Quadratically constrained quadratic problem

$$\begin{aligned}
&\min_{x\in\mathbb{R}^N} x^T C x \\
&\text{s.t} \qquad\qquad\qquad\qquad x^T F : x \geq g_i, i = 1, \dots, p \\
&\qquad\qquad\qquad\qquad\qquad x^T B : x \leq b_i, i =, \dots, l \\
&\qquad\qquad\qquad\qquad\qquad x^T H : x = h, i = 1, \dots, k
\end{aligned}$$

The matrices are real and symmetric.

Remember, that $f(x) = x^T C x$ is convex if $c \succeq 0$.

What makes the feasible set convex?

$$\{x \in \mathbb{R}^N \,:\, x^T F x \leq 1\}$$

If $F \succeq 0$ the set is convex. Is the following set convex?

$$\{x \in \mathbb{R}^N \,:\, x^T F x \geq 2\}$$

No, as this set is alle points outside of the circle with radius 2. If the inequality is flipped the set will be convex.

Let C and F be PSD

$$\begin{aligned}
\min \qquad & x^T C x \\
\text{s.t.} \qquad & x^T F x \geq 1
\end{aligned}$$

This problem is not convex.

$$\begin{aligned}
\min_{x\in\mathbb{R}^N} \qquad & x^T C x \\
\text{s.t.} \qquad & x_i^2 = 1 i = 1, \dots, p
\end{aligned}$$

The set is discrete as only few points fulfill the properties. The problem is not convex.

Trace of a matrix
The trace tr of a matrix is the sum of the elements on the main diagonal.

$$x^T C x = \operatorname{tr}(x^T C x)$$

The tr-operator is invariant to cyclic permutation:

$$\operatorname{tr}(ABCD) = \operatorname{tr}(DABC)$$

This means that

$$x^T C = \operatorname{tr}(x^T C x) = \operatorname{tr}(x x^T C)$$

Notice that $xx^T \in \mathbb{R}^{N\times N}$ and $x^T x \in \mathbb{R}$. Let $w = xx^T$, which gives $\operatorname{rank}(w) = 1$.

Making a non-convex problem convex

$$\begin{aligned}
&\min_{x\in\mathbb{R}^N} && x^T C x \\
&\text{s.t.} && x^T F x \geq g \\
&\Downarrow && \\
&\min_{w\in\mathbb{R}^{N\times N}} && \operatorname{tr}(Cw) \\
&\text{s.t.} && \operatorname{tr}(Fw) \geq g \\
& && w \succeq 0 \\
& && \operatorname{rank}(w) = 1
\end{aligned}$$

The second problem is linear in w but the probem is just as hard as the first problem as $\operatorname{rank}(w) = 1$. If $\operatorname{rank}(w) = 1$ is skipped, it is called semidefinite relaxation:

$$\begin{aligned}
&\min_{w\in\mathbb{R}^{N\times N}} && \operatorname{tr}(Cw) \\
&\text{s.t.} && \operatorname{tr}(Fw) \geq g \\
& && w \succeq 0
\end{aligned}$$

The problem is that $\operatorname{rank}(w)$ is usually large and the problem therefore does not give the same solutions as the original problem.

Let $\operatorname{rank}(w) = r$.

$$\begin{aligned}
w &= \sum_{i=1}^{r} \lambda_i v_i v_i^T \\
&\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \\
&v_i,\ i = 1, \ldots, r
\end{aligned}$$

Then $w \approx \lambda_1 \lambda_1 v_1^T$. This means that $x \approx \sqrt{\lambda_1} v_1 \sqrt{\lambda_1} v_1^T$.

Let $S^{N\times N}$ be the set of all symmetric $N \times N$-matrices.
Let $\xi \in \mathbb{R}^N \sim \mathcal{N}(0, Q)$.

$$\begin{aligned} \min \qquad & E\left[\xi^T C \xi\right] \\ \text{s.t.} \qquad & E[\xi^T F \xi] \geq g \\ & Q \succeq 0 \end{aligned}$$

Remember that $E\left[\xi\xi^T\right]$ and $\xi^T C \xi = \operatorname{tr}(C\xi\xi^T)$. As both are linear:

$$\begin{aligned} \operatorname{tr}\left(E\left[\xi^T C \xi\right]\right) &= E\left[\operatorname{tr}\left(\xi^T C \xi\right)\right] \\ &= E\left[\operatorname{tr}\left(c\xi\xi^t\right)\right] \\ &= \operatorname{tr}\left(CE\left[\xi\xi^T\right]\right) \\ &= \operatorname{tr}(CQ) \end{aligned}$$

The problem is now

$$\begin{aligned} \min_Q \qquad & \operatorname{tr}(CQ) \\ \text{s.t.} \qquad & \operatorname{tr}(FQ) \geq g \\ & Q \succeq 0 \end{aligned}$$

And $Q = w$.