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Poisson process

We are observing events happening over time / area / space etc.
We say that the occurances / happenings observed in a time scale follow a poisson process they satisfy following assumptions

① No. of occurances in disjoint time int. are independent

② The probability of a single occurance during a small interval is prop. to length of interval

$x(h) \rightarrow$ no. of occurcerences in interval of length (h)

$P(x(h) = 1) = \lambda h$

③ Prob. of more than one occurance in a small time interval is negligable

$x(t) \rightarrow$ no. of occurances in an interval of length t

Under assumption (1) - (3),

$$P_n(t) = \frac{e^{-\lambda t} (\lambda t)^n}{n!}, \quad n = 0, 1, 2 \ldots$$

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$X(t) \rightarrow$ no. of occurances in the interval of length t

$P_n(t) = P(X(t) = n)$
$= P($ occurances in interval $(0, t])$

$P_1(h) = \lambda h + o(h)$

$P_2(h) + P_3(h) + \dots = o(h)$

$1 - P_0(h) - P_1(h) = o(h)$

Under these three assumptions, we have to show

$$P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!} \qquad n = 0, 1, 2 \dots$$

Eg:- Suppose students enter class at a rate of 10 per min.

(i) What is the probability that no student entered in one minute period?

(ii) What is the probability that 20 students entered in a 5 min period?

$\lambda = 10$

$P_0(1) = e^{-\lambda t} = e^{-10} \approx 0.000045$

P (no. student entered in 6sec) $= e^{-60}$

$$P_{20}(5) = \frac{e^{-\lambda t}(\lambda t)^{20}}{20!} = \frac{e^{-50}(50)^{20}}{20!}$$

eg :- Suppose natural disasters take place in ~~an~~ 3 in an year. Probability that there is a disaster once in (i) 6 months (ii) 4 months

A) Unit : 1yr $\lambda = 3$

$$P_0\left(\frac{1}{2}\right) = e^{-\lambda t} = e^{-3/2} \approx 0.22$$

$$P_0\left(\frac{1}{3}\right) = e^{-\lambda t}(\lambda t) = e^{-1} \approx 0.37$$

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad x = 0, 1, 2 \cdots$$

$$\mu_1' = E(x) = \sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!} = \lambda$$

$$\mu_2' = E(x^2) = E\, x(x-1) + E(x)$$
$$= \lambda^2 + \lambda$$

$$\mu_2 = var(x) = \mu_2' - \mu_1'^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

$$\mu_3' = \lambda^3 + 3\lambda^2 + \lambda, \quad \mu_3 = \lambda$$

$$\mu_4' = \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda, \quad \mu_4 = \lambda + 3\lambda^2$$

$$\beta_1 = \frac{\mu_3}{\mu_2^{3/2}} = \frac{\lambda}{\lambda^{3/2}} = \frac{1}{\lambda^{1/2}} > 0$$

So poisson distribution is +vely skewed

$\beta_2 = \frac{\mu_4}{\mu_2^2} - 3$

MGF : $M_X(t) = E(e^{tx})$
$= e^{\lambda(e^t - 1)}$

Poisson distn. as a limiting form of Binomial distn.

$p \to 0 \quad \ni \quad np \to \lambda$

$P_X(x) \to \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, \ldots$

Due to uniqueness property of mgf, it follows that Binomial distn. converges to poisson distn.

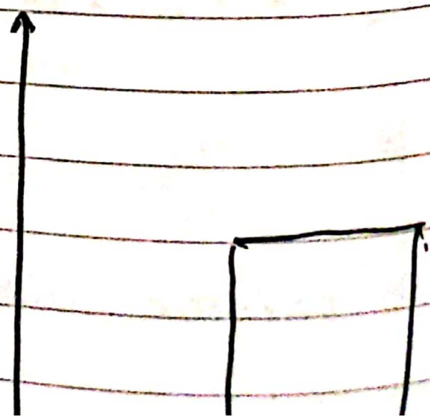
Special Cont. Distributions

(1) Uniform Distribution : If density fn. is uniform / constant over an interval, it's called cont. uniform distribution.

$$f_X(x) = \begin{cases} k & a < x < b \\ 0 & \text{else} \end{cases}$$

$k = \frac{1}{(b-a)}$

Also called rect. distribution.



