Point Processes in 2D

In this study note we introduce the concept of a two-dimensional (2D) *point process*. We cover only the simplest classes of point processes, namely *binomial point processes* and *Poisson point processes*. If you are interested in getting to know about other classes of point processes (or perhaps point processes in more than 2D), then browse the literature listed at the end of this note or ask the lecturer for recommendations on books, papers, tutorials, etc.

1.1 Observing "random" point patterns

Assume that yesterday's weather was horrible with lots of rain, strong winds and lightning (fair assumption in Denmark!). Meteorologists have equipment for monitoring and recording the exact locations of individual lightning strikes. A certain meteorologist creates a map with the locations of all lightning strikes recorded yesterday within some fixed geographical region (rectangular, 10km by 5km). The complete map is shown in Figure 1.1.

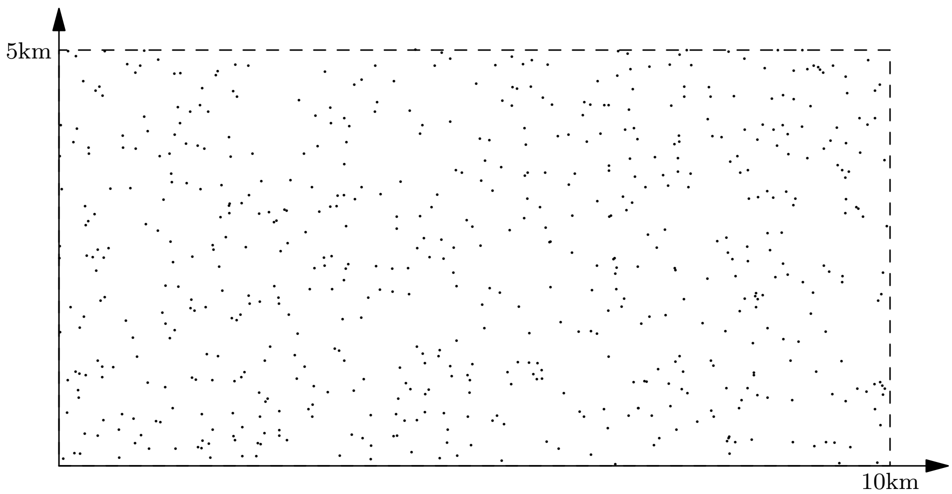


Figure 1.1: Locations of lightning strikes within a fifty square kilometer region. A total of 613 lightning strikes were recorded.

The meteorologist wonders if it makes sense to think of this particular point pattern as being random. More precisely, he wonders if the observed point pattern in Figure 1.1 can be thought of as a *realization* of some random mechanism.

Exercise 1. What do you think, is the meteorologist on the right track? Assume that in two months from now the weather will turn out similar as yesterday (i.e. rain, wind, and lightning). What will happen if the meteorologist creates a new map for the same rectangular region? Where will lightning strike exactly? How many strikes in total?

The meteorologist is particularly interested in the following type of questions: Do the locations of lightning strikes tend to cluster or do they spread out regularly? Are certain regions more likely to be hit than others (e.g. if a region contains tall metallic obstacles or hills)? How likely is it that some single square kilometer sub-region is not going to be hit at all?
Now, replace the meteorologist with a biologist and replace the locations of lightning strikes with locations of trees of some particular type. The biologist is similarly interested in knowing if the locations of trees tend to cluster (local seed spreading) or if there is some kind of repulsion going on (survival of the fittest). Are there certain pronounced regions which do not contain any trees at all? If so, why could that be?
Finally, replace the biologist with a telephone network operator and replace the locations of trees with the locations of active mobile users within some fixed communication cell.

Exercise 2. What kind of questions do you think a network operator would like to ask? Think in terms of system operability, connectivity, coverage, throughput rates, interference levels, user quality of service, etc. Next, think of examples of random point patterns which you would be likely to encounter within your own field of study. What questions would you be interested in being able to answer?

Point patterns show up everywhere and so far we have mentioned three examples. In most applications the observer of such point patterns is not directly interested in the exact point locations themselves. The observer is more likely to be interested in what can be inferred from these locations about some underlying mechanism that governs where the points occur. In a nutshell, this has to do with statistical estimation theory. However, in order to apply such statistical tools we need a mathematical modeling framework for random point patterns. In fact, as we shall see in Section 1.8, stochastic models of point patterns are very important in their own right. In particular, they can be used as building blocks for generating ordinary random processes (our goal in this note).

