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05/02/2022 ECE 210 HW14

2. Given transfer functions $\hat{H}(s)$ determine inverse Laplace transforms $h(t)$ and BIBO stability

a) $\hat{H}(s) = \dfrac{s+3}{(s+2)(s-4)}$ → proper, rational, distinct poles (✓, ✓, ✓)

$s = -2, 4$

$\hookrightarrow \dfrac{A_1}{s+2} + \dfrac{A_2}{s-4}$

$A_1 = \hat{H}(s)(s+2)\Big|_{s=-2} = \dfrac{(s+3)\cancel{(s+2)}}{\cancel{(s+2)}(s-4)} = \dfrac{s+3}{s-4}\Big|_{s=-2} = \dfrac{-1}{6}$

$A_2 = \hat{H}(s)(s-4)\Big|_{s=4} = \dfrac{(s+3)\cancel{(s-4)}}{(s+2)\cancel{(s-4)}} = \dfrac{s+3}{s+2}\Big|_{s=4} = \dfrac{7}{6}$

$\hat{H}(s) = \dfrac{-1}{6(s+2)} + \dfrac{7}{6(s-4)}$

Table 7.2

$e^{pt}u(t) \Leftrightarrow \dfrac{1}{s-p}$

$$\boxed{h(t) = -\tfrac{1}{6}e^{-2t}u(t) - \tfrac{7}{6}e^{4t}u(t) \Leftrightarrow \mathcal{L}^{-1}}$$

$s = -2, 4$ → BIBO Unstable due to a pole not located on the left of the plane ($s=4$)

b) $\hat{H}(s) = \dfrac{2}{s(s-4)^2}$ → $s=0$, $s=4$ repeated $\Rightarrow \left[\dfrac{A_0}{s} + \dfrac{A_1}{(s-4)} + \dfrac{A_2}{(s-4)^2}\right]$

$A_n = \hat{f}(s)(s-p_1)^n\big|_{s=p_1}$, other terms: $A_{n-m} = \dfrac{1}{m!}\dfrac{d^m}{ds^m}\left[\hat{f}(s)(s-p_1)^n\right]\Big|_{s=p_1}$

$= \left[\dfrac{\frac{2}{(0-4)^2}}{s} + \dfrac{\frac{-1}{8}}{(s-4)} + \dfrac{\frac{2}{-4}}{(s-4)^2}\right]$ @ $s=0$, @ $s=4$

$A_1 = A_{2-1} = \dfrac{1}{1!}\dfrac{d}{ds}\hat{H}(s)(s-4)^2\Big|_{s=4}$

$= \dfrac{d}{ds}\left[\dfrac{2(s-4)^2}{s(s-4)^2}\right] = \dfrac{d}{ds}\left[\dfrac{2}{s}\right]$

$= \dfrac{s(0) - 2(1)}{s^2} = \dfrac{-2}{s^2}\Big|_{s=4}$

$= \dfrac{-2}{16} = -\dfrac{1}{8}$

$\hat{H}(s) = \dfrac{1}{8s} - \dfrac{1}{8(s-4)} - \dfrac{1}{2(s-4)^2}$

$$\boxed{h(t) = \tfrac{1}{8}u(t) - \tfrac{1}{8}e^{4t}u(t) - \tfrac{1}{2}te^{4t}u(t) \rightarrow s=0 \rightarrow \text{BIBO Unstable}}$$

(2)

2c) $\hat{H}(s) = \dfrac{s^2+4s+4}{(s+1)(s+2)} = \dfrac{(s+2)^2}{(s+1)(s+2)} = \dfrac{s^2+4s+4-s-2}{s^2+3s+2}$

$= 1 - \dfrac{s+2}{(s+1)(s+2)} \to \mathcal{L}^{-1} \to h(t) = \delta(t) - \mathcal{L}^{-1}\left\{\dfrac{s+2}{(s+1)(s+2)}\right\}$

$A_1 = \dfrac{(s+2)(s+1)}{(s+1)(s+2)}\Big|_{s=-1} = 1$, $A_2 = \dfrac{(s+2)(s+2)}{(s+1)(s+2)}\Big|_{s=-2} = 0$

