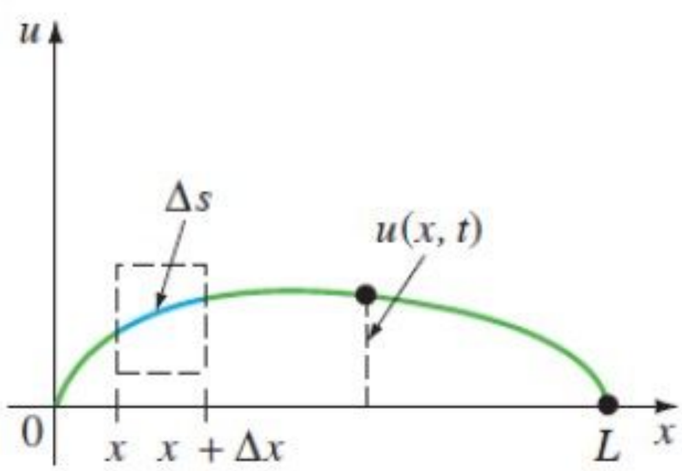
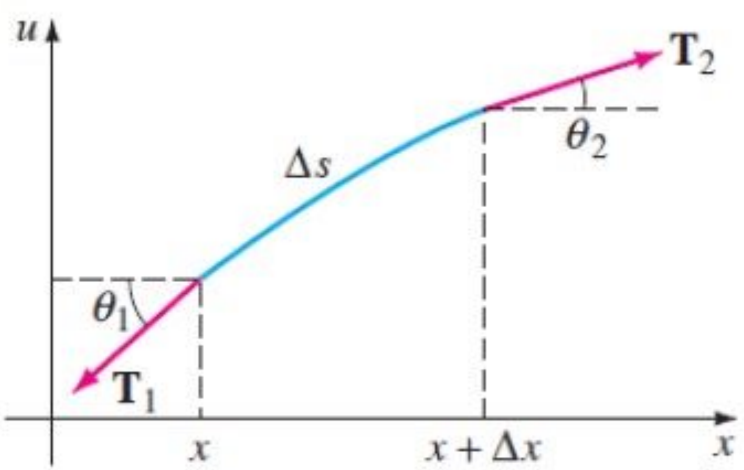
Wave Equation: Initial and Boundary Conditions



(a) Segment of string



Assumptions:

- No external force.
- Tension force is large compared to gravity and is the same at all points.
- Slope of the curve is very small at all points.
- Vertical displacement $\ll$ string length.
- String has mass per unit length ρ.

Initial Conditions:
Provide the initial displacement u and velocity u_t at time $t = 0$.

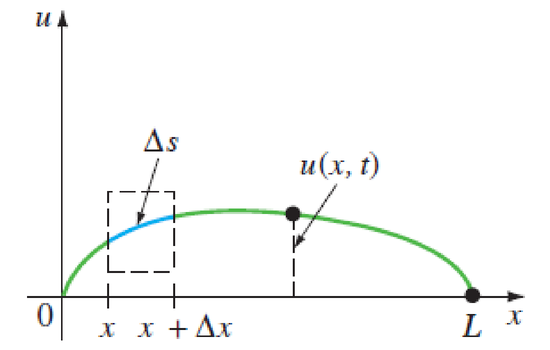
$$u(x,0) = f(x), \; u_t(x,0) = g(x), \; 0 < x < L$$

Boundary Conditions:
At the end points $x = 0$ and $x = L$, give the constraints on u, u_x, or $u_x + hu$. Usually in the scenario of strings, the boundary conditions are

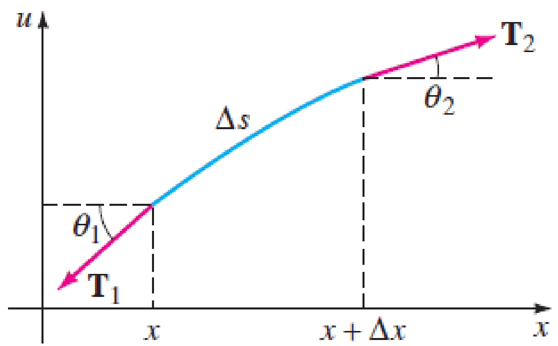
$$u(0,t) = 0, \; u(0,L) = 0, \; t > 0 \quad \text{Both ends are fixed.}$$

$$u_x(0,t) = 0, \; u_x(0,L) = 0, \; t > 0 \quad \text{Free-ends condition}$$

Wave Equation: Boundary-Value Problems



(a) Segment of string



Examples:

