Problem 1

a) $Z_T = 2-2j \qquad Z_L = Z_T^* = 2+2j \qquad P = \dfrac{|V_T|^2}{8R_T} \qquad V_T = 6+j$

$$P = \frac{(6^2+1^2)}{8\cdot 2} = \frac{37}{16}\ W$$

b) $y(t) = |H(2)| \cos\left(2t + \frac{\pi}{6} + \angle H(2)\right) = 2\cos\left(2t + \frac{\pi}{3}\right)$

c) $H(2) = \sqrt{2}\ \angle \pi/4 \qquad y(t) = 2\cos\left(2t + \frac{\pi}{2}\right)$

d) $F_3 = j \quad (w_0 = \frac{1}{6}\ rad/sec)$

Problem 4

a) $w_0 = 4$ rad/sec

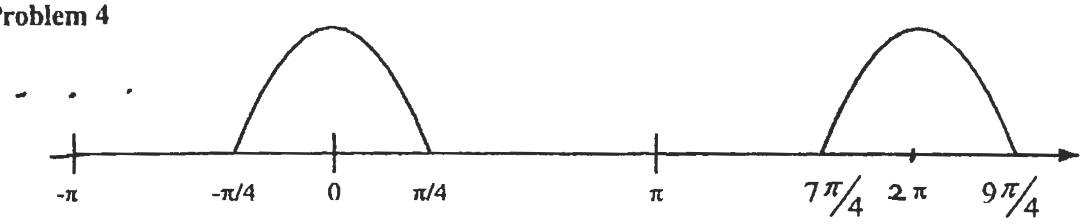
$$F_n = \frac{2-j8n}{\pi(1-16n^2)}$$

b) $F_n = \dfrac{1}{\frac{\pi}{2}} \displaystyle\int_0^{\pi/2} \sin t\, e^{-jn4t}\, dt = \frac{2}{\pi}\int_0^{\pi/2} \frac{e^{jt}+e^{-jt}}{2} \cdot e^{-jn4t}\, dt$

c) $w_0 = 2$ rad/sec only F_0 F_1 F_{-1} will go through.

$$F_0 + \sum_{n=1}^{\infty} 2|F_n| \cos(2nt+\theta_n) \to H(w) \to y(t) = 0.504 + 2\cdot\frac{0.504}{\sqrt{1+16}} \cos\left(2t - \tan^{-1}4 - \frac{1}{3}\right)$$

Problem 4



The periodic signal shown above is given by $f(t) = \begin{cases} \cos(2t), & -\pi/4 < t < \pi/4 \\ 0, & \pi/4 < t < 7\pi/4 \end{cases}$

(a) What is the period T and the fundamental frequency ω_o ?

(+4) $T =$ 2π, $\omega_o =$ 1

(b) Is the complex Fourier coefficient F_n
(circle the correct answer)

(+4) totally real, totally imaginary, or both parts non-zero?

(c) Write the integral equation for F_n (leave in integral form):

(+4) $F_n = \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} \cos(2t)\, e^{-jnt}\, dt = \frac{1}{\pi}\int_0^{\pi/4} \cos(2t)\cos(nt)\,dt$

(d) Compute F_0:

(+4) $F_0 = \frac{1}{\pi}\int_0^{\pi/4} \cos(2t)\,dt = \frac{1}{\pi}\, \frac{\sin(2t)}{2}\Big]_0^{\pi/4} = \frac{1}{2\pi}\{\sin(\pi/2) - \sin(0)\} = \frac{1}{2\pi}\{1-0\}$

$F_0 = \frac{1}{2\pi}$

(e) Compute F_1 :

$F_1 = \frac{1}{\pi}\int_0^{\pi/4} \cos(2t)\cos(t)\,dt = \frac{1}{2\pi}\int_0^{\pi/4} [\cos(t) + \cos(3t)]\,dt = \frac{1}{2\pi}\left[\sin(t) + \frac{1}{3}\sin(3t)\right]_0^{\pi/4}$

