{h}'(s) = \bar{F_i}'(s) - \bar{F_o}'(s)$$

$$\bar{h}'(s) = \frac{\bar{F_i}'(s) - \bar{F_o}'(s)}{As}$$

Step 4 : Make the Block Diagram



Transfer fn. of an Isothermal CSTR



$F_i, F \rightarrow$ Volumetric flow rates

Comp. Mass Balance,

$$\frac{dC_A}{dt} = \frac{F_i}{V}(C_{Ai} - C_A) - K_0 e^{-E/RT} C_A$$

↳ non-isothermal

For isothermal, $\frac{dC_A}{dt} = \frac{F_i}{V}(C_{Ai} - C_A) - KC_A$

$$\frac{dC_A}{dt} + \left(\frac{F_i}{V} + K\right) C_A = \frac{F_i}{V} C_{Ai}$$

$$\tau = \frac{V}{F_i}$$

$$\frac{dC_A}{dt} + \left(\frac{1}{\tau} + K\right) C_A = \frac{1}{\tau} C_{Ai}$$

In terms of deviation variables,

$$\frac{dC_A'}{dt} + \left(\frac{1}{\tau} + K\right) C_A' = \frac{1}{\tau} C_{Ai}'$$

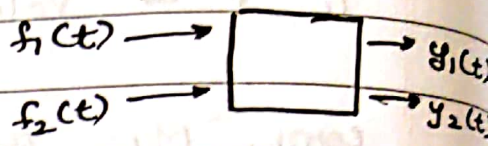
L- Transform, $s\,\bar{C}_A'(s) + \left(\frac{1}{\tau} + K\right) \bar{C}_A'(s) = \frac{1}{\tau} \bar{C}_{Ai}'(s)$

$$G(s) = \frac{\bar{C}_A'(s)}{\bar{C}_{Ai}'(s)} = \frac{1/\tau}{s + \left(\frac{1}{\tau} + K\right)}$$

$$\bar{C}_{Ai}'(s) \longrightarrow \boxed{G(s) = \frac{1/\tau}{s + (1/\tau + K)}} \longrightarrow \bar{C}_A'(s)$$

TF of a MIMO system,

$$\frac{dy_1}{dt} = a_{11} y_1 + a_{12} y_2 + b_{11} f_1 + b_{12} f_2$$



$$\frac{dy_2}{dt} = a_{21} y_1 + a_{22} y_2 + b_{21} f_1 + b_{22} f_2$$

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$y_1, y_2 \rightarrow$ Deviation variables

Taking L-transform and rearranging,

$$\bar{y}_1(s) = \frac{(s-a_{22}) b_{11} + a_{12} b_{21}}{P(s)} \bar{f}_1(s) + \frac{(s-a_{22}) b_{12} + a_{12} b_{22}}{P(s)} \bar{f}_2(s)$$

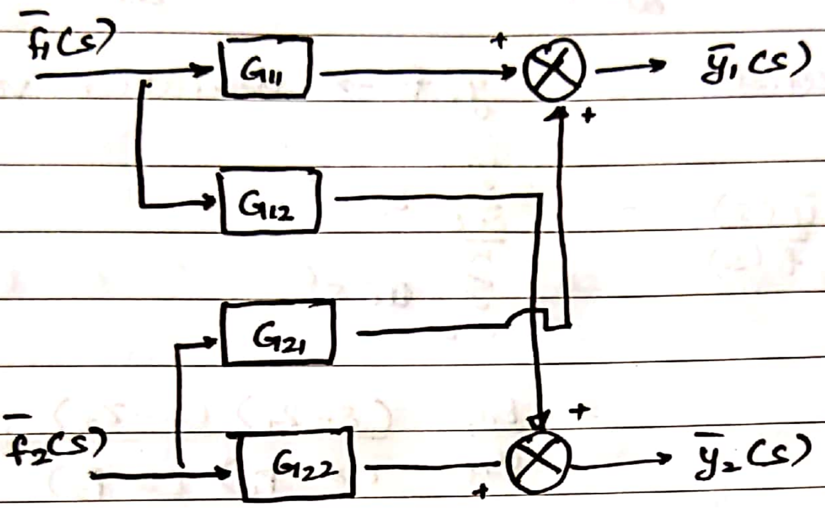
$$\bar{y}_2(s) = \frac{(s-a_{11}) b_{21} + a_{21} b_{11}}{P(s)} \bar{f}_1(s) + \frac{(s-a_{11}) b_{22} + a_{21} b_{12}}{P(s)} \bar{f}_2(s)$$

