- Inverse Laplace transform
  - Causal signals have unique transforms

$$f(t) \overset{\mathcal{L}}{\longleftrightarrow} \hat{F}(s)$$

  - It is known that

Table 11.1

$\delta(t) \leftrightarrow 1$
$e^{pt}u(t) \leftrightarrow \frac{1}{s-p}$ ← p is a pole (pole @ s=p)
$te^{pt}u(t) \leftrightarrow \frac{1}{(s-p)^2}$
$t^n e^{pt}u(t) \leftrightarrow \frac{n!}{(s-p)^{n+1}}$

  - Can we use this information to obtain the inverse Laplace transform of more general functions without having to do integration in the complex plane?

- Inverse Laplace transform - cont.
  - Consider the subset of transforms that are proper rational functions:

$$\hat{F}(s) = \frac{N(s)}{D(s)}$$

  - N(s) and D(s) are polynomials in s with

degree of N(s) < degree of D(s)

  - So

$$\hat{F}(s) = \frac{N(s)}{(s-p_1)(s-p_2)\ldots(s-p_n)}$$

partial fraction expansion (PFE)

  - Can we write it as

$$\hat{F}(s) = \frac{A_1}{s-p_1} + \frac{A_2}{s-p_2}+\ldots+\frac{A_n}{s-p_n}\ ?$$

- Inverse Laplace transform - Distinct poles
  - If all the poles of $\hat{F}(s)$ are distinct, how can we write it as

non-repeated

$$\hat{F}(s) = \frac{A_1}{s-p_1} + \frac{A_2}{s-p_2} + \ldots + \frac{A_n}{s-p_n}\ ?$$

A_1:

$$\underbrace{(s-p_1)\hat{F}(s)}_{\text{evaluate at } s=p_1} = A_1 + \frac{A_2\cancel{(s-p_1)}^{0}}{(s-p_2)} + \ldots \frac{A_n\cancel{(s-p_1)}^{0}}{(s-p_n)} \Rightarrow A_1 = (s-p_1)\hat{F}(s)\Big|_{s=p_1}$$

$$A_n = (s-p_n)\hat{F}(s)\Big|_{s=p_n}$$

- Inverse Laplace transform - Distinct poles - Example # 10

  - Determine the inverse Laplace transform of

proper? ✓
rational? ✓
poles are distinct? ✓

$$\hat{F}(s) = \frac{s+3}{s^2+3s+2} = \frac{s+3}{(s+1)(s+2)}$$

poles @ $s = -1, -2$

$p_1 = -1, \; p_2 = -2$

$$\hat{F}(s) = \frac{s+3}{(s+1)(s+2)} = \frac{A_1}{s+1} + \frac{A_2}{s+2}$$

$$A_1 = \hat{F}(s)(s+1)\Big|_{s=-1} = \frac{(s+3)\cancel{(s+1)}}{\cancel{(s+1)}(s+2)} = \frac{s+3}{s+2}\Big|_{s=-1} = \frac{-1+3}{-1+2} = 2$$

$$A_2 = \hat{F}(s)(s+2)\Big|_{s=-2} = \frac{(s+3)\cancel{(s+2)}}{(s+1)\cancel{(s+2)}} = \frac{s+3}{s+1}\Big|_{s=-2} = -1$$

$$\hat{F}(s) = \underset{p=-1}{\frac{2}{s+1}} - \underset{p=-2}{\frac{1}{s+2}} \xrightarrow{\mathcal{L}^{-1}} 2e^{-t}u(t) - e^{-2t}u(t) = f(t)$$

$$e^{pt}u(t) \Leftrightarrow \frac{1}{s-p} \quad \text{Table 11.1}$$

• Inverse Laplace transform - Distinct poles - Example # 10

- Determine the inverse Laplace transform of

$$\hat{F}(s) = \frac{s+3}{s^2+3s+2} = \frac{s+3}{(s+1)(s+2)} = \frac{A_1}{s+1} + \frac{A_2}{s+2}$$

Using "thumb" rule:

1) To get A_1, put your thumb over $(s+1)$, because A_1 has it $\Rightarrow$

$\Rightarrow \frac{s+3}{(s+1)(s+2)} \Rightarrow$ then evaluate @ $s=-1$ $\Rightarrow \frac{-1+3}{-1+2} = 2 = A_1$

thumb :)

