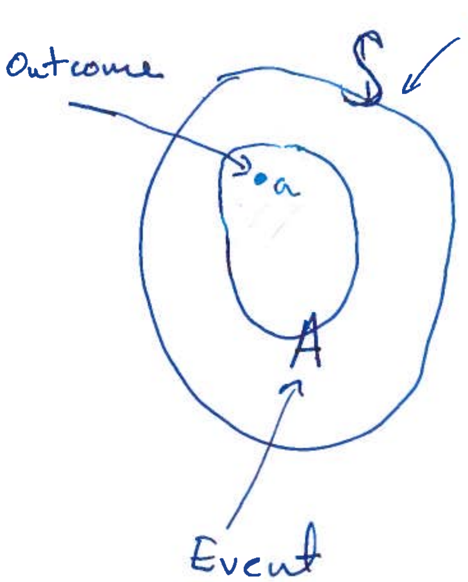
TPe 8/9-2014

Stochastic Processes

Lecture 1: Recap of random variables and vectors

Probability Space $(\mathcal{S}, \mathcal{A}, \mathbb{P})$



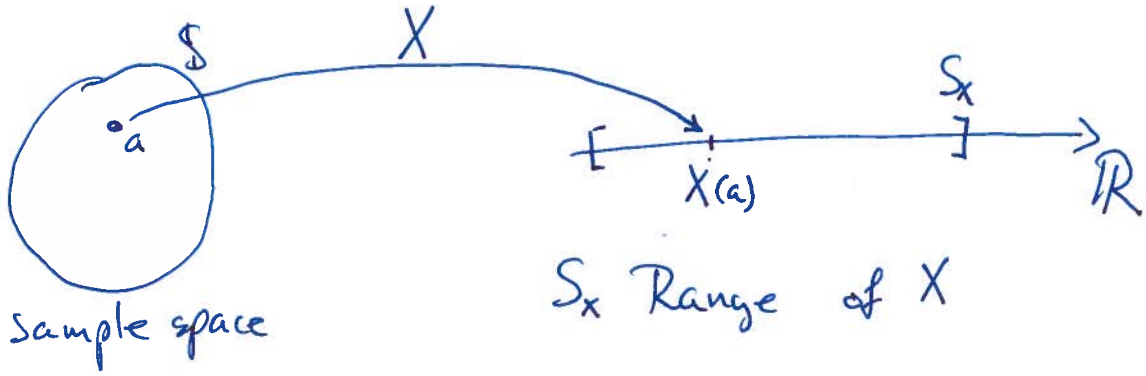
$\mathbb{P}$ is a function that assigns a number in the interval $[0,1]$ to each event in the (σ-algebra) $\mathcal{A}$.

$$\mathbb{P}(\mathcal{S}) = 1 \quad , \quad \mathbb{P}(\phi) = 0$$

For A, B disjoint, $\mathbb{P}(A \cup B) = P(A) + P(B)$.

Random Variable

is a mapping (or function) from $\mathcal{S}$ to a real number in the range S_X



- A "random variable" is not a variable, but a function!
- We often suppress the explicit mention of $\mathcal{S}$ and (a) and say

  "...the random variable X"

Continuous Random Variable

- A continuous r.v. has a continuous (uncountable) set as range.

  Ex: $S_X = \mathbb{R}$, $S_X = [0,1]$

- Characterized by the prob. density function (pdf) p_X

$$\mathbb{P}(a \leq X \leq b) = \int_a^b p_X(x)\, dx$$

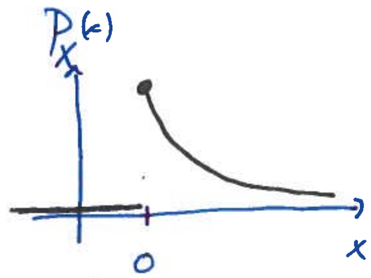
- p_X is non-negative and integrates to 1.

$$\int_{-\infty}^{\infty} p_X(x)\, dx = 1, \quad p_X(x) \geq 0.$$

Examples of R.V.s

Exponential r.V.:

$$P_X(x) = \begin{cases} \lambda \exp(-\lambda x), & x \geq 0 \\ 0 & x < 0 \end{cases}$$

