ECE 210 (AL2) - ECE 211 (E)

Chapter 5

Frequency Response $H(\omega)$ of LTI Systems

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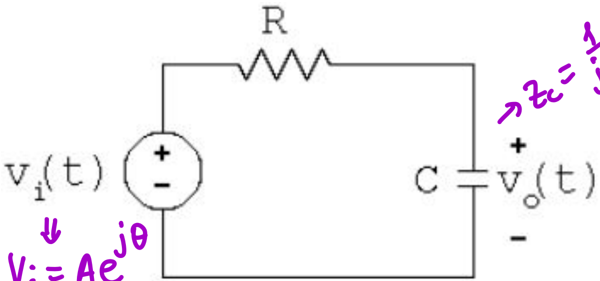
Chapter objectives

- Understand the meaning and application of an LTI system's frequency response
- Be able to obtain the frequency response of an LTI system
- Know the properties of the frequency response of an LTI system
- Use the frequency response of an LTI system obtain the system's response to co-sinusoidal inputs
- Use the frequency response of an LTI system obtain the system's response to multifrequency co-sinusoidal inputs

- Example #1



$\Downarrow$

$V_i = Ae^{j\theta}$

$\rightarrow Z_c = \frac{1}{j\omega C} = \frac{1}{j2C}\ \Omega$

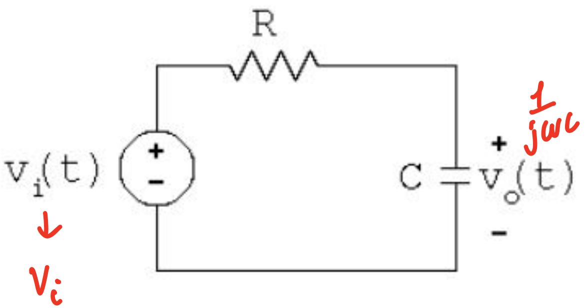
$$V_o = V_i \frac{\frac{1}{j2C}}{R + \frac{1}{j2C}} = V_i \frac{1}{j2RC+1} = \frac{|V_i|\, e^{j\angle V_i}}{\sqrt{1^2 + (2RC)^2}\, e^{j\tan^{-1}\left(\frac{2RC}{1}\right)}}$$

- Let $v_i(t) = A\cos(2t + \theta)$
- Determine $v_{o_{ss}}(t)$

$$V_o = \frac{\overbrace{|V_i|}^{A}}{\sqrt{1 + 4R^2C^2}}\, e^{j\left(\underbrace{\angle V_i}_{\theta} - \tan^{-1}(2RC)\right)}$$

$$v_o(t) = \frac{A}{\sqrt{1 + 4R^2C^2}} \cos\left(2t + \theta - \tan^{-1}(2RC)\right)\ V$$

- Example #1



$$V_o = \frac{V_i \frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = V_i \frac{1}{j\omega RC + 1} = V_o$$

H(ω) frequency response

- What if we change the frequency of the input?

$$V_o = \frac{|V_i|}{\sqrt{1+\omega^2 R^2 C^2}} e^{j(\measuredangle V_i - \tan^{-1}(\omega RC))}$$

$$v_{o_{ss}}(t) = |V_i| \frac{1}{\sqrt{1+\omega^2 R^2 C^2}} \cos(\omega t + \measuredangle V_i - \tan^{-1}(\omega RC))\ V$$

