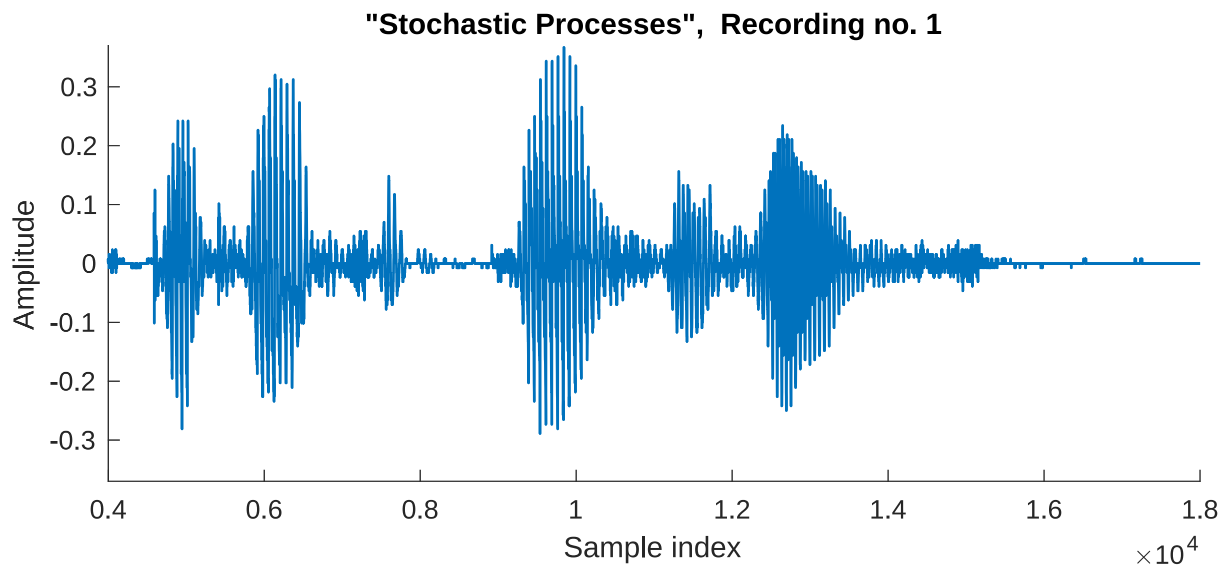
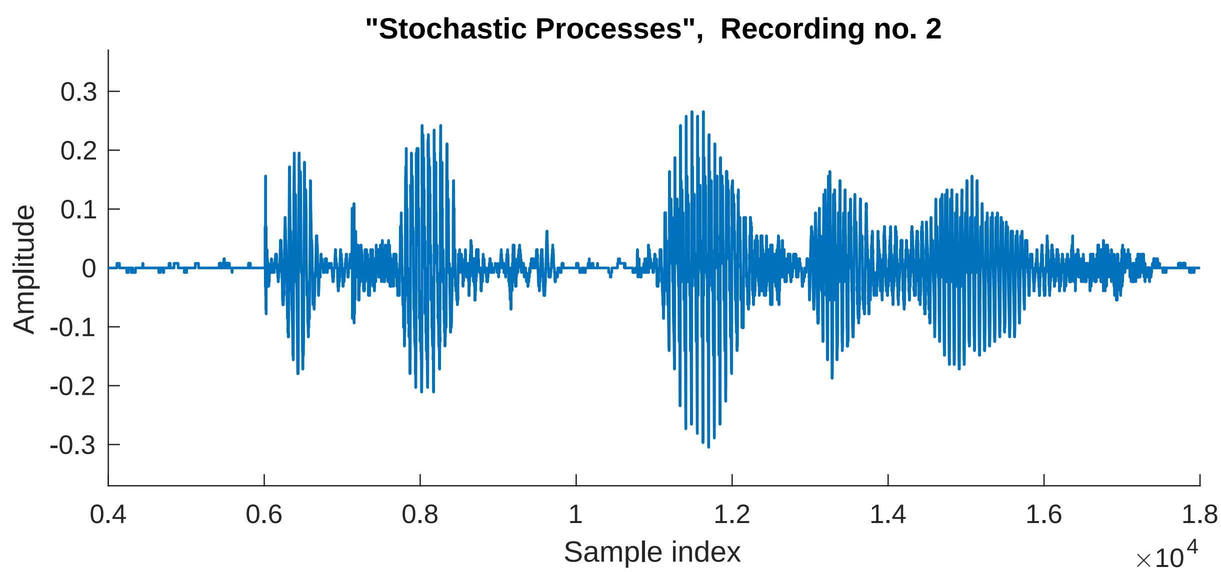
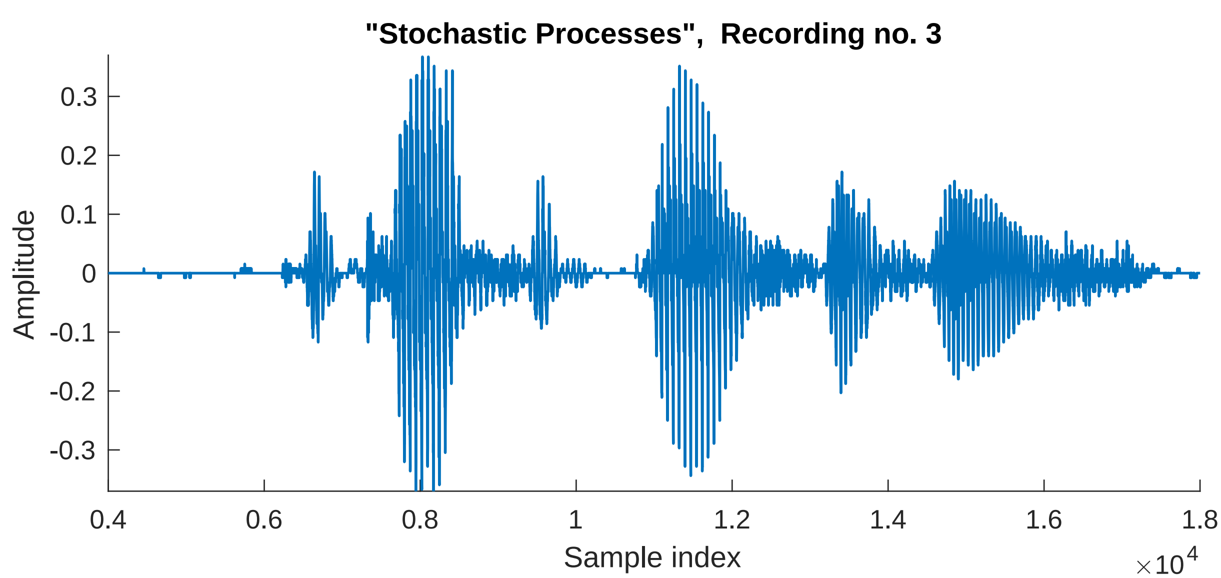
Stochastic Processes (or Random Processes)

