## Function of RV

Theorem : Let $X$ be a R.V defined on $(\Omega, B, P)$. Let $g: R \to R$ be a measurable fn. Then $Y = g(X)$ is also a R.V

Theorem : Given a R.V $X$ with cdf $F_X()$ the distribution of R.V $Y = g(X)$, where $g$ is measurable, can be determined

Eg :- Let $X$ be a R.V with cdf $F_X()$. Let
$Y_1 = aX + b$, $a \neq 0$, $b \in R$
$Y_2 = |X|$, $Y_3 = X^2$, $Y_4 = \log_e X$ $(X > 0)$, $Y_5 = e^X$
$Y_6 = \max(X, 0)$

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Incase $X$ is a discrete r.v. $P(x_i)$, we can consider

$$g : \{x_1, x_2 \dots\} \to \{y_1, y_2 \dots\}$$

$$P(Y = y_j) = P(g(X) = y_j)$$
$$= \sum_{g(x_i) = y_j} P(X = x_i) = \sum_{g(x_i) = y_j} P_X(x_i)$$

Eg :- $P_X(-2) = \frac{1}{5}$, $P_X(-1) = \frac{1}{6}$, $P_X(0) = \frac{1}{5}$, $P_X(1) = \frac{1}{15}$

$P_X(2) = \frac{11}{30}$

A) $\cancel{Y}$ $Y = X^2 \longrightarrow 0, 4, 1$

$$P_Y(0) = P_X(0) = \frac{1}{5}$$

$$P_Y(1) = P_X(-1) + P_X(1) = \frac{7}{30}$$

$$P_Y(4) = P_X(-2) + P_X(2) = \frac{17}{30}$$

Theorem : Let $X$ be a cont. R.V with pdf $f_X()$. Let $y = g(x)$ is diff. fn for all $x$ and either $g'(x) > 0 \ \forall x$ or $g'(x) < 0 \ \forall x$. Then $Y = g(X)$ is a cont. R.V with pdf

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

where range of $y$ is determined from range of $x$

In case $g$ is strictly decreasing $\left( \frac{dg^{-1}}{dy}(y) < 0 \right)$

So pdf is $f_Y(y) = - f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y)$

$$= f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

Let $X$ have a weibull distn:

$$f_X(x) = \begin{cases} 6x^2 e^{-2x^3} & x > 0 \\ 0 & x \le 0 \end{cases}$$

Let $y = x^3$

$x = y^{1/3}$ $\qquad \frac{dx}{dy} = \frac{1}{3} y^{-2/3}$

Pdf of $y$ is,

$$f_Y(y) = 6y^{2/3} e^{-2y} \cdot \frac{1}{3} y^{-2/3}$$

$$= \begin{cases} 2e^{-2y} & y > 0 \\ 0 & y \le 0 \end{cases}$$

$U = e^{-2y}$

$-\frac{1}{2} \ln u = y$ $\qquad \frac{dy}{du} = \frac{-1}{2u}$

$f_U(u) = +\cancel{2u} \times \frac{+1}{\cancel{2u}} = 1$

$$f_U(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

# Probability Integral Transform

Let $X$ be a cont. R.V with cdf $F_X()$..
Define R.V,

$$Y = F_X(X)$$

Then $Y$ has a uniform distn: on the interval $[0, 1]$

Conversly if $Y$ has $U[0,1]$ distn: and $F$ is a cdf of a cont. R.V, then $X = F^{-1}(y)$ has a cd $F$

Theorem :

Let $X$ be a cont. R.V with pdf $f_X(x)$ and $y = g(x)$ be diff. and let $g_i^{-1}(y)$ $i = 1, \dots k$ $k_i$ be $k$ inverse images. Then the pdf $y$

$$f_Y(y) = \sum_{i=1}^{k} f(g_i^{-1}(y)) \left| \frac{\partial}{\partial y} g_i^{-1}(y) \right|$$

Let $X \sim N(0,1)$, $Y = X^2$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad -\infty < x < \infty$$

$$x = -\sqrt{y}, \quad x = \sqrt{y}$$

$$\frac{dx}{dy} = \frac{-1}{2\sqrt{y}}, \quad \frac{dx}{dy} = \frac{1}{2\sqrt{y}}$$

$$\left| \frac{dx}{dy} \right| = \frac{1}{2\sqrt{y}}$$

So the pdf of $Y$ is

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y/2} \cdot \frac{1}{2\sqrt{y}} + \frac{1}{\sqrt{2\pi}} e^{-y/2} \frac{1}{\sqrt{2y}}$$

$$= \begin{cases} \frac{1}{\sqrt{2\pi y}} e^{-y} & y > 0 \\ 0 & y \le 0 \end{cases}$$

