Random Variables

- Mathematical models for probabilistic experiments with numerical output.

Ex: Roll a dice and note down the number of dots

3, 3, 6, ...

Discrete R.V.

$S_X = \{1, 2, \ldots, 6\}$

Ex: Roll a dice and note down the the distance from this point:

$S_X = \mathbb{R}_+$

×

11

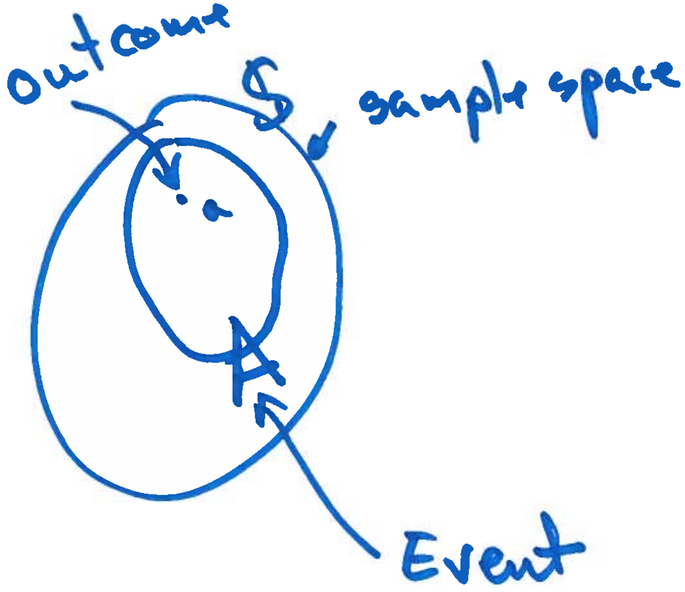
10

7,5

⋮

Continuous R.V.

Probability Space $(S, \mathcal{A}, \mathbb{P})$



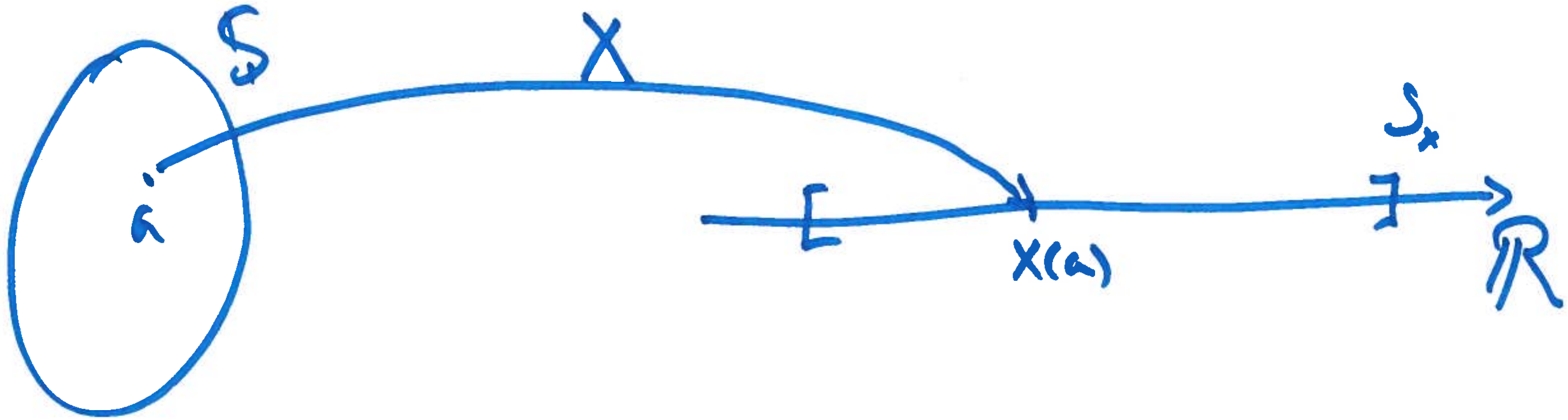
$\mathcal{A}$: σ- algebra

$\mathbb{P}$: Function that assigns a number in $[0,1]$ to each event in $\mathcal{A}$

$$\mathbb{P}(S) = 1 \qquad \mathbb{P}(\emptyset) = 0$$

For A, B disjoint, $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

<u>Random Variable</u>: X is a mapping (or function) from an outcome in S to a real number in the range S_X:



- A random variable is not a variable but a function!

