Probability

Experiment : Observing something happen or conducting something under certain conditions.

Sample Space : The set of all possible outcomes of a random experiment is called a sample space (Ω, S)

Eg : ① Tossing of a coin $\Omega = \{H, T\}$
② Rolling of a dice $\Omega = \{1, 2, 3, 4, 5, 6\}$

Event : Any subset of a sample space is an event.

$\phi \rightarrow$ Impossible Event
$S, \Omega \rightarrow$ Sure Event

Union $\rightarrow E \cup F$
↳ happening of either E or F or both

$A_1 \cup A_2 \cup A_3 \dots \cup A_\infty = \bigcup_{i=1}^{\infty} A_i$
↳ Occurance of atleast one of A_i

Intersection $E \cap F$
↳ Simultaneous occurance of both, E and F

$\bigcap_{i=1}^{\infty} A_i \rightarrow$ Simultaneous occurance

→ If $\bigcup_{i=1}^{\infty} A_i = \Omega$ then $A_1, A_2 \dots A_n$ are called exhaustive events

→ If $A \cap B = \phi$, then A and B are called disjoint or mutually exclusive events.

If $A_1, A_2 \dots A_n$ are any events and $A_i \cap A_j = \phi$ $i \neq j$ then these are called pairwise disjoint events.

$A^c \rightarrow$ not happening of A

Classical defn: of probability

Suppose a random expt. has N possible outcomes which are mutually exclusive, exhaustive equally likely. Let M of these be favourable to happening of an event E. The probability of event E is defined by

$$P(E) = \frac{M}{N}$$

→ N need not be infinite

→ Definition is circular in nature, as it uses the term equally likely, implying that outcomes are with equal probability.

Relative Frequency

If a random expt. is repeated n times and an event E occurs a_n times in these n trails then

$$P(E) = \lim_{n \to \infty} \frac{a_n}{n}$$

- Actual observations on the random experiment may not be available
- The probility of an event may be zero but the event may be occuring. Eg : $a_n = \log n$
  Then $\frac{a_n}{n} \to 0$

Similarly an event may not be happening but the probability may be one, eg. $a_n = n \sin(\frac{1}{n})$

Axiomatic Definition

Let Ω be a sample space ; B be a set (class) of events. i.e B is a class of subsets of Ω

(i) $E \in B \Rightarrow E^c \in B$
(ii) If $E_1, E_2 \ldots \in B$ then $\bigcup_{i=1}^{\infty} E_i \in B$

Then B is called a σ- field or σ- algebra of subsets of Ω. A consequence of this defn. is that B is closed under the operation of countable intersections, differences etc.

Ω - Sample
B - σ- algebra / σ- field of subsets of Ω
↳ event space

A said function $P: B \to R$ is said to satisfy the ~~b e~~ be a probability fn: if it satisfies the following three axioms:

P1 : $P(E) \geq 0 \quad \forall E \in B$ is probability is always non-negative

P2 : $P(\Omega) = 1$

P3 : Let $\{E_i\}$ be a sequence of pairwise disjoint sets in B, then $P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$

This is countable additivity

Some properities of probability

① $P(\phi) = 0$

Proof :

Let $E_1 = E_2 = E_3 \cdots = E_n = \phi$

$P(\Omega) = P(\Omega) + P(\phi) + P(\phi) + \cdots$

$1 = 1 + P(\phi) + \cdots \quad \Rightarrow \quad P(\phi) = 0$

③ For any finite collection $\{E_1, E_2, E_3,$

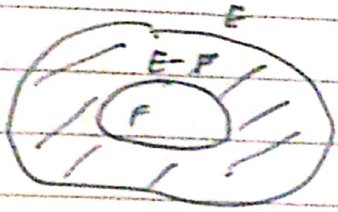
Let us take $E_{n+1} = E_{n+2} = \cdots = \phi$

The $P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i) + P(\phi) + P(\phi) \cdots$

