6 Generating functions

- Define generating function
- Prove the generating function for a sum of random variables
- Explain the above for the sum of iid varaibels with common pgf
- Prove that a thinng of a Poisson process is a Poisson process

Definition 3.23. Let X nonnegative and integer valued. The function

$$G_X(s) = E[s^X], \qquad 0 \leq s \leq 1 \tag{1}$$

is called the *probability generating function* of X.

The pgf can be calculated by

$$G_X(s) = \sum_{k=1}^{\infty} s^k p_X(k), \qquad 0 \leq s \leq 1 \tag{2}$$

Proposition 3.38. Let $X_1, X_2, \ldots, X_n$ be independent random variables with pgfs $G_1, G_2, \ldots, G_n$, respectively and let $S_n = X_1 + X_2 + \ldots + X_n$. Then S_n has pgf

$$G_{S_n}(s) = G_1(s)G_2(s)\ldots G_n(s), \qquad 0 \leq s \leq 1 \tag{3}$$

Bevis
Since $X_1, \ldots, X_n$ are independent so are $s^{X_1}, \ldots, s^{X_1}$ for each s in $[1, 0]$, and we get

$$\begin{aligned} G_{S_n}(s) &= E[s^{X_1+X_2+\ldots+X_n}] \\ &= E[s^{X_1}]E[s^{X_2}]\ldots E[s^{X_n}] \\ &= G_{X_1}(s)G_{X_2}(s)\ldots G_{X_n}(s) \end{aligned}$$

Explain **Proposition 3.39.**
States that the sum S_N of N independent identical distributions with commen pgf G_X has pgf

$$G_{S_N}(s) = G_N(G_X(s)) \tag{4}$$

Proposition 3.44. The thinned process is a Poisson process with rate λp.

Bevis

Proposition 3.39 can be used as observations in two disjoint intervals are independent. Consider an interval of length t, letting $X(t)$ be the total number of points and $X_p(t)$ be the number of observed points in the interval. Then,

$$X_p(t) = \sum_{k=1}^{X(t)} I_k \tag{5}$$

where $I_k = 1$ if kth point observed and 0 otherwise. From proposition 3.39 the pgf of $X_p(t)$ is

$$G_{X_p}(s) = G_{X(t)}(G_I(s)) \tag{6}$$

where

$$G_{X(t)}(s) = \mathrm{e}^{\lambda t(s-1)} \tag{7}$$

and as $I \sim \text{Bern}(p)$ we get

$$\begin{aligned} G_I(s) &= \sum_{k=1}^{\infty} p_X(x) s^x \\ &= p_X(0)s^0 + p_X(1)s^1 \\ &= (1-p) + ps \end{aligned}$$

Therefore

$$\begin{aligned} G_{X_p}(s) &= G_{X(t)}(G_I(s)) = \mathrm{e}^{\lambda t(G_I(s)-1)} = \mathrm{e}^{\lambda t(1-p+ps-1)} \\ &= \mathrm{e}^{\lambda tp(s-1)} \end{aligned}$$

which is the pgf of a Poisson distribution with parameter λtp.