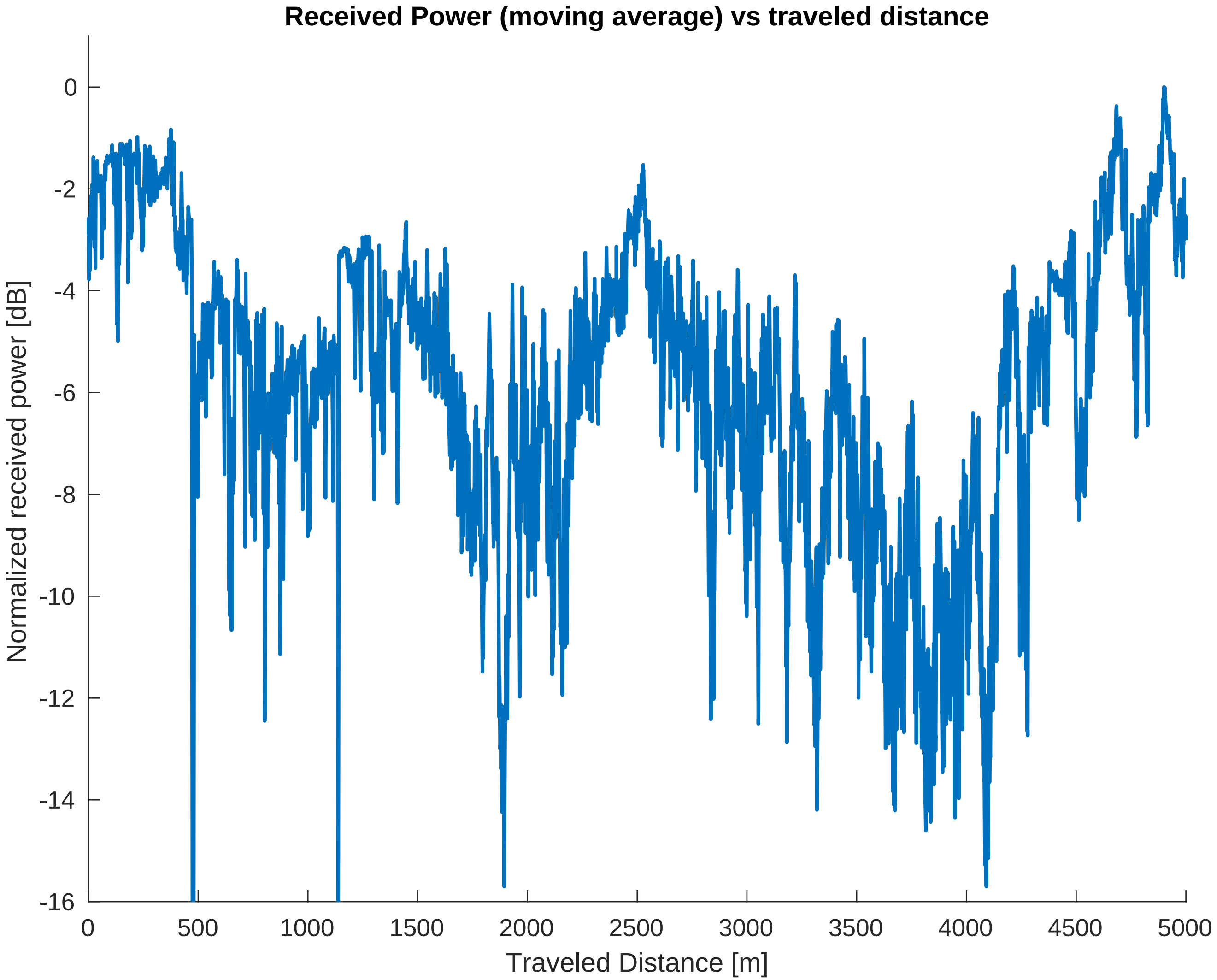
Study: Probability models for signals/functions of time and space.

- Examples:
  - Speech signal
  - Received Signal Power
  - Accelerometer data
  - Noise signal ~~[illegible]~~

