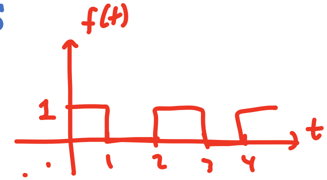
- Properties of exponential Fourier series coefficients

  - Recall f(t) with period T = 2s, given by

$$f(t) = \begin{cases} 1 & t \in [0,1) \\ 0 & t \in [1,2) \end{cases} = \frac{1}{2} + \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \frac{1}{jn\pi} e^{jn\pi t}$$

$\omega_0 = \frac{2\pi}{T} = \pi$ rad/s

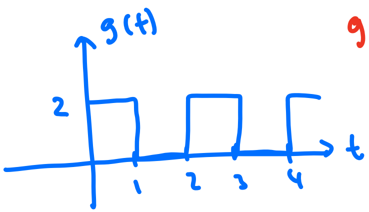
with $F_n = \begin{cases} \frac{1}{jn\pi} & n \text{ odd} \\ \frac{1}{2} & n = 0 \\ 0 & n \text{ even}, n \neq 0 \end{cases}$



  - Consider now

$$g(t) = \begin{cases} 2 & t \in [0,1) \\ 0 & t \in [1,2) \end{cases}$$ with period T = 2s , $\omega_0 = \frac{2\pi}{T} = \pi$ rad/s

What are the exponential Fourier series coefficients G_n?



$g(t) = 2f(t)$

$$2f(t) = 2\sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} 2F_n e^{jn\omega_0 t}$$

$$g(t) = \sum_{n=-\infty}^{\infty} G_n e^{jn\omega_0 t}$$

$$\Rightarrow G_n = 2 \cdot F_n = \begin{cases} \frac{2}{jn\pi} & n \text{ odd} \\ 1 & n = 0 \\ 0 & n \text{ even}, n \neq 0 \end{cases}$$

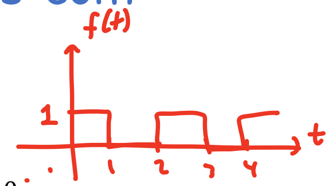
Amplitude scaling property:

$$K f(t) \Leftrightarrow K \cdot F_n$$

• Properties of exponential Fourier series coefficients-cont

- Recall f(t) with period T = 2s, given by

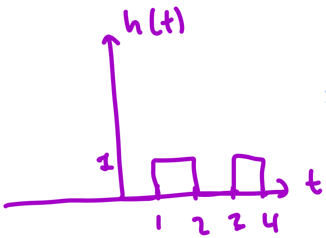
$$f(t) = \begin{cases} 1 & t \in [0,1) \\ 0 & t \in [1,2) \end{cases} = \frac{1}{2} + \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \frac{1}{jn\pi} e^{jn\pi t} \qquad \text{with } F_n = \begin{cases} \frac{1}{jn\pi} & n \text{ odd} \\ \frac{1}{2} & n = 0 \\ 0 & n \text{ even}, n \neq 0 \end{cases}$$



- Consider now

$$h(t) = \begin{cases} 0 & t \in [0,1) \\ 1 & t \in [1,2) \end{cases} \qquad \text{with period } T = 2s \Rightarrow \omega_0 = \frac{2\pi}{T} = \pi \text{ rad/s}$$

What are the exponential Fourier series coefficients H_n?



$$h(t) = f(t-1) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0(t-1)} = \sum_{n=-\infty}^{\infty} \underbrace{F_n e^{-jn\omega_0}}_{H_n} \cdot e^{jn\omega_0 t}$$

$$h(t) = f(t-t_0) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0(t-t_0)} = \sum_{n=-\infty}^{\infty} F_n \cdot e^{jn\omega_0(-t_0)} e^{jn\omega_0 t}$$

$$\Rightarrow \boxed{H_n = F_n e^{-jn\omega_0 t_0}}$$

$$h(t) = \sum_{n=-\infty}^{\infty} H_n e^{jn\omega_0 t}$$

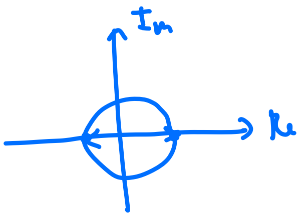
- Properties of exponential Fourier series coefficients-cont

$$h(t) = f(t-1) \Rightarrow H_n = F_n e^{-jn\omega_0} = F_n \cdot e^{-jn\pi}$$

- Time-shift property:

$$h(t) = f(t - t_0) \qquad H_n = F_n e^{-jn\omega_0 t_0}$$

$$H_n = \begin{cases} \frac{1}{jn\pi} \cdot (-1) = -\frac{1}{jn\pi} & n \text{ odd} \\ \frac{1}{2} \cdot (1) = \frac{1}{2} & n = 0 \\ 0 & n \text{ even}, n \neq 0 \end{cases}$$



n even $\quad e^{-jn\pi} = 1$

n odd $\quad e^{-jn\pi} = -1$

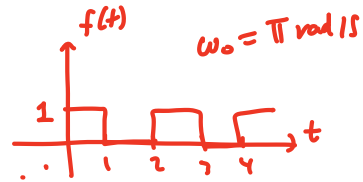
$$F_n = \begin{cases} \frac{1}{jn\pi} & n \text{ odd} \\ \frac{1}{2} & n = 0 \\ 0 & n \text{ even}, n \neq 0 \end{cases}$$

$$h(t) = \frac{1}{2} + \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \left(-\frac{1}{jn\pi}\right) e^{jn\pi t}$$

