Exercise 1
Let $y_1, \ldots, y_n$ be real numbers, and $\bar{y} = (y_1 + \ldots y_2)/n$ their sample mean. Show that for any $\mu \in \mathbb{R}$,

$$\sum_{i=1}^{n}(y_i - \mu)^2 = \sum_{i=1}^{n}(y_1 - \bar{y})^2 + n(\bar{y} - \mu)^2 \tag{1}$$

Omskriv

$$\begin{aligned}\sum_{i=1}^{n}((y_i - \bar{y}) + (\bar{y} - \mu))^2 &= \sum_{i=1}^{n}((y_i - \bar{y})^2 + (\bar{y} - \mu)^2 + 2(\bar{y} - \mu)(y_i - \bar{y})) \\ &= \sum_{i=1}^{n}((y_i - \bar{y})^2 + 2(\bar{y} - \mu)(y_i - \bar{y}) + n(\bar{y} - \mu)^2\end{aligned}$$

Da $\sum_{i=1}^{n} y_i = n\bar{y}$ fås

$$\sum_{i=1}^{n}((y_i - \bar{y}) + (\bar{y} - \mu))^2 = \sum_{i=1}^{n}(y_i - \bar{y})^2 + n(\bar{y} - \mu)^2$$

Da der blot er lagt et nul til på venstresiden er vi i mål.

Exercise 2
Let $Y_1, \ldots, Y_n$ be i.i.d. zero-one variables with probability parameter $\theta \in [0, 1]$.

1. Show that $t(Y_1, \ldots, Y_n) = Y_1 + \ldots Y_n$ is a sufficient statistic for θ.
2. Specify the distribution of $t(Y_1, \ldots, Y_1)$.

Den sufficiente statistik skal opfylde

$$f_Y(y, \theta) = h(y)g(t(y), \theta) \tag{2}$$

For at finde frem til dette skrives produktet af Bernoullifordelingerne.

$$f_{y_1 \ldots y_n}(y_1, \ldots, y_n; \theta) = \theta^{\sum_{i=1}^{n} y_i}(1 - \theta)^{1 - \sum_{i=1}^{n} y_i}$$

Dermed er det på den ønskede form, og 1 er bevist. Fordelingen er en binomialfordeling.

Exercise 3
Let EXP(μ) denote the exponential distribution with mean $\mu > 0$. Consider

the parametric statistical model given by $Y \sim \text{Exp}(\mu)$ and $\mu \in (0, \infty)$. So Y has density

$$f(y; \mu) = \frac{\mathrm{e}^{(-y/\mu)}}{\mu}, \qquad y > 0 \tag{3}$$

Show that for any observed value of $Y = y > 0$ we have that

1. The log-likelihood function is

$$l(\mu; y) = -\frac{y}{\mu} - \log \mu \tag{4}$$

   Der tages logaritmen af likelihood funktionen:

$$\begin{aligned} l(\mu; y) &= \log L(\mu; y) = \log f_Y(y, \mu) \\ &= \log \frac{\mathrm{e}^{-y/\mu}}{\mu} \\ &= -\frac{y}{\mu} - \log \mu \end{aligned}$$

2. The score function is

$$S(\mu; y) = \frac{y}{\mu^2} - \frac{1}{\mu} \tag{5}$$

   Likelihood funktionen partialdifferentieres (brug log-likelihood):

$$\begin{aligned} S(\mu, y) &= \frac{\partial}{\partial \mu} L(\mu; y) = -\frac{\partial}{\partial \mu} \left(\frac{y}{\mu} - \log \mu \right) \\ &= \frac{y}{\mu^2} - \frac{1}{\mu} \end{aligned}$$

3. The observed information is

$$j(\mu; y) = 2\frac{y}{\mu^3} - \frac{1}{\mu^2} \tag{6}$$

   Definitionen siger, at

$$\begin{aligned} j(\mu; y) &= -\frac{\partial^2}{\partial \mu \partial \mu^T} l(\mu; y) \\ &= -\frac{\partial}{\partial \mu^T} \left(\frac{y}{\mu^2} - \frac{1}{\mu} \right) \\ &= 2\frac{y}{\mu^3} - \frac{1}{\mu} \end{aligned}$$

4. The Fisher information is
$$i(\mu) = \frac{1}{\mu^2} \tag{7}$$
Brug definitionen:
$$\begin{aligned}
i(\mu) &= E_\mu\left[j(\mu;Y)\right] = E_\mu\left[2\frac{y}{\mu^3} - \frac{1}{\mu^2}\right] = 2E_{\mu^2}\left[\frac{y}{\mu^3} - \frac{1}{\mu^2}\right] \\
&= 2\frac{1}{\mu^3}E_\theta[y] - \frac{1}{\mu^2} = 2\frac{1}{\mu^3}\mu - \frac{1}{\mu^2} \\
&= 2\frac{1}{\mu^2} - \frac{1}{\mu^2} \\
&= \frac{1}{\mu^2}
\end{aligned}$$