Detection Theory

Example: A radar-based target detector outputs a signal

$$X(n) = s + W(n), \qquad\qquad n = 1, 2, \ldots$$

where

$$s = \begin{cases} 0, & \text{when no target is detected} \\ 1, & \text{when allied target is detected} \\ 2, & \text{when enemy target is detected} \end{cases}$$

and $w(n \overset{i.i.d.}{=\sim} \mathcal{N}(0,1)$. An operator of a defense system observes

$$\mathbf{X} = \begin{bmatrix} X(1) \\ X(2) \end{bmatrix} = \begin{bmatrix} 1.9 \\ 1.1 \end{bmatrix}$$

Should a missile be shot at the target?

Elements of detection theory

- Set of K hypotheses: $\mathcal{H}\{h_0, h_1, \ldots, h_{K-1}\}$
- True (but unknown) hypthesis: H
- Vector of observations: $\mathbf{X} = [x_0, x_2, \ldots, x_N]^T$
- Prior information: $P_k = P(H = h_k), \qquad k = 0, \ldots, K-1$

Decision Rule

$$\hat{H}(\mathbf{X}) : \qquad \hat{H} : \text{range}(\mathbf{X}) \to \mathcal{H}$$

$\hat{H}$ should be designed to be optimal according to a criterion. For example to make the right decision as often as possible.

Maximum a-posteriori (MAP) rule

The probability of a correct decision given $\mathbf{X} = \mathbf{x}$ is:

$$P(\hat{H}(\mathbf{x}) = H | \mathbf{X} = \mathbf{x})$$

Examine the a-posteriori probabilities (APP) of H:

$$P(H = h_k | \mathbf{X} = \mathbf{x}) = \frac{f_{\mathbf{X}|H}(\mathbf{x}|H = h_k) P_k}{f_{\mathbf{X}}(\mathbf{x})}$$

where $f_{\mathbf{X}|H}(\mathbf{x}|H = h_k)$ is called the likelihod for h_k and $f_{\mathbf{X}}(\mathbf{x}) = \sum_{k=0}^{K-1} f_{\mathbf{X}|H}(\mathbf{x})$.
MAP-rule

$$\begin{aligned}\hat{H}_{MAP}(\mathbf{x}) &= \arg\max_{h_k \in \mathcal{H}} P(H = h_k|\mathbf{x}) \\ &= \arg\max_{h_k \in \mathcal{H}} \frac{f_{\mathbf{X}|H}(\mathbf{x}|H = h_k)P_k}{f_{\mathbf{X}}(\mathbf{x})}\end{aligned}$$

As we are only interested in the argument and $f_{\mathbf{X}}(\mathbf{x})$ is a positive constant we can negalect it:

$$\hat{H}_{MAP}(\mathbf{x}) = \arg\max_{h_k \in \mathcal{H}} f_{\mathbf{X}|H}(\mathbf{x}|H = h_k)P_k$$

If P_k is constant for all k this can furthermore be ignored. Probability of error:

$$P_e = 1 - P(\hat{H} = H) = 1 - \int_{\mathbf{X}} P(\hat{H} = H|\mathbf{x}) f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}$$

Insert the MAP-rule:

$$P_e = 1 - P(\hat{H}_{MAP} = H) = 1 - \int_{\mathbf{X}} P(\hat{H}_{MAP} = H|\mathbf{x}) f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}$$

If the MAP-rule maximizes the part in the integral then it minimizes P_e.

Example (continued)

$$X(n)|H = h_k \overset{i.i.d.}{\sim} \mathcal{N}(s_k, 1), \qquad s_k = \begin{cases} 0, & k = 0 \\ 1, & k = 1 \\ 2, & k = 2 \end{cases}$$

$$\begin{aligned}f_{\mathbf{X}|H}(\mathbf{x}|h_k) &= f_{X(1)|H}(x(1)|h_k) f_{X(2)|H}(x(2)|h_k) \\ &= \frac{1}{\sqrt{2\pi}} \exp\left(-(x(1) - s_k)^2\right) \frac{1}{\sqrt{2\pi}} \exp\left(-(x(2) - s_k)^2\right)\end{aligned}$$

The MAP-rule:

$$\begin{aligned}H_{MAP}(\mathbf{X}) &= \arg\max_{h_k \in \mathcal{H}} f_{\mathbf{X}|H}(\mathbf{x}|h_k)P_k \\ &= \arg\max_{h_k \in \mathcal{H}} \frac{1}{2\pi} \exp\left(-(x(1) - s_k)^2 - (x(2)s_k)^2\right) P_k \\ &= \arg\max_{h_k \in \mathcal{H}} -(x(1) - s_k)^2 - (x(2) - s_k) + \log P_k, \quad \text{Applying log} \\ &= \arg\min_{h_k \in \mathcal{H}} (x(1) - s_k)^2 + (x(2) - s_k)^2 - \log P_k\end{aligned}$$

Expected cost
Compute the expected cost of deciding for hypothesis h_k:

$$C(\hat{H} = h_k|\mathbf{X}) = \sum_{k'=0}^{K=1} C_{kk'} P(H = k' + \mathbf{X})$$

Notice the difference between k and k'. We want to minimize the expected cost:

$$\hat{H}_{Bayes}(\mathbf{x}) = \underset{h_k \in \mathcal{H}}{\arg\min} C(h_k|\mathbf{x})$$

Binary decisions
Only 2 hypotheses:
$K = 2$, $\mathcal{H} = \{h_0, h_1\}$
This gives

$$\begin{aligned} \hat{H}_{ML}(\mathbf{x}) &= \underset{h_k \in \mathcal{H}}{\arg\max} f_{\mathbf{X}|H}(\mathbf{x}|h_k) \\ &= \begin{cases} h_0, & \frac{f_{\mathbf{X}|H}(\mathbf{x}|h_0)}{f_{\mathbf{X}|H}(\mathbf{x}|h_1} \geq 1 \\ h_1, & \text{otherwise} \end{cases} \end{aligned}$$

Likelihood ratio

$$\Lambda(\mathbf{x}) = \frac{f_{\mathbf{X}|H}(\mathbf{x}|h_0)}{f_{\mathbf{X}|H}(\mathbf{x}|h_1)}$$