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# Jointly distributed RV

$X_1 \rightarrow$ Marks in course1

$X_2 \rightarrow$ Marks in course2

$\vdots$

$X_k \rightarrow$ Marks in course k

$X = (X_1, X_2 \cdots X_k) : \Omega \rightarrow R^k$

↳ k dimensional R.V

CDF of a random vector

$$F_X(x) = P(X_1 \le x_1, \cdots X_k \le x_k)$$

X :

Let us first take the $k = 2$ ~~$X$~~ $(X, Y)$

$$F_{X,Y}(x, y) = P(X \le x, Y \le y) \quad \forall x, y \in R^2$$

Properities :

① $\lim\limits_{x \to -\infty} F_{X,Y}(x, y) = 0$

② $\lim\limits_{y \to -\infty} F_{X,Y}(x, y) = 0$

③ $\lim\limits_{x \to +\infty} F_{X,Y}(x, y) = F_Y(y)$

④ $\lim\limits_{y \to \infty} F_{X,Y}(x, y) = F_X(x)$

⑤ $F(\cdot, \cdot)$ is non-decreasing in each of it's arguments

⑥ $F(\cdot, \cdot)$ is cont. from right in each of it's arguments

## Discrete case

Suppose both X and Y are discrete. Then the joint pmf $P_{X,Y}(x_i, y_j)$ is defined as

(i) $P(X = x_i, Y = y_j) = P_{X,Y}(x_i, y_j)$

(ii) $P_{X,Y}(x_i, y_j) \geq 0$

(iii) $\sum \sum P_{X,Y}(x_i, y_j) = 1$

Eg:- 10 computers in a showroom

5 (good)

2 (defective monitors)

3 (defective CPU)

Suppose 2 computers are selected at random

$X \rightarrow$ no. of computers with DM

$Y \rightarrow$ no. of computers with DC

$$P_{X,Y}(0,0) = \frac{^5C_2}{^{10}C_2} = \frac{2}{9}$$

$$P_{X,Y}(0,1) = \frac{^5C_1 \, ^3C_1}{^{10}C_2} = \frac{1}{3}$$

| X \ Y | 0 | 1 | 2 | $P_X$ |
|---|---|---|---|---|
| 0 | 10/45 | 15/45 | 3/45 | 28/45 |
| 1 | 10/45 | 6/45 | 0 | 16/45 |
| 2 | 1/45 | 0 | 0 | 1/45 |
| $P_Y$ | 21/45 | 21/45 | 3/45 | |

$$P(X \leq 1, Y \leq 1) = P(0,0) + P(0,1) + P(1,0) + P(1,1)$$
$$= 41/45$$

Marginal pmf of x is

$$P_x(x_i) = \sum_{y_i \in Y} P_{x,y}(x_i, y_j)$$

The marginal pmf of Y is

$$P_Y(y_j) = \sum_{x_i \in X} P_{x,y}(x_i, y_j)$$

The conditional pmf of x ~~is~~ given ~~by~~ $Y = y_j$

$$P_{x, Y=y_j}(x_i \mid y_j) = P(X = x_i \mid Y = y_j)$$

$$= \frac{P_{x,y}(x_i, y_j)}{P_Y(y_j)}$$

Similarly the cond. pmf of Y given $x = x_i$

$$P_{Y, x=x_i}(y_j \mid x_i) = \frac{P_{x,y}(x_i, y_j)}{P_x(x_i)}$$

Find the conditional pmf of $Y \mid X = 0$

$$P_{Y|x=0}(0 \mid 0) = \frac{P_{x,y}(0,0)}{P_x(0)} = \frac{10/45}{28/45} = \frac{10}{28}$$

$$P_{Y|x=0}(1 \mid 0) = \frac{P_{x,y}(0,\cancel{0})}{P_x(0)} = \frac{15}{28}$$

$$P_{Y|x=0}(2 \mid 0) = \frac{P_{x,y}(0,2)}{P_x(0)} = \frac{3}{28}$$

Let $(X,Y)$ be jointly distributed cont. RV with joint pdf $f_{X,Y}(x,y)$

(i) $f_{X,Y}(x,y) \geq 0 \quad \forall \quad (x,y) \in \mathbb{R}^2$

(ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy\,dx$

The marginal pdf of X is

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

and the marginal pdf of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

Conditional pdf of $X|Y=y$ is

$$f_{X|Y=y} = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$f_{Y|X=x} = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