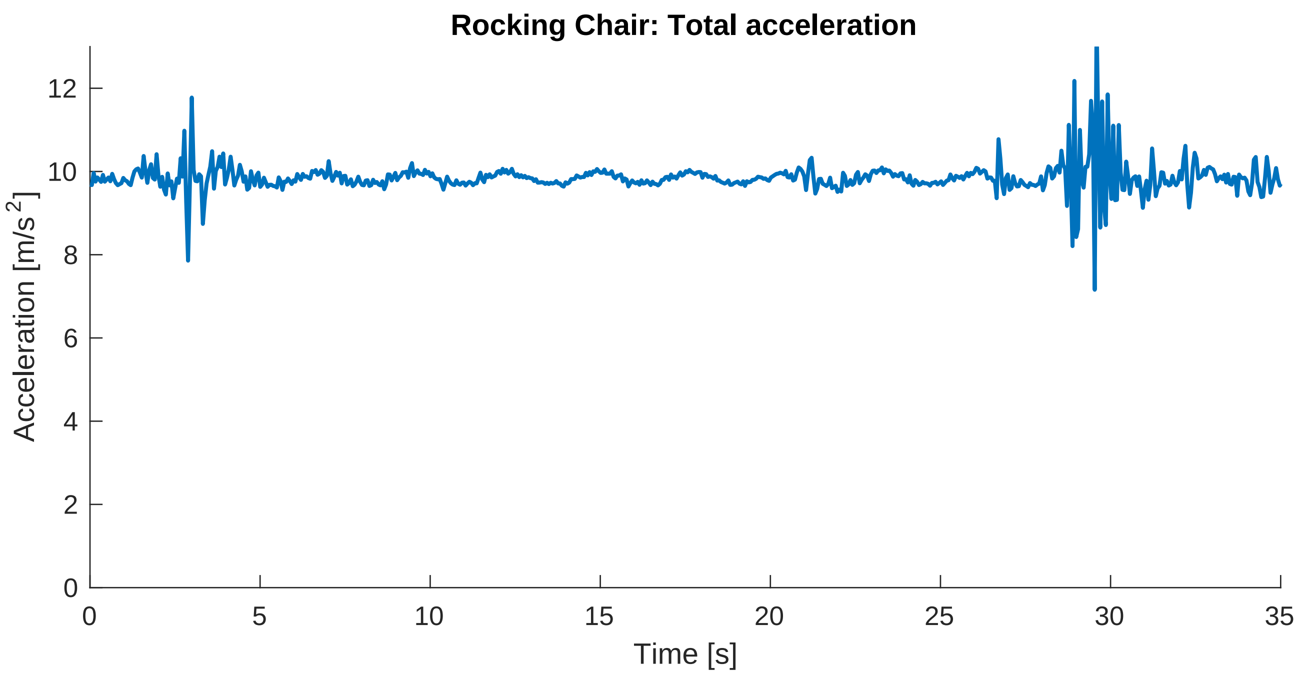

"Stochastic Processes", Recording no. 1
Amplitude
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
Sample index
×10^4


"Stochastic Processes", Recording no. 2
Amplitude
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
Sample index
×10^4

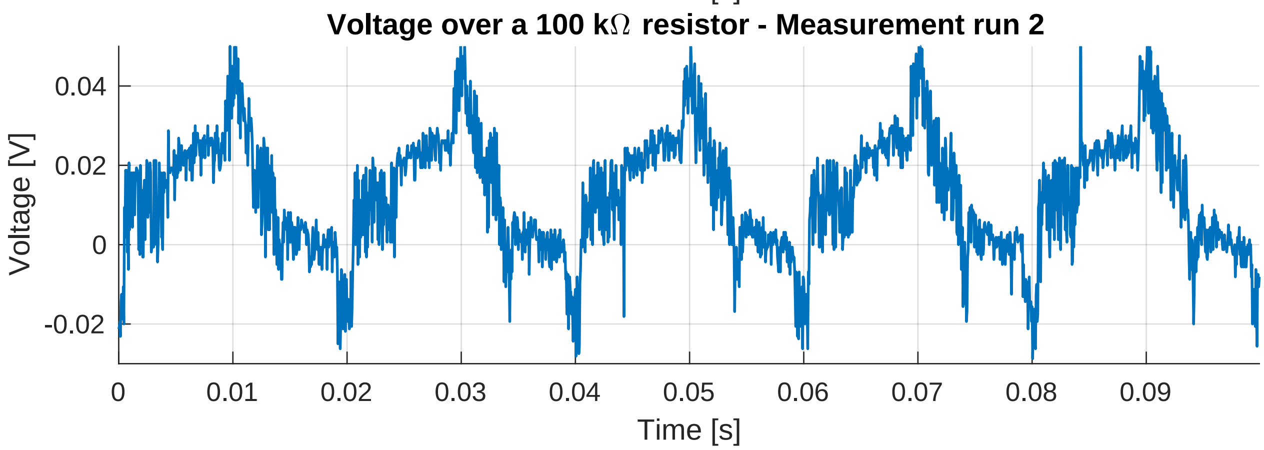

"Stochastic Processes", Recording no. 3
Amplitude
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
Sample index
×10^4



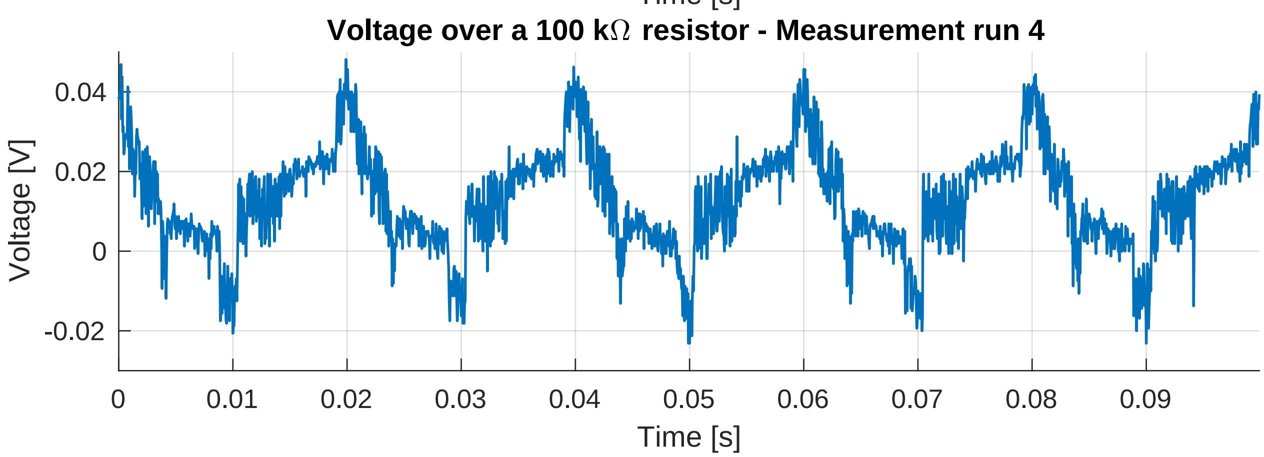
Data from: Diversity Measurement Camapaign, Telenor-RATE, 1998


