University of Illinois

Solutions

1. Determine the Laplace transform $\hat{F}(s)$, and the ROC, of the following signals $f(t)$. In each case identify the corresponding pole locations where $|\hat{F}(s)|$ is not finite.
Solution:

(a) $f(t) = u(t) - u(t-8)$

$$\hat{F}(s) = \int_{0^-}^{+\infty} [u(t) - u(t-8)]\, e^{-st} dt = \int_0^8 e^{-st} dt$$
$$= \boxed{\frac{1-e^{-8s}}{s}}$$

$$\boxed{\begin{array}{l} \text{Pole}: \; s = -\infty + j\omega \\ \text{ROC}: \; \sigma = Re\{s\} > -\infty \end{array}}$$

(b) $f(t) = te^{2t}u(t)$

$$\hat{F}(s) = \int_{-\infty}^{\infty} te^{(2-s)t} dt = \frac{1}{(s-2)^2},$$

The integral above converges only if $\sigma = \mathrm{Re}\{s\} > 2$, which defines the region of convergence (ROC). There is a double pole at $s = 2$.

(c) $f(t) = te^{-4t} + \delta(t) + u(t-2)$

$$\hat{F}(s) = \int_{0^-}^{\infty} te^{(-4-s)t} dt + \int_{0^-}^{\infty} \delta(t) e^{-st} dt + \int_2^{\infty} e^{-st} dt$$
$$= \frac{1}{(s+4)^2} + 1 + \frac{1}{s}(e^{-2s} - 0)$$
$$= 1 + \frac{1}{(s+4)^2} + \frac{1}{s} e^{-2s} \longrightarrow \text{Converge only if } \sigma > 0$$

↳ Converge only if $\sigma > -4$

$$\boxed{\begin{array}{l} \hat{F}(s) = 1 + \frac{1}{(s+4)^2} + \frac{1}{s} e^{-2s} \\ \text{ROC}: \; \sigma > 0 \\ \text{poles}: \; s = -4 \text{ (double)}, \; s = 0 \end{array}}$$

There is also a hidden pole at $s = -\infty + j\omega$.
$\lim_{s\to-\infty} \hat{F}(s) = \lim_{s\to-\infty} \left\{1 + \frac{1}{(s+4)^4} + \frac{1}{s}e^{-2s}\right\} = \lim_{s\to-\infty} \left\{1 + \frac{1}{s}e^{-2s}\right\} =$
$1 + \lim_{s\to-\infty} \left\{-2e^{-2s}\right\} = -\infty$

(d) $f(t) = e^{2t}\cos(t)u(t)$.

$$f(t) = e^{2t}\cos(t)u(t).$$

$$= e^{2t}\cdot\frac{1}{2}\left[e^{jt} + e^{-jt}\right] u(t)$$

$$= \left[\frac{1}{2}e^{(2+j)t} + \frac{1}{2}e^{(2-j)t}\right] u(t)$$

$$\hat{F}(s) = \frac{1}{2}\int_0^{\infty}\left(e^{(2+j-s)t} + e^{(2-j-s)t}\right) dt$$

$$= \frac{1}{2}\,\frac{-1}{2+j-s} + \frac{1}{2}\,\frac{-1}{2-j-s}$$

$$= \boxed{\frac{s-2}{(s-2)^2+1}} \qquad \boxed{\begin{array}{l}\text{ROC}: \sigma > 2\\ \text{poles}: s = 2 \pm j\end{array}}$$