$$E(X) = \int_a^b \frac{x}{b-a}\,dx = \frac{a+b}{2}$$

$$\mu_k' = \frac{b^{k+1} - a^{k+1}}{(k+1)(b-a)}$$

$$F_X(x) = \begin{cases} 0 & x \le a \\ (x-a)/(b-a) & a < x < b \\ 1 & x \geq b \end{cases}$$

$$M_X(t) = E(e^{tX}) = \int_a^b \frac{e^{xt}}{b-a}\,dx$$

$$= \frac{e^{bt} - e^{at}}{t(b-a)} \qquad t \neq 0$$

$$= 1 \qquad \text{if } t = 0$$

special case : $a = 0, b = 1$

$X \sim U(0,1)$

$$f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{else} \end{cases}$$

$$M_X(t) = \begin{cases} \frac{e^t - 1}{t} & t \neq 0 \\ 1 & t = 0 \end{cases}$$

Consider a poisson process $X(t)$ with root λ

Let Y be the time for first occurance.

$P(Y > y) =$ P(no occurance in $(0, y]$)

$$= P(X(y) = 0) = e^{-\lambda y} \qquad y > 0$$

$$1 \qquad y \le 0$$

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$$F_Y(y) = 1 - P(Y > y) = \begin{cases} 0 & y \leq 0 \\ 1 - e^{-\lambda y} & y > 0 \end{cases}$$

So the pdf of Y is

$$f_Y(y) = \begin{cases} 0 & y < 0 \\ \lambda e^{-\lambda y} & y \geq 0 \end{cases}$$

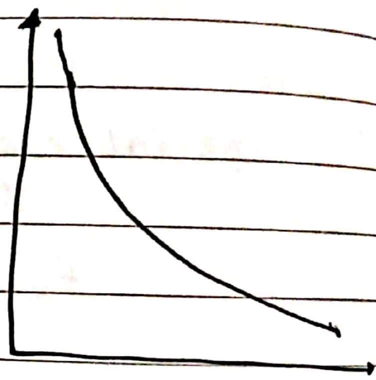
$$\mu_k' = E(Y^k) = \int_0^\infty y^k \lambda e^{-\lambda y}\, dy$$

$$= \frac{\lambda\, \Gamma(k+1)}{\lambda^{k+1}} = \frac{k!}{\lambda^k}, \quad k = 1, 2 \ldots$$

$$\mu_1' = \frac{1}{\lambda} = E(Y)$$

$$\mu_2' = \frac{2}{\lambda^2} \Rightarrow \quad \mu_2 = \frac{1}{\lambda^2}$$

$$\mu_3 = \frac{2}{\lambda^3} \qquad \mu_4 = \frac{1}{\lambda^4}$$



$$\beta_1 = \frac{\mu_3}{\mu_2^{3/2}} = 2 > 0$$

Always positively skewed.

$$\beta_2 = \frac{\mu_4}{\mu_2^2} - 3 = 6 > 0$$

Memoryless Property of Exponential Distribution

Let $Y \sim \exp(\lambda)$

$$P(Y > a) = e^{-\lambda a}$$

$$P(\underbrace{Y > a+b}_{A} \mid \underbrace{Y > b}_{B}) = \frac{P(Y > a+b)}{P(Y > b)} \qquad a > 0$$

$$= \frac{e^{-\lambda(a+b)}}{e^{-\lambda b}} = e^{-\lambda a} = P(Y > a)$$

M·G·F : $M_X(Y) = E(e^{ty})$

$$= \int_0^\infty e^{ty} \lambda e^{-\lambda y}\, dy$$

$$= \frac{\lambda}{\lambda - t}, \quad 0 < t < \lambda$$

Eg:- The time to failure (in moths) x of light bulbs produced at two plants A & B. obeys exponential dist^{n} with means 5 and 2 respectively. A company buys bulbs from both plants but 3 times from B as compared to from A. What is the prob. that a randomnly selected bulb will have a life atleast 5 months.

