Exam: Stochastic Processes

Date and Time: Friday January 6, 2017, 9:00–13:00.

This entire problem set contains **4 pages**. Please make sure that you have received all pages.

The exam is graded according to your answer as a whole: both quantity and quality count. We value concise arguments showing your command of the topics. Simply answering "Yes." or "No." will not do that!

We recommend that you read through each problem thoroughly before starting to solve it. Should you happen to get stuck at some point, we recommend that you continue and anyway try to solve the rest. You always have the opportunity to sketch or explain how you would have continued if you hadn't got stuck.

It is allowed to use books, lecture notes, your own notes, calculators and computers during the exam. Communication to others during the exam is not allowed—therefore, the use of internet is strictly forbidden.

Problem 1: Stationary Processes?

Consider the following four stochastic processes:

$$X(n) \overset{i.i.d}{\sim} \mathcal{N}(2,1) \tag{1}$$

$$Y_1(n) = \frac{X(n-2)+X(n-3)+X(n+5)}{n} \tag{2}$$

$$Y_2(n) = 3X(n) - 5X(n-2) + 6 \tag{3}$$

$$Y_3(n) = \begin{cases} X(n) & n = 0,2,4,6,\ldots \\ X(n-1) & n = 1,3,5,7,\ldots \end{cases} \tag{4}$$

1.1 Which of these four processes are **not** white sense stationary (WSS)? Justify why.

1.2 For those processes which are WSS, calculate their mean function and autocorrelation function (ACF). Make a sketch of the ACF as well.

1.3 Compute and sketch the PSD of the WSS processes in the problem.

1.4 Are any of the processes strict sense stationary (SSS)? Justify your answer.

1.5 Provide an example (different from the processes in the problem) of a process which is WSS but is not SSS.

Next assume that you are provided with a set of data of length $N = 1000$ samples. The data is known to belong to a realization of either the process $X(n)$ in (1) or the process $Y_2(n)$ in (3).

1.6 Explain two different ways in which you could process the data in order to determine whether it is a realization of $X(n)$ or a realization of $Y_2(n)$.

Problem 2: Estimation of Temperature in Coloured Noise

An electronic circuit designed to measure a temperature T outputs a discrete-time voltage signal $X(n)$ modeled as:

$$X(n) = sT + W(n), \qquad n = 1, 2, \dots, N. \tag{5}$$

Here the known constant s represents the sensitivity of the temperature sensor. The electronic components produce some noise, which is modeled as the WSS process

$$W(n) = aW(n-1) + U(n), \qquad n = 1, 2, \dots, N. \tag{6}$$

with $\{U(n)\} \overset{i.i.d}{\sim} \mathcal{N}(0, \sigma_U^2)$ and $W(0) \sim \mathcal{N}(0, \frac{\sigma_U^2}{1-a^2})$. The noise process is assumed to be independent of the temperature.

2.1 What type of process is the process $W(n)$ in (6)?

2.2 The process $W(n)$ in (6) can be seen as the output of a linear time-invariant system to a random input $U(n)$. Calculate the transfer function of the system and use it to compute the power spectral density (PSD) of $W(n)$.

2.3 Make an approximate sketch of the PSD for the values $a = 1/2$ and $a = -1/2$. Based on the sketched PSDs, comment on the temporal correlation properties of the process for these two specific values for a.

We now want to estimate the value of the temperature T based on $N = 2$ measurements of the output voltage $X(n)$ (expressed in millivolt, mV). The temperature is (in degree Celsius, °C) assumed to be uniformly distributed between 0 °C and 24 °C, i.e., $T \sim \mathcal{U}(0, 24)$, and the sensor has a sensitivity $s = 10\,\mathrm{mV}/°\mathrm{C}$. In addition, the noise process parameters are $a = 0.5$ and $\sigma_U^2 = 10 \cdot (1 - a^2)\,\mathrm{mV}^2$.

