- Inverse Laplace transform - Example # 14

  - Determine the inverse Laplace transform of

$$\hat{F}(s) = \frac{s}{s+1} = \frac{s+1-1}{s+1} = 1 - \frac{1}{s+1}$$

$$\downarrow \mathcal{L}^{-1}$$

$$f(t) = \delta(t) - e^{-t}u(t)$$

$$\frac{d}{dt}\delta(t) = \underbrace{\delta'(t)}_{\text{doublet}}$$

$$f(t) * \delta'(t) = f'(t)$$

$$\hat{G}(s) = \frac{s^2}{s+1} = s\left(\frac{s}{s+1}\right) = s\left(\frac{s+1-1}{s+1}\right) = s\left(1 - \frac{1}{s+1}\right)$$

$$\downarrow \mathcal{L}^{-1}$$

$$g(t) = \frac{d}{dt}\left(\delta(t) - e^{-t}u(t)\right)$$

- Inverse Laplace transform - improper rational functions
  - What if $\hat{F}(s)$ is not proper?
  - Case 3:

$$\hat{F}(s) = \frac{s^n + Q(s)}{s^n + P(s)} = \frac{s^n + P(s) - P(s) + Q(s)}{s^n + P(s)} =$$

with degree P(s),Q(s) < n

$$= 1 - \underbrace{\left(\frac{P(s) - Q(s)}{s^n + P(s)} \right)}_{\text{proper}}$$

$$\downarrow \mathcal{L}^{-1}$$

$$f(t) = \delta(t) - \mathcal{L}^{-1}\left\{ \frac{P(s) - Q(s)}{s^n + P(s)} \right\}$$

- # Inverse Laplace transform - Example # 15
  - Determine the inverse Laplace transform of

$$\hat{F}(s) = \frac{s^2+1}{s^2+3s+2} = \frac{s^2+3s+2-3s-2+1}{s^2+3s+2} =$$

$$= 1 - \frac{3s+1}{s^2+3s+2}$$

$$\downarrow \mathcal{L}^{-1}$$

$$f(t) = \delta(t) - \mathcal{L}^{-1}\left\{\frac{3s+1}{s^2+3s+2}\right\} =$$

$$= \delta(t) - \mathcal{L}^{-1}\left\{\frac{5}{s+2} - \frac{2}{s+1}\right\} =$$

$$= \delta(t) - \left(5e^{-2t}u(t) - 2e^{-t}u(t)\right)$$

- Inverse Laplace transform - improper rational functions
  - What if $\hat{F}(s)$ is not proper?

$f(t) \Leftrightarrow \hat{F}(s)$

$f(t-t_0) \Leftrightarrow \hat{F}(s)e^{-t_0 s}$

$t_0 \geq 0$

  - Case 4:

for $a>0$

$$\hat{F}(s) = e^{-as}\frac{N(s)}{D(s)}$$

$\downarrow$

time shift to the right by a

$$g(t) = \mathcal{L}^{-1}\left\{\frac{N(s)}{D(s)}\right\}$$

then time shift to the right by a

$$f(t) = g(t-a)$$

- # Inverse Laplace transform - Example # 16

  - Determine the inverse Laplace transform of

$$\hat{F}(s) = e^{-2s}\frac{1}{s+1} \quad \mathcal{L}^{-1}$$

time shift
to the right by 2

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}u(t)$$

$$f(t) = e^{-(t-2)}u(t-2)$$

- s-domain analysis of LTIC systems

  - Recall that if the input to an LTIC system is $\hat{F}(s)$ and the output is $\hat{Y}(s)$, then

$$\vec{F}(s) \rightarrow \boxed{\vec{H}(s)} \rightarrow \vec{Y}_{zs}(s) = \vec{F}(s)\vec{H}(s) \qquad \hat{H}(s) = \frac{\widehat{Y_{zs}}(s)}{\hat{F}(s)}$$

← transfer function
only works for z-s solution

  - It is know that the transfer functions of lumped element LTIC circuits are rational:

$$\hat{H}(s) = \frac{\hat{N}(s)}{\hat{D}(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

  - Hence,

$$\vec{H}(s) = \frac{\widehat{Y_{zs}}(s)}{\hat{F}(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

$$\widehat{Y_{zs}}(s)(s^n + a_1 s^{n-1} + \cdots + a_n) = \hat{F}(s)(b_0 s^m + b_1 s^{m-1} + \cdots + b_m)$$

- s-domain analysis of LTIC systems-cont

$$\widehat{Y_{zs}}(s)(s^n + a_1 s^{n-1} + \cdots + a_n) = \hat{F}(s)(b_0 s^m + b_1 s^{m-1} + \cdots + b_m)$$

$$s^n \widehat{Y_{zs}}(s) + a_1 s^{n-1} \widehat{Y_{zs}}(s) + \cdots + a_n \widehat{Y_{zs}}(s) = b_0 s^m \hat{F}(s) + b_1 s^{m-1} \hat{F}(s) + \cdots + b_m \hat{F}(s))$$

$\mathcal{L}^{-1}$

$$\frac{d^n}{dt^n} y_{zs} + a_1 \frac{d^{n-1}}{dt^{n-1}} y_{zs} + \ldots + a_n y_{zs} = b_0 \frac{d^m}{dt^m} f + b_1 \frac{d^{m-1}}{dt^{m-1}} f + \cdots + b_m f$$

- s-domain analysis of LTIC systems-cont
  - What if we want the full solution, not just $y_{zs}(t)$?