$$P(s) = s^2 - (a_{11} + a_{22}) s - (a_{12} a_{21} - a_{11} a_{22})$$

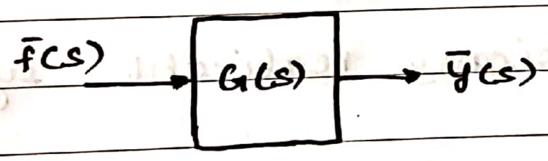
Characteristic Polynomial

$$\bar{y}_1(s) = G_{11}(s) \bar{f}_1(s) + G_{12}(s) \bar{f}_2(s)$$
$$\bar{y}_2(s) = G_{21}(s) \bar{f}_1(s) + G_{22}(s) \bar{f}_2(s)$$

$$\begin{bmatrix} \bar{y}_1(s) \\ \bar{y}_2(s) \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} \bar{f}_1(s) \\ \bar{f}_2(s) \end{bmatrix}$$



Poles and zeros of a TF



$$G(s) = \frac{\bar{y}(s)}{\bar{f}(s)} = \frac{Q(s)}{P(s)}$$

Zeros : The roots of the polynomial $Q(s)$ are the zeros of the system or TF

Poles : The roots of the polynomial $P(s)$ are the poles of a system or TF

At the poles of the system, the TF becomes ∞

At the zeros of a system, the TF becomes 0

General form of a TF

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots a_0 y = b_m \frac{d^m f}{dt^m} + \dots + b_0 f$$

$a, b \rightarrow$ const. , $y, f \rightarrow$ deviation variables

$$G(s) = \frac{\bar{y}(s)}{\bar{f}(s)} = \frac{\sum_{i=0}^{m} b_i s^i}{\sum_{i=0}^{n} a_i s^i}$$

$$= \frac{b_m}{a_n} \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

$z_i \rightarrow$ roots , $p_i \rightarrow$ poles

$n \geq m$ (physically realizable systems)

$$G(s) = \frac{Q(s)}{P(s)} = \frac{Q(s)}{(s-p_1)(s-p_2)(s-p_3)^m (s-p_4)(s-p_4^*) \ast (s}$$

$$G(s) = \frac{c_1}{s-p_1} + \frac{c_2}{s-p_2} + \left\{ \frac{c_{31}}{s-p_3} + \frac{c_{32}}{(s-p_3)^2} + \dots \frac{c_{3m}}{(s-p_3)^m} \right\} + \frac{c_4}{s-p_4} + \frac{c_4^*}{s-p_4^*} + \frac{c_5}{s-p_5}$$

① For real and distinct poles

$$G(s) = \frac{c_1}{s-p_1} + \frac{c_2}{s-p_2}$$

$$L^{-1}[G(s)] = c_1 e^{p_1 t} + c_2 e^{p_2 t}$$

Date ______
Page ______

$P_1 < 0 \qquad c_1 e^{P_1 t} \to 0 \quad$ as $\quad t \to \infty$

$P_2 > 0 \qquad c_2 e^{P_2 t} \to \infty \quad$ as $\quad t \to \infty$

② Multiple Real Poles

$$G(s) = \frac{c_{31}}{s - P_3} + \frac{c_{32}}{(s-P_3)^2} + \cdots \frac{c_{3m}}{(s-P_3)^m}$$

$$L^{-1}[G(s)] = \left(c_{31} + \frac{c_{32}}{1!} t + \frac{c_{33}}{2!} t^2 + \cdots \frac{c_{3m}}{(m-1)!} t^{m-1} \right) e^{P_3 t}$$

$$= W e^{P_3 t}$$

$P_3 > 0, \quad e^{P_3 t} \to \infty \quad$ as $\quad t \to \infty$

$P_3 < 0, \quad e^{P_3 t} \to 0 \quad$ as $\quad t \to \infty$

$P_3 = 0, \quad e^{P_3 t} = 1 \quad$ as $\quad t \to \infty$

$\qquad W \to \infty \quad$ as $\quad t \to \infty$

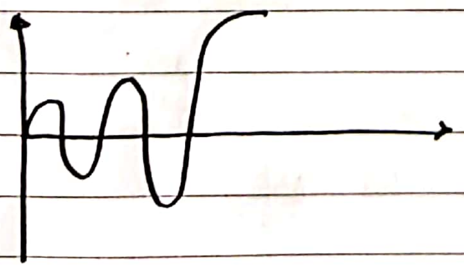
③ Complex conjugate poles

$$G(s) = \frac{c_4}{s - P_4} + \frac{c_4^*}{s - P_4^*} \qquad P_4 = \alpha + j\beta \qquad P_4^* = \alpha - j\beta$$