Both ends are fixed:

$$a^2 u_{xx} = u_{tt}, \quad 0 < x < L, \quad t > 0$$ Wave equation

$$u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0$$ Boundary condition

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 < x < L$$ Initial condition

Free Ends:

$$a^2 u_{xx} = u_{tt}, \quad 0 < x < L, \quad t > 0$$ Wave equation

$$u_x(0,t) = 0, \quad u_x(L,t) = 0, \quad t > 0$$ Boundary condition

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 < x < L$$ Initial condition

Boundary Conditions Associated with the Wave Equation

- We have discussed the one-dimensional transverse vibrations of a string. A few other types of important vibrations are:
  1. Sound waves (longitudinal waves)
  2. Electromagnetic waves of light and electricity
  3. Vibrations in solids (longitudinal, transverse, and torsional)
  4. Probability waves in quantum mechanics
  5. Water waves (transverse waves)
  6. Vibrating string (transverse waves)

- We will discuss some of the various types of BCs that are associated with physical problems of this kind.

  We will stick to one-dimensional problems where the BCs (linear ones) are generally grouped in to one of three kinds:

  1. Controlled end points (first kind)

$$u(0,t) = g_1(t)$$
$$u(L,t) = g_2(t)$$

$\boxed{2.}$ Force given on the boundaries (second kind)

$$u_x(0, t) = g_1(t)$$
$$u_x(L, t) = g_2(t)$$

$\boxed{3.}$ Elastic attachment on the boundaries (third kind)

$$u_x(0, t) - \gamma_1 u(0, t) = g_1(t)$$
$$u_x(L, t) - \gamma_2 u(L, t) = g_2(t)$$

