Basics of probability

Define

- (Event)
- Axioms of probability
- Conditional probability

Prove

- Law of Total Probability
- Bayes' Formula

Definition 1.2 (Event). A subset of S, $A \subseteq S$, is called an *event*.

Definition 1.3 (Axioms of Probability). A *probability measure* is a function P, which assigns to each event A a number $P(A)$ satisfying

- **(a)** $0 \leq P(A) \leq 1$
- **(b)** $P(S) = 1$
- **(c)** If $A_1, A_2, \ldots$ is a sequence of *pairwise disjoint* event, that is, if $i \neq j$, then $A_i \cap A_j = \emptyset$, then

$$P\left(\cup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} P(A_k) \tag{1}$$

Definition 1.4. (Conditional probability). Let B be an event such that $P(B) > 0$. For any event A, denote and define the *conditional probability of A given B* as

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \tag{2}$$

Theorem 1.1 (Law of Total Probability). Let $B_1, B_2, \ldots$ be a sequence of events such that

(a) $P(B_x) > 0$ for $k = 1, 2, \ldots$

(b) B_i and B_j are disjoint whenever $i \neq j$

(c) $S = \cup_{k=1}^{\infty} B_k$

Then, for any event A, we have

$$P(A) = \sum_{k=1}^{\infty} P(A|B_k)P(B_k) \tag{3}$$

Bevis
Ved den distibutive lov for uendelige fællesmængder fås

$$A = A \cap B = \cup_{k=1}^{\infty}(A \cap B_k) \tag{4}$$

Siden $A \cap B_1, A \cap B_2, \ldots$ er parvis disjunkte og derfor kan summeres fås

$$P(A) = \sum_{k=1}^{\infty} P(A \cap B_k) = \sum_{k=1}^{\infty} P(A|B_k)P(B_k) \tag{5}$$

Proposition 1.11 (Bayes' Formula). Under the same assumptions as in the law of total probability and if $P(A) > 0$, then for any event B_j, we have

$$P(B_j|A) = \frac{P(A|B_j)P(B_j)}{\sum_{k=1}^{\infty} P(A|B_k)P(B_k)} \tag{6}$$

Bevis
Fra loven om samlet sandsynlighed (law of total probability) kan nævneren skrives om, således

$$P(B_j|A) = \frac{P(A|B_j)P(B_j)}{P(A)} \tag{7}$$

som kan omskrives til

$$P(B_j|A)P(A) = P(A|B_j)P(B_j) \tag{8}$$

Dette er sandt, da loven om betinget sandsynlighed giver, at de to udtryk er ens.

Proposition 1.3. Let P be a probability measure on some sample space S and let A and B be events. Then

(a) $P(A^c) = 1 - P(A)$

(b) $P(A \setminus B) = P(A) - P(A \cap B)$

(c) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(d) If $A \subseteq B$, then $P(A) \leq P(B)$

TABLE 1.1 Basic Set Operations and Their Verbal Description

Notation	Mathematical Description	Verbal Description
$A \cup B$	The union of A and B	A or B (or both) occurs
$A \cap B$	The intersection of A and B	Both A and B occur
A^c	The complement of A	A does not occur
$A \setminus B$	The difference between A and B	A occurs but not B
$\emptyset$	The empty set	Impossible event

Proposition 1.1. Let A, B, and C be events. Then

(a) ***(Distributive Laws)***
$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$
$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$

(b) ***(De Morgan's Laws)***
$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c$$

Proposition 1.5. Let $A_1, A_2, \ldots, A_n$ be a sequence of n events. Then

$$\begin{aligned} P\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} P(A_k) \\ - \sum_{i<j} P(A_i \cap A_j) \\ + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) \\ \vdots \\ + (-1)^{n+1} P(A_1 \cap A_2 \cap \cdots \cap A_n) \end{aligned}$$

TABLE 1.2 Choosing k Out of n Objects

	With Replacement	Without Replacement
With regard to order	n^k	$n(n-1)\cdots(n-k+1)$
Without regard to order	$\binom{n-1+k}{k}$	$\binom{n}{k}$