amplitude response $|H(\omega)|$

phase response $\measuredangle H(\omega)$

$$H(\omega) = |H(\omega)| e^{j \measuredangle H(\omega)}$$

- Frequency response of LTI systems

output phasor

$$V_o = V_i H(\omega)$$

input phasor

- Frequency response $H(\omega)$

$$V_i = |V_i| e^{j(\angle V_i)}$$

$$V_o = |V_i| |H(\omega)| e^{j(\angle V_i + \angle H(\omega))}$$

real signal $\rightarrow$ $\boxed{LTI}$ $\rightarrow$ real signal

$|V_i|\ \cancel{H(\omega)} \cos(\ldots\ )$

$$v_i(t) = |V_i| \cos(\omega t + \angle V_i)$$

$$v_o(t) = |V_i| |H(\omega)| \cos(\omega t + \angle V_i + \angle H(\omega))$$

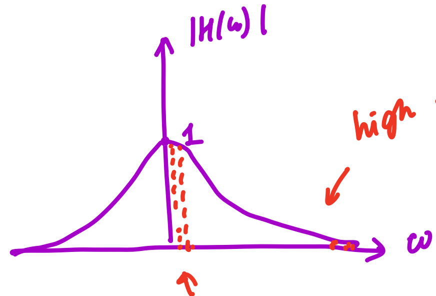
Note: works the same way if an input is a sin :)

• Frequency response of LTI systems-cont

$$v_o(t) = |V_i||H(\omega)|\cos(\omega t + \angle V_i + \angle H(\omega))$$

- Amplitude response $|H(\omega)|$ $= \dfrac{1}{\sqrt{1+\omega^2R^2C^2}}$

$$H(\omega) = \frac{1}{1+j\omega RC}$$



high frequencies are very attenuated

low frequencies pass through almost untouched

$\Downarrow$ low-pass filter

- Phase response $\angle H(\omega)$

$$\frac{e^{j0}}{e^{j\tan^{-1}\left(\frac{\omega RC}{1}\right)}}$$

$$\angle H(\omega) = -\tan^{-1}(\omega RC)$$

- Example #1-cont

- Recall $|H(\omega)| = \frac{1}{\sqrt{1+(\omega RC)^2}}$ $\quad \angle H(\omega) = -\tan^{-1}(\overset{3}{\omega} RC)$

- Determine $v_o(t)$ if $\underbrace{v_i(t) = 2\cos(\overset{\overset{3}{\omega}}{3}t + \frac{\pi}{3})}$

$v_o(t) = 2 \cdot |H(3)| \cos\left(3t + \pi/3 + \angle H(3)\right) = 2 \frac{1}{\sqrt{1+9R^2C^2}} \cos(3t + \pi/3 - \tan^{-1}(3RC))$ V

- Determine $v_o(t)$ if $v_i(t) = 3\sin(\overset{\omega}{6}t + \frac{\pi}{6})$ $= 3\cos(6t + \pi/6 - \pi/2)$

$v_o(t) = 3 \cdot |H(6)| \cos\left(6t + \pi/6 - \pi/2 + \angle H(6)\right) =$

$= 3 \cdot |H(6)| \sin\left(6t + \pi/6 + \angle H(6)\right)$

$v_o(t) = 3 \cdot |H(6)| \sin\left(6t + \pi/6 + \angle H(6)\right)$

• Frequency response of LTI systems-cont

- More generally, for LTI systems with

input $f(t) = \text{Re}\,\{F e^{j\omega t}\}$

output $y(t) = \text{Re}\,\{Y e^{j\omega t}\}$

We have:

output phasor ↙ input phasor ↙

$$\underbrace{Y = FH(\omega)}$$

$$F = |F| e^{j(\angle F)}$$

$$Y = |F||H(\omega)| e^{j(\angle F + \angle H(\omega))}$$

$$f(t) = |F| \cos(\omega t + \angle F)$$

$$y(t) = |F||H(\omega)| \cos(\omega t + \angle F + \angle H(\omega))$$

$$f(t) \rightarrow \boxed{LTI} \rightarrow y(t)$$

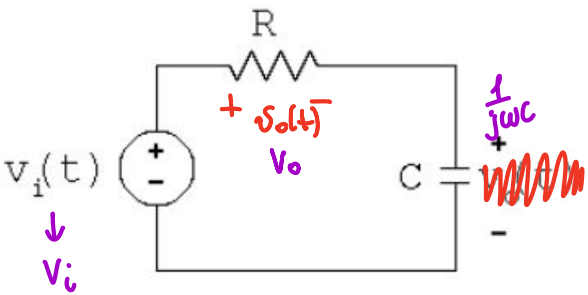
$$\downarrow$$

$$F \rightarrow \boxed{H(\omega)} \rightarrow Y = F \cdot H(\omega)$$

$$\Downarrow$$

$$\boxed{H(\omega) = \frac{Y}{F}}$$

- Example #2



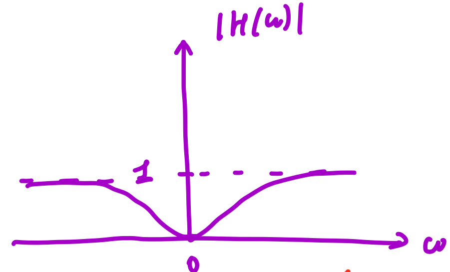
$$V_o = V_i \frac{R}{R + \frac{1}{j\omega C}} = V_i \underbrace{\frac{j\omega RC}{j\omega RC + 1}}_{H(\omega)}$$

$$Y = F \cdot H(\omega)$$

- Let $v_i(t) = A\cos(\omega t + \theta)$
- Determine $H(\omega)$, $|H(\omega)|$ and $\angle H(\omega)$)

$$|H(\omega)| = \frac{|\omega| RC}{\sqrt{1 + \omega^2 R^2 C^2}}$$

$$\angle H(\omega) = \begin{cases} \pi/2 - \tan^{-1}(\omega RC) & \text{if } \omega > 0 \\ -\pi/2 - \tan^{-1}(\omega RC) & \text{if } \omega < 0 \\ 0 & \text{if } \omega = 0 \end{cases}$$

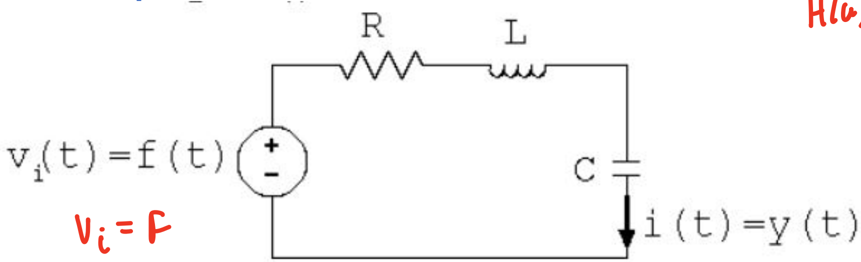


high frequencies pass through almost untouched

$\Downarrow$

high-pass filter

- Example #3



$V_i = F$

$Y = I$

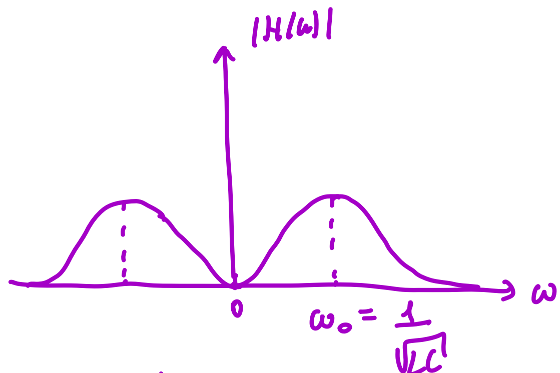
$$H(\omega) = \frac{Y}{F} = \frac{I}{V_i}$$

$$I = \frac{V_i}{R + j\omega L + \frac{1}{j\omega C}} = \frac{j\omega C\, V_i}{j\omega RC - \omega^2 LC + 1}$$

$H(\omega)$

- Let $v_i(t) = A\cos(\omega t + \theta)$
- Determine $H(\omega)$, $|H(\omega)|$ and $\angle H(\omega)$)

$$|H(\omega)| = \frac{|\omega| C}{\sqrt{(1-\omega^2 LC)^2 + (\omega RC)^2}}$$



band-pass filter

- Example #4
  - Consider the ODE

$$\frac{dy}{dt} + y = 3f(t)$$

$\downarrow$ phasors

$$j\omega Y + Y = 3F$$

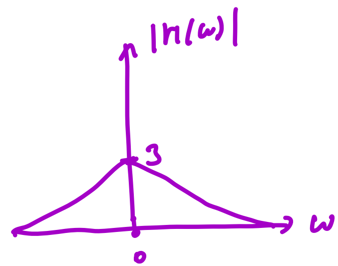
$$Y = \frac{3F}{j\omega + 1}$$

  - Let $\mathrm{f}(t) = A\cos(\omega t + \theta)$
  - Determine $H(\omega), |H(\omega)|$ and $\angle H(\omega))$