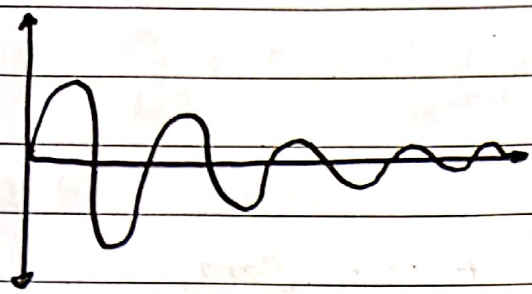
$$L^{-1}[G(s)] = W e^{\alpha t} \sin(\beta t + \phi)$$

$\alpha > 0, \quad e^{\alpha t} \to \infty \quad$ as $\quad t \to \infty \qquad e^{\alpha t} \sin(\beta t + \phi)$

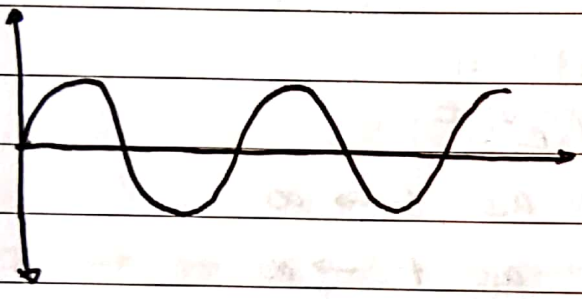
grows to inf in oscillating manner



$\alpha < 0$ $\quad e^{\alpha t} \to 0$ as $t \to 0$ $\quad e^{\alpha t} \sin(\beta t + \phi)$ decays to zero in an oscillating manner



$\alpha = 0$ $\quad e^{\alpha t} \to 1$ as $t \to \infty$ $\quad e^{\alpha t} \sin(\beta t + \phi)$ oscillates cont. with const. amplitude



④ Poles at the origin

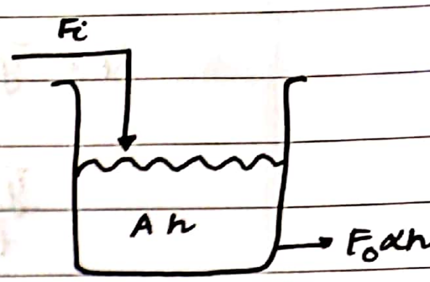
$P_5 = 0$, $\quad G(s) = \dfrac{C_5}{s}$

$L^{-1}[G(s)] = C_5$

A system is stable if all poles of it's TF lie in left of imaginary axis.

eg:-

$$\bar{h}'(s) = \frac{1}{AS}\bar{F_i}'(s) - \frac{1}{AS}\bar{F_o}'(s)$$



$$A\frac{dh}{dt} = F_i - F_o = F_i - \alpha h$$

$$A\frac{dh}{dt} + \alpha h = F_i$$

$$A\frac{dh'}{dt} + \alpha h' = F_i'$$

$$G(s) = \frac{\bar{h}'(s)}{\bar{F_i}(s)} = \frac{1}{sA+\alpha}$$

$$s = -\alpha/A \quad (\text{stable})$$

Dynamics of first order systems

Input f → Process → output (y)

$$a_1\frac{dy}{dt} + a_0 y = b\,f(t)$$

$a, b \rightarrow$ const.

$f \rightarrow$ input, $y \rightarrow$ output

$$\frac{a_1}{a_0}\frac{dy}{dt} + y = \frac{b}{a_0}f(t)$$

Case 1 : $a_1 \neq 0$

Time const. of the process

$$\tau_p = \frac{a_1}{a_0}, \quad K_p = \frac{b}{a_0} \rightarrow \text{steady state gain}$$