$$P(X > 5) = P(X > 5 \mid A)\, P(A) + P(X > 5 \mid B)\, P(B)$$

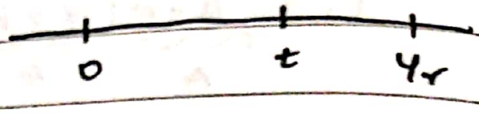
$$f_{X/A}(x) = \frac{1}{5} e^{-x/5} \quad x > 0$$

$$f_{x/B}(x) = \frac{1}{2} e^{-x/2} \quad x > 0$$

$$P(X > 5) = e^{-1} \cdot \frac{1}{4} + e^{-5/2} \cdot \frac{3}{4}$$

$$\approx 0.1534$$

Further consider a Poisson process $X(t)$ with rate λ. Let Y_r denote the time for rth occurance. $(r \geq 1)$. We want distn. of Y_r



$$P(Y_r > t)$$

$$= P(X(t) \leq r-1)$$

$$= \sum_{j=0}^{r-1} P(X(t) = j)$$

$$= \sum_{j=0}^{r-1} \frac{e^{-\lambda t}(\lambda t)^j}{j!}$$

So,

$$F_{Y_r}(t) = \begin{cases} 0 & t < 0 \\ 1 - \sum \frac{e^{-\lambda t}(\lambda t)^j}{j!} & t > 0 \end{cases}$$

So the pdf of Y_r is given by

$$f_{Y_r}(t) = -\frac{d}{dt}\left[e^{-\lambda t} + \lambda t\, e^{-\lambda t} + \frac{(\lambda t)^2}{2!} e^{-\lambda t} + \cdots + \frac{(\lambda t)^{r-1}}{(r-1)!} e^{-\lambda t}\right]$$

$$= -\frac{d}{dt}\left[\cancel{-\lambda e^{-\lambda t}} + \cancel{\lambda e^{-\lambda t}} - \cancel{\lambda^2 t e^{-\lambda t}} + \cancel{\lambda^2 t e^{-\lambda t}} - \frac{\lambda^r t^{r-1}}{(r-1)!} e^{-\lambda t}\right]$$

$$= \frac{1}{(r-1)!} t^{r-1} e^{-\lambda t} \qquad t > 0$$

so pdf of Y_r is

$$f_{Y_r}(t) = \begin{cases} \frac{1}{\Gamma r} \lambda^r t^{r-1} e^{-\lambda t} & t > 0 \\ 0 & t \leq 0 \end{cases}$$

This is gamma or erlang distn:

$$\mu_k' = E(Y_r^k) = \int_0^\infty t^k \frac{1}{\Gamma r} e^{-\lambda t} dt \cdot t^{k+r-1}$$

$$= \frac{\lambda^r}{\Gamma r} \int_0^\infty t^{k+r-1} e^{-\lambda t} dt$$

$$= \frac{\lambda^r}{\Gamma r} \frac{\Gamma k+r}{\lambda^{k+r}} = \frac{\Gamma(k+r)}{\Gamma r} \frac{1}{\lambda^k}, \quad k = 1, 2 \ldots$$

$$\mu_1' = E(Y_r) = \frac{r}{\lambda} \qquad \mu_2' = \frac{r(r+1)}{\lambda^2}$$

$$\mu_2 = var(Y_r) = \frac{r}{\lambda^2}$$

Eg :- The CPU time req. on a system is a R.V X having gamma distribution with mean 40, s.d 20 and any job less than 20s is called a short job. what is probability that out of 5 rand. selected jobs atleast 2 are short jobs.

A → 0.1429

$$\text{mgf}(f_r) = \left(\frac{\lambda}{\lambda - t}\right)^r \quad t < \lambda$$

Weibull Distribution : A cont. r.v. is said to have a weibull distn: with parame α and β (>0) if it has pdf given by

$$f_x(x) = \begin{cases} \alpha\beta\, x^{\beta-1} e^{-\alpha x^{\beta}} & x>0, \ \alpha>0, \beta>0 \\ 0 & x \le 0 \end{cases}$$

$$F_x(x) = \begin{cases} 1 - e^{-\alpha x} & x>0 \quad \alpha>0, \beta>0 \\ 0 & x \le 0 \end{cases}$$

$$\mu_k' = \frac{\Gamma\left(\frac{k+\beta}{\beta}\right)}{\alpha^{k/\beta}}$$

$$\mu_1' = \alpha^{-1/\beta}\, \Gamma(\beta+1)/\beta$$

Let x denote the life of a system/ component.