2) To get A_2, put your thumb over $(s+2)$ because A_2 has it $\Rightarrow$

$\frac{s+3}{(s+1)(s+2)} \Rightarrow$ then evaluate @ $s=-2$ $\Rightarrow \frac{-2+3}{-2+1} = -1 = A_2$

- Inverse Laplace transform - Distinct poles - Example # 11

  - Determine the inverse Laplace transform of

✓ proper? ✓ rational? distinct poles?

$$\hat{H}(s) = \frac{1}{s(s^2+2s+2)} = \frac{1}{s(s-(-1+j))(s-(-1-j))}$$

poles @ $s = -1 \pm j$; 0.

complex poles come in conjugate pairs

$$\hat{H}(s) = \frac{A_0}{s} + \frac{A_1}{s-(-1+j)} + \frac{A_2}{s-(-1-j)} =$$

A_0 @ $s=0$ A_1 @ $s=-1+j$ A_2 @ $s=-1-j$

$$= \frac{\frac{1}{0^2+2(0)+2}}{s} + \frac{\frac{1}{(-1+j)(-1+j-(-1-j))}}{s-(-1+j)} + \frac{\frac{1}{(-1-j)(-1-j-(-1+j))}}{s-(-1-j)} =$$

$$= \frac{\frac{1}{2}}{s} + \frac{\frac{1}{2(-j-1)}}{s-(-1+j)} + \frac{\frac{1}{2(j-1)}}{s-(-1-j)} \Rightarrow A_1 = A_2^*$$

$\downarrow \mathcal{L}^{-1}$

$$h(t) = \frac{1}{2}u(t) + \underbrace{\frac{1}{2(-j-1)}e^{(-1+j)t}u(t) + \frac{1}{2(j-1)}e^{(-1-j)t}u(t)}_{(i)}$$

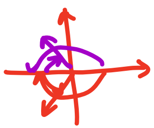
• Inverse Laplace transform - Distinct poles - Example # 11

- Determine the inverse Laplace transform of

$$\hat{H}(s) = \frac{1}{s(s^2 + 2s + 2)}$$

Rewrite complex coeff. in expon. form:

$$\frac{1}{2(-j-1)} = \frac{1}{\sqrt{4+4}} \frac{e^{j0}}{e^{j(-\frac{3\pi}{4})}} = \frac{1}{2\sqrt{2}} e^{j\frac{3\pi}{4}}$$



$$-\pi + \tan^{-1}\left(\frac{-2}{-2}\right) = -\frac{3\pi}{4}$$

$$\pi + \tan^{-1}\left(-\frac{2}{2}\right) = \frac{3\pi}{4}$$

$$\frac{1}{2(j-1)} = \frac{1}{2\sqrt{2}} e^{-j\frac{3\pi}{4}}$$

$$(i) = e^{-t}\left(\frac{1}{2\sqrt{2}} e^{j\frac{3\pi}{4}} \cdot e^{jt} + \frac{1}{2\sqrt{2}} e^{-j\frac{3\pi}{4}} \cdot e^{-jt}\right) u(t) = \frac{1}{\sqrt{2}} e^{-t} \cos\left(t + \frac{3\pi}{4}\right) u(t)$$

$$h(t) = \frac{1}{\sqrt{2}} e^{-t} \cos\left(t + \frac{3\pi}{4}\right) u(t) + \frac{1}{2} u(t)$$

- Inverse Laplace transform - Distinct poles - Example # 11
  - Determine the inverse Laplace transform of

$$\hat{H}(s) = \frac{1}{s(s^2 + 2s + 2)}$$

- Inverse Laplace transform - Repeated poles
  - What if the poles of $\hat{F}(s)$ are repeated?

$$\hat{F}(s) = \frac{N(s)}{(s - p_1)^{n}}$$

multiplicity n

$$\hat{F}(s) = \frac{A_1}{s-p_1} + \frac{A_2}{(s-p_1)^2} + \dots \frac{A_n}{(s-p_1)^n}$$

For n-th term:

$$A_n = \hat{F}(s)(s-p_1)^n \Big|_{s=p_1}$$

For the other terms:

$$A_{n-m} = \underbrace{\frac{1}{m!}}_{③} \overbrace{\frac{d^m}{ds^m} \underbrace{\hat{F}(s)(s-p_1)^n}_{①}}^{②} \Big|_{s=p_1} \; ④$$