$h(t) = \delta(t) - \mathcal{L}^{-1}\left\{\dfrac{1}{s+1}\right\} = \boxed{\delta(t) - e^{-t}u(t) = h(t)}$

$\boxed{s = -1, -2 \to \text{BIBO Stable}}$

2d) $\hat{H}(s) = \dfrac{s^3}{s^2+4} \to \begin{array}{r} s \\ s^2+4 \enclose{longdiv}{s^3+0} \\ -s^3+4s \\ \hline 0-4s \end{array} \to s - \dfrac{4s}{s^2+4} \to \mathcal{L}^{-1}$

$\hookrightarrow s = \pm 2j$

$= \mathcal{L}^{-1}\{s\} - \mathcal{L}^{-1}\left\{\frac{4s}{s^2+4}\right\} = \delta'(t) - \mathcal{L}^{-1}\left\{4 \cdot \dfrac{s}{s^2+(2)^2}\right\}$

$\uparrow \omega_0$

$\boxed{\begin{array}{l} \Rightarrow h(t) = \delta'(t) - 4\cos(2t)u(t) \\ \Rightarrow \text{BIBO Stable} \quad s = \pm 2j \end{array}}$

③

2e) $\hat{H}(s) = \frac{e^{-3s}}{(s+1)(s+2)} = e^{-3s} \cdot \frac{1}{(s+1)(s+2)} \rightarrow \frac{A_1}{s+1} + \frac{A_2}{s+2}$

$\hookrightarrow s = -1, -2$

$A_1 : \left. \frac{s+1}{(s+1)(s+2)} \right|_{s=-1} = \frac{1}{s+2} = \frac{1}{1} = 1$

$A_2 : \left. \frac{(s+2)}{(s+1)(s+2)} \right|_{s=-2} = \frac{1}{s+1} = \frac{1}{-1} = -1$

$\rightarrow \left(\frac{1}{s+1} - \frac{1}{s+2} \right) e^{-3s} \rightarrow$ time shift by 3

$$\boxed{\begin{array}{l} h(t) = \left(e^{-1(t-3)} - e^{-2(t-3)}\right) u(t) \rightarrow h(t) = \left(e^{3-t} - e^{6-2t}\right) u(t) \\ \hookrightarrow s = -1, -2 \Rightarrow \text{BIBO Stable} \end{array}}$$

2f) $\hat{H}(s) = \frac{s+1}{s^2+3s+2} = \frac{s+1}{(s+1)(s+2)} = \frac{A_1}{s+1} + \frac{A_2}{s+2}$, $s = -1, -2$

$A_1 = \left. \frac{(s+1)(s+1)}{(s+1)(s+2)} \right|_{s=-1} = \left. \frac{(s+1)}{(s+2)} \right|_{s=-1} = 0$

$A_2 = \left. \frac{(s+1)(s+2)}{(s+1)(s+2)} \right|_{s=-2} = 1$

$\left.\begin{array}{l} A_1 = 0 \\ A_2 = 1 \end{array}\right\} \frac{1}{(s+2)} \rightarrow \mathcal{L}^{-1}$

$$\boxed{\hookrightarrow h(t) = e^{-2t} u(t) \quad , s = -1, -2 \Rightarrow \text{BIBO Stable}}$$

Input to Laplace Domain. Multiply this conversion by the Transfer function to obtain $\hat{Y}(s)$. Expand with Partial fractions, then use Inverse Laplace Transform, referencing Table 7.5, producing the LTIC system zero-state response in Time Domain

3. $\hat{H}(s) = \dfrac{1}{s^2+9}$, $f(t) = \cos(2t)\,u(t)$ Table 7.5

$\cos(\omega_0 t)\,u(t) \Leftrightarrow \dfrac{s}{s^2+\omega_0^2}$

~~Table 7.11~~

~~$h(t) = \frac{1}{3}\sin(3t)\,u(t)$~~

$\hat{F}(s) = \dfrac{s}{s^2+4} \rightarrow \hat{Y}(s) = \dfrac{s}{(s^2+4)(s^2+9)} \rightarrow$ P.F.D.