2. For each of the following Laplace transforms $\hat{F}(s)$, determine the inverse Laplace transform $f(t)$.
Solution:

(a) $\hat{F}(s) = \frac{s+3}{(s+2)(s+4)}$

$$\hat{F}(s) = \frac{s+3}{(s+2)(s+4)} = \frac{K_1}{s+2} + \frac{K_2}{s+4}$$

$$K_1 = \frac{(-2)+3}{(-2)+4} = \frac{1}{2}, \qquad K_2 = \frac{(-4)+3}{(-4)+2} = \frac{1}{2}$$

$$\hat{F}(s) = \frac{1}{2(s+2)} + \frac{1}{2(s+4)}$$

$$\boxed{f(t) = \frac{1}{2}e^{-2t}u(t) + \frac{1}{2}e^{-4t}u(t)}$$

(b) $\hat{F}(s) = \frac{s^2}{(s+2)(s+4)}$

$$\hat{F}(s) = \frac{s^2}{(s+2)(s+4)} = 1 + \frac{-6s-8}{(s+2)(s+4)}$$

$$= 1 + \frac{K_1}{s+2} + \frac{K_2}{s+4}$$

b) $$\hat{F}(s) = \frac{s^2}{(s+2)(s+4)} = 1 + \frac{-6s-8}{(s+2)(s+4)}$$

$$= 1 + \frac{K_1}{s+2} + \frac{K_2}{s+4}$$

$$K_1 = \frac{-6(-2)-8}{-2+4} = 2$$

$$K_2 = \frac{-6(-4)-8}{-4+2} = -8$$

$$\hat{F}(s) = 1 + \frac{2}{s+2} + (-8)\frac{1}{s+4}$$

$$\boxed{f(t) = \delta(t) + 2e^{-2t}u(t) - 8e^{-4t}u(t)}$$

(c) $\hat{F}(s) = \frac{s^2+2s+1}{(s+1)(s+2)}$

$$\hat{F}(s) = \frac{s^2+2s+1}{(s+1)(s+2)} = \frac{s+1}{s+2}$$

$$= 1 - \frac{1}{s+2}$$

$$\boxed{f(t) = \delta(t) - e^{-2t}u(t)}$$

3. Sketch the amplitude response $|H(\omega)|$ and determine the impulse response $h(t)$ of the LTIC systems having the following transfer functions:
Solution:

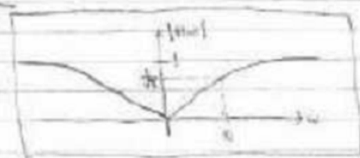
   (a) $\hat{H}(s) = \frac{s}{s+10}$

$$\hat{H}(s) = \frac{s}{s+10} = 1 + \frac{s-(s+10)}{s+10} = 1 - \frac{10}{s+10}$$

$$\boxed{h(t) = \delta(t) - 10e^{-10t}u(t)}$$

$$H(\omega) = \hat{H}(j\omega) = \frac{j\omega}{j\omega+10}$$

$$|H(\omega)| = \frac{|\omega|}{\sqrt{\omega^2+100}}$$



   (b) $\hat{H}(s) = \frac{s}{s^2+3s+2}$

$$\hat{H}(s) = \frac{s}{s^2+3s+2} = \frac{s}{(s+1)(s+2)} = \frac{K_1}{s+1} + \frac{K_2}{s+2}$$

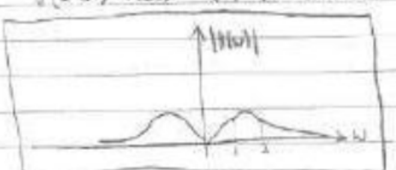
$$K_1 = \frac{(-1)}{(-1)+2} = -1$$

$$K_2 = \frac{(-2)}{(-2)+1} = 2$$

$$\hat{H}(s) = \frac{-1}{s+1} + \frac{2}{s+2} \qquad \boxed{h(t) = -e^{-t}u(t) + 2e^{-2t}u(t)}$$

$$H(\omega) = \hat{H}(j\omega) = \frac{j\omega}{(j\omega)^2+3j\omega+2} = \frac{j\omega}{2-\omega^2+3j\omega}$$

$$|H(\omega)| = \frac{|\omega|}{\sqrt{(2-\omega^2)^2+(3\omega)^2}} = \frac{|\omega|}{\sqrt{4-4\omega^2+\omega^4+9\omega^2}} = \frac{|\omega|}{\sqrt{\omega^4+5\omega^2+4}}$$