## Independence of RV

We say that RV $X, Y$ are independently distributed if

$$F_{X,Y}(x,y) = F_X(x) \cdot F_Y(y) \quad \forall (x,y) \in \mathbb{R}^2$$

If $X, Y$ are discrete, the cond. is

$$P_{X,Y}(x_i, y_j) = P_X(x_i)\, P_Y(y_j) \;\forall (x_i, y_j)$$

If $X, Y$ are cont.,

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y) \;\forall (x,y) \in \mathbb{R}^2$$

## Joint Expectation

$$E\, g(X,Y) = \sum_{x_i, y_j}\sum g(x_i, y_j)\, P_{X,Y}(x_i, y_j)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{X,Y}(x,y)\, dx\, dy \longleftrightarrow dy\, dx$$

Provided that the series / Integral on the right are absolutely convergent.

## Product Moments

$\mu'_{r,s} = E(X^r Y^s)$

$\rightarrow (r,s)^{th}$ non central pdt. moment

Special cases

$\mu'_{1,0} = \mu_X \qquad \mu'_{0,1} = \mu_Y$

$\mu'_{1,1} = \mu_{X,Y}$

$\mu_{r,s} = E((X-\mu_X)^r (Y-\mu_Y)^s]$

$\mu_{1,1} = E((X-\mu_X)(Y-\mu_Y))$

$\hookrightarrow$ Covariance $(X,Y)$

$= E(XY) - E(X)E(Y)$

Let R.V's $X, Y$ be independent. Suppose they are cont. with $f_X(x)$, $f_Y(y)$

$E(X^r Y^s) = E(X^r)E(Y^s)$

In particular, if $X$ and $Y$ are independent
$\text{cov}(X,Y) = 0$

$\sigma_X^2 = \text{var}(X)$, $\sigma_Y^2 = \text{var}(Y)$, $\sigma_{XY} = \text{Cov}(X,Y)$

Co-efficient of correlation

$$\rho_{X,Y} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

Theorem :

For any jointly distributed RV $(X, Y)$, $-1 \le \rho_{X,Y} \le 1$

Thus $\rho_{X,Y}$ is a measure of linear relationship b/w R.V's X and Y

If $\rho_{X,Y} = 0$ we say that $X, Y$ are linearly uncorrelated.

If $X, Y$ are independent then $\rho_{X,Y} = 0$ But the converse isn't true.

Eg :- ① Find the $\operatorname{corr}(X, Y)$ if $(X, Y)$ is jointly cont. with pdf

$$f(x,y) = \begin{cases} x+y & 0<x<1, \quad 0<y<1 \\ 0 & \text{else} \end{cases}$$

$$E(XY) = \int_0^1 \int_0^1 xy(x+y)\,dx\,dy = 1/3$$

$$f_X(x) = \int_0^1 (x+y)\,dy = x + 1/2 \qquad 0<x<1$$

$$f_Y(y) = \int_0^1 (x+y)\,dy = y + 1/2 \qquad 0<y<1$$

$$E(X) = \int_0^1 x\left(x+\frac{1}{2}\right) dx = 7/12$$

$$E(Y) = 7/12$$

$$E(X^2) = \int_0^1 x^2\left(x+\frac{1}{2}\right) dx = 5/12$$

$$V(X) = \frac{5}{12} - \frac{49}{144} = 11 \ldots = V(Y)$$

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$$\text{cov}(X,Y) = E(XY) - E(X)E(Y)$$

$$= \frac{1}{3} - \frac{49}{144} = -\frac{1}{144}$$

$$\rho_{X,Y} := \frac{\text{cov}(X,Y)}{\sqrt{\text{var}\,X}\,\sqrt{\text{var}\,Y}} = -\frac{1}{11}$$

The joint mgf of $X, Y$ is

$$M_{X,Y}(s,t) = E\left(e^{sX+tY}\right)$$

→ If $X, Y$ are independent

$$M_{X,Y}(s,t) = M_X(s)\,M_Y(t)$$

$$M_{X+Y}(t) = M_X(t)\,M_Y(t)$$

Bivariate Normal Distn:

A cont. jointly distributed R.V $(X,Y)$ is said to have a bivariate normal distn. if it's pdf is

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\, e^{-\frac{1}{2}Q}$$

$$Q = \frac{1}{1-\rho^2}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \left(\frac{y-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right)\right]$$

$(x,y) \in \mathbb{R}^2$

$$Q = \left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \left\{\frac{y-\mu_2}{\sigma_2} - \rho\left(\frac{x-\mu_1}{\sigma_1}\right)\right\}^2$$

Thus

$$\frac{1}{\sigma_1\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma_1}\right)^2}$$

$$e^{-\frac{1}{2\sigma_2^2(1-\rho^2)}\left[y - \left\{\mu_2 + \rho\sigma_2\left(\frac{x-\mu}{\sigma_1}\right)\right\}\right]^2}$$

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$$f_X(x) = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma_1}\right)^2}$$

So $X \sim N(\mu_1, \sigma_1^2)$

$$f_{Y|X=x}(y|x) = \frac{1}{\sigma_2\sqrt{1-\rho^2}\sqrt{2\pi}} e^{-\frac{1}{2\sigma_2^2(1-\rho^2)}\left[y - \left\{\mu_2 + \rho\sigma_2\left[\frac{x-\mu_1}{\sigma_1}\right.\right.\right.}$$

So

$$Y|X=x \sim N\left(\mu_2 + \rho\sigma_2\left(\frac{x-\mu_1}{\sigma_1}\right), \sigma_2^2(1-\rho^2)\right)$$

Similarly,

$$f_Y(y) = \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu_2}{\sigma_2}\right)^2}$$

So $Y \sim N(\mu_2, \sigma_2^2)$

$$f_{X|Y=y}(x,y) = \frac{1}{\sigma_1\sqrt{1-\rho^2}\sqrt{2\pi}} e^{-\frac{1}{2\sigma_1^2(1-\rho^2)}\left[x - \left\{\mu_1 + \rho\sigma_1\left(\frac{y-\mu_2}{\sigma_2}\right)\right\}\right]^2}$$

$x \in \mathbb{R}$

$$X|Y=y \sim N\left(\mu_1 + \rho\sigma_1\left(\frac{y-\mu_2}{\sigma_2}\right), \sigma_1^2(1-\rho^2)\right)$$

Theorem:

$(X,Y)$ have bivariate normal distn. iff the marginal distn. of $X, Y$ and the cond. distn. of $X$ given $Y=y$ and $Y$ given $X=x$ are univariate normal

$(X,Y) \sim BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$

Eg:- $(X,Y) \sim BVN(6, 4, 1, 0.25, 0.1)$

$X \sim N(6,1)$ , $Y \sim N(4, 0.25)$

$Y | X = x \sim N\left(4 + (0.1)(0.5)\left(\frac{x-6}{1}\right), 0.25(1-0.01)\right)$

$X | Y = y \sim N\left(6 + 0.1(1)\left(\frac{y-4}{0.5}\right), (1-0.01)\right)$

$E(g(X,Y)) = E^{Y} E^{X|Y}\{g(X|Y)|Y\}$

Let $(X,Y)$ be jointly distributed R.V's and $g: \mathbb{R}^2 \to \mathbb{R}$ be a measurable fn:

$$E\{g(X,Y)\} = E^{Y} E(g(X,Y)|Y)$$
$$= E^{X} E(g(X,Y)|X)$$

Provided expectation exists

Let $(X,Y) \sim BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$

$X \sim N(\mu_1, \sigma_1^2)$ $\quad Y \sim N(\mu_2, \sigma_2^2)$

$E(X|Y=y) = \mu_1 + \rho\sigma_1\left(\frac{y-\mu_2}{\sigma_2}\right)$ $\quad \sigma_1^2(1-\rho^2) = V(X|Y=y)$

$Y|X=x \sim N\left(\mu_2 + \rho\sigma_2\left(\frac{x-\mu_1}{\sigma_1}\right), \sigma_2^2(1-\rho^2)\right)$