Assume $f(t)$ is causal.

$$\frac{d^n}{dt^n}y + a_1\frac{d^{n-1}}{dt^{n-1}}y + \ldots + a_n y = b_0\frac{d^m}{dt^m}f + b_1\frac{d^{m-1}}{dt^{m-1}}f + \cdots + b_m f$$

$\downarrow \mathcal{L}$

$$s^n\hat{Y} - s^{n-1}y(0^-) - s^{n-2}y'(0^-) - s^{n-3}y''(0^-) - \ldots - y^{(n-1)}(0^-) +$$

$$+ a_1\left(s^{n-1}\hat{Y} - s^{n-2}y(0^-) - s^{n-3}y'(0^-) - s^{n-4}y''(0^-) - \ldots - y^{(n-2)}(0^-)\right) +$$

$$\ldots a_n\hat{Y} =$$

$$= b_0\left(s^m\hat{F} - s^{m-1}\underbrace{\cancel{f(0^-)}}_{0} - \ldots - \underbrace{\cancel{f^{(m-1)}(0^-)}}_{0}\right] + \ldots$$

$$+ \ldots b_m\hat{F}$$

- s-domain analysis of LTIC systems-cont

$$\hat{Y}\left[s^n + a_1 s^{n-1} + .. + a_n\right] - y(0^-)\left[s^{n-1} + a_1 s^{n-2} + .. + a_{n-1}\right] -$$

$$- y'(0^-)\left[s^{n-2} + a_1 s^{n-3} + .. + a_{n-2}\right] - ..$$

$$- y^{(n-1)}(0^-) =$$

$$= \hat{F}\left[b_0 s^m + b_1 s^{m-1} + .. + b_m\right]$$

$$\hat{Y} = \frac{\hat{F}\left[b_0 s^m + b_1 s^{m-1} + .. + b_m\right] + y(0^-)\left[s^{n-1} + .. + a_{n-1}\right] + .. + y^{(n-1)}(0^-)}{s^n + a_1 s^{n-1} + .. + a_n}$$

full solution

$\hat{Y}_{ZS}$

characteristic polynomial

$\hat{Y}_{ZI}$

$$\hat{Y} = \hat{Y}_{ZS} + \hat{Y}_{ZI}$$

- s-domain analysis of LTIC systems-cont

$$\hat{Y} = \underbrace{\hat{F}\cdot\hat{H}}_{\hat{Y}_{zs}} + \underbrace{\frac{y(0^-)(s^{n-1} + \dots + a_{n-1}) + \dots + y^{(n-1)}(0^-)}{s^n + \dots + a_n}}_{\hat{Y}_{zI}}$$

$$\hat{F} \rightarrow \boxed{\hat{H}} \rightarrow \hat{Y}_{zs} = \hat{F}\cdot\hat{H}$$

$$s^n + a_1 s^{n-1} + \dots + a_n = (s-p_1)(s-p_2)\dots(s-p_n)$$

charac. polynomial

characteristic poles

$p_1, p_2, \dots, p_n$

Note: before cancellation!

Characteristic modes

$t^k e^{p_1 t}, e^{p_2 t} \dots, e^{p_n t}$

correspond. exp. of char. poles

- s-domain analysis of LTIC systems - Example # 17
  - Consider an LTIC system described by the following ODE:

$$\frac{d^2}{dt^2}y + 5\frac{d}{dt}y + 4y = 2f(t)$$

where

– f(t) = u(t)

– y (0^-) = 1

– y' (0^-) = 0

√ • Determine the characteristic poles and characteristic modes of the system

√ • Determine $\hat{H}(s)$ and h(t)

√ • Determine if the system is BIBO stable

- Determine $y_{ZI}(t), y_{ZS}(t)$ and y(t)

- s-domain analysis of LTIC systems - Example # 17-cont
  - Consider an LTIC system described by the following ODE:

$$\frac{d^2}{dt^2}y + 5\frac{d}{dt}y + 4y = 2f(t) \rightarrow s^2 + 5s + 4$$

  - Determine the characteristic poles and characteristic modes of the system

$\downarrow \mathcal{L}$

$$\left(s^2\hat{Y} - s^{2-1}y(0^-) - s^0 y'(0^-)\right) + 5\left(s\hat{Y} - y(0^-)\right) + 4\hat{Y} = 2\hat{F}$$

$$\hat{Y}\underbrace{\left(s^2+5s+4\right)}_{\text{charact. polynomial}} = 2\hat{F} + y(0^-)(s+5) + y'(0^-)$$

$= (s+4)(s+1)$

charact. poles: $p_1 = -4$; $p_2 = -1$

charact. modes: e^{-4t}, e^{-t}

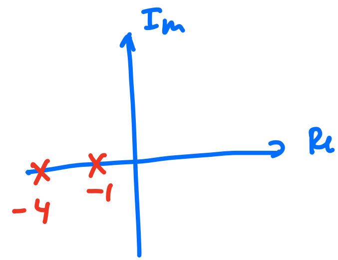
- s-domain analysis of LTIC systems - Example # 17-cont

$$\hat{Y}(s) = \frac{2\hat{F} + \cancel{y(0^-)(s+5) + y'(0^-)}}{(s+4)(s+1)} \quad = 0 \text{ for z-s.}$$

  - Determine $\hat{H}(s)$ and h(t)
  - Determine if the system is BIBO stable

$$\hat{Y}_{zs} = \hat{F}\cdot\hat{H} \Rightarrow \frac{2\hat{F}}{(s+4)(s+1)}$$

$\hat{H}(s)$



BIBO stable

$$h(t) = \mathcal{L}^{-1}\{\hat{H}(s)\}$$