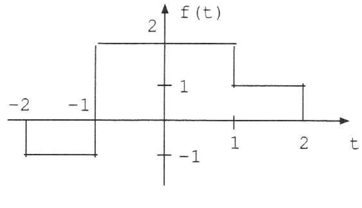
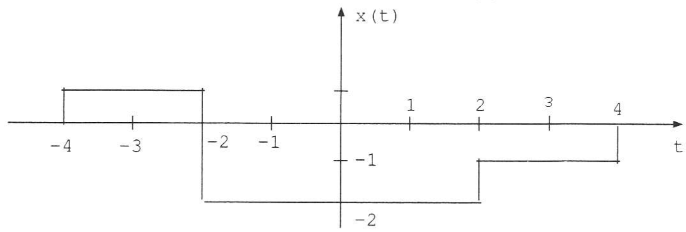
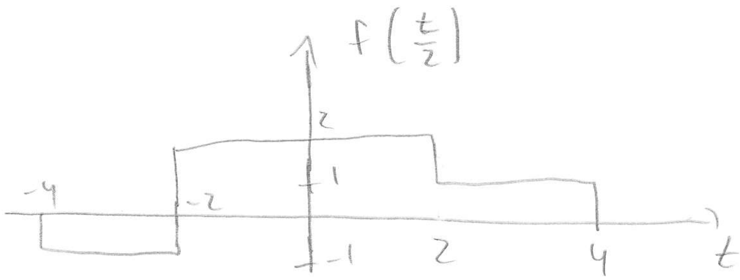
1. (10 pts) Let $f(t)$, plotted below, have Fourier transform $F(\omega)$.



(a) (4 pts) Obtain the Fourier transform of $x(t)$, plotted below, in terms of $F(\omega)$.

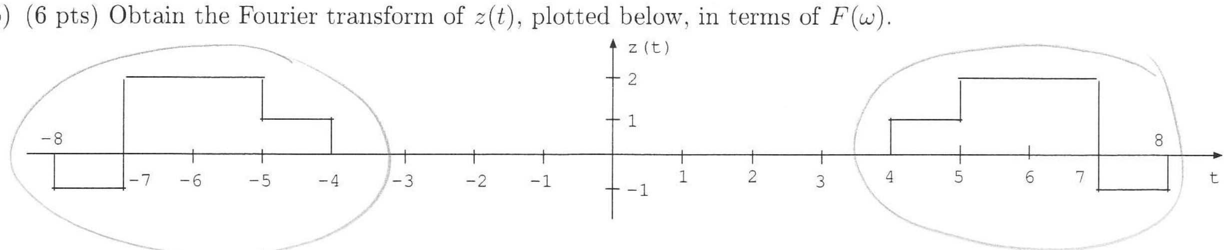




So $x(t) = -f\left(\frac{t}{2}\right)$. By table 3, properties 1 and 7:

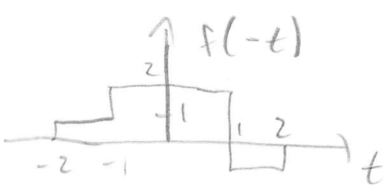
$X(\omega) =$ $-\left[\frac{1}{|1/2|} F\left(\frac{\omega}{1/2}\right)\right] = \boxed{-2F(2\omega) = X(\omega)}$

(b) (6 pts) Obtain the Fourier transform of $z(t)$, plotted below, in terms of $F(\omega)$.



$= f(t+6)$

$= f(-(t-6))$

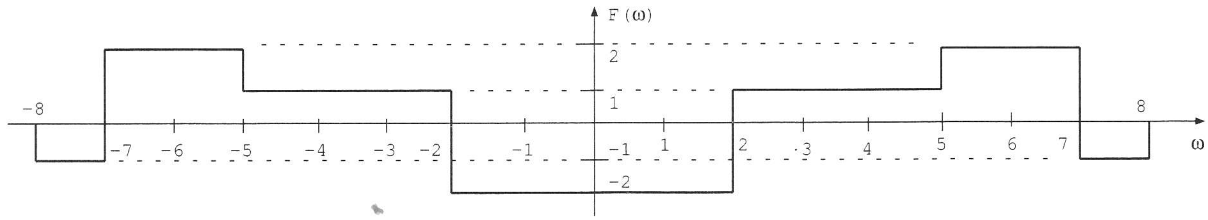


$\Rightarrow z(t) = f(t+6) + f(-(t-6))$

By properties 7 and 8 in table 3:

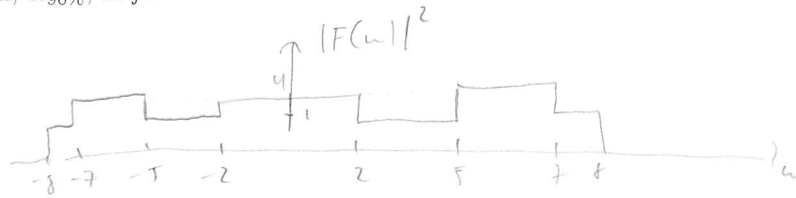
$Z(\omega) =$ $F(\omega)e^{j\omega 6} + \frac{1}{|-1|} F\left(\frac{\omega}{-1}\right) e^{-j\omega 6} = \boxed{F(\omega)e^{j\omega 6} + F(-\omega)e^{-j\omega 6}}$

2. (15 pts) Let $f(t)$ have Fourier transform $F(\omega)$, plotted below.



(a) (5 pts) Obtain the 90% energy bandwidth, $\Omega_{90\%}$, of f.

$W = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega$



$= \frac{1}{2\pi}$ area of $|F(\omega)|^2$

$= \frac{1}{2\pi}\left[(1)8 + (4)8\right] = \frac{40}{2\pi} = \frac{20}{\pi}$

$\Omega_{90\%} \Rightarrow 0.9W = \int_{-\Omega_{90\%}}^{\Omega_{90\%}} |F(\omega)|^2 d\omega$ or $0.1W = 2\int_{\Omega_{90\%}}^{8} |F(\omega)|^2 d\omega$

$\frac{1}{10}\left(\frac{1}{2}\right)\left(\frac{20}{\pi}\right) = \frac{1}{2\pi}$ area between $\Omega_{90\%}$ and 8

$2 =$ area $\div$ $\Omega_{90\%}$ and 8

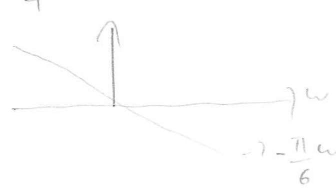
last square has area 1, so $4x = 1$

remaining $\Rightarrow x = \frac{1}{4} \Rightarrow$

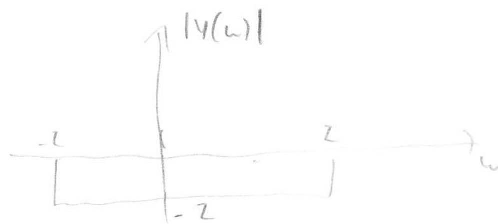
$\Rightarrow \Omega_{90\%} = 8 - 1 - \frac{1}{4} = 6\frac{3}{4}$

$\Omega_{90\%} =$ $\boxed{6\ 3/4 \text{ rad/s}}$

(b) (10 pts) Let $f(t)$ be the input to an LTI system with frequency response $H(\omega) = rect(\frac{\omega}{4})e^{-j\frac{\pi}{6}\omega}$. Obtain the zero-state output $y_{zs(t)}$.



$\angle H(\omega) = \angle Y(\omega)$



$Y(\omega) = F(\omega)H(\omega) = -2\,rect\left(\frac{\omega}{4}\right)e^{-j\frac{\pi}{6}\omega}$ → time shift of $-\frac{\pi}{6}$

$\mathcal{F}^{-1}\downarrow$

$\frac{2}{\pi}\,\text{sinc}(2t)$

$y_{zs(t)}$ $\boxed{-\frac{4}{\pi}\,\text{sinc}\left(2\left(t-\frac{\pi}{6}\right)\right)}$

problem 1 cont'd

(c) Let $f(t) = e^{-t}u(t-2)$. Obtain its Fourier transform $F(\omega)$, evaluated at $\omega = 1$.