$$= \frac{A_1 s + A_0}{s^2+4} + \frac{A_3 s + A_2}{s^2+9} \rightarrow \frac{s\,(s^2+4)(s^2+9)}{(s^2+4)(s^2+9)}$$

$$= \frac{(A_1 s + A_0)(s^2+4)(s^2+9)}{(s^2+4)} + \frac{(A_3 s)(s^2+4)(s^2+9)}{(s^2+9)}$$

$$\hookrightarrow s = (A_1 s + A_0)(s^2+9) + (A_3 s + A_2)(s^2+4)$$
$$= A_1 s^3 + 9A_1 s + A_0 s^2 + 9A_0 + A_3 s^3 + 4A_3 s + A_2 s^2 + 4A_2$$

$$s = s^3(a_1+a_3) + s^2(a_0+a_2) + s(9a_1+4a_3) + (9a_0+4a_2)$$

$9a_0 + 4a_2 = 0$, $9a_1 + 4a_3 = 1$, $a_0 + a_2 = 0$, $a_1 + a_3 = 0 \rightarrow a_0 = 0, a_2 = 0, a_1 = \frac{1}{5}, a_3 = -\frac{1}{5}$

$$\hookrightarrow \frac{(\frac{1}{5})s\ 0}{s^2+4} + \frac{(-\frac{1}{5})s + 0}{s^2+9} = \frac{s}{5(s^2+4)} - \frac{s}{5(s^2+9)} \rightarrow \mathcal{L}^{-1}$$

$$= \frac{1}{5}\left[\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\}\right] = \frac{1}{5}\left[\cos(2t) - \cos(3t)\right]u(t)$$

$$\boxed{\hookrightarrow y_{zs}(t) = \left[\frac{\cos(2t) - \cos(3t)}{5}\right]u(t)}$$

4. For each LTIC system, determine the transfer function ($\hat{H}(s)$) characteristic poles and modes, zero state and zero-input responses ($\hat{Y}_{zs}(s)$ and $\hat{Y}_{zi}(s)$), BIBO/asymptotic/marginal stability

a) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 8y(t) = 6f(t)$, $y(0^-) = 0$, $y'(0^-) = 1$, $f(t) = e^{-3t}u(t)$

$\mathcal{L}$

$\hookrightarrow s^2 + 2s - 8$ $\quad\hookrightarrow (s^2\hat{Y} - s^{2-1}y(0^-) - s^0 y'(0^-)) + 2(s\hat{Y} - y(0^-)) - 8\hat{Y} = 6\hat{F}$

? $\hookrightarrow \hat{Y}(s^2+2s-8) = 6\hat{F} + y(0^-)(s+2) + y'(0^-)$

$s^2+2s-8 = (s+4)(s-2)$ $\rightarrow$ characteristic poles at $p_1 = -4, p_2 = 2$
$\hookrightarrow s = -4, 2$ $\quad$ characteristic modes: e^{-4t}, e^{2t}

$\hookrightarrow \hat{Y}(s)_{zi} = \frac{1}{(s+4)(s-2)}$, $\hat{Y}(s)_{zs} = \frac{6}{(s+4)(s-2)(s+3)}$

$\hat{H}(s) = \frac{6}{(s+4)(s-2)}$ $\quad$ BIBO Unstable Asymptotically Stable and Marginally Stable ?

b) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y(t) = 2f(t)$, $y(0^-) = 1$, $y'(0^-) = 1$, $f(t) = \delta(t)$

$\hookrightarrow s^2 + 2s + 1 \rightarrow (s+1)^2$

$\hookrightarrow s = -1$ $\rightarrow$ characteristic poles at $s = -1$ repeated
characteristic modes: e^{-t}, te^{-t}

$\hat{H}(s) = \frac{-1}{(s+1)^2}$, $\hat{Y}(s)_{zi} = \frac{(s+3)}{(s+1)^2}$, $\hat{Y}(s)_{zs} = \frac{-2}{(s+1)^2}$

BIBO Stable $\quad$ Marginally Unstable ?