$\boxed{1.}$ Controlled End Points

We are now involved with problems like

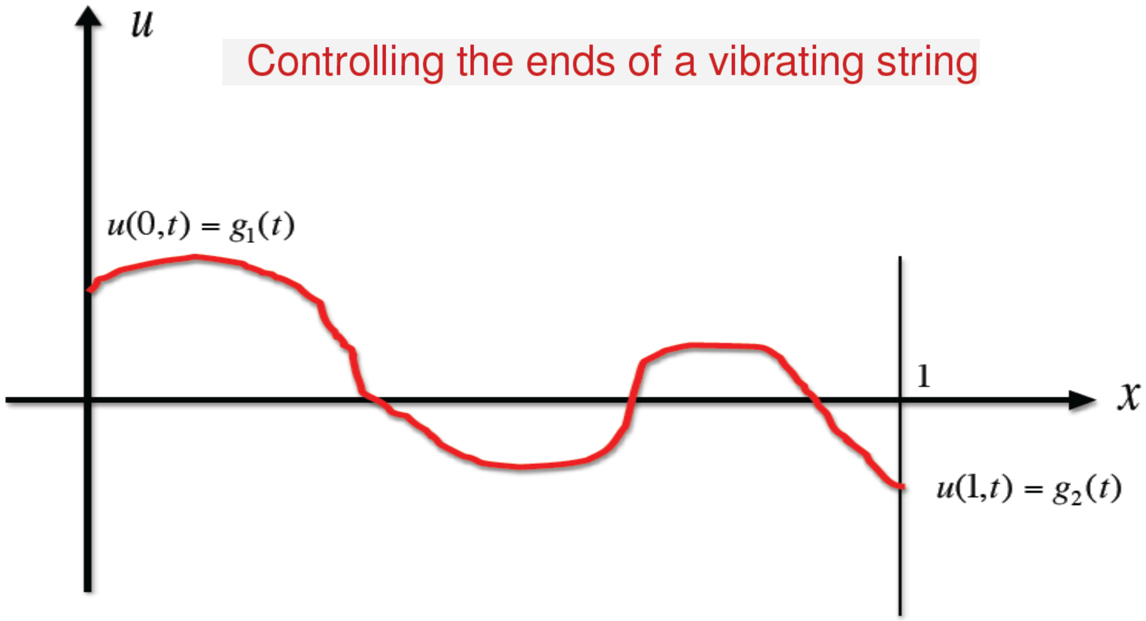
Problem

To find the function $u(x, t)$ that satisfies

$$\text{PDE:} \quad u_{tt} = c^2 u_{xx}, \quad 0 < x < 1, \quad 0 < t < \infty$$

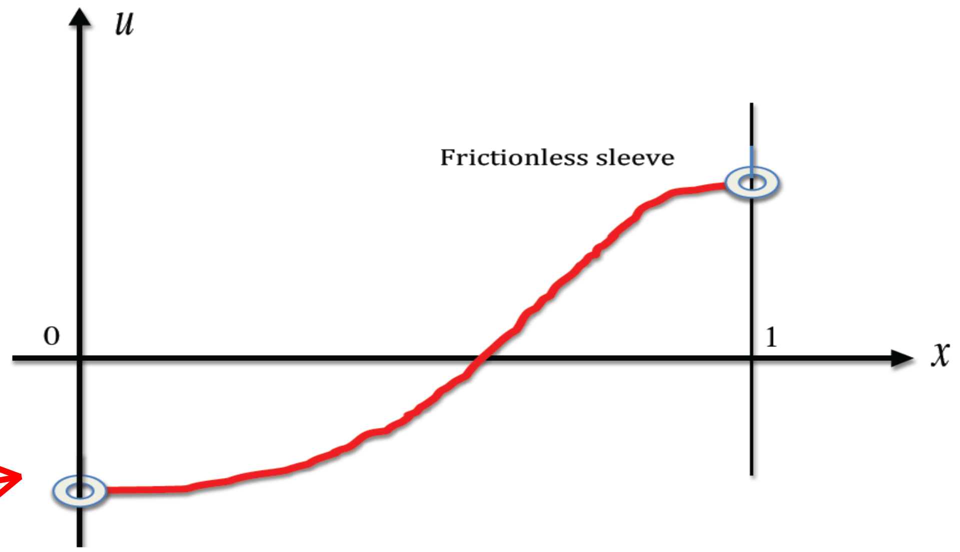
$$\text{BCs:} \quad \begin{cases} u(0, t) = g_1(t) \\ u(1, t) = g_2(t) \end{cases} \quad 0 < t < \infty$$

$$\text{ICs:} \quad \begin{cases} u(x, 0) = f(x) \\ u_t(x, 0) = g(x) \end{cases} \quad 0 \leqslant x \leqslant 1$$



where we control the end points so that they move in a given manner.

Free BC on the string



$\boxed{2.}$ Force Given on the Boundaries

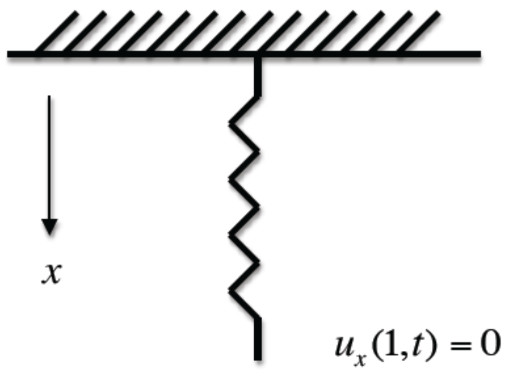
Inasmuch as the vertical forces on the string at the left and right ends are given by $Tu_x(0, t)$ and $Tu_x(L, t)$, respectively, by allowing the ends of the string to slide vertically on frictionless, the BCs become

$$u_x(0, t) = 0$$
$$u_x(L, t) = 0$$

BCs are presented in the following two examples:

$\boxed{a)}$ Free end of a longitudinally vibrating spring

Consider a vibrating spring with the bottom end unfastened



$\boxed{b)}$ Forced end of a vibrating spring

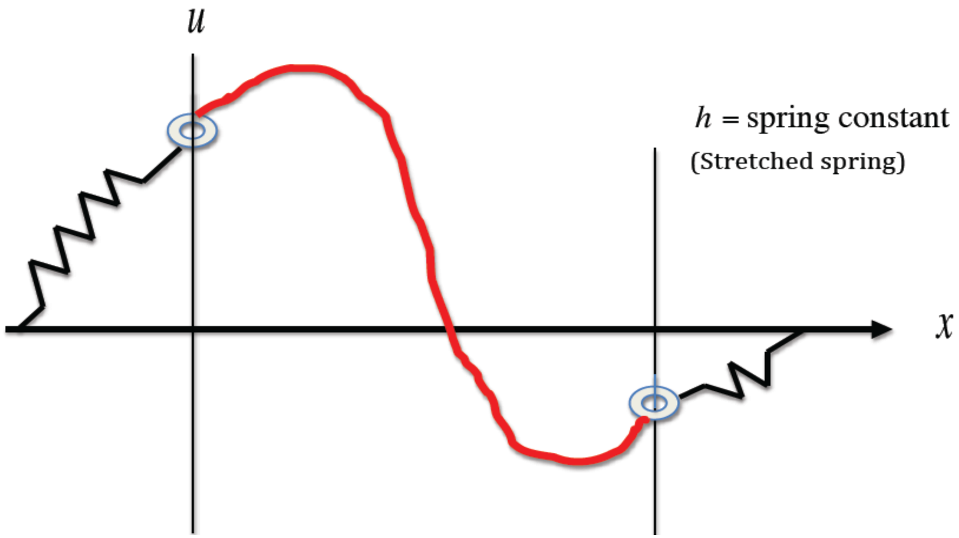
- If a force of $v(t)$ dynes is applied at the end $x = 1$ (a positive force is measured downward), then the BC would be

$$u_x(1, t) = \frac{1}{k} v(t) \qquad (k \text{ is Young's modulus})$$

- In the case of a forced BC, the ends of the string (or spring) are not required to maintain a given position, but the force that's applied tends to move the boundaries in the given direction.

3. Elastic Attachment on the Boundaries

Consider finally a violin string whose ends are attached to an elastic arrangement



The spring attachments at each end give rise to vertical forces proportional to the displacements

$$\text{Displacement at the left end} = u(0,t)$$
$$\text{Displacement at the right end} = u(L,t)$$

Setting the vertical tensions of the spring at the two ends

$$\begin{aligned}\text{Upward tension at the left end} &= Tu_x(0,t)\\ \text{Upward tension at the right end} &= -Tu_x(L,t)\end{aligned} \quad (T = \text{string tension})$$

equal to these displacements (multiplied by the spring constant h) gives us our desired BCs:

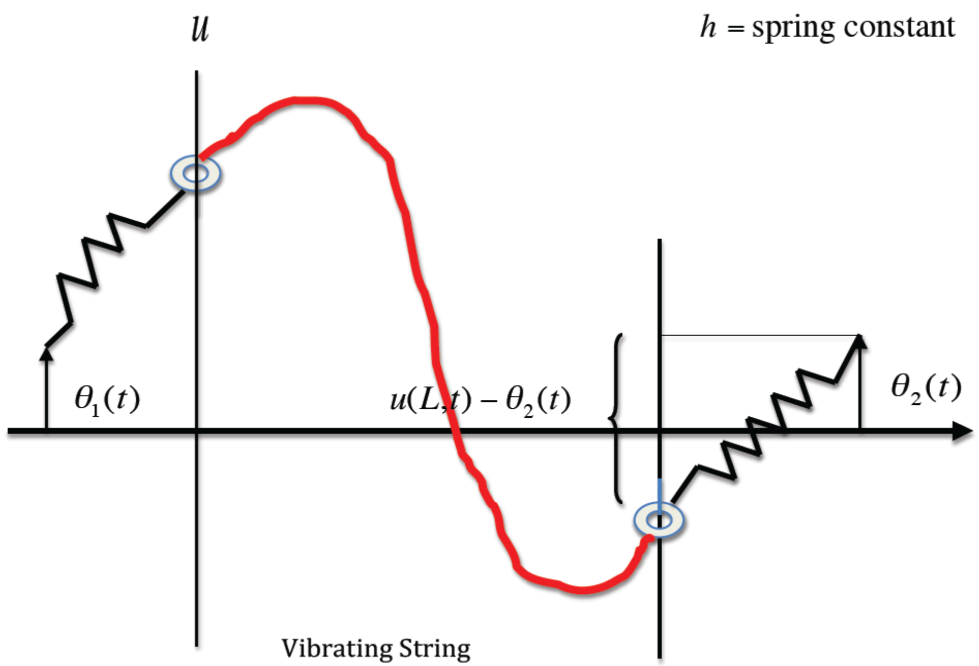
$$u_x(0,t) = \frac{h}{T}u(0,t)$$
$$u_x(L,t) = -\frac{h}{T}u(L,t)$$

Remark

Note that $u(0,t)$ positive means that $u_x(0,t)$ is positive, while if $u(L,t)$ is positive, then $u_x(L,t)$ is negative.

