ECE 210 (AL2)

Chapter 10

Impulse Response, Stability, Causality, and LTIC Systems

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Chapter objectives

- Understand the meaning of an LTI system's impulse response and its relation to the frequency response
- Understand and test for BIBO stability
- Understand and test for causality of systems and signals

• Convolution and impulse response

- Recall that

$$F(\omega) \rightarrow \boxed{\text{LTI with } H(\omega)} \rightarrow Y(\omega) = F(\omega)H(\omega)$$

- What is $F(\omega)$ or $H(\omega)$ doesn't exist?

$$\updownarrow$$

$$f(t) \rightarrow \boxed{\text{LTI with } h(t)} \rightarrow y(t) = f(t) * h(t)$$

- Q: How to get h(t) if we do not know it?
  - Recall that

$$\delta(t) \rightarrow \boxed{\text{LTI with } h(t)} \rightarrow y(t) = \delta(t) * h(t) = h(t)$$

h(t) is the impulse response

- Convolution and unit-step response

  - Consider

$$u(t) \longrightarrow \boxed{\text{LTI with h(t)}} \longrightarrow \underbrace{y(t) = u(t) * h(t)}$$

y(t) is the *unit-step* response

$$\frac{dy(t)}{dt} = \left(\frac{d}{dt}u(t)\right) * h(t) =$$

$$= \delta(t) * h(t) = h(t)$$

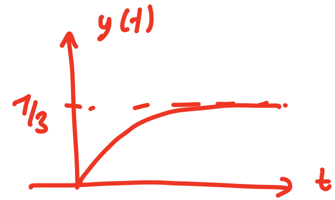
$$\boxed{\frac{dy(t)}{dt} = h(t)}$$

  - Q: How to get h(t) from y(t) ?

$$y(t) = f(t) * h(t)$$

$$\frac{dy(t)}{dt} = \frac{d}{dt}f(t) * h(t) = f(t) * \frac{d}{dt}h(t)$$

• Unit-step response - Example # 1



- Let the unit-step response of an LTI system be

$$y(t) = \frac{1}{3}(1 - e^{-3t})u(t)$$

- Determine h(t) :

$$u(t) \rightarrow \boxed{h(t)} \rightarrow y(t) = \frac{1}{3}\left(1 - e^{-3t}\right)u(t)$$

@t=0

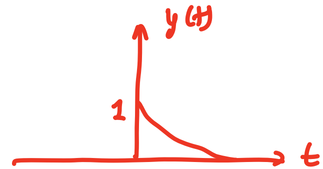
$$h(t) = \frac{dy(t)}{dt} = \frac{1}{3}\left(\underbrace{(1 - e^{-3t})\,\delta(t)}_{\text{sampling}} + 3e^{-3t}u(t)\right) =$$

$$= \frac{1}{3}\left(\underbrace{\underbrace{(1 - e^{-3(0)})}_{0}\,\delta(t)}_{0} + 3e^{-3t}u(t)\right) = e^{-3t}u(t)$$

- # Unit-step response - Example # 2

  - Let the unit-step response of an LTI system be

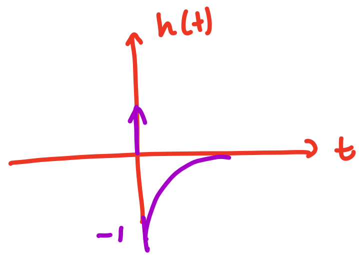
$$y(t) = e^{-t}u(t)$$



  - Determine h(t) :

$$u(t) \rightarrow \boxed{h(t)} \rightarrow y(t) = e^{-t}u(t)$$

$$h(t) = \frac{dy(t)}{dt} = \underbrace{e^{-t}\delta(t)} - e^{-t}u(t) = \underbrace{e^{-0}}_{1}\delta(t) - e^{-t}u(t)$$



- Bounded input-bounded output (BIBO) stability
  - Consider

general

$f(t) \rightarrow$ system $\rightarrow y(t)$

  - In a bounded input-bounded output stable system

bounded $\rightarrow$ system $\rightarrow$ bounded

for any bounded f(t) $\quad |f(t)| \le C_1 \Rightarrow |y(t)| \le C_2$

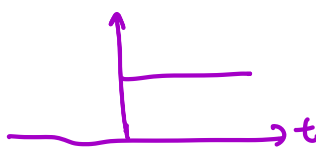
Note: BIBO doesn't care what happens to unbounded inputs