Rocking Chair: Total acceleration
Acceleration [m/s²]
Time [s]
0
2
4
6
8
10
12
0
5
10
15
20
25
30
35

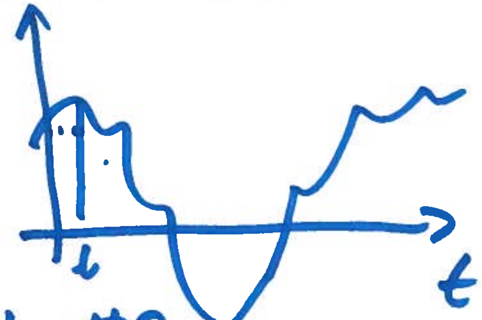

Voltage over a 100 kΩ resistor - Measurement run 1
Voltage [V]
0.04
0.02
0
-0.02
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
Time [s]


Voltage over a 100 kΩ resistor - Measurement run 2
Voltage [V]
0.04
0.02
0
-0.02
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
Time [s]

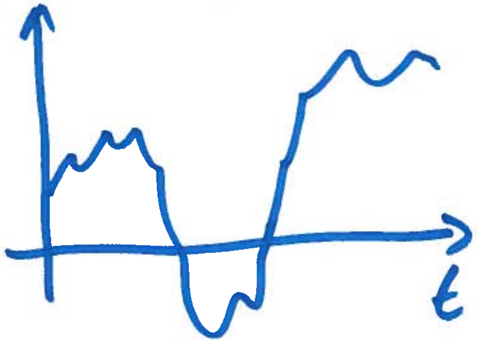

Voltage over a 100 kΩ resistor - Measurement run 3
Voltage [V]
0.04
0.02
0
-0.02
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
Time [s]


Voltage over a 100 kΩ resistor - Measurement run 4
Voltage [V]
0.04
0.02
0
-0.02
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
Time [s]