2.4 Compute the coefficients of the linear minimum mean squared error (LMMSE) estimator of T given the observations $\mathbf{X} = [X(1), X(2)]^T$.
Hints: Recall that the ACF of the WSS process $W(n)$ defined in (6) is given by $R_W(k) = \frac{\sigma_U^2}{1-a^2} a^{|k|}$. In addition, the inverse of any invertible 2×2 matrix is given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

2.5 Calculate the mean square error (MSE) of the estimator.

2.6 Do you think this estimation problem is suitable for solving with a Kalman filter? Why or why not?

Problem 3: Communication in Random Forest

We consider a communication system consisting of a transmitter and a receiver placed in a forest. Figure 1 illustrates how the transmitter and receiver are placed some known distance d apart. The direct line between the transmitter and receiver, or line-of-sight (LOS) is also marked. First we are concerned with modeling the probability for LOS to be undisturbed by the trees. For simplicity we shall consider only trees growing between the transmitter and receiver, i.e. in between the two parallel dashed lines indicated in the plot.

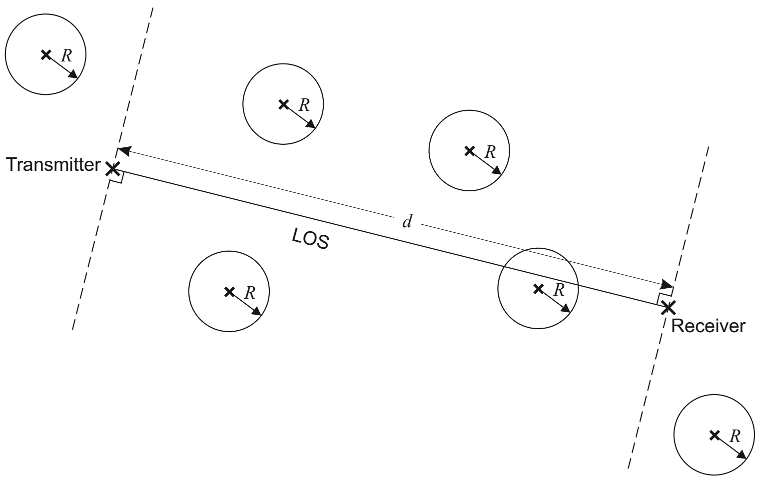

Figure 1: The considered communication setup. In this particular case, the LOS is blocked by one tree trunk.

We consider the following two-dimensional model: Tree trunks are assumed circular with radius R and center points of the trunks of the trees follow a homogeneous Poisson point process with intensity ϱ.

3.1 Compute the probability of a tree trunk having a center point exactly at the LOS?

For the rest of this exercise, we consider the LOS to be blocked whenever it is intersected by one or more tree trunks.

3.2 Calculate the average number of trees blocking the LOS for a given distance d.
Hint: Sketch first the region in which the center point of a tree trunk should lie in order to intersect the LOS.

3.3 Compute the probability P_{LOS} that no tree trunks intersect the LOS, i.e. the probability that this region is void. Sketch this probability as a function of d.

For the rest of this exercise we consider the case where the distance is $d = 100\,\text{m}$, the forrest density $\rho = 0.1\,\text{m}^{-2}$ and the trunk radius is $R = 0.01\,\text{m}$. In this case, $P_{\text{LOS}} \approx 0.14$.

The communication system should be able to detect whether or not the signal LOS is blocked. The receiver can measure the logarithm (in units of dB) of the received signal power (relative to some reference) denoted by X which is distributed according to

$$X \sim \mathcal{N}(\mu(d), 9) \quad \text{where} \quad \mu(d) = \begin{cases} 40\log_{10}\left(\frac{1}{d}\right), & \text{if LOS is not blocked} \\ 40\log_{10}\left(\frac{1}{d}\right) - 0.9d, & \text{if LOS is blocked} \end{cases} \tag{7}$$

3.4 Formulate the maximum likelihood decision rule for LOS detection based on a single observation of X.

3.5 Formulate the decision rule to minimize the probability of error given the P_{LOS}.