Continuous stochastic variables

Define and present

- Continuous random variable
- Expected value
- Variance
- Expected value of a function

Prove

- Expected value of unif
- Variance of unif

Definition 2.5. (Continuous random variable) If the cdf F is a continuous function, then X is said to be a *continuous random variable*.

Definition 2.9 (Expected value) Let X be a continuous random variable with pdf f. The *expected value* of X is defined as

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx \tag{1}$$

Defintion 2.10. (Variance) Let X be a random variable with expected value μ. The *variance* of X is defined as

$$\mathrm{Var}[X] = E[(X-\mu)^2] \tag{2}$$

Proposition 2.12.(Expected value of function) Let X be a random variable with pdf f_X, and let $g : \mathrm{R} \to \mathrm{R}$ be any function. Then

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx \tag{3}$$

Proposition 2.13. (Expected value and variance of unif) If $X \sim \mathrm{unif}[a,b]$, then

$$E[X] = \frac{a+b}{2} \qquad \text{and} \qquad \mathrm{Var}[X] = \frac{(b-a)^2}{12} \tag{4}$$

Bevis
Calculating $E[X]$:

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x f(x)\, dx = \int_a^b x \frac{1}{b-a}\, dx \\ &= \frac{1}{b-a} \int_a^b x\, dx = \frac{b^2 - a^2}{2(b-a)} \\ &= \frac{b+a}{2} \end{aligned}$$

Calculating $E[X^2]$:

$$\begin{aligned} E[X^2] &= \int_a^b x^2 f(x)\, dx = \frac{1}{b-a} \int_a^b x^2\, dx \\ &= \frac{b^3 - a^3}{3(b-a)} = \frac{(b-a)(a^2+ab+b^2)}{3(b-a)} \\ &= \frac{a^2+ab+b^2}{3} \end{aligned}$$

This gives variance:

$$\begin{aligned} \mathrm{Var}[X] &= \frac{a^2+ab+b^2}{3} - \left(\frac{a+b}{2}\right)^2 \\ &= \frac{a^2+ab+b^2}{3} - \frac{a^2+ab+b^2}{4} \\ &= \frac{4(a^2+ab+b^2)}{12} - \frac{3(a^2+2ab+b^2)}{12} \\ &= \frac{(b-a)^2}{12} \end{aligned}$$