If the two spring attachments are displaced according to the functions $\theta_1(t)$ and $\theta_2(t)$, we would have the nonhomogeneous BCs

$$u_x(0,t) = \frac{h}{T}\left[u(0,t) - \theta_1(t)\right]$$
$$u_x(L,t) = -\frac{h}{T}\left[u(L,t) - \theta_2(t)\right]$$



Vibrating String

Remarks

- Another BC not discussed today occurs when the vibrating string experiences a force at the ends proportional to the string velocity (and in the opposite direction). Here, we have the BC (at the left end)

$$Tu_x(0,t) = -\beta u_t(0,t)$$

- A nonlinear elastic attachment at the left end of the string would be

$$Tu_x(0,t) = \phi\left[u(0,t)\right]$$

where $\phi(u)$ is an arbitrary function of u; for example

$$Tu_x(0,t) = -hu^3(0,t)$$

- If a mass m is attached to the lower end of a longitudinally vibrating string, the BC would be

$$mu_{tt}(L,t) = -ku_x(L,t) + mg$$

Wave Equation: a Boundary-Value Problem

$$\boxed{\begin{array}{lll} \text{Solve } u(x,t): & a^2 u_{xx} = u_{tt}, \quad 0 < x < L, \quad t > 0 & \\ \text{subject to}: & u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0 & \\ & u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 < x < L & \end{array}}$$

We focus on solving the above BVP (both ends are fixed).

Step 1: Separation of variables:
Assume that the solution $u(x,t) = X(x)T(t)$, $X, T \neq 0$. Then,

$$a^2 u_{xx} = u_{tt} \implies a^2 X'' T = X T'' \implies \frac{X''}{X} = \frac{T''}{a^2 T} = -\lambda$$

$$\implies \begin{cases} X'' + \lambda X & = 0 \\ T'' + a^2 \lambda T & = 0 \end{cases}$$

The 2 homogeneous boundary conditions become $X(0) = X(L) = 0$.

Solve in the x-Dimension and Find λ

$$\boxed{\begin{aligned} &\text{Solve :} \quad X'' + \lambda X = 0, \quad T'' + a^2 \lambda T = 0 \\ &\text{subject to :} \quad X(0) = 0, \quad X(L) = 0 \\ &\qquad X(x)\,T(0) = f(x), \quad X(x)\,T'(0) = g(x), \quad 0 < x < L \end{aligned}}$$

Step 2: λ remains to be determined. What values should λ take?

1. $\lambda = 0$: $X(x) = c_1 + c_2 x$. $X(0) = X(L) = 0 \implies c_1 = c_2 = 0$.
2. $\lambda = -\alpha^2 < 0$: $X(x) = c_1 e^{-\alpha x} + c_2 e^{\alpha x}$.
   Plug in $X(0) = X(L) = 0$, we get $c_1 = c_2 = 0$.
3. $\lambda = \alpha^2 > 0$: $X(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$.
   Plug in $X(0) = X(L) = 0$, we get $c_1 = 0$, and $c_2 \sin(\alpha L) = 0$. Hence, $c_2 \neq 0$ only if $\alpha L = n\pi$.

Since $X \neq 0$, pick $\boxed{\lambda = \frac{n^2 \pi^2}{L^2}, \ n = 1, 2, \ldots} \implies X(x) = c_2 \sin \frac{n\pi}{L} x$.