- We suppress the explicit mention of S an (a) and say "... the random variable X".

Discrete R.V.

S_X: Countable

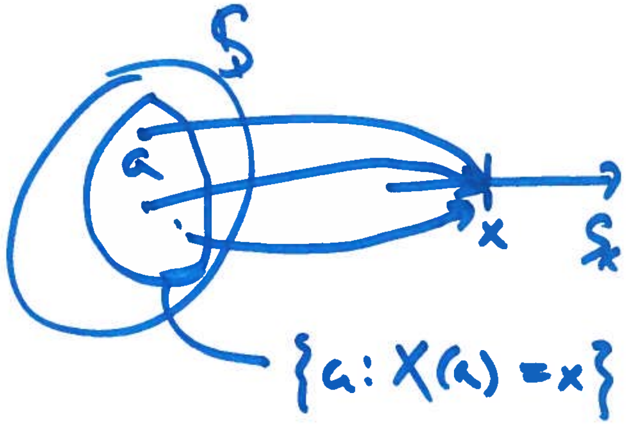
Ex: $\{1, 2, 3, 4, 5, 6\}$

$\mathbb{N}$

$\{0, 1\}$

Probability Mass Function (pmf)

$$P(x) = \mathbb{P}(\{a : X(a) = x\})$$



$$\Rightarrow \sum_{x \in S_X} P(x) = 1$$

$$P(x) = 0 \quad , x \notin S_X$$

Examples of discrete R.V.s

- Uniform discrete: $P(x) = \begin{cases} \text{const} & \text{for } x \in S_x \\ 0 & \text{for } x \notin S_x \end{cases}$

- Bernoulli: $P(x) = \begin{cases} p, & x = 1 \\ 1-p, & x = 0 \end{cases}$ $\qquad S_x = \{0, 1\}$

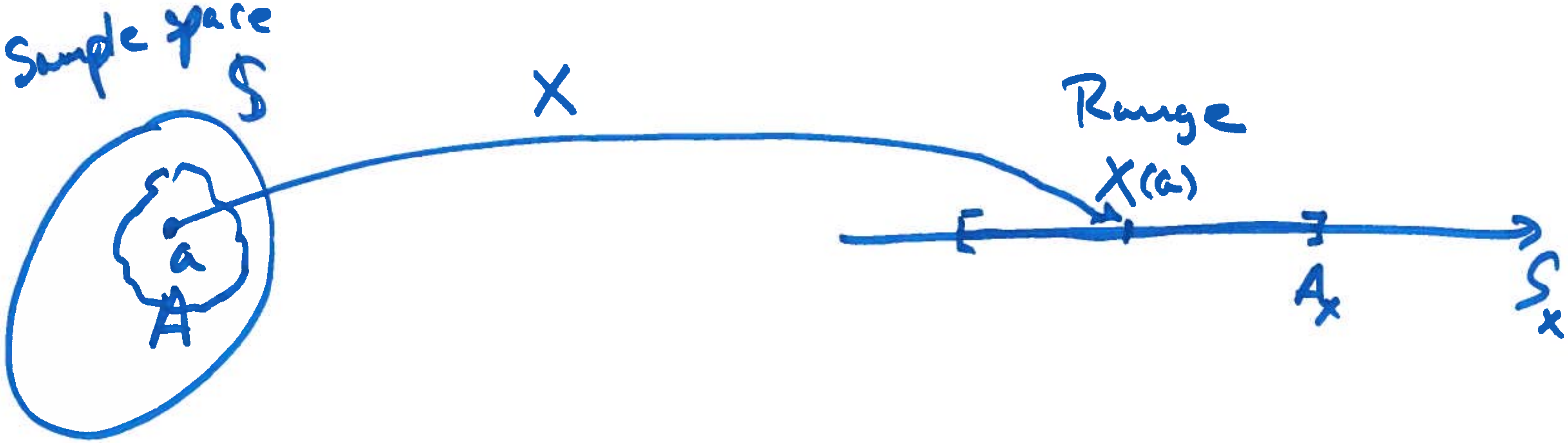
- Binomial:

$$P(x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad S_x = \{0, 1, \ldots, n\}$$