gain $= \frac{\Delta \text{output}}{\Delta \text{input}}$

$$K_p = \frac{y_s}{f_s}$$

$$\tau_p\frac{dy}{dt} + y = K_p f(t)$$

$y, f \rightarrow$ deviation variables

$y(0) = f(0) = 0$

$$\tau_p s\,\bar{y}(s) + \bar{y}(s) = K_p\,\bar{f}(s)$$

$$\frac{\bar{y}(s)}{\bar{f}(s)} = \frac{K_p}{\tau_p s + 1} = G(s)$$

Case 2 : $a_0 = 0$

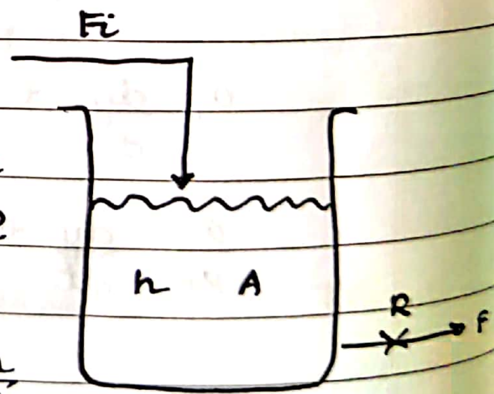
$$a_1 \frac{dy}{dt} = b\,f(t)$$

$$\frac{dy}{dt} = \frac{b}{a_1} f(t) = K_p'\,f(t)$$

L- Transform : $s\,y'(s) = \frac{b}{a_1}\,\bar{f}(s)$

$$G(s) = \frac{\bar{y}(s)}{\bar{f}(s)} = \frac{K_p'}{s}$$

Liquid Tank System



F is linearly related to hydrostatic pressure of h through resistance R

$$F = \frac{\text{Driving force for flow}}{\text{Resistance to flow}} = \frac{h}{R}$$

Model : $A\frac{dh}{dt} = F_i - F_o$

$$A\frac{dh}{dt} + \frac{h}{R} = F_i$$

$$AR\frac{dh}{dt} + h = F_i R$$

$$AR\frac{dh'}{dt} + h' = RF_i' \quad , \quad \tau_p = AR \quad K_p = R$$

$$\tau_p \frac{dh'}{dt} + h' = K$$

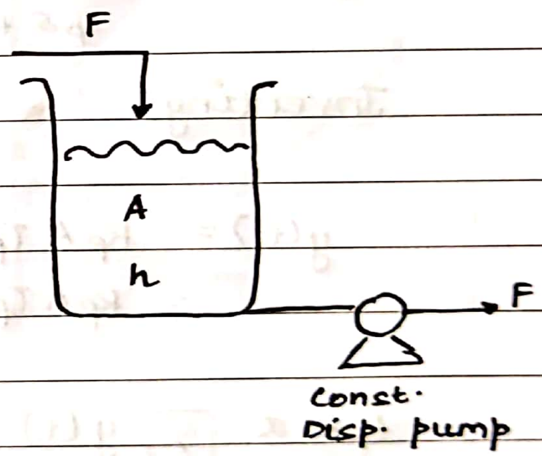
L- Transform : $\tau_p s \bar{h}'(s) + \bar{h}'(s) = K_p \bar{F_i}'(s)$

$$\frac{\bar{h}'(s)}{\bar{F_i}'(s)} = \frac{K_p}{s\tau_p + 1} = \frac{R}{ARs + 1} = G(s)$$

storage capacitance

$\tau_p = AR \rightarrow$ Resistance to flow

Pure Capacitive system



Model

$$A\frac{dh}{dt} = F_i - F$$

At SS : $A\frac{dh_s}{dt} = 0 = F_{is} - F$

$$A\frac{dh'}{dt} = F_i'$$

L- Transform : $As \cdot \bar{h}'(s) = \bar{F_i}'(s)$

$$\frac{\bar{h}'(s)}{\bar{F_i}'(s)} = \frac{1}{As} = \frac{1/A}{s} = \frac{K_p'}{s}$$

Dynamic Behavior : First Order Process (Ramp Input)

$$G(s) = \frac{\bar{y}(s)}{\bar{f}(s)} = \frac{K_p}{\tau_p s+1}$$

Ramp Input : $f(t) = At$, $\bar{f}(s) = A/s^2$

$$\bar{y}(s) = \frac{K_p A}{s^2(\tau_p s+1)} \quad (\because \bar{f}(s) = A/s^2)$$

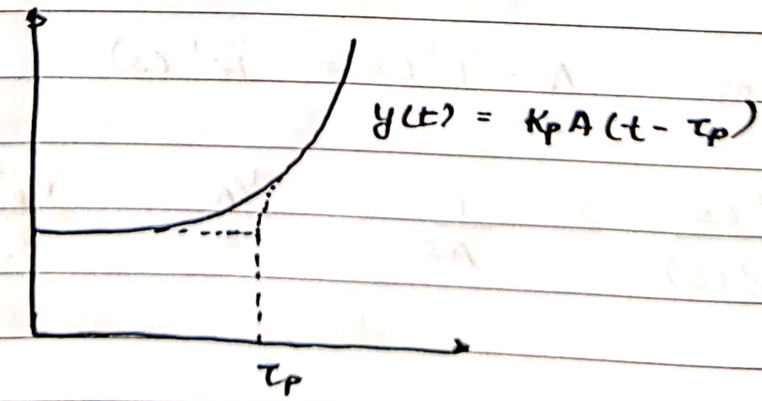
$$= \frac{c_1}{\tau_p s+1} + \frac{c_2}{s} + \frac{c_3}{s^2}$$

$$\bar{y}(s) = \frac{K_p A \tau_p^2}{\tau_p s+1} - \frac{K_p A \tau_p}{s} + \frac{K_p A}{s}$$

Inverting,

$$y(t) = K_p A \tau_p \left(e^{-t/\tau_p} + t/\tau_p - 1\right)$$
$$= K_p A \tau_p e^{-t/\tau_p} + K_p A (t-\tau_p)$$

$t \to \infty$, $y(t) \to K_p A (t-\tau_p)$



Ramp Input yields a ramp output with slope $K_p A$.