Asymptotically Stable

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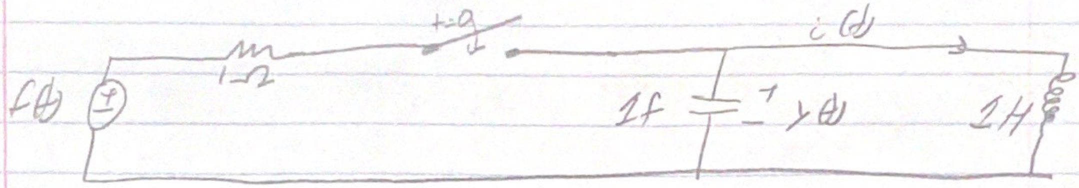
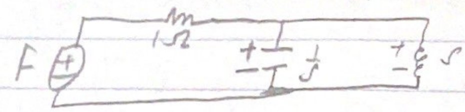
5. $f(t) = e^{2t}\,V$, $y(0^-) = 2V$, $i(0^-) = 0$



a) show for $t>0$ that $\hat{H}(s) = \frac{\hat{Y}(s)}{F(s)} = \frac{s}{s^2+s+1}$: Laplace Domain Circuit



$Z_L = s$

$Z_C = \frac{1}{s}$

Node-Voltage: $\frac{F-Y}{1} = \frac{Y}{(1/s)} + \frac{Y}{s}$, $F-Y = Ys + \frac{Y}{s}$, $F-Y = \frac{s^2Y+Y}{s}$

$sF - sY = s^2Y + Y$

$sF = s^2F + sY + Y \rightarrow s = \frac{Y}{F}(s^2+s+1) \rightarrow \boxed{\hat{H}(s) = \frac{\hat{Y}(s)}{F(s)} = \frac{s}{s^2+s+1}}$

b) Characteristic Poles/Modes: $s^2+s+1 \rightarrow s = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm 3j}{2}$

↳ characteristic poles @ $s = -\frac{1}{2} + \frac{3}{2}j,\ -\frac{1}{2} - \frac{3}{2}j$

↳ characteristic modes @ $e^{\left(\frac{-1+3j}{2}\right)t}$ and $e^{\left(\frac{-1-3j}{2}\right)t}$

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5c) determine $y_{zs}(t)$ for $t>0$

$f(t) = e^{3t} \rightarrow \hat{F}(s) = \frac{1}{s-3} \rightarrow \hat{Y}_{zs}(s) = \hat{F}(s)\hat{H}(s) = \left(\frac{1}{s-3}\right)\left(\frac{s}{s^2+s+1}\right)$

$= \frac{s}{(s-3)(s^2+s+1)}$

$\hookrightarrow \frac{A}{s-3} + \frac{Bs+C}{s^2+s+1} = \frac{s}{(s-3)(s^2+s+1)}$

$\hookrightarrow A(s^2+s+1) + (Bs+C)(s-3) = s \rightarrow s^2(A+B) + s(A-3B+C) + A-3C = s,\ A+B=0$

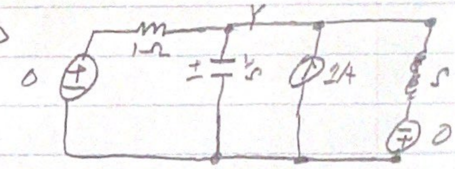
$A-3B+C=1,\ A-3C=0 \rightarrow A = 3/13,\ B = -3/13,\ C = 1/13$

$\hookrightarrow \hat{Y} = \frac{3/13}{s-3} + \frac{1/13 - 3s/13}{s^2+s+1} \rightarrow$ complete the square $\rightarrow \hat{Y}_{zs}(s) = \frac{3/13}{s-3} + \frac{1/13 - 3s/13}{(s+1/2)^2 + 3/4}$

$= \frac{3/13}{s-3} + \left(\frac{5}{26}\left(\frac{1}{(s+1/2)^2 + 3/4}\right) + \left(\frac{s+1/2}{(s+1/2)^2}\right)\right) \rightarrow y_{zs}(t) = \mathcal{L}^{-1}\{\hat{Y}_{zs}(s)\}$

$$\boxed{\hookrightarrow y_{zs}(t) = \frac{3}{13} e^{3t} u(t) + \left(\frac{5e^{-t/2}}{13\sqrt{3}} \cdot \sin\left(\frac{\sqrt{3}}{2} t\right) - \frac{3e^{-t/2}}{13} \cdot \cos\left(\frac{\sqrt{3}}{2} t\right)\right)(u(t)) \text{ Volts}}$$

