Stochastic Processes

Session 12 – Lecture

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Outline for Session 12 – Lecture

2D Point Processes

Binomial Point Processes

Poisson Point Processes

Shot Noise Processes and Campbell's Theorem

ILOs for Session 12

After having attended this lecture and solved the exercises you should be able to:

- Give examples of practical occurrences of random point patterns from your own field of study.
- Explain intuitively what a region count is and discuss its main properties.
- Relate the interpretation of an intensity function to the interpretation of a probability density function (pdf). Discuss similarities and distinctions.
- Simulate realizations of 2D binomial and Poisson point processes.

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2D Point Processes

Binomial Point Processes

Poisson Point Processes

Shot Noise Processes and Campbell's Theorem

Definition of a Point Process

A point process X is a random, countable collection of points sitting in some region S (of the line, plane, sphere, etc). A realization X is a set of points:

$$X = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \dots\} \qquad \boldsymbol{x}_i \in S$$

Both the total number $N(S)$ of points in X and their values may be random.

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In lecture exercises:

Ex1: Sketch a few realizations of a 1D point process

Ex2: Sketch a few realizations of a 2D point process

Ex3: Give one or two examples of a phenomenon that can be described by either 1D or 2D process from your field of study

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- Ex1: Sketch a few realizations of a 1D point process
- Ex2: Sketch a few realizations of a 2D point process
- Ex3: Give one or two examples of a phenomenon that can be described by either 1D or 2D process from your field of study

Remark: We focus here in on 2D point processes for convenience. However all the results from 2D processes carry over to the 1D case.

Region Counts

The region count $N_X(B)$ is the number of points from X falling in some region $B \subseteq S$:

$$N_X(B) = |X \cap B| = \sum_{\boldsymbol{x} \in X} \mathbb{1}[\boldsymbol{x} \in B].$$

$N_X(B)$ is a discrete random variable with range $0, 1, 2, \ldots$ and probability mass function depending on the particular of point process.

Ex4: Sketch a realization of a 2D point point process and draw two overlapping regions A and B. Then find the values $N_X(A)$, $N_X(B)$, $N_X(A \cup B)$ and $N_X(A \cap B)$.

Ex5: Argue for the following facts

- $N_X(\emptyset) = 0$.
- $N_X(A \cup B) = N_X(A) + N_X(B)$ for disjoint sets A and B (this is, $A \cap B = \emptyset$).

Intensity Measures

Since the region count $N_X(B)$ is a random variable, we find its expected value $\mathbb{E}[N_X(B)]$. By doing so, we obtain a deterministic function of the region B

$$\mu_X(B) = \mathbb{E}[N_X(B)] = \mathbb{E}\left[\sum_{\boldsymbol{x} \in X} \mathbb{1}[\boldsymbol{x} \in B]\right]$$

which we call the *intensity measure* of X.

If the distribution of the region count $N_X(B)$ is known, we can calculate the intensity function as

$$\mu_X(B) = \mathbb{E}[N_X(B)] = \sum_{n=0}^{\infty} n \Pr(N_X(B) = n).$$

Unfortunately, the distribution of region counts $N_X(B)$ is often unknown. Instead the intensity measure can in most relevant cases be defined via an *intensity function*.

Intensity Functions

In most cases we can express the intensity measure as an integral of a non-negative locally integrable function called the *intensity function* ϱ_X

$$\mu_X(B) = \int_B \varrho_X(\boldsymbol{x}) d\boldsymbol{x}$$

The value $\varrho_X(\boldsymbol{x})$ can be interpreted as the mean number of points per unit area in a small neighborhood of $\boldsymbol{x}$.

When the intensity function of a process X is constant ($\varrho_X(\boldsymbol{x}) = \varrho_0$ for all $\boldsymbol{x} \in S$), X is a *homogeneous* point process[1]. Otherwise, X is *inhomogeneous*.

[1] For homogeneous point processes, the value of the intensity function ϱ_0 can be interpreted as the mean number of points per unit area

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Binomial Point Processes

A *binomial point process* $X \sim binomialPP(S, k, f)$ is a collection of k points drawn iid. according to a pdf $f(\boldsymbol{x})$ on S, i.e.

$$X = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k\} \overset{iid}{\sim} f(\boldsymbol{x}) \tag{1}$$

Intensity measure and function:

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Ex6: What is $\mu_X(S)$ in this case?

Ex7: Let A, B be disjoint sets; $N_X(A)$ and $N_X(B)$ are dependent random variables - do you see why?.

Binomial Point Processes — About the Name

The name of the binomial point process indicates the property that region counts are binomial random variables.