1.2 Mathematical framework for point processes

We desire a simple and convenient mathematical theory for our random point patterns. To achieve this we restrict ourselves along the way. For example, we cannot handle if there are so many points that we can no longer count them. Moreover, the theory is simpler if points cannot fall directly on top of each other.

Definition 1. A two-dimensional (2D) *point process* is a random countable collection of points in the cartesian plane $\mathbb{R}^2$.

After having read the above definition we would probably like to ask the following questions. What does it really mean that a collection is *random* and what does it mean that a collection is *countable*?

Example 1. The collection of *natural numbers* $\mathbb{N} = \{1, 2, 3, \ldots\}$ is countable. An arbitrary collection is countable if each of its members can be associated uniquely with a number in $\mathbb{N}$. The idea is that the members of a countable collection can be counted one at a time. The counting procedure is allowed to never end but every member must eventually be associated with a natural number.

Exercise 3. Is $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ a countable collection? What about the closed interval $[0, 1] \subset \mathbb{R}$, is this a countable collection?

Exercise 4. Consider a collection of points obtained by drawing 75 points uniformly in the square $[-5, 5] \times [-5, 5]$. Is it a random collection? Is it countable? Think about how to simulate such collections and write a (Matlab) script for this purpose.

Exercise 5. Now, consider a collection of points constructed as follows. First draw a Poisson distributed random number L with mean 75. Given L, then draw L points uniformly in the square $[-5, 5] \times [-5, 5]$. Is this a random collection? Is it countable? Think about what happens from one realization to another. Compare with the construction from the previous exercise.

Historically, one-dimensional (1D) point processes were the first to be considered. The 1D space was almost exclusively used to represent *time*, e.g. the entire real line $\mathbb{R}$ or the set of positive reals $[0, \infty)$.

Exercise 6. Qualitatively, how does a 1D point process realization look like? Is there something very special about the 1D case, something that is not really possible in 2D? Sketch a few figures with your own example realizations of 1D point processes. Discuss where such a 1D random point pattern could happen to emerge in practice. What do you think a 1D point process could be used to represent? Occurrences of earthquakes for instance?

1.3 Convenient restrictions and notation

From a mathematical point of view it is convenient to add a few restrictions to Definition 1 on page 3. Specifically, we limit ourselves to consider only point processes which are *locally finite* and *simple*. These two conditions are of technical kind and by default we assume that they are fulfilled with probability one. That a point process is locally finite means that only a *finite* number of points are falling in every *bounded* region of $\mathbb{R}^2$. That a point process is simple means that no two points of the process coincide.

Example 2. The square region $[-1,1] \times [-1,1] \subset \mathbb{R}^2$ is bounded. The first quadrant $[0,\infty) \times [0,\infty) \subset \mathbb{R}^2$ is an unbounded region. The diagonal $\{(x_1,x_2) : x_1 = x_2\} \subset \mathbb{R}^2$ is also not bounded.

Example 3. The collection $\{(\frac{\cos(n)}{\sqrt{n}}, \frac{\sin(n)}{\sqrt{n}}) : n \in \mathbb{N}\} \subset \mathbb{R}^2$ is countable but indeed not random. However, it is not locally finite either. Do you see why? If not, try drawing it.

In the rest of this chapter we use X to denote a locally finite and simple point process defined on a space $S \subseteq \mathbb{R}^2$. Do not confuse X with a random variable or a random process. Now, X is a random countable collection of points in S. The 2D space S could be the whole $\mathbb{R}^2$, the unbounded subset $\mathbb{R} \times [0,\infty)$ or perhaps the bounded rectangle $[a,b] \times [c,d]$. Since the point process X is assumed to be simple it follows that X has no repetitions of points. Accordingly, the individual realizations of X can be seen as countable *sets* of points

$$\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \dots\} \qquad \boldsymbol{x}_i \in S. \tag{1.1}$$

In (1.1) we use boldface notation for the individual points to stress the fact that each point $\boldsymbol{x}_i$ is a 2D vector. The set in (1.1) can be either finite or countably infinite. The dummy index i is used only to distinguish points and to indicate countability. *We emphasize that the dummy index i is not used to indicate any ordering of the points.* Notice also that we now make use of the term "set" instead of collection. If X had not been a simple point process the set notation in (1.1) would be useless and misleading since occurrences of multiple points would be disregarded. For example, the set $\{4,1,1,2,1,3,3\}$ is the same as the set $\{1,2,3,4\}$. Right?

