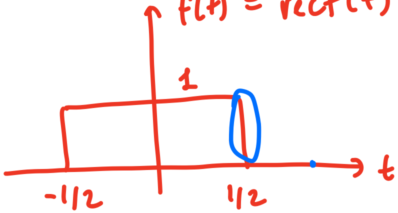
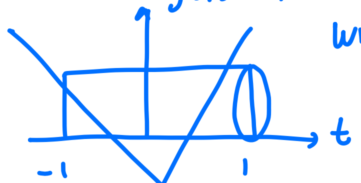
- Time scaling and time shifting of functions

$f(t) = \mathrm{rect}(t)$



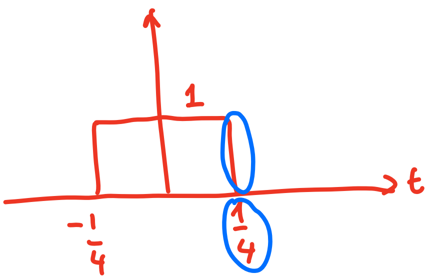
$g(t) = f(2t)$

wrong way !



$g(1) = f(2\cdot(1)) = f(2)$

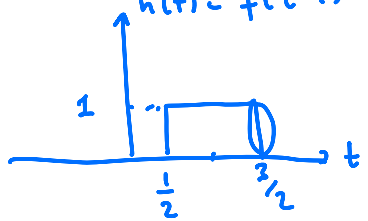
$g(t) = f(2t)$



$g\left(\frac{1}{4}\right) = f\left(2\cdot\left(\frac{1}{4}\right)\right) = f\left(\frac{1}{2}\right)$

shifting :

$h(t) = f(t-1)$

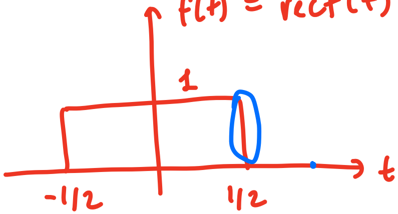


$h\left(\frac{3}{2}\right) = f\left(\frac{3}{2}-1\right) = f\left(\frac{1}{2}\right)$

- Time scaling and time shifting simultaneously

Always do time scaling first, then shift relative to that time scaling.

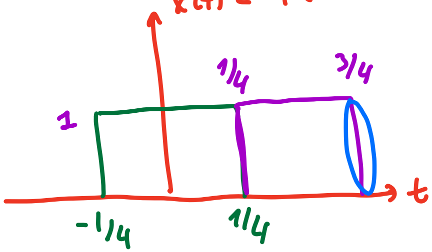
$f(t) = \mathrm{rect}(t)$



① ② time scaling

③ shift

$$x(t) = f(2t-1) = f\left(2\left(t-\frac{1}{2}\right)\right)$$



$$x\left(\frac{3}{4}\right) = f\left(2\cdot\left(\frac{3}{4}\right)-1\right) = f\left(\frac{1}{2}\right)$$

$$y(t) = f(-t+3) = f(-(t-3))$$

- Parseval's theorem
  - Define the energy content of *f(t)* as

$$W = \int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

→ energy spectrum

total signal energy

energy signals – signals with finite energy

- Parseval's theorem - Example # 7
  - ~~Define~~ Determine the energy content of $f(t) = 10e^{-t}u(t)$

$$W = \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{0}^{\infty} (10e^{-t})^2 dt =$$

$$= 100 \int_{0}^{\infty} e^{-2t} dt = 100 \cdot \frac{e^{-2t}}{-2} \Big|_{0}^{\infty} = -50(0-1) = 50$$

$$W = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{100}{1^2+\omega^2} d\omega =$$

$$= \frac{50}{\pi} \int_{-\infty}^{\infty} \frac{1}{1+\omega^2} d\omega = \frac{50}{\pi} \tan^{-1}(\omega) \Big|_{-\infty}^{\infty} =$$

$$= \frac{50}{\pi}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 50$$

Table 7.2.

$$e^{-at}u(t) \leftrightarrow \frac{1}{a+j\omega}, \quad a>0$$

$$10e^{-t}u(t) \leftrightarrow \frac{1 \cdot 10}{1+j\omega}$$

describes the width of the energy spectrum

- Energy bandwidth - Low-pass signals
  - 3dB bandwidth

positive frequency

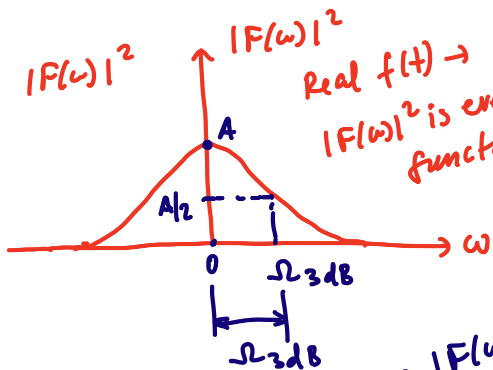
$\omega = \Omega = 2\pi \cdot B$

[rad/s] [Hz]

the energy is concentrated at low frequencies

beyond which energy spectrum $|F(\omega)|^2$ is very small.