$\Rightarrow P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$

So probability is finitely additive

③ Let $F \subset E$



$E = E \cup (E - F)$

$P(E) = P(E) + P(E - F)$

$\Rightarrow P(F) \leq P(E)$ — ①

and also $P(E-F) = P(E) - P(F)$

Probability is a monotonic fn:

④ since $\phi \subset E \subset \Omega$

for any $E \in B$

$\Rightarrow P(\phi) \leq P(E) \leq P(\Omega)$

$\Rightarrow 0 \leq P(E) \leq 1$ for any event

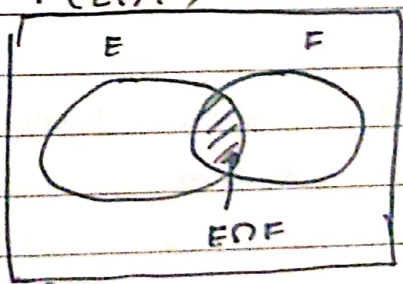
⑤ $\Omega = A \cup A^c$ for any $A \in B$

$P(\Omega) = P(A) + P(A^c)$

$1 = P(A) + P(A^c) \Rightarrow P(A^c) = 1 - P(A)$

⑥ Addition Rule : Let $E, F \in B$, Then

$P(E \cup F) = P(E) + P(F) - P(E \cap F)$



$E \cup F = E \cup (F - (E \cap F))$

So

$P(E \cup F) = P(E) + P(F - (E \cap F))$

$\therefore P(E \cup F) = P(E) + P(F) - P(E \cap F)$

⑦ General addition rule.

For any event $E_1, \cdots E_n \in B$, $P(\bigcup_{i=1}^{n} E_i)$

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i) - \sum_{i<j}\sum P(E_i \cap E_j) + \sum_{i<j<k}\sum\sum P(E_i \cap E_j \cap E_k)$$

$$+ (-1)^{n+1} P\left(\bigcap_{i=1}^{n} E_i\right)$$

⑧ For $E_1, \ldots, E_n \in B$

$$P\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{i=1}^{n} P(E_i)$$

(Sub additivity of probability fn.)

⑨ For any countable sequence $\{E_i\}$ in B,

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} P(E_i)$$

⑩ Bonfesoni Inequality: For any events $E_1, \ldots, E_n \in B$

$$\sum_{i=1}^{n} P(E_i) - \sum_{i<j}^{n}\sum^{n} P(E_i \cap E_j)$$
$$\leq P\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{i=1}^{n} P(E_i)$$

⑪ Boole's Inequality: Let $\{E_n\} \in B$. Then

$$P\left(\bigcap_{i=1}^{\infty} E_i\right) \geq 1 - \sum_{i=1}^{\infty} P(E_i^c)$$

Eg:- Suppose there are n students in a class. Assume $n \leq 365$ and no student has a bday on 29th Feb. What is the probability that no two students share a common bday

A) $A \rightarrow$ No two students share a common b'day

$P(A) = \dfrac{{}^{365}P_n}{(365)^n}$ assuming all dates to be equally likely.

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Eg: let there be 7 balls

A) ① ② ③ ④ ⑤ ⑥ ⑦

One empty cell can be selected in 7 ways. Now one cell will have two balls. This all can be selected in $^6C_1 = 6$ ways

Two balls can be selected out of 7 in $^7C_2$ ways. Remaining 5 balls can be placed in 5 cells in $5!$ ways.

So req. probability $= \dfrac{7 \times 6 \times {}^7C_2 \times 5!}{7^7}$

$= \dfrac{2160}{16807} \approx 0.1285$

Eg:- A pair of fair dice is tossed. If an odd sum or a sum of 4 appears, we stop. Else the toss is continued. What is probability that 4 appears.