• Properties of exponential Fourier series coefficients-cont

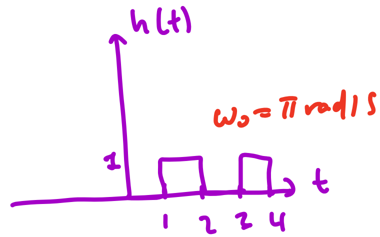
- Addition property: If f(t) and h(t) have the same ω_0, then

$$x(t) = f(t) + h(t) \qquad\qquad X_n = F_n + H_n$$

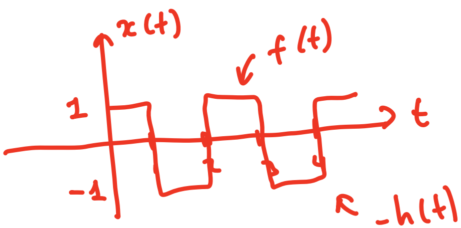


- Let

$$x(t) = \begin{cases} 1 & t \in [0,1) \\ -1 & t \in [1,2) \end{cases} \qquad \text{with period } T = 2s$$



What are the exponential Fourier series coefficients X_n?



$$x(t) = f(t) - h(t)$$

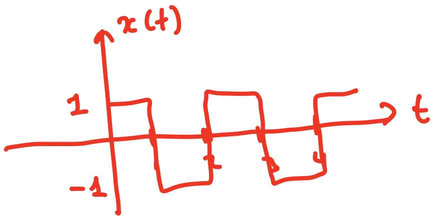
$$X_n = F_n - H_n = \begin{cases} \frac{1}{jn\pi} - \left(-\frac{1}{jn\pi}\right) = \frac{2}{jn\pi} & n \text{ odd} \\ \frac{1}{2} - \frac{1}{2} = 0 & n = 0 \\ 0 - 0 = 0 & n \text{ even}, \ n \neq 0 \end{cases}$$

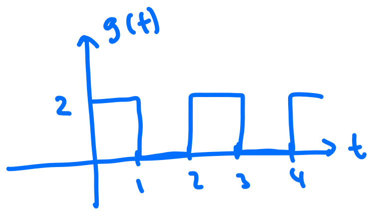
• Properties of exponential Fourier series coefficients-cont

- Another way to get X_n by recalling:

$$g(t) = \begin{cases} 2 & t \in [0,1) \\ 0 & t \in [1,2) \end{cases} \qquad G_n = \begin{cases} \frac{2}{jn\pi} & n \text{ odd} \\ 1 & n = 0 \\ 0 & n \text{ even}, n \neq 0 \end{cases}$$

$$x(t) = \begin{cases} 1 & t \in [0,1) \\ -1 & t \in [1,2) \end{cases}$$ with period $T = 2s$





$$x(t) = g(t) - 1 \Rightarrow X_n = \begin{cases} G_n - 1 & n = 0 \\ G_n - 0 & \text{else} \end{cases} =$$

only DC term is affected

$$= \begin{cases} \frac{2}{jn\pi} - 0 = \frac{2}{jn\pi} & n \text{ odd} \\ 1 - 1 = 0 & n = 0 \\ 0 - 0 = 0 & n \text{ even}, n \neq 0 \end{cases}$$

- Properties of exponential Fourier series coefficients-cont

	Name:	Condition:	Property:
1	√ Scaling	Constant K	$K\,f(t) \leftrightarrow K\,F_n$
2	√ Addition	$f(t) \leftrightarrow F_n,\ g(t) \leftrightarrow G_n,\ \ldots$	$f(t) + g(t) + \ldots \leftrightarrow F_n + G_n + \ldots$
3	√ Time shift	Delay t_o	$f\left(t - t_o\right) \leftrightarrow F_n e^{-jn\omega_o t_o}$
4	Derivative	Continuous $f(t)$	$\frac{df}{dt} \leftrightarrow jn\omega_o F_n$
5	√ Hermitian	Real $f(t)$	$F_{-n} = F_n^*$
6	Even function	$f(-t) = f(t)$	$f(t) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n\omega_o t\right)$
7	Odd function	$f(-t) = -f(t)$	$f(t) = \sum_{n=1}^{\infty} b_n \sin\left(n\omega_o t\right)$
8	Average power		$P \equiv \frac{1}{T}\int_T \lvert f(t)\rvert^2\,dt = \sum_{n=-\infty}^{\infty} \lvert F_n\rvert^2$