$$H(\omega) = \frac{Y}{F} = \frac{3}{1+j\omega}$$

$$|H(\omega)| = \frac{3}{\sqrt{1+\omega^2}}$$

$$\angle H(\omega) = 0 - \tan^{-1}(\omega) = -\tan^{-1}(\omega)$$



low-pass filter

- General first-order filters and second
  - Low-pass

    $|H(0)| = 1$

    $|H(\infty)| = 0$

    $$|H(\omega)| \approx \frac{1}{\sqrt{1+\omega^2}}$$

  - High-pass

    $|H(0)| = 0$

    $|H(\infty)| = 1$

    $$|H(\omega)| \approx \frac{\omega}{\sqrt{1+\omega^2}}$$

  - Band-pass

    $|H(0)| = 0$

    $|H(\infty)| = 0$

    in between $\neq 0$

    $$|H(\omega)| \approx \frac{\omega}{\sqrt{\omega^2+(1-\omega^2)^2}}$$

  - $H(\omega)$ is only meaningful for dissipative LTI systems

- Properties of $H(\omega)$ for real-valued LTI systems

- Conjugate symmetry

$$H(\omega) = H^*(-\omega)$$

$j\omega L \qquad (j(-\omega)L)^* = -j\cdot(-\omega)L = j\omega L$

- Even amplitude response

$$|H(\omega)| = |H(-\omega)|$$

- Odd phase response

$$\angle H(\omega) = -\angle H(-\omega)$$

$$-\angle H(\omega) = \angle H(-\omega)$$

- Properties of $H(\omega)$-cont
  - Real-valued DC response

$$H(0) \in \mathbb{R}$$

  - Steady-state response to complex exponential

$$f(t) = A\, e^{j(\omega t + \theta)} \qquad H(\omega)\, A\, e^{j(\omega t+\theta)} = y(t)$$

$$\longrightarrow \boxed{H(\omega)} \longrightarrow$$

• Example #5

- What if there are multiple frequencies?
- Let $f(t) = \cos\left(\overset{\omega_1}{\frac{1}{2}}t\right) + 3\cos(\overset{\omega_2}{2}t + \frac{\pi}{4}) + 2\sin(\overset{\omega_3}{3}t)$

be the input to the LTI system with input-output rule $\underbrace{y(t) = \frac{d}{dt}f(t)}$

$\downarrow$ phasors

$Y = j\omega F$

- Determine $y_{ss}(t)$

① $H(\omega)$, $|H(\omega)|$, $\measuredangle H(\omega)$

$H(\omega) = \frac{Y}{F} = j\omega$

$|H(\omega)| = |\omega|$

$$\measuredangle H(\omega) = \begin{cases} \pi/2 & \omega > 0 \\ -\pi/2 & \omega < 0 \\ 0 & \omega = 0 \end{cases}$$

- Example #5-cont

$$f(t) = \underbrace{1\cos\left(\frac{1}{2}t\right)}_{\omega_1} + \underbrace{3\cos\left(2t + \frac{\pi}{4}\right)}_{\omega_2} + \underbrace{2\sin(3t)}_{\omega_3}$$

$$y(t) = \frac{d}{dt}f(t)$$

$$|H(\omega)| = |\omega|$$

$$\measuredangle H(\omega) = \begin{cases} \pi/2 & \omega > 0 \\ -\pi/2 & \omega < 0 \\ 0 & \omega = 0 \end{cases}$$

- Determine $y_{ss}(t)$

$$y_{ss}(t) = 1 \cdot |H(\omega_1)| \cos\left(\frac{1}{2}t + \measuredangle H(\omega_1)\right) +$$

$$+ 3 \cdot |H(\omega_2)| \cos\left(2t + \frac{\pi}{4} + \measuredangle H(\omega_2)\right) +$$