A) E → odd sum appears. These are total 18 elements in E

let F → sum is 4, F = {(1, 3), (3, 1), (2, 2)}

P(4 appears) = P(4 appears in 1^{st} + 2^{nd} + ... trail)

$= \dfrac{1}{12} + \dfrac{15}{36} \times \dfrac{1}{12} + \left(\dfrac{15}{36}\right)^2 \times \dfrac{1}{12} + \cdots$

$= \dfrac{1}{7}$

Conditional Probability

Let $(\Omega, \mathcal{B}, P)$ be a probability space. let B be any event with $P(B) > 0$. For any event $A \in \mathcal{B}$, we can define the conditional probability of A given that B has already occured is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$\Rightarrow \quad P(A \cap B) = P(B)\, P(A|B)$$

If we assume $P(A) > 0$, then

$$P(A \cap B) = P(A)\, P(B|A)$$

Let $A_1, A_2 \dots A_n \in \mathcal{B}$ with $P(\bigcap_{i=1}^{n} A_i) > 0$.
Then

$$P(\bigcap_{i=1}^{n} A_i) = P(A_1)\, P(A_2|A_1)\, P(A_3|A_1 \cap A_2) \dots \dots P(A_n | \bigcap_{i=1}^{n-1} A_i)$$

Let $B_1, B_2 \dots B_n$ be pairwise disjoint events with $B = \bigcup_{i=1}^{n} B_i$ and $P(B_j) > 0$, $j = 1 \dots n$

Then $P(A \cap B) = \sum P(A|B_j)\, P(B_j)$

$$(A \cap B) = A \cap (\bigcup_{j=1}^{n} B_j) = \bigcup_{j=1}^{n} (A \cap B_j)$$

$$P(A \cap B) = \sum_{j=1}^{n} P(A \cap B_j)$$

$$= \sum_{j=1}^{n} P(A/B_j)\, P(B_j)$$

When $B = \Omega \Rightarrow B_1, B_2 \cdots B_n$ are exhaustive events, then

$$P(A) = \sum_{j=1}^{n} P(A/B_j)\, P(B_j)$$

This is theorm of total probability

Example : A computer manfacturer processes IC's from suppliess B_1, B_2, B_3 with 40% from B_1, 30% from B_2 and 30% from B_3. Suppose 1% of supply from B_1 is defective, 5% from B_2, 10% from B_3 are randomnly selected IC from the stock is defective.

A) $A \rightarrow$ IC is defectere

$$P(A) = \sum_{j=1}^{3} P(A|B_j)\, P(B_j)$$

$$= 0.01 \times 0.4 + 0.05 \times 0.3 + 0.1 \times 0.3$$

$$= 0.049$$

Baye's Theorem

Let $B_1, B_2 \cdots B_m$ be pairwise disjoint exhaustive events $(\cup B_i = \Omega)$ with $P(B_j) > 0 \; \forall j$

Let $A \in B$, with $P(A) > 0$. Then

$$P(B_i|A) = \frac{P(A|B_i)\, P(B_i)}{\sum_{j=1}^{n} P(A|B_j)\, P(B_j)}$$

Independence of Events

If $P(A|B) = P(A)$

$\Rightarrow P(A \cap B) = P(A) \cdot P(B)$

$\Rightarrow P(B|A) = P(B)$

So, events A and B are said to be independent (statistically) if

$$P(A \cap B) = P(A) \cdot P(B)$$

Similarly A, B, C are independent if

$P(A \cap B) = P(A) \cdot P(B)$

$P(B \cap C) = P(B) \cdot P(C)$

$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$

$A_1, A_2 \cdots A_n$ are independent if

$P(A_i \cap A_j) = P(A_i) P(A_j) \quad \forall \; i<j$

$P(A_i \cap A_j \cap A_k) = P(A_i) P(A_j) P(A_k) \quad \forall \; i<j<k$

$$\cdots \quad P\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} P(A_i)$$

Consider all families with two children and assumes that boys and girls are equally likely

(i) If a family is chosen at random and is found to have a boy what is the probability that the other one is also boy?

(ii) If a child is chosen at random from these families and is found to be a boy what is the probability that the other child in the family is a boy?

A) (i) $\Omega = \{ (b,b), (b,g), (g,b), (g,g) \}$

A → family has a boy
B → second child is also a boy.