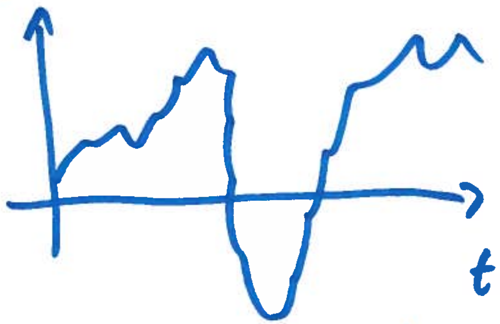
Realization #1



Real. #2

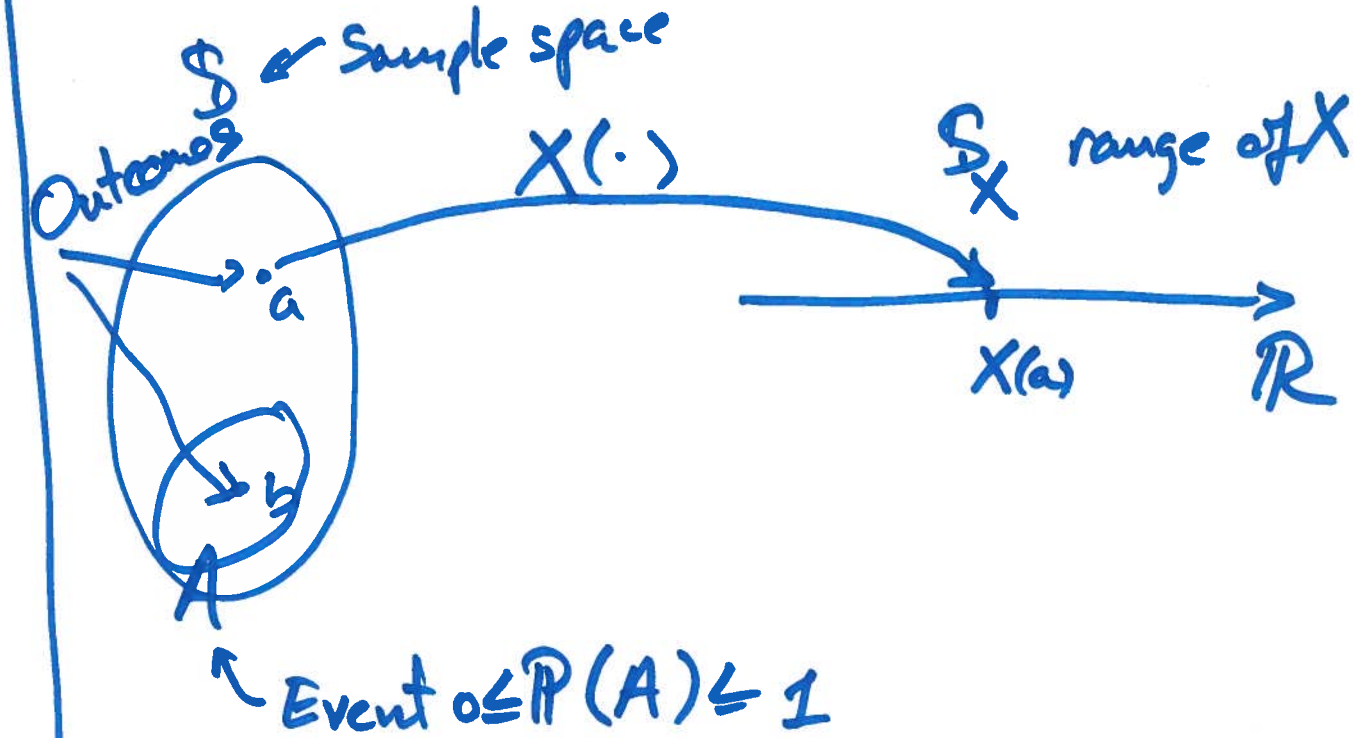


Real. #3

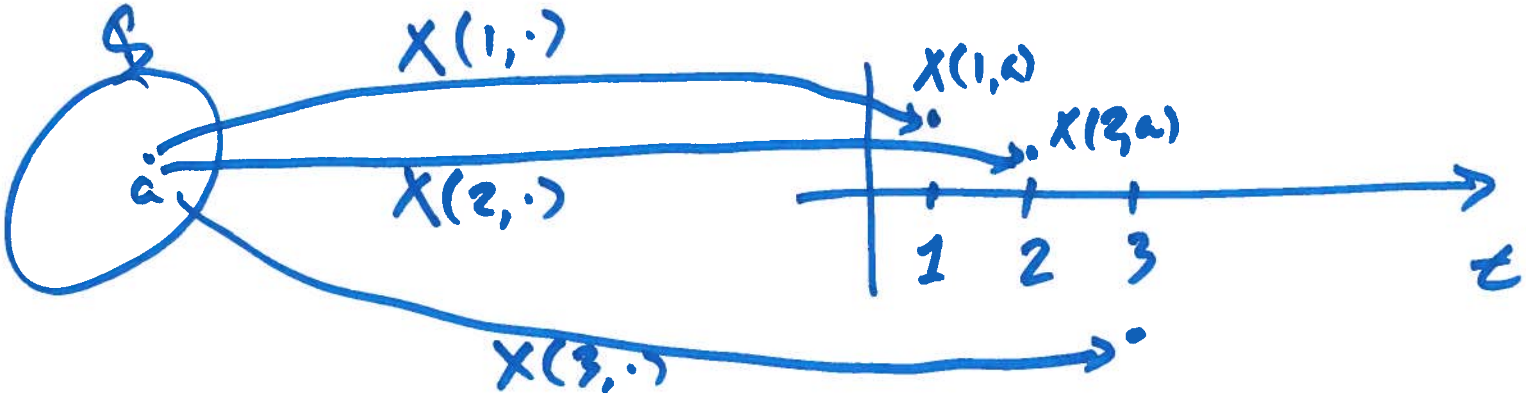


How to model random signals?

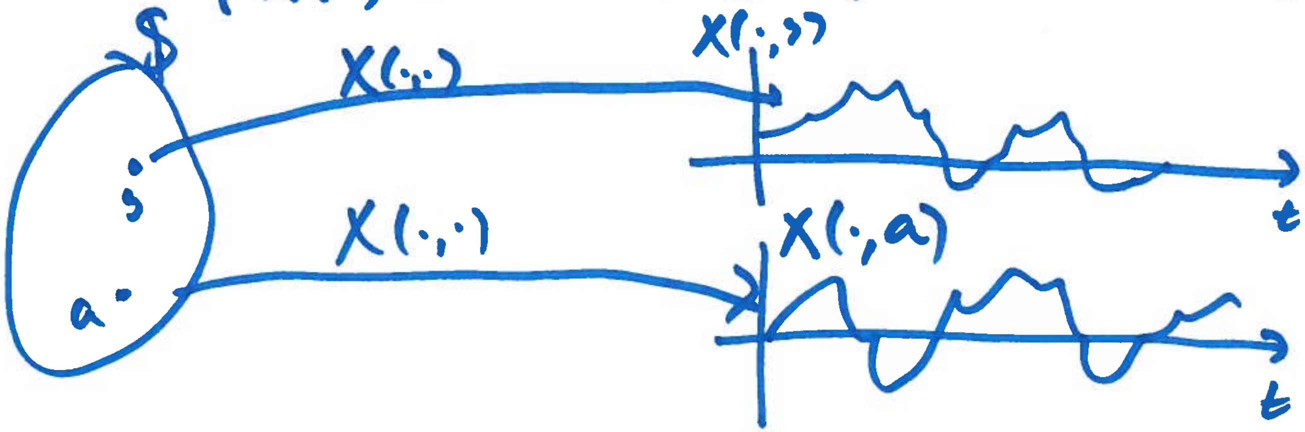
Stochastic/random variable



Def. 1: A <u>stochastic process</u> is a collection $\{X(t;\cdot) : t \in T\}$ of random variables indexed by t



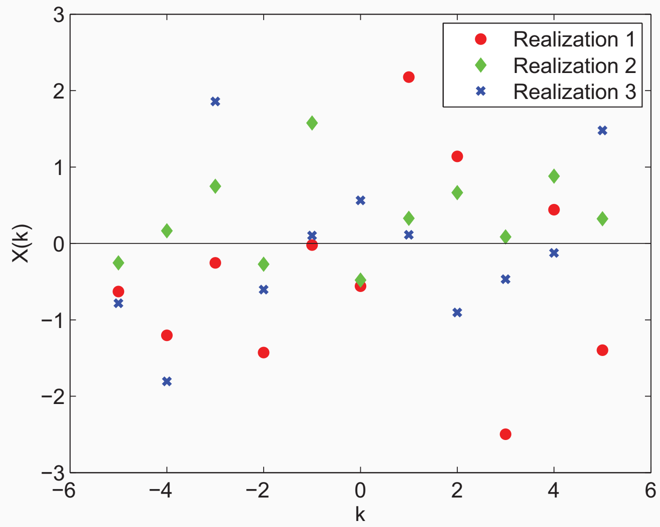
Def. 2: A <u>stochastic process</u> is a collection $\{X(\cdot, s) : s \in S\}$ of deterministic functions of time indexed by outcome s



Two examples:

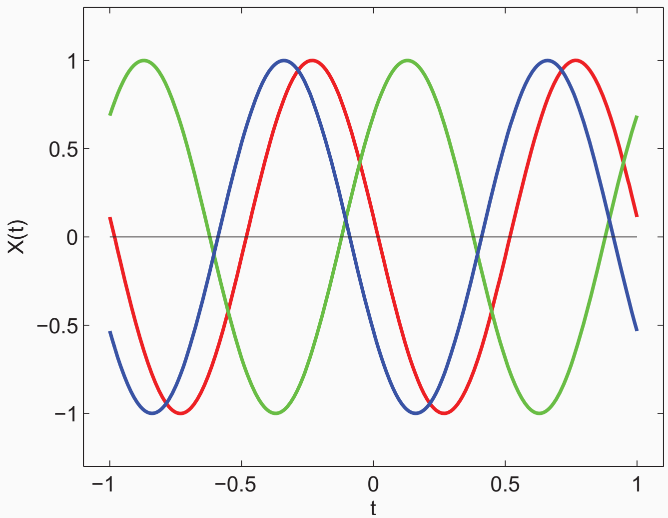
Example: (first definition)

$$\{X(k;\cdot)\ :\ k \in \mathbb{Z}\}$$
$$\mathbb{S} = ???$$
$$X(k) = X(k;\cdot) \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$$



Example: (second definition)

$$\{X(\cdot;s)\ :\ s \in \mathbb{S}\}$$
$$\mathbb{S} = [-\pi, \pi]$$
$$\text{Random variable } \Theta \sim \mathcal{U}(-\pi, \pi)$$
$$X(t) = X(\cdot;\theta) = \sin(2\pi t + \theta)$$



Remark: In some "toy-examples" we can write up the sample space $\mathbb{S}$ explicitly. In realistic cases, this can be hard or impossible.

