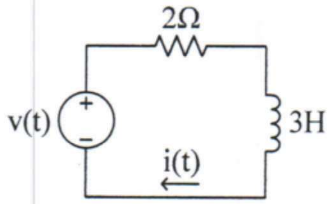
Problem 3

(a) The circuit below has frequency response $H(\omega) = \frac{I}{V}$.



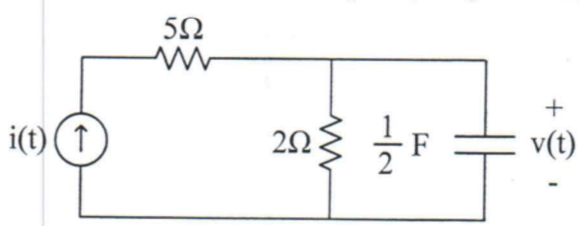
$V = IZ \Rightarrow \frac{I}{V} = \frac{1}{Z}$

$Z = 2 + j3\omega$

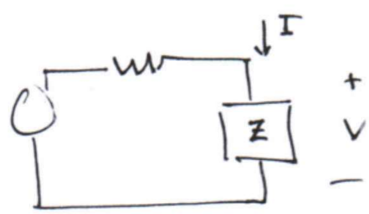
i) Determine $H(\omega)$: $\frac{1}{2 + j3\omega}$

ii) Determine the maximum value for $|H(\omega)|$: $\frac{1}{2}$ at $\omega = 0$

(b) The circuit below has frequency response $H(\omega) = \frac{V}{I}$.



$V = IZ \Rightarrow \frac{V}{I} = Z$



$Z = \dfrac{2 \cdot \frac{1}{\frac{1}{2} j\omega}}{2 + \frac{1}{j\frac{1}{2}\omega}} = \dfrac{2}{1 + j\omega}$

i) Determine $H(\omega)$: $\frac{2}{1 + j\omega}$

ii) Determine $|H(\omega)|$ as $\omega \to \infty$: 0

iii) Determine $\angle H(\omega)$ as $\omega \to \infty$: $-\frac{\pi}{2}$

$\lim_{\omega \to \infty} \angle \frac{2}{1+j\omega} = \lim_{\omega \to \infty} \angle 2 - \angle 1 + j\omega$

$= 0 - \frac{\pi}{2}$

(c) An LTI system with input f(t) and output y(t) is described by the ODE

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 3y(t) = f(t)$$

$$(j\omega)^2 Y + j2\omega Y + 3Y = F$$

The frequency response for the system $H(\omega) = \frac{Y}{F}$.

$$\frac{Y}{F} = \frac{1}{(j\omega)^2 + j2\omega + 3}$$

i) Determine $H(\omega)$: $\frac{1}{-\omega^2 + 3 + j2\omega}$

ii) Determine the value of ω ($\omega > 0$) such that $\angle H(\omega) = -\frac{\pi}{4}$: 1

$$\angle H = -\angle\, -\omega^2 + 3 + j2\omega$$

$$\angle H = -\frac{\pi}{4} \text{ when } \angle \text{ denominator } = \frac{\pi}{4}$$

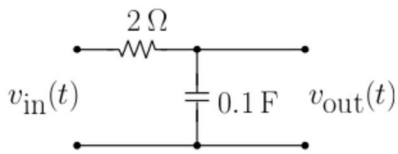
when Re{denominator} = Im{denominator}

$$3 - \omega^2 = 2\omega \implies \omega = 1$$

1. Problem 1 (25 points)

These problems are intended to be quick and easy:

(a) For the following circuit.



(3 pt)

i. Find the frequency response $H(\omega)$.

Sol:
Phasor voltage divider: $H(\omega) = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C}+R} = \frac{1}{1+j\omega RC} = \frac{1}{1+j\omega RC} = \frac{1}{1+j\omega \cdot 0.2}$

(1 pt)

ii. Is $H(\omega)$ the frequency response of a high pass, low pass or a band pass filter?

Sol:
Lowpass ($H(0) = 1$, $H(\omega \to \infty) = 0$, monotonically decreasing)

(3 pt)

iii. If $v_{in}(t) = 2 + \cos(5t + \frac{\pi}{4})$, then $v_{out}(t) =$?

