

Analog Signal Processing

Thursday, October 21, 8:45-10pm

Exam II

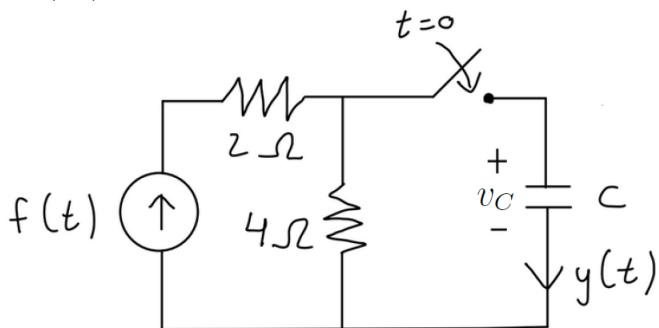
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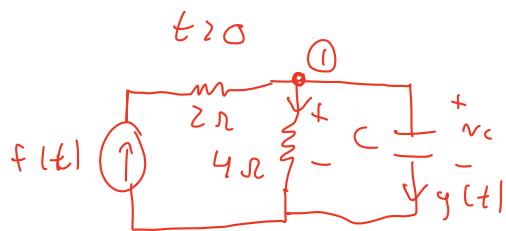
<p>Clearly PRINT your name in CAPITAL LETTERS.</p> <p>This is a closed book and closed notes exam.</p> <p>Calculators are not allowed.</p> <p>To get credit, please SHOW all your work and simplify your answers.</p> <p>Write your final answers in the spaces provided.</p> <p>All answers should INCLUDE UNITS whenever appropriate.</p> <p>The exam is printed double-sided.</p>	<p>DO NOT write in these spaces.</p> <p>Problem 1 (25 points):_____</p> <p>Problem 2 (25 points):_____</p> <p>Problem 3 (25 points):_____</p> <p>Problem 4 (25 points):_____</p> <p>Total: (100 points):_____</p>
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1. (25 pts) Consider the LTI circuit below with current $f(t) = \cos(\omega t) + 3 \sin(\omega t)$ A and voltage $v_C(0^-) = v_0$ V.



It is known that the current $y(t) = 2 \cos(2t) + \sin(2t) + Be^{-2t}$ A for $t > 0$.

- (a) [5 pts] Write the ODE that governs this LTI system for $t > 0$ in terms of $f(t)$, C , $v_c(t)$ and ω .



$\text{KCL } @ \textcircled{1} :$

$$f(t) = \frac{v_c}{4} + \dot{v}_c$$

$$f(t) = \frac{v}{4} + C \frac{dv}{dt}$$

$$\frac{1}{C} f(t) = \frac{dv}{dt} + \frac{1}{4C} v$$

$$\boxed{\frac{1}{C} f(t) = \frac{dv}{dt} + \frac{1}{4C} v}$$

- (b) [3 pts] Determine the value of $C = \boxed{\frac{1}{8}}$

The exponential term in $y(t)$ is the homogeneous solution: $Ae^{-t/2} = Be^{-2t} \rightarrow T = \frac{1}{2} = R_f C = 4C$

$$\frac{1}{2R_f} = \frac{1}{4C} \Rightarrow R_f = 4C$$

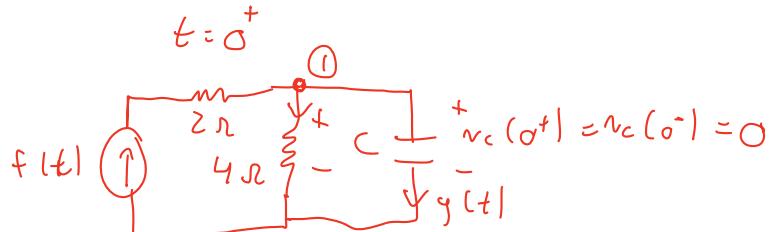
$$\rightarrow C = \frac{1}{2(4)} = \frac{1}{8}$$