What is "small"?



Real $f(t) \rightarrow |F(\omega)|^2$ is even function

$$\frac{|F(\Omega_{3dB})|^2}{|F(0)|^2} = \frac{1}{2}$$

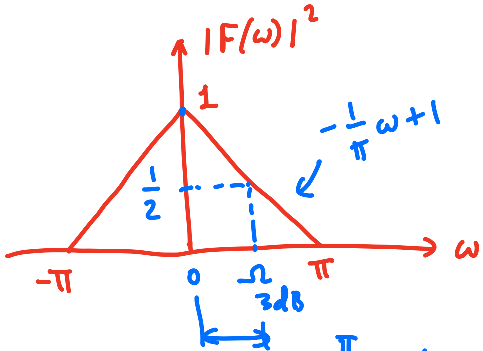
where $|F(\omega)|^2$ fall to 1/2 of $|F(0)|^2$ at DC

Why "3dB" name?

$$10\log_{10}\left(\frac{|F(\Omega_{3dB})|^2}{|F(0)|^2}\right) = -3dB$$

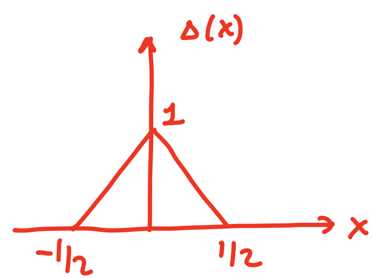
- Energy bandwidth - Low-pass signals - Example # 8
  - Determine the 3dB bandwidth of the signal with energy spectrum

$$|F(\omega)|^2 = \Delta(\frac{\omega}{2\pi})$$



$\Omega_{3dB} = \frac{\pi}{2}$ rad/s

unit triangle



$$\frac{|F(\omega)|^2}{|F(0)|^2} = \boxed{\frac{1}{2} = \frac{-\frac{1}{\pi}\omega + 1}{1}}$$

$\Downarrow$

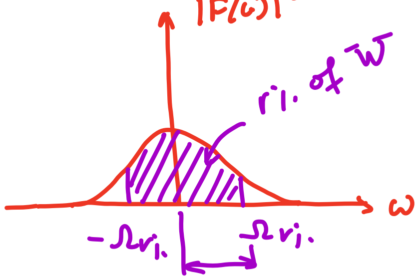
solve for ω :

$$\omega = \Omega_{3dB} = \frac{\pi}{2} \text{ rad/s}$$

- Energy bandwidth - Low-pass signals-cont
  - r% bandwidth

define $\Omega_{r\%}$ as freq. below which a certain fraction of total energy is concentrated

specific % of overal energy



① Total energy:

$$\bar{W} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

②

$$\frac{r}{100}\cdot\bar{W} = \frac{1}{2\pi}\int_{-\Omega_{r\%}}^{\Omega_{r\%}} |F(\omega)|^2 d\omega$$

or by symmetry, just do one side:

$$\frac{r}{100}\cdot\frac{\bar{W}}{2} = \frac{1}{2\pi}\int_{0}^{\Omega_{r\%}} |F(\omega)|^2 d\omega$$

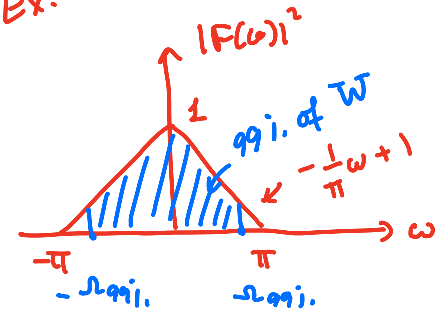
$\Downarrow$

solve for $\Omega_{r\%}$.

- Energy bandwidth - Low-pass signals - Example # 9
  - Determine the 99% bandwidth of the signal with energy spectrum

$$|F(\omega)|^2 = \Delta(\frac{\omega}{2\pi})$$

Ex. 7.13



① $\overline{W} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega =$

$= \frac{1}{2\pi}(\text{area of } \triangle) = \frac{1}{2\pi}\cdot\frac{(2\pi\cdot 1)}{2} = \frac{1}{2}$

② $\frac{99}{100}\cdot\frac{\overline{W}}{2} = \frac{1}{2\pi}\int_{0}^{\Omega_{r\%}} |F(\omega)|^2 d\omega =$

$= \frac{1}{2\pi}\int_{0}^{\Omega_{r\%}} \left(-\frac{1}{\pi}\omega + 1\right) d\omega$

$\Omega_{r\%} = \frac{9\pi}{10}$ or $\frac{11\pi}{10}$ rad/s