Dynamic Behaviour : Pure capacitive Process (step Input)

$$a_1 \frac{dy}{dt} + a_0 y = b f(t)$$

For, $a_0 = 0$, $\frac{\bar{y}(s)}{\bar{f}(s)} = \frac{K_p'}{s}$ ... Pure capacitive

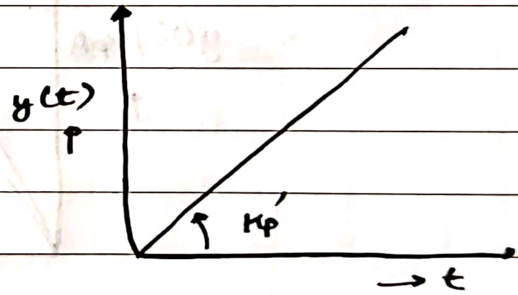
$f(t) = 1$ .... unit step change

$\bar{f}(s) = 1/s$

$\bar{y}(s) = \frac{K_p'}{s^2}$, $\bar{f}(s) = 1/s$

Inverting, $y(t) = K_p' t$ ... linear eqn.

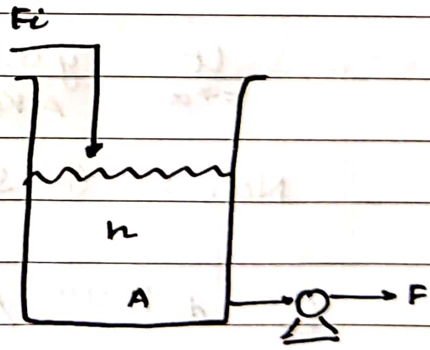
$t \to \infty$, $y(t) \to \infty$



Non self regulating systems

$F_i \uparrow$, $h\uparrow$ Flooding

$F_i \downarrow$, $h\downarrow$ Tank becomes Empty



If there is no pump, with $F_i\uparrow$, $h\uparrow$, then hydrostatic pressure increases, thereby increasing F Equilibrium is established (New ss). If there is no pump the process is self regulating.

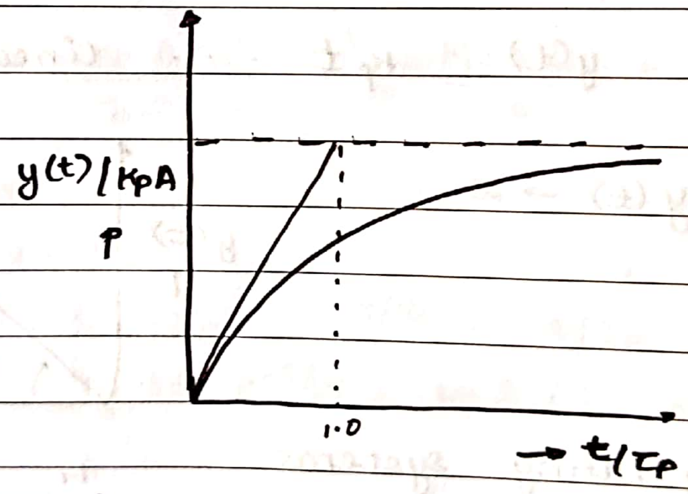
Dynamic Behavior : First order system (Step change)

$$G(s) = \frac{\bar{y}(s)}{\bar{f}(s)} = \frac{K_p}{\tau_p s + 1}$$

$f(t) = A$, $\bar{F}(s) = A/s$

$$\bar{y}(s) = \frac{K_p A}{s(\tau_p s + 1)} = K_p A \left[\frac{1}{s} - \frac{\tau_p}{\tau_p s + 1} \right]$$

$$y(t) = K_p A \left[1 - e^{-t/\tau_p} \right]$$



$$\lim_{t \to \infty} \frac{y(t)}{A K_p} = 1$$

Note : ① Self - Regulating Process

② $$\left. \frac{d\left[y(t)/AK_p \right]}{d(t/\tau_p)} \right|_{t=0} = \left. e^{-t/\tau_p} \right|_{t=0} = 1$$

Initial rate of change of $y(t)$ - maintained - response would remain its final value in one time const.