Time t can be:

- Discrete : $t \in \mathbb{Z} = \{\ldots, -2, -1, 0, +1, +2, \ldots\}$
  - $\longrightarrow$ Discrete-time random process
- Continuous : $t \in \mathbb{R}$
  - $\longrightarrow$ Continuous-time random process

We call t for "time" but it could be other entities:

- distance
- space
- frequency

$\vdots$

IID Processes

Def: An IID process is a discrete-time process $\{X(k)\} = \{X_k\}$ with independent and identically distributed (IID) samples.

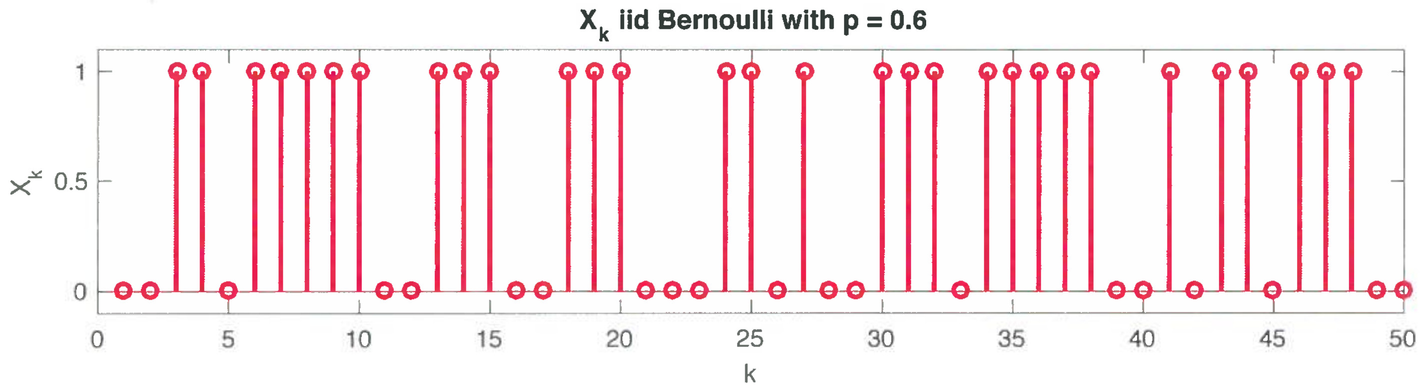
Notation: $X_k \overset{iid}{\sim} \text{pmf}$ or $X_k \overset{iid}{\sim} \text{pdf}$

IID processes are

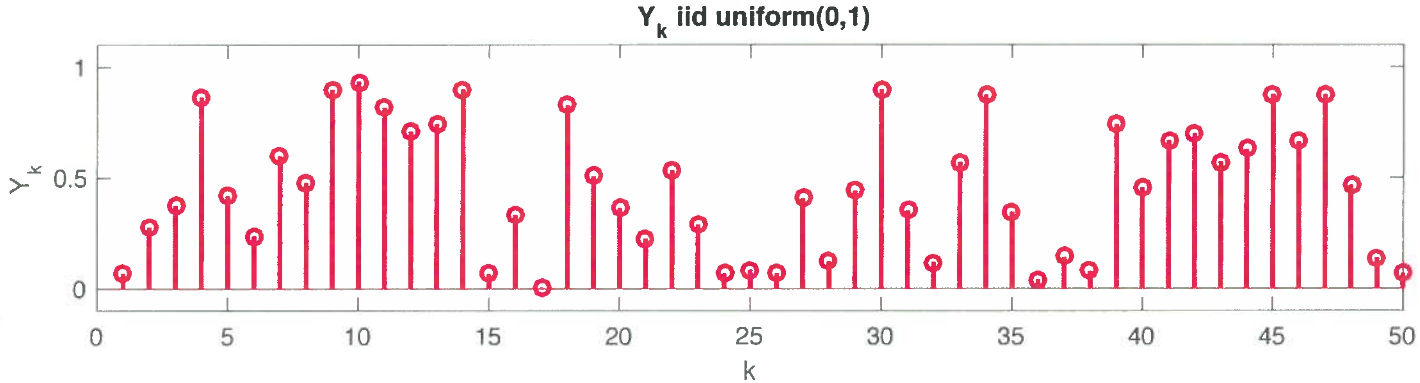
- Simple to analyze
- Easy to simulate
- Used as building blocks

Examples of IID Processes

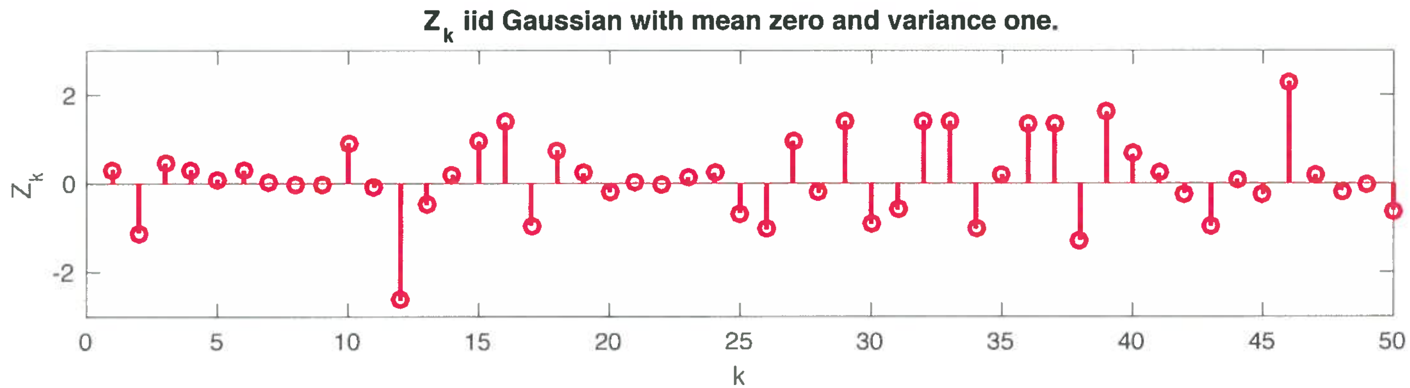
$X_k \overset{iid}{\sim} \text{Bernoulli}(p)$



