Definition 8.1: Markov chain
A discrete Markov chain with state space S (countable) is a sequence of random variables $X_0, X_1, X_2, \ldots$ such that:

$$P(X_{n+1} = j/X_0 = i_0, X_1 = i_1, \ldots X_1 = i) = P(X_{n+1} = j/X_n = i) \qquad (1)$$

for $m = 0, 1, 2, \ldots$; $i_0, i_1, \ldots, i, j \in S$.

- Assume that the Markov chain i time-homogeneous: $\forall n \in \mathbb{N}$, $P(X_{n+1}/X_n = i) = P(X_1 = j/X_0 = i)$.
- The transition probability: $P_{ij} = P(X_1 = j/X_0 = i)$
- $P = \{P_{ij}\}_{ij \in S}$ is alled the transition probability matrix (size SxS).
- $\forall i \in S : \sum_{j \in S} P_{ij} = 1$
- The n-step transition probability: $P_{ij}^{(n)} = P(X_n = j/X_0 = i)$

Proposition: $P^{(n)} = P^n$
The n-step transition probability matric is equal to the transition probability matrix P to the power n. **The Chapman-Kolmogorov equation**

$$\begin{aligned} P^{(n+m)} &= P^{(n)} P^{(m)} \\ P_{ij}^{(n+m)} &= \sum_{k \in S} P_{ij}^{(n)} P_{ij}^{(m)} \end{aligned}$$

Example 8.1
You start playing a roulette game where at each turn you bet 1 dollar and win with probability $\frac{18}{38}$.

$$X_n = \{\text{My fortune at time } n\} \qquad (2)$$

By independence of the game (assumed) check the Markov property. Write the transition matrix:

$$P = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ \frac{20}{38} & 0 & \frac{18}{38} & \ldots & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & & & & \end{bmatrix} \qquad (3)$$

We say that 0 is absorbant.

Examples 8.2/8.3/8.5: Genotype

Let $S = \{AA, aa, aA\}$ be the genotype of a plant. At each time n cross the plant with itselp: $X_n = \{\text{the genotype at time } n\}$. The genotype depends only of the parents.
The transition matrix:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 1 \end{bmatrix} \tag{4}$$

The n-step transitioj matrix then becomes

$$P^{(n)} = \begin{bmatrix} 1 & 0 & 0 \\ \left(1 - \left(\frac{1}{2}\right)^n\right)/2 & \left(\frac{1}{2}\right)^n & \left(1 - \left(\frac{1}{2}\right)^n\right)/2 \\ 0 & 0 & 1 \end{bmatrix} \tag{5}$$

On/off-system

$$\begin{cases} 0 \text{ system is off} \\ 1 \text{ system is on} \end{cases} \tag{6}$$

Let p be the proability that it turns/remains on and q the opposite. This gives transition matrix

$$P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} \tag{7}$$

The resulting n-step transition matrix is then

$$P^{(n)} = \frac{1}{p+q} \begin{bmatrix} q & p \\ q & o \end{bmatrix} \tag{8}$$

Definition 8.2+8.3

- If it exists $n \in \mathbb{N}$ such that $P_{ij}^{(n)} > 0$ we say that j is accessible (within $i \to j$).
- If $i \to j$ and $j \to i$ we say that i and j communicate.
- If all states S in a Markov chain communicate we say the chain is irreducible.
- The communicating property is an equivalence relation.

Definition 8.4
For $i \in S:\ T_i = \min\{n \geq 1 :\ X_n = i\}$. This is the time of the first vist of i.

- We say that i is recurrent if $P(T_i < \infty / X_0 = i) = 1$.
- We say that i is transcient if $P(T_i < \infty / X_0 = i) < 1$.

Corollary 8.1
In an irreducible Markov chain all the states are all transcirent or all recurrent.

Corollary 8.2+8.3
If S is finite

- i is transcient $\Leftrightarrow \exists j$, $i \to j$, $j \not\to i$.
- there is at least one recurrent state.

Proposition 8.1
A state i is

- transcient if and only if $\sum_{n=1}^{\infty} P_{ij}^{(n)} < \infty$.
- recurent if and only if $\sum_{n=1}^{\infty} P_{ij}^{(n)} = \infty$.

Definition 8.5
Let P be the transition matrix of a Markov chain with state space S. A disitribution $\pi = (\pi_s, s \in S)$ is called a stationary distribution og the Markov chain if

$$\pi P = \pi. \tag{9}$$

- $\pi_s = P(s \text{ happens})$.
- $\pi_j = \sum_{i \in S} P_{ij} \pi_i$, $\forall i, j \in S$.
- $S \sum_{j \in S} \pi_j = 1$
- If $X_n \sim \pi \Rightarrow X_{n+1} \sim \pi$, $X_{n+2} \sim \pi$.

The stationary distribution is also called the invariant distribution and the equilibrium distribution.

Proposition 8.2
If a Markov chain is irreducible and there exists a stationary distribution it is unique.

Proposition 8.3
If S is finite and the Markov chain is irreducible there exists a unique stationary distribution.

Definition 8.6
Let i be recurrent.

- i is said to be positive recurrent if $E(T_i) < \infty$.

- i is said to be null recurrent if $E(T_i) = \infty$.

Corollary 8.6
For an irreducible Markov chain there is 3 possibilities:

- All the states are positive recurrent.
- All the states are null recurrent.
- All the states are transcient.

Definition 8.4
For an irreducible Markov chain: $\{X_n\}_{n\in\mathbb{N}}$. A stationary distribution exists $\Leftrightarrow$ X_n is positive recurrent for all $n \in \mathbb{N}$. In that case it is unique and $\pi_i > 0$, $\forall j \in S$.

Definition 8.7
Let $p_{ij}^{(n)}$ be the n-step transition of a Markov chain such that

$$P_{ij}^{(n)} \underset{m\to\infty}{\to} q_{ij},\ \forall i, j \in S. \tag{10}$$

We call $q = (q_0, q_1, \ldots)$ the limit distribution.

Definition 8.8
The period of a state i is $d(i) = gcd(n \geq 1,\ P_{ij}^{(n)} > 0)$.

- If $d(i) = 1$ the state is aperiodic.
- If $d(i) > 1$ the state is periodic.

Theorem 8.1
For an irreducible positive recurrent and aperiodic Markov chain with stationary distribution π and n-step transition probability $P_{ij}^{(n)}$:

$$P_{ij}^{(n)} \to \pi_i,\ \forall i, j \in S \tag{11}$$