4. Determine the zero-state responses of the systems defined in the previous problem (3) to a causal input $f(t) = u(t)$. Use $y(t) = h(t) * f(t)$ or find the inverse Laplace transform of $\hat{Y}(s) = \hat{H}(s)\hat{F}(s)$, whichever is more convenient.
Solution:
First, note that the Laplace transform of the unit step function $f(t) = u(t)$ is given by $\hat{F}(s) = \frac{1}{s}$.

   (a) It is likely easiest to solve for $y(t)$ by finding the inverse Laplace transform of $\hat{Y}(s) = \hat{H}(s)\hat{F}(s)$.

$$\hat{Y}(s) = \hat{H}(s)\hat{F}(s) = \frac{1}{s+10}.$$

   This gives the simple inverse Laplace transform $y(t) = e^{-10t}u(t)$.

   (b) Again, $y(t)$ can be solved for easily as the inverse Laplace transform of $\hat{Y}(s) = \hat{H}(s)\hat{F}(s)$.

$$\hat{Y}(s) = \hat{H}(s)\hat{F}(s) = \frac{1}{s^2+3s+2} = \frac{1}{(s+1)(s+2)} = \frac{K_1}{s+1} + \frac{K_2}{s+2}.$$

   We solve for the coefficients K_1, K_2 by partial fractions expansion. Rewriting the expression above,

$$1 = K_1(s+2) + K_2(s+1),$$

then

$$
\begin{aligned}
K_1 &= \left.\frac{1}{s+2}\right|_{s=-1} = 1, \\
K_2 &= \left.\frac{1}{s+1}\right|_{s=-2} = -1.
\end{aligned}
$$

Therefore,

$$\hat{Y}(s) = \frac{1}{s+1} - \frac{1}{s+2},$$

and $y(t) = e^{-t}u(t) - e^{-2t}u(t)$.

5. Given the frequency response $H(\omega)$, below, determine the system transfer function $\hat{H}(s)$ and impulse response $h(t)$.

Solution:

(a) $H(\omega) = \frac{j\omega}{(1+j\omega)(2+j\omega)}$

a) $H(\omega) = \frac{j\omega}{(1+j\omega)(2+j\omega)}$

$$\boxed{\hat{H}(s) = \frac{s}{(s+1)(s+2)}} = \frac{-1}{s+1} + \frac{2}{s+2}$$

$$\boxed{h(t) = -e^{-t}u(t) + 2e^{-2t}u(t)}$$

(b) $H(\omega) = \frac{j\omega}{1-\omega^2+j\omega}$.

b) $H(\omega) = \frac{j\omega}{1-\omega^2+j\omega} = \frac{j\omega}{1+(j\omega)^2+j\omega}$

$$\boxed{\hat{H}(s) = \frac{s}{s^2+s+1}}$$

poles: $s^2+s+1=0 \Rightarrow s = \frac{-1\pm\sqrt{-3}}{2} = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$

$$\hat{H}(s) = \frac{K_1}{s-(-\frac{1}{2}+j\frac{\sqrt{3}}{2})} + \frac{K_2}{s-(-\frac{1}{2}-j\frac{\sqrt{3}}{2})}$$

$$K_1 = \frac{(-\frac{1}{2}+j\frac{\sqrt{3}}{2})}{(-\frac{1}{2}+j\frac{\sqrt{3}}{2})-(-\frac{1}{2}-j\frac{\sqrt{3}}{2})} = \frac{-\frac{1}{2}+j\frac{\sqrt{3}}{2}}{j\sqrt{3}} = \frac{1}{2}+j\frac{1}{2\sqrt{3}}$$