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{1/4}{3/4} = \frac{1}{3}$$

(ii) $\Omega = \{ bb, bg, gb, gg \}$

A → child is body $P(A) = 1/2$
B → child has a brother $P(A \cap B) = 1/4$

$$P(B|A) = \frac{1/4}{1/4} = \frac{1}{2}$$

2) There are two types of tubes in an electronic gadget. It will cease to fn: iff one of each kind is defective. The prob that these are defective tube of first kind is 0.1, the second kind is 0.2. It is known that two tubes are defective. What is the probability that the gadget still works?

A) A → Two tubes are defective
B → Gadget still works

$$P(A) = 0.1^2 + 0.2^2 + 2(0.1)(0.2)$$
$$= 0.09$$

$$P(A \cap B) = 0.1^2 + 0.2^2 = 0.05$$

$$P(B|A) = 5/9$$

3) All the bolts in a machine come from either factory A or factory B (both have same chance). The % of defective bolts is 5% from A and 1% from B. Two bolts are inspected

i) If first is found to be defective what is probability that second is also good?

(ii) If first is ~~defective~~ good what is probab that second is good?

A) (i) $G_1 \rightarrow$ first bolt is good

$G_2 \rightarrow$ Second bolt is good

$$P(G_1) = P(\text{first good} \mid A)\, P(A) + P(\text{first good} \mid B)\, P(B)$$

$$= 0.95 \times \frac{1}{2} + 0.99 \times \frac{1}{2} = 0.97$$

$$P(G_1 \cap G_2) = P(\text{both good} \mid A)\, P(A) + P(\text{both good} \mid B)\, P(B)$$

$$= 0.95^2 \cdot \frac{1}{2} + 0.99^2 \cdot \frac{1}{2} = \cancel{202970}$$

$$P(G_2 \mid G_1) = \frac{P(G_1 \cap G_2)}{P(G_1)} = 0.9704$$

$$> 0.97$$

(ii) $D_1 \rightarrow$ first defective

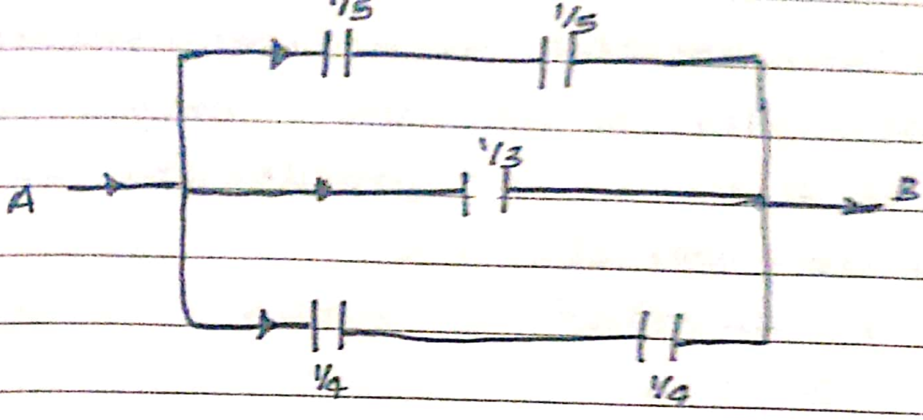
$D_2 \rightarrow$ second defective

$$P(D_1) = 0.05 \cdot \frac{1}{2} + 0.01 \cdot \frac{1}{2} = 0.03$$

$$P(D_1 \cap D_2) = 0.05^2 \cdot \frac{1}{2} + 0.01^2 \cdot \frac{1}{2}$$

$$P(D_2 \mid D_1) = \frac{P(D_1 \cap D_2)}{P(D_1)} = \frac{13}{300} > 0.03$$

4)



An electric network looks as in the above figure, where the numbers indicate the probabilities of failure for various links, which are all independent. What is the probability that the ckt. is closed?