$Y_k \overset{iid}{\sim} \mathcal{U}(a,b)$



$Z_k \overset{iid}{\sim} \mathcal{N}(0,\sigma^2)$



Description of a Stochastic Process

Full description: Specify all N^{th} order joint prob. distributions of samples !

$N=1$: $\mathbb{P}(X(t,\cdot) \leq x)$ for all t, all x

$N=2$: $\mathbb{P}(X(t_1,\cdot) \leq x_1 ,\ X(t_2,\cdot) \leq x_2)$ for all t_1, t_2, x_1, x_2

$\vdots$

$\mathbb{P}(X(t_1) \leq x_1 , \dots , X(t_N) \leq x_N)$ for all $t_1, t_2, \dots, t_N$, $x_1, x_2, \dots, x_N$

Hard to do ! (in most cases...)

Example: Full description of IID process.

For $\{X_k\}$ an IID process with $X_k \overset{iid}{\sim} F_X$,

The joint pmf of N samples $X_{k_1}, X_{k_2}, \dots, X_{k_N}$ reads:

$$\mathbb{P}(X_{k_1} \le x_1, X_{k_2} \le x_2, \dots, X_{k_N} \le x_N)$$

$$= \prod_{n=1}^{N} \mathbb{P}(X_{k_n} \le x_n) \qquad \text{(independence)}$$

$$= \prod_{n=1}^{N} F_X(x_n) \qquad \text{(identically)}$$

Partial description of a stochastic process $X(t)$

- The mean: $\mu_X(t) := \mathbb{E}[X(t)]$

First-order charact. of $X(t)$

- The variance: $\sigma_X^2(t) := Var(X(t)) = \mathbb{E}[(X(t)-\mu_X(t))^2]$

Second-order characterization

- The covariance function:

$$C_X(t_1, t_2) := \mathbb{E}[(X(t_1)-\mu_X(t_1)) \cdot (X(t_2)-\mu_X(t_2))]$$
$$= Cov(X(t_1), X(t_2))$$

Example: <u>1^{st} & 2^{nd}-order characterization of IID process</u>

$$X_k \overset{iid}{\sim} \mathcal{N}(0,1)$$

Mean: $\mu_X(k) = \mathbb{E}[X_k] = 0$ for all k.

Covariance function:

$$C_X(k,l) = \mathbb{E}[(X_k - \mu_k)\cdot(X_l - \mu_l)]$$

$$= \begin{cases} \mathbb{E}[(X_k - \mu_k)^2] = Var(X_k) = 1 & (k=l) \\ \underbrace{\mathbb{E}[(X_k - \mu_k)]}_{=0} \cdot \underbrace{\mathbb{E}[X_l - \mu_l]}_{=0} = 0 & k \neq l. \end{cases}$$

Shift - invariance:

- For $\{X_k\} \overset{iid}{\sim} P_X$ the pdf of any sample X_k is the same. In particular it is shift invariant:

$$P_{X_k}(x) = P_X(x) = P_{X_{k+\ell}}(x), \qquad \ell \in \mathbb{Z}$$

- Also <u>joint</u> pdfs are shift invariant.

For two samples:

$$P_{X_n X_m}(x, x') = P_X(x) P_X(x') = P_{X_{n+\ell} X_{m+\ell}}(x, x'), \quad \ell \in \mathbb{Z}$$

For N samples $\underline{X} = [X_1, \dots, X_N]$, $\quad \underline{X}' = [X_{1+\ell}, \dots, X_{N+\ell}]$

$$P_{\underline{X}}(\underline{x}) = \prod_{n=1}^{N} P_X(x_n) = P_{\underline{X}}(\underline{x}').$$

Strict Sense Stationary (SSS) Processes

- Want to narrow down the class of stochastic processes
- Systems are often "time-invariant"
- Idea: Consider only the class of processes where the CDFs do not change if time is shifted.

$$\mathbb{P}(X(t_1) \le x_1) = \mathbb{P}(X(t_1+\tau) \le x_1)$$

and

$$\mathbb{P}(X(t_1) \le x_1, X(t_2) \le x_2) = \mathbb{P}(X(t_1+\tau) \le x_1, X(t_2+\tau) \le x_2)$$

and all N'th order CDFs should be time-invariant:

$$\mathbb{P}(X(t_1) \le x_1, \dots, X(t_N) \le x_N) = \mathbb{P}(X(t_1+\tau) \le x_1, \dots, X(t_N+\tau) \le x_N)$$

Strict Sense Stationary (SSS) Processes

<u>Def</u>: $\{X(t)\}$ is SSS if all N-point joint CDFs are invariant to arbitrary time-shifts:

$$\mathbb{P}(X(t_1) \le x_1, \ldots, X(t_N) \le x_N) = \mathbb{P}(X(t_1+\tau) \le x_1, \ldots, X(t_N+\tau) \le x_N)$$

for arbitrary N

<u>Rmk</u>: SSS is hard to show as it requires access to the full characterization (N-point CDF's).