we define

$R(t) = P(x > t)$ as the reliability of the system at time t

$= P(x > t)$

Inst. Failure Rate of system at time t

$$= \lim_{h \to 0} \frac{1}{h} P(\underbrace{t < x \le t+h}_{A} \mid \underbrace{x > t}_{B})$$

$$= \lim_{h \to 0} \frac{1}{h} \frac{P(t < x \le t+h)}{P(x > t)}$$

$$= \lim_{h \to 0} \left\{ \frac{F_x(t+h) - F_x(t)}{h} \right\} \frac{1}{1 - F_x(t)}$$

$$= \frac{f_x(t)}{1 - F_x(t)} = \frac{f_x(t)}{1 - F_x(t)}$$

$$= \frac{f_x(t)}{R_x(t)} = H_x(t)$$

$\downarrow$

Hazard rate at time t

$$H_x(t) = -\frac{d}{dt}(\log(1 - F_x(t)))$$

$$\log(1 - F_x(t)) = -\int H_x(t)\,dt + c$$

$$1 - F_x(t) = k e^{-\int H_x(t)\,dt}$$

So for a cont. R.V x describing life of a system, there is a one-to-one correspondance b/w the distribution

For exponential distribution :

$$f_x(t) = \begin{cases} \lambda e^{-\lambda t} & t > 0 \\ 0 & \text{else} \end{cases}$$

$$F_x(t) = 1 - e^{-\lambda t}$$

So $R_x(t) = e^{-\lambda t}$, $H_x(t) = \lambda$

So exponential distn., failure rate is const.

Consider Weibull distn.

$$f_X(t) = \alpha \beta t^{\beta-1} e^{-\alpha t^\beta}, \quad t > 0$$

$$F_X(t) = 1 - e^{-\alpha t^\beta} \qquad R_X(t) = e^{-\alpha t^\beta}$$

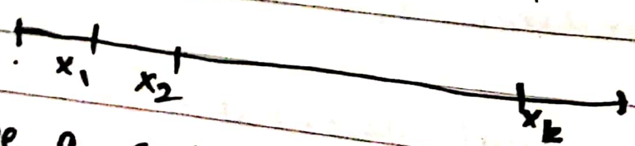
So for $\beta = 1$, it's exponential distn.
For $\beta > 1$, it goes to zero faster than exponential rate
For $\beta < 1$, it goes to zero slower than exponential rate.

$$H_X(t) = \frac{f_X(t)}{R_X(t)} = \frac{\alpha\beta t^{\beta-1} \cancel{e^{-\alpha t^\beta}}}{\cancel{e^{-\alpha t^\beta}}} = \alpha\beta t^{\beta-1}$$

For $\beta = 1$, $H_X(t) = \alpha$ constant
For $\beta > 1$, $H_X(t)$ is increasing in t, so the lifetime has increasing failure rate (IFR)

For $\beta < 1$, $H_X(t)$ is decreasing in t
So the lifetime has decreasing failure rate (DFR)

Relibility of Series system



Suppose a system has k comp. connected in Series. and comp. lives be $X_1, \dots X_k$

So the reliability of system at time t

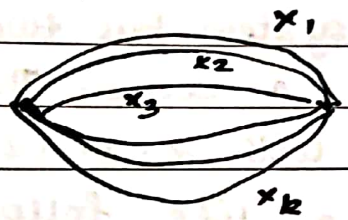
$$R_X(t) = P(X > t)$$
$$= P(X_1 > t, \cdots X_k > t)$$

If comp. lives are assumed to independent

$$R_X(t) = \prod_{i=1}^{k} P(X_i > t) = \prod_{i=1}^{k} R_{X_i}(t)$$

Reliability of a Parallel System

Suppose a system has k independent comp. connected in parallel

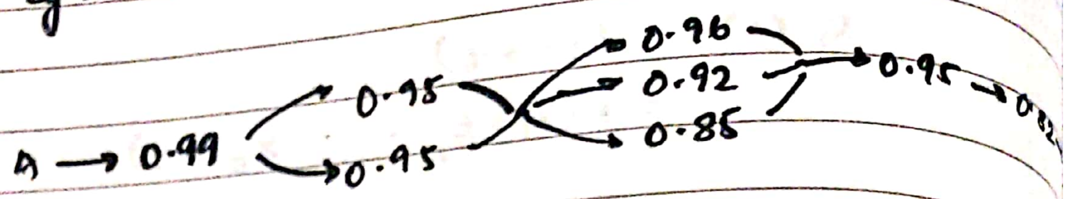


System Reliability

$$R_X(t) = P(X > t) = 1 - P(X \le t)$$
$$= 1 - P(X_1 \le t, X_2 \le t \cdots X_k \le t)$$
$$= 1 - \prod_{i=1}^{k} P(X_i \le t)$$
$$= 1 - \prod_{i=1}^{k} \{1 - R_{X_i}(t)\}$$

Thus, in a series reliability decreases with addn. of comp. In a parallel system the reliability increases with addn. of comp.