- (c) [3 pts] Determine the value of $\omega = \boxed{2 \text{ rad/s}}$

Frequency of $f(t)$ & $y(t)$ must be the same

$$\rightarrow \omega = 2 \text{ rad/s}$$

(d) [4 pts] If $v_0 = 0V$, determine the value of $B = \boxed{-1}$ Amperes



$$v_{RN} = v_c = 0 \rightarrow i_{RN} = 0 \rightarrow i_c(0^+) + i_{RN}(0^+) = f(0^+)$$

$$y(t) = 2\cos(\omega t) + 3\sin(\omega t) + Be^{-2t} \quad i_c(0^+) = \cos(\omega) + 3\sin(\omega) \\ \approx 1$$

$$\rightarrow y(0^+) = 2\cos(0) + 3\sin(0) + B = 2+B$$

!!

$$i_c(0^+) = 1 \rightarrow 1 = 2+B \rightarrow B = -1$$

(e) [4 pts] Determine steady-state current phasor $Y = \boxed{2-j \text{ A}}$

$$y(t) = \underbrace{2\cos(\omega t) + 3\sin(\omega t)}_{\text{Steady-state}} + \underbrace{Be^{-2t}}_{\text{transient}}$$

[phasor]

$$y_{ss}(t) = 2\cos(\omega t) + \cos(\omega t + \frac{\pi}{2})$$

$$y = 2 + e^{-j\frac{\pi}{2}} = 2-j$$

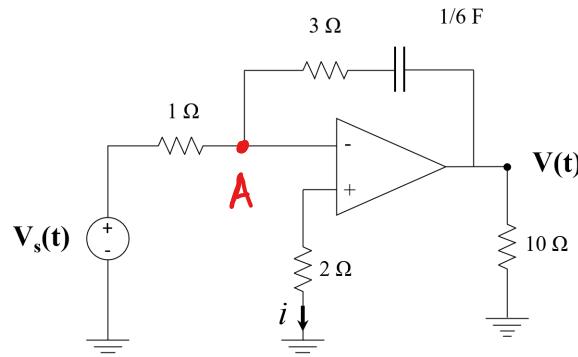
(f) [5 pts] If instead of $f(t) = \cos(\omega t) + 3\sin(\omega t) \text{ A}$, the input current source in this LTI system is given by $f_1(t) = 2\cos(\omega t - 1) + 6\sin(\omega t - 1) \text{ A}$, determine the steady-state response, $y_1(t)$.

$$\text{Notice } f_1(t) = 2 f\left(t - \frac{1}{\omega}\right)$$

$$\text{LT I} \rightarrow y_1(t) = 2 y_{ss}\left(t - \frac{1}{\omega}\right) \\ = 4\cos\left(2\left(t - \frac{1}{\omega}\right)\right) + 2\sin\left(2\left(t - \frac{1}{\omega}\right)\right) \\ = 4\cos\left(2t - \frac{2}{\omega}\right) + 2\sin\left(2t - \frac{2}{\omega}\right)$$

$$y_1(t) = \boxed{4\cos(2t-1) + 2\sin(2t-1) \text{ Amperes}}$$

2. (25 pts) The output voltage in the circuit below is given by $V(t) = 4\cos(2t)$. Using the phasor method and ideal op-amp approximations determine:



$$\omega = 2$$

$$Z_c = -\frac{j}{2 \cdot \frac{1}{6}} = -j3$$

$$V = 4$$

- (a) [18 pts] The input voltage $V_s(t)$ and express it in terms of real-valued functions only.

$$i_+ = i_- = 0;$$

$V_+ = V_- = 0$; (no current flows through 2 Ω resistor)

$$\text{KCL at } A: \frac{V_s - 0}{1} = \frac{0 - V}{3 - 3j}; \quad V_s = -\frac{4}{3 - 3j} = -\frac{4}{3} \cdot \frac{1}{1-j}; \quad \frac{1-j}{\sqrt{2} \cdot e^{-j\frac{\pi}{4}}} = \frac{\sqrt{1+1}}{\sqrt{2}} = \sqrt{2}$$

$$\frac{2\sqrt{2}}{3} \cdot \cos\left(2t + \frac{5\pi}{4}\right)$$

$$V_s = -\frac{4}{3} \cdot \frac{1}{\sqrt{2}} \cdot e^{\frac{j\pi}{4}} = -\frac{2\sqrt{2}}{3} \cdot e^{\frac{j\pi}{4}}$$

$$\frac{2\sqrt{2}}{3} \cdot \cos\left(2t - \frac{3\pi}{4}\right) \xrightarrow{\text{Same}} -\frac{2\sqrt{2}}{3} \cdot \cos\left(2t + \frac{5\pi}{4}\right) V$$

