Lecture 49, Tuesday, April 26, 2022

- Inverse Laplace transform of improper rational functions

  - Case 1 ($k \geq 0$):

$$\hat{F}(s) \;=\; \frac{s^{k+n}}{s^n + a_1 s^{n-1} + \cdots + a_n} \;=\; s^{k+n} \underbrace{\left(\frac{1}{s^n + a_1 s^{n-1} + \cdots + a_n}\right)}_{=\,\hat{G}(s)} \;=\; s^{k+n}\hat{G}(s) \;=\; \overbrace{s^{k+n}}^{\xrightarrow{\mathcal{L}^{-1}}\ \frac{d^{k+n}}{t^{k+n}}} \;\underbrace{\hat{G}(s)}_{\xrightarrow{\mathcal{L}^{-1}}\ g(t)}$$

$$\Rightarrow \;\; f(t) \;=\; \tfrac{d^{k+n} g(t)}{t^{k+n}}$$

  - Case 2 (degree of $P(s) < n$):

$$\hat{F}(s) \;=\; \frac{s^n}{s^n + P(s)} \;=\; \frac{s^n + P(s) - P(s)}{s^n + P(s)} \;=\; 1 + \underbrace{\left(\frac{-P(s)}{s^n + P(s)}\right)}_{=\,\hat{G}(s)} \;=\; \overbrace{1}^{\xrightarrow{\mathcal{L}^{-1}}\ \delta(t)} \;+\; \underbrace{\hat{G}(s)}_{\xrightarrow{\mathcal{L}^{-1}}\ g(t)}$$

$$\Rightarrow \;\; f(t) \;=\; \delta(t) + g(t)$$

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– Case 3 (degree $P(s), Q(s) < n$):

$$\hat{F}(s) \;=\; \frac{s^n + Q(s)}{s^n + P(s)} \;=\; \frac{s^n + P(s) - P(s) + Q(s)}{s^n + P(s)} \;=\; 1 + \underbrace{\left(\frac{-P(s) + Q(s)}{s^n + P(s)}\right)}_{=\,\hat{G}(s)} \;=\; \overbrace{1}^{\xrightarrow{\mathcal{L}^{-1}}\ \delta(t)} + \underbrace{\hat{G}(s)}_{\xrightarrow{\mathcal{L}^{-1}}\ g(t)}$$

$$\Rightarrow\; f(t) \;=\; \delta(t) + g(t)$$

– Case 4 ($a > 0$):

$$\hat{F}(s) \;=\; e^{-as}\frac{N(s)}{D(s)} \;=\; e^{-as}\underbrace{\left(\frac{N(s)}{D(s)}\right)}_{=\,\hat{G}(s)} \;=\; \overbrace{e^{-as}}^{\xrightarrow{\mathcal{L}^{-1}}\ \text{time shift}} \quad \underbrace{\hat{G}(s)}_{\xrightarrow{\mathcal{L}^{-1}}\ g(t)}$$

$$\Rightarrow\; f(t) \;=\; g(t-a)$$

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- The transfer functions of lumped element LTIC systems are rational:

$$\hat{H}(s) \;=\; \frac{\hat{Y}_{zs}(s)}{\hat{F}(s)} \;=\; \frac{\hat{N}(s)}{\hat{D}(s)} \;=\; \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

$$\Rightarrow\; \hat{Y}_{zs}(s)\left(s^n + a_1 s^{n-1} + \cdots + a_n\right) \;=\; \hat{F}(s)\left(b_0 s^m + b_1 s^{m-1} + \cdots + b_m\right)$$

$$\xrightarrow{\mathcal{L}^{-1}} \quad \frac{d^n}{dt^n} y_{zs} + a_1 \frac{d^{n-1}}{dt^{n-1}} y_{zs} + \cdots + a_n y_{zs} \;=\; b_0 \frac{d^m}{dt^m} f + b_1 \frac{d^{m-1}}{dt^{m-1}} f + \cdots + b_m f$$

- The full solution satisfies the ODE:

$$\frac{d^n}{dt^n} y + a_1 \frac{d^{n-1}}{dt^{n-1}} y + \cdots + a_n y \;=\; b_0 \frac{d^m}{dt^m} f + b_1 \frac{d^{m-1}}{dt^{m-1}} f + \cdots + b_m f$$

$$\begin{aligned} \xrightarrow{\mathcal{L}} \; \hat{Y}(s)\left(s^n + a_1 s^{n-1} + \cdots + a_n\right) \;-\; y\left(0^-\right)\left(s^{n-1} + a_1 s^{n-2} + \cdots + a_{n-1}\right) \\ -\; y'\left(0^-\right)\left(s^{n-2} + a_1 s^{n-3} + \cdots + a_{n-2}\right) \;-\; \ldots \;-\; y^{(n-1)}\left(0^-\right) \\ =\; \hat{F}(s)\left(b_0 s^m + b_1 s^{m-1} + \cdots + b_m\right) \end{aligned}$$

$$\Rightarrow \quad \hat{Y}(s) \;=\; \frac{\hat{F}(s)\left(b_0 s^m + \cdots + b_m\right) \;+\; y\left(0^-\right)\left(s^{n-1} + \cdots + a_{n-1}\right) \;+\; \ldots \;+\; y^{(n-1)}\left(0^-\right)}{\left(s^n + a_1 s^{n-1} + \cdots + a_n\right)}$$