# Stochastic Processes

## Session 5 — Lecture

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# Outline for Session 5 — Lecture

ILOs for Session 5

The autocorrelation and its properties

Weak sense stationary (WSS) processes

Gaussian WSS processes are SSS

Autocorrelation quiz

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## ILOs for Session 5

After attending this lecture and solving the exercises you should be able to:

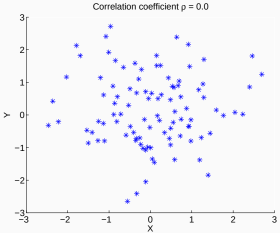
- Explain the meaning of an ACF to a fellow student.
- Compute the ACF for simple stochastic process using its definition.
- Use the theoretical properties of any ACF as sanity checks of your derivations.
- Determine whether or not a stochastic process is WSS.
- Relate the definitions of SSS and WSS.
- Know (without hesitation and computation) the ACF of an iid. process.

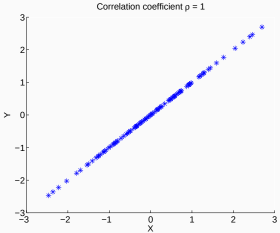
# Statistical dependency revealed via second-order characteristics.

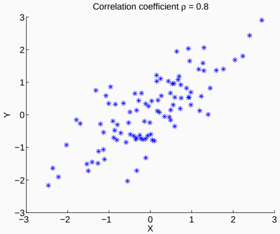
**Example:**

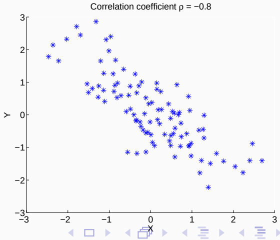
$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim \mathcal{N}\left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \right)$$

Correlation coefficient $\rho = \mathrm{Cov}(X, Y)/\sqrt{Var(X) \cdot Var(Y)}$



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# Definition of autocorrelation function (ACF)

Statistical dependency revealed via second-order characteristics.

For a random process $X(t)$, the second-order characteristics are captured by:

**Autocorrelation function (ACF):**[1]

$$R_X(t_1, t_2) := \mathbb{E}\big[X(t_1)X(t_2)\big]$$

An autocorrelation function is measuring statistical dependency among the random variables of a stochastic process

[1] [Kay] uses the term "autocorrelation sequence" (ACS) when discussing discrete-time processes and reserve the term "autocorrelation function" (ACF) for the continuos time case. We do not distinguish in the name.

## Properties of the autocorrelation function for general processes

**Non-negative at $t_1 = t_2$:**

$$R_X(t_1, t_1) \geq 0 \qquad \textit{(Show this!)}$$

**Symmetric function:**

$$R_X(t_1, t_2) = R_X(t_2, t_1) \qquad \textit{(Show this!)}$$

**Relation to covariance function:**

$$R_X(t_1, t_2) = C_X(t_1, t_2) + \mu_X(t_1)\mu_X(t_2) \qquad \textit{(Show this!)}$$

(Recall that $C_X(t_1, t_2) := \mathbb{E}[(X(t_1) - \mu_X(t_1)) \cdot (X(t_2) - \mu_X(t_2))]$.)

**Relation to the (auto-)correlation coefficient:**

$$\rho_X(t_1, t_2) := \frac{C_X(t_1, t_2)}{\sqrt{C_X(t_1, t_1)C_X(t_2, t_2)}} \in [-1, 1]$$

# Outline for Session 5 — Lecture

# Strict-sense stationarity is too restrictive in practice

**Recall the definition of SSS:**

**Definition of SSS:**
A stochastic process $X(t)$ is said to be *strict-sense stationary* (SSS) if for each $n \in \mathbb{N}$ and any $\tau \in \mathbb{T}$:

$$P\Big(X(t_1) \leq x_1, \ldots, X(t_n) \leq x_n\Big) = P\Big(X(t_1 + \tau) \leq x_1, \ldots, X(t_n + \tau) \leq x_n\Big)$$

$$\forall t_1, t_2, \ldots, t_n \in \mathbb{T} \qquad \text{and} \qquad \forall x_1, x_2, \ldots, x_n \in \mathbb{R}$$

**In words:**
The process $X(t)$ is SSS if *all* of its $n$'th-order joint cdf's are invariant under *arbitrary* time shifts.

**SSS is too strict for practical purposes:**

- It is usually hard to verify that all $n$'th-order joint cdf's are shift invariant.
- It is very rarely fulfilled in practice.

Therefore we seek a "weaker" stationarity definition.

