Exercises for Session 18 - Stochastic Processes

Exercise 1

A random electrical signal varies over time according to the expression

$$S(n) = \frac{1}{2}S(n-1) + U(n), \qquad n = 1, 2, \ldots$$

where $U(n) \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma_U^2)$ and $S(0) \sim \mathcal{N}(0, 1)$. At time n=1, a measurement device is connected to provide observations of the electrical signal that read

$$X(n) = S(n) + W(n), \qquad n = 1, 2, \ldots$$

Unfortunately, the device introduces a large measurement noise, especially right after connecting it. This is modeled by the noise process $W(n)$, which has zero-mean, Gaussian samples with decreasing variance $\sigma_W^2(n) = (1/2)^n$. The samples of $W(n)$ are statistically independent, and they are uncorrelated with the samples of $S(n)$ and $U(n)$.
We want to use a Kalman filter to estimate the values of $S(n)$, $n = 1, 2, \ldots$ from the observations $X(n)$.

- Write up the system and channel models for the Kalman filter estimating the samples of $S(n)$, and draw their corresponding block diagrams. Will you need a scalar or a vector Kalman filter?
- Draw the block diagram of the Kalman filter for the given problem. Write the expressions of the prediction and updating steps (including the Kalman gain) at time instant $n+1$, assuming estimates for time instant n have already been obtained. Do the same for the associated mean square errors.
- Calculate (by hand!) the predictions, updates and Kalman gains corresponding to the time instants $n = 1$ and $n = 2$. (Find the result as a function of the observations X(1) and X(2)).
- As n increases, the variance of the observation noise $W(n)$ decreases drastically. As n approaches infinity, the variance $\sigma_W^2(n)$ will approach zero. How will this show in the expressions of the Kalman gain and the obtained mean square error?

Exercise 2

For the scalar Kalman filter, show that the mean square error of the estimates after the updating step is always smaller or equal than the mean square error after the prediction step. In other words, prove that

$$R(n+1|n+1) \leq R(n+1|n).$$

When are these MSEs equal? Intuitively, does this make sense? What is the Kalman gain for this case?