$E(Y|X=x) = \mu_2 + \rho\sigma_2\left(\frac{x-\mu_1}{\sigma_1}\right)$

$V(Y|X=x) = \sigma_2^2(1-\rho^2)$

$Cov(X,Y) = \rho\sigma_1\sigma_2$ , $Corr(X,Y) = \rho$

The joint mgf of $(X, Y)$

$$M_{X,Y}(s,t) = e^{[\mu_1 s + \mu_2 t + \frac{1}{2}\sigma_1^2 s^2 + \frac{1}{2}\sigma_2^2 t^2 + \rho\sigma_1\sigma_2 st]}$$

Theorem : Let $(X,Y) \sim BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$
Then $X, Y$ are independent $\Longleftrightarrow$ $\rho = 0$

Random Vectors

$$X = (X_1, \cdots X_k) : \Omega \to R^k$$

The joint cdf of $X$ is

$$F_X(x) = P(X_1 \le x_1, X_2 \le x_2 \cdots X_k \le x_k)$$

Properities

1. Inorder to get joint cdf of a subset $(X_{i_1}, \cdots X_{i_r})$, $1 \le r < k$ we take $\lim x_j \to \infty$ for $j \ne i_1, \cdots i_r$

2. $\lim_{x_i \to -\infty} F_X(x) = 0 \quad \forall\ i = 1, \cdots k$

3. $F$ is non-decreasing in each of it's arguments.

4. $F$ is cont. from right in each of it's arguments

Incase $(X_1, \cdots X_k)$ is jointly discrete, we have joint pmf $P_X(x)$ satisfying

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(i) $0 \leq p_X(x) \leq 1 \quad \forall \; x \in R^k$

(ii) $\sum\sum \cdots \sum p_X(x) = 1$

(iii) $P_X(x) = P(X_1 = x_1 \cdots X_k = x_k)$

Let $(X_1, \cdots X_k)$ be jointly cont. with pdf $f_X(x)$. Then $f_X(x)$ satisfies

(i) $f_X(x) \geq 0 \quad \forall \; x \in R^k$

(ii) $\int\int\int \cdots f_X(x) \, dx_1 \cdots dx_k = 1$

(iii) $P(X \in A) = \int\int\int_A \cdots f_X(x) \, dx_1 \cdots dx_k$

The joint mgf of $X = (X_1, \cdots X_k)$ is

$$M_X(t) = E\left[e^{(t_1 X_1 + \cdots t_k X_k)}\right]$$

$t = (t_1 \cdots t_k)$

If $X_1, \cdots X_k$ are independently distributed

$$M_Y(t) = \prod_{i=1}^{k} M_{X_i}(t) \quad \text{where } Y = \sum_{i=1}^{k} X_i$$

Additive Properities of some distributions

① Let $X_1 \cdots X_k$ be i.i.d, r.v with
$X_i \sim Bin(n_i, p)$, $i = 1 \cdots k$
Then $Y = \sum_{i=1}^{k} X_1 \sim Bin\left(\sum_{i=1}^{k} n_i, p\right)$

② $X_1 \cdots X_k$ be i.i.d, poisson R.V.'s with
$X_i \sim P(\lambda_i)$, $i = 1 \cdots k$
$Y = \sum_{i=1}^{k} X_i \sim P\left(\sum_{i=1}^{k} \lambda_i\right)$

③ $X_1 \cdots X_k$ be i.i.d, Geo $(p)$. Then
$Y = \sum_{i=1}^{k} X_i \sim$ Neg Bin $(k, p)$

④ Let $X_1 \cdots X_k$ be i.i.d. $Exp(\lambda)$

Then $Y = \sum_{i=1}^{k} X_i \sim$ Gamma $(k, \lambda)$

Linearity property of ND

Let $X_1, X_2 \cdots X_k$ be independent normal R.V.'s with $X_i \sim N(\mu_i, \sigma_i^2)$ $i = 1, \cdots k$

Let $Y = \sum_{i=1}^{k} (a_i X_i + b_i)$

Then $Y \sim N\left(\sum (a_i \mu_i + b_i), \sum a_i^2 \sigma_i^2\right)$

Transformation of RV

Let $f : R^n \to R^m$ be a measurable fn.