A) Denote three paths by E_1, E_2, E_3 as working

$$P\left(\bigcup_{i=1}^{3} E_i\right) = 1 - P\left(\left(\cup E_i\right)^c\right)$$

$$= 1 - \prod_{i=1}^{3} P(E_i^c)$$

Both working

$$P(E_2^c) = \frac{1}{3} \qquad P(E_1^c) = 1 - \left(\frac{4}{5}\right)^2$$

$$P(E_3) = 1 - \left(\frac{3}{4}\right)^2$$

$$\therefore \; P\left(\bigcup_{i=1}^{3} E_i\right) = \frac{379}{400}$$

Random Variables

A random variable is a real valued function defined on the sample space. Let (Ω, B, P) be a probability space.
A random variable

$$X : \Omega \to R$$
$$B \to C$$
$$P \to Q$$

?? $P(A) \to Q(E)$

$$(\Omega, B, P) \xrightarrow{X} (R, C, Q)$$

Eg :- Suppose a fair coin is tossed once.

$\Omega = \{H, T\}$

$B = \{\phi, \{H\}, \{T\}, \Omega\}$

$P(\phi) = 0 \qquad P(\{H\}) = \frac{1}{2} \qquad P(\{T\}) = \frac{1}{2}$

$X \to$ no. of heads

$X(H) = 1 \quad , \quad X(T) = 0$

$\{0, 1\}$

$C = \{\phi, \{1\}, \{0\}, R\}$

$Q(\{1\}) = \frac{1}{2} \qquad Q(\{0\}) = \frac{1}{2}$

$P(X = 1) = \frac{1}{2} \qquad P(X = 0) = \frac{1}{2}$

Types of Random Variables

A random variable is said to be discrete if it takes finite or countable number of values.

For eg, the number of wins in a series of games, number of children in a family, no. of deaths in a hospital.

If the random variable takes a value over an interval it is said to be cont. r.v.
For example, age of problem, length of a string

To describe prob. distr. of a R.V we have following cases.

① **Probability Mass Function**

If a R.V. X is discrete and it takes values $x_1, x_2, \dots \in X$ it's probability distribution is described by a fn. called probability mass fn. $P_X(x)$ if it satisfies

① $P(X = x_i) = P_X(x_i), \quad x_i \in X$

② $0 \le P_X(x_i) \le 1 \quad \forall \; x_i \in X$

③ $\sum_{x_i \in X} P_X(x_i) = 1$

Eg:- Consider tossing of two fair dice. Let X = sum on the two dice. $X \to 2, 3, \dots, 12$

Then,

$P_X(2) = P(X=2) = \frac{1}{36}, \quad P_X(3) = \frac{2}{36}$

Eg :- Suppose there are 5 ATM's in an office and two are working. A person randomly selects three machines. Let x be the no. of working marchines in his selection. Find probability distribution of x.

$x \to 0, 1, 2$

$$P_x(0) = \frac{^3C_3}{^5C_3} = \frac{1}{10}$$

$$P_x(1) = \frac{^2C_1 \cdot {^3C_2}}{^5C_3} = \frac{6}{10}$$

$$P_x(2) = \frac{^3C_1 \; {^2C_2}}{^5C_3} = \frac{3}{10}$$

Continuous R.V : The probability distribution of a cont. R.V. x is described by a prob. density fn: $f_x(x)$ satisfying :

(i) $f_x(x) \geq 0 \quad \forall \; x \in R$

(ii) $\int_{-\infty}^{\infty} f_x(x)\,dx = 1$

(iii) $P(a < x < b) = \int_a^b f_x(x)\,dx$

Eg :- $f_x(x) = \begin{cases} 1/3 & 1 < x < 4 \\ 0 & \text{otherwise} \end{cases}$

$$\int_{-\infty}^{\infty} f_x(x)\,dx = \int_1^4 \frac{1}{3}\,dx = 1$$

$$P(2 < x < 3) = \int_2^3 \frac{1}{3}\,dx = \frac{1}{3}$$

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$$f_X(x) = \begin{cases} 2x & 0 < 0 < 1 \\ 0 & \end{cases}$$

$$\int_0^1 2x \, dx = 1$$

$$P\left(\frac{1}{4} < x < \frac{1}{3}\right) = \int_{1/4}^{1/3} 2x \, dx$$

$$= \frac{1}{9} - \frac{1}{16} = \frac{7}{16}$$

For cont. R.V the probability of a point is zero i.e

$$P(X = c) = 0 \quad \forall \; c \in \mathbb{R}$$

Cummulative Distribution function of a R.V:
For a R.V X, cdf $F_X(x)$ is defined as

$$F_X(x) = P(\{\omega : X(\omega) \leq x\})$$
$$= P(X \leq x)$$

Properities of CDF :