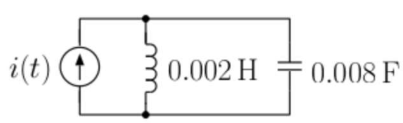
Sol:
$v_{\text{in}}(t)$ composed of two co-sinusoidal components with frequencies 0 and 5:

ω_o	V_{in} phasor	$H(\omega_o)$	V_{out} phasor $= V_{\text{in}} H(\omega_o)$	$v_{\text{out}}(t, \omega_o)$
0	$2e^{j0}$	1	2	2
5	$1e^{j\frac{\pi}{4}}$	$\frac{1}{1+j} = \frac{e^{-j\frac{\pi}{4}}}{\sqrt{2}}$	$\frac{1}{\sqrt{2}} e^{j(\frac{\pi}{4}-\frac{\pi}{4})} = \frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}} \cos(5t+0)$

$\therefore v_{\text{out}}(t) = 2 + \frac{1}{\sqrt{2}} \cos(5t)$

(2 pt)

(b) What is the resonance frequency of the following circuit.



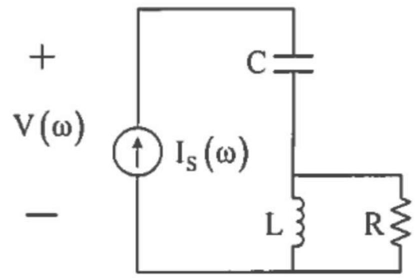
Sol:
$\omega_{\text{resonance}} = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{16 \cdot 10^{-6}}} = \frac{1}{4 \cdot 10^{-3}} = 250 \, \frac{\text{rad}}{\text{sec}}$.
Also,
total impedance: $Z = Z_L \parallel Z_C = \frac{1}{\frac{1}{j\omega L} + j\omega C} = \frac{j\omega L}{1-\omega^2 LC}$.
Then $Z \to \infty$, when $\omega \to \frac{1}{\sqrt{LC}} = 250 \, \frac{\text{rad}}{\text{sec}}$.

Problem 3 (25 points)

All parts of this problem are independent.

a) (5 points) In the circuit below, $H(\omega) = V(\omega)/I_S(\omega)$. Find $H(\omega)$. Your answer may include the variables, R, C, L and ω.

$$Z_{equivalent} = \frac{1}{j\omega C} + \frac{j\omega LR}{j\omega L + R}.$$



$$H(\omega) = \frac{V(\omega)}{I_s(\omega)} = Z_{equivalent} = \frac{1}{j\omega C} + \frac{j\omega LR}{j\omega L + R}.$$

$H(\omega) = \frac{1}{j\omega C} + \frac{j\omega LR}{j\omega L + R}.$

b) A particular circuit is characterized by $H(\omega) = \dfrac{15j\omega}{25j\omega + (1 - \omega^2 64)}$

i) (4points) What is the frequency ω_p, at which $|H(\omega)|$ is maximum?

$$|H(\omega)| = \frac{15\omega}{\sqrt{(25\omega)^2 + (1-\omega^2 64)^2}}$$

$\omega_p = \frac{1}{8}$

$|H(\omega)|$ will maximize when $(1-\omega^2 64)^2$ is at its minimum which is 0.

$1 - \omega^2 64 = 0 \qquad \omega^2 64 = 1$

$\Rightarrow \omega_p = \frac{1}{8}$

ii) (4 points) What is $|H(\omega_p)|$?

$$|H(\omega_p)| = \frac{15\omega}{25\omega} = \frac{3}{5}$$

$|H(\omega_p)| = \frac{3}{5}$.

Problem 3 (continued)

c) A circuit has input $f(t)$, output $y(t)$, and frequency response $H(\omega)$ given by

$$H(\omega) = \frac{a + j\omega b}{c + j\omega d}$$

You may answer each of the following sub-parts in terms of the variables a, b, c, and d if you wish. You may assume that a, b, c, and d are all positive real constants.

i) (4 points) If $f(t) = 6\cos(35t)$, then $y(t) = A\cos(35t + \theta)$ for what value of A?

A = $\frac{6\sqrt{a^2+(35b)^2}}{\sqrt{c^2+(35d)^2}}$

$A = 6 \times |H(\omega)|$

$= 6 \cdot \frac{\sqrt{a^2+(35b)^2}}{\sqrt{c^2+(35d)^2}}$

ii) (4 points) If $f(t) = 6\cos(35t)$, then $y(t) = A\cos(35t + \theta)$ for what value of θ?

