Stochastic Processes
Session 17

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Standard form and Elements of an Estimation Problem

Standard Form of an **Estimation Problem:**

$$\text{Given } \boldsymbol{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix} \text{ estimate } \theta.$$

Elements:

- A set of random observations: $\boldsymbol{X} = [X_1, X_2, \ldots, X_N]^{\mathsf{T}}$
- A parameter θ whose value we want to estimate
- A model relating X to θ: e.g. the conditional pdf $p(\boldsymbol{X}|\theta)$
- Possibly, some prior information on θ: $p(\theta)$
- An estimator $g(\cdot)$ yielding the estimate: $\hat{\theta} = g(\boldsymbol{X})$
- Typically, $g(\cdot)$ is designed to satisfy and *optimality criterion*

How to Solve It (An Estimation Problem)

To solve an estimation problem, we follow these steps:

1. **Identify the elements of the problem:**
   —What are the unknowns (θ) and the data ($\boldsymbol{X}$)?
   —How do they relate?

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2. **Extract all information available from the problem:**
   —What information is provided (μ_θ, σ_θ^2, $p(\theta)$, $p(\boldsymbol{X}|\theta)$,...)?
   —What information can we obtain ($\boldsymbol{\mu}_{\boldsymbol{X}}$,$\boldsymbol{C}_{\boldsymbol{X}\theta}$,$\boldsymbol{C}_{\boldsymbol{XX}}$,$p(\theta|\boldsymbol{X})$,...)?

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3. Based on the info at hand, **choose and estimator and compute it:**
   —If $p(\theta|\boldsymbol{X})$ is known, we can(?) compute the MMSE estimator.
   —If only means, variances and covariances are known, we can use the LMMSE estimator.

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   —Compute bias and MSE of the estimator.
   —Implement and run Monte Carlo simulations.
   —Evaluate complexity.

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5. **Are we satisfied?:**
—*NO*: go back to previous steps and refine.
—*YES*: We are done!

LMMSE Estimation – Advantages of LMMSE Estimation

The LMMSE estimator has some nice properties which makes it widely used:

- It is *very simple to implement*: once the optimal coefficients have been calculated only $N+1$ multiplications and additions are needed to compute an estimate.
- The optimal coefficients depend only on *first-order moments* (μ_θ and $\boldsymbol{\mu_X}$) and *second-order central moments* ($\boldsymbol{C_{X\theta}}$ and $\boldsymbol{C_{XX}}$) of the parameter of interest θ and the observations $\boldsymbol{X}$, and not on their full joint distribution.
- With the additional knowledge of σ_θ^2, the *theoretical MSE* of the LMMSE estimates can be *easily evaluated*.
- The LMMSE estimator is the fundamental building block for the *Kalman filter*.