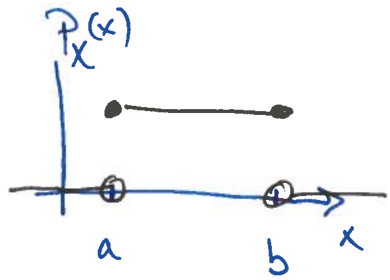


Short-hand: $X \sim Exp(\lambda)$

→ Often seen in queing theory

Uniform r.V.:

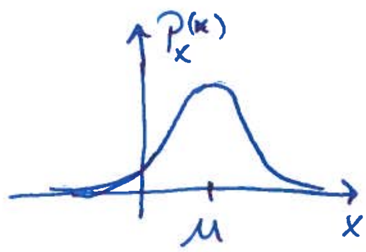
$$P_X(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$



Short-hand: $X \sim \mathcal{U}(a,b)$

→ Most computer systems can generate uniform pseudo r.v. so they often appear in simulations.
→ Round-off errors are approximately uniform

Gaussian (or Normal) r.v.:



$$P_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

Shorthand: $X \sim \mathcal{N}(\mu, \sigma^2)$

→ Electric noise is often Gaussian
→ Sums of many (independent) r.V.s are approx. Gaussian (CLT)

Expectation operator

Def: The expectation of a function of a r.v. is

$$E[g(X)] = \int g(x)\, p_X(x)\, dx$$

Special Cases:

Mean: Expectation of X : $\mu = E[X] = \int x\, p_X(x)\, dx$

Variance: Expectation of $(X-\mu)^2$:

$$\sigma^2 = \mathrm{Var}(X) = E[(X-\mu)^2]$$

$$= \int (x-\mu)^2\, p_X(x)\, dx$$

Mean square (second moment): Expect. of X^2 :

$$E[X^2] = \int x^2\, p_X(x)\, dx$$

Remark: $E[X^2] = \sigma^2 + \mu^2$

Expectation is a linear operator!

$$\begin{cases} E[g_1(X) + g_2(X)] = E[g_1(X)] + E[g_2(X)] \\ E[a \cdot g(X)] = a \cdot E[g(X)] \end{cases}$$

Prove this!

