Exam: Stochastic Processes

Date and Time: Tuesday January 13, 2015, 9:00–13:00.

This entire problem set contains **4 pages**. Please make sure that you have received all pages.

The exam is graded according to your answer as a whole: both quantity and quality count. We value concise arguments showing your command of the topics. Simply answering “Yes.” or “No.” will not do that!

We recommend that you read through each problem thoroughly before starting to solve it. Should you happen to get stuck at some point, we recommend that you continue and anyway try to solve the rest. You always have the opportunity to sketch or explain how you would have continued if you hadn’t got stuck.

It is allowed to use books, lecture notes, your own notes, calculators and computers during the exam. Communication to others during the exam is not allowed—therefore, the use of internet is strictly forbidden.

Problem 1:

Consider the stochastic process $Y(n)$ defined as

$$Y(n) = X(n) + 2X(n-1) - X(n-2)$$

where $X(n)$ is a WSS process with known mean μ_X, autocorrelation function $R_X(k)$, and power spectrum $S_X(f)$.

1.1 $Y(n)$ may be viewed as the output of a certain LTIV filter with $X(n)$ applied as an input. Find the impulse response and transfer function of the filter.

1.2 Compute:

a) The mean of $Y(n)$ in terms of μ_X.

b) The autocorrelation function of $Y(n)$ in terms of $R_X(k)$.

c) The power spectrum of $Y(n)$ in terms of $S_X(f)$.

1.3 Show that $Y(n)$ is WSS.

From now on, suppose that $\{X(n)\} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,3)$.

1.4 Which type of process is $Y(n)$ in this case? List all its parameters.

1.5 Compute $S_X(f)$ and $S_Y(f)$ for this case. Sketch the two power spectra.

1.6 Compute the variance of $Y(n)$, i.e. σ_Y^2.

1.7 Design the transfer function of a whitening filter such that when $Y(n)$ is applied as input, the power spectrum of the output $Z(n)$ is $S_Z(f) = 1$.

Problem 2:

Consider the observation of a two-dimensional random variable $\boldsymbol{X}$ defined as

$$\boldsymbol{X} = \boldsymbol{a}\theta + \boldsymbol{W}$$

where $\boldsymbol{W} = [W_1, W_2]^{\mathrm{T}} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I})$ is white Gaussian noise, $\theta \sim \mathcal{U}(-2, 4)$ is an unknown parameter and $\boldsymbol{a} = [1, -2]^{\mathrm{T}}$. In addition, $\boldsymbol{W}$ and θ are statistically independent. We want to use the observations $\boldsymbol{X}$ to estimate the value of the unknown variable θ with an estimator of the form

$$\hat{\theta} = h_0 + \boldsymbol{h}^{\mathrm{T}}\boldsymbol{X}$$

where $\boldsymbol{h} = [h_1, h_2]^{\mathrm{T}}$.

2.1 Find the value of the coefficients h_0, h_1 and h_2 that minimize the mean square error of the estimate $\hat{\theta}$.
Hint: The inverse of any invertible 2×2 matrix is given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

2.2 Calculate the value of the mean square error of the estimator, i.e. compute $\mathbb{E}[(\theta - \hat{\theta})^2]$.

2.3 Knowing that the estimator $\hat{\theta}$ is the linear minimum mean square error (LMMSE) estimator of θ, what can you say about the value of the following expectations:

a) $\mathbb{E}[(\theta - \hat{\theta})(X_1 - 3X_2 + 4)]$,
b) $\mathbb{E}[(\theta - \hat{\theta})(X_1^2 - \exp(X_2))]$.

2.4 Now assume that $\tilde{\theta}$ is the (non-linear) MMSE estimator of θ based on the observations $\boldsymbol{X}$. What can you say about the expectations:

a) $\mathbb{E}[(\theta - \tilde{\theta})(X_1 - 3X_2 + 4)]$,
b) $\mathbb{E}[(\theta - \tilde{\theta})(X_1^2 - \exp(X_2))]$.

Next, assume that, instead of a single variable θ, we want to estimate the samples of a random process defined as

$$\theta(n) = c\theta(n-1) + U(n), \qquad n = 1, 2, \ldots$$

where $\{U(n)\} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, 2)$ and $c = 0.5$ is a known constant. To do so, we can observe the outcome of the process $\boldsymbol{X}(n)$, defined as

$$\boldsymbol{X}(n) = \boldsymbol{a}\theta(n) + \boldsymbol{W}(n), \qquad n = 1, 2, \ldots$$

with $\{\boldsymbol{W}(n)\} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\boldsymbol{0}, \boldsymbol{I})$ and $\boldsymbol{a}$ is defined as above.

2.5 Draw the block diagram of the Kalman Filter implementation of an LMMSE estimator for this problem. Include as well the system and observation models in the diagram, indicating the dimensions of each of the vectors and matrices involved.

2.6 Write up the prediction and update equations of the Kalman filter for the above problem.

Problem 3:

In this problem we consider a simplistic stochastic model of the output voltage of a Geiger-Müller tube (GM-tube) used as a sensing element of the Geiger counter for detection of ionizing radiation. The output of the GM-tube is continuous-time voltage signal which we denote by $X(t)$. An ionized particle entering the GM-tube at time τ results in a contribution to the output signal by a pulse $g(t-\tau)$. The particular pulse-shape depends on the specific GM-tube. However, for simplicity we shall assume it to be a rectangular pulse of duration a:

$$g(t) = \begin{cases} 1, & 0 \le t \le a \\ 0 & \text{otherwise.} \end{cases}$$

Ignoring a possible constant voltage (DC-offset) in $X(t)$, the output of a GM-tube can be modeled as a sum of pulses as

$$X(t) = \sum_{\tau \in \mathcal{T}} g(t-\tau)$$

where the set $\mathcal{T} = \{\tau_1, \tau_2, \dots\}$ are the set of occurrence times of ionized particles in the detector. With good approximation $\mathcal{T}$ can be assumed to be a 1D Poisson point process with intensity $\varrho(t)$. Thus we have modeled $X(t)$ as a shot noise process. We assume that the GM-tube is covered in shielding material which is removed at time $t = 0$. Thus the intensity exhibits a jump at $t = 0$:

$$\varrho(t) = \begin{cases} \varrho_0, & t > 0 \\ 0, & t \le 0. \end{cases}$$

3.1 Make qualitative sketches of realizations of $\mathcal{T}$ and $X(t)$. Make one sketch for a small and another for large values of ϱ_0.

3.2 Use Campbell's theorem to calculate the mean of the output signal $X(t)$, i.e. to compute $E[X(t)]$. Make a sketch of $E[X(t)]$.

3.3 Compute the probability mass function of the number of particles hitting the detector within the time interval $[t_1, t_2]$ for $t_1, t_2 > 0$.

3.4 Suppose that a particle hits the detector at time $t = 1$ resulting in a contribution $g(t-1)$ to $X(t)$. Assuming that the pulse duration is less than one ($a < 1$), what is the probability that this pulse overlaps with pulses resulting from other particles hitting the detector?
Hint: It may be easier to first compute the probability of no overlap.

3.5 Explain how to simulate the points of $\mathcal{T}$ that fall in the interval $[0, 1]$ by writing a pseudo-code or drawing a flow-chart.

3.6 State a necessary condition on $\varrho(t)$ for $X(t)$ be a WSS process.
Hint: A process should have constant mean to be WSS.