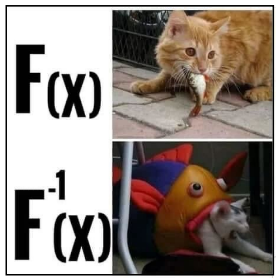
ECE 210 (AL2)

Chapter 7

Fourier Transform and LTI System Response to Energy Signals



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Spring 2022

Chapter objectives

- Understand the significance and interpretation of Fourier transform and the inverse Fourier transform
- Apply properties of Fourier transform to determine effect of basic signal processing
- Be able to calculate the energy of a signal both in time and frequency
- Be able to determine energy bandwidth of a signal
- Understand the effect of LTI systems, via $H(\omega)$, on signals via their Fourier transform

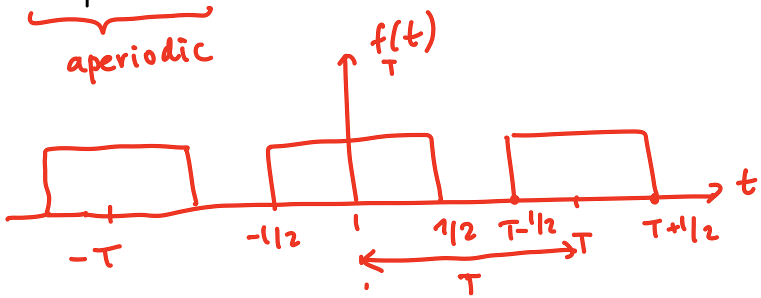
- Recall that in an LTI system, the output to a periodic input is also periodic:

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \rightarrow \overset{H(\omega)}{\boxed{\text{LTI}}} \rightarrow y(t) = \sum_{n=-\infty}^{\infty} H(n\omega_0) F_n e^{jn\omega_0 t}$$

where

$$\boxed{F_n = \frac{1}{T}\int_T f(t)e^{-jn\omega_0 t}dt}$$

- What if f(t) is $\underbrace{\text{not periodic?}}_{\text{aperiodic}}$



Let $T \to \infty$

- Fourier transform

$$f_T(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} \Big(\underbrace{\frac{1}{T} \int_{-T/2}^{T/2} f(s) e^{-jn\omega_0 s} ds}_{F_n} \Big) e^{jn\omega_0 t} =$$

$$T = \frac{2\pi}{\omega_0}$$

$$= \sum_{n=-\infty}^{\infty} \frac{\omega_0}{2\pi} \int_{-\pi/\omega_0}^{\pi/\omega_0} f(s) e^{-jn\omega_0 s} ds \cdot e^{jn\omega_0 t} =$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \Big(\underbrace{\int_{-\pi/\omega_0}^{\pi/\omega_0} f(s) e^{-jn\omega_0 s} ds}_{\text{define as } F(n\omega_0)} \Big) e^{jn\omega_0 t} \cdot \omega_0 =$$

$$\int_{-\infty}^{\infty} f(t)\, dt = \lim_{\Delta t \to 0} \sum_{n=-\infty}^{\infty} f(n\Delta t) \Delta t$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} F(n\omega_0) e^{jn\omega_0 t} \cdot \omega_0 \xrightarrow{\omega_0 \to 0}$$

$$\rightarrow \boxed{\frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega = f(t)}$$

- Fourier transform pairs

frequency domain

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \in \mathbb{C}$$

Fourier transform

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

time domain

inverse Fourier transform

$$f(t) \longleftrightarrow F(\omega)$$

unique transform pair

$Y(\omega) = F(\omega) H(\omega)$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \rightarrow \boxed{\text{LTI}}_{H(\omega)} \rightarrow y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) F(\omega) e^{j\omega t} d\omega$$

- # Existence of Fourier transform

  - Can we represent any function f(t) in terms of a Fourier transform?

Yes, if f(t) is absolutely integrable (A.I.):

$$\int_{-\infty}^{\infty} |f(t)|dt < \infty$$ has to be finite

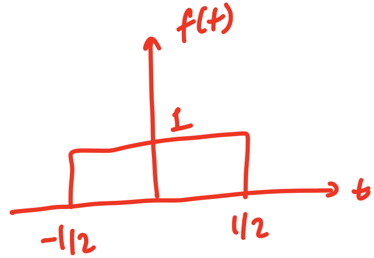
A.I → F.T. exists for sure

Also, some functions that are not A.I. do have a Fourier transform.

- Fourier transform - Example # 1

  - Determine the Fourier transform of

$$f(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & \text{else} \end{cases} \quad := rect(t)$$



$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt =$$

$$= \int_{-1/2}^{1/2} (1)\, e^{-j\omega t}\, dt = \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-1/2}^{1/2} =$$

$$= \frac{e^{-j\omega \frac{1}{2}} - e^{-j\omega \cdot (-\frac{1}{2})}}{-j\omega} = \frac{2\left(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right)}{2\, j\omega} = \frac{2\sin\left(\frac{\omega}{2}\right)}{\omega} = \frac{\sin\left(\frac{\omega}{2}\right)}{\omega/2} = \operatorname{sinc}\left(\frac{\omega}{2}\right)$$

