

Analog Signal Processing**Thursday, October 21, 8:45-10pm****Exam II**

Last Name (capitalized):	Solutions
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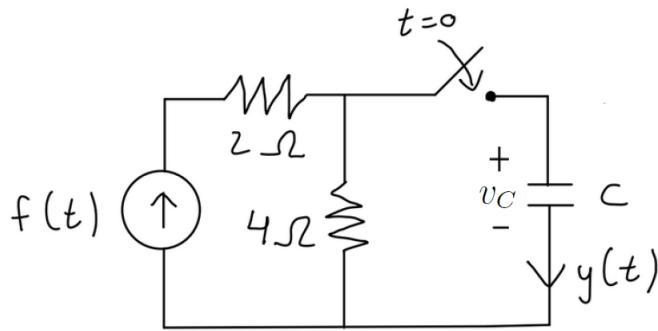
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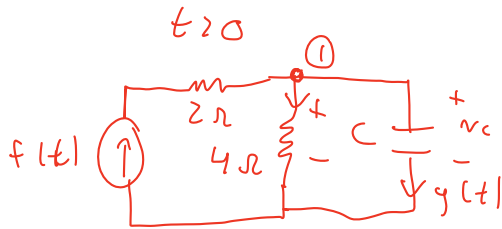
<p>Clearly PRINT your name in CAPITAL LETTERS.</p> <p>This is a closed book and closed notes exam.</p> <p>Calculators are not allowed.</p> <p>To get credit, please SHOW all your work and simplify your answers.</p> <p>Write your final answers in the spaces provided.</p> <p>All answers should INCLUDE UNITS whenever appropriate.</p> <p>The exam is printed double-sided.</p>	<p style="text-align: center;">DO NOT write in these spaces.</p> <p>Problem 1 (25 points):_____</p> <p>Problem 2 (25 points):_____</p> <p>Problem 3 (25 points):_____</p> <p>Problem 4 (25 points):_____</p> <p>Total: (100 points):_____</p>
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1. (25 pts) Consider the LTI circuit below with current $f(t) = \cos(\omega t) + 3 \sin(\omega t)$ A and voltage $v_C(0^-) = v_0$ V.



It is known that the current $y(t) = 2 \cos(2t) + \sin(2t) + Be^{-2t}$ A for $t > 0$.

- (a) [5 pts] Write the ODE that governs this LTI system for $t > 0$ in terms of $f(t)$, C , $v_C(t)$ and ω .



KCL @ ①.

$$f(t) = \frac{v_C}{4} + \dot{v}_C$$

$$f(t) = \frac{v}{4} + C \frac{dv}{dt}$$

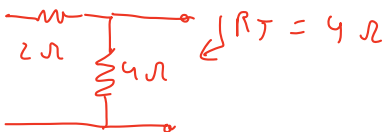
$$\frac{1}{C} f(t) = \frac{dv}{dt} + \frac{1}{4C} v$$

$$\text{ODE} = \frac{1}{C} f(t) = \frac{dv}{dt} + \frac{1}{4C} v$$

- (b) [3 pts] Determine the value of $C =$

$$\frac{1}{8} \text{ F}$$

The exponential term in $y(t)$ is the homogeneous solution: $Ae^{-t/\tau} = Be^{-2t} \rightarrow \tau = \frac{1}{2} = RC = 4C$



$$\rightarrow C = \frac{1}{2(4)} = \frac{1}{8}$$

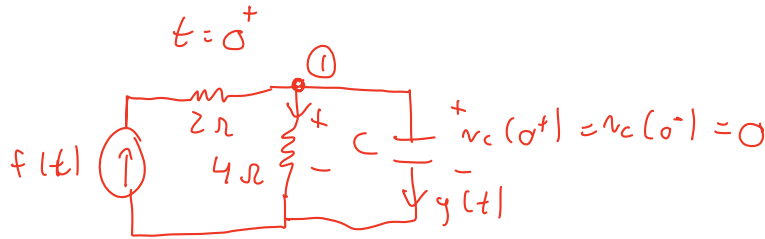
- (c) [3 pts] Determine the value of $\omega =$

$$2 \text{ rad/s}$$

Frequency of $f(t)$ & $y(t)$ must be the same

$$\rightarrow \omega = 2 \text{ rad/s}$$

(d) [4 pts] If $v_0 = 0V$, determine the value of $B = \boxed{-1}$ Amperes



$$v_u = v_c = 0 \rightarrow i_{4\Omega} = 0 \rightarrow i_c(0^+) + i_{4\Omega}(0^+) = f(0^+)$$