(i) $\lim_{x \to -\infty} F(x) = 0$

(ii) $\lim_{x \to \infty} F(x) = 1$

(iii) If $x_1 < x_2$, $F(x_1) \leq F(x_2)$

F - cont. from right every point. i.e

$$\lim_{h \to 0} F(x+h) = F(x)$$

Conversely if a fn. F satisfies the above prop, then it is cdf of some RV, X

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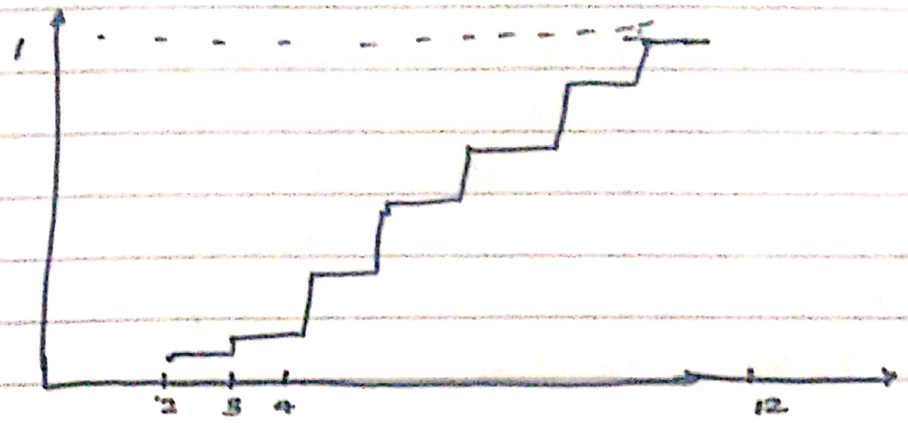
If X is a cont. R.V, then the relationship between pdf and cdf is

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$$ and

$$\frac{d}{dx} F_X(x) = f_X(x)$$

For dice problem $X \rightarrow$ sum

$$F_X(x) = \begin{cases} 0, & x < 2 \\ \frac{1}{36}, & 2 \le x < 3 \\ \frac{3}{36} & 3 \le x < 4 \\ \frac{6}{36} & 4 \le x < 5 \\ \frac{10}{36} & 5 \le x < 6 \\ \frac{15}{36} & 6 \le x < 7 \\ \frac{21}{36} & 7 \le x < 8 \\ \frac{26}{36} & 8 \le x < 9 \\ \frac{30}{36} & 9 \le x < 10 \\ \frac{33}{36} & 10 \le x < 11 \\ \frac{35}{36} & 11 \le x < 12 \\ 1 & x \geq 12 \end{cases}$$



So cdf of a discrete R.v is a step-fn: and the size of discont. at finite or countably infinite no. of points

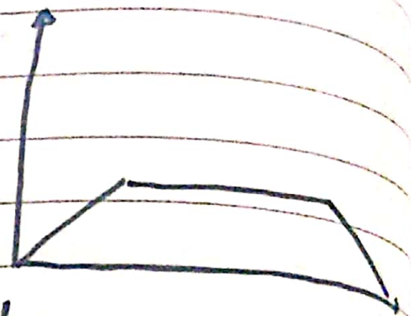
$$f_X(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \begin{cases} 0 & x \le 0 \\ \int_0^x 2t\,dt & 0 < x \le 1 \\ \int_0^1 2t\,dt & x \geq 1 \end{cases}$$

$$= \begin{cases} 0 & x \le 0 \\ x^2 & 0 \le x < 1 \\ 1 & x \geq 1 \end{cases}$$

$$\frac{d}{dx} F_X(x) = \begin{cases} 2x & 0 < x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Eg:- $f_X(x) = \begin{cases} x/2 & 0 < x < 1 \\ 1/2 & 1 \le x < 2 \\ (3-x)/2 & 2 \le x < 3 \\ 0 & \text{otherwise} \end{cases}$