$$t/\tau_p = 1$$

③ $\frac{y(t)}{A K_p} = 1 - e^{-t/\tau_p}$

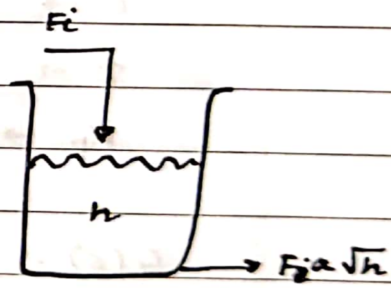
$t/\tau_p = 1$, $\frac{y(t)}{A K_p} = 0.632$

Time	$t = \tau_p$	$2\tau_p$	$3\tau_p$	$4\tau_p$
$\frac{y(t)}{A K_p}$	0.632	0.865	0.95	0.98

④ Gain $= \frac{\Delta \text{ Output}}{\Delta \text{ Input}} = \frac{K_p A}{A} = K_p$

Variable Time const. (τ_p) vs Gain (K_p)

$F = \alpha\sqrt{h}$



$A \frac{dh}{dt} + \alpha\sqrt{h} = F_i$

$\Rightarrow A \frac{dh'}{dt} + \frac{\alpha}{2\sqrt{h_s}} h' = F_i'$

$\tau_p \frac{dh'}{dt} + h' = K_p F_i'$ $\rightarrow$ $\tau_p = \frac{2A\sqrt{h_s}}{\alpha}$

$K_p = \frac{2\sqrt{h_s}}{\alpha}$

τ_p, K_p aren't constant in this case

Dynamics of 2^{nd} order systems

Process : $a_2 \frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = b f(t)$

$a, b \rightarrow$ constants $\quad y \rightarrow$ output

$f \rightarrow$ input

$$\frac{a_2}{a_0} \frac{d^2y}{dt^2} + \frac{a_1}{a_0} \frac{dy}{dt} + y = \frac{b}{a_0} f(t)$$

$$\tau^2 \frac{d^2y}{dt^2} + 2\psi\tau \frac{dy}{dt} + y = K_p f(t)$$

$\tau^2 = \frac{a_2}{a_0}$, $\quad 2\psi\tau = \frac{a_1}{a_0}$, $\quad K_p = \frac{b}{a_0}$

$\tau \rightarrow$ Natural period of oscillation

$\psi \rightarrow$ Damping factor

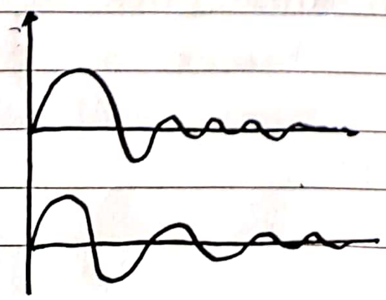
$K_p \rightarrow$ static gain

$$G(s) = \frac{\bar{y}(s)}{\bar{f}(s)} = \frac{K_p}{\tau^2 s^2 + 2\psi\tau s + 1}$$

Damping factor

Small value of ψ - Little Damping

- large oscillations



large Damping

small Damping

$\psi = 0$ Belongs to Undamped system
$\psi < 0$ Unstable system

For all stable systems, $\psi > 0$

Systems with 2^{nd} order or higher order dynamics

① Process consist of two or more capacities in series.
② 1^{st} order system + Controller $\rightarrow$ 2^{nd} order
③ Few processes are inherently 2^{nd} order systems.

Types of second order system Response

$$G(s) = \frac{K_p}{\tau^2 s^2 + 2\psi\tau s + 1} \quad \ldots$$

Poles : $\tau^2 s^2 + 2\tau\psi s + 1 = 0$

$$P = \frac{-2\psi\tau \pm \sqrt{(2\psi\tau)^2 - 4\tau^2}}{2\tau^2}$$

$$P_1 = -\frac{\psi}{\tau} + \frac{\sqrt{\psi^2 - 1}}{\tau} \qquad P_2 = -\frac{\psi}{\tau} - \frac{\sqrt{\psi^2 - 1}}{\tau}$$

If $\psi > 1$, Two distinct real poles (overdamped)
If $\psi = 1$, Two real and equal poles (Critically damped)
If $\psi < 1$, Two complex conjugate poles (underdamped)

Step Response Time Domain soln:

$$G(s) = \frac{\bar{y}(s)}{\bar{f}(s)} = \frac{K_p}{\tau^2 s^2 + 2\psi\tau s + 1}$$

$$f(t) = A \quad , \quad \bar{f}(s) = A/s$$

$$\bar{y}(s) = \frac{K_p A}{s(\tau^2 s^2 + 2\psi\tau s + 1)}$$

Overdamped $(\psi > 1)$

$$y(t) = K_p A \left[1 - e^{-\psi t/\tau} \left\{ \cosh \frac{\sqrt{\psi^2 - 1}}{\tau} t + \frac{\psi}{\sqrt{\psi^2 - 1}} \sinh \frac{\sqrt{\psi^2 - 1}\, t}{\tau} \right\}\right]$$

