7 Limit theorems

- Define sample mean
- Calculate expected value and variance of it
- Prove Law of Large Numbers
- Draw graph of a roll with a fair die

Proposition 2.14. (Chebyshevs ulighed). Let X be any random variable with mean μ and variance σ^2. For any constant $c > 0$, we have

$$P(|X - \mu| \geq c\sigma) \leq \frac{1}{c^2} \tag{1}$$

Bevis
The continuous case. Fix c and let $B = \{x \in \mathbb{R} : |x - \mu| \geq c\sigma\}$. We get

$$\begin{aligned}\sigma^2 = E[(X-\mu)^2] &= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx \\ &\geq \int_B (x-\mu)^2 f(x)\,dx \geq c^2\sigma^2 \int_B f(x)\,dx = c^2\sigma^2 P(X \in B)\end{aligned}$$

Sample mean

$$\overline{X} = \frac{1}{n}\sum_{k=1}^{n} X_k \tag{2}$$

Expectance and variance of sample mean Let X_k have mean μ and variance σ^2.

$$\begin{aligned}E[\overline{X}] &= E\left[\frac{1}{n}\sum_{k=1}^{n} X_k\right] = \frac{1}{n}\sum_{k=1}^{n} E[X_k] = \mu \\ \mathrm{Var}[\overline{X}] &= \mathrm{Var}\left[\frac{1}{n}\sum_{k=1}^{n} X_k\right] = \sum_{k=1}^{n} \frac{1}{n^2}\mathrm{Var}[X_k] = \frac{\sigma^2}{n}\end{aligned}$$

Theorem 4.1. (The Law of Large Numbers) Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with mean μ, and let $\overline{X}$ be their sample mean. Then, for every $\varepsilon > 0$

$$P(|\overline{X} - \mu| > \varepsilon) \to 0 \text{ as } n \to \infty \tag{3}$$

Bevis
Assume X_k has finite variance $\sigma^2 < \infty$. Apply Chebyshev's to $\overline{X}$ and let $c = \varepsilon\sqrt{n}/\sigma$. Since $E[\overline{X}]$ and $\mathrm{Var}[\overline{X}] = \sigma^2/n$, we get

$$P(|\overline{X} - \mu| > \varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2} \to 0 \text{ as } n \to \infty \tag{4}$$

We say that $\overline{X}$ converges in probability to μ and write (med et P over pilen)

$$\overline{X} \to \mu \text{ as } n \to \infty \tag{5}$$

Corollary 4.1 Experiment with event A occuring with probability p. Repeat and let S_n be times we get A in n trials and let $f_n = S_n/n$. Then

$$f_n \to p \text{ as } n \to \infty \text{ (in probability)} \tag{6}$$

Bevis
Define indicators

$$I_k = \begin{cases} 1 & \text{if we get } A \text{ in the } k\text{th trial} \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } k = 1, 2, \ldots, n \tag{7}$$

The I_k are i.i.d. and they have mean $\mu = p$ (Bernoulli distribution). Since f_n is the sample mean, the law of large numbers gives

$$f_n \to p \text{ as } n \to \infty \text{ (in probability)} \tag{8}$$