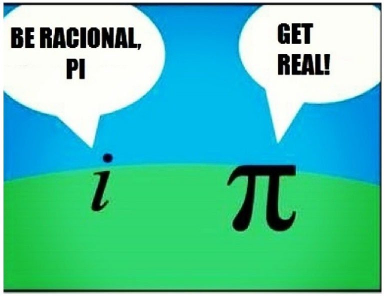
ECE 210 (AL2) - ECE 211 (E)

Appendix A

Complex numbers



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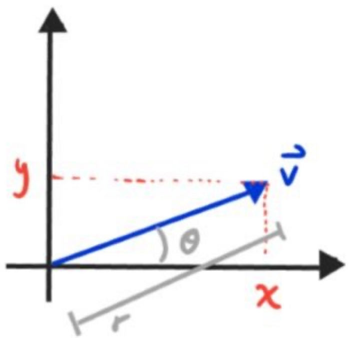
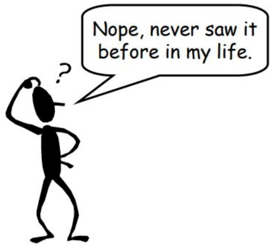
• Recall: 2-D vector $\vec{v}$





(rectangular)

• In Cartesian coordinates: $\vec{v}$ = (x,y)

• In polar coordinates: $\vec{v}$ = (r,θ)

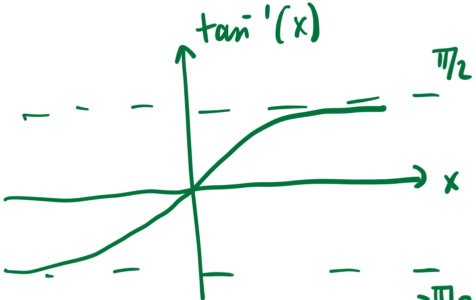
polar → rectangular

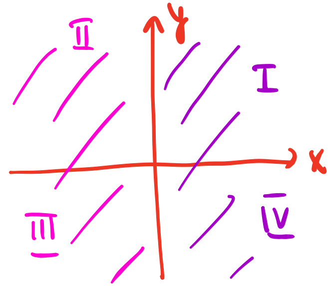
$x = r\cos\theta$

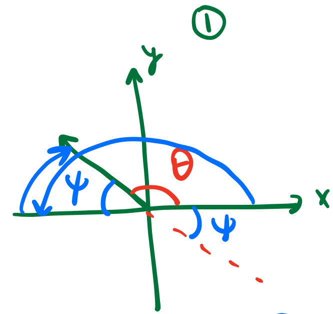
$y = r\sin\theta$

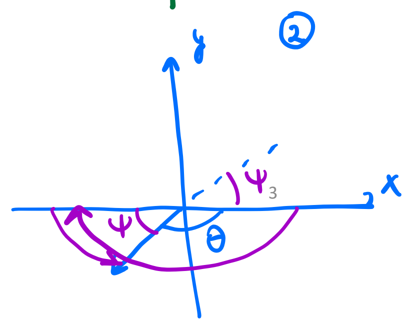
$$r = \sqrt{x^2 + y^2}$$

$$\theta = \begin{cases} \tan^{-1}\left(\frac{y}{x}\right) & x > 0 \\ \pi + \tan^{-1}\left(\frac{y}{x}\right) & x<0,\ y>0 \quad ① \\ -\pi + \tan^{-1}\left(\frac{y}{x}\right) & x<0,\ y<0 \quad ② \end{cases}$$

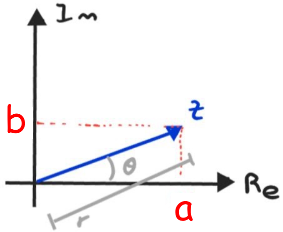








- Now, a complex number Z



- In Cartesian (rectangular form): $\underbrace{Z = a + jb}$ = (a,b)

  j = $\sqrt{-1}$ - imaginary unit ; $j^2 = -1$

  a = Re$\{Z\} \in \underline{\mathbb{R}}$ $\qquad \frac{1}{j} = -j$

  b = Im$\{Z\} \in \underline{\mathbb{R}}$

- In exponential form: Z = r$e^{j\theta}$

  r = $|Z|$

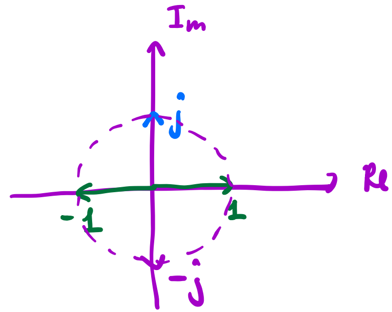
  θ = ∡ Z

- Special cases

$$1\,e^{j0} = 1 \qquad 0 = \theta$$

$$1\,e^{\pm j\pi} = -1 \qquad e^{j\pi} = -1, \quad e^{-j\pi} = -1$$

$$1\,e^{\pm j\pi/2} = \pm j$$



- Euler's identity: $\overset{-}{e}^{j\theta} = \cos\theta \overset{-}{+} j\sin\theta$

$$\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}$$

- Consider two complex numbers:

$Z_1 = a_1 + jb_1 = r_1 e^{j\theta_1}$

$Z_2 = a_2 + jb_2 = r_2 e^{j\theta_2}$

- Addition/subtraction:

$$\mathbf{Z_1 \pm Z_2 = (a_1 + jb_1) \pm (a_2 + jb_2) = (a_1 \pm a_2) + j(b_1 \pm b_2)}$$

- Multiplication:

$$\mathbf{Z_1 Z_2 = (r_1 e^{j\theta_1})(r_2 e^{j\theta_2}) = (r_1 r_2)\, e^{j(\theta_1+\theta_2)}}$$

$x^2 \cdot x^3 = x^{2+3}$

- Division:

$$\mathbf{\frac{Z_1}{Z_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \left(\frac{r_1}{r_2}\right) e^{j(\theta_1-\theta_2)}}$$

$\frac{x^2}{x^3} = x^{2-3}$

- Powers:

$$\mathbf{Z_1^n = (r_1 e^{j\theta_1})^n = r_1^n e^{jn\theta_1}}$$

- Complex conjugate: $\mathbf{Z_1^* = (r_1 e^{j\theta_1})^* = r_1 e^{-j\theta_1} = (a_1 + jb_1)^* = a_1 - jb_1}$

$j \to -j$

• Example: Let $Z_1 = -\sqrt{3} + j$ and $Z_2 = \sqrt{2}e^{j\frac{\pi}{4}}$. Determine

↓ convert into rect.

$e^{j\theta} = \cos\theta + j\sin\theta$

• $Z_3 = Z_1 + Z_2 =$

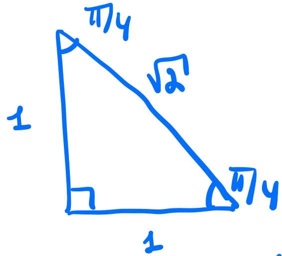
$= -\sqrt{3} + j + 1 + j = (1-\sqrt{3}) + j2$

$= \sqrt{2}\left(\cos(\pi/4) + j\sin(\pi/4)\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right) =$

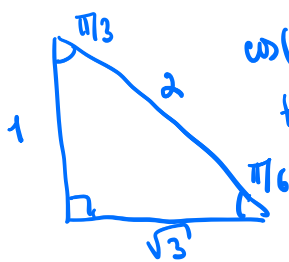
$= 1 + j$

$\cos = \frac{\text{adj}}{\text{hyp}}$, $\sin = \frac{\text{oppos}}{\text{hyp}}$,

$\tan = \frac{\text{oppos}}{\text{adj}}$.



$\cos(\pi/4) = \frac{1}{\sqrt{2}}$



$\cos(\pi/3) = \frac{1}{2}$

$\tan(\pi/6) = \frac{1}{\sqrt{3}}$

$z_2 = \sqrt{2}\, e^{j\pi/4}$

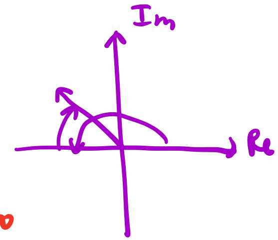
$z_1 = -\sqrt{3} + j = 2e^{j\frac{5\pi}{6}}$

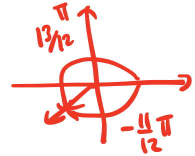
- $Z_4 = Z_1 Z_2 = \sqrt{2} \cdot e^{j\pi/4} \cdot 2e^{j\frac{5\pi}{6}} = 2\sqrt{2}\, e^{j(\pi/4 + \frac{5\pi}{6})} = 2\sqrt{2}\, e^{j\frac{13\pi}{12}} = 2\sqrt{2}\, e^{-j\frac{11\pi}{12}}$

  we want to keep angle between $-\pi$ and π

  exp: $\sqrt{a^2+b^2} = \sqrt{1+3} = 2$

  $\theta = \pi + \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = \pi - \pi/6 = \frac{5\pi}{6}$





- $Z_5 = \frac{Z_1}{Z_2} = \frac{2e^{j\frac{5\pi}{6}}}{\sqrt{2}\, e^{j\pi/4}} = \frac{2}{\sqrt{2}} e^{j(\frac{5\pi}{6} - \pi/4)} = \sqrt{2}\, e^{j\frac{7\pi}{12}}$

- $Z_7 = Z_1^* = -\sqrt{3} - j = 2e^{-j\frac{5\pi}{6}}$