Critically Damped $(\psi = 1)$

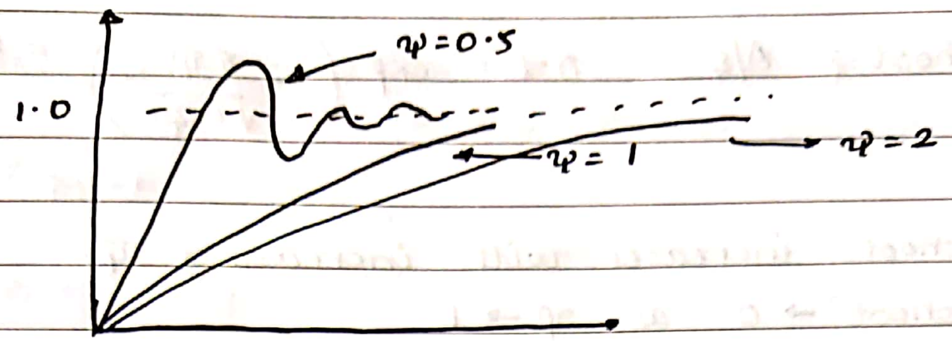
$$y(t) = K_p A \left[1 - e^{-t/\tau}(1 + t/\tau)\right]$$

Under Damped $(\psi < 1)$

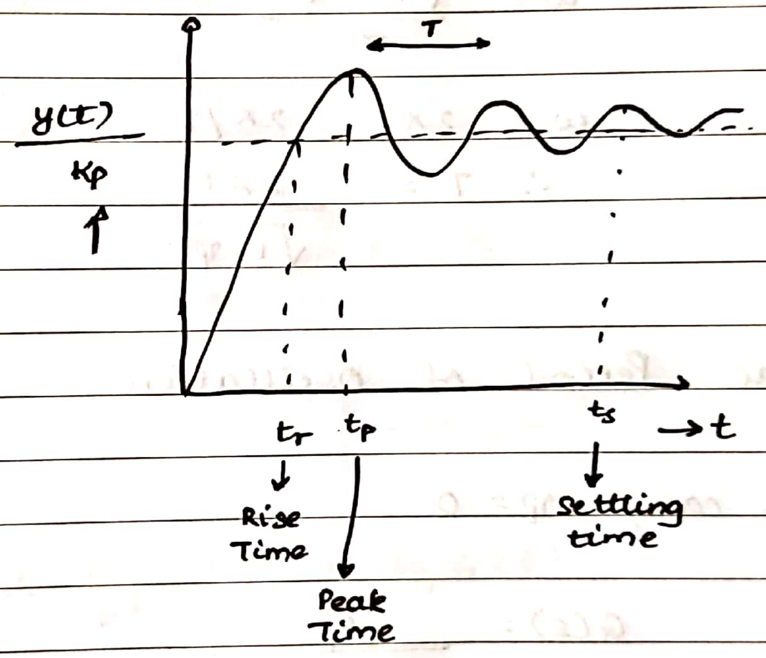
$$y(t) = K_p A \left[1 - e^{-\psi t/\tau} \left(\cos \frac{(1-\psi^2)^{1/2}}{\tau} t + \frac{\psi}{\sqrt{\psi^2 - 1}} \sin \frac{\sqrt{\psi^2 - 1}}{\tau}\right.\right.$$

$$\Rightarrow y(t) = K_p A \left[1 - \frac{1}{\sqrt{1 - \psi^2}} e^{-\psi t/\tau} \sin(\omega t + \phi)\right]$$

$$\omega = \frac{\sqrt{1 - \psi^2}}{\tau} \qquad \phi = \tan^{-1}\left(\frac{\sqrt{1 - \psi^2}}{\psi}\right)$$



Underdamped Response



Rise Time: It is the time req. for the output y to first reach it's final steady state value.

Peak Time: It is the time req. for the output to reach it's first maximum value.

Response Time: It is the time req. for some output to come within some prescribed band of final steady state value.

Decay Ratio: C/A $\quad D.R = \exp\left(\frac{-2\pi\psi}{\sqrt{1-\psi^2}}\right)$

Overshoot : A/B $\quad O \cdot S = \exp\left(\frac{-\pi \psi}{\sqrt{1-\psi^2}}\right)$ = (Decay Ratio)

Overshoot increases with increasing ψ.

