ECE 210/211 **Spring 2021**

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Analog Signal Processing

Thursday, March 25, 7-8.50pm

Exam II – Solutions

- **Exam duration**: The midterm exams will be **110 minutes** (1 hour and 50 minutes) long, **including** scan/upload time, so we strongly advise you to finish your exam with plenty of time to spare, in case you run into issues scanning/uploading. We will **deduct** 20 points per late minute after those first 110 minutes, and Gradescope will automatically stop accepting exams after those 115 minutes. **No** late exams will be accepted and you will get a **zero** in the exam if it is not uploaded on time (no exceptions).
- **No collaboration allowed**: You are **not allowed** to share or collaborate on this exam and all work should be your own; otherwise, an Academic Integrity report will be filed against you and sanctions will be applied.
- **Closed notes**: you cannot use the textbook, nor any notes; otherwise, an Academic Integrity report will be filed against you and sanctions will be applied.
- **Calculations**: Calculators and other electronic ways to do calculations, like Wolfram alpha, are **not allowed and neither** is searching online; otherwise, an Academic Integrity report will be filed against you and sanctions will be applied.
- **Solving the exam**: you **must** solve the exam in blank sheets of paper. Tablets are **not allowed** for writing and you may not print the exam; otherwise, an Academic Integrity report may be filed against you and sanctions may be applied.
- An Academic Integrity report will be filed against you and sanctions will be applied for unauthorized actions.
- **Solution uploads**: Make sure that your scans or photos are legible and that you correctly assign each solution to its question, or you will be **deducted** at least 5% of the corresponding problem part. - Instructions for uploading your solutions to Gradescope were provided by you, as well as opportunitites to practice doing so. You **MUST** remain in the proctoring session until you are finished uploading, even if the proctor says you can leave before that. We will **not accept** your exam if you leave the proctoring session before finishing your submission to Gradescope, and you will get a **zero** in the exam.
- **Proctoring**: This course uses the College of Engineering CBTF Online for its exams. You must already have reviewed those policies and be aware of them. Any violation/disregard of CBTF policy may result in your exam not being accepted (giving a zero) or in an Academic Integrity report being filed against you with sanctions applied, depending on the situation. If you have any issue during an exam, please inform the proctor immediately. Work with the proctor to resolve the issue at the time before logging off.
- **Show all your work and simplify** your answers. Answers with no explanation/justification/work will be given **little/no credit**, and an Academic Integrity report may be filed against you and sanctions may be applied.
- **Box your answers**.
- Answers **should** include units if appropriate.

1. (1 pt) Sign acknowledging you will abide by this course's policies, CBTF policies and the University's Academic Integrity policies or face sanctions for not doing so. If your solutions upload does not include your signature, your exam will NOT be graded, resulting in a zero.

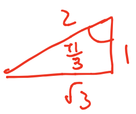
sign: ________________________

2. (24 pts) The two parts of this phasor problem are unrelated.

(a) [08s pts] Determine the amplitude, $A > 0$, and the phase shift, θ (in rad), of the signal $f(t) = \cos(\pi t) + \sin(\pi t) - \sqrt{2}\cos\left(\pi t + \frac{\pi}{4}\right) + \frac{1}{\sqrt{3}}\cos\left(\pi t - \frac{\pi}{3}\right)$ when written as a single cosine $f(t) = A\cos(\pi t + \theta)$.

↳ all terms have $\omega = \pi$ rad/s, use super-position of phasors:

$$F = 1e^{j0} + 1e^{-j\frac{\pi}{2}} - \sqrt{2}e^{j\frac{\pi}{4}} + \frac{1}{\sqrt{3}}e^{-j\frac{\pi}{3}} = 1 - j - \sqrt{2}\left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{3}}\left(\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)$$



$$= 1 - j - 1 - j + \frac{1}{2\sqrt{3}} - \frac{j}{2} = \frac{1}{2\sqrt{3}} - j\frac{5}{2}$$

$$= \sqrt{\left(\frac{1}{2\sqrt{3}}\right)^2 + \left(\frac{5}{2}\right)^2}\, e^{j\tan^{-1}\left(-\frac{5}{2}/\frac{1}{2\sqrt{3}}\right)}$$

$$= \sqrt{\frac{1}{12} + \frac{25}{4}}\, e^{j\tan^{-1}(-5\sqrt{3})} = \sqrt{\frac{76}{12}}\, e^{-j\tan^{-1}(5\sqrt{3})}$$

