Stochastic Processes, Session 13 – Group Work
2D Point Processes

Problem 1: Binomial and Poisson Random Variables

Find the commands to draw binomial and Poisson distributed random numbers in Matlab, and read the help.

a) Generate i.i.d. samples from a binomial distribution with $N = 10$ (number of trials) and $p = 0.3$. Generate a histogram and compare it with the probability mass function of the distribution (*Hint:* check the command `binopdf`).

b) Generate i.i.d. samples from a poisson distribution with mean $\mu = 3$. Generate a histogram and compare it with the probability mass function of the distribution (*Hint:* check the command `poisspdf`).

c) Next, write a program that plots the probability mass function for a binomial distribution with configurable number of trials N and success probability $p = 3/N$. Plot it for larger and larger values of N, and compare the results to the probability mass function generated in b).

Are you surprised by the results? In fact, the poisson distribution with mean μ can be seen as the limit of a binomial distribution with number of trials N and success probability $p = \mu/N$ when $N \to \infty$.

Problem 2: Binomial and Poisson Point Processes

Consider a 2D point process given by $X \sim \texttt{BinomialPP}(S, 12, f)$ with $S = [0,2] \times [0,2]$ and

$$f(x_1, x_2) = \begin{cases} 1/4, & (x_1, x_2) \in S \\ 0, & \text{otherwise.} \end{cases}$$

a) Simulate and plot realizations of the point process X.

b) Now, consider a region $B_1 = [0,1] \times [0,1] \subseteq S$. Discuss what the distribution of region count $N_X(B_1)$ is, write up its probability mass function and plot it.

c) Extend your program to compute the region count $N_X(B_1)$ for each realization. Draw a large number of realizations of the point process and plot a histogram of the obtained region counts. Does the histogram fit with the results of the previous question?

d) Next, include in your program code for computing the region count of $B_2 = S \setminus B_1$. Draw a large number of realizations and make a scatter plot of the obtained region counts $N_X(B_1)$ and $N_X(B_2)$. Can you observe any correlation between the two region counts?

Consider now instead a second point process given by $Y \sim \texttt{PoissonPP}(S, \varrho_Y)$. This is a homogeneous Poisson point process, defined also on S, and with a constant intensity function $\varrho_Y(y_1, y_2) = \varrho_0$.

e) Write a program to generate and plot realizations of the point process Y, for any value of ϱ_0.

f) Set the value of ϱ_0 so that the process Y has, in average, the same number of points as X.

g) Again, consider a region $B_1 = [0,1] \times [0,1] \subseteq S$. What is the distribution of region count $N_Y(B_1)$ now? Write up its probability mass function and plot it.

h) Draw a large number of realizations of the point process and plot a histogram of the obtained region counts $N_Y(B_1)$. Does the histogram fit with the results of the previous question?

i) With the same definition of $B_2 = S \setminus B_1$ as before, make a scatter plot of the obtained region counts $N_Y(B_1)$ and $N_Y(B_2)$. Compare your results to those you obtained for process X.

Problem 3: Inhomogeneous Poisson Point Process

Consider the inhomogeneous 2D poisson point process $X \sim \texttt{PoissonPP}\big(S, \varrho_X\big)$, with $S = [0,1] \times [0,1]$ and intensity function

$$\varrho_X(\boldsymbol{x}) = \varrho_X(x_1, x_2) = 30 \cdot x_1 \mathbb{1}[0 \leq x_1 \leq 1] \cdot \mathbb{1}[0 \leq x_2 \leq 1].$$

a) What is the expected number of points falling in S?

b) Qualitatively, how will realizations of X look like? Sketch (by hand) a few figures of these realizations.

c) Write a script in Matlab to draw realizations of this inhomogeneous Poisson point process. *Hint:* The inverse transformation method can be used for this. Also notice that x_1 and x_2 are independent.

d) Now, change the intensity function to be $\widetilde{\varrho}_X(\boldsymbol{x}) = \varrho_X(\boldsymbol{x})\mathbb{1}[x_1^2 + x_2^2 \leq 1]$, where $\varrho_X(\boldsymbol{x})$ is the old intensity function given above. Sketch first the new space $\widetilde{S}$. Then simulate this modified Poisson point process $\widetilde{X}$.

Problem 4: Shot Noise and Campbell's theorem

For this problem you need to download the script `shotNoise.m` from Moodle.

a) Go through `shotNoise.m` and discuss every line. Write up the shot noise model and the intensity function assumed in the script.

b) Run the script and discuss the output. Try different values for ϱ_0 and observe what happens. Write down your observations.

c) Discuss the effect of ϱ_0 on the estimated mean and on the Normal Probability Plot. Try to connect what you see to other theorems?

d) Derive the expected value for the $Z(t)$ using Campbell's theorem and plot it on top of the estimated mean.