- Poisson:

$$P(x) = e^{-\lambda} \cdot \frac{\lambda^x}{x!} \qquad S_x = \{0, 1, 2, \ldots\} = \mathbb{N}_0$$

Probability of X "belonging to A_X"



Event

$$A = \{a : X(a) \in A_X\} \qquad \mathbb{P}(A)$$

Convention

$$\mathbb{P}(X \in A_X) = \mathbb{P}(A) = \mathbb{P}(\{a : X(a) \in A_X\})$$

similarly:

$$\mathbb{P}(X > c),\ \mathbb{P}(X < c),\ \mathbb{P}(X = c), \ldots$$

Continuous R.V.

S_x : Uncountable (continuous)

Ex: $S_x = \mathbb{R}_+$
$S_x = \mathbb{R}$
$S_x = [0, 2)$

$$\mathbb{P}(\{s : a \leq X(s) \leq b\}) = \mathbb{P}(a \leq X \leq b)$$

$$= \int_a^b p(x)\, dx$$

probability density function (pdf).

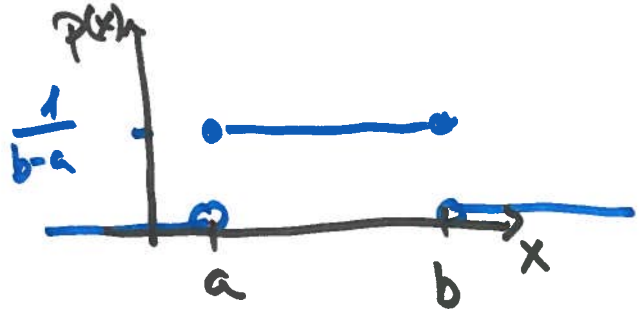
Pdf is nonnegative and integrates to 1:

$$p(x) \geq 0 \quad x \in S_x \quad , \quad \int_{-\infty}^{+\infty} p(x)\, dx = 1$$

Examples of continuous RVs:

Uniform:

$$p(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

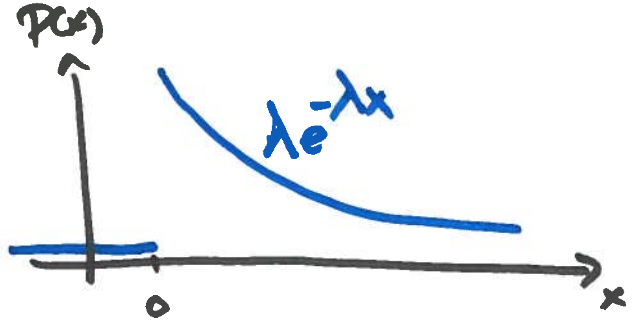


$X \sim U([a,b])$

→ Most computer systems can generate uniform pseudo RV.
→ Round-off errors are approximately uniform.

Exponential:

$$p(x) = \begin{cases} \lambda e^{-\lambda x} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$$

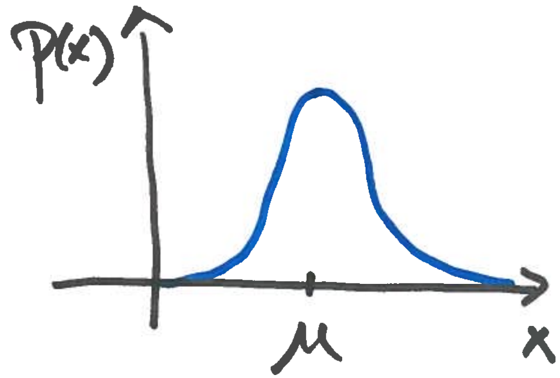


$X \sim Exp(\lambda)$

→ Often seen in queing theory.

Gaussian (or Normal):

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$



Shorthand: $X \sim \mathcal{N}(\mu, \sigma^2)$

→ Electrical noise

→ Sums of many (independent) RVs are approx Gaussian (CLT).