$= \frac{1}{2\pi}\left[\sin(\pi/4) + \frac{1}{3}\sin(3\pi/4) - \sin(0) - \frac{1}{3}\sin(0)\right] = \frac{1}{2\pi}\left(\frac{1}{\sqrt{2}} + \frac{1}{3}\frac{1}{\sqrt{2}}\right) = \frac{4}{3\sqrt{2}}\cdot\frac{1}{2\pi} = \frac{\sqrt{2}}{3\pi}$

(+4) Alternative solution:

$F_1 = \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} \cos(2t)\, e^{-jt}\,dt = \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} \frac{(e^{j2t} + e^{-j2t})}{2}\, e^{-jt}\,dt = \frac{1}{4\pi}\int_{-\pi/4}^{\pi/4} (e^{jt} + e^{-j3t})\,dt$

$= \frac{1}{4\pi}\left(\frac{e^{jt}}{j} + \frac{e^{-j3t}}{-3j}\right)\Big]_{-\pi/4}^{\pi/4} = \frac{1}{2\pi}\left\{\sin(\pi/4) + \frac{1}{3}\sin\left(\frac{3\pi}{4}\right)\right\}$

$= \frac{1}{2\pi}\left\{\frac{1}{\sqrt{2}} + \frac{1}{3}\frac{1}{\sqrt{2}}\right\} = \frac{1}{2\pi}\,\frac{4}{3\sqrt{2}} = \frac{\sqrt{2}}{3\pi}$

$F_1 = \frac{\sqrt{2}}{3\pi}$

(f) Given input Fourier coefficient $F_n = 1/n$ for all n $(n>0)$ system response $H(\omega) = \frac{1}{1+j\omega}$, and fundamental frequency $\omega_o = 2$, write the equation for the output Fourier coefficient Y_n in **polar form**.

(+5) $Y_n = \frac{1}{n}\cdot H(2n) = \frac{1}{n}\,\frac{1}{(1+j2n)} = \frac{1}{n\sqrt{1+4n^2}}\, e^{-j\,\mathrm{atan}(2n)}$

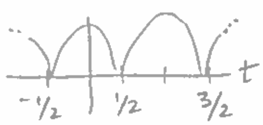
4. (25 pts) The two parts in this problem are unreleated.

(a) Consider the following periodic function

$$f(t) = cos^2(\pi t) + sin(2\pi t + \frac{\pi}{2})$$

i. What is the fundamental frequency ω_0 and period T of $f(t)$?

$\cos^2(\pi t)$



$\underbrace{\quad}_{1 = T = \frac{2\pi}{\omega}}$

$\omega = 2\pi.$

$\sin(2\pi t + \pi/2)$

$\uparrow \; w = \frac{2\pi}{T}$

So $T = 1$ sec

* both terms have the same frequency, so it is the fundamental freq. of $f(t)$.

$\omega_0 =$ 2π [rad/s]

$T =$ 1 [sec]

ii. $f(t)$ can be expressed as an exponential Fourier series, where $f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$, what is F_n for $n = 0, 1, -1, 2, -2$?

Express each term in exponential form via Eulers identity:

$F_0 =$ 1/2

$F_1 =$ 3/4

$F_{-1} =$ 3/4

$F_2 =$ 0

$F_{-2} =$ 0

$$\sin(2\pi t + \pi/2) = \frac{1}{2j}\left[e^{j(2\pi t + \pi/2)} - e^{-j(2\pi t + \pi/2)}\right]$$

$$= \frac{1}{2j}\left[j e^{j2\pi t} - (-j) e^{-j2\pi t}\right]$$

$$= \frac{1}{2} e^{j2\pi t} + \frac{1}{2} e^{-j2\pi t} = \cos(2\pi t)$$

$$\cos^2(\pi t) = \left[\frac{1}{2}\left(e^{j\pi t} + e^{-j\pi t}\right)\right]^2 = \left[\frac{1}{4}\left(e^{j2\pi t} + 2e^{j\pi t}e^{-j\pi t} + e^{-j2\pi t}\right)\right]$$

$$= \frac{1}{2} + \frac{1}{4}e^{j2\pi t} + \frac{1}{4}e^{-j2\pi t} = \frac{1}{2} + \frac{1}{2}\cos(2\pi t)$$

so $f(t) = \underset{\uparrow F_0}{\frac{1}{2}} + \underset{\uparrow F_1}{\frac{3}{4}} e^{\underset{n=1}{j2\pi t}} + \underset{\uparrow F_{-1}}{\frac{3}{4}} e^{\underset{n=-1}{-j2\pi t}} = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$ where $F_{|n| \geq 2} = 0$

