Limit Theorems
Let $Y_1, Y_2, \dots, Y_n$ be random variables with cdfs $F_1, F_2, \dots, F_n$. Let Y be a random variable with cdf F.

The weak law of large numbers
We say that $\{Y_n\}_{n\in\mathbb{N}}$ converges in probability to Y if

$$\forall\epsilon > 0, \qquad P(|Y_n - Y| > \epsilon) \to 0 \tag{1}$$

Noted $Y_n \overset{P}{\to} Y$.

Almost surely
We say that $\{Y_n\}_{n\in\mathbb{N}}$ converges almost surely to Y if

$$P(\lim_{n\to\infty} |Y_n - Y| = 0) = 1 \tag{2}$$

Noted $Y_n \overset{a.s.}{\to} Y$.

We say that Y converges in distribution to Y if

$$F_n(x) \to_{n\to\infty} F(x) \text{ for all } x \tag{3}$$

Noted $Y \overset{d}{\simeq} Y$.

To illustrate the strenght of the above:

$$\{Y_n \overset{a.s.}{\to} Y\} \Rightarrow \{Y_n \overset{P}{\to} Y\} \Rightarrow \{Y_n \overset{d}{\to} Y\} \tag{4}$$

The law of large numbers
Let $X_1, X_2, \dots X_n$ be iid random variables

$$\overline{X} := \overline{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_l. \tag{5}$$

This is a reasonable approximation to the mean of $X_1(E[X_1] = \mu)$.

$$E[\overline{X}] = \frac{1}{n}\sum_{k=1}^{n} E[X_k] = \frac{1}{n} n\mu = \mu \tag{6}$$

The strong law of large numbers

$$\overline{X}_n \overset{a.s.}{\to} \mu \tag{7}$$

Corollary 4.1
Consider an experiment where the event A occurs with probability p. Let this experiment be repeated independently. We note $X_k = 1_{\text{event } A \text{ occurs at } k\text{-th trial}}$. Then $\overline{X}_n \xrightarrow{P} P$. We call $\overline{X}_n = f_n$ the relative frequeny of A.

Corollary 4.2
Let g be a continuous function R $\to$ R. Then

$$g(\overline{X}_n) \xrightarrow{P} g(\mu), \text{ and } g(\overline{X}_n) \xrightarrow{a.s.} g(\mu) \tag{8}$$

Theorem 4.2: The Central Limit Theorem
Recall, that Φ is the cdf of $\mathcal{N}(0,1)$. It describes convergence to the standard normal distribution:

$$\sqrt{n}\frac{\overline{X} - \mu}{\sigma} \xrightarrow{d} \mathcal{N}(0,1) \tag{9}$$

Remarks:

- The CLT does not depend on the disribution of X_1.
- The speed of convergence in "$\sqrt{n}$" is optimal.
- Direct consequence of the CLT:

$$P(|\overline{x} - \mu| > \epsilon) = P\left(\sqrt{n}\frac{|\overline{X}\mu|}{\sigma} > \frac{\sqrt{n}\epsilon}{\sigma}\right) \simeq 2\left(1 - \Phi\left(\frac{\sqrt{n}\epsilon}{\sigma}\right)\right) \tag{10}$$

Proposition 4.1: The Delta Method
Let g be a function from R to R such that $g'(\mu) \neq 0$. Then

$$\sqrt{n}\frac{g'(\overline{X}_n) - g(\mu)}{\sqrt{\sigma^2[g'(\mu)]^2}} \xrightarrow{d} \mathcal{N}(0,1) \tag{11}$$