Special Cases:

Mean: Expectation of X : $\mu = \mathbb{E}[X]$

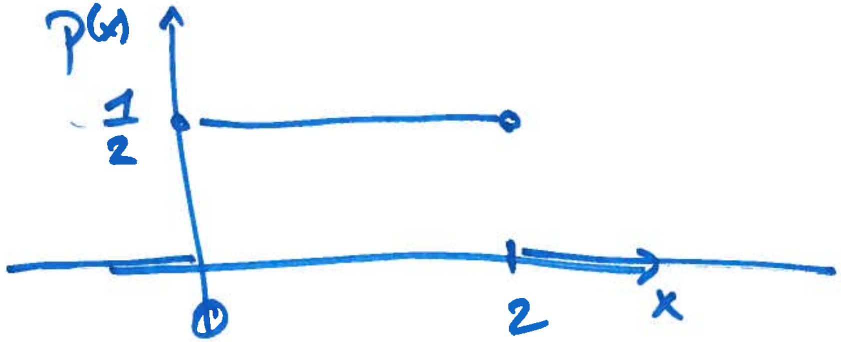
Variance: Expectation of $(X-\mu)^2$: $\sigma^2 = \mathrm{Var}(X) = \mathbb{E}[(X-\mu)^2]$
$= \mathbb{E}[X^2] - \mu^2$

Mean Square (Second moment): Expectation of X^2 : $\mathbb{E}[X^2]$

- $\mathbb{E}[X^2] = \sigma^2 + \mu^2$

Example:

$$X \sim U([0,2])$$



$$\mu = \mathbb{E}[X] = \int_{-\infty}^{\infty} x\, p(x)\, dx = \int_0^2 x\, \frac{1}{2}\, dx = 1$$

$$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 p(x)\, dx = \int_0^2 x^2\, \frac{1}{2}\, dx = \frac{4}{3}$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mu^2 = \frac{4}{3} - 1 = \frac{1}{3}$$

Expectation Operator $\mathbb{E}[\cdot]$:

Def: The expectation of a function g of a RV is

$$\mathbb{E}[g(X)] = \begin{cases} \sum_{x} g(x) P(x) & , \; X \text{ discrete} \\ \int_{-\infty}^{\infty} g(x) p(x)\, dx & , \; X \text{ continuous} \end{cases}$$

Expectation is a linear operator:

$$\begin{cases} \mathbb{E}[g_1(X) + g_2(X)] = \mathbb{E}[g_1(X)] + \mathbb{E}[g_2(X)] \\ \mathbb{E}[a\, g(X)] = a \cdot \mathbb{E}[g(X)] \end{cases}$$

Remark: Variance is not linear!

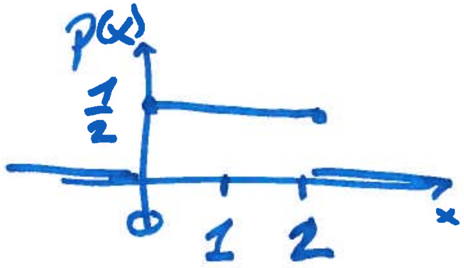
$$\mathrm{Var}(aX) = a^2\, \mathrm{Var}(X)$$

$$\mathrm{Var}(-X) = \mathrm{Var}(-1 \cdot X) = (-1)^2 \cdot \mathrm{Var}(X) = \mathrm{Var}(X).$$

$$\mathrm{Var}(X+X) = \mathrm{Var}(2X) = 4\, \mathrm{Var}(X).$$

Example:

$$X \sim \mathcal{U}([0,2])$$



$$\mu = \mathbb{E}[X] = \int_0^2 x \; \frac{1}{2} \, dx = \frac{1}{2} \left[\frac{1}{2} x^2 \right]_0^2 = \frac{1}{4} \cdot 2^2 = 1$$

$$\mathbb{E}[X^2] = \int_0^2 x^2 \; \frac{1}{2} \, dx = \frac{1}{2} \cdot \left[\frac{1}{3} x^3 \right]_0^2 = \frac{1}{6} \cdot 2^3 = \frac{8}{6} = \frac{4}{3}$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mu^2 = \frac{4}{3} - 1^2 = \frac{1}{3}$$