Optimering foregår tit under begrænsninger. F.eks.

$$\min f(x) \qquad s.t. \, x \in C \tag{1}$$

Dette problem er ækvivalent med

$$\min f(x) + T_C(x) \qquad I_C = \begin{cases} 0 & \text{hvis } x \in C \\ \infty & \text{ellers} \end{cases} \tag{2}$$

Subgradient
Lineær tangent som underestimerer funktionen for alle x. Det er en funktion g, som opfylder

$$\frac{f(x) - f(x')}{x - x'} \geq g(x')$$
$$f(x) \geq f(x') + g(x')(x - x)$$

Gradient
Gradienten er en vektor med alle partielle afledede af en funktion.

$$\nabla f(x) = \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \vdots \end{bmatrix} \tag{3}$$

Hessia er den afledede af gradienten – giver en kvadratisk matrix.

$$\nabla(\nabla f(x))^T = \begin{bmatrix} \frac{\partial^2}{\partial x_1^2} & \cdots & \cdots \\ \frac{\partial^2}{\partial x_2 \partial x_1} & \ddots & \\ \vdots & & \frac{\partial^2}{\partial x_N^2} \end{bmatrix} \tag{4}$$

Taylorrækker
Gradienten bruges

$$f(x + \delta) = f(x) + \nabla f(x)^T \delta + \frac{1}{2} \delta^T H(x) \delta + o(\|\delta\|_2^2) \tag{5}$$

Hvor $H(x)$ er Hessia.

Local minimizer $x^\star$
Hvis der gælder, at

$$\|f(x) - f(x^\star)\|_2 < \varepsilon \tag{6}$$

kan erstattes af

$$f(x) - f(x^\star) < \varepsilon \tag{7}$$

så kaldes $x^\star$ en lokal løsning.

Feasible direction

$$x + \delta \qquad \text{new point} \tag{8}$$

where $\delta = \alpha d$, $\alpha \in \mathbb{R}$, $d \in \mathbb{R}^N$.
d – direction vector
α – step size

Let C be the feasible set

$$x + \alpha d \in C \tag{9}$$

d is a feasible direction at x for any range $\alpha \leq \beta$.

Theorem
If $f \in \mathrm{C}^1$ and $x^\star$ is a local minimizer, then $(\nabla f(x))^T d \geq 0$ for any feasible direction d.

Proof Begin at $x^\star$ and take step d.

$$f(x^\star + d) \approx f(x^\star) + (\nabla f(x^\star)^T d \tag{10}$$

Assume $(\nabla f(x^\star)^T d < 0$. Then the left hand side will be smaller than the local minimum $f(x^*)$ which is contradictionary.

Convex function
A convex function complies with

$$(1-\alpha) f(x_1) + \alpha f(x_2) \geq f((1-\alpha)x_1 + \alpha x_2), \qquad \forall \alpha \in [0,1] \tag{11}$$

A convex function has a non decreasing slope, equivalent with the second order derivative being positive.

A vector function (taking vector arguments and scalar values) is convex at x if the Hessian matrix of the function at x is positive semi definite.