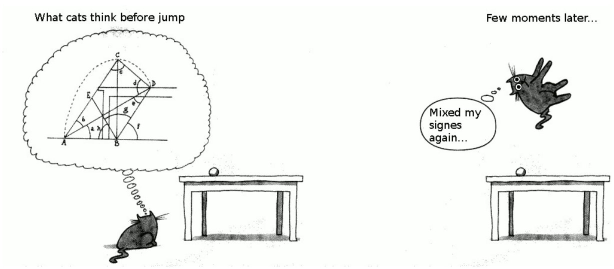
# ECE 210 (AL2)

## Chapter 11

## Laplace Transform, Transfer Function, and LTIC System Response




Olga Mironenko
University of Illinois
Spring 2022

# Chapter objectives

- Perform Laplace transforms
- Understand the region of convergence of Laplace transform and its relation to BIBO stability
- Understand the relation between Laplace transform and Fourier transform
- Understand and apply properties of Laplace transform
- Perform inverse Laplace transform of rational functions using partial fraction expansion
- Perform s-domain system analysis
- Obtain the general response of LTIC circuits and systems
- Perform analysis of LTIC system combinations

# • Laplace transform

- Recall that

$$f(t) \rightarrow \boxed{\text{LTIC with h(t)}} \rightarrow y(t) = \int_{0}^{\infty} f(t-\tau)\underbrace{h(\tau)}_{=0 \text{ for } t<0} d\tau$$

$= f(t) * h(t)$

- Recall also that

$$e^{j\omega t} \rightarrow \boxed{\text{LTI with h(t)} \leftrightarrow H(\omega)} \rightarrow y(t) = e^{j\omega t} H(\omega)$$

$e^{j\omega t}$ is an eigenfunction of H(ω)

- jω is purely imaginary, extend to include real part: s = σ + jω

real part

$$e^{st} \rightarrow \boxed{\text{LTIC with h(t)}} \rightarrow y(t) = \int_{0}^{\infty} e^{s(t-\tau)} h(\tau) d\tau =$$

$$= \int_0^\infty e^{st} \cdot e^{-s\tau} h(\tau) d\tau = e^{st} \underbrace{\int_{0^-}^{\infty} h(\tau) e^{-s\tau} d\tau}_{\hat{H}(s)}$$

→ Laplace Transform of h(t)

$$e^{st} \rightarrow \boxed{\text{LTIC with h(t)}} \rightarrow e^{st} \hat{H}(s)$$

# • Laplace transform - cont

- Define the Laplace transform of a function f(t):

$$\hat{F}(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt$$

*includes possible impulses at t = 0

*s = σ + jω ∈ $\mathbb{C}$

- In particular, if h(t) is the impulse response of an LTIC system:

$$\hat{H}(s) = \int_{0^-}^{\infty} h(t)e^{-st}dt$$ is the *transfer function* of the system

$f(t)$ → $\boxed{\text{LTIC with } h(t) \leftrightarrow \hat{H}(s)}$ → $y(t) = f(t) * h(t)$

ℒs

$\hat{F}(s)$ → $\boxed{\text{LTIC with } h(t) \leftrightarrow \hat{H}(s)}$ → $\hat{Y}(s) = \hat{F}(s)\hat{H}(s)$

ℒs

Similar to

$$H(\omega) = \frac{Y(\omega)}{F(\omega)}$$

$$\hat{H}(s) = \frac{\hat{Y}(s)}{\hat{F}(s)}$$

- Laplace transform - cont.
  - Notice that if σ = 0, then $\underbrace{s = \sigma + j\omega}$ = $\underbrace{j\omega}$

$$\underbrace{\hat{F}(j\omega)}_{} = \int_{0^-}^{\infty} \underbrace{f(t)e^{-j\omega t}dt}_{} = \underbrace{F(\omega)}_{}$$

"s

* if $\hat{F}$ exists at s = jw

* if f(t) is causal

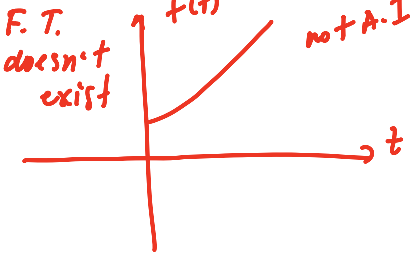
not A.I.

$$\hat{H}(\sigma + j\omega) = \int_{0^-}^{\infty} \overbrace{h(t)}^{\text{not A.I.}} \underbrace{e^{-\sigma t}}_{\text{maybe A.I.}} e^{-j\omega t}dt$$

maybe A.I.