NOTE! $f(t)$ is real so $F_{-n} = F_n^*$

NOTE! if integrate, use orthogonal property: $\frac{1}{T}\int_T e^{jkt} e^{-jmt} = \begin{cases} 1 & k=m \\ 0 & k \neq m \end{cases}$

$$F_n = \frac{1}{1}\int_0^1 f(t) e^{-jn2\pi t} dt = \underbrace{\int_0^1 \tfrac{1}{2} e^{j0} dt}_{= 1/2 \; (n=0)} + \underbrace{\int_0^1 \tfrac{3}{4} e^{j2\pi t} e^{-jn2\pi t} dt}_{= 3/4 \; (\text{if } n=1)} + \underbrace{\int_0^1 \tfrac{3}{4} e^{-j2\pi t} e^{-jn2\pi t} dt}_{= 3/4 \; (\text{if } n=-1)}$$

(b) The periodic function $f(t) = |sin(t)|$ can be expressed in Fourier series form as

$$f(t) = |sin(t)| = \sum_{n=-\infty}^{\infty} \underbrace{\boxed{\frac{2}{\pi}\frac{1}{1-4n^2}}}_{=F_n} e^{jn2t}$$

ω_0

Let $f(t)$ be input to an LTI system having frequency response

$$H(\omega) = \frac{j\omega}{1+j\omega}$$

Then the output $y(t)$ is also periodic and can be expressed in Fourier Series form as

$$y(t) = \sum_{n=-\infty}^{\infty} Y_n e^{jn2t}$$

i. Write the expression for the $n = 2$ coefficient, Y_2. You do not have to simplify your answer.

$Y_2 = \left(\frac{-2}{15\pi}\right)\left(\frac{4j}{1+4j}\right)$

$Y_n = F_n H(n\omega_0)$ $\quad F_n = \frac{2}{\pi}\frac{1}{1-4n^2}$

$Y_2 = F_2 H(2\omega_0)$ $\quad \omega_0 = 2$

$= \left(\frac{2}{\pi}\frac{1}{1-4(2)^2}\right)\left(\frac{j(2\omega_0)}{1+j(2\omega_0)}\right) = \frac{2}{-15\pi}\cdot\frac{4j}{1+4j}$

ii. Determine whether the following statements are true or false. Briefly justify your answer.

TRUE / FALSE: The $\underbrace{\text{DC component of the output}}_{=Y_0}$ from this system is zero $\underbrace{\text{regardless of the input.}}$

$Y_0 = F_0 H(0\cdot\omega_0) = F_0 H(0) = 0$ when $\begin{cases} F_0 = 0 \text{ ... irrelevant here} \\ H(0) = 0 \text{ . here: } H(0) = \frac{j0}{1+j0} = \frac{0}{1} \end{cases}$

TRUE / FALSE: This system acts as a $\underbrace{\text{band pass filter}}_{\text{has } H(0)=0 \text{ AND } H(\infty)=0}$.

this system: $|H(\omega)| = \frac{\omega}{1+\omega^2}$ so $|H(\infty)| = \frac{\infty}{1+\infty^2} = 1$ so it's a HIGH PASS, not a band pass.

TRUE / FALSE: $\underbrace{y(t) \text{ is real-valued}}$ when $f(t)$ is real valued..

so $Y_{n-} \overset{?}{=} Y_n^*$ $\quad Y_{n-} = F_{n-} H(-n\omega_0)$ since $F_{n-} = F_n^*$ ($f(t)$ real)

$= F_n^* H(n\omega_0)^* = Y_n^*$ and since $H(-\omega) = H(\omega)^*$ (see below)

TRUE / FALSE: $H(-\omega) = H(\omega)^*$ ONLY when f(t) is real-valued.

for this system: $H(-\omega) = \frac{j(-\omega)}{1+j(-\omega)} = \frac{(-j)\omega}{1+(-j)\omega} = H(\omega)^*$

so, the Hermitian property holds regardless of the input