1.4 Region counts

A natural and intuitively appealing way of exploring the properties of a point process X is to count the number of points falling in different regions. Accordingly, for any set $B \subseteq S$ consider the *region count*

$$\begin{aligned} N_X(B) &= \text{"the number of points from } X \text{ falling in } B\text{"} && (1.2)\\ &= |X \cap B| && (1.3)\\ &= \sum_{\boldsymbol{x} \in X} \mathbb{1}[\boldsymbol{x} \in B], && (1.4) \end{aligned}$$

where $|\cdot|$ in (1.3) denotes *set cardinality* (not absolute value) and where $\mathbb{1}[\cdot]$ in (1.4) denotes an ordinary *indicator function*. For *fixed* and *bounded* $B \subseteq S$, the region count $N_X(B)$ is a discrete random variable with range $\{0, 1, 2, 3, \ldots\}$. This property is due to our default assumption of X being locally finite. The interplay between X and N_X is illustrated in Figure 1.2.

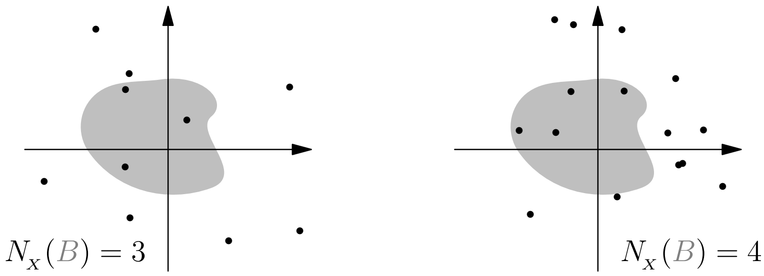


Figure 1.2: Two realizations of a point process X inducing different values of the associated region count $N_X(B)$. Here, B is a fixed potato-shaped region.

The probability distribution of $N_X(B)$ depends on the region B via its overall area, its location, its orientation, and so on. The region B can be very complicated but certain general properties of the region counts are easily established.

Exercise 7. Let X be a point process on $S \subseteq \mathbb{R}^2$. Show that $N_X(\emptyset) = 0$ where $\emptyset$ denotes the empty set. Furthermore, show that if $A, B \subseteq S$ are *disjoint* then $N_X(A \cup B) = N_X(A) + N_X(B)$. *Hint:* Make a drawing at first and use your intuition to argue for the two properties of N_X. Afterwards, show that the properties are satisfied by direct use of (1.4).

In general, various complicated regions can be build up by simpler ones by use of set operations for which the behavior of N_X is well-understood.

1.5 Intensity measures and intensity functions

We have just learned that for any fixed and bounded region $B \subseteq S$, the region count $N_X(B)$ is a non-negative integer-valued random variable. By forming the expected value of this random variable we get a deterministic function of the region B (i.e. a function which takes as input a set and outputs a number).

Definition 2. The *intensity measure* μ_X of X is defined as

$$\mu_X(B) := \mathbb{E}\big[N_X(B)\big] = \mathbb{E}\left[\sum_{\boldsymbol{x} \in X} \mathbb{1}[\boldsymbol{x} \in B]\right], \qquad B \subseteq S. \tag{1.5}$$

Using the definition of expected values, the intensity measure in (1.5) can "in principle" be computed as

$$\mu_X(B) = \mathbb{E}\big[N_X(B)\big] = \sum_{n=0}^{\infty} n\Pr\big(N_X(B) = n\big).$$

The snag is just that this approach is often not possible. In most cases we simply don't know the probability distribution of $N_X(B)$. Luckily, the intensity measure $\mu_X(B)$ can nearly always be expressed in terms of integrating another non-negative function across the region B.

Definition 3. If the intensity measure μ_X in (1.5) can be written as

$$\mu_X(B) = \int_B \varrho_X(\boldsymbol{x})\mathrm{d}\boldsymbol{x}, \qquad B \subseteq S, \tag{1.6}$$

for some *locally integrable* function $\varrho_X : S \to [0, \infty)$, then ϱ_X is called the *intensity function* of X.

