Definition 3.23: The probability generating function
Let X be nonnegative and integer valued ({0,1,2,...}). The function

$$G_X(s) = E[s^X],\ 0 \leq s \leq 1 \tag{1}$$

is called the *probability generating function* (pgf) of X.

Remark: $G_X(s) = \sum_{k=0}^{\infty} s^k P(X = k)$.

Properties

a) $G_X(0) = P(X = 0)$

b) $P(X = k) = \frac{G_X^{(k)}(0)}{k!}$, $k = 0, 1, \ldots$

c) $E[X] = G'_X(1)$, $\quad$ $\mathrm{Var}[X] = G''_X(1) + G'_X(1) - [G'_X(1)]^2$

d) If $X_1, \ldots, X_n$ are independent with pgf $G_1, \ldots, G_2$, then the pgf of $S_n = X_1 + X_2 + \ldots + X_n$ is

$$G_{S_n}(s) = G_{X_1}(s)G_{X_2}(s) \ldots G_{X_n}(s). \tag{2}$$

In particular, if $X_1, \ldots, X_n$ are iid:

$$G_{S_n}(s) = [G_{X_1}(s)]^n \tag{3}$$

e) Let $X_1, X_2 \ldots$ be iid random variables with values $\in \{0, 1, \ldots\}$ and pgf G_X. If N is a random variable independent of $X_1, X_2, \ldots$ then the pgf of $S_N = X_1 + \ldots 0 X_N$ is

$$G_{S_n}(s) = G_N(G_X(s)) \tag{4}$$

It follow that

$$E[S_N] = E[N]E[X_1]; \qquad \mathrm{Var}[S_N] = E[N]\mathrm{Var}[X_1] + \mathrm{Var}??? \tag{5}$$

Definition 3.24: The moment generating function
Let X be a random variable. The function

$$M_X(t) = E[\mathrm{e}^{tX},\ t \in \mathbb{R} \tag{6}$$

is called the *moment generating funtion* (mgf) of X.

a) $M_X(0) = 1$

b) $M_X(t)$ can be infinite, we say that M_X exists if $M_X(t) < \infty$, for all t.

c) If X is continuous

$$M_X(t) = \int_{-\infty}^{\infty} \mathrm{e}^{tx} f_X(x)\, dx \tag{7}$$

d) If X is discrete with values $\in \{0, 1, \ldots\}$

$$M_X(t) = G_X(\mathrm{e}^t) \tag{8}$$

Properties

a) $M_{aX+b}(t) = \mathrm{e}^{bt} M_X(at)$

b) $E[X^n] = M_X^{(n)}(0), \qquad n = 0, 1, \ldots$ and assume that M_X is differentiable in a neighbourhood of 0.

c) Let $X_1, X_2, \ldots X_n$ be independent random variables with mgf $M_1, M_2, \ldots$ then $M_{S_m}(t) = M_{X_1}(t) M_{X_2}(t) \ldots M_{X_n}(t)$

Definition 3.25: The Poisson process
A Poisson process with rate $\lambda > 0$ is a point process in the line where the time between teo consecutive points are iid random variables$\sim$Exp(λ).

The Poisson process consists of the points $\{T_1, T_1 + T_2, T_1 + T_2 + T_3, \ldots\}$.

Example: It is used to model the time of appearance of an accident e.g. earthquakes.

Notation: Let $X(t) = \{\text{number of points in } [0, t]\}$.

The process is called Poisson because

$$X(t) \sim \text{Poi}(\lambda t) \tag{9}$$

Calcultation of point in an interval:

$$P(X(t) = k) = \mathrm{e}^{-\lambda t} \frac{(\lambda t)^k}{k!} \tag{10}$$

Proposition 3.43 Given $X(I) = n$ the joint distribution of the n points is the same as the distribution of n iid $unif(I)$.

Thinning
Thinning is removing points from a Poisson process; a Poisson process X is thinned with probability $p \in (0, 1)$. We call X_p the *thinned process*.

The thinned process X_p is still a Poisson process and has rate λp.

Proposition
The superposition of two independent Poisson processes is a Poisson process