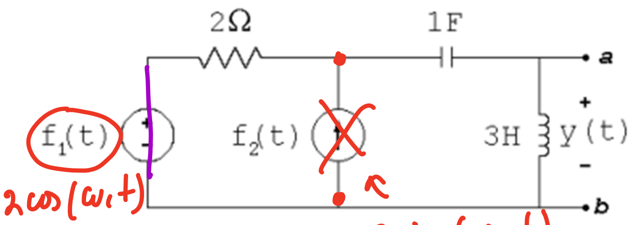
$$+ 2 \cdot |H(\omega_3)| \sin\left(3t + \measuredangle H(\omega_3)\right) =$$

$$= 1 \cdot \frac{1}{2}\cos\left(\frac{1}{2}t + \frac{\pi}{2}\right) + 3 \cdot 2\cos\left(2t + \frac{\pi}{4} + \frac{\pi}{2}\right) + 2 \cdot 3\sin\left(3t + \frac{\pi}{2}\right)$$

- Example #6

$\omega_1 = \frac{1}{3}$ rad/s $\omega_2 = 1$ rad/s

Consider the circuit below, where $f_1(t) = 2\cos\left(\frac{1}{3}t\right)\ V$ and $f_2(t) = 3\sin(t)\ A$.



look at effect of each source individually, and add in time domain!

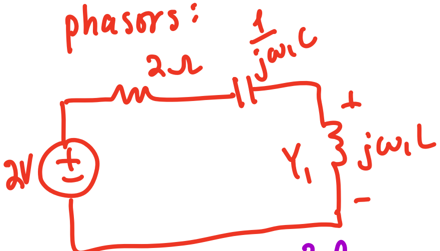
$y_{ss}(t) = y_{ss_1}(t) + y_{ss_2}(t)$

$3\sin(\omega_2 t) =$
$= 3\cos(\omega_2 t - \pi/2)$

- Determine $y_{ss}(t)$

Only $f_1(t)$:

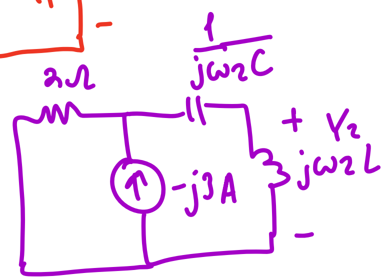
phasors:



$Y_1 = \frac{1}{\sqrt{2}} e^{j\frac{3\pi}{4}} \rightarrow y_{ss_1}(t) = \ldots$

Only f_2:

phasors:



$Y_2 = \frac{9}{\sqrt{2}} e^{-j\pi/4} \rightarrow y_{ss_2}(t) = \ldots$

$y_{ss}(t) = y_{ss_1}(t) + y_{ss_2}(t)$

- In general

- What if there are multiple frequencies?

$$\mathsf{f}(t) = \sum_n c_n \cos(\omega_n t + \theta_n) + \sum_m F_m e^{j\omega_m t} + \sum_k b_k \sin(\omega_k t + \psi_k)$$

$$f(t) \rightarrow \boxed{H(\omega)} \rightarrow y(t) = \sum_n |H(\omega_n)| c_n \cos(\omega_n t + \theta_n + \measuredangle H(\omega_n)) +$$

$$+ \sum_m H(\omega_m) F_m e^{j\omega_m t} + \sum_k |H(\omega_k)| b_k \sin(\omega_k t + \psi_k + \measuredangle H(\omega_k))$$

- Decibel amplitude response

$$|H(\omega)|_{dB} = 10\log_{10}(|H(\omega)|^2) = 20\log|H(\omega)|$$

- Small differences are emphasized

$$\text{if } |H(\omega)| = \begin{cases} \frac{1}{\sqrt{2}} & \text{then } |H(\omega)|_{dB} = \text{-3dB} \\ 1 & \text{then } |H(\omega)|_{dB} = \text{0dB} \\ \sqrt{2} & \text{then } |H(\omega)|_{dB} = \text{3dB} \\ 2 & \text{then } |H(\omega)|_{dB} = \text{6dB} \end{cases}$$

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