Table 2: Fourier series properties

Table 6.3

- LTI system response to periodic inputs
  - Recall that in an LTI system

$H(\omega)$

$$e^{j\omega t} \rightarrow \boxed{\text{LTI}} \rightarrow H(\omega)e^{j\omega t}$$

$$\cos(\omega t + \theta) \rightarrow \boxed{\text{LTI}} \rightarrow |H(\omega)|\cos(\omega t + \theta + \angle H(\omega))$$

so that

$$\text{f(t)} = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \rightarrow \boxed{\text{LTI}} \rightarrow \text{y(t)} = \sum_{n=-\infty}^{\infty} H(n\omega_0)F_n e^{jn\omega_0 t}$$

or

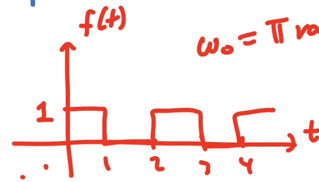
like phasors : $F_n \rightarrow \boxed{H(\omega)} \rightarrow H(n\omega_0)F_n$

$$\text{f(t)} = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t + \theta) \rightarrow \boxed{\text{LTI}} \rightarrow \text{y(t)} = \text{H}(0)\frac{c_0}{2} + \sum_{n=1}^{\infty} |H(n\omega_0)| c_n \cos(n\omega_0 t + \theta_n + \angle H(n\omega_0))$$

- LTI system response to periodic inputs-example #7

- Recall the periodic function f(t) with period T = 2s

$$f(t) = \begin{cases} 1 & t \in [0,1) \\ 0 & t \in [1,2) \end{cases} = \frac{1}{2} + \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \frac{1}{jn\pi} e^{jn\pi t}$$



$\omega_0 = \pi$ rad/s

- Let f(t) be the input to an LTI system with frequency response

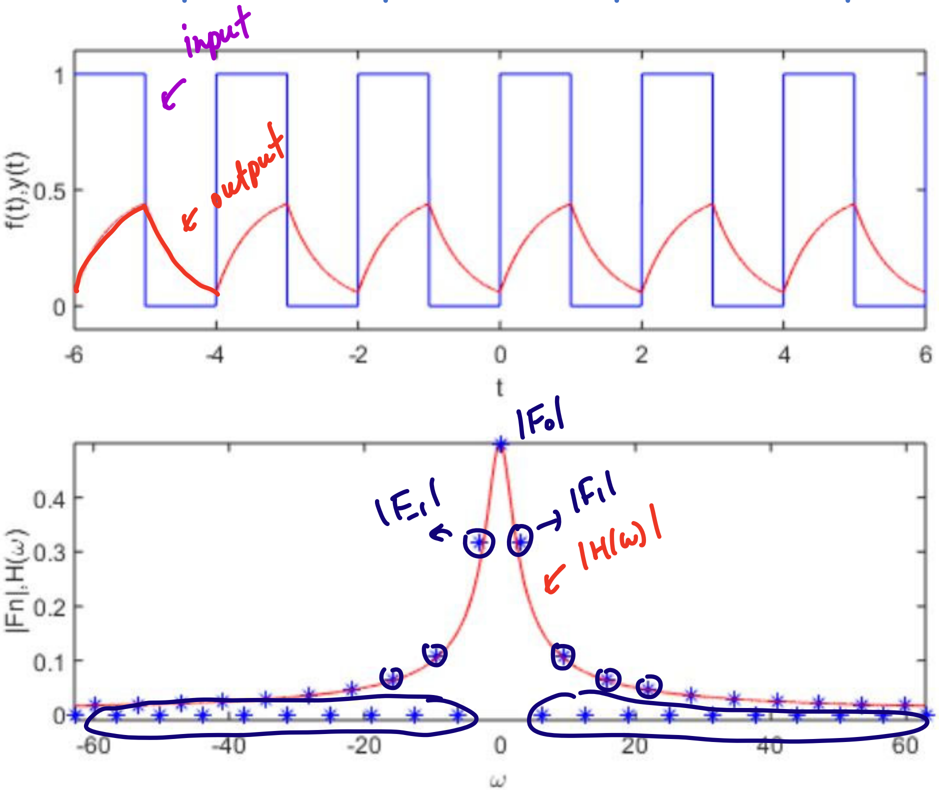
$$H(\omega) = \frac{1}{2 + j\omega} \quad \omega \to n\pi \qquad |H(\omega)| = \frac{1}{\sqrt{4 + \omega^2}} \qquad \angle H(\omega) = -\tan^{-1}\left(\frac{\omega}{2}\right)$$

low-pass filter

- Determine the steady state output, y(t):

$$y(t) = \frac{1}{2} \cdot \underbrace{H(0)}_{\frac{1}{2}} + \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \left(\frac{1}{jn\pi}\right) \cdot H(\overset{n\omega_0}{n\pi})\, e^{jn\pi t} = \frac{1}{4} + \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \frac{1}{jn\pi} \cdot \left(\frac{1}{2+jn\pi}\right) e^{jn\pi t}$$

- LTI system response to periodic inputs-example #7-cont



$F_n \to \boxed{H(\omega)} \to$

$Y_n = F_n \cdot H(n\omega_0)$