$X = (x_1, \cdots x_n)$

$Y_1 = f_1(x_1 \cdots x_n)$

$\vdots$

$Y_m = f_m(x_1, \cdots x_n)$

$Y = (Y_1, \cdots Y_m)$

$Y$ is m-dimensional R.V

For specific types of fn's sometimes mgf is useful. For eg, in finding distn. of sums of independent R.V's

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Eg :- Let $x_1, \dots x_n \overset{i.i.d}{\sim} N(\mu, \sigma^2)$, $Y = \Sigma x_i$

$$M_Y(t) = \prod_{i=1}^{n} M_{x_i}(t) = \prod_{i=1}^{n} (e^{\mu t + \sigma^2 t^2/2})$$

$$= e^{n\mu t + \frac{1}{2} n\sigma^2 t^2} \quad \text{i.e } Y \sim N(n\mu, n\sigma^2)$$

$$\bar{x} = \frac{Y}{n} = \frac{\Sigma x_i}{n} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

If R.V's are discrete, we may have to use pmf directly to derive pmf of transformed variables

Eg :- Let $x, Y \overset{i.i.d}{\sim} B(n, p)$

$U = X + Y \sim Bin(2n, p)$

$V = X - Y$, $v \to -n, -(n-1), \dots -1, 0, 1 \dots n$

$$P(V = v) = P(X - Y = v) = P(X = v + Y)$$

$$= \sum_{y=0}^{n} P(X = v+y, Y = y)$$

$$= \sum \binom{n}{v+y} p^{v+y} (1-p)^{n-v-y} \binom{n}{y} p^y (1-p)^{n-y}$$

$$= \sum \binom{n}{v+y} \binom{n}{y} p^{v+2y} (1-p)^{2n-v-2y}$$

$v + y = 0, 1, \dots n$

② $U = \frac{x}{y+1}$ $\quad V = y+1$

$v \to 1, 2 \cdots (n+1)$

$U \to 0, 1, \frac{1}{2} \cdots \frac{1}{n+1}$

$\vdots$

$n, \frac{n}{2} \cdots \frac{n}{n+1}$

$$P(U = u, V = v) = P(x = uv, y = v-1)$$
$$= P(x = uv)\, P(y = v-1)$$
$$= \binom{n}{nv} p'$$

③ Let $(x, y)$ have joint pdf

| $y \backslash x$ | $-1$ | $0$ | $1$ |
|---|---|---|---|
| $-2$ | $1/6$ | $1/12$ | $1/6$ |
| $1$ | $1/6$ | $1/12$ | $1/6$ |
| $2$ | $1/12$ | $0$ | $1/12$ |

$U = |x| \quad v = y^2$

$U \to 0, 1 \quad v \to 1, 4$

| $v \backslash u$ | $0$ | $1$ |
|---|---|---|
| $1$ | $1/12$ | $1/3$ |
| $4$ | $1/12$ | $1/2$ |

CDF Approach

let $(X, Y)$ have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} \frac{1+xy}{4} & |x|<1, |y|<1 \\ 0 & \text{else} \end{cases}$$

$U = X^2, \quad V = Y^2$

Joint cdf of $(U, V)$, $f_{X,Y}(x,y) = \begin{cases} \frac{1+xy}{4} & |x|<1 \\ 0 & |y|<1 \end{cases}$

$$F_{U,V}(u,v) = P(U \le u, V \le v)$$
$$= P(-\sqrt{u} \le X \le \sqrt{u}, -\sqrt{v} \le y \le \sqrt{v})$$
$$= \int_{-\sqrt{v}}^{\sqrt{v}} \int_{-\sqrt{u}}^{\sqrt{u}} \left(\frac{1+xy}{4}\right) dx\, dy = \sqrt{u}\sqrt{v}$$

Theorem : Let $X = (X_1 \cdots X_n)$ be cont. RV with joint pdf $f_X(x)$, $x = (x_1, \cdots x_n)$

a) let $u_i = g_i(x)$, $i = 1 \cdots n$
$u = (u_1, \cdots u_n) : R^n \to R^n$ be one to one

let $x_i = h_i(u)$, $i = 1, \cdots n$ be inverses

b) Let the fn: and inverse be cont.

c) Assume partial derivatives $\frac{\partial x_i}{\partial u_j}$ $i,j = 1 \cdots n$

exist and cont.

