ECE 210 (AL2) - ECE 211 (E)

Chapter 6

Fourier Series and LTI System Response to Periodic Signals

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Chapter objectives

- Identify periodic signals and obtain their periods and fundamental frequencies
- Understand the significance and interpretation of Fourier series and its coefficients
- Apply properties of Fourier series to determine effect of basic signal processing
- Understand the effect of LTI systems, via H(ω), on periodic signals via their Fourier series and its coefficients
- Be able to calculate the average power of a periodic signal both in time and frequency
- ~~Understand Total Harmonic Distortion (THD) and how to calculate it~~

- LTI system response to co-sinusoids and complex exponentials
  - Recall

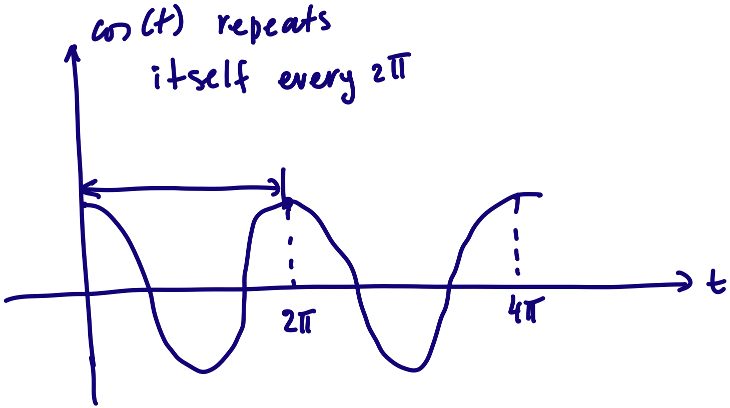
$$A\cos(\omega t+\theta) \longrightarrow \boxed{H(\omega)} \longrightarrow |H(\omega)|\,A\cos(\omega t+\theta+\measuredangle H(\omega))$$

$$A e^{j(\omega t+\theta)} \longrightarrow \boxed{H(\omega)} \longrightarrow H(\omega)\,A e^{j(\omega t+\theta)}$$

  - What about other periodic signals?

- Periodic signals

$\cos(t)$ repeats itself every $n\,2\pi$



- Periodic signals-cont

A signal is periodic if there exists a delay t_0 to such that

$$f(t) = f(t - t_0)$$

for all t.

The smallest such delay is called the *period* of the signal, denoted by *T*.

The fundamental frequency of the signal is defined as $\boxed{\omega_0 = \frac{2\pi}{T}}$ in rad/s.

T and ω_0 are inversely proportional to each other.

- Periodic signals – example #1

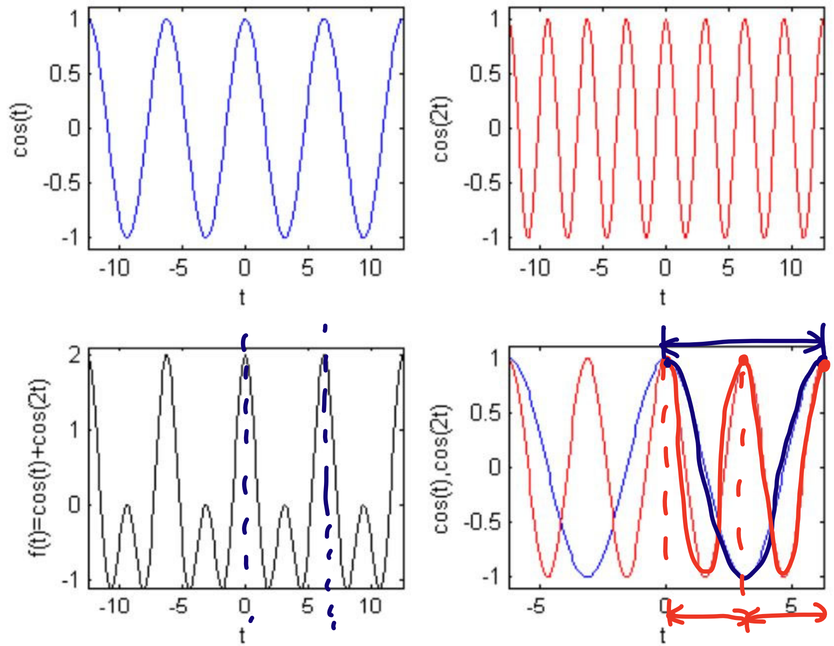
$\cos(t)$ is periodic

$\cos(2t)$ is also periodic

Is f(t) = $\cos(t) + \cos(2t)$ also periodic?

