Stochastic Processes, Session 10 — Group Work
AR, MA and ARMA Processes

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10 Session 10

Go through the exercises below. Allow yourself the time to reflect over your results and discuss them with other students! Use the book for inspiration and for further information.

10.1 Investigating the ARMA(1,1) process

Consider an ARMA(1,1) process defined via the recurrence relation

$$X(n) = aX(n-1) + bZ(n-1) + Z(n), n = 1, 2, 3, \dots \tag{1}$$

where a and b are some known parameters and $\{Z(n)\} \overset{iid.}{\sim} \mathcal{N}(0, \sigma_Z^2)$.

- Prepare an argument for why $X(n)$ is a Gaussian process. Find another group (not sitting at the same table) and compare your argument to theirs.
- For this item, assume that $b = 0$, so that $\{X(n)\}$ is an AR(1) process. Compute the mean and variance functions for $\{X(n)\}$, assuming $X(0) = 0$. How should $X(0)$ be distributed in order to make the process WSS?[1]
- Compute the PSD for $\{X(n)\}$ using the result derived in the lecture note. Write a program to plot the PSD.
- Investigate the behavior of the PSD of $\{X(n)\}$ for specific settings of the parameters a, b, σ_Z^2 including those in Table 1. Do so by both inspecting the equation and comparing this to the plotted values. To keep track of your observations, make your own version of Table 1.
- Simulate and plot a few realizations of $\{X(n)\}$ for the each of the settings in Table 1. You may also find it interesting to inspect the corresponding periodograms and compare these to the theoretical PSDs.

[1] Although slightly more complex, the same calculation can be done even if we assume $b \neq 0$. How should the initial samples $X(0)$ and $Z(0)$ be drawn in that case?

σ_Z^2	a	b	Sketch of the PSD	Comments/observations
1	0	$\frac{1}{2}$		
1	$\frac{1}{2}$	0		
1	2	$\frac{1}{2}$		
1	$\frac{1}{2}$	2		
1	1	1		
1	$\frac{1}{2}$	$-\frac{1}{2}$		
3	$\frac{1}{2}$	$\frac{1}{2}$		
$\vdots$	$\vdots$	$\vdots$		

Table 1: Table for ARMA(1,1) investigations. Make your own with enough space for sketching the PSDs.

10.2 Second Order Autoregressive Process

An autoregressive (AR) process is a special case of an ARMA process. Here we consider the 2nd order AR process, i.e. the AR(2) process, $X(n)$ obtained by filtering a white Gaussian process $Z(n) \overset{\text{i.i.d}}{\sim} \mathcal{N}(0, \sigma_Z^2)$ through a linear time-invariant system with input-output relationship:

$$X(n) = Z(n) + \phi_1 X(n-1) + \phi_2 X(n-2). \tag{2}$$

An AR(2) process has three parameters: ϕ_1 and ϕ_2, and σ_Z^2.

- Write up the expression of the power spectral density of the output process, $S_X(f)$. You may use the expression from the lecture notes.

While deriving the PSD of an AR process is a relatively simple operation, finding its autocorrelation function requires a bigger effort. Closed-form expressions cannot be found except for the simplest types of AR processes. Instead, the ACF can be calculated recursively. For an AR(2) process, we have the recursion

$$R_X(k) = \phi_1 R_X(k-1) + \phi_2 R_X(k-2) + \sigma_Z^2 \delta(k), \quad k \geq 0. \tag{3}$$

- Write up (3) for values $k = \{0, 1, 2\}$ and show that $R_X(0)$, $R_X(1)$, and $R_X(2)$ can be found by solving the following system of equations:

$$\begin{bmatrix} -1 & \phi_1 & \phi_2 \\ \phi_1 & \phi_2 - 1 & 0 \\ \phi_2 & \phi_1 & -1 \end{bmatrix} \begin{bmatrix} R_X(0) \\ R_X(1) \\ R_X(2) \end{bmatrix} = \begin{bmatrix} -\sigma_Z^2 \\ 0 \\ 0 \end{bmatrix}. \tag{4}$$

Assume that the AR(2) process in (2) has parameters $\sigma_Z^2 = 1$, $\phi_1 = 0.9$ and $\phi_2 = -0.5$. With these parameters, we will write a program that recursively calculates the ACF of the process up to an arbitrary time-lag k.

- Calculate the ACF of the process, $R_X(k)$ for $|k| = 0, 1, 2$ by solving the system of equations in (4) numerically.

- Next, use the obtained values to calculate the ACF for $|k| > 2$ by recursively applying (3).

- Plot the calculated ACF and PSD of the process in MATLAB.
- To check that your calculations so far are correct, generate realizations of the process and use them to estimate its ACF and PSD. Compare the estimates to the ACF and PSD you plotted before. (You may need to average the estimates obtained from several realizations of the process to obtain conclusive results).

10.3 Building AR Processes with a Target ACF

The set of equations obtained in (3) is called the Yule-Walker equations for an AR(2) process. The general Yule-Walker equations for an AR(P) process read

$$R_X(k) = \sum_{i=1}^{P} \phi_i R_X(k-i) + \sigma_Z^2 \delta(k), \quad k \geq 0. \tag{5}$$

Using these equations, one can find the values of $R_X(k)$ for any autoregressive process, given the parameters σ_Z^2 and the coefficients ϕ_i, $i = 1, 2, \ldots, P$. The reverse process is also doable: given an ACF specified by $R_X(k)$, $k = 0, 1, \ldots, P$, they can be used to find the parameters σ_Z^2 and ϕ_i, $i = 1, 2, \ldots, P$ of an AR(P) process that has the desired ACF for $|k| \leq P$. In this way, using an AR(P), one can construct random processes $X(n)$ with ACF $R_X(k)$ that is exactly equal to any arbitrary ACF selected for $|k| < P$. We shall see in this exercise how to the parameters of the AR process ($\phi_1, \ldots, \phi_P$ and σ_Z^2) are obtained from the Yule Walker equations.

Assume that we want to construct a random process that has an ACF $R_X(k)$ given by

$$R_X(0) = 8, \quad R_X(1) = R_X(-1) = 2, \quad R_X(2) = R_X(-2) = 0,$$
$$R_X(3) = R_X(-3) = 2, \quad R_X(4) = R_X(-4) = -4.$$

for $|k| \leq 4$, and we do not care about the ACF for values $k > 4$. As stated above, we can do so by using an AR process of order $P = 4$.

- Write up the Yule-Walker equations in (5) for $k = 0, 1, \ldots, 4$ and $P = 4$. Now you have a system with 5 equations, in which the values of the ACF for $|k| \leq 4$ are known. How many unknown variables do you have in this system of equations?
- Solve the system of equations to find the parameters of the desired AR process. (Hint: Write it up in matrix-vector form and solve it with MATLAB. It is easiest to solve first for the bottom 4 equations, then use the first one to find the missing unknown. Notice that the matrix describing the bottom 4 equations in the system is a Toeplitz matrix, so the MATLAB command `toeplitz` will come in handy!).
- With the found parameters, generate realizations of the AR process and check that the realizations show the right properties. Again, averaging of the estimates obtained over multiple realizations may be needed to be fairly certain.