Opgave 94
Let X be lognormal with parameters $\mu = 0$ and $\sigma^2 = 1$. Find **(a)** $P(X \leq 2)$, **(b)** $P(X^2 \leq 2)$, **(c)** $P(X > E[X])$, and **(d)** the median m of X.

(a) Gør som i eksempel 2.44

$$P(X \leq 2) = P(\log X \leq \log 2) = \Phi\left(\frac{\log 2 - 0}{\sqrt{1}}\right) \approx 0.755$$

(b) Omskriv $X^2 \leq 2$ til $X \leq \sqrt{2}$.

$$P(X \leq \sqrt{2}) = P(\log X \leq \log \sqrt{(2)}) = \Phi\left(\frac{\log(\sqrt{2}) - 0}{\sqrt{1}}\right) \approx 0.633 \tag{1}$$

(c) Beregn $E[X]$.

$$E[X] = \mathrm{e}^{0+1^2/2} = \mathrm{e}^{1/2} \tag{2}$$

Beregn derefter $P(X \leq E[X])$.

$$P(X > \mathrm{e}^{1/2} = P(\log X > \log \mathrm{e}^{1/2}) = 1 - P(\log X \leq \frac{1}{2}) = 1 - \Phi\left(\frac{1}{2}\right) \approx 0.31 \tag{3}$$

(d) Medianen beregnes ved e^{μ}.

$$m = \mathrm{e}^{\mu} = \mathrm{e}^0 = 1 \tag{4}$$

Opgave 73
The element *nobelium* has a half-life of 58 min. Let X be the lifetime of an individual nobelium atom. Find **(a)** $P(X > 30)$, **(b)** $P(X \leq 60|X > 30)$, and **(c)** $E[X]$ and $\mathrm{Var}[X]$.

(a) Find λ ved hjælp af halveringstiden.

$$P(X > 58) = \frac{1}{2} = \mathrm{e}^{-58\lambda} \Rightarrow \lambda = \frac{\log 2}{58} \tag{5}$$

Beregn derefter $P(X > 30)$.

$$P(X > 30) = \mathrm{e}^{-30 \log 2/58} \approx 0.6987 \tag{6}$$

(b) Brug definition 1.4.

$$P(X < 60|X \geq 30) = \frac{P(X < 60 \cap X \geq 30)}{P(X \geq 40)} = \frac{\mathrm{e}^{\frac{-30 \log 2}{58}} - \mathrm{e}^{\frac{-60 \log 2}{58}}}{\mathrm{e}^{\frac{-30 \log 2}{58}}} \approx 0.30127 \tag{7}$$

(c) Brug definitioner.

$$
\begin{aligned}
E[X] &= \frac{1}{\lambda} = \frac{58}{\log 2} \approx 83.7 \\
\mathrm{Var}[X] &= \frac{1}{\lambda^2} = \frac{58^2}{\log 2^2} \approx 7002
\end{aligned}
$$

Opgave 81

Two species of fish have weights that follow normal distributions. Species A has mean 20 and standard deviation 2; species B has mean 40 and standard deviation 8. Which is more extreme: a 24-pound A-fish or a 48-pound B-fish?

Der skal beregnes en Z-score.

$$
Z = \frac{X - \mu}{\sigma} \tag{8}
$$

Jo højere Z-score, jo lavere sandsynlighed.

$$
\begin{aligned}
Z_A &= \frac{24 - 20}{2} = 2 \\
Z_B &= \frac{48 - 40}{8} = 1
\end{aligned}
$$

En 24-punds A-fisk er mindst sandsynlig.

Opgave 92

Let (X, Y) be uniformly distributed on the triangle with corners in $(0,0)$,$(0,1)$, and $(1,0)$. **(a)** Compute the correlation coefficient $\rho(X, Y)$. **(b)** If you have done (a) correctly, the value of $\rho(X,Y)$ is negative. Explain intuitively.

(a) Brug definitioner.

$$
\rho(X, Y) = \frac{\mathrm{Cov}[X, Y]}{\sqrt{\mathrm{Var}[X]\mathrm{Var}[Y]}} \tag{9}
$$

Kovarians beregnes.

$$
\begin{aligned}
\mathrm{Cov}[X, Y] = E[XY] - E[X]E[Y] &= \int_0^1 \int_0^{1-x} 2xy \, dy \, dx - \int_0^1 \int_0^{1-x} 2x \, dy \, dx \\
&= \frac{1}{12} - \frac{1}{9} = -\frac{1}{36}
\end{aligned}
$$

Og varians.

$$\begin{aligned}\mathrm{Var}[X] = E[X^2] - (E[X])^2 &= \int_0^1 \int_0^{1-x} 2x^2 \, dy \, dx - \left(\int_0^1 \int_0^{1-x} 2x \, dy \, dx \right)^2 \\ &= \frac{1}{6} - \frac{1}{9} = \frac{1}{18}\end{aligned}$$

Slutteligt

$$\mathrm{Cor}[X, Y] = \frac{-\frac{1}{36}}{\frac{1}{18}} = -\frac{1}{2} \tag{10}$$

(b) Korrelationskoefficienten er negativ, da y generelt aftager, når x vokser.

Opgave 107

Let (X, Y) be bivariate normal with means 0, and variances 1, and correlation coefficient $\rho > 0$, and let $U = X + Y, V = X - Y$. What is the joint distribution of (U, V).