$$\operatorname{sinc}(x) = \frac{\sin(x)}{x}$$

$$rect(t) \Leftrightarrow \operatorname{sinc}\left(\frac{\omega}{2}\right)$$

• Fourier transform - Example # 1-cont

- Determine the Fourier transform of

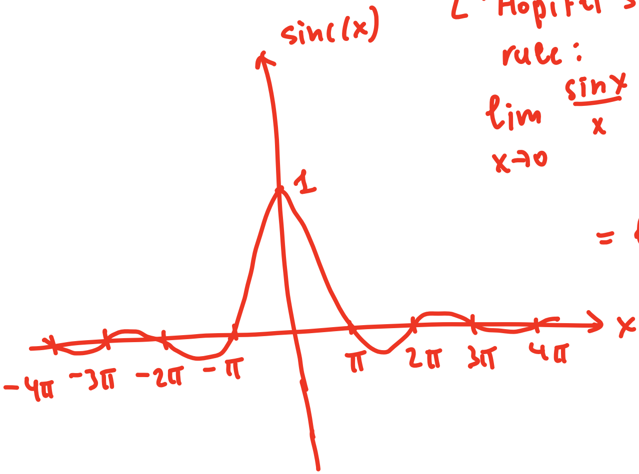
$$f(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & \text{else} \end{cases} \quad := rect(t)$$

$$\text{sinc}(x) = \frac{\sin(x)}{x}$$

zeroes if $\sin(x) = 0 \Rightarrow$

$x = n\pi$

$n \neq 0$



L'Hopital's rule:

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\frac{d}{dx}\sin x}{\frac{d}{dx}x} =$$

$$= \lim_{x \to 0} \frac{\cos x}{1} = 1$$

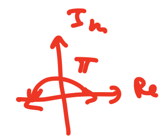
$F(\omega) = \text{sinc}\left(\frac{\omega}{2}\right)$

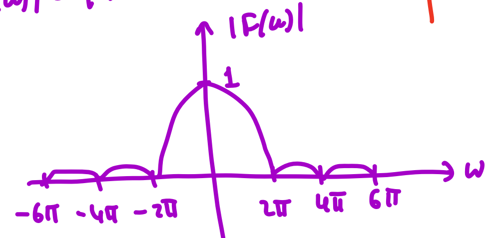
$\sin\left(\frac{\omega}{2}\right) = 0$

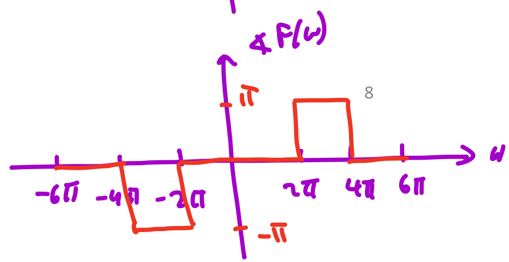
$\frac{\omega}{2} = n\pi$

$\omega = n2\pi$

$|F(\omega)| = \left|\text{sinc}\left(\frac{\omega}{2}\right)\right|$



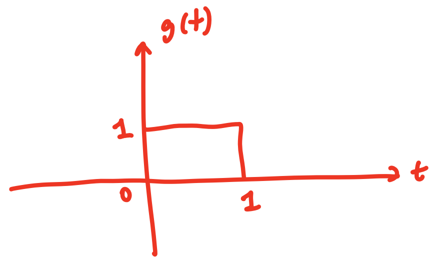


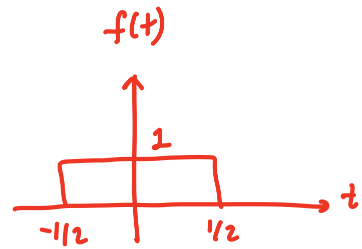


- # Fourier transform - Example # 2

  - Determine the Fourier transform of

$$\overset{g(t)}{\cancel{f(t)}} = \begin{cases} 1 & 0 < t < 1 \\ 0 & \text{else} \end{cases}$$





$F(\omega) = \text{sinc}\left(\frac{\omega}{2}\right)$

$g(t) = f\left(t - \frac{1}{2}\right) \rightarrow$

$$G(\omega) = F(\omega)\, e^{j\omega\left(-\frac{1}{2}\right)} =$$

$$= F(\omega)\, e^{-j\frac{\omega}{2}} = \text{sinc}\left(\frac{\omega}{2}\right) e^{-j\frac{\omega}{2}}$$

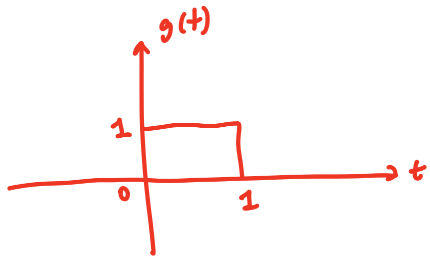
Time shift property:

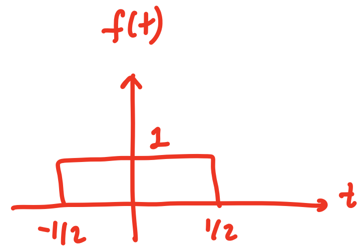
$f(t) \leftrightarrow F(\omega)$

$g(t) = f(t - t_0) \leftrightarrow G(\omega) = F(\omega)\, e^{j\omega(-t_0)}$

- Fourier transform - Example # 2-cont
  - Determine the Fourier transform of

$$f(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & \text{else} \end{cases}$$





$F(\omega) = \text{sinc}\left(\frac{\omega}{2}\right)$

$g(t) \leftrightarrow G(\omega) = F(\omega)\, e^{-j\frac{\omega}{2}}$

$|G(\omega)| = |F(\omega)| = \left|\text{sinc}\left(\frac{\omega}{2}\right)\right|$

$\measuredangle G(\omega) = \measuredangle F(\omega) + \measuredangle e^{-j\frac{\omega}{2}} = \measuredangle F(\omega) - \frac{\omega}{2}$