θ = $\tan^{-1}\left(\frac{35b}{a}\right) - \tan^{-1}\left(\frac{35d}{c}\right)$

$\theta = \angle H(\omega) + \angle f(t) = \angle H(\omega) + 0$

$= \tan^{-1}\left(\frac{35b}{a}\right) - \tan^{-1}\left(\frac{35d}{c}\right)$

iii) (4 points) If $f(t)=10$, what is $y(t)$?

$\omega = 0$

$H(0) = \frac{a + j0b}{c + j0d} = \frac{a}{c}$

y(t) = $10\frac{a}{c}$

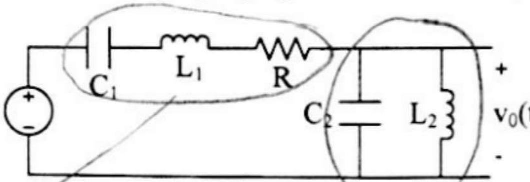
$|y(t)| = |H(0)| \cdot |f(t)| = 10\frac{a}{c}$

$\angle y(t) = \angle f(t) + \angle H(0) = \angle f(t) = 0$

$y(t) = 10\frac{a}{c}$

Problem 3

(a) Determine the frequency response $H(\omega)=\dfrac{V_o(\omega)}{V_S(\omega)}$ of the following circuit. Give the answer in terms of ω, C_1, C_2, L_1, L_2 and R. Do not simplify your answer.



$$H(\omega) = \dfrac{\dfrac{j\omega L_2}{1-\omega^2 L_2 C_2}}{\dfrac{j\omega L_2}{1-\omega^2 L_2 C_2} + \dfrac{1}{j\omega C_1} + j\omega L_1 + R}$$

$$Z_S = \frac{1}{j\omega C_1} + j\omega L_1 + R \qquad Z_p = \frac{\left(\frac{1}{j\omega C_2}\right)(j\omega L_2)}{\frac{1}{j\omega C_2} + j\omega L_2} = \frac{j\omega L_2}{1-\omega^2 L_2 C_2}$$

$$V_o = V_s \frac{Z_p}{Z_p + Z_s} \Rightarrow H(\omega) = \frac{Z_p}{Z_p + Z_s}$$

(b) A linear system with the input f(t) and output y(t) is described by ODE

$$3\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y(t) = 2\frac{df}{dt} + f(t)$$

Determine the frequency response $H(\omega)=\dfrac{Y(\omega)}{F(\omega)}$ of the system.

$$\frac{d}{dt} \to j\omega \quad \Rightarrow \quad 3(j\omega)^2 Y + 2(j\omega)Y + Y = 2(j\omega)F + F$$

$$Y[-3\omega^2 + 2j\omega + 1] = F(2j\omega + 1)$$

$$\frac{Y}{F} = \frac{1+j2\omega}{1-3\omega^2+j2\omega}$$

$$H(\omega) = \frac{1+j2\omega}{1-3\omega^2+j2\omega}$$

(c) Given an input $f(t) = 2\cos\omega_1 t + \sin(\omega_2 t + \theta_2)$ and $H(\omega)=\dfrac{1+j\omega}{2+j\omega}$ determine the steady-state response $y_{ss}(t)$ of the system $H(\omega)$.

$$|H(\omega)| = \sqrt{\frac{1^2+\omega^2}{2^2+\omega^2}} = \sqrt{\frac{1+\omega^2}{4+\omega^2}}$$

$$A\cos(\omega t + \theta) \to \boxed{H(\omega)} \to A|H(\omega)|\cos\left(\omega t + \theta + \measuredangle H(\omega)\right)$$

$$\measuredangle H(\omega) = \tan^{-1}\left(\frac{\omega}{1}\right) - \tan^{-1}\left(\frac{\omega}{2}\right)$$

$$y_{ss}(t) = 2\sqrt{\frac{1+\omega_1^2}{4+\omega_2^2}}\cos\left(\omega_1 t + \tan^{-1}(\omega_1) - \tan^{-1}\left(\frac{\omega_1}{2}\right)\right) + \sqrt{\frac{1+\omega_2^2}{4+\omega_2^2}}\sin\left(\omega_2 t + \theta_2 + \tan^{-1}(\omega_2) - \tan^{-1}\left(\frac{\omega_2}{2}\right)\right)$$