Solve in t-Dimension and Superposition

$$\begin{array}{|l|}\hline \text{Solve}: \quad X'' + \lambda X = 0, \quad T'' + a^2 \lambda T = 0 \\ \text{subject to}: \quad X(0) = 0, \quad X(L) = 0 \\ \qquad X(x)\,T(0) = f(x), \quad X(x)\,T'(0) = g(x), \quad 0 < x < L \\ \hline \end{array}$$

Step 3: Once we fix $\lambda = \frac{n^2\pi^2}{L^2}$, $n = 1, 2, \ldots$, we obtain

$$X(x) = c_2 \sin\left(\frac{n\pi}{L} x\right), \quad T(t) = c_3 \cos\left(\frac{n\pi a}{L} t\right) + c_4 \sin\left(\frac{n\pi a}{L} t\right)$$

$$\Longrightarrow \; u_n(x,t) = \left\{ A_n \cos\left(\frac{n\pi a}{L} t\right) + B_n \sin\left(\frac{n\pi a}{L} t\right) \right\} \sin\left(\frac{n\pi}{L} x\right),$$

$$(A_n := c_2 c_3, \; B_n := c_2 c_4)$$

$\Longrightarrow \; u(x,t) := \sum_{n=1}^{\infty} u_n(x,t)$ is a solution, by the superposition principle.

Plug in Initial Condition, Revoke Fourier Series, and Done

$$\boxed{\begin{array}{lll} \text{Solve } u(x,t): & a u_{xx} = u_{tt}, \quad 0 < x < L, \quad t > 0 \\ \text{subject to}: & u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0 \\ & u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 < x < L \end{array}}$$

Step 4: Plug in the initial conditions and find $\{A_n, B_n \mid n = 1, 2, \ldots\}$.

$$u(x,0) = f(x), \quad u(x,t) = \sum_{n=1}^{\infty} \left\{ A_n \cos\left(\frac{n\pi a}{L} t\right) + B_n \sin\left(\frac{n\pi a}{L} t\right) \right\} \sin\left(\frac{n\pi}{L} x\right)$$

$$\implies f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L} x\right), \quad 0 < x < L$$

From the Fourier sine series expansion on $(0, L)$, we get

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) \, dx.$$

Plug in Initial Condition, Revoke Fourier Series, and Done

$$\boxed{\begin{array}{lll} \text{Solve } u(x,t): & a u_{xx} = u_{tt}, \quad 0 < x < L, \quad t > 0 \\ \text{subject to}: & u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0 \\ & u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 < x < L \end{array}}$$

Step 4: Plug in the initial conditions and find $\{A_n, B_n \mid n = 1, 2, \ldots\}$.

$$u_t(x,0) = g(x), \quad u(x,t) = \sum_{n=1}^{\infty} \left\{ A_n \cos\left(\frac{n\pi a}{L} t\right) + B_n \sin\left(\frac{n\pi a}{L} t\right) \right\} \sin\left(\frac{n\pi}{L} x\right)$$

$$\Longrightarrow g(x) = \sum_{n=1}^{\infty} B_n \frac{n\pi a}{L} \sin\left(\frac{n\pi}{L} x\right), \quad 0 < x < L$$

From the Fourier sine series expansion on $(0, L)$, we get

$$B_n \frac{n\pi a}{L} = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi}{L} x\right)\, dx. \Longrightarrow B_n = \frac{2}{n\pi a} \int_0^L g(x) \sin\left(\frac{n\pi}{L} x\right)\, dx.$$

Final Solution

Solve $u(x,t)$: $\quad au_{xx} = u_{tt}, \quad 0 < x < L, \quad t > 0$

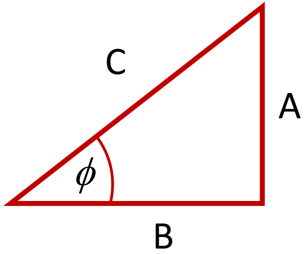
subject to : $\quad u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0$

$\quad u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 < x < L$

Step 5: The final solution is

$$u(x,t) = \sum_{n=1}^{\infty} \left\{ A_n \cos\left(\frac{n\pi a}{L} t\right) + B_n \sin\left(\frac{n\pi a}{L} t\right) \right\} \sin\left(\frac{n\pi}{L} x\right)$$

$$= \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi a}{L} t + \phi_n\right) \sin\left(\frac{n\pi}{L} x\right)$$



$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) \, dx, \quad B_n = \frac{2}{n\pi a} \int_0^L g(x) \sin\left(\frac{n\pi}{L} x\right) \, dx$$

$$C_n = \sqrt{A_n^2 + B_n^2}, \quad \sin\phi_n = \frac{A_n}{C_n}, \; \cos\phi_n = \frac{B_n}{C_n}$$