Definition 4. If the intensity function ϱ_X is constant across the entire space S, then X is called a *homogeneous* point process. If ϱ_X is not constant on S, then X is said to be *inhomogeneous*.

By now we have introduced something called the intensity measure as well as the intensity function of a point process. Do not confuse these two quantities with one another, the former is simply obtained by integrating the latter. *The shape of the intensity function ϱ_X indicates where points from X are more likely to occur.* The integral of ϱ_X across some region $B \subseteq S$ gives the expected number of points from X falling within B, i.e. the intensity function specifies the mean value structure of the region count $N_X(B)$. If X is a homogeneous point process such that $\varrho_X(\boldsymbol{x}) = \varrho_0$ for all $\boldsymbol{x} \in S$, then the non-negative constant ϱ_0 has a plain and simple interpretation: The average number of points per unit area.

Exercise 8. As mentioned above, the shape of the intensity function ϱ_X indicates where points from X are more likely to occur. This is very similar to the interpretation of an ordinary probability density function (pdf) of a random variable. Apart from a shift in notation, does (1.6) look familiar to you? Discuss the similarities as well as the distinctions between the intensity function of a point process and the pdf of an ordinary random variable.

1.6 The binomial point process

Definition 5. Let f be a pdf on $S \subseteq \mathbb{R}^2$ and fix an integer $k \in \mathbb{N}$. A point process X consisting of k points drawn i.i.d. according to f is called a *binomial point process.* We denote this by writing $X \sim \texttt{binomialPP}\big(S, k, f\big)$.

Let's start out by looking at a simple example.

Example 4. Let S be the bounded rectangle $[a, b] \times [c, d]$ and let f be the uniform density on S, i.e. $f(\boldsymbol{x}) = \frac{1}{(b-a)(d-c)}$ for every $\boldsymbol{x} \in S$. We fix $k = 75$ and plot two different example realizations in Figure 1.3. We have used the parameters $a = c = -5$ and $b = d = 5$. Does the construction seem familiar? Recall Exercise 4 on page 3.

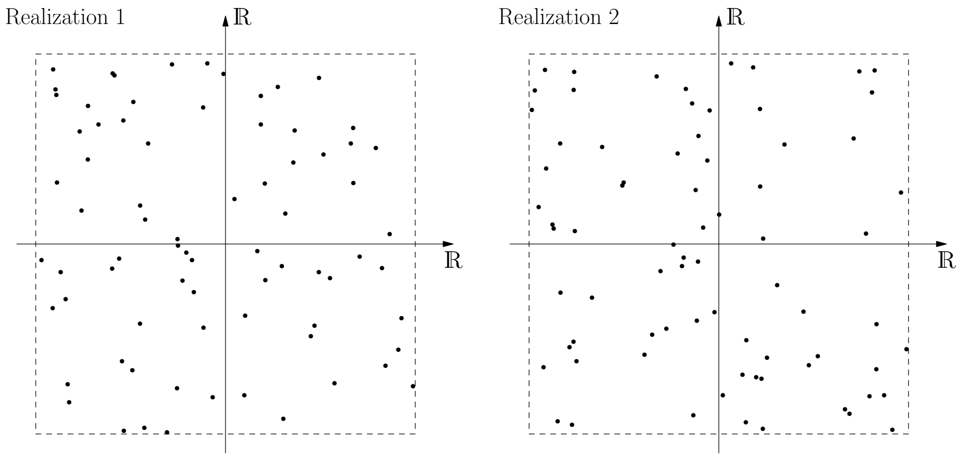


Figure 1.3: Two example realizations of $X \sim \texttt{binomialPP}\big([-5, 5]^2, 75, \frac{1}{100}\big)$.

At a first glance, one may wonder why the construction in Definition 5 is called a binomial point process. The answer has to do with the probability distribution of an arbitrary region count.

Exercise 9. Let $X \sim \texttt{binomialPP}\big(S, k, f\big)$ where $S \subseteq \mathbb{R}^2$ and f is some arbitrary pdf on S. Argue that the region count $N_X(B)$, $B \subseteq S$, has a binomial distribution (think of a coin tossing experiment) and identify the two parameters of this discrete probability distribution. Does it make sense that the *success probability* depends on B?