a) Assume Jacobian of transformation

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \cdots & \frac{\partial x_1}{\partial u_n} \\ \vdots & & & \\ \frac{\partial x_n}{\partial u_1} & - & - & \frac{\partial x_n}{\partial u_n} \end{vmatrix} \neq 0$$

in the range of transformation

Then the R. vector $U = (U_1 \cdots U_n)$ is cont. and has joint pdf

$$f_U(x) = f_X(h_1(x) \cdots h_n(x)) \, |J|$$

Eg:- Let $X_1, X_2, X_3 \overset{i.i.d}{\sim} \exp(1)$, so joint pdf of $X = (X_1, X_2, X_3)$

$$f_X(x) = \prod_{i=1}^{3} f_{X_i}(x_i) = e^{-(x_1+x_2+x_3)}$$

Let $Y = (Y_1, Y_2, Y_3)$

$$Y_1 = X_1 + X_2 + X_3 \quad Y_2 = \frac{X_1 + X_2}{X_1 + X_2 + X_3}, \quad Y_3 = \frac{X_1}{X_1 + X_2}$$

$$x_1 = y_1 y_2 y_3$$
$$x_2 = y_1 y_2 (1 - y_3)$$
$$x_3 = y_1 (1 - y_2)$$

$$|J| = -y_1^2 y_2$$

So joint pdf of $Y = (y_1, y_2, y_3)$ is

$$f_Y(y) = \begin{cases} y_1^2 y_2 e^{-y_1} \\ 0 \end{cases}$$

Marginal pdf of $y_1$ is

$$f_{y_1}(y_1) = \begin{cases} \frac{1}{2} y_1^2 e^{-y_1} \\ 0 \end{cases} \qquad \text{Gamma}(3,1)$$

Marginal pdf of $y_2$ is

$$f_{y_2}(y_2) = \begin{cases} 2y_2 \\ 0 \end{cases} \qquad \text{Beta}(2,1)$$

Marginal pdf of $y_3$ is

$$f_{y_3}(y_3) = \begin{cases} 1 \\ 0 \end{cases} \qquad U(0,1)$$

Since $f_y(y) = \pi f_y(y_i)$, $y_1, y_2, y_3$ are also independent.

2) Let $X, Y \overset{i.i.d}{\sim} U(0,1)$

$U = X + Y, \quad V = X - Y$

$$f_{X,Y} = \begin{cases} 1 \\ 0 \end{cases} \qquad x = \frac{u+v}{2}, \quad y = \frac{u-v}{2}$$

$$|J| = \frac{-1}{2}$$

$$f_{u,v}(u,v) = \begin{cases} \frac{1}{2} & 0 < u+v < 2, \; 0 < u-v < 2 \\ & 0 < u < 2, \; -1 < v < 1 \\ 0 & \end{cases}$$

Marginal pdf of $u$

$$f_u(u) = \begin{cases} u & 0 < u < 1 \\ 2-u & 1 < u < 2 \\ 0 & \text{else} \end{cases}$$

Marginal pdf of $v$

$$f_v = \begin{cases} 1+v & -1 < v \leq 0 \\ 1-v & 0 < v < 1 \\ 0 & \text{else} \end{cases}$$

The sum and diff of two independent $U(0,1)$ R.V's have triangular distn.

Theorem : Let $X = (X_1 \ldots X_n)$ be a cont. R. vector with joint pdf $f_X(x)$ and let $u : R^n \to R^n$, $u = (u_1 \ldots u_n)$ $u_i = g_i(x)$, $i = 1 \ldots n$. Suppose that for each $u$ the transformation $g = (g_1 \cdots g_n)$ has finite no. $k = k(u)$ inverses. Suppose that $R^n$ can be partitioned into $k$ disjoint sets $A_1 \ldots A_k$ such that transformation $g$ from $A_i$ into $R^n$ is one to one with inverse transformation. $g$ from $A_i$ into $R^n$ is one - one with inverse transformation.

$$x_1 = h_{1i}(u) \;\ldots\; x_n = h_{ni}(u) \quad i = 1 \ldots k$$

Suppose that first order partial derivatives are cont. and each Jacobian

$$J_i = \begin{vmatrix} \frac{\partial h_{1i}}{\partial u_1} & \cdots & \frac{\partial h_{1i}}{\partial u_n} \\ \vdots & & \\ \frac{\partial h_{ni}}{\partial u_1} & \cdots & \frac{\partial h_{ni}}{\partial u_n} \end{vmatrix} \neq 0$$

in the range of transformation. Then the Joint pdf of $U = (U_1 \cdots U_n)$ is

$$f_U(u) = \sum f_X(h_{1i}(x) \cdots h_{ni}(x)) \, |J_i|$$