5d) determine $y_{zI}(t)$ for $t>0$ $\rightarrow$ Laplace Domain $\rightarrow$



Node-Voltage @ Y: $1 + \frac{0-Y}{1} = sY + \frac{Y}{s}$

$\hookrightarrow 1 - Y = sY + \frac{Y}{s},\ 1 = sY + \frac{Y}{s} + Y$

$\hookrightarrow 1 = \left(\frac{s^2+s+1}{s}\right)Y \rightarrow \hat{Y}_{zI}(s) = \frac{s}{s^2+s+1} \rightarrow y_{zs}(t) = \mathcal{L}^{-1}\{\hat{Y}_{zI}(s)\}$

$= \mathcal{L}^{-1}\left\{\frac{s}{s^2+s+1}\right\}$

complete the square

$\hookrightarrow \frac{s}{s^2+s+1} = \frac{s+1/2}{(s+1/2)^2 + 3/4} - \frac{1/2}{(s+1/2)^2 + 3/4}$

$\hookrightarrow \frac{s+1/2}{(s+1/2)^2 + 3/4} - \frac{1/2}{(s+1/2)^2 + 3/4}$

$\mathcal{L}^{-1} = \left(e^{-t/2} \cdot \cos\left(\frac{\sqrt{3}}{2} t\right) - e^{-t/2} \cdot \frac{1}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2} t\right)\right)(u(t))$

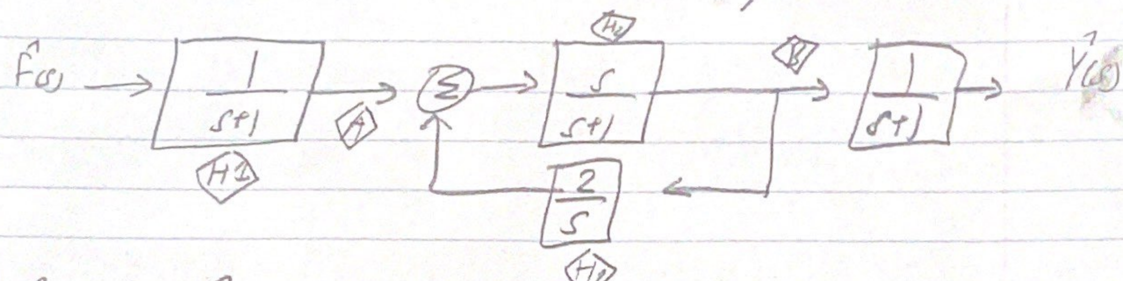
$$\boxed{\hookrightarrow y_{zI}(t) = e^{-t} \cdot \cos\left(\frac{\sqrt{3}}{2} t\right) u(t) - e^{-t} \cdot \frac{1}{\sqrt{3}} \cdot \sin\left(\frac{\sqrt{3}}{2} t\right) u(t) \text{ Volts}}$$

5e) $y(t)$ for $t>0$: $y(t) = y_{zs}(t) + y_{zi}(t)$

$$\boxed{y(t) = \frac{3}{13}e^{3t}u(t) + \frac{10}{13}e^{-t/2}u(t)\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{8}{13\sqrt{3}}e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)} \text{ Volts}$$

5f) BIBO Stable, Asymptotically Stable, Marginally Unstable

6. Determine the transfer function and BIBO stability



$\hat{Y}(s) = \hat{F}(s) \cdot \hat{H}(s) \rightarrow$ Take intermediate outputs A and B

$\hookrightarrow B = (A + BH_3) * H_2$, $B = AH_2 + BH_2H_3$, $B(1 - H_2H_3) = AH_2$

$$\frac{B}{A} = \frac{H_2}{1 - H_2H_3} = \left(\frac{s}{s+1}\right) \div \left(1 - \frac{2}{s}\left(\frac{s}{s+1}\right)\right) = \frac{\frac{s}{s+1}}{1 - \frac{2}{s+1}}$$

$\hookrightarrow B/A = \frac{s}{s-1} \rightarrow$ for the system

$$\hookrightarrow \hat{H}(s) = \left(\frac{1}{s+1}\right)\left(\frac{s}{s-1}\right)\left(\frac{1}{s+1}\right) \rightarrow \boxed{\hat{H}(s) = \frac{s}{(s+1)^2(s-1)}}$$

Poles at $s = -1$, $s = 1$
BIBO Unstable