$A =$ $\sqrt{\frac{19}{3}} = \frac{\sqrt{57}}{3}$

$\theta =$ $-\tan^{-1}(5\sqrt{3})$ rad

(b) [16 pts] Consider a dissipative LTI system with input-output relation given by the ODE $\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = 2sin(2t + \frac{\pi}{3})$. Determine the output phasor, Y, in exponential form, and the steady state solution, $y_{ss}(t)$.

↳ transform into phasors: $(j\omega)^3 Y + 3(j\omega)^2 Y + 3(j\omega)Y + Y = 2e^{j\left(\frac{\pi}{3} - \frac{\pi}{2}\right)}$

↳ $\omega = 2$ rad/s

$$\to Y = \frac{2e^{j\left(\frac{2-3}{6}\right)\pi}}{(j2)^3 + 3(j2)^2 + 3j2 + 1} = \frac{2e^{-j\frac{\pi}{6}}}{-j8 - 12 + j6 + 1} = \frac{2e^{-j\frac{\pi}{6}}}{-11 - j2}$$

3rd quadrant ⇒ adjust angle by $-\pi$

$$= \frac{2e^{-j\frac{\pi}{6}}}{\sqrt{(-11)^2 + (-2)^2}\, e^{j\left(-\pi + \tan^{-1}\left(\frac{2}{11}\right)\right)}}$$

$$= \frac{2}{\sqrt{125}}\, e^{j\left(-\frac{\pi}{6} + \pi - \tan^{-1}\left(\frac{2}{11}\right)\right)} = \frac{2}{5\sqrt{5}}\, e^{j\left(\frac{5\pi}{6} - \tan^{-1}\left(\frac{2}{11}\right)\right)}$$

$$\to y_{ss}(t) = |Y|\cos(2t + \angle Y) = \frac{2}{5\sqrt{5}}\cos\left(2t + \frac{5\pi}{6} - \tan^{-1}\left(\frac{2}{11}\right)\right)$$

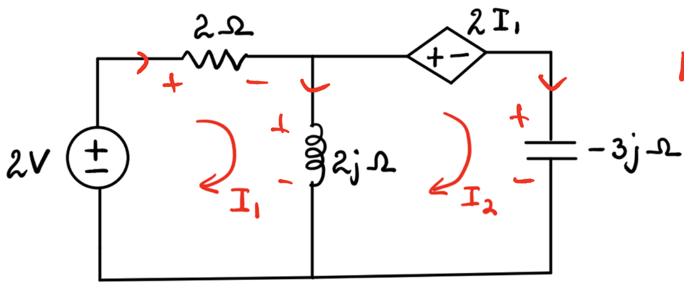
$Y =$ $\frac{2}{5\sqrt{5}}\, e^{j\left(\frac{5\pi}{6} - \tan^{-1}\left(\frac{2}{11}\right)\right)} = \frac{2\sqrt{5}}{25}\, e^{j\left(\frac{5\pi}{6} - \tan^{-1}\left(\frac{2}{11}\right)\right)}$

$y_{ss}(t) =$ $\frac{2}{5\sqrt{5}}\cos\left(2t + \frac{5\pi}{6} - \tan^{-1}\left(\frac{2}{11}\right)\right)$

$= \frac{2\sqrt{5}}{25}\cos\left(2t + \frac{5\pi}{6} - \tan^{-1}\left(\frac{2}{11}\right)\right)$

3. The two parts of this problem are unrelated..

(a) [6 pts] In the following circuit, use the loop-current method to obtain a set of two equations, in terms of loop currents I_1 and I_2, that can be solved to get I_1 and I_2. You do not need to solve the equations.