- (b) [3 pts] The current in the 2Ω resistor and express it in terms of real-valued functions only .

$$i_{2\Omega}(t) = \underline{\hspace{10cm}} \quad \textcircled{O} \text{ A (ideal OP-AMP)}$$

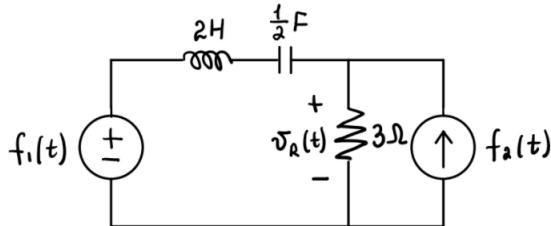
- (c) [2 pts] The average absorbed power in the 1Ω resistor, $P_{1\Omega} = \underline{\hspace{10cm}} \frac{4}{9} W$

$$P = \frac{1}{2} \frac{|V|^2}{R} = \frac{1}{2} \cdot \frac{\frac{8}{9}}{1} = \frac{4}{9} W$$

$$V = -\frac{2\sqrt{2}}{3}; \quad R = 1;$$

- (d) [2 pts] The average absorbed power in the capacitor, $P_C = \underline{\hspace{10cm}} \textcircled{O} W$

3. (25 points) Consider the circuit below, where the output $y(t) = v_R(t)$.



(a) [6 pts] Determine the frequency response $H_1(\omega)$, considering only the input voltage source, $f_1(t)$.

$$F_1 \xrightarrow{\text{parallel combination}} \frac{j\omega 2\Omega}{j\omega^2 + j\omega} = \frac{2}{j\omega} \Omega$$

$$H(\omega) = \frac{Y}{F}$$

$$Y_1 = F_1 \frac{3}{j\omega 2 + \frac{2}{j\omega} + 3} = F_1 \frac{3j\omega}{-2\omega^2 + 2 + 3j\omega} = F_1 \frac{3j\omega}{2 - 2\omega^2 + 3j\omega} = H_1(\omega)$$

$$H_1(\omega) = \frac{3j\omega}{2 - 2\omega^2 + 3j\omega}$$

(b) [5 pts] Determine the steady-state output voltage $v_{R_1}(t)$ given $f_1(t) = 3 \cos(2t) + 2 \sin\left(\frac{t}{2}\right)$ V.

$$v_{R_1}(t) = 3 \cdot |H_1(2)| \cos(2t + \angle H_1(2)) + 2 \cdot |H_1(\frac{1}{2})| \sin\left(\frac{1}{2}t + \angle H_1(\frac{1}{2})\right) \quad \textcircled{E}$$

$$H_1(2) = \frac{6j}{-6 + 6j} = \frac{j}{-1 + j} = \frac{e^{j\pi/2}}{\sqrt{2}} e^{j(\pi + \tan^{-1}(-1))}$$

$$|H_1(2)| = \frac{1}{\sqrt{2}}$$



$$\angle H_1(2) = \frac{\pi}{2} - \pi + \frac{\pi}{4} = -\frac{\pi}{4}$$

$$v_{R_1}(t) = \frac{3}{\sqrt{2}} \cos(2t - \frac{\pi}{4}) + \frac{2}{\sqrt{2}} \sin\left(\frac{1}{2}t + \frac{\pi}{4}\right) \checkmark$$

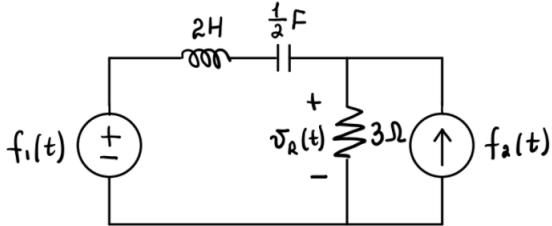
$$\omega_1 = 2 \text{ rad/s} \quad \omega_2 = \frac{1}{2} \text{ rad/s}$$

$$H_1\left(\frac{1}{2}\right) = \frac{\frac{3}{2}j}{\frac{3}{2} + \frac{3}{2}j} = \frac{j}{1+j} = \frac{e^{j\pi/2}}{\sqrt{2}} e^{j(\pi + \tan^{-1}(1))}$$

$$|H_1(\frac{1}{2})| = \frac{1}{\sqrt{2}} \quad \angle H_1(\frac{1}{2}) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

