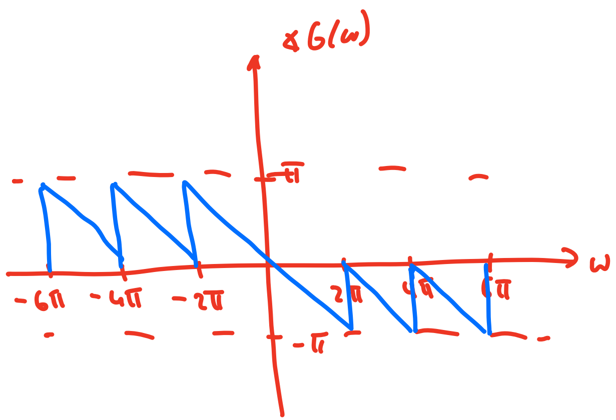
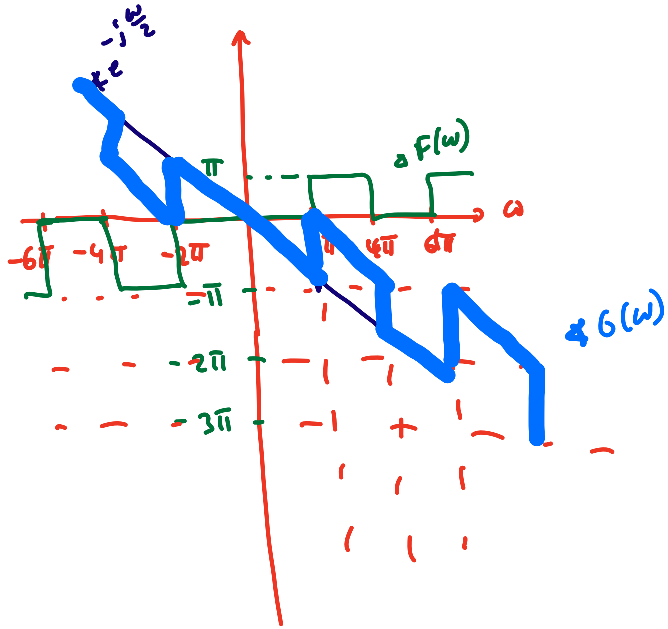
• Fourier transform - Example # 2-cont

- Determine the Fourier transform of

$$f(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & \text{else} \end{cases}$$





• Fourier transform - Properties Table 7.1

	Name:	Condition:	Property:
1	V Amplitude scaling	$f(t) \leftrightarrow F(\omega)$, constant K	$Kf(t) \leftrightarrow KF(\omega)$
2	Addition	$f(t) \leftrightarrow F(\omega)$, $g(t) \leftrightarrow G(\omega)$, $\cdots$	$f(t) + g(t) + \cdots \leftrightarrow \underline{F(\omega) + G(\omega)} + \cdots$
3	V Hermitian	Real $f(t) \leftrightarrow F(\omega)$	$F(-\omega) = F^*(\omega)$
4	V Even	Real and even $f(t)$	Real and even $F(\omega)$
5	V Odd	Real and odd $f(t)$	Imaginary and odd $F(\omega)$

- Fourier transform – Properties-cont
  - Time-scaling. For real valued *a:*

$$f(t) \Leftrightarrow F(\omega) \qquad \underbrace{f(at)}_{=\,g(t)} \Leftrightarrow \ ? \ \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

$$F\left(\frac{\omega}{a}\right) = \int_{-\infty}^{\infty} f(t)\, e^{-j\frac{\omega}{a}t}\, dt$$

$$G(\omega) = \int_{-\infty}^{\infty} g(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} f(\underbrace{at}_{=\,s})\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} f(s)\, e^{-j\omega\left(\frac{s}{a}\right)} \frac{ds}{a} = \frac{1}{a}\int_{-\infty}^{\infty} f(s)\, e^{-j\frac{\omega}{a}\cdot s}\, ds =$$

$s = at \Rightarrow t = \frac{s}{a}$

$ds = a\,dt$

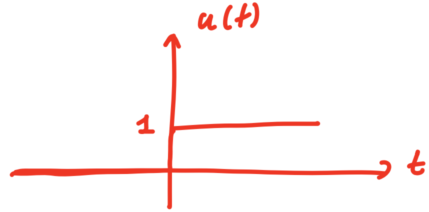
assume $a > 0$

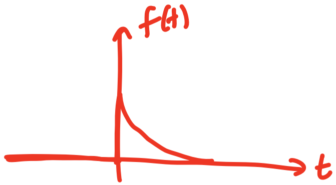
$$= \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

works for both $a > 0$, $a < 0$

- Fourier transform - Example # 3
  - For a > 0 obtain the Fourier transform of

$$f(t) = \begin{cases} e^{-at} & t > 0 \\ 0 & \text{else} \end{cases} = e^{-at} u(t) \rightarrow \text{unit-step function}$$