- Inverse Laplace transform - Repeated - Example # 12

  - Determine the inverse Laplace transform of

$$\hat{F}(s) = \frac{6s-1}{(s+1)^2(s-2)}$$

repeated pole @ $s=-1$

pole @ $s=2$

$$A_{n-m} = \underbrace{\frac{1}{m!}}_{③} \overbrace{\frac{d^m}{ds^m} \underbrace{\hat{F}(s)(s-p_1)^n}_{①}}^{②} \Bigg|_{s=p_1} ④$$

$$\hat{F}(s) = \frac{A_0}{s-2} + \frac{A_1}{s+1} + \frac{A_2}{(s+1)^2} =$$

$$= \frac{\frac{6(2)-1}{(2+1)^2}}{s-2} + \frac{\frac{6(-1)-1}{-1-2}}{(s+1)^2}$$

@ $s=2$: $A_0 = \frac{11}{9}$ @ $s=-1$: $A_2 = \frac{7}{3}$

$$A_1 = \frac{1}{1!}\frac{d}{ds}\hat{F}(s)(s+1)^2\Bigg|_{s=-1} = \frac{6(-1-2)-(6(-1)-1)(1)}{(-1-2)^2} = -\frac{11}{9}$$

$A_1 \to n-m$, $n=2$, $m=1$

$$\hat{F}(s)(s+1)^2 = \frac{6s-1}{s-2} \Rightarrow \frac{d}{ds}\left(\frac{6s-1}{s-2}\right) = \frac{1}{1!}\frac{6(s-2)-(6s-1)(1)}{(s-2)^2}\Bigg|_{s=-1}$$

$$\hat{F}(s) = \frac{\frac{11}{9}}{s-2} + \frac{7/3}{(s+1)^2} - \frac{11/9}{s+1} \xrightarrow{\mathcal{L}^{-1}} \frac{11}{9}e^{2t}u(t) + \frac{7}{3}te^{-t}u(t) - \frac{4}{9}e^{-t}u(t) = f(t)$$

• Inverse Laplace transform - improper rational functions

- What if $\hat{F}(s)$ is not proper? Degree of a num. at least as big as a degree of a denom.

- Case 1:

for $k \geq 0$

$$\hat{F}(s) = \frac{s^{k+n}}{s^n + a_1 s^{n-1} + \cdots + a_n} = s^{k+n} \underbrace{\left(\frac{1}{s^n + \ldots + a_n} \right)}_{\text{proper}}$$

$\downarrow \mathcal{L}^{-1}$

Recall

$f(t) \leftrightarrow \hat{F}(s)$

$\frac{d}{dt} f(t) \overset{\mathcal{L}}{\leftrightarrow} s \hat{F}(s)$

$$f(t) = \frac{d^{k+n}}{dt^{k+n}} \mathcal{L}^{-1} \left\{ \frac{1}{s^n + \ldots + a_n} \right\}$$

① Take $\mathcal{L}^{-1}$

② Take derivative(s)

- Inverse Laplace transform - Example # 13
  - Determine the inverse Laplace transform of

$$\hat{F}(s) = \frac{s}{s+1} = s\left(\frac{1}{s+1}\right)$$

$$\Big\downarrow \mathcal{L}^{-1}$$

$$f(t) = \frac{d}{dt}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} =$$

$$= \frac{d}{dt}\left(e^{-t}u(t)\right) =$$

$$= e^{-t}\delta(t) - e^{-t}u(t) =$$

$$= \delta(t) - e^{-t}u(t)$$

• Inverse Laplace transform - improper rational functions

- What if $\hat{F}(s)$ is not proper?

- ~~Case 2~~: option 2

$$\hat{F}(s) = \frac{s^n}{s^n + P(s)} = \frac{s^n + P(s) - P(s)}{s^n + P(s)} =$$

with degree P(s) < n

$$= 1 - \underbrace{\frac{P(s)}{s^n + P(s)}}_{\text{proper}}$$

$$\downarrow \mathcal{L}^{-1}$$

$$f(t) = \frac{d^k}{dt^k}\left(\delta(t) - \mathcal{L}^{-1}\left\{\frac{P(s)}{s^n + P(s)}\right\}\right)$$

Note: if numerator is s^{n+k} for $k>0$ first

factor out s^k : $\frac{s^{n+k}}{s^n + P(s)} = s^k \left(\frac{s^n}{s^n + P(s)}\right)$

we will take k derivative at the end