Multiple Random Variables / Random Vectors (RVs)

Ex: Trow a dice and record the distance to this point:

x

and the number of dots.

dist.	dots
1	2
3	4
5	1
⋮	⋮

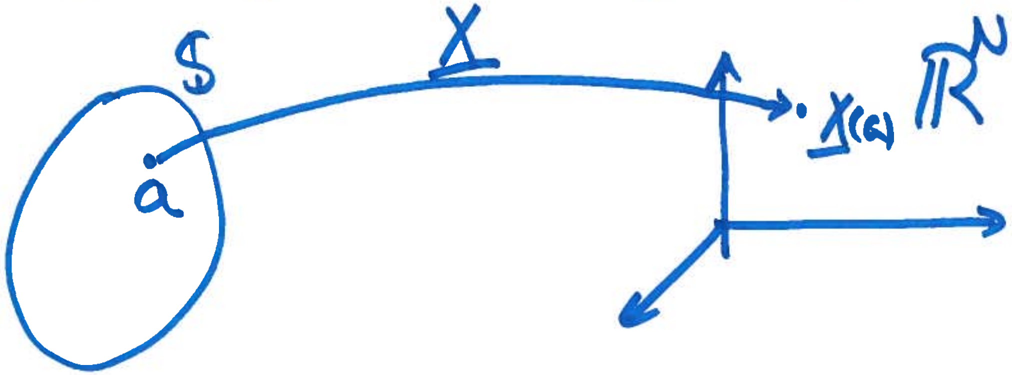
Ex: Trow a dice and record the distances to these two points

x 1 x 2

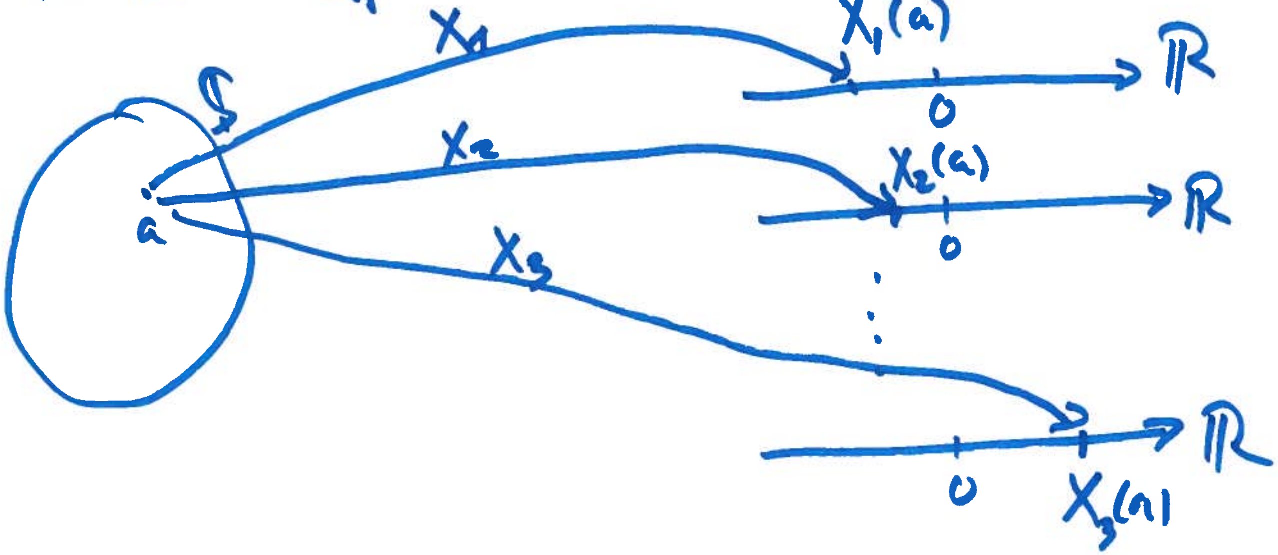
dist1	dist2
0.6	10.5
12,1	2.2
13,0	2.7
⋮	⋮

Random Vector $\underline{X}$:

- function from sample space S to range in $\mathbb{R}^N$:



- a vector of N scalar random variables

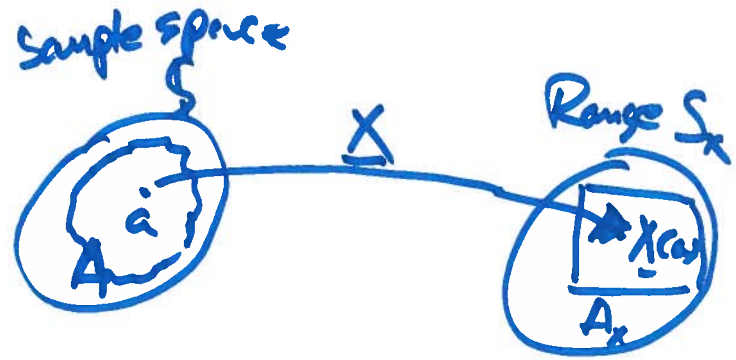


$$\underline{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix}.$$

Both views lead to the same math. object.

Joint pdf of cont. random vector (r.v.) $\underline{X}$

- $\underline{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_N \end{bmatrix}$ has joint pdf $p_{\underline{X}}(\underline{x}) = p_{X_1,\dots,X_N}(x_1,\dots,x_N)$



- Relation to probability:

$$\mathbb{P}(\underline{X} \in A_x) = \mathbb{P}(A)$$

$$= \mathbb{P}(\{a : X(a) \in A_x\})$$

$$= \int_{A_x} p_{\underline{X}}(\underline{x})\, d\underline{x} = \int \cdots \int_{A_x} p_{X_1,\dots,X_N}(x_1,\dots,x_N)\, dx_1 \cdots dx_N$$

- $\mathbb{P}(X \in S_x) = \mathbb{P}(S) = 1 \Rightarrow \int p_X(x)\, dx = 1, \; p_X(x) \geq 0.$

Example: N-variate Gaussian (Normal)

The pdf of an N-variate Gaussian vector $\underline{X}$ with mean $\underline{\mu}$ and covariance $\underline{\underline{C}}$:

$$P_{\underline{X}}(\underline{x}) = \frac{1}{\sqrt{2\pi}^N \det(\underline{\underline{C}})} \cdot \exp\left(-\frac{1}{2}(\underline{x}-\underline{\mu})^T \underline{\underline{C}}^{-1} (\underline{x}-\underline{\mu})\right)$$

Short-hand notation $\underline{X} \sim \mathcal{N}(\underline{\mu}, \underline{\underline{C}})$

Expectation operator (continuous case)

Def:

$$\mathbb{E}[g(\underline{X})] = \int g(\underline{x})\, p_{\underline{X}}(\underline{x})\, d\underline{x}$$

Ex: Mean: $g(\underline{X}) = \underline{X}$

$$\mu_{\underline{X}} = \mathbb{E}[\underline{X}] = \begin{bmatrix} \mathbb{E}[X_1] \\ \vdots \\ \mathbb{E}[X_N] \end{bmatrix}$$

Covariance matrix $\left(\underline{g}(\underline{x}) = (\underline{x} - \mu_{\underline{x}})(\underline{x} - \mu_{\underline{x}})^T \right)$

$$C_{\underline{X}} = \mathbb{E}\left[(\underline{X} - \mu_{\underline{X}})(\underline{X} - \mu_{\underline{X}})^T\right] = \mathbb{E}[XX^T] - \mu_x \mu_X^T$$

$$= \mathrm{Cov}(\underline{X})$$

Expectation is a *linear operator*

(also in the vector case)

For functions g_1, g_2 and constant a:

$$\begin{cases} \mathbb{E}\left[g_1(\underline{X}) + g_2(\underline{X}) \right] = \mathbb{E}\left[g_1(\underline{X}) \right] + \mathbb{E}\left[g_2(\underline{X}) \right] \\ \mathbb{E}\left[a\, g_1(\underline{X}) \right] = a\, \mathbb{E}\left[g_1(\underline{X}) \right]. \end{cases}$$

Marginal pdfs

- For r.v. $\underline{X} = [X_1, \ldots, X_N]^T$ we can obtain the pdf of X_i by integrating out the other variables in the joint pdf.