Eg :- Find the reliability of the system



$$R_x(t) = (0.99)\{1-(1-0.95)^2\}\{1-(1-0.96)(1-0.9\ldots)(1-0.85)\} \times 0.95 \times 0.\ldots$$

$= 0.7689$

~~Beta Di~~

8) Suppose a system has two independent comp. connected in a series. The life of first comp. is weibull with $\alpha = 0.006$, $\beta = 0.5$. The second has a life following exp. distn. with $\eta = 25000$hrs.

(i) What is ~~probability~~ the reliability at 2500hrs

(ii) What is the probability that the system will fail before 2000hrs?

(iii) What is the system reliability of comp. are connected in parallel?

$X_1 \rightarrow$ first. comp. life

$$R_{X_1}(t) = e^{-\alpha t^{\beta}} = e^{-0.006 t^{1/2}}$$

$X_2 \rightarrow$ second comp. life

$$R_{X_2}(t) = e^{-t/25000}$$

System reliability at time t

$$R_X(x) = e^{-0.006\sqrt{t}} \cdot e^{-t/25000}$$

$$R_X(2500) = e^{-0.006 \times 50} \, e^{-0.1}$$

$$\simeq 0.67$$

$$\cancel{B_X} P(X < 2000) = 1 - R_X(2000)$$

$$= 1 - e^{-0.006 \times \sqrt{2000}} \, e^{-2/25}$$

If they are connected in parallel

$$R_X(2500) = 1 - \left(1 - e^{-0.006 \times 50}\right)\left(1 - e^{-0.1}\right)$$

$$\simeq 0.98$$

Beta Distribution

A R.V X is said to have a Beta distn: with parameters α, β (>0) if it has pdf

$$f_X(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

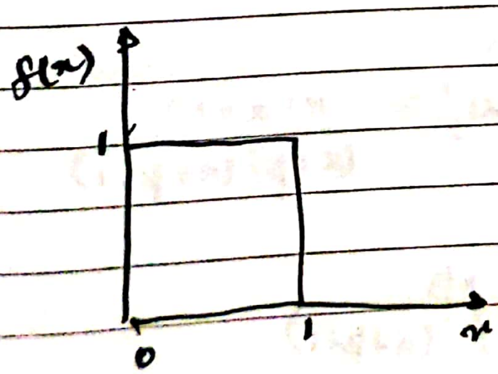
$$0 < x < 1$$

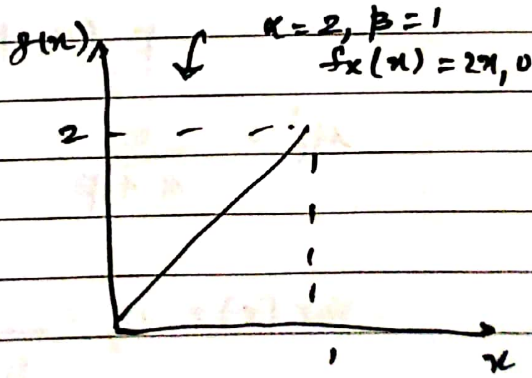
$$\alpha > 0, \ \beta > 0$$

If $\alpha = \beta = 1$, this a $U(0,1)$ distn:

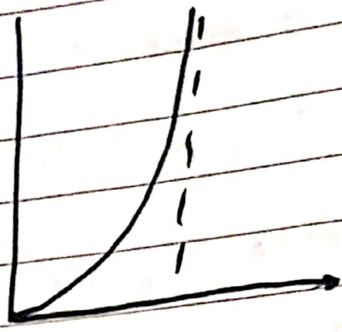
$\alpha = 2, \beta = 1$

$f_X(x) = 2x, \ 0 < x < 1$





(iii)

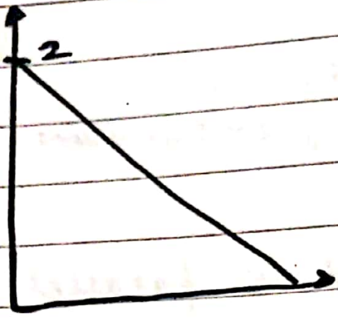


$\alpha = 3,\ \beta = 1$

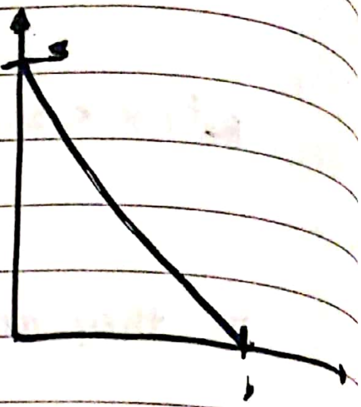
$f(x) = 3x^2$

$0 < x < 1$

(iv)