A) $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$

$= 0 \qquad x \le 0$

$= \int_0^x \frac{t}{2}\,dt \qquad 0 \le x \le 1$

$= \int_0^1 \frac{t}{2}\,dt + \int_1^x \frac{1}{2}\,dt \qquad 1 \le x < 2$

$= \int_0^1 \frac{t}{2}\,dt + \int_1^2 \frac{1}{2}\,dt + \int_2^x \frac{3-t}{2}\,dt \qquad 2 \le x$

$= \int_0^1 \frac{t}{2}\,dt + \int_1^2 \frac{1}{2}\,dt + \int_2^3 \frac{3-t}{2}\,dt \qquad x \ge 3$

so,

$F_X(x) = 0 \qquad x \le 0$

$= x^2/4 \qquad 0 < x \le 1$

$= \frac{1}{4} + \frac{x-1}{2}, \qquad 1 \le x < 2$

$= \frac{3}{4} + \left\{ -\frac{(3-t)^2}{4} \right\} \Big|_2^x \qquad 2 \le x < 3$

$= 1 \qquad x \ge 3$

Eg:- $F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1/2 \\ 1 & x > 1/2 \end{cases}$

A) $\frac{d}{dx} F_X(x)$ This isn't cont. from right $x = 1/2$.
So not a cdf

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x < 1/2 \\ 1 & x \geq 1/2 \end{cases}$$

$$\left.\begin{array}{l} f_X(x) = 1, \quad 0 < x < 1/2 \\ P(X = 1/2) = 1/2 \end{array}\right\} \text{Mixed R.V}$$

If a R.V, is partly discrete and partly cont., it is said to be a mixed R.V.

Let X be discrete with pmf $P_X(x_i)$,

$$E(X) = \sum_{x_i \in X} x_i \, P_X(x_i)$$

provided the series on the right is absolutely convergent.

Incase of cont. R.V

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

provided the integral on the right is absolutely convergent.

Eg :- Sum on two dice

$$E(X) = 2 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} + 4 \cdot \frac{3}{36} + 5 \cdot \frac{4}{36} + 6 \cdot \frac{5}{36}$$

$$+ 7 \cdot \frac{6}{36} + 8 \cdot \frac{5}{36} + 9 \cdot \frac{4}{36} + \dots + 12 \cdot \frac{1}{36}$$

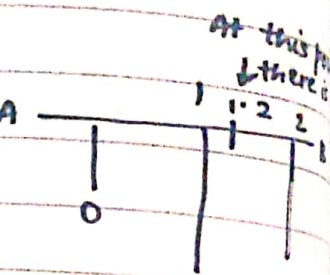
$$= \frac{252}{36} = 7$$

↑ Avg sum

Working ATM eg.

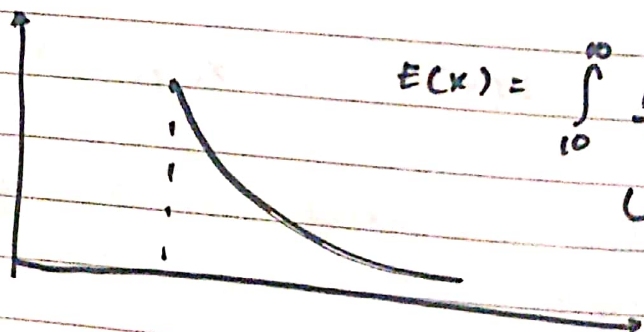
$$E(X) = 0 \cdot \frac{1}{10} + 1 \cdot \frac{6}{10} + 2 \cdot \frac{3}{10}$$

$$= \frac{6}{5} = 1.2$$



$$f_X(x) = \begin{cases} 10/x^2 & x > 10 \\ 0 & \text{otherwise} \end{cases}$$

E(x) → doesn't ex



$$E(X) = \int_{10}^{\infty} \frac{10}{x} \, dx$$

↳ doesn't converge