Stochastic Processes, Session 4. Group Work

Monte Carlo Simulation of Stochastic Processes

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Session 4

Go through the exercises below. Allow yourself the time to reflect over your results and discuss them with other students! Use the book for inspiration and for further information.

4.1 Interview your group mates

For this exercise, we will work in pairs (groups of three are also OK). Use 10 minutes to ask yourself at least three relevant questions about the concepts of discrete and continuous random variables, random processes, expected values, strict-sense stationary and i.i.d. random processes. Make sure you can find the answer to your questions in the book or the course slides/notes. Write your questions, as shortly and clearly as possible, and pass them on to your group mate. Use the next 10 minutes to attempt to answer each others questions. Examples of such could be:

- What is the covariance function of an i.i.d. random process?
- Name two examples of continuous probability distributions.

4.2 I.i.d. Processes

Make a program that can stem plot N samples of a realization of the standard discrete time Gaussian i.i.d. process

$$\{X_k\} \overset{i.i.d.}{\sim} \mathcal{N}(0,1) \tag{1}$$

and from the uniform i.i.d. process

$$\{Y_k\} \overset{i.i.d.}{\sim} \mathcal{U}(0,1). \tag{2}$$

Now, simulate an exponential i.i.d. process $\{Z_k\}$ by transforming $\{Y_k\}$.[1] Discuss differences and similarities between the realizations of $\{X_k\}$, $\{Y_k\}$, and $\{Z_k\}$.

Derive and/or write up the mean and variance functions:

$$\begin{aligned} \mu_X(k) &= \mathbb{E}[X_k] & \sigma_X^2(k) &= \mathrm{Var}(X_k) \\ \mu_Y(k) &= \mathbb{E}[Y_k] & \sigma_Y^2(k) &= \mathrm{Var}(Y_k) \\ \mu_Z(k) &= \mathbb{E}[Z_k] & \sigma_Z^2(k) &= \mathrm{Var}(Z_k) \end{aligned}$$

[1] Read again [Kay] Section 10.9 if you don't know how to do this.

Modify your program to draw a large number of such realizations, and then estimate the mean and variance functions. Compare your estimates to the theoretical values. How many realizations do you need to get accurate estimates?

Now change you program such that $\mu_X(k) = \mu_Y(k) = a$ and $\sigma_X^2(k) = \sigma_Y^2(k) = b$, where a and b are arbitrary input parameters to your program.

4.3 Sum of Two Independent Random Variables

In this exercise, we want to find out about the distribution of a random variable obtained by adding two independent random variables. First, consider the following variables:

$$\begin{aligned}
X_1 &\sim \mathcal{N}(0,1), \\
X_2 &\sim \mathcal{N}(0,1), \\
Y &= X_1 + X_2.
\end{aligned}$$

Write a script that generates multiple realizations of the variables. Use the obtained realizations to estimate and plot their pdfs.[2] From the obtained results, what seems to be the distribution of Y? Does this confirm what you would expect from the theory?

Next, repeat the same procedure but with the following variables:

$$\begin{aligned}
U_1 &\sim \mathcal{U}(-\sqrt{3}, +\sqrt{3}), \\
U_2 &\sim \mathcal{U}(-\sqrt{3}, +\sqrt{3}), \\
Z &= U_1 + U_2.
\end{aligned}$$

How does the pdf of the transformed random variable Z look in this case? If you are surprised by the results, you should review again Section 12.6 in Kay's book, in particular Example 12.8 and the generalization that follows.

To finalize, repeat the exercise, only this time adding up two uniform random variables with different ranges.

4.4 Cosines With Random Phases

Consider the continuous time process

$$X(t) = \cos(t + \Theta), \quad t \in \mathbb{R}, \quad \Theta \sim \mathcal{U}(-\pi, \pi). \tag{3}$$

Simulate and plot some realization of $X(t)$. Since $X(t)$ is a continuous-time process you will need to sample in order to plot it in Matlab.

Compute the mean and variance functions for $X(t)$. Check your results with Monte Carlo simulations.

[2] You can estimate pdfs from realizations of random variables in several ways using MATLAB. You may use histograms, which provide useful information about the pdf's shape (try out the commands `histogram` and `histfit`. A slightly more useful tool can be the command `ksdensity`. Read the MATLAB documentation about these commands and try them out!