$f(t) = e^{-(t+2-2)}u(t-2) = e^{-2}e^{-(t-2)}u(t-2)$ $=g(t)$ $=G(\omega)$

Entry #1 in Table 7.2 $u(t)e^{-at} \leftrightarrow \frac{1}{a+j\omega} \Rightarrow e^{-t}u(t) \leftrightarrow \frac{1}{1+j\omega}$

Time-shift property (#8) in Table 7.1 $h(t) = f(t-t_0) \leftrightarrow H(\omega) = e^{-j\omega t_0}F(\omega)$

$f(t) = e^{-2}g(t-2) \Rightarrow F(\omega) = e^{-2}e^{-j\omega 2}G(\omega) = \frac{e^{-2(1+j\omega)}}{1+j\omega}$

$F(1) = \frac{e^{-2(1+j)}}{1+j}$

Also
$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$
$= \int_2^{\infty} e^{-t}e^{-j\omega t}\,dt = \frac{e^{-2(1+j\omega)}}{1+j\omega}$

$$\boxed{F(1) = \frac{e^{-2(1+j)}}{1+j}}$$

(d) The function f(t) has Fourier transform $F(\omega) = 3\pi\, e^{-3\omega}u(\omega)$. Indicate (circle) which of the following functions corresponds to the function f(t), and explain why (no points if no valid reason).

(a) $f(t) = \frac{3}{2}\frac{1}{3-t}$

(b) $f(t) = \frac{3}{2}\frac{1}{3-jt}$

(c) $f(t) = \frac{3}{2}(e^{3t} + e^{-3t})$

(d) $f(t) = \frac{3}{2}(e^{j3t} + e^{-j3t})$

(e) $f(t) = \frac{3}{2j}(e^{j3t} + e^{-j3t})$

Why?

See part 1(b) for similarity $\frac{1}{3-jt} \leftrightarrow 2\pi e^{-3\omega}u(\omega)$.

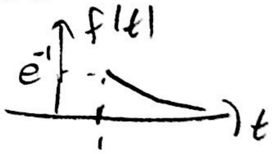
multiply by $\frac{3}{2}$ $\quad \frac{3}{2}\frac{1}{3-jt} \leftrightarrow 3\pi e^{-3\omega}u(\omega)$

Also $f(t) = \int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega = \int_0^{\infty} 3\pi e^{-3\omega}e^{j\omega t}\,d\omega = \frac{3}{2}\frac{1}{3-jt}$

Also, $|F(\omega)|$ is not an even function so $f(t)$ is not real. The only $f(t)$ above which is not real is (b).

(b) Obtain the Fourier transform of

i) $f(t) = e^{-t}u(t-1)$



$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt = \int_{1}^{\infty} e^{-t} e^{-j\omega t}\, dt = \left. \frac{e^{-t(1+j\omega)}}{-(1+j\omega)} \right|_{1}^{\infty} = 0 + \frac{e^{-(1+j\omega)}}{1+j\omega}$$

OR

$$f(t) = e^{-1} e^{-(t-1)} u(t-1) \quad e^{-1} e^{-j\omega} \frac{1}{1+j\omega}$$

$$e^{-t}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1+j\omega}$$

$$g(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega t_0} G(\omega)$$

F(ω) = $\boxed{\dfrac{e^{-(1+j\omega)}}{1+j\omega}}$

ii) $g(t) = \dfrac{1}{1+j(t-1)} + \dfrac{1}{1-j(t-1)}$

(Hint: Do not combine the fractions)

From Table $f(t) = e^{-t}u(t) \longleftrightarrow F(\omega) = \dfrac{1}{1+j\omega}$

From Table symmetry property $f(t) \overset{\mathcal{F}}{\longleftrightarrow} F(\omega)$

$F(t) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi f(-\omega)$

$$\Rightarrow \frac{1}{1+jt} \longleftrightarrow 2\pi e^{-(-\omega)} u(-\omega) = 2\pi e^{\omega} u(-\omega)$$

By similar reasoning on $e^{t}u(-t) \overset{\mathcal{F}}{\longleftrightarrow} \dfrac{1}{1-j\omega}$

$$\frac{1}{1-j\omega} \longleftrightarrow 2\pi e^{-\omega} u(-(-\omega)) = 2\pi e^{-\omega} u(\omega)$$

By time shift $f(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega t_0} F(\omega)$

G(ω) = $2\pi e^{-j\omega}\left[e^{\omega}u(-\omega) + e^{-\omega}u(\omega)\right]$

Problem 1

a) Find the Fourier transform $F(\omega)$ and the energy of the following signal $f(t)$.

$$f(t)=\left[u(t-1)-u(t-3)\right]e^{-t}$$

$F(\omega) =$ ________________ (6 points)