Recall



(c) [6 pts] Determine the frequency response $H_2(\omega)$, considering only the input current source, $f_2(t)$.

$$\begin{aligned}
 & \text{Circuit diagram: } \boxed{\text{parallel branch}} \quad \text{with } \frac{j\omega 2\pi}{3} \text{ in series with } 3\Omega. \\
 & Y_2 = F_2 \left(\frac{Z_P}{3} \right) \cdot 3 = F_2 \cdot Z_P = F_2 \left(\frac{3(2-2\omega^2)}{2-2\omega^2 + 3j\omega} \right) \\
 & Z_P = \frac{3 \cdot (j\omega 2 + \frac{3}{j\omega})}{3 + j\omega 2 + \frac{3}{j\omega}} = \frac{3(-2\omega^2 + 2) \cdot j\omega}{j\omega(3j\omega - 2\omega^2 + 2)} \\
 & H_2(\omega) = \frac{3(2-2\omega^2)}{2-2\omega^2 + 3j\omega}
 \end{aligned}$$

(d) [5 pts] Determine the steady-state output voltage $v_{R_2}(t)$ given $f_2(t) = 3 \cos(2t) + 2 \sin\left(\frac{t}{2}\right)$ A.

$$\begin{aligned}
 H_2(j) &= \frac{-18}{-6+6j} = \frac{3}{1-j} = \frac{3e^{j0}}{\sqrt{2} e^{j\tan'(-1)}} \quad H_2\left(\frac{1}{2}\right) = \frac{9/2}{\frac{3}{2} + \frac{3}{2}j} = \frac{3}{1+j} = \frac{3e^{j0}}{\sqrt{2} e^{j\tan'(1)}} \\
 |H_2(j)| &= \frac{3}{\sqrt{2}} \quad |H_2\left(\frac{1}{2}\right)| = \frac{3}{\sqrt{2}} \\
 \angle H_2(j) &= \pi/4 \quad \angle H_2\left(\frac{1}{2}\right) = -\pi/4
 \end{aligned}$$

$$v_{R_2}(t) = \frac{9}{\sqrt{2}} \cos(2t + \pi/4) + \frac{6}{\sqrt{2}} \sin\left(\frac{1}{2}t - \pi/4\right) V$$

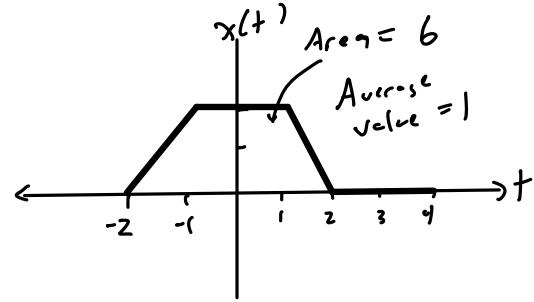
(e) [3 pts] Determine the steady-state output voltage $v_R(t)$.

$$\begin{aligned}
 v_R(t) &= v_{R_1}(t) + v_{R_2}(t) = \frac{3}{\sqrt{2}} \cos(2t - \pi/4) + \frac{3}{\sqrt{2}} \sin\left(\frac{1}{2}t + \pi/4\right) + \\
 &+ \frac{9}{\sqrt{2}} \cos(2t + \pi/4) + \frac{6}{\sqrt{2}} \sin\left(\frac{1}{2}t - \pi/4\right) V
 \end{aligned}$$

4. (25 points) The two parts of this problem are unrelated.

(a) Let:

$$x(t) = \begin{cases} 2t + 4 & -2 < t \leq -1 \\ 2 & -1 < t \leq 1 \\ -2t + 4 & 1 < t \leq 2 \\ 0 & 2 < t \leq 4 \end{cases}$$



be a periodic signal, which repeats at time $t = 4\text{s}$.

