Conditional probability What is the probability of A given B?

$$\#A \text{ knowing } B = \frac{\#A \cap B}{\#B} \tag{1}$$

Definition (Conditional probability) Let B be an event with $P(B) > 0$. For ant event $A \in S$

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \tag{2}$$

Putting in the same way as the axioms of probability:

Proposition For any event B with $P(B) > 0$, $P(\cdot/B)$ is a probability

- For $A \in S$, $0 \leq P(A/B) \leq 1$
- $P(S/B) = 1$
- $P(\cup_{k=1}^{+\infty} A_k/B) = \sum_{k=1}^{+\infty} P(A_k/B)$ for $A_i \cap A_j = \emptyset$ for $i < j$.

Example Roll a die and observe the number

$$A = \{\text{an odd number}\} \tag{3}$$
$$B = \{\text{at least 4}\} \tag{4}$$

This gives

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{6}}{\frac{3}{6}} = \frac{1}{3} \tag{5}$$

Properties

$$P(A^c/B) = 1 - P(A/B) \tag{6}$$
$$P(A/\text{\c{}}B = P(A/B) - P(A \cap c/B) \tag{7}$$
$$P(A \cup c/B) = P(A/B) + P(c/B) - P(A \cap c/B) \tag{8}$$
$$\text{If } A \in c,\ P(A/B) \leq P(c/B) \tag{9}$$

Independence
Let A and B be two events
A is independent of B if $P(A \cap B) = P(A)P(B)$
Remark: Assume $P(B) > 0$.
$P(A/B) = \frac{P(A \cap B)}{P(b)} = \frac{P(A)P(B)}{P(B)} = P(A)$

If A is independant of B then B is independant of A.

Example
Choose a card from a deck (52 cards)

$$A = \{ace\} \tag{10}$$
$$B = \{heart\} \tag{11}$$

Are A and B independant?
The probability of drawing ace of hearts:

$$P(A \cap B) = 1/52 \tag{12}$$

The probability of A:

$$P(A) = 4/52 \tag{13}$$

The probability of B:

$$P(B) = 13/52 = 1/4 \tag{14}$$

We have $P(A \cap B) = P(A)P(B)$ so A and B are independant.

Definition
Three events A, B and C are said to be mutually independant if

a) They are pairwise independant
b) $P(A \cap B \cap C) = P(A)P(B)P(C)$

Definition
For $n = 1$ and $A_1, \ldots, A_n \in S$ the events $A_1, \ldots, A_n$ are mutually independant if

a) they are pairwise independant

b) any combination of A's is mutually independant

Proposition (Law of Total Probability)
For any events $B_1, B_2, \ldots$ such that

- $P(B_i) > 0$ for $i = 1, 2, \ldots$
- $B_i \cap B_j = \emptyset$ for $i \neq j$ and $i, j = 1, 2, \ldots$
- $\cup_{i=1}^{+\infty} B_i = S$

$B_1, B_2, \ldots$ is called a partition of S. And for any event A, $P(A) = \sum_{i=1}^{+\infty} P(A/B_i)P(B_i)$ $\blacksquare$