$\varepsilon =$ ________________ (6 points)

$$\int_1^3 e^{-t} e^{-j\omega t} dt = \int_1^3 e^{-(1+j\omega)t} dt$$

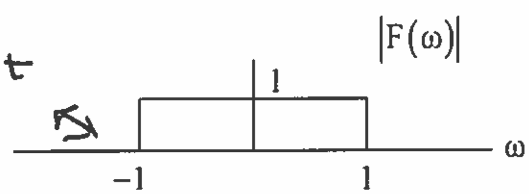
$$\frac{e^{-(1+j\omega)} - e^{-3(1+j\omega)}}{1+j\omega}$$

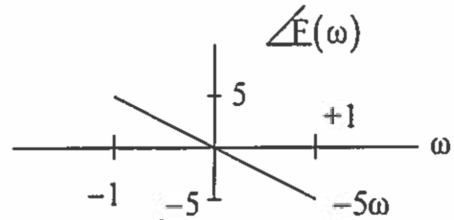
$$\mathcal{E} = \int f^2 dt = \int_1^3 e^{-2t} dt$$

$$= \left(e^{-2} - e^{-6}/2\right)^1 \ J$$

b) Fourier transform of a signal $f(t)$ is given as

$\frac{1}{\pi} \text{Sinc} t$





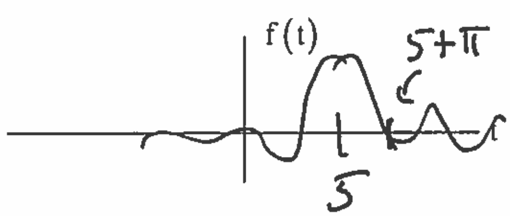
$\text{Sinc} Wt \Leftrightarrow \frac{\pi}{W} \text{rect} \frac{\omega}{W}$

$F(\omega) e^{-j5\omega} \Leftrightarrow f(t-5)$

i) Find the inverse Fourier transform $f(t)$. Simplify your answer. Sketch $f(t)$, label axis carefully.

$\sin t = 0 \qquad t = n\pi$

$f(t) =$ $\frac{1}{\pi} \text{Sinc}(t-5)$ (7 points)



ii) Find the 90% energy bandwidth of $f(t)$.

BW = 0.9 r/sec (6 points)

$$\mathcal{E} = \frac{1}{2\pi} \cdot 2 \cdot 1 = \frac{1}{\pi}$$

$$0.9 \cdot \frac{1}{\pi} = \frac{1}{2\pi} \cdot X = 0.9$$

1. (25 pts) Given the Fourier transform pair $f(t) = e^{-t}u(t)$ and $F(\omega) = \frac{1}{1+j\omega}$, obtain the Fourier transforms of the following functions:

(a) $g(t) = \frac{d^2 f}{dt^2}$. Obtain $G(\omega) =$ $-\frac{w^2}{1+jw}$

$$\frac{d^2f}{dt^2} \leftrightarrow (jw)^2 F(w) = \frac{-w^2}{1+jw}$$

(b) $h(t) = e^{-t}u(t-2)$. Obtain $H(\omega) =$ $\frac{1}{1+jw} e^{-2-j2w}$

$$h(t) = e^{-2} e^{-(t-2)} u(t-2)$$

$$H(w) = e^{-2} \frac{1}{1+jw} e^{-j2w}$$

(c) $x(t) = e^{-t}rect\left(\frac{t}{4}\right)$. Obtain $X(\omega) =$ $\frac{1}{1+jw}\left(e^{2+j2w} - e^{-2-j2w}\right)$

$$x(t) = e^{-t}\left(u(t+2) - u(t-2)\right)$$

$$= e^{2} e^{-(t+2)} u(t+2) - e^{-2} e^{-(t-2)} u(t-2)$$

$$X(w) = \frac{1}{1+jw}\left(e^{2+j2w} - e^{-2-j2w}\right)$$

(d) $y(t) = \frac{1}{1+j(t+1)}$. Obtain $Y(\omega) =$ $2\pi e^{w} u(-w) e^{jw}$

For $f(t) \leftrightarrow F(w)$ pair, apply symmetry property:

$$F(t) \leftrightarrow 2\pi f(-w)$$

$$\frac{1}{1+jt} \leftrightarrow 2\pi e^{w} u(-w)$$

$$\frac{1}{1+j(t+1)} \leftrightarrow 2\pi e^{w} u(-w) e^{jw}$$