Exercise 10. Determine the intensity function ϱ_X for a general binomial point process $X \sim \texttt{binomialPP}\big(S, k, f\big)$. *Hint:* Recall and make use of (1.5) and (1.6).

Exercise 11. Let $X \sim \texttt{binomialPP}\big(S, k, f\big)$ and let $B \subset S$ be some fixed region. Argue whether or not the two region counts $N_X(B)$ and $N_X(S \backslash B)$ are independent random variables (draw it). Intuitively, are $N_X(B)$ and $N_X(S \backslash B)$ positively correlated, negatively correlated or uncorrelated?

1.7 The Poisson point process

In the following, the binomial point process enters directly in a two-step definition of the Poisson point process. This two-step definition is convenient since it gives a direct procedure for simulation of Poisson point processes, e.g. in Matlab.

Definition 6. A point process X on $S \subseteq \mathbb{R}^2$ is called a *Poisson point process* with intensity function ϱ_X if:

i) For any region $B \subseteq S$ with $\mu_X(B) = \int_B \varrho_X(\boldsymbol{s})\mathrm{d}\boldsymbol{s} < \infty$ the associated region count $N_X(B)$ has a Poisson distribution with parameter $\mu_X(B)$, i.e.

$$\Pr\big(N_X(B) = k\big) = \exp\big(-\mu_X(B)\big)\frac{\big(\mu_X(B)\big)^k}{k!}, \qquad k = 0, 1, 2, \ldots$$

ii) Given that $N_X(B) = k \in \mathbb{N}$, then these k points form a binomial point process on B such that

$$X \cap B \sim \texttt{binomialPP}\big(B, k, f_B\big), \qquad f_B(\boldsymbol{x}) = \mathbb{1}[\boldsymbol{x} \in B]\frac{\varrho_X(\boldsymbol{x})}{\mu_X(B)}.$$

We denote this by writing $X \sim \texttt{PoissonPP}\big(S, \varrho_X\big)$.

For a Poisson point process the individual region counts are Poisson distributed random variables. Hence the name of the process. An important property of the Poisson point process is that if $B_1, B_2, \ldots, B_n \subset S$ are fixed disjoint regions, then the corresponding region counts $N_X(B_1), N_X(B_2), \ldots, N_X(B_n)$ are mutually independent random variables. This property could as well have been used instead of part *ii*) in Definition 6, but the definition would then not directly tell us how to simulate the Poisson point process.

Example 5. Recall Exercise 5 on page 3. This construction is in fact a homogeneous Poisson point process X on $S = [-5, 5] \times [-5, 5]$ with $\varrho_X(\boldsymbol{x}) = \varrho_0 = \frac{3}{4}$. With our notation from above we have $X \sim \texttt{PoissonPP}\big([-5, 5] \times [-5, 5], \frac{3}{4}\big)$.

Exercise 12. Consider part *i*) in Definition 6. What happens with the region count $N_X(B)$ if $\mu_X(B) = 0$? Will it affect part *ii*) and how?

Exercise 13. Let $X \sim \texttt{PoissonPP}\big([0, 2]\times[0, 1], \varrho_0\big)$ for some constant $\varrho_0 > 0$. That is, X is a homogeneous Poisson point process on the bounded rectangle $S = [0, 2] \times [0, 1]$. What is the expected number of points from X falling in the region $B = [0, 1] \times [0, 1]$? What is the probability that X has no points in S at all?

Exercise 14. Let's think in terms of computer simulation. In principle, what steps would you need to carry out if you wanted to simulate the point process $X \sim \texttt{PoissonPP}\big(\mathbb{R}^2, 1\big)$? What is $\mu_X(S)$ in this case?

1.8 Applications of point processes

In this section we show how point process models such as those from the previous two sections can be used as building blocks for generating ordinary random processes. Numerous text books on probability theory and random processes cover topics like *the Poisson counting process* and *queuing theory*. In the following we cover these topics as well but our treatment is most likely different from what you will find in standard text books.

The Poisson counting process

Let $Y \sim \texttt{PoissonPP}\big([0,\infty), \varrho_0\big)$ be a one-dimensional (1D) homogeneous Poisson point process. We use the symbol Y to stress the fact that we are now dealing with a 1D point process. Then, consider an ordinary random process $Z(\cdot)$ defined as

$$Z(t) := \sum_{y \in Y} \mathbb{1}[y \le t], \qquad t \ge 0. \tag{1.7}$$

By definition, $Z(\cdot)$ is a continuous-time staircase alike random process with jumps at every point of Y. It is often referred to as a Poisson counting process. When writing $Z(\cdot)$ we mean the entire random process and when writing $Z(t)$ it means that time t is considered fixed. Hence, $Z(t)$ is a random variable.