(v)



$\alpha = 1,\ \beta = 2$

$f_X(x) = 2(1-x)$

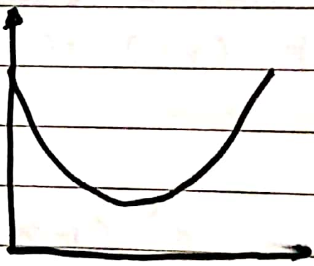
$0 < x < 1$

$\alpha = 1,\ \beta = 3$

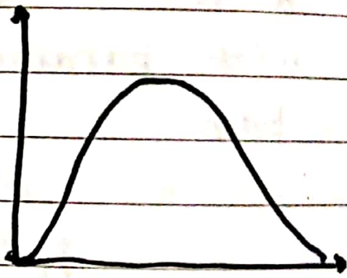
$f_X(x) = 3(1-x)^2$

$0 < x < 1$

(vi) $\alpha = \beta < 1$



(vii) $\alpha = \beta > 1$



so β- distn: can be used to model various kinds of datasets.

$$\mu_k' = \frac{\beta(\alpha+k, \beta)}{\beta(\alpha, \beta)}$$

$$\mu_1' = \frac{\alpha}{\alpha+\beta} \qquad \mu_2' = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

$$Var(x) = \mu_2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

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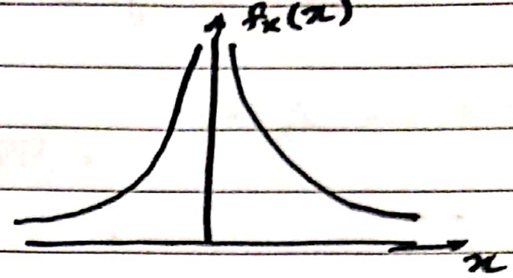
Double Exponential (or) Laplace Distn:

$$f_X(x) = \frac{1}{2\sigma} e^{-\frac{|x-\mu|}{\sigma}}$$

$-\infty < x < \infty$
$-\infty < \mu < \infty$
$\sigma > 0$

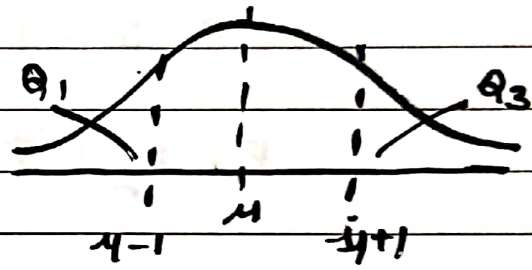
$E(X) = \mu$
$Med(X) = \mu$
$Var(X) = 2\sigma^2$



Cauchy Distn:

$$f_X(x) = \frac{1}{\pi} \cdot \frac{1}{1+(x-\mu)^2}$$

$x \in R, \mu \in R$



Mean doesn't exist

$Med(X) = \mu$

$$F_X(x) = \frac{1}{\pi} \int_{-\infty}^{x} \frac{1}{1+(t-\mu)^2}\, dx$$

$$= \frac{1}{\pi}\left[\tan^{-1}(x-\mu) + \frac{\pi}{2}\right]$$

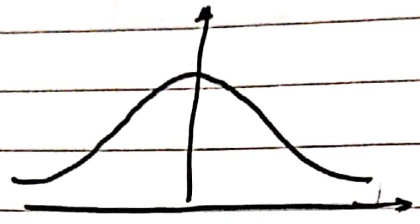
So $F_X(x) = \frac{1}{2}$ when $x = \mu$. So μ is median

Similarly, $x = \mu - 1 = Q_1$
$x = \mu + 1 = Q_3$

Normal Distn:

A cont. R.V X is said to have a normal distn: with mean μ and variance σ^2 if pdf is given by

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad x \in R, \quad \mu \in R, \quad \sigma > 0$$



$$I = \int_{-\infty}^{\infty} f_X(x)\,dx$$

$$z = \frac{x-\mu}{\sigma}$$

$\therefore I = 1$

$$E\left(\frac{X-\mu}{\sigma}\right)^k = \int_{-\infty}^{\infty} z^k \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,dz$$