$$y(t) = 2\cos(2t) + \sin(2t) + Be^{-2t}$$

$$i_c(0^+) = \cos(0) + 3\sin(0) = 1$$

$$\rightarrow y(0^+) = 2\cos(0) + \sin(0) + B = 2 + B$$

$$\parallel$$

$$i_c(0^+) = 1 \rightarrow 1 = 2 + B \rightarrow B = -1$$

(e) [4 pts] Determine steady-state current phasor $Y = \boxed{2-j}$ A

$$y(t) = \underbrace{2\cos(2t) + \sin(2t)}_{\text{steady-state}} + \underbrace{Be^{-2t}}_{\text{transient}}$$

↓ phase-

$$y_{ss}(t) = 2\cos(2t) + \cos(2t + \frac{\pi}{2})$$

$$y = \underbrace{2}_{\downarrow} + e^{-j\frac{\pi}{2}} = 2 - j$$

(f) [5 pts] If instead of $f(t) = \cos(\omega t) + 3\sin(\omega t)$ A, the input current source in this LTI system is given by $f_1(t) = 2\cos(\omega t - 1) + 6\sin(\omega t - 1)$ A, determine the steady-state response, $y_1(t)$.

$$\text{Notice } f_1(t) = 2 f\left(t - \frac{1}{\omega}\right)$$

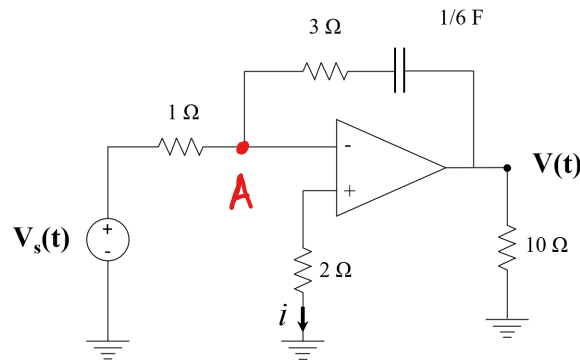
$$\text{LTI} \rightarrow y_1(t) = 2 y_{ss}\left(t - \frac{1}{\omega}\right)$$

$$= 4\cos\left(2\left(t - \frac{1}{\omega}\right)\right) + 2\sin\left(2\left(t - \frac{1}{\omega}\right)\right)$$

$$= 4\cos\left(2\left(t - \frac{1}{\omega}\right)\right) + 2\sin\left(2\left(t - \frac{1}{\omega}\right)\right)$$

$$y_1(t) = \boxed{4\cos(2t - 1) + 2\sin(2t - 1) \text{ Amperes}}$$

2. (25 pts) The output voltage in the circuit below is given by $V(t) = 4\cos(2t)$. Using the phasor method and ideal op-amp approximations determine:



$$\omega = 2$$

$$Z_c = -\frac{j}{2 \cdot \frac{1}{6}} = -j \cdot 3$$

$$V = 4$$

- (a) [18 pts] The input voltage $V_s(t)$ and express it in terms of real-valued functions only.

$$i_+ = i_- = 0;$$

$$V_+ = V_- = 0; \text{ (no current flows through } 2\Omega \text{ resistor)}$$

$$1-j \rightarrow r = \sqrt{1+1} = \sqrt{2}$$

$$\phi = \tan^{-1}(-1)$$

$$\frac{1}{\sqrt{2} \cdot e^{-j\frac{\pi}{4}}}$$

$$\text{KCL @ A: } \frac{V_s - 0}{1} = \frac{0 - V}{3 - 3j}; \quad V_s = -\frac{4}{3 - 3j} = -\frac{4}{3} \cdot \frac{1}{1 - j};$$

$$V_s = -\frac{4}{3} \cdot \frac{1}{\sqrt{2}} \cdot e^{j\frac{\pi}{4}} = -\frac{2\sqrt{2}}{3} \cdot e^{j\frac{\pi}{4}}$$

$$\frac{2\sqrt{2}}{3} \cdot \cos\left(2t + \frac{5\pi}{4}\right)$$

$$\frac{2\sqrt{2}}{3} \cdot \cos\left(2t - \frac{3\pi}{4}\right)$$

Same

$$-\frac{2\sqrt{2}}{3} \cdot \cos\left(2t + \frac{\pi}{4}\right) \text{ V}$$

$$V_s(t) =$$

- (b) [3 pts] The current in the 2Ω resistor and express it in terms of real-valued functions only.