- Laplace transform - Example # 1

  - Determine the Laplace transform of $f(t) = e^t u(t)$




F.T. doesn't exist

not A.I.

$$\hat{F}(s) = \int_{0^-}^{\infty} f(t)\, e^{-st}\, dt = \int_{0}^{\infty} e^{t} e^{-st}\, dt =$$

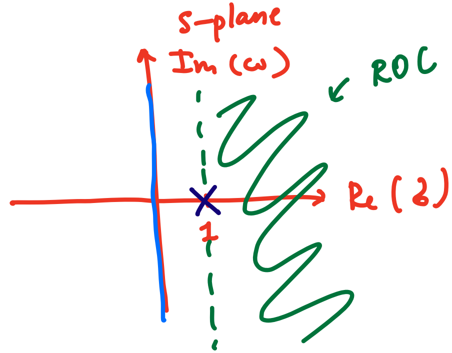
$$= \int_{0}^{\infty} e^{t(1-s)}\, dt = \frac{e^{t(1-s)}}{1-s}\Bigg|_{0}^{\infty}$$

for converg. we need $1-\sigma<0 \Rightarrow \sigma>1$

$$e^{t(1-s)} = e^{t(1-(\sigma+j\omega))} = e^{\underbrace{t(1-\sigma)}_{<0}} \cdot e^{-j\omega t}$$

if $\sigma>1$ $\quad \boxed{\hat{F}(s) = \frac{0-1}{1-s} = \frac{1}{s-1}}$

pole @ $s=1$




Can't get $F(\omega)$ from $\hat{F}(j\omega)$, because $\hat{F}$ doesn't exist at $s=j\omega$

Region of convergence (ROC) is the region in the S-plane, where $\hat{F}(s)$ converges

ROC $\{ s : \sigma = \mathrm{Re}\{s\} > 1 \}$

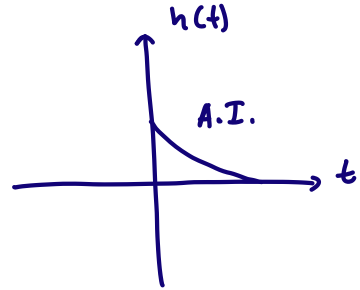
Pole is a location on s-plane where $|\hat{F}(s)| \to \infty$

Zero: where $\hat{F}(s)=0$ "O"

- Laplace transform - Example # 1
  - Is $f(t) = e^t u(t)$ A.I.?

- Laplace transform - Example # 2

  - Determine the Laplace transform of $h(t) = e^{-t}u(t)$




Recall $F(\omega) = \dfrac{1}{1+j\omega}$

$$\tilde{F}(s) = \int_{0^-}^{\infty} f(t)\, e^{-st}\, dt = \int_0^{\infty} e^{-t} e^{-st}\, dt = \int_0^{\infty} e^{-t(1+s)}\, dt =$$

$$= \left. \frac{e^{-t(1+s)}}{-(1+s)} \right|_0^{\infty}$$

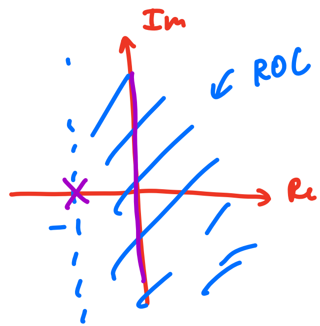
$$e^{-t(1+s)} = e^{-t(1+\sigma+j\omega)} = e^{\overbrace{-t(1+\sigma)}^{\to 0}} e^{-j\omega t}$$

$\Rightarrow 1+\sigma > 0$

$\sigma > -1$

if $\underline{\sigma > -1}$ $\quad \tilde{F}(s) = \dfrac{0-1}{-(1+s)} = \dfrac{1}{s+1}$

pole @ $s = -1$



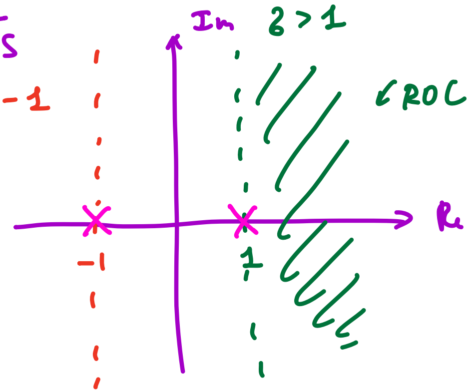
- Laplace transform - Example # 3
  - Determine the Laplace transform of $h(t) = \underbrace{e^{-t}u(t)}_{\downarrow \mathcal{L}\ \frac{1}{1+s}} + \underbrace{e^{t}u(t)}_{\downarrow \mathcal{L}\ \frac{1}{s-1}}$

$\sigma > -1$

$$G(s) = \frac{1}{s+1} + \frac{1}{s-1} = \frac{s-1+s+1}{(s+1)(s-1)} = \frac{2s}{(s+1)(s-1)}$$

poles @ $s = \pm 1$

if $\underline{\sigma > 1}$




Both terms should exist

- Laplace transform - cont

  - The region of convergence of the Laplace transform is the region in the complex plane to the right of the rightmost pole (not including $\infty$)

  - If there is a pole at $s = \infty$, then the Laplace transform has a term proportional to $s$ (or increasing with $s$.)