9 Stochastic simulation

- Maybe talk about number generation
- Define cdf
- List requirements for inverse cdf
- Prove the below proposition
- Prove the Inverse Transformation Method

Det ønskes at simulere $U \sim \text{unif}[0,1]$.

- Approksimér [0,1] ved $\{0, \frac{1}{m}, \dots, \frac{m-1}{m}, 1\}$ for store m
- Ønskes at simulere $U = \frac{Y}{m}$, $Y \sim \text{unif}\{0, 1, \dots, m\}$
- Lad Y_0 være givet ved et seed
  Lad $Y_{n+1} = aY_n + b$
- Så er sekvensen $Y_1, Y_2, \dots$ en sekvens af i.i.d. RVer $\sim \text{unif}\{0, 1, \dots, m\}$

Any discrete distribution X which assumes values $x_1, x_2, \dots$ with probability $p_1, p_2, \dots$ can be simulated by parting $U \sim \text{unif}[0,1]$ into subintervals of the p's and assigning $X = x_k$ if U lands in the interval $[p_{k-1}, p_k]$.

Proposition 5.1 Consider the pmf p on the range $\{x_1, x_2, \dots\}$ and let

$$F_0 = 0, \quad F_k = \sum_{j=1}^{k} p(x_k), \qquad k = 1, 2, \dots \tag{1}$$

Let $U \sim \text{unif}[0,1]$ and let $X = x_k$ if $Fk - 1 < U \leq F_k$. Then X has pmf p.
Bevis
Note that $X = x_k$ if and only if $U \in (F_{k-1} F_k]$, which has probability

$$P(X = x_k) = P(F_{k-1} < U \leq F_k) = F_k - F_{k-1} = p(x_k), \qquad k = 1, 2, \dots \tag{2}$$

If the range is finite $\{x_1, x_2, \dots, x_n\}$ we get $F_n = 1$.

Proposition 5.2 (The Inverse Transformation Method) Let F be a distribution function that is continuous and stricly increasing. Furhter, let $U \sim \text{unif}[0,1]$ and define then random variable $Y = F^{-1}(U)$. Then Y has dsitribution function F.

Bevis
Let F_y be distribution function of Y and let x be in range of Y:

$$\begin{aligned} F_Y(x) &= P(F^{-1}(U) \leq x) \\ &= P(U \leq F(x)) = F_U(F(x)) = F(x) \end{aligned}$$

Proposition 5.3 (The Rejection Method)

1. Generate Y and $U \sim \text{unif}[0,1]$ independent of each other.
2. If $U \leq \frac{f(Y)}{cg(Y)}$, set $X = Y$. Otherwise return to step 1.

The random variable X generated by the algorithm has pdf f.

Bevis
Make sure the algorithm terminates. Calculate the probability of succes:

$$\begin{aligned} P\left(U \leq \frac{f(Y)}{cg(Y)}\right) &= \int_R P\left(U \leq \frac{f(Y)}{cg(Y)}\right) g(y)\, dy \\ &= \int_R \frac{f(y)}{cg(y)} g(y)\, dy = \frac{1}{c}\int_R f(y)\, dy = \frac{1}{c} \end{aligned}$$

where we used the independence of U and Y and the fact that $U \sim \text{unif}[0,1]$. The probability of succes is $1/c$. Now we want to show that the conditional distribution of Y is the same as the distribution of X. Use conditional probabilty:

$$P\left(Y \leq x | U \leq \frac{f(Y)}{cg(Y)}\right) = cP\left(Y \leq x, U \leq \frac{f(Y)}{cg(Y)}\right) \tag{3}$$

With independence we get joint pdf $f(u,y) = g(y)$.

$$\begin{aligned} cP\left(Y \leq x, U \leq \frac{f(Y)}{cg(Y)}\right) &= c\int_{-\infty}^{x}\int_0^{f(y)/cg(y)} g(y)\, dy \\ &= c\int_{-\infty}^{x} \frac{f(y)}{cg(y)} g(y)\, dy = P(X \leq x) \end{aligned}$$