$$i_{2\Omega}(t) = 0 \text{ A (ideal OP-AMP)}$$

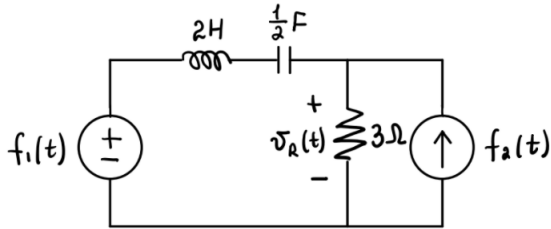
- (c) [2 pts] The average absorbed power in the 1Ω resistor, $P_{1\Omega} = \frac{4}{9} \text{ W}$

$$P = \frac{1}{2} \frac{|V|^2}{R} = \frac{1}{2} \cdot \frac{8}{9} = \frac{4}{9} \text{ W}$$

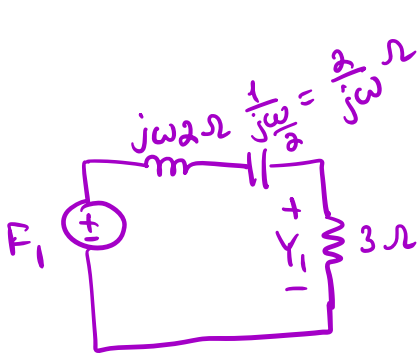
$$V = -\frac{2\sqrt{2}}{3}; \quad R = 1;$$

- (d) [2 pts] The average absorbed power in the capacitor, $P_C = 0 \text{ W}$

3. (25 points) Consider the circuit below, where the output $y(t) = v_R(t)$.



- (a) [6 pts] Determine the frequency response $H_1(\omega)$, considering only the input voltage source, $f_1(t)$.



$$H(\omega) = \frac{Y}{F}$$

$$Y_1 = F_1 \frac{3}{j\omega 2 + \frac{2}{j\omega} + 3} = F_1 \frac{3j\omega}{-2\omega^2 + 2 + 3j\omega} = F_1 \frac{3j\omega}{\underbrace{2 - 2\omega^2 + 3j\omega}_{H_1(\omega)}}$$

$$H_1(\omega) = \frac{3j\omega}{2 - 2\omega^2 + 3j\omega}$$

- (b) [5 pts] Determine the steady-state output voltage $v_{R_1}(t)$ given $f_1(t) = 3 \cos(2t) + 2 \sin(\frac{1}{2}t)$ V.

$$v_{R_1}(t) = 3 \cdot |H_1(2)| \cos(2t + \angle H_1(2)) + 2 |H_1(\frac{1}{2})| \sin(\frac{1}{2}t + \angle H_1(\frac{1}{2})) \quad \oplus$$

$$H_1(2) = \frac{6j}{-6 + 6j} = \frac{j}{-1 + j} = \frac{e^{j\pi/2}}{\sqrt{2} e^{j(\pi + \tan^{-1}(-1))}}$$

$$|H_1(2)| = \frac{1}{\sqrt{2}}$$



$$\angle H_1(2) = \frac{\pi}{2} - \pi + \pi/4 = -\pi/4$$

$$\omega_1 = 2 \text{ rad/s} \quad \omega_2 = \frac{1}{2} \text{ rad/s}$$

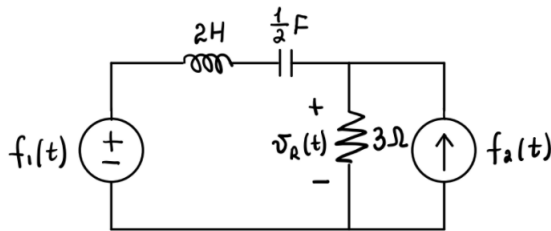
$$H_1(\frac{1}{2}) = \frac{\frac{3}{2}j}{\frac{3}{2} + \frac{2}{2}j} = \frac{j}{1+j}$$

$$= \frac{e^{j\pi/2}}{\sqrt{2} e^{j\tan^{-1}(1)}}$$

$$|H_1(\frac{1}{2})| = \frac{1}{\sqrt{2}} \quad \angle H_1(\frac{1}{2}) = \frac{\pi}{2} - \frac{\pi}{4} = \pi/4$$

$$v_{R_1}(t) \ominus \frac{3}{\sqrt{2}} \cos(2t - \pi/4) + \frac{2}{\sqrt{2}} \sin(\frac{1}{2}t + \pi/4) \text{ V}$$

Recall



(c) [6 pts] Determine the frequency response $H_2(\omega)$, considering only the input current source, $f_2(t)$.