If $k \in$ odd, then $\mu_k = 0$

So $E\left(\frac{X-\mu}{\sigma}\right) = 0 \Rightarrow E(X) = \mu$

So μ represents the mean of normal distn: So all odd ordered central moments of a normal distn: vanish
In particular $\mu_3 = 0$, $\beta_1 = 0$

For $k = 2m$

$$\mu_{2m} = 2\sigma^{2m} \int_0^\infty z^{2m} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$= (2m-1)(2m-3)\cdots 5\cdot 3\cdot 1\cdot \sigma^{2m}$$

So $\mu_2 = \sigma^2$ i.e σ^2 represents variance of normal distn:

$$\mu_4 = 3\sigma^4$$

Measure of kurtosis $\beta_2 = \frac{\mu_4}{\mu_2^2} - 3 = 0$

Med $(x) = \mu$
Mode $(x) = \mu$

Mgf of normal distn:

$$M_x(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

Linearity property of Normal Distn:

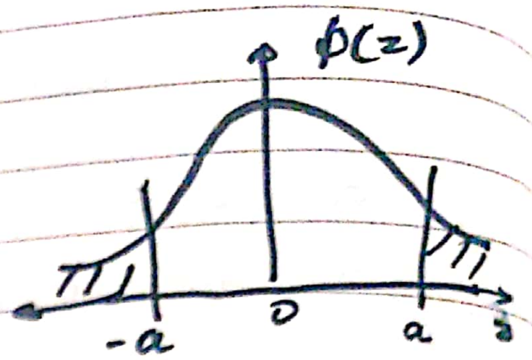
Theorem : Let $x \sim N(\mu, \sigma^2)$ and $Y = ax+b$, $a \neq 0$, $b \in R$. Then $Y \sim N(a\mu + b, a^2\sigma^2)$

If $x \sim N(\mu, \sigma^2)$ then $Z = \frac{x-\mu}{\sigma} \sim N(0,1)$

This is called std. normal distn:

The pdf of Z is

$$f_Z(z) = \phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad z \in \mathbb{R}$$



$$\phi(-z) = \phi(z) \quad \forall z$$

The cdf of Z is

$$\Phi(z) = \int_{-\infty}^{z} \phi(t)\, dt$$

Due to ~~symett~~ symmetry of pdf about 0, we get

$$\Phi(-a) = 1 - \Phi(a)$$
$$\Phi(-a) + \Phi(a) = 1$$
$$\therefore \Phi(0) = 1/2$$

We can use this transformation of any R.V to a std normal R.V for evaluating probabilities related to any normal distn.

Let $X \sim N(\mu, \sigma^2)$

$$P(a < X \leq b) = P(X \leq b) - P(X \leq a)$$
$$= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

Let $Z \sim N(0, 1)$

Eg:- A distance runner completes a one mile race in time $X \sim N(\mu, \sigma^2)$ where $\mu = 241s$ and $\sigma = 2s$. What is the probability that this runner will take less than 4 min? or more than 3 min 55 sec

A) $P(X < 240) = P\left(Z < \frac{240 - 241}{2}\right)$

$= P(Z < -0.5)$

$= \phi(-0.5) = 0.3085$

$P(X > 235) = P\left(Z > \frac{235 - 241}{2}\right) = P(Z > -3)$
$= P(Z < 3)$

$= \phi(3) = 0.9987$

Poisson Approximation to Normal

Let $X \sim P(\lambda)$

As $\lambda \to \infty$ the distn: of $Z = \frac{X - \lambda}{\sqrt{\lambda}} \to N(0,1)$

Consider mgf of Z

After solving,

$M_Z(t) = e^{t^2/2}$ as $\lambda \to \infty$

$\hookrightarrow$ MGF of $N(0,1)$ distn:

Let $X \sim Bin(n, p)$

$Z = \frac{X - np}{\sqrt{npq}}$ As $n \to \infty$
$Z \to N(0,1)$

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The probability that a patient recovers from a rare blood disease is 0.4. If 100 persons are treated what is the probability that less than 30 survive?