The probability of a given point $\boldsymbol{x_i} \in X$ being inside a region $B \subseteq S$ is

$$\Pr(\boldsymbol{x_i} \in B) = \int_B f(\boldsymbol{x}) d\boldsymbol{x}.$$

The region counts $N_X(B)$ follow a binomial distribution with number of trials k and success probability p, i.e.

$$\Pr(N_X(B) = n) = \left(\begin{array}{c} k \\ n \end{array} \right) p^n (1-p)^{k-n}, \qquad p = \int_B f(\boldsymbol{x}) d\boldsymbol{x}$$

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Poisson Point Processes

A point process X on $S \subseteq \mathbb{R}^2$ is called a *Poisson Point Process* with intensity function ϱ_X ($X \sim \text{PoissonPP}(S, \varrho_X)$) if two conditions are satisfied:

- For any region $B \subseteq S$ with $\mu_X(B) = \int_B \varrho(\boldsymbol{x}) d\boldsymbol{x} < \infty$, the region count $N_X(B)$ is Poisson distributed with mean $\mu_X(B)$:

$$\Pr(N_X(B) = k) = \exp(-\mu_X(B)) \frac{(\mu_X(B))^k}{k!}.$$

- Conditioned on $N_X(B) = k$, these k points form a binomial point process on B:

$$X \cap B \sim \text{binomialPP}(B, k, f_B), \qquad \text{with } f_B(\boldsymbol{x}) = \mathbb{1}[\boldsymbol{x} \in B] \frac{\varrho(\boldsymbol{x})}{\mu_X(B)}.$$

An important property of Poisson point processes is that, if two regions $A, B \subset S$ are disjoint ($A \cap B = \emptyset$), then their region counts $N_X(A)$ and $N_X(B)$ are independent random variables.

This fact implies that to simulate a Poisson processes can be done by partitioning S and draw points in the subsets independently.

The Poisson Point Process is a Limit of Binomial Point Process

Consider the process $X \sim \text{binomialPP}(S, k, f)$.
It is possible to increase $k \to \infty$ while keeping the mean count constant $\mu_X(A) = \lambda$ for a region A by either shrinking A as k increases or by expanding S and thus reducing f.
In this case, the region count converges to a Poisson pmf.
Since $\lambda = p \cdot k$, we have $p = \lambda/k$, and thus

$$\begin{aligned}
\Pr(N(A) = n) &= \binom{k}{n} \left(\frac{\lambda}{k}\right)^n \left(1 - \frac{\lambda}{k}\right)^{k-n} \\
&= \binom{k}{n} \left(\frac{\lambda}{k}\right)^n \times \left(1 - \frac{\lambda}{k}\right)^{k} \times \left(1 - \frac{\lambda}{k}\right)^{-n}
\end{aligned}$$

Taking the limit of each factor, we obtain the Poisson pmf:

$$\begin{aligned}
\lim_{k\to\infty} \Pr(N(A) = n) &= \underbrace{\lim_{k\to\infty} \binom{k}{n} \left(\frac{\lambda}{k}\right)^n}_{=\frac{\lambda^n}{n!}} \times \underbrace{\lim_{k\to\infty} \left(1 - \frac{\lambda}{k}\right)^{k}}_{=e^{-\lambda}} \times \underbrace{\lim_{k\to\infty} \left(1 - \frac{\lambda}{k}\right)^{-n}}_{=1} \\
&= \frac{\lambda^n}{n!} \exp(-\lambda).
\end{aligned}$$

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Shot Noise Processes

Point processes are often used as the foundation to build other types of random processes. One example of these are *shot noise processes*.

A shot noise process is a continuous-time random process constructed as[2]

$$Z(t) = \sum_{y \in Y} h(t - y)$$

where

- Y is a one-dimensional homogeneous Poisson point process, and
- $h(t)$ is a deterministic, real-valued function.

Shot noise processes can be used to model many different phenomena arising in science and engineering. See the lecture notes for some examples.

[2] This is only the most basic type of shot noise processes. This definition can be generalized to N dimensional processes, by using other types of point processes Y or, even, non-deterministic functions $h(t)$.

Campbell's Theorem

A useful tool to operate with shot noise processes is *Campbell's theorem*. When the theorem applies, it provides an easy way to compute the expected value of a function summed over a point process with a given intensity function. The theorem states:

- Let X be a point process on S with intensity function ϱ_X. Then for any function $g : S \to \mathbb{R}$, the random variable $\sum_{\boldsymbol{x} \in X} g(\boldsymbol{x})$ has expected value

$$\mathbb{E}\left[\sum_{\boldsymbol{x} \in X} g(\boldsymbol{x})\right] = \int_S g(\boldsymbol{x}) \varrho_X(\boldsymbol{x}) d\boldsymbol{x}$$

  provided that the integral on the right-hand side exists.

If we apply the above result to the shot noise process we defined in the previous slide, we get

$$\mathbb{E}[Z(t)] = \mathbb{E}\left[\sum_{y \in Y} h(t-y)\right] = \int_S h(t-y) \varrho_Y(y) dy.$$