KVL @ (1):

$$-2 + 2I_1 + 2j(I_1 - I_2) = 0$$

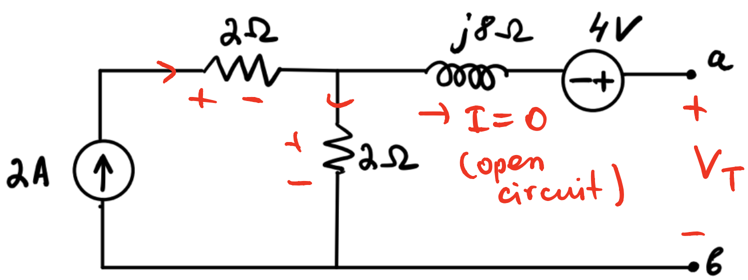
KVL @ (2):

$$-2j(I_1 - I_2) + 2I_1 + (-3j)I_2 = 0$$

$(2+j2)$ I_1 + $(-2j)$ I_2 = 2

$(2-j2)$ I_1 + $(-j)$ I_2 = 0

(b) [10 pts] Consider the circuit shown below (use this circuit for parts b, c and d)



$$-V_{2\Omega} - 4V + V_T = 0$$

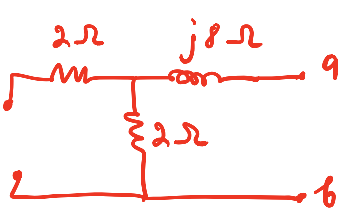
$$V_{2\Omega} = 2 \cdot 2 = 4V$$

$$\Downarrow$$

$$V_T = 4 + 4 = 8V$$

Find the Thevenin voltage V_T and Thevenin impedance Z_T between the terminals a and b.

Z_T: source suppression:



$$\Rightarrow Z_T = 2 + j8\,\Omega$$

V_T = $8V$ Z_T = $2+j8\,\Omega$

(c) [5 pts] What is available average power P_a of the network?

$$P_a = \frac{|V_T|^2}{8R_T} = \frac{8^2}{8 \cdot 2} = \frac{64}{16} = 4W$$

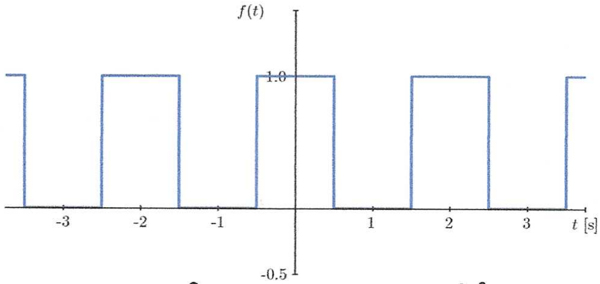
$P_a =$ $4W$

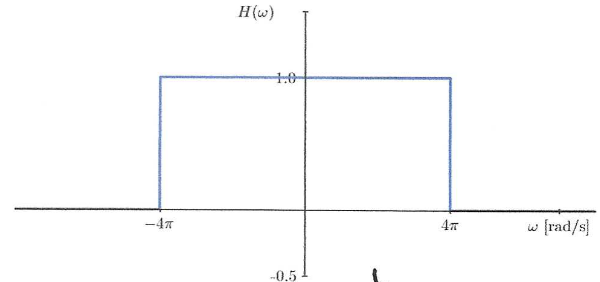
(d) [4 pts] If a load is attached to the network between the terminals a and b, what must its impedance Z_L be for the maximum power to be delivered from the network?

$$Z_L = Z_T^* = 2 - j8\,\Omega$$

$Z_L =$ $2 - j8\,\Omega$

4. (25 pts) The periodic signal $f(t)$ with period $T = 2$s is the input to an LTI system with frequency response $H(\omega)$, given below. Find the output $y(t)$ for the system with the given input. NOTE: express it in terms of real-valued functions only.