# First and second moment properties of SSS processes:

**Recall that** a SSS process $X(t)$ has the following properties:
The *mean and variance are constants*:

$$\mu_X(t) = \mu_X \quad \text{and} \quad \sigma_X^2(t) = \sigma^2$$

The covariance $C_X(t, t+\tau)$ *is a function of the time lag* $\tau$ only:

$$C_X(t, t+\tau) = C_X(\tau)$$

The latter property carries over to the autocorrelation (show this):

$$R_x(t, t+\tau) = R_x(\tau)$$

**Warning:** While a SSS process has constant mean, variance and a covariance depending only on the time difference, these properties are not enough to ensure that a process is SSS!
**Motivation:** Often we care only about first and second moments. Thus we consider the class of processes with the same first and second moment properties as SSS processes.

# Weak-sense (or wide-sense) stationary (WSS) processes

We consider processes with the same first and second moment properties as holds for SSS processes. We call such processes weak-sense stationary (WSS):

**Definition:**
A stochastic process $X(t)$ is said to be *weak-sense stationary* (WSS) if:

1) $\mathbb{E}[X(t)] = \mu_X$, i.e. *no* dependency on time $t$
2) $\mathbb{E}\big[X(t)X(t+\tau)\big] = R_X(\tau)$, i.e. dependency *only* on the time lag $\tau$

**In plain words:**
The process $X(t)$ is WSS if it has constant mean and its autocorrelation function depends only on the time lag (not on absolute time instances).

## SSS versus WSS

We see from our definition of WSS that:

$$\boxed{X(t) \text{ is SSS} \quad \Longrightarrow \quad X(t) \text{ is WSS}}$$

*The converse is not true!*

# Properties of the autocorrelation function for WSS processes

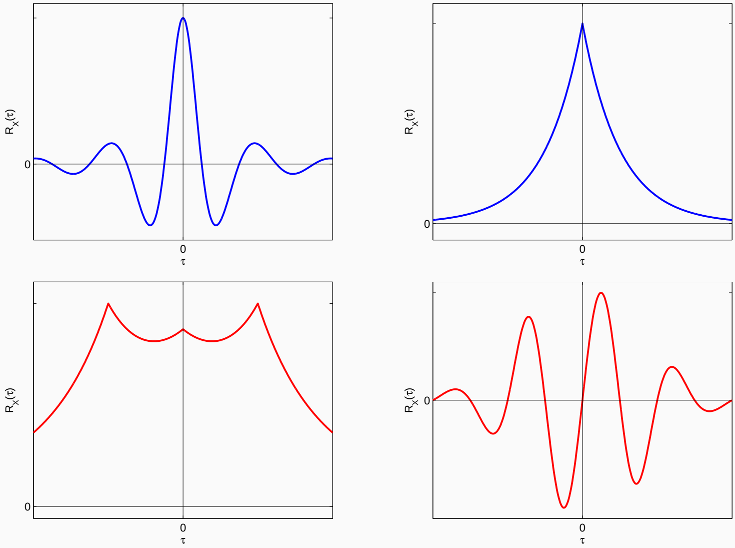
The ACF of a WSS process has a number of properties useful for checking your calculations:

1) $R_x(0) \geq 0$
2) $R_x(-\tau) = R_x(\tau) = R_x(|\tau|)$ for all $\tau \in \mathbb{T}$
3) $|R_x(\tau)| \leq R_x(0)$ for all $\tau \in \mathbb{T}$
4) $R_x(0) = \mathbb{E}[X(t)^2] = \sigma_X^2 + \mu_X^2$

In plain words this means that $R_x(\tau)$:

1) is nonnegative at zero *(When does the case $R_x(0) = 0$ occur?)*
2) is an even function (i.e. symmetric around zero)
3) has maximum magnitude at zero
4) The value at zero is the mean square of $X(t)$

# Graphical examples (and counterexamples)



# Positive semi-definiteness of the ACF is necessary and sufficient

Not all functions qualify as the autocorrelation function of a WSS process.

**Definition:**
A function $g : \mathbb{R} \to \mathbb{C}$ is called *positive semi-definite* if for each $n \in \mathbb{N}$ and $\forall t_1, t_2, \ldots, t_n \in \mathbb{R}$ and $\forall a_1, a_2, \ldots, a_n \in \mathbb{C}$ it holds that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i g(t_i - t_j) a_j^* \geq 0$$

which can be stated in matrix form as $\mathbf{a}^{*T}\mathbf{G}\mathbf{a} \geq 0$.

$R_X(\tau)$ is the autocorrelation function of a WSS random process $X(t)$
*if and only if*
$R_X(\tau)$ is a positive semi-definite function

The positive definiteness property of an autocorrelation function leads to all previously mentioned properties!