Suppose that X is a r.v. with mean 1 and variance 2

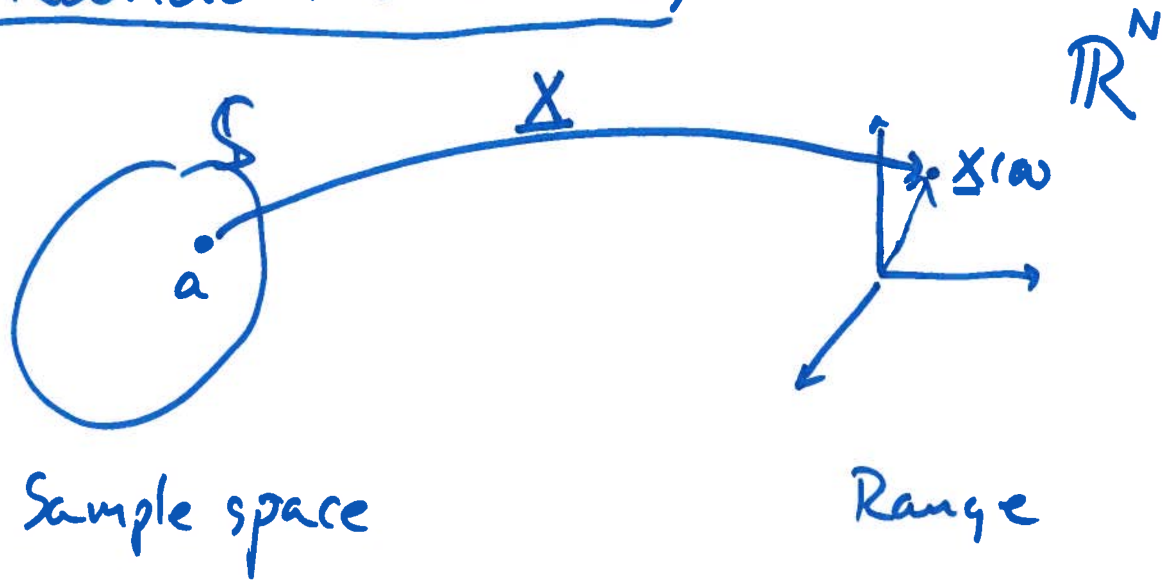
Compute $E[5X^2 - 4X + 2]$

Variance is not linear!

Show that

- $Var(aX) = a^2 \, Var(X)$
- $Var(-X) = Var(X)$
- $Var(X+X) = 4 \, Var(X)$

Multiple Random Variables
(Random Vectors)



$$\underline{X} : S \rightarrow \mathbb{R}^N \quad : \quad a \mapsto \underline{X}(a)$$

- A random vector is a function from the sample space to a range in $\mathbb{R}^N$
- Alternatively $\underline{X}$ may be seen as a vector of N random vars $X_1, \ldots, X_N$

$$\underline{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_N \end{bmatrix}$$

Joint pdf of random vector (r.v.)

- A continuous r.v. is described by the joint pdf

$$p_{\underline{X}}(\underline{x}) = p_{X_1,\dots,X_N}(x_1,\dots,x_N)$$

which is related to the prob. of $\underline{X} \in A$:

$$\mathbb{P}(\underline{X} \in A) = \iint\cdots\int_A p_{X_1,\dots,X_N}(x_1,\dots,x_N)\, dx_1 \cdots dx_N$$

$$= \int_A p_{\underline{X}}(\underline{x})\, d\underline{x}$$

(The vector notation is just a shorthand, $d\underline{x} = dx_1 dx_2 \cdots dx_N$)

Expectation operator

- We define the expectation of $g(\underline{X})$ as:

$$E[g(\underline{X})] = \int g(\underline{x})\, p_{\underline{X}}(\underline{x})\, d\underline{x}$$

- Mean vector $\quad (g(\underline{x}) = \underline{x})$:

$$\underline{\mu}_{\underline{X}} = E[\underline{X}] = \int \underline{x}\, p_{\underline{X}}(\underline{x})\, d\underline{x} = \begin{bmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_N] \end{bmatrix}$$

- Covariance matrix $\left(g(\underline{x}) = (\underline{x} - \underline{\mu}_{\underline{x}})(\underline{x} - \underline{\mu}_{\underline{x}})^T \right)$

$$C_{\underline{X}} = E\left[(\underline{X} - \underline{\mu}_{\underline{X}})(\underline{X} - \underline{\mu}_{\underline{X}})^T\right]$$

$$= E[\underline{X}\,\underline{X}^T] - \underline{\mu}_{\underline{X}}\,\underline{\mu}_{\underline{X}}^T$$

we use the notation $\mathrm{Cov}(\underline{X})$.

Covariance Matrix in different forms

- For r.v. $\underline{X}$ with mean $\underline{\mu}$, the covariance can be written in a different form:

$$C_{\underline{X}} = E\left[(\underline{X}-\underline{\mu})(\underline{X}-\underline{\mu})^T\right]$$

$$= E\begin{bmatrix} (X_1-\mu_1)(X_1-\mu_1) & (X_1-\mu_1)(X_2-\mu_2) & (X_1-\mu_1)(X_3-\mu_3) \\ (X_2-\mu_2)(X_1-\mu_1) & (X_2-\mu_2)(X_2-\mu_2) & (X_2-\mu_2)(X_3-\mu_3) \\ (X_3-\mu_3)(X_1-\mu_1) & (X_3-\mu_3)(X_2-\mu_2) & (X_3-\mu_3)(X_3-\mu_3) \end{bmatrix}$$

$$= \begin{bmatrix} Var(X_1) & Cov(X_1,X_2) & cov(X_1,X_3) \\ Cov(X_2,X_1) & Var(X_2) & Cov(X_2,X_3) \\ cov(X_3,X_1) & cov(X_3,X_2) & Var(X_3) \end{bmatrix}$$

- Diagonal holds the variances of $X_1, \dots, X_W$
- Off-diagonals holds pair-wise covariances

- The covariance matrix is positive semi-definite.