$T = 2s \qquad \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi \text{ rad/s}$

$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\pi t}$

$F_n = \frac{1}{2}\int_{-1}^{1} f(t) e^{-jn\pi t} dt = \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-jn\pi t} dt$

$= \frac{1}{2}\left[\frac{1}{-jn\pi} e^{-jn\pi t}\Big|_{-1/2}^{1/2}\right]$

$= \frac{1}{2}\left[\frac{1}{-jn\pi}\left(e^{-jn\pi/2} - e^{jn\pi/2}\right)\right]$

$= \frac{1}{2}\left[\frac{1}{-jn\pi}\left(-j2\sin\left(\frac{n\pi}{2}\right)\right)\right]$

$\underline{F_n = \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right)}$

$F_0 = \frac{1}{2}\int_{-1}^{1} f(t)\, dt \Rightarrow \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} dt$

$F_0 = \frac{1}{2}\left(t\Big|_{-1/2}^{1/2}\right) = \frac{1}{2}\left(\frac{1}{2} - \left(-\frac{1}{2}\right)\right)$

$F_0 = \frac{1}{2}(1) \Rightarrow \underline{F_0 = \frac{1}{2}}$

$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right)e^{jn\pi t} + \frac{1}{-n\pi}\sin\left(-\frac{n\pi}{2}\right)e^{-jn\pi t}\right)$

$C_0 = 2|F_0| = 2\left(\frac{1}{2}\right) = 1 \qquad \theta_n = 0$ (F_n always real)

$C_n = 2|F_n| = \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$

$\underline{f(t) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos(n\pi t)}$

$y(t) = \frac{1}{2}|H(\omega=0)| + \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)|H(n\omega_0)|\cos(n\pi t)$

From $H(\omega)$, only $n=0,1,2,3,4$ pass through

$\sin\left(\frac{2\pi}{2}\right) = \sin\left(\frac{4\pi}{2}\right) = 0 \qquad \sin\left(\frac{\pi}{2}\right) = 1 \qquad \sin\left(\frac{3\pi}{2}\right) = -1$

only have $n = 0, 1, 3$

$$\boxed{y(t) = \frac{1}{2} + \frac{2}{\pi}\cos(\pi t) - \frac{2}{3\pi}\cos(3\pi t)}$$

Regular exam

[7 points] 5a) A linear system with input $f(t)$ and output $y(t)$ is described by the ODE

$$y(t)+4\frac{dy}{dt}+2\frac{d^2y}{dt^2}=\frac{df}{dt}+2\frac{d^2f}{dt^2}$$

Determine the frequency response $H(\omega)$ of the system.

Solution:

$$Y+4j\omega Y+2(j\omega)^2Y=j\omega F+2(j\omega)^2F$$
$$(1+4j\omega-2\omega^2)Y=(j\omega-2\omega^2)F$$
$$H(\omega)=\frac{Y}{F}=\frac{j\omega-2\omega^2}{1+4j\omega-2\omega^2}$$

[10 points] Part b) A linear system has the frequency response $H(\omega)=\frac{j\omega}{2+j\omega}$.

Determine the system steady-state output $y(t)$ for input $f(t)=3\sin(2t)+5$.

Solution:

$$H(0)=0\Rightarrow|H(0)|=0.$$
$$H(2)=\frac{2j}{2+2j}\Rightarrow|H(2)|=\frac{2}{2\sqrt{2}}=\frac{\sqrt{2}}{2},\angle H(2)=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$$

$$y(t)=|H(2)|\times 3\sin(2t+\angle H(2))+|H(0)|\times 5=\frac{3\sqrt{2}}{2}\sin\left(2t+\frac{\pi}{4}\right)$$

[8 points] Part c) For each one of the following functions of t, determine whether or not the function is periodic. If periodic, find its period; if not periodic, explain why.

i) $f(t)=\cos(\sqrt{2}t)+\sin(2\sqrt{2}t)$

ii) $g(t)=e^{j2t}+e^{4t}$

Solution:

i) $f(t)$ is periodic, because $\cos(\sqrt{2}t)$ is periodic with period $T_1=\frac{2\pi}{\sqrt{2}}$, $\sin(2\sqrt{2}t)$ is periodic with period $T_2=\frac{2\pi}{2\sqrt{2}}$, and $\frac{T_1}{T_2}=2$ is a rational number.

The period of $f(t)$ is $T=T_1=2T_2=\sqrt{2}\pi$.

ii) $g(t)$ is not periodic because e^{j2t} is periodic but e^{4t} is not periodic.