# Proof of necessity

It is easy to prove that an autocorrelation function is positive semi-definite: Define a random variable

$$Y := \sum_{i=1}^{n} a_i X(t_i) =: \mathbf{a}^T \mathbf{X}$$

for arbitrary $n$ and $a_i$'s. Trivially $\mathbb{E}[|Y|^2] \geq 0$ and therefore

$$0 \leq \mathbb{E}[|Y|^2] = \mathbb{E}[\mathbf{a}^{*T} \mathbf{X}^* \mathbf{X}^T \mathbf{a}] = \mathbf{a}^{*T} \mathbb{E}[\mathbf{X}\mathbf{X}^T] \mathbf{a} = \mathbf{a}^{*T} \mathbf{R}_{\mathbf{X}} \mathbf{a}$$

Entry $(i, j)$ of $\mathbf{R}_{\mathbf{X}}$ is $R_x(t_i - t_j)$, hence the $R_x(\tau)$ is positive semidefinite.

It is much harder to prove the converse, i.e. to prove that for any positive semidefinite function there exist a stochastic process with this as autocorrelation. The proof is out of scope of this course.

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# Gaussian WSS processes are SSS

Let $X(t)$ be a Gaussian WSS processes with covariance function $C_X(\tau)$ and define the two vectors

$$\mathbf{X} := \begin{bmatrix} X(t_1) \\ X(t_2) \\ \vdots \\ X(t_n) \end{bmatrix} \sim \mathcal{N}(\boldsymbol{\mu}_{\mathbf{X}}, \boldsymbol{C}_{\mathbf{X}}) \quad \text{and} \quad \mathbf{X}' := \begin{bmatrix} X(t_1+\tau) \\ X(t_2+\tau) \\ \vdots \\ X(t_n+\tau) \end{bmatrix} \sim \mathcal{N}(\boldsymbol{\mu}_{\mathbf{X}'}, \boldsymbol{C}_{\mathbf{X}'}).$$

The vector $\mathbf{X}$ is Gaussian with mean vector $\boldsymbol{\mu}_{\mathbf{X}} = \mathbb{E}[\mathbf{X}]$ and covariance matrix

$$\boldsymbol{C}_{\mathbf{X}} := [(\mathbf{X} - \boldsymbol{\mu}_{\mathbf{X}})(\mathbf{X} - \boldsymbol{\mu}_{\mathbf{X}})^T].$$

Similarly, $\mathbf{X}'$ has mean $\boldsymbol{\mu}_{\mathbf{X}'}$ and covariance $\boldsymbol{C}_{\mathbf{X}'}$

To prove that $X(t)$ is SSS we check that $\boldsymbol{\mu}_{\mathbf{X}} = \boldsymbol{\mu}_{\mathbf{X}'}$ and $\boldsymbol{C}_{\mathbf{X}} = \boldsymbol{C}_{\mathbf{X}'}$ hold.

# Gaussian WSS processes are SSS (contd.)

Since $X(t)$ is WSS and thus have shift invariant mean and covariance functions:

$$\boldsymbol{\mu}_{\mathbf{X}} = \begin{bmatrix} \mu_X(t_1) \\ \mu_X(t_2) \\ \vdots \\ \mu_X(t_n) \end{bmatrix} = \begin{bmatrix} \mu_X \\ \mu_X \\ \vdots \\ \mu_X \end{bmatrix} = \begin{bmatrix} \mu_X(t_1+\tau) \\ \mu_X(t_2+\tau) \\ \vdots \\ \mu_X(t_n+\tau) \end{bmatrix} = \boldsymbol{\mu}_{\mathbf{X}'}$$

$$\begin{aligned} \boldsymbol{C}_{\mathbf{X}} &= \mathbb{E}[(\mathbf{X}-\boldsymbol{\mu}_{\mathbf{X}})(\mathbf{X}-\boldsymbol{\mu}_{\mathbf{X}})^T] \\ &= \begin{bmatrix} C_X(0) & C_X(t_1-t_2) & \ldots & C_X(t_1-t_n) \\ C_X(t_2-t_1) & C_X(0) & \ldots & C_X(t_2-t_n) \\ \vdots & \vdots & \ddots & \vdots \\ C_X(t_n-t_1) & C_X(t_n-t_2) & \ldots & C_X(0) \end{bmatrix} \\ &= \mathbb{E}[(\mathbf{X}'-\boldsymbol{\mu}_{\mathbf{X}'})(\mathbf{X}'-\boldsymbol{\mu}_{\mathbf{X}'})^T] = \boldsymbol{C}_{\mathbf{X}'} \end{aligned}$$

We conclude that

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|---|
| If $X(t)$ is Gaussian and WSS it is also SSS. |

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# Quiz: Which realizations belong to which ACF?