Handwritten solution for (c):

Looking into the terminals from the right (from the current source F_2), the impedance is Z_p :

$$Z_p = \frac{3 \cdot (j\omega 2 + \frac{2}{j\omega})}{3 + j\omega 2 + \frac{2}{j\omega}} = \frac{3(-2\omega^2 + 2) \cdot j\omega}{j\omega(3j\omega - 2\omega^2 + 2)}$$

$$H_2(\omega) = \frac{3(2 - 2\omega^2)}{2 - 2\omega^2 + 3j\omega}$$

Also shown: $V_2 = F_2 \left(\frac{Z_p}{3} \right) \cdot 3 = F_2 \cdot Z_p = F_2 \left(\frac{3(2 - 2\omega^2)}{2 - 2\omega^2 + 3j\omega} \right)$

(d) [5 pts] Determine the steady-state output voltage $v_{R_2}(t)$ given $f_2(t) = 3 \cos(2t) + 2 \sin\left(\frac{t}{2}\right)$ A.

Handwritten solution for (d):

For $\omega = 2$:

$$H_2(2) = \frac{-18}{-6 + 6j} = \frac{3}{1-j} = \frac{3e^{j0}}{\sqrt{2}e^{j \tan^{-1}(-1)}}$$

$$|H_2(2)| = \frac{3}{\sqrt{2}}$$

$$\angle H_2(2) = \pi/4$$

For $\omega = \frac{1}{2}$:

$$H_2\left(\frac{1}{2}\right) = \frac{9/2}{\frac{3}{2} + \frac{3}{2}j} = \frac{3}{1+j} = \frac{3e^{j0}}{\sqrt{2}e^{j \tan^{-1}(1)}}$$

$$|H_2\left(\frac{1}{2}\right)| = \frac{3}{\sqrt{2}}$$

$$\angle H_2\left(\frac{1}{2}\right) = -\pi/4$$

$$v_{R_2}(t) = \frac{9}{\sqrt{2}} \cos(2t + \pi/4) + \frac{6}{\sqrt{2}} \sin\left(\frac{1}{2}t - \pi/4\right) \text{ V}$$

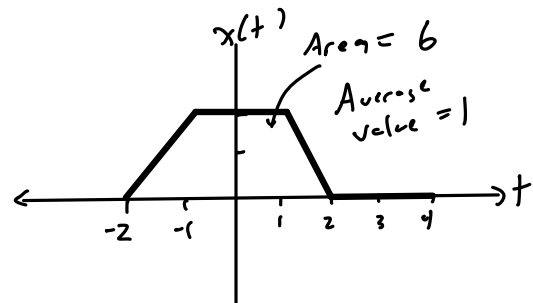
(e) [3 pts] Determine the steady-state output voltage $v_R(t)$.

$$v_R(t) = v_{R_1}(t) + v_{R_2}(t) = \frac{3}{\sqrt{2}} \cos(2t - \pi/4) + \frac{2}{\sqrt{2}} \sin\left(\frac{1}{2}t + \pi/4\right) + \frac{9}{\sqrt{2}} \cos(2t + \pi/4) + \frac{6}{\sqrt{2}} \sin\left(\frac{1}{2}t - \pi/4\right) \text{ V}$$

4. (25 points) The two parts of this problem are unrelated.