Standing Waves

The final solution

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi a}{L} t + \phi_n\right) \sin\left(\frac{n\pi}{L} x\right)$$

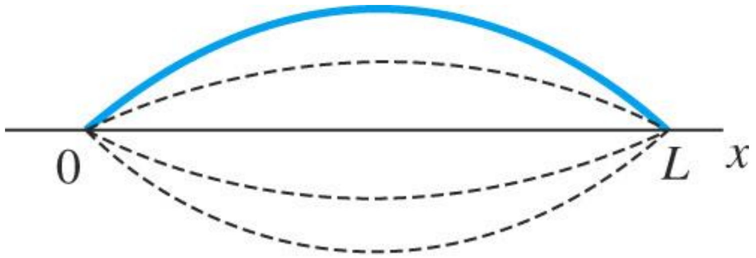
is a linear combination of **standing waves** or **normal modes**

$$u_n(x,t) = \boxed{\color{blue}{C_n} \boxed{\color{blue}{\sin\left(\frac{n\pi a}{L} t + \phi_n\right)}} \sin\left(\frac{n\pi}{L} x\right)}, \quad n = 1, 2, \ldots$$

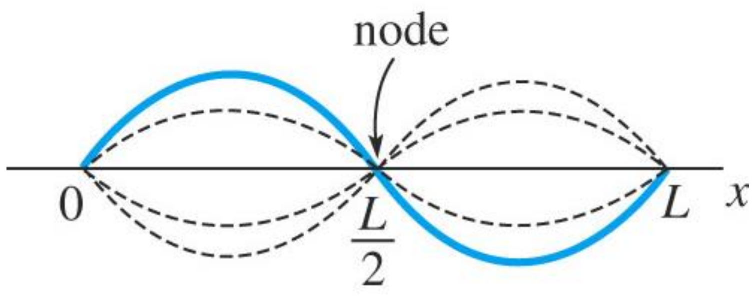
For a normal mode n, at a fixed location x, the string moves with

- time-varying amplitude $\boxed{C_n \sin\left(\frac{n\pi}{L} x\right)}$
- frequency $f_n := \frac{n\pi a/L}{2\pi} = \frac{na}{2L}$

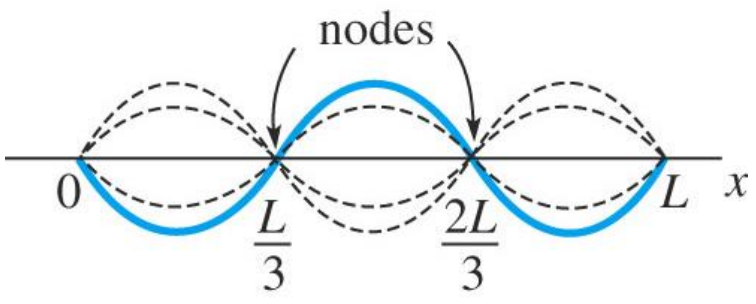
Fundamental Frequency: $f_1 := \frac{\pi a/L}{2\pi} = \frac{a}{2L}$



(a) First standing wave



(b) Second standing wave



(c) Third standing wave

- When n = 1, $u_1(x, t)$ is called the first standing wave, the first normal mode or the fundamental mode of vibration. The frequency $f_1 = a/2L$ of the first normal mode is called the fundamental frequency or first harmonic. See Fig.

$$f_1 = \frac{a}{2L} = \frac{1}{2L}\sqrt{\frac{T}{\rho}}$$

- So there are three ways to increase the frequency of a sound, producing a “higher” note:

Increase *a* (continuous changes) Tuning

Increase *n* (discrete changes) Overblowing Playing harmonics

$$\text{frequency} = \frac{an}{2L}$$

Decrease *L* (continuous changes) Shortening Fretting

- With wind instruments, it’s not always true that if L is the length of the tube, $u(0, t) = u(L, t) = 0$. This is true with the flute (because the pressure doesn’t change at the ends).

