4 Two random variables – independence

Define

- Discrete random vector
- Joint pmf

Prove

- Marginal pmfs
- Independence if and only if

Definition 3.3. (Discrete random vector) If X and Y are discrete random variables, then (X, Y) is called a *discrete random vector*.

Definition 3.4. (Joint pmf) If (X, Y) is discrete with range $\{(x_j, y_k) : j, k = 1, 2, \ldots\}$, the function

$$p(x_j, y_k) = P(X = x_j, Y = y_k) \tag{1}$$

is called the *joint pmf* of (X, Y).

Proposition 3.2 (Marginal pmfs) If (X, Y) has joint pmf p, then the marginal pmfs of X and Y are

$$\begin{aligned} p_X(x_j) &= \sum_{k=1}^{\infty} p(x_j, y_k), \qquad j = 1, 2, \ldots \\ p_Y(y_k) &= \sum_{j=1}^{\infty} p(x_j, y_k), \qquad k = 1, 2, \ldots \end{aligned}$$

Bevis

$$\begin{aligned} p_X(x_j) = P(X = x_j) &= P\left(\{X = x_j\} \cap \{\cup_{k=1}^{\infty} Y = y_k\}\right) \\ &= P\left(\cup_{k=1}^{\infty} \{X = x_j, Y = y_k\}\right) \\ &= \sum_{k=1}^{\infty} P(X = x_j, Y = y_k) = \sum_{k=1}^{\infty} p(x_j, y_k) \end{aligned}$$

Definition 3.8. (Independence) The random variables X and Y are said to be *independent* if

$$P(X \in A, Y \in B) = P(X \in A)P(Y \in B) \tag{2}$$

for alle $A, B \subseteq \mathbb{R}$.

Proposition 3.9. (Independence if and only if) Suppose that (X, Y) is discrete with joint pmf p. Then X and Y are independent if and only if

$$p(x, y) = p_X(x)p_Y(y) \tag{3}$$

for all $x, y \in \mathbb{R}$.

Bevis
Suppose X and Y independent. Let $A = \{x\}$ and $B = \{y\}$ so that

$$p(x, y) = P(X = y, Y = y) = P(X = x)P(Y = y) = p_X(x)p_Y(y) \tag{4}$$

Conversely, suppose $p(x, y) = p_X(x)p_Y(y)$ and take the subsets A and B of $\mathbb{R}$.

$$\begin{aligned} P(X \in A, Y \in B) &= \sum_{x \in A} \sum_{y \in B} p(x, y) = \sum_{x \in A} p_X(x) \sum_{y \in B} p_Y(y) \\ &= P(X \in A)P(Y \in B) \end{aligned}$$

Both "if" and "only if" have been shown.