# 5 Two random variables – conditional expectation

- Define conditional expectation
- Prove Law of Total Expectation
- $E[Y] = E[E[Y|X]]$
- $\text{Var}[Y|X] = E[Y^2|X] - (E[Y|X])^2$
- Prove special variance

**Definition 3.12.** Suppose $X$ and $Y$ are jointly continuous. We define

$$E[Y|X = x] = \int_{-\infty}^{\infty} y f_Y(y|x)\, dy \tag{1}$$

**Proposition 3.6** Let $X$ and $Y$ be jointly continuous. Then

$$f_Y = \int_{-\infty}^{\infty} f_Y(y|x|) f_X(x)\, dx \tag{2}$$

**Proposition 3.17. (Law of Total Expectation)** Suppose that $X$ and $Y$ are jointly continuous. Then

$$E[Y] = \int_{-\infty}^{\infty} E[Y|X = x] f_X(x)\, dx \tag{3}$$

**Bevis**
By definition of expected value and proposition 3.6

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\, dy = \int_{-\infty}^{\infty} y f_Y(y|x) f_X(x)\, dx \tag{4}$$

Change order of integration

$$E[Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_Y(y|x)\, dy f_X(x)\, dx \tag{5}$$

The inner integral is the definition of $E[Y|X = x]$.

**Corollary 3.5.**

$$E[Y] = E\left[E[Y|X]\right] \tag{6}$$

**Corollary 3.7.**

$$\mathrm{Var}[Y|X] = E[Y^2|X] - (E[Y|X])^2 \tag{7}$$

**Proposition 3.19.**

$$\mathrm{Var}[Y] = \mathrm{Var}[E[Y|X]] + E[\mathrm{Var}[Y|X]] \tag{8}$$

**Bevis**

Take the expected value on corollary 3.7:

$$\begin{aligned} E[\mathrm{Var}[Y|X]] &= E[E[Y^2|X]] - E[(E[Y|X])^2] \\ &= E[Y^2] - E[(E[Y|X])^2] \end{aligned}$$

Use the fact, that

$$\mathrm{Var}[X] = E[X^2] - (E[X])^2 \tag{9}$$

and corollary 3.5 to write

$$\begin{aligned} \mathrm{Var}[E[Y|X]] &= E[(E[Y|X)^2] - (E[E[Y|X])^2 \\ &= E[(E[Y|X])^2] - (E[Y])^2 \end{aligned}$$

Add the two equations:

$$\begin{aligned} E[\mathrm{Var}[Y|X]] + \mathrm{Var}[E[Y|X]] &= E[Y^2] - E[(E[Y|X])^2] + E[(E[Y|X])^2] - (E[Y])^2 \\ &= E[Y^2] - (E[Y])^2 \\ &= \mathrm{Var}[Y] \end{aligned}$$

Gør opmærksom på hvor vigtigt resultatet er.