(a) Let:

$$x(t) = \begin{cases} 2t + 4 & -2 < t \leq -1 \\ 2 & -1 < t \leq 1 \\ -2t + 4 & 1 < t \leq 2 \\ 0 & 2 < t \leq 4 \end{cases}$$



be a periodic signal, which repeats at time $t = 4$ s.

i. [2 pts] The period of $x(t)$ is $T = \underline{6 \text{ s}}$

ii. [3 pts] The average value of $x(t)$ is $x_{av} = \underline{1}$

$$\begin{aligned} x_{av} &= \frac{1}{T} \int_{-2}^4 x(t) dt \\ &= \frac{1}{6} \left[\int_{-2}^{-1} (2t+4) dt + \int_{-1}^1 2 dt + \int_1^2 (-2t+4) dt \right] \\ &= \frac{1}{6} \left[t^2 + 4t \Big|_{-2}^{-1} + 2t \Big|_{-1}^1 + (-t^2 + 4t) \Big|_1^2 \right] \\ &= \frac{1}{6} \left[(1-4) - (4-8) + (2(1) - 2(-1)) + (-4+8) - (-1+4) \right] \\ &= \frac{1}{6} \left[-3 + 4 + 2 + 2 + 4 - 3 \right] = \frac{1}{6} [6] = 1 \end{aligned}$$

iii. [2 pts] Now let us define a different periodic signal $y(t) = dx(t)/dt$. Express $y(t)$ as a piecewise function like it is done for $x(t)$ above.

$$y(t) = \begin{cases} 2 & -2 < t \leq -1 \\ 0 & -1 < t \leq 1 \\ -2 & 1 < t \leq 2 \\ 0 & 2 < t \leq 4 \end{cases}$$

- iv. [5 pts] Determine the Fourier coefficients Y_n for all n .

$$\begin{aligned}\omega_0 &= \frac{2\pi}{T} = \frac{2\pi}{6} = \frac{\pi}{3} \\ Y_n &= \frac{1}{T} \int_T y(t) e^{-jn\omega_0 t} dt \\ &= \frac{1}{6} \left[\int_{-2}^{-1} 2 e^{-jn\frac{\pi}{3}t} dt + \int_{-1}^2 (-2) e^{-jn\frac{\pi}{3}t} dt \right] \\ &= \frac{1}{6} \left[\frac{2}{-jn\frac{\pi}{3}} e^{-jn\frac{\pi}{3}t} \Big|_{-2}^{-1} + \frac{(-2)}{-jn\frac{\pi}{3}} e^{-jn\frac{\pi}{3}t} \Big|_{-1}^2 \right] \\ &= \frac{1}{6} \left[\frac{j6}{n\pi} (e^{jn\frac{\pi}{3}} - e^{jn\frac{2\pi}{3}}) - \frac{j6}{n\pi} (e^{-jn\frac{2\pi}{3}} - e^{-jn\frac{\pi}{3}}) \right] \\ &= \frac{j}{n\pi} [e^{jn\frac{\pi}{3}} + e^{-jn\frac{\pi}{3}} - e^{jn\frac{2\pi}{3}} - e^{-jn\frac{2\pi}{3}}] \\ Y_n &= \frac{j2}{n\pi} (\cos \frac{n\pi}{3} - \cos \frac{2n\pi}{3}), \quad Y_0 = 0\end{aligned}$$

- v. [5 pts] Use the derivative property to determine the Fourier coefficients X_n for all n .

$$\begin{aligned}y(t) &= \frac{dx(t)}{dt} \leftrightarrow jn\omega_0 X_n = Y_n \\ X_n &= \frac{Y_n}{jn\omega_0} \\ X_n &= \frac{Y_n}{jn\frac{\pi}{3}} \quad \text{except } X_0 = 1\end{aligned}$$

- (b) Consider the function $q(t) = 2\sin(2t) + 3\cos(4t)$,

- i. [2 pts] Its fundamental frequency is $\omega_0 = 2 \text{ rad/s}$

- ii. [5 pts] Express the function in exponential Fourier series form.

$$\begin{aligned}q(t) &= \frac{2}{j2} (e^{j2t} - e^{-j2t}) + \frac{3}{2} (e^{j4t} + e^{-j4t}) \\ q(t) &= \sum_{n=-2}^2 Q_n e^{jn\omega_0 t} \quad \begin{matrix} Q_1 = \frac{1}{j} \\ Q_{-1} = -\frac{1}{j} \\ Q_{\pm 2} = \frac{3}{2} \\ Q_n = 0 \text{ else} \end{matrix}\end{aligned}$$

- iii. [5 pts] Its average power is $P_q = 6.5 \text{ W}$

$$\begin{aligned}\sum_{n=-2}^2 |F_n|^2 &= |F_1|^2 + |F_{-1}|^2 + |F_2|^2 + |F_{-2}|^2 \\ &= 1 + 1 + \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2 = 2 + \frac{9}{2} \\ &= 6.5 \text{ W}\end{aligned}$$