i. [2 pts] The period of $x(t)$ is $T = \underline{\hspace{2cm}} 6\text{s} \underline{\hspace{2cm}}$

ii. [3 pts] The average value of $x(t)$ is $x_{av} = \underline{\hspace{2cm}} 1 \underline{\hspace{2cm}}$

$$\begin{aligned} x_{av} &= \frac{1}{T} \int_{-2}^4 x(t) dt \\ &= \frac{1}{6} \left[\int_{-2}^{-1} (2t+4) dt + \int_{-1}^1 2 dt + \int_1^2 (-2t+4) dt \right] \\ &= \frac{1}{6} \left[\left. t^2 + 4t \right|_{-2}^{-1} + 2t \Big|_{-1}^1 + \left. (-t^2 + 4t) \right|_1^2 \right] \\ &= \frac{1}{6} [(1-4) - (4-8) + (2(1)-2(-1)) + (-4+8) - (-1+4)] \\ &= \frac{1}{6} [-3 + 4 + 2 + 2 + 4 - 3] = \frac{1}{6}[6] = 1 \end{aligned}$$

iii. [2 pts] Now let us define a different periodic signal $y(t) = dx(t)/dt$. Express $y(t)$ as a piecewise function like it is done for $x(t)$ above.

$$y(t) = \begin{cases} 2 & -2 < t \leq -1 \\ 0 & -1 < t \leq 1 \\ -2 & 1 < t \leq 2 \\ 0 & 2 < t \leq 4 \end{cases}$$

iv. [5 pts] Determine the Fourier coefficients Y_n for all n .

$$\begin{aligned} \omega_0 &= \frac{2\pi}{T} = \frac{2\pi}{6} = \frac{\pi}{3} \\ Y_n &= \frac{1}{T} \int_T y(t) e^{-jnw_0 t} dt \\ &= \frac{1}{6} \left[\int_{-2}^{-1} 2 e^{-jn\frac{\pi}{3}t} dt + \int_1^2 (-2) e^{-jn\frac{\pi}{3}t} dt \right] \\ &\approx \frac{1}{6} \left[\frac{2}{-jn\frac{\pi}{3}} e^{-jn\frac{\pi}{3}t} \Big|_{-2}^{-1} + \frac{(-2)}{-jn\frac{\pi}{3}} e^{-jn\frac{\pi}{3}t} \Big|_1^2 \right] \\ &= \frac{1}{6} \left[\frac{j6}{n\pi} (e^{jn\frac{\pi}{3}} - e^{jn2\pi/3}) - \frac{j6}{n\pi} (e^{-jn2\pi/3} - e^{-jn\pi/3}) \right] \\ &= \frac{j}{n\pi} \left[e^{jn\pi/3} + e^{-jn\pi/3} - e^{jn2\pi/3} - e^{-jn2\pi/3} \right] \\ Y_n &= \frac{j^2}{n\pi} (\cos \frac{n\pi}{3} - \cos \frac{2n\pi}{3}), \quad Y_0 = 0 \end{aligned}$$

v. [5 pts] Use the derivative property to determine the Fourier coefficients X_n for all n .

$$y(t) = \frac{d x(t)}{dt} \Leftrightarrow jn\omega_0 X_n = Y_n$$

$$X_n = \frac{Y_n}{jn\omega_0}$$

$$X_n = \frac{\frac{Y_n}{jn\pi/3}}{} \quad \text{except } X_0 = 1$$

(b) Consider the function $q(t) = 2\sin(2t) + 3\cos(4t)$,

i. [2 pts] Its fundamental frequency is $\omega_0 = 2 \text{ rad/s}$

ii. [5 pts] Express the function in exponential Fourier series form.

$$q(t) = \frac{2}{j2} (e^{j2t} - e^{-j2t}) + \frac{3}{2} (e^{j4t} + e^{-j4t})$$

$$= -j \quad = j$$

$$Q_1 = \frac{1}{j}, \quad Q_{-1} = -\frac{1}{j}, \quad Q_{\pm 2} = \frac{3}{2}$$

$$Q_n = 0 \quad \text{else}$$

$$q(t) = \sum_{n=-2}^2 Q_n e^{jn\omega_0 t}$$

iii. [5 pts] Its average power is $P_q = 6.5 \text{ W}$

$$\begin{aligned} \sum_{n=-2}^2 |F_n|^2 &= |F_1|^2 + |F_{-1}|^2 + |F_2|^2 + |F_{-2}|^2 \\ &= 1 + 1 + \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2 = 2 + \frac{9}{2} \\ &= 6.5 \text{ W} \end{aligned}$$