Maximum-Margin Classifier Exercises

Exercises

1. Suppose that with respect to some optimal decision line, $(0,0)$ is the (positive) support vector for the positively classified training vectors, while $(2,0)$ and $(0,-2)$ are its (negative) support vectors for the negatively classified training vectors. Provide the equation of the optimal decision line and its maximum margin.

2. If the positively classified training vectors are

$$(-2,3),(-3,4),(0,0),(-5,-1),$$

while the negatively classified training vectors are

$$(6,-3),(8,-4),\ \text{and}\ (4,-2),$$

then compute an optimal decision line that separates this data. Compute w and b for this line.

3. For the previous exercise, provide an equation $\overline{w}\cdot\overline{x}=b$ for the optimal-decision surface so that $\overline{w}\cdot\overline{x}_+=b+1$, while $\overline{w}\cdot\overline{x}_-=b-1$, where $\overline{x}_+$ and $\overline{x}_-$ are the repsective positive and negative support vectors.

4. A function $f:\mathcal{R}^n\to\mathcal{R}$ is called **convex** iff, for all vectors $\overline{x},\overline{y}\in\mathcal{R}^n$ and $\alpha\in[0,1]$,

$$f(\alpha\overline{x}+(1-\alpha)\overline{y})\leq\alpha f(\overline{x})+(1-\alpha)f(\overline{y}).$$

Geometrically, this means that any line segment between two points on the graph of f will lie above the graph of f. Prove algebraically that $f(\overline{x})=|\overline{x}|^2$ is convex. Hint: $|\overline{x}|^2=(\overline{x}\cdot\overline{x})^2$.

5. Find values for x, y, and α that prove $f(x)=x^3$ is *not* a convex function over the set of real numbers.

Exercise Solutions

1. The optimal decision line is perpendicular to the vector
$$\overline{w} = (0,0) - (1,-1) = (-1,1)$$
and passes through the midpoint $(1/2,-1/2)$ of the line segment $\overline{(0,0)(1,-1)}$. Hence,
$$b = (-1,1)\cdot(1/2,-1/2) = -1,$$
and the equation is
$$(-1,1)\cdot(x,y) = -x+y = -1.$$
Finally, support vector $(0,0)$ has a distance of $|\overline{w}\cdot\overline{0} - (-1)|/|\overline{w}| = 1/\sqrt{2}$ from the line. Therefore, the maximum margin is $2(1/\sqrt{2}) = \sqrt{2}$.

2. After plotting the points, we see that the two support vectors are $P = (0,0)$ and $Q = (4,-2)$. The distance between these two points is $2\sqrt{5}$. Moreover, since the line that passes through the midpoint of $\overline{PQ}$ and is perpendicular to $\overline{PQ}$ separates the two classes, this line has a maximum margin equal to $2\sqrt{5}$, and is thus an optimal decision line. Moreover, since it passes through the midpoint $(2,-1)$ and is perpendicular to the vector
$$\overline{w} = (-4,2) = (0,0) - (4,-2),$$
it has offset
$$b = (2,-1)\cdot(-4,2) = -10.$$
Therefore, the equation of the line is
$$\overline{w}\cdot\overline{x} = (-4,2)\cdot(x,y) = -4x+2y = -10.$$

3. From the previous exercise we have $\overline{x}_+ = (0,0) = \overline{0}$ and $\overline{x}_- = (4,-2)$. Moreover, since we desire that $\overline{w}\cdot\overline{0} = 0 = b+1$, it follows that $b = -1$. And since the equation found in the previous exercise is $-4x+2y = -10$, we get the new equation by dividing both sides by 10:
$$\frac{-2x}{5} + \frac{y}{5} = -1.$$
Finally, notice that $(-2/5,1/5)\cdot(4,-2) = -2 = b-1$, which is the desired value. Therefore, $\overline{w} = (-2/5,1/5)$ and $b = -1$ are the desired parameters for the optimal decision line.

4.
$$\begin{gathered}
f(\alpha\overline{x} + (1-\alpha)\overline{y}) = |\alpha\overline{x} + (1-\alpha)\overline{y}|^2 = (\alpha\overline{x} + (1-\alpha)\overline{y})\cdot(\alpha\overline{x} + (1-\alpha)\overline{y}) = \\
\alpha^2(\overline{x}\cdot\overline{x}) + (1-\alpha)^2(\overline{y}\cdot\overline{y}) + 2\alpha(1-\alpha)(\overline{x}\cdot\overline{y}) \leq \\
\alpha f(\overline{x}) + (1-\alpha)f(\overline{y}) = \alpha|\overline{x}|^2 + (1-\alpha)|\overline{y}|^2 \Leftrightarrow \\
2\alpha(1-\alpha)(\overline{x}\cdot\overline{y}) \leq (\alpha-\alpha^2)|\overline{x}|^2 + ((1-\alpha)-(1-\alpha)^2)|\overline{y}|^2 \Leftrightarrow \\
2\alpha(1-\alpha)(\overline{x}\cdot\overline{y}) \leq \alpha(1-\alpha)|\overline{x}|^2 + \alpha(1-\alpha)|\overline{y}|^2 \Leftrightarrow \\
2(\overline{x}\cdot\overline{y}) \leq |\overline{x}|^2 + |\overline{y}|^2 \Leftrightarrow \\
|\overline{x}|^2 + |\overline{y}|^2 - 2(\overline{x}\cdot\overline{y}) = (\overline{x}-\overline{y})\cdot(\overline{x}-\overline{y}) = |\overline{x}-\overline{y}|^2 \geq 0,
\end{gathered}$$
which is true.

5. Let $x = -2$, $y = 2$, and $\alpha = 3/4$. Then
$$f((3/4)(-2)+(1/4)(2)) = f(-1) = -1 > (3/4)f(-2)+(1/4)f(2) = (3/4)(-8)+(1/4)(8) = -4.$$