• Periodic signals – example #1-cont

Is f(t) = $\cos(t) + \cos(2t)$ also periodic? Yes → How to find T and ω_0?



First time they both repeat itself together is the period of f(t)

- Periodic signals – example #1-cont

$T_1 = \frac{2\pi}{\omega_1} = 2\pi\,s$

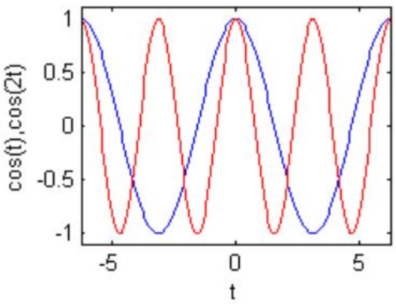
$T_2 = \frac{2\pi}{\omega_2} = \pi\,s$

Is f(t) = $\cos(t) + \cos(2t)$ also periodic?

$\omega_1 = 1\ rad/s$

$\omega_2 = 2\ rad/s$

What are T and ω_0 for f(t)?



n	nT_1	nT_2
1	2π	π
2	4π	2π

$\rightarrow T = 2\pi\,s$

$\omega_0 = \frac{2\pi}{T} = 1\ rad/s$

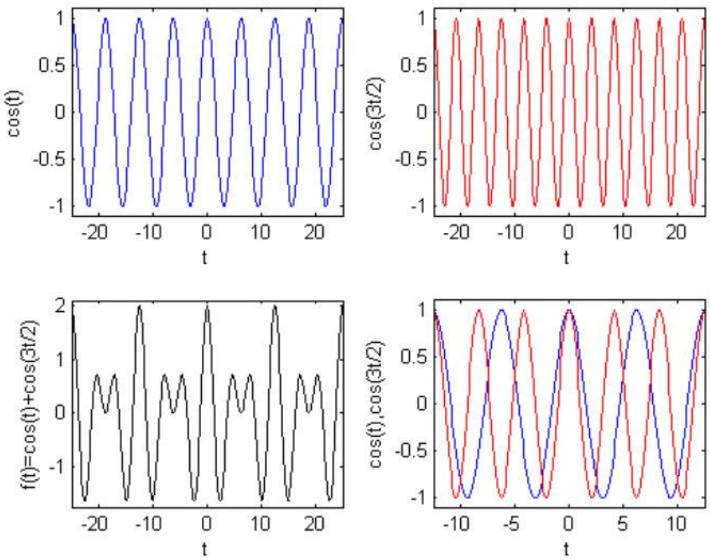
$\frac{\omega_1}{\omega_2} = \frac{1}{2} \in \mathbb{Q} \rightarrow$ periodic

$\omega_0 = GCD(1,2) = 1\ rad/s$

$\frac{T_1}{T_2} = \frac{2\pi}{\pi} = 2 \in \mathbb{Q}$

- Periodic signals – example #2

Is g(t) = $\cos(t) + \cos(\frac{3}{2}t)$ also periodic?



- Periodic signals – example #2-cont

$T_1 = \frac{2\pi}{\omega_1} = 2\pi\, s$

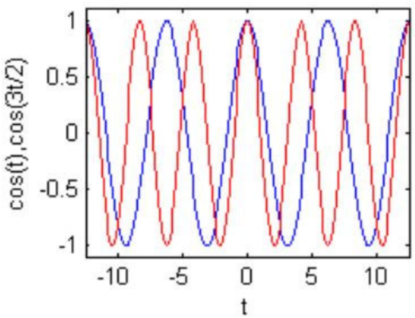
$T_2 = \frac{2\pi}{\omega_2} = \frac{4\pi}{3}\, s$

$\omega_1 = 1\, rad/s$

Is f(t) = $\cos(t) + \cos(\frac{3}{2}t)$ also periodic?