<u>Ex</u>: - Marginal for X_1:

$$P_{X_1}(x_1) = \int \cdots \int P_{\underline{X}}(\underline{x})\, dx_2\, dx_3 \ldots dx_N$$

- Marginal for X_1 and X_N:

$$P_{X_1 X_N}(x_1, x_N) = \int \cdots \int P_{\underline{X}}(\underline{x})\, dx_2 \cdots dx_{N-1}.$$

Independent Random Variables

Def: R.Vs are called independent if their joint pdf factorizes into marginals:

$$p_{\underline{X}}(\underline{x}) = p_{X_1}(x_1) \cdot p_{X_2}(x_2) \cdots p_{X_N}(x_N)$$

$$\Updownarrow$$

$X_1, \ldots, X_N$ are independent.

Covariance Matrix in different forms

$$C_{\underline{x}} = \mathbb{E}\left[(x-\mu_x)(x-\mu_x)^T\right] \qquad \mu_x = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \end{bmatrix}$$

$$= \mathbb{E}\left[XX^T\right] - \mu_x \mu_x^T$$

$$= \mathbb{E}\begin{bmatrix} (x_1-\mu_1)(x_1-\mu_1) & (x_1-\mu_1)(x_2-\mu_2) & (x_1-\mu_1)(x_3-\mu_3) & \cdots \\ (x_2-\mu_2)(x_1-\mu_1) & (x_2-\mu_2)(x_2-\mu_2) & (x_2-\mu_2)(x_3-\mu_3) & \\ (x_3-\mu_3)(x_1-\mu_1) & (x_3-\mu_3)(x_2-\mu_2) & (x_3-\mu_3)(x_3-\mu_3) & \\ \vdots & & & \end{bmatrix}_{N\times N}$$

$$= \begin{bmatrix} \mathrm{Var}(X_1) & \mathrm{Cov}(X_1,X_2) & \mathrm{Cov}(X_1,X_3) & \cdots \\ \mathrm{Cov}(X_2,X_1) & \mathrm{Var}(X_2) & \mathrm{Cov}(X_2,X_3) & \\ \mathrm{Cov}(X_3,X_1) & \mathrm{Cov}(X_3,X_2) & \mathrm{Var}(X_3) & \\ \vdots & & & \end{bmatrix}_{N\times N}$$

Independence enables factorization of expectation of products.

For independent X_1, X_2,

$$\mathbb{E}[X_1 \cdot X_2] = \iint x_1 x_2 \, p_{X_1 X_2}(x_1, x_2) \, dx_1 \, dx_2$$

$$= \iint x_1 x_2 \, p_{X_1}(x_1) \, p_{X_2}(x_2) \, dx_1 \, dx_2$$

$$= \int x_1 \, p_{X_1}(x_1) \, dx_1 \cdot \int x_2 \, p_{X_2}(x_2) \, dx_2$$

$$= \mathbb{E}[X_1] \cdot \mathbb{E}[X_2]$$

Independence $\overset{\not\Leftarrow}{\Rightarrow}$ zero covariance

For R.V.s X_1, X_2, the covariance is

$$\mathrm{cov}(X_1, X_2) = \mathbb{E}[(X_1-\mu_1)(X_2-\mu_2)] \underset{\substack{\uparrow \\ \text{linearity of } \mathbb{E}[\cdot]}}{=} \mathbb{E}[X_1 X_2] - \mu_1 \mu_2 .$$

For X_1 and X_2 independent,

$$\begin{aligned}\mathrm{Cov}(X_1, X_2) &= \mathbb{E}[X_1] \cdot \mathbb{E}[X_2] - \mu_1 \mu_2 \\ &= \mu_1 \cdot \mu_2 - \mu_1 \mu_2 = 0.\end{aligned}$$

Rmk: Zero covariance $\not\Rightarrow$ Independence

Independence ⟹ diagonal covariance matrix

$X_1, \dots, X_N$ independent , $\underline{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_N \end{bmatrix}$

$$C_{\underline{X}} = \begin{bmatrix} \mathrm{Var}(X_1) & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \mathrm{Var}(X_N) \end{bmatrix}$$

I.I.D. : Independent Identically Distributed r.v.s

$$p_{\underline{X}}(\underline{x}) = p_{X_1}(x_1) \cdots p_{X_N}(x_N) \qquad \underline{\text{and}} \qquad p_{X_1}(x) = p_{X_2}(x) = \cdots = p_{X_N}(x)$$

$$\text{IID} \Rightarrow \qquad C_{\underline{X}} = c \cdot \underline{\underline{I}} = \begin{bmatrix} c & & 0 \\ & \ddots & \\ 0 & & c \end{bmatrix}.$$