A) $X \rightarrow$ survivors $\qquad \sum_{j=0}^{29} \binom{100}{j} (0.4)^j (0.6)$

Then $X \sim Bin(100, 0.4)$

$np = 40, \quad npq = 24, \quad \sqrt{npq} = 4.899$

$$Z = \frac{X-40}{4.899} \approx N(0,1)$$

$$P(\underset{\le 29}{X < 30}) \simeq P(X \le 29.5)$$

$$= P\left(Z \le \frac{29.5-40}{4.899}\right) = P(Z \le -2.14)$$
$$= 0.01624$$

Suppose thefts occur ~~at~~ in hostels like poisson process with $\lambda \doteq 1/2$ / day. What is the probability of not more than 10 thefts in a month? Not less than 17 a month.

$$\sum_{j=0}^{10} e^{-15} \frac{(15)^j}{j!}$$

For a month $x = 15 \qquad \frac{X-15}{\sqrt{15}} \rightarrow N(0,1)$

$$P(X \le 10) \simeq P(X \le 10.5)$$
$$= P\left(\frac{X-15}{\sqrt{15}} \le \frac{10.5-15}{\sqrt{15}}\right) = \phi(-1.16)$$
$$= 0.123$$

$$P(X \ge 17) = 1 - P(X \le 16) \simeq 1 - P(X \le 16.5) = 1 - \phi(0.39)$$
$$0.3483$$

LogNormal Distribution

Let $Y \sim N(\mu, \sigma^2)$

The $X = e^Y$ is said to have a lognormal distn.

$$f_X(x) = \frac{1}{\sigma x\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(\log x - \mu)^2} \qquad x > 0$$

$$\mu \in R, \sigma > 0$$

$$E(X) = E(e^Y) = M_Y(1) = e^{\mu + \sigma^2/2}$$

$$E(X^2) = E(e^{2Y}) = M_Y(2) = e^{2\mu + 2\sigma^2}$$

$$V(X) = e^{2\mu + \sigma^2}(e^{\sigma^2} - 1)$$

$$\mu_k' = E(X^k) = E(e^{kY}) = M_Y(k)$$
$$= e^{\mu k + \sigma^2 k^2/2}$$

eg:- The demand X of a certain item follows a log normal ~~to~~ distn. with mean 7.43 and variance 0.56. Find $P(X > 8)$

A) $\mu_1' = 7.43$, $\mu + \frac{\sigma^2}{2} = 2.0055$

$$\mu_2 = \mu_2' - \mu_1'^2 = 0.56 + (7.43)^2$$

$$2\mu + 2\sigma^2 = 4.0211$$

$\therefore \mu \cong 2$, $\sigma \cong 0.1$ $\qquad \log_e X \sim N(2, (0.1)^2)$

$$P(X > 8) = P(\log X > \log 8)$$
$$= P\left(\frac{\log X - 2}{0.1} > \frac{\log 8 - 2}{0.1}\right)$$

Function of RV

Theorem : Let X be a R.V defined on (Ω, B, P). Let $g : R \to R$ be a measurable fn. Then $Y = g(X)$ is also a R.V

Theorem : Given a R.V X with cdf $F_X()$ the distribution of R.V $Y = g(X)$, where g is measurable, can be determined

Eg :- Let X be a R.V with cdf $F_X()$. Let
$Y_1 = aX + b$, $a \neq 0$, $b \in R$
$Y_2 = |X|$, $Y_3 = X^2$, $Y_4 = \log_e X$ $(X > 0)$, $Y_5 = e^X$
$Y_6 = \max(X, 0)$

Incase X is a discrete r.v. $P(x_i)$, we can consider

$$g : \{x_1, x_2 \dots\} \to \{y_1, y_2 \dots\}$$

$$P(Y = y_j) = P(g(X) = y_j)$$
$$= \sum_{g(x_i) = y_j} P(X = x_i) = \sum_{g(x_i) = y_j} P_X(x_i)$$

Eg :- $P_X(-2) = \frac{1}{5}$, $P_X(-1) = \frac{1}{6}$, $P_X(0) = \frac{1}{5}$, $P_X(1) = \frac{1}{15}$

$P_X(2) = \frac{11}{30}$

A) $\cancel{Y \leftarrow}$ $Y = X^2 \longrightarrow 0, 4, 1$

$$P_Y(0) = P_X(0) = \frac{1}{5}$$

$$P_Y(1) = P_X(-1) + P_X(1) = \frac{7}{30}$$

$$P_Y(4) = P_X(-2) + P_X(2) = \frac{17}{30}$$

Theorem : Let X be a cont. R.V with pdf $f_X()$. Let $y = g(x)$ is diff. fn: for all x and either $g'(x) > 0 \ \forall x$ or $g'(x) < 0 \ \forall x$. Then $Y = g(X)$ is a cont. R.V with pdf

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

where range of y is determined from range of X

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