$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt = \int_0^{\infty} e^{-at} \cdot e^{-j\omega t}\, dt = \int_0^{\infty} e^{-(a+j\omega)t}\, dt =$$

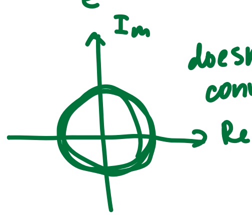
$$= \left. \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right|_0^{\infty} = \frac{0-1}{-(a+j\omega)} = \frac{1}{a+j\omega} = F(\omega)$$

$$\boxed{e^{-at} u(t) \Leftrightarrow \frac{1}{a+j\omega}, \; a>0}$$

$$e^{-(a+j\omega)t} = e^{-at} \cdot e^{-j\omega t}$$

if $a=0$

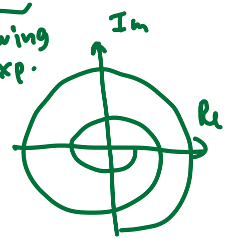
$e^{-j\omega t}$



doesn't converge

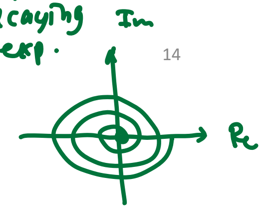
if $a<0$

$\underbrace{e^{-at}}_{\text{growing exp.}} \cdot e^{-j\omega t}$



if $a>0$

$\underbrace{e^{-at}}_{\text{decaying exp.}} \cdot e^{-j\omega t}$

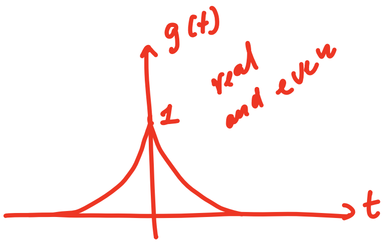


- Fourier transform - Example # 4
  - For a > 0, obtain the Fourier transform of

$$g(t) = e^{-a|t|}$$

Time scaling + addition properties

$$f(at) \leftrightarrow \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$



$$g(t) = e^{-a|t|} = f(t) + f(-t)$$

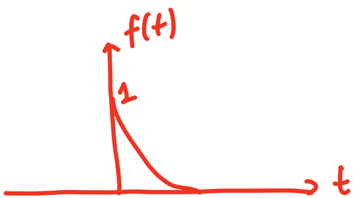
F.T.

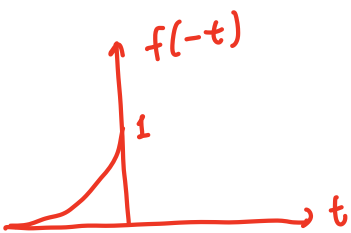
$$G(\omega) = F(\omega) + \mathcal{F}\{f(-t)\} =$$

$$= F(\omega) + \frac{1}{|-1|} F\left(\frac{\omega}{-1}\right) =$$

$$= \frac{1}{a+j\omega} + \frac{1}{a+j(-\omega)} =$$

$$= \boxed{\frac{2a}{a^2+\omega^2} = G(\omega)}$$





- # Fourier transform - Example # 5
  - Recall that for a > 0:

$$g(t) = e^{-a|t|} \qquad G(\omega) = \frac{2a}{a^2 + \omega^2}$$

  - Obtain the Fourier transform of

$$h(t) = \frac{2a}{a^2 + t^2}$$

t

$H(\omega)$ - ?

Symmetry property:

$f(t) \Leftrightarrow F(\omega)$

$F(t) \Leftrightarrow 2\pi f(-\omega)$

time		frequency
$e^{-a\|t\|}$	$t=-\omega$, $\omega=t$, $\times 2\pi$	$\frac{2a}{a^2+\omega^2}$
$\frac{2a}{a^2+t^2}$		$2\pi e^{-a\|-\omega\|} = 2\pi e^{-a\|\omega\|}$ $2\pi f(-\omega)$
$\text{rect}(t)$	$t=-\omega$, $\omega=t$, $\times 2\pi$	$\text{sinc}\left(\frac{\omega}{2}\right)$
$\text{sinc}\left(\frac{t}{2}\right)$		$2\pi\, \text{rect}(-\omega)$

- Fourier transform – Properties-cont

  - Symmetry:

$$f(t) \longleftrightarrow F(\omega) \qquad F(t) \longleftrightarrow 2\pi f(-\omega)$$

$$F(t) \longleftrightarrow \int_{-\infty}^{\infty} F(t)\, e^{-j\omega t}\, dx$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(x)\, e^{jxt}\, dx = f(t)$$

$$f(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(x)\, e^{jx(-\omega)}\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)\, e^{-jt\omega}\, dt$$

$\omega = x$

$d\omega = dx$

$t = -\omega$

$x = t$

$dx = dt$

$$\underbrace{f(-\omega)\cdot 2\pi}_{} = \underbrace{\int_{-\infty}^{\infty} F(t)\, e^{-j\omega t}\, dt}_{\text{"}\mathcal{F}\{F(t)\}}$$