Exercise 15. Sketch a few different example realizations of the 1D point process Y. Sketch the corresponding realizations of $Z(\cdot)$, e.g. in the range $t \in [0, 20]$. Explain what will happen if ϱ_0 is selected larger. Express $Z(t)$ as a certain region count $N_Y(B_t)$ for some suitably chosen region B_t and argue that $Z(t)$ is Poisson distributed with mean parameter $\varrho_0 t$. Is $Z(\cdot)$ a wide-sense stationary (WSS) process?

Queuing theory

Queuing theory deals with arrival times of customers and service times at counters, e.g. humans at checkout lines in supermarkets. Stochastic models of queues are widely used for analyzing the behavior of time-shared computer and communication systems.

Exercise 16. What kind of questions do you think are typically sought to be answered in applications involving queues? *Hint:* Think of concepts such as queue lengths and customer waiting times.

Denote by p the pdf of some non-negative continuous random variable (exponential, gamma, Weibull, chi-square, etc.). Then, consider the 2D inhomogeneous Poisson point process $X \sim \texttt{PoissonPP}\big(\mathbb{R} \times [0,\infty), \varrho_X\big)$ where the intensity function has the form

$$\varrho_X(\boldsymbol{x}) = \varrho_X(x_1, x_2) = \lambda p(x_2), \qquad (x_1, x_2) \in \mathbb{R} \times [0, \infty).$$

Each random point $\boldsymbol{x} = (x_1, x_2) \in X$ has two components and the interpretation of each component is as follows:

$$x_1 = \text{the random time instance where a new customer enters}$$
$$x_2 = \text{the random service time needed to process this customer.}$$

Notice that the intensity function ϱ_X is such that it does not vary with its first argument x_1. This means that customers keep arriving with constant intensity λ all day long. On the other hand, ϱ_X has a functional dependency on its second argument x_2 such that the average service time of any customer is given by the expected value associated with the pdf p.

Exercise 17. For each random point $\boldsymbol{x} = (x_1, x_2) \in X$, what is the interpretation of the random time instance $x_1 + x_2$?

We now use the 2D point process X to form a random process similar to the one in (1.7). Specifically, our construction now reads

$$Z(t) := \sum_{\boldsymbol{x} \in X} \mathbb{1}[x_1 \leq t,\ x_1 + x_2 > t], \qquad t \in \mathbb{R}. \tag{1.8}$$

and this continuous-time random process is jumping both up and down. It models the behavior of the so-called $M/G/\infty$ queue. The random variable $Z(t)$ gives the instantaneous queue length at time t (do you see why?). Thus, $\mathbb{E}\big[Z(t)\big]$ is the average queue length at time t.

Exercise 18. Sketch one example realization of the 2D point process X. Sketch the corresponding realization of $Z(t)$, e.g. in the range $t \in [-20, 20]$. Look carefully at (1.8) and express $Z(t)$ as a certain region count $N_X(B_t)$ for some[1] appropriately chosen region $B_t \subset \mathbb{R} \times [0, \infty)$. Recall Definition 6 and argue that $Z(t)$ is a Poisson distributed random variable. Finally, try to calculate

$$\mathbb{E}\big[Z(t)\big] = \mathbb{E}\big[N_X(B_t)\big] = \mu_X(B_t) = \int_{B_t} \varrho_X(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \iint_{B_t} \lambda p(x_2)\mathrm{d}x_1\mathrm{d}x_2,$$

and discuss whether you find it reasonable that this mean function does not depend on time t.

Final remark: It can be shown that the random process $Z(\cdot)$ in (1.8) is in fact strict-sense stationary (SSS).

Shot noise and Campbell's theorem

In its simplest form, a *shot-noise* random process $Z(t)$ is a continuous-time random process constructed from two components, namely:

[1] The correct region B_t has an unbounded triangular shape.

- a one-dimensional homogeneous Poisson point process Y, and
- a fixed real-valued function $h(t)$.

