8 Markov chains

- Define Markov chains
- Make example of on/off
- Communication
- Irreducibility
- Recurrence
- Stationary distribution
- Convergence

Definition 8.1. (Markov chain). Let $X_0, X_1, \ldots$ be a sequence of discrete random variables, taking values in som set S and that are such that

$$P(X_{n+1} = j | X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i) = P(X_{n+1} = j | X_n = i) \quad (1)$$

for all $i, j, i_0, \ldots i_{n-1}$ in S and all n. The sequence $\{X_n\}$ is the called a *Markov chain.*

$$p_{ij} = P(X_{n+1} | X_n = j) \quad (2)$$

On/Off example Probability p of switching on and q of switching of.

$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} \quad (3)$$

Limit distribution

$$\lim_{n\to\infty} P^{(n)} = \frac{1}{p+q} \begin{pmatrix} q & p \\ q & p \end{pmatrix} \quad (4)$$

Definition 8.2. (Communication). If $p_{ij}^{(n)} > 0$ for som n, we say that state j is accessible from state i written $i \to j$. If both ways, they communicate.

Definition 8.3. (Irreducibility). If all states in S communicate with each other, the Markov chain is said to be *irreducible.*

Definition 8.4. (Recurrence). Consider a state $i \in S$ and τ_i the number of steps to first visit i:

$$\tau_i = \min\{n \geq 1 : X_n = i\} \quad (5)$$

$\tau_i = \infty$ never visited

Recurrent if $P_i(\tau_i < \infty) = 1$

Transcient if $P_i(\tau < \infty) < 1$

Definition 8.8 (Periodicity). The *period* of state i is defined as

$$d(i) = gcd\{n \geq 1 : p_{ij}^{(n)} > 0\} \tag{6}$$

If $d(i) = 1$, i is said to be aperiodic; otherwise periodic.

Definition 8.5 (Stationary distribution). Let P be the transition matrix of a Markov chain with state space S. A probability distribution $\pi = (\pi_1, \pi_2, \ldots)$ on S satisfying

$$\pi P = \pi \tag{7}$$

is called a stationary distribution of the chain.

Theorem 8.1 (Convergence). Consider an irreducible, positive recurrent, and aperiodic Markov chain with stationary distribution π and n-step transition matrix $p_{ij}^{(n)}$. Then

$$p_{ij}^{(n)} \to \pi_j \text{ as } n \to \infty \tag{8}$$

for all $i, j \in S$.