$$K_2 = \frac{(-\frac{1}{2}-j\frac{\sqrt{3}}{2})}{(-\frac{1}{2}-j\frac{\sqrt{3}}{2})-(-\frac{1}{2}+j\frac{\sqrt{3}}{2})} = \frac{1}{2}-j\frac{1}{2\sqrt{3}}$$

$$\hat{H}(s) = \frac{\frac{1}{2}+j\frac{1}{2\sqrt{3}}}{s-(-\frac{1}{2}+j\frac{\sqrt{3}}{2})} + \frac{\frac{1}{2}-j\frac{1}{2\sqrt{3}}}{s-(-\frac{1}{2}-j\frac{\sqrt{3}}{2})}$$

$$
\begin{aligned}
h(t) &= (\tfrac{1}{2}+j\tfrac{1}{2\sqrt{3}})e^{(-\frac{1}{2}+j\frac{\sqrt{3}}{2})t}u(t) + (\tfrac{1}{2}-j\tfrac{1}{2\sqrt{3}})e^{(-\frac{1}{2}-j\frac{\sqrt{3}}{2})t}u(t) \\
&= e^{-\frac{1}{2}t}u(t)\left[(\tfrac{1}{2}+j\tfrac{1}{2\sqrt{3}})e^{j\frac{\sqrt{3}}{2}t} + (\tfrac{1}{2}-j\tfrac{1}{2\sqrt{3}})e^{-j\frac{\sqrt{3}}{2}t}\right] \\
&= e^{-\frac{1}{2}t}u(t)\cdot 2\,\mathrm{Re}\left\{(\tfrac{1}{2}+j\tfrac{1}{2\sqrt{3}})e^{j\frac{\sqrt{3}}{2}t}\right\} \\
&= e^{-\frac{1}{2}t}u(t)\,\mathrm{Re}\left\{(1+j\tfrac{1}{\sqrt{3}})(\cos(\tfrac{\sqrt{3}}{2}t)+j\sin(\tfrac{\sqrt{3}}{2}t))\right\}
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
&= e^{-\frac{1}{2}t}u(t)\left(\cos(\tfrac{\sqrt{3}}{2}t) - \tfrac{1}{\sqrt{3}}\sin(\tfrac{\sqrt{3}}{2}t)\right) \\
&\approx 1.15\,e^{-\frac{1}{2}t}\cos(\tfrac{\sqrt{3}}{2}t+\tfrac{\pi}{6})\,u(t)
\end{aligned}
}
$$

6. Determine whether the LTIC systems with the following transfer functions are BIBO stable and explain why or why not.
Solution:

(a) $\hat{H}_1(s) = \frac{s^3+1}{(s+2)(s+4)}$

a) $\hat{H}_1(S) = \frac{S^3+1}{(S+2)(S+4)}$

pole at $+\infty$ $\Rightarrow$ $\boxed{\text{unstable}}$

(b) $\hat{H}_2(s) = 2 + \frac{s}{(s+1)(s-2)}$

b) $\hat{H}_2(S) = 2 + \frac{S}{(S+1)(S-2)}$

pole at $S=2$ $\Rightarrow$ $\boxed{\text{unstable}}$

(c) $\hat{H}_3(s) = \frac{s^2+4s+6}{(s+1+j6)(s+1-j6)}$

c) $\hat{H}_3(S) = \frac{S^2+4S+6}{(S+1+j6)(S+1-j6)}$

poles in LHP $\Rightarrow$ $\boxed{\text{stable}}$

(d) $\hat{H}_4(s) = \frac{1}{s^2+16}$

d) $\hat{H}_4(S) = \frac{1}{S^2+16}$

poles on w-axis. System is marginally stable

But not BIBO stable $\Rightarrow$ $\boxed{\text{unstable}}$

(e) $\hat{H}_5(s) = \frac{s-2}{s^2-4}$.

e) $\hat{H}_5(S) = \frac{S-2}{S^2-4} = \frac{1}{S+2}$

$\Rightarrow$ $\boxed{\text{stable}}$ due to pole-zero cancellation