$u(t) \longrightarrow \boxed{h(t)=u(t)} \longrightarrow y(t) = u(t) * u(t) =$

LTI

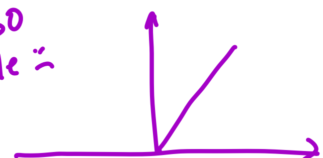
$= t\,u(t)$

$\Downarrow$ system is not BIBO stable :(



bounded

$|u(t)| \le 1$



unbounded

- BIBO stability and LTI systems
  - If the system is LTI, then it is BIBO if and only if its impulse response is absolutely integrable.

$$\text{BIBO stable} \iff \int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

Assuming $h(t)$ is a function. If not, use original definition

check the system, not the input!

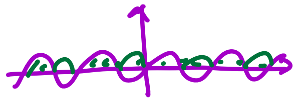
Note: do not check boundness of $h(t)$, just A.I.!

- BIBO stability and LTI systems - Example # 3

  - Determine which of the following impulse responses correspond to BIBO stable systems:

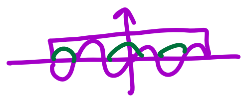
Check $\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$

X not BIBO stable 1. $h(t) = \sin(\omega_0 t)$



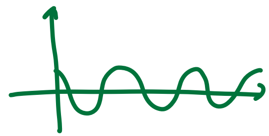
2. $h(t) = \sin(\omega_0 t) rect(t)$

V ☺ BIBO stable



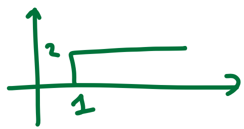
3. $h(t) = \cos(\omega_0 t)$u(t)

X BIBO stable



4. $h(t) = 2u(t-1)$

X BIBO stable



5. $h(t) = \delta(t-1)$

V BIBO stable

$$f(t) \rightarrow \boxed{h(t)} \rightarrow y(t) = f(t) * \delta(t-1) = f(t-1)$$

($h(t) = \delta(t-1)$)

$$|f(t)| \le c_1 \Rightarrow |f(t-1)| \le c_1$$

- Causality and LTIC systems
  - Consider

general

$$f(t) \rightarrow \boxed{\text{system}} \rightarrow y(t)$$

- If the output y(t) does not depend on the future of the input f(t), then the system is *causal.*

This has to be true for any input f(t)

- If the output y(t) does depend on the future of the input f(t), then the system is *non-causal* (unrealizable).

- Causality and LTI systems
  - If the system is LTI, then it is causal if and only if its impulse response has the following property:

causal $\Longleftrightarrow$ $h(t) = 0$ for $t < 0$

Note: only if $h(t)$ is a function. If not, use original definition.

$$y(t) = \int_{-\infty}^{\infty} h(\tau) f(t-\tau)\, d\tau$$

t_1

$$y(t_1) = \int_{-\infty}^{\infty} \underbrace{h(\tau)}_{\substack{= \\ 0 \\ \text{for} \\ \tau < 0}} f(t_1-\tau)\, d\tau$$

$\underline{\tau > 0}$: past values of the input

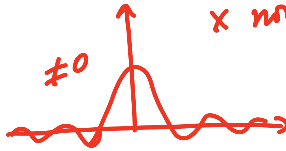
$\underline{\tau = 0}$: present values of the input

$\underline{\tau < 0}$: future values of the input

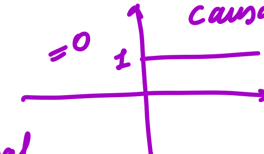
• Causality and LTI systems - Example # 4

- Determine which of the following impulse responses correspond to causal systems:

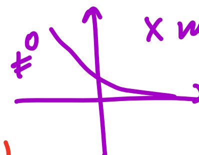
1. $h(t) = \text{sinc}(\omega_0 t)$
2. $h(t) = u(t)$
3. $h(t) = e^{-t}$
4. $h(t) = e^{-t}u(t-1)$
5. $h(t) = \delta(t)$