Example: IID process is SSS

For $\{X_k\} \overset{iid}{\sim} F_x$:

$$\mathbb{P}(X_{k_1} \leq x_1, \ldots, X_{k_N} \leq x_N) = \prod_{n=1}^{N} F_X(x_n)$$

$$= \mathbb{P}(X_{k_1+\ell} \leq x_1, \ldots, X_{k_N+\ell} \leq x_N)$$

for arbitrary $\ell \in \mathbb{Z}$ $\Longrightarrow$ X_k is SSS.

Mean, Variance & Covariance of SSS process

Let $\{X(t)\}$ be SSS.

Then $F_{X(t)} = F_{X(t+\tau)} = F_{X(0)} \Rightarrow$ $\mu_X(t) = \mu_X(0) =: \mu_X$, $\sigma_X^2(t) = \sigma_X^2(0) =: \sigma_X^2$

Furthermore: $F_{X(t_1)X(t_2)} = F_{X(t_1+\tau)X(t_2+\tau)}$

$= F_{X(t_1-t_2)X(0)}$

$$\Rightarrow C_X(t_1, t_2) = C_X(t_1 - t_2, 0) =: C_X(t_1 - t_2)$$

Hence, C_X depends on $t_1 - t_2$ only.

We have shown:

$$\{X(t)\} \text{ SSS} \begin{array}{c} \Longrightarrow \\ \not\Longleftarrow \end{array} \left\{ \begin{array}{l} \mu_X(t) = \mu_X \\ \sigma_X^2(t) = \sigma_X^2 \\ C_X(t_1, t_2) = C_X(t_1 - t_2) \end{array} \right\} \bigstar$$

Question: Is opposite true? ie $\bigstar \overset{?}{\Longrightarrow}$ SSS

No!

Example: Non-SSS process

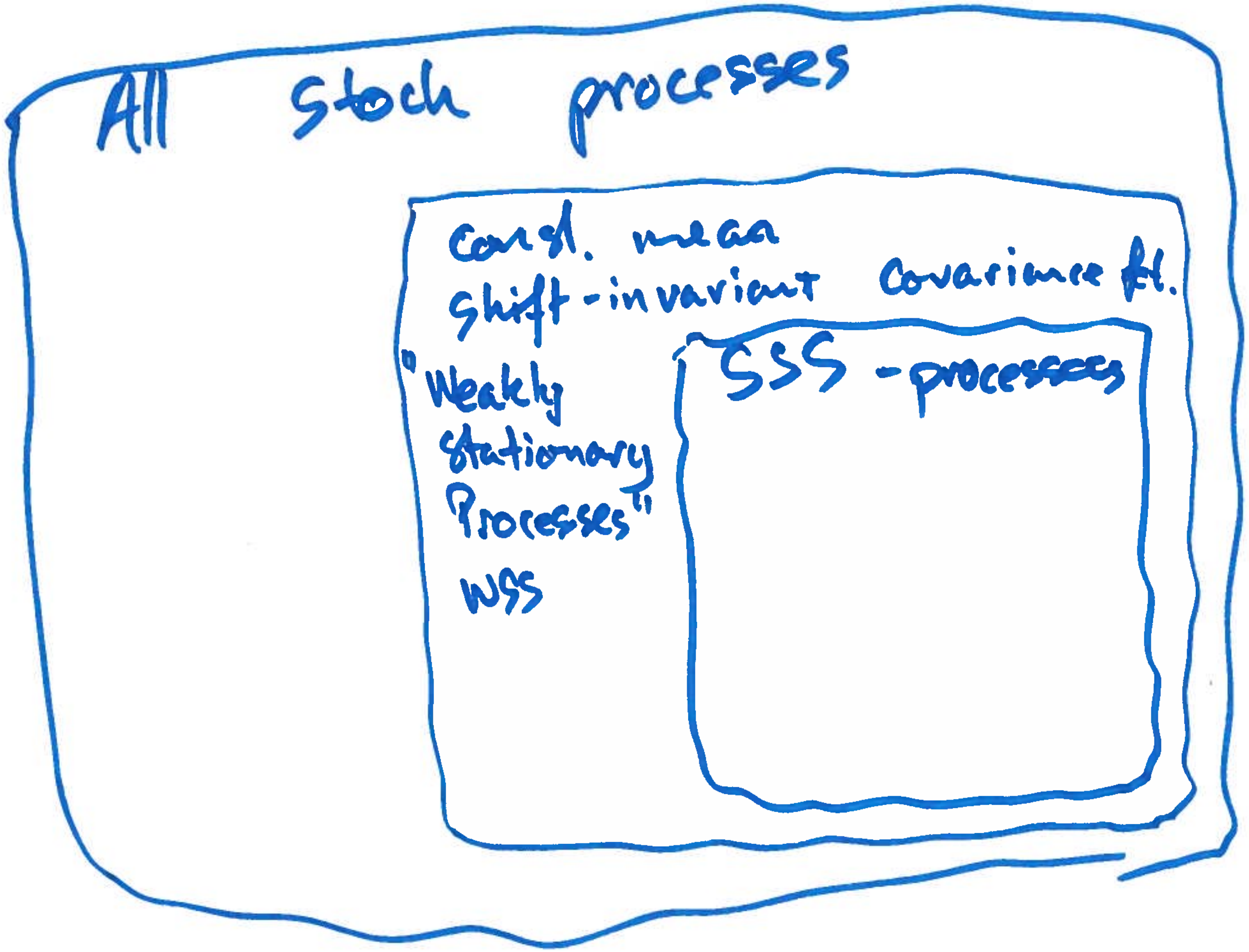
$$X_k \overset{\text{independt}}{\sim} \begin{cases} \mathcal{U}(-\sqrt{3}, \sqrt{3}), & k \text{ odd} \\ \mathcal{N}(0,1), & k \text{ even} \end{cases}$$

- $\mu_X(k) = 0$

Kronecker's delta

- $C_X(k_1, k_2) = \begin{cases} 1, & k_1 = k_2 \\ 0, & k_1 \neq k_2 \end{cases} = \delta(k_1 - k_2)$

- But $\{X_k\}$ is not SSS since $F_{X_1} \neq F_{X_2}$.


All stoch processes
const. mean
shift-invariant covariance fct.
"Weakly stationary Processes"
WSS
SSS-processes