However, what happens with some instruments (like the sax) is that $u_x(0, t) = u_x(L, t) = 0$, which means that the $X(x)$ functions are cosine terms rather than sines. Still more bizarre is the behavior of the clarinet. It has boundary conditions

$$u(0,t)=0$$
$$u_x(L,t)=0$$

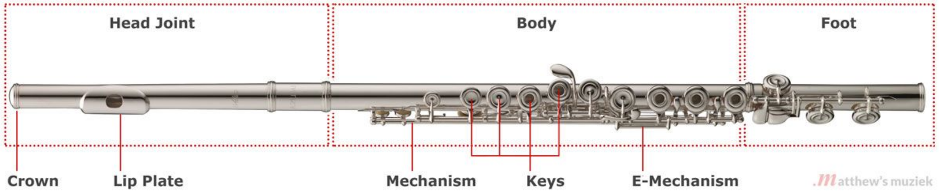
The harmonics we hear on a string instrument and the overtones on a wind instrument are actually produced by **standing waves**, which are solutions of the form

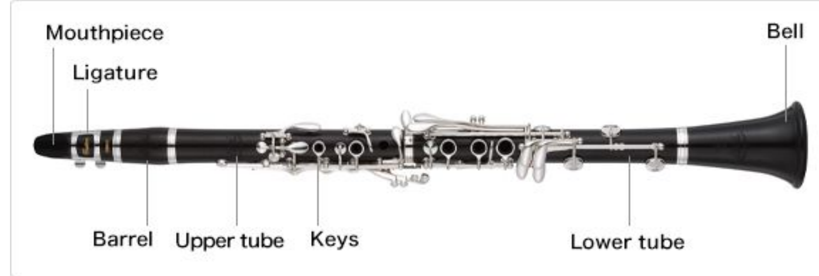
$$b\sin\frac{n\pi x}{L}\cos\frac{an\pi t}{L}$$

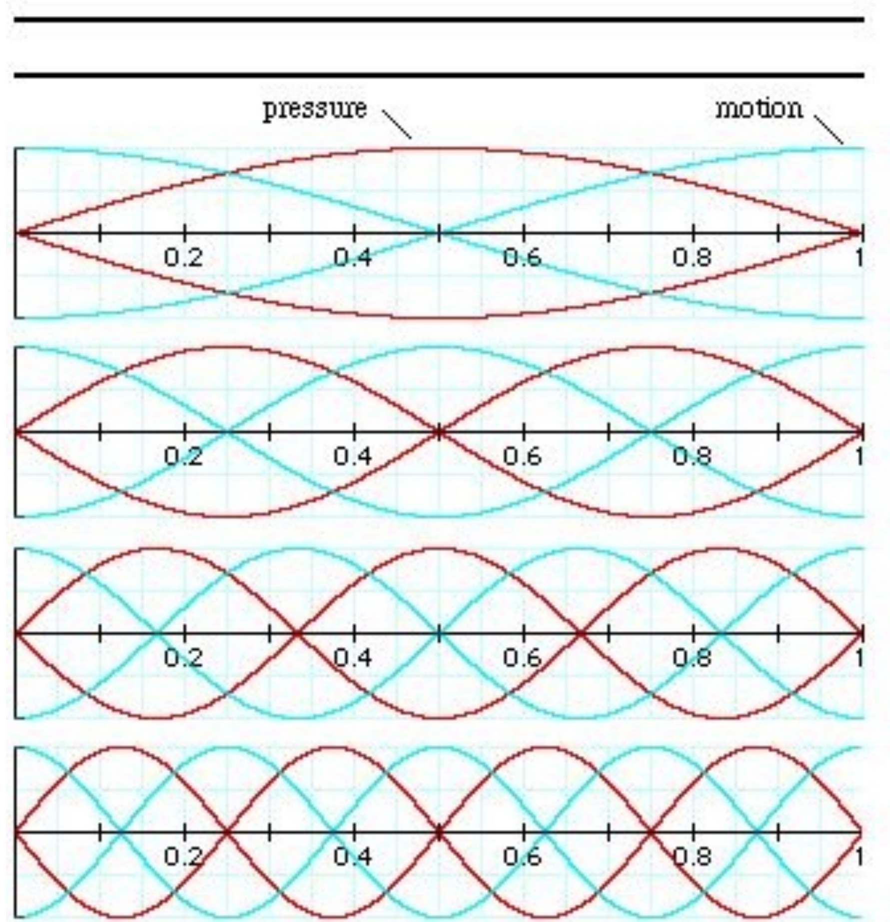
Sketch some standing waves