Specifically, consider a linear time-invariant (LTIV) system with impulse response $h(t)$ and let the input to this system be a (Dirac) impulse-train with impulses located at the random point occurrences of Y. The output from this LTIV system is the shot-noise random process $Z(t)$, i.e.

$$Z(t) = \sum_{y \in Y} h(t - y), \qquad t \in \mathbb{R}.$$

Accordingly, the shot-noise random process $Z(t)$ is a superposition of randomly time-shifted versions on the deterministic function $h(t)$.

Shot-noise comprises a surprisingly flexible class of continuous-time random processes. We can pick different functions $h(t)$, we can change the type of point process Y as well as its intensity function. Among others, shot-noise random processes can be used to model and represent (see graphical illustrations in Figure 1.4 on page 12):

- counting processes, birth-death processes, queuing systems,
- received signals in wireless communication systems, and
- waveforms arising as a superposition of charged particles emitted from a certain source (light, current, radioactivity, etc.).

A very useful tool for analyzing shot-noise random processes (and random variables of shot-noise type) is *Campbell's Theorem.*

Campbell's Theorem. Let X be a point process on $S \subseteq \mathbb{R}^2$ with intensity function ϱ_X. Then for any real or complex-valued function $g : S \to \mathbb{R}$ (or $\mathbb{C}$), the random variable

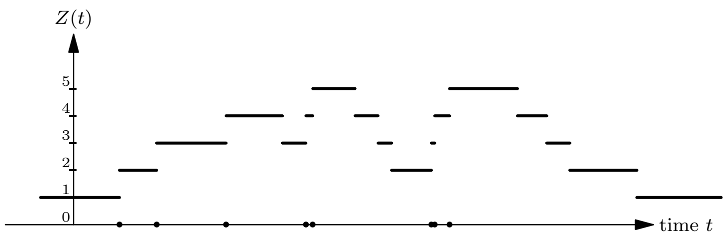
$$\sum_{\boldsymbol{x} \in X} g(\boldsymbol{x})$$

has expected value

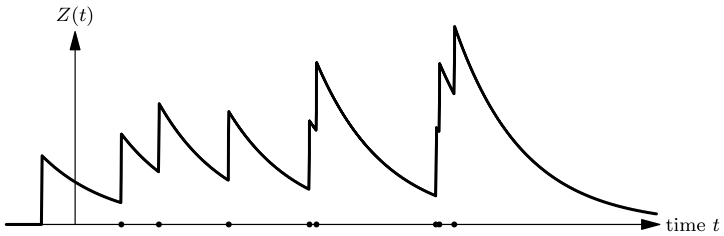
$$\mathbb{E}\Big[\sum_{\boldsymbol{x} \in X} g(\boldsymbol{x})\Big] = \int_S g(\boldsymbol{x}) \varrho_X(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}, \tag{1.9}$$

provided that the integral on the right exists.

Exercise 19. Use Campbell's Theorem to calculate the mean $\mathbb{E}\big[Z(t)\big]$ of the shot-noise random processes defined in (1.7) and in (1.8).



(a) $h(t)$ ="rectangular box"



(b) $h(t) = \exp(-t)$

Figure 1.4: Two examples of shot-noise random processes driven by the same underlying point process.

1.9 Final remark: 1D versus 2D

In section 1.2 we mentioned that 1D point processes were the first to be considered (historically). Indeed, the 1D approach may at first glance appear more attractive and simpler compared to our 2D approach (and compared to higher dimensional generalizations as well). However, there is one very peculiar feature of $\mathbb{R}$ which has no straightforward analogue in $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^d$ in general. *The real line has a natural ordering of its members.* For this single reason it is highly recommended to always think of (and relate to) the 2D case when dealing with point processes. Despite the peculiar ordering feature in 1D, it is crucial to keep in mind that this setup comprises a very important special case. However, the general theory of point processes is easier to comprehend if we initially develop it without relying on features which are valid only for the 1D case. *This is the very reason why we have chosen the 2D case as our reference approach.*

1.10 Further reading

- David R. Cox and Valerie Isham, "Point Processes", Chapman & Hall, 1980.
- John F. C. Kingman, "Poisson Processes", Oxford University Press, 1993.
- Adrian J. Baddeley, "Spatial Point Processes and their Applications" (in "Stochastic Geometry - Lecture Notes in Mathematics"), Springer, 2007.