Overshoot $\rightarrow 0$ as $\psi \rightarrow 1$

Period of Oscillation

$$\omega = \frac{\sqrt{1-\psi^2}}{\tau}$$

$$\omega = 2\pi f = 2\pi / T$$

$$\therefore T = \frac{2\pi\tau}{\sqrt{1-\psi^2}}$$

Natural Period of Oscillation

In this case, $\psi = 0$

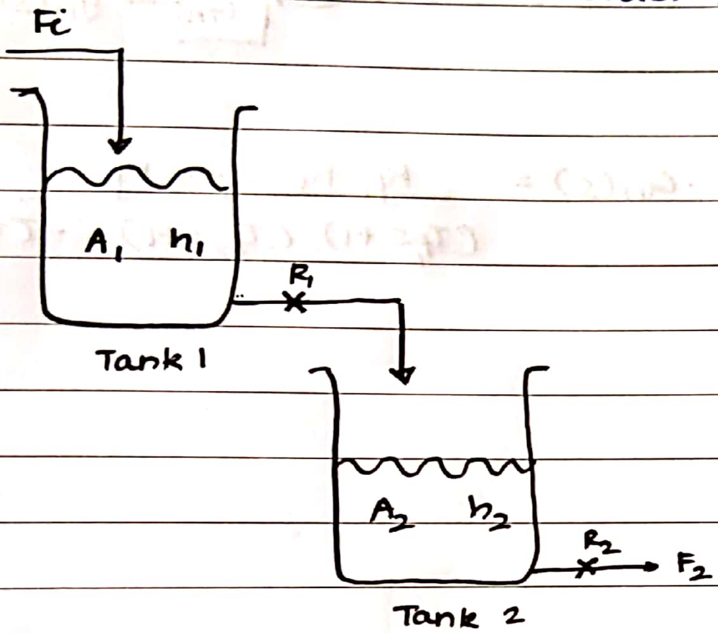
$$G(s) = \frac{K_p}{\tau^2 s^2 + 2\psi\tau s + 1}$$

$$= \frac{K_p'}{\tau^2 s^2 + 1} = \frac{K_p/\tau^2}{\left(s-\frac{i}{\tau}\right)\left(s+\frac{i}{\tau}\right)}$$

$$\omega_n = \frac{1}{\tau}, \quad T_n = \frac{2\pi}{\omega_n} = 2\pi\tau$$

Multicapacity Processes as second order system

$$2^{nd}\text{ Order system}\begin{cases}\text{Two } 1^{st}\text{ order sys in series}\\ 1^{st}\text{ order sys + controller}\end{cases}$$



Non-Interacting System

$$\bar{F}_i(s) \longrightarrow \boxed{G_1} \xrightarrow{y_1(s)} \boxed{G_2} \longrightarrow \bar{y}_2(s)$$

$$G_1 : \tau_{p_1}\frac{dy_1}{dt} + y_1 = K_{p_1} f_1(t)$$

$$G_2 : \tau_{p_2}\frac{dy_2}{dt} + y_2 = K_{p_2} \cancel{\$}\, y_2(t)$$

$$G_1(s) = \frac{\bar{y}_1(s)}{\bar{f}_1(s)} = \frac{K_{p_1}}{\tau_{p_1}s+1}$$

$$G_2(s) = \frac{\bar{y}_2(s)}{\bar{y}_1(s)} = \frac{K_{p_2}}{\tau_{p_2}s+1}$$

$$G_0(s) = \frac{\bar{y}_2(s)}{\bar{f}_1(s)} = G_1(s)\,G_2(s) = \frac{K_{p_1}K_{p_2}}{(\tau_{p_1}s+1)(\tau_{p_2}s+1)}$$

$$= \frac{K_p'}{(\tau')^2 s^2 + 2\varphi'\tau' s + 1}$$

$$\tau' = \tau_{p_1}\tau_{p_2} \quad ; \quad 2\varphi'\tau' = \tau_{p_1} + \tau_{p_2} \quad ; \quad K_p' = K_{p_1}K_{p_2}$$

Poles : $P_1 = -1/\tau_{p_1}$ $\quad P_2 = -1/\tau_{p_2}$ $\quad$ (Overdamped ($\tau_{p_1} \neq \tau_{p_2}$)
(Critically Damped ($\tau_{p_1} = \tau_{p_2}$)

$$\rightarrow \boxed{G_1} \rightarrow \boxed{G_2} \rightarrow \cdots\cdots \rightarrow \boxed{G_n} \rightarrow Y_n(s)$$

$$G_0(s) = G_1(s)\cdots G_n(s) = \frac{K_{p_1} K_{p_2} \cdots K_{p_n}}{(\tau_{p_1}s+1)(\tau_{p_2}s+1)\cdots(\tau_{p_n}s+1)}$$