4.5 N-variate Gaussian Random Vectors

Simulate a zero mean N variate Gaussian random vector $\mathbf{X} = [X_1, X_2, \ldots, X_N]^T$ with covariance matrix[3]

$$\mathbf{C}_\mathbf{X} := \mathbb{E}[(\mathbf{X} - \boldsymbol{\mu}_\mathbf{X})(\mathbf{X} - \boldsymbol{\mu}_\mathbf{X})^T] \tag{4}$$

$$= \begin{bmatrix} 1 & h & h^2 & \ddots & \ddots & h^{N-1} \\ h & 1 & h & h^2 & \ddots & h^{N-2} \\ h^2 & h & 1 & h & \ddots & h^{N-3} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ h^{N-1} & h^{N-2} & \ldots & \ldots & h & 1 \end{bmatrix} \tag{5}$$

Notice that for $\mathbf{C}_\mathbf{X}$ to be a covariance matrix, it should be positive semi-definite. What is the valid interval for the parameter h such that $\mathbf{C}_\mathbf{X}$ is positive semi-definite? Compare realizations for different values of h.

4.6 Moving Average (MA(1)) Process

Later in the course we will look at discrete time Moving Average (MA) processes. The MA(1) process is a special case of an MA process defined as:

$$V_k = X_k + X_{k+1}, \tag{6}$$

with the i.i.d. process $\{X_k\}$ defined as in (1).

Compute the mean and variance functions for $\{V_k\}$. Estimate the mean and variance functions for $\{V_k\}$ using Monte Carlo simulation.

Compute the covariance function for $\{V_k\}$. Is $\{V_k\}$ an i.i.d. process?

[3] Recap [Kay] Sections 12.11 and 14.9 if needed.

4.7 Extra: The Cauchy Challenge

Let $X \sim \mathcal{N}(0,1)$, i.e. X is a standard Gaussian/normal random variable, and denote by f_X the pdf of X. Consider also a continuous random variable Y with pdf

$$f_Y(y) = \frac{1}{\pi(1+y^2)}, \qquad y \in \mathbb{R}.$$

The two pdf's are shown jointly in the figure below.

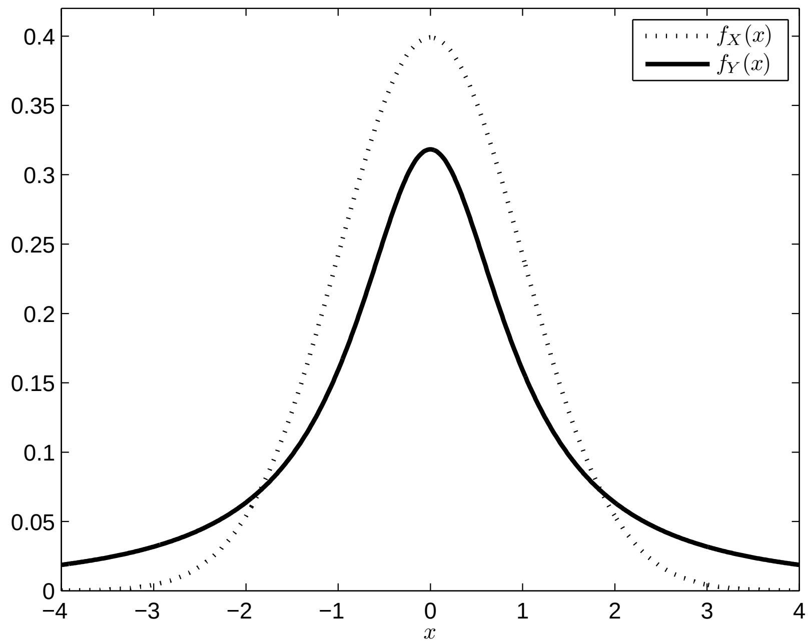

Figure 1: Joint plot of $f_X(x)$ and $f_Y(x)$ in the range $-4 \le x \le 4$.

We know that $\mathbb{E}[X] = 0$ and we are not surprised by this fact since f_X is symmetric around zero. Now, since f_Y is symmetric around zero as well, what is your immediate and intuitive guess of $\mathbb{E}[Y]$?

The distribution inherited by Y is called the *Cauchy distribution*. Verify that f_Y is indeed a probability density function (i.e. it is nonnegative and integrates to unity) and afterwards calculate $\mathbb{E}[Y]$ using the definition of expectation.[4] If you are surprised, you should conduct a small experiment in Matlab. To draw observations of Y, let $Z \sim \mathcal{U}\big(-\frac{\pi}{2}, \frac{\pi}{2}\big)$ and set $Y = \tan(Z)$. Draw an i.i.d. sample $Y_1, \dots, Y_N$ and estimate the mean for increasing values of N.

[4] You may use the antiderivatives:

$$\int \frac{1}{1+y^2} dy = \arctan(y) \qquad \text{and} \qquad \int \frac{2y}{1+y^2} dy = \ln(1+y^2), \qquad y \in \mathbb{R}.$$