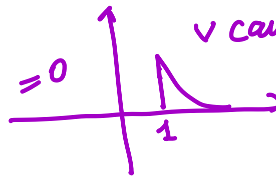
$\neq 0$ x non-causal



$= 0$ causal ✓



$\neq 0$ x non-causal



$= 0$ ✓ causal

$y(t) = f(t-1)$ ↑ past ⇒ causal

$\delta(t-1)$

$f(t) \rightarrow \boxed{\delta(t)} \rightarrow y(t) = f(t) * \delta(t) = f(t)$

$\delta(t+1)$

same t, present ⇒ causal

$y(t) = f(t+1)$

↑ future ⇒ non-causal

- Causality

  - An LTIC system is
    - Linear
    - Time-invariant
    - Causal

  - A signal f(t) is causal if it could be an LTIC impulse response

$f(t) = 0 \quad t < 0$

$\delta(t)$, $u(t-1)$ — causal signals

$u(t+1)$, $\delta(t+2)$ — non-causal signals

• LTIC systems - Example # 5

- For each of the following systems, determine if they are linear, time-invariant, BIBO stable and/or causal:

  Z-S

1. $y(t) = \sin(t+3)f(t)$ ← Yes ✓.

A

① Z-S linear :

② T.I :

$f(t) \rightarrow \square \rightarrow y(t)$

$y(t) = \sin(t+3) f(t)$ ↙ f only changes this t

y changes both ts ⇒ not the same shift ⇒ not T.I. ✗

$f_1(t) = f(t-t_d) \rightarrow \square \rightarrow y_1(t) = y(t-t_d)$

③ BIBO : $y(t) = \underbrace{\sin(t+3)}_{|\sin(t+3)| \le 1} \underbrace{f(t)}_{|f(t)| \le C}$ ⇒ $|y(t)| \le C$ ⇒ BIBO stable ∴ ✓

④ Causal : $y(t) = \sin(t+3) f(t)$ same t, present ⇒ causal ✓

• LTIC systems - Example # 5-cont

- For each of the following systems, determine if they are linear, time-invariant, BIBO stable and/or causal:

2. $y(t) = f((t-1)^2)$

① Z-S linear: ✓

② T.I: $y(t) = f((t-1)^2)$

$f(t) \to \square \to y(t) = f((t-1)^2)$

$f_1(t) \to \square \to y_1(t) \overset{?}{=} y(t-t_d)$

$f_1(t) = f(t-t_d)$

$f_1((t-1)^2) = f((t-1)^2 - t_d)$

$y(t-t_d) = f((t-t_d-1)^2)$

not T.I. ✗

③ BIBO:

$y(t) = f((t-1)^2)$

amplitude is not affected $\Rightarrow$ BIBO stable ✓

④ Causal:

$y(-3) = f((-3-1)^2)$

$y(-3) = f(16)$

$16 > -3 \Rightarrow$ future of $f(t) \Rightarrow$ non-causal ✗

• LTIC systems - Example # 5-cont

- For each of the following systems, determine if they are linear, time-invariant, BIBO stable and/or causal:

3. $y(t) = f^2(t)$

Is linear: ↑ as a function of f, this is not a line ⇒ X

T.I.: $y(t) = f^2(t)$

same time ⇒ T.I. V

BIBO stable: $y(t) = f^2(t)$

$|f(t)| \leq c \Rightarrow |f^2(t)| \leq c^2 \Rightarrow$ V

Causal: $y(t) = f^2(t)$

same t, present ⇒ causal V

• LTIC systems - Example # 5-cont

- For each of the following systems, determine if they are linear, time-invariant, BIBO stable and/or causal:

$$f(t) \to \underbrace{\boxed{LTI}}_{h(t)} \to y(t) = f(t) * h(t)$$

4. $y(t) = f(t) * \underbrace{u(t-1)}_{h(t)}$

$\Rightarrow$ Linear + T.I. ✓

$f^2(t)$

↖ it breaks linearity

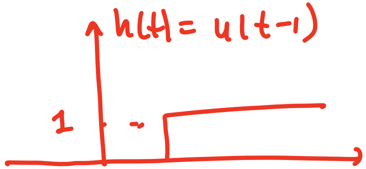
$f(t^2)$

↖ it breaks T.I.

BIBO stable : if LTI

BIBO $\Leftrightarrow \int_{-\infty}^{\infty} |h(t)|\,dt < \infty$ A.I.

$h(t) = u(t-1)$



$\Rightarrow$ not A.I. $\Rightarrow$ ✗

causal : if LTI :

causal $\Leftrightarrow$ $h(t) = 0 \quad t < 0$ ✓

Chapter objectives

- Understand the meaning of an LTI system's impulse response and its relation to the frequency response
- Understand and test for BIBO stability
- Understand and test for causality of systems and signals