$\omega_2 = \frac{3}{2}\, rad/s$

What are T and ω_0 for g(t)?



n	nT_1	nT_2
1	2π	$\frac{4\pi}{3}$
2	4π	$8\pi/3$
3	6π	$\frac{12\pi}{3} = 4\pi$

$\rightarrow T = 4\pi\, s$

$\omega_0 = \frac{2\pi}{T} = \frac{1}{2}\, rad/s$

$\frac{\omega_1}{\omega_2} = \frac{1 \cdot 2}{3} \in \mathbb{Q} \rightarrow$ periodic

$\omega_0 = GCD(1, \frac{3}{2}) = \frac{1}{2}\, rad/s$

$\frac{T_1}{T_2} = \frac{2\pi \cdot 3}{4\pi} = \frac{3}{2} \in \mathbb{Q}$

For more terms look at all ratios. For ex. $\frac{\omega_1}{\omega_2}, \frac{\omega_2}{\omega_3}, \frac{\omega_1}{\omega_3}$

- Sums of periodic signals

  - For a sum of periodic signals to be periodic, all possible ratios of the individual fundamental frequencies must be rational numbers (fractions). $\frac{\text{integer}}{\text{Integer}}$

  - Same holds for the individual periods (basically, each must complete an integer number of cycles when they meet again).

  - There must exist a frequency $\widehat{\omega}$ such that all individual frequencies, ω_i, are integer multiples of $\widehat{\omega}$. The largest such frequency is the fundamental frequency ω_0.

    $$\omega_0 = GCD(\omega_1, \omega_2)$$

  - Similarly, there must exist a time duration $\hat{T}$ which is an integer multiple of all individual periods, T_i. The smallest such time duration is the period T.

    $$T = LCM(T_1, T_2)$$

- Sums of periodic signals – examples

Determine if the following signals are periodic:

- $f_1(t) = \cos(\pi t) + \cos(2t)$

$\frac{\omega_1}{\omega_2} = \frac{\pi}{2} \notin \mathbb{Q} \rightarrow$ not periodic

- $f_2(t) = \cos(\pi t) + \sqrt{2}\cos(\frac{5}{2}\pi t)$

$\frac{\omega_1}{\omega_2} = \frac{\pi \cdot 2}{5\pi} = \frac{2}{5} \in \mathbb{Q} \rightarrow$ periodic

$\omega_0 - ?$

$T - ?$

$\omega_0 = GCD(\pi, \frac{5\pi}{2}) = \frac{\pi}{2}$ rad/s

$T = \frac{2\pi}{\omega_0} = 4s$

- Sums of periodic signals - cont

For a fixed ω_0

$e^{j\omega_0 t}$ is periodic

$+$

$e^{j2\omega_0 t}$?

$+$

$e^{-j2\omega_0 t}$?

$+$

$e^{j3\omega_0 t}$?

$\frac{\omega_1}{\omega_2} = \frac{\omega_0}{2\omega_0} = \frac{1}{2} \in \mathbb{Q} \rightarrow$ periodic

$\frac{\omega_1}{\omega_3} = \frac{\omega_0}{-2\omega_0} = -\frac{1}{2}$ ✓ $\frac{\omega_2}{\omega_3} = \frac{2\omega_0}{-2\omega_0} = -1$ ✓

$e^{jn\omega_0 t}$? ← integer

$\sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$? → always periodic

← constant (can be complex)

↑ integer multiple of ω_0

- ## Exponential Fourier series
  - If f(t) is periodic with fundamental frequency ω_0, then we can express it as

$$\text{f(t)} = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

F_n is complex BUT independent of time.

will change depending on the function

  - Distinct functions with the same ω_0 will have distinct sets of F_n

sin(t) and cos(t)

⇓

same ω_0, but will have different F.S. coefficients

  - How to get the coefficients F_n?