Expectation is linear!

- also in the vector case

For functions g_1, g_2,

$$\begin{cases} E[g_1(\underline{X}) + g_2(\underline{X})] = E[g_1(\underline{X})] + E[g_2(\underline{X})] \\ E[a g(\underline{X})] = a\, E[g(\underline{X})]. \end{cases}$$

$\boxed{\text{For } \underline{X} = [X_1, X_2]^T \text{ with } E[\underline{X}] = [2,3]^T \text{ compute } E[3X_1 + 5X_2]}$

$\boxed{\text{For } \underline{X} = [X_1, \dots, X_N]^T \text{ with } E[\underline{X}] = \underline{\mu} \text{ compute } E[\underline{X}^T \underline{v}] \text{ where } \underline{v} \text{ is a known (deterministic) vector}}$

$\boxed{\text{For } \underline{X} = \underline{S} + \underline{W} \text{ with } E[\underline{S}] = [1, 1, \dots, 1]^T \text{ and } E[\underline{W}] = \underline{0}, \text{ compute } E[\underline{X}].}$

Example: N-variate Gaussian (or Normal)

The pdf of N-variate Gaussian vector $\underline{X}$ with mean $\underline{\mu}$ and covariance $\underline{\underline{C}}$ reads

$$P_{\underline{X}}(\underline{x}) = \frac{1}{\sqrt{2\pi}^N \cdot \sqrt{\det(\underline{\underline{C}})}}$$

$$\cdot \exp\left(-\frac{1}{2}(\underline{x}-\underline{\mu})^T \underline{\underline{C}}^{-1}(\underline{x}-\underline{\mu})\right)$$

Short-hand: $\underline{X} \sim \mathcal{N}(\underline{\mu}, \underline{\underline{C}})$.

Marginal pdfs

- For a r.v. $\underline{X} = [X_1, \ldots, X_N]^T$, we can obtain the pdf of X_i by integrating out the other variables.

- E.g. marginal for X_1:

$$P_{X_1}(x_1) = \int \cdots \int P_{\underline{X}}(\underline{x})\, dx_2\, dx_3 \cdots dx_N$$

- Marginal for X_1 and X_N:

$$P_{X_1 X_N}(x_1, x_N) = \int \cdots \int P_{\underline{X}}(\underline{x})\, dx_2 \cdots dx_{N-1}$$

Independent r.v.s

R.v.s are independent if their joint pdf factorizes into marginals:

$$p_{\underline{X}}(\underline{x}) = p_{X_1}(x_1)\, p_{X_2}(x_2) \cdots p_{X_N}(x_N)$$

$$\Updownarrow$$

$X_1, \ldots, X_N$ are independent.

For independent X_1, X_2,

$$E[X_1 \cdot X_2] = \iint x_1 \cdot x_2 \; p_{X_1}(x_1)\, p_{X_2}(x_2)\, dx_1\, dx_2$$

$$= \int x_1 p_{X_1}(x_1)\, dx_1 \cdot \int x_2\, p_{X_2}(x_2)\, dx_2$$

$$= E[X_1] \cdot E[X_2].$$

- independence allow us to factorize the expectation of products.

Independence $\Rightarrow$ diag covariance.

For $X_1, \dots, X_N$ independent:

$$[C_{\underline{X}}]_{ij} = E[(X_i - \mu_i)(X_j - \mu_j)]$$

$$\overset{\text{independence}}{=} E[X_i - \mu_i] \cdot E[X_j - \mu_j] \qquad i \neq j$$

$$= 0.$$

$$[C_{\underline{X}}]_{ii} = E[(X_i - \mu_i)(X_i - \mu_i)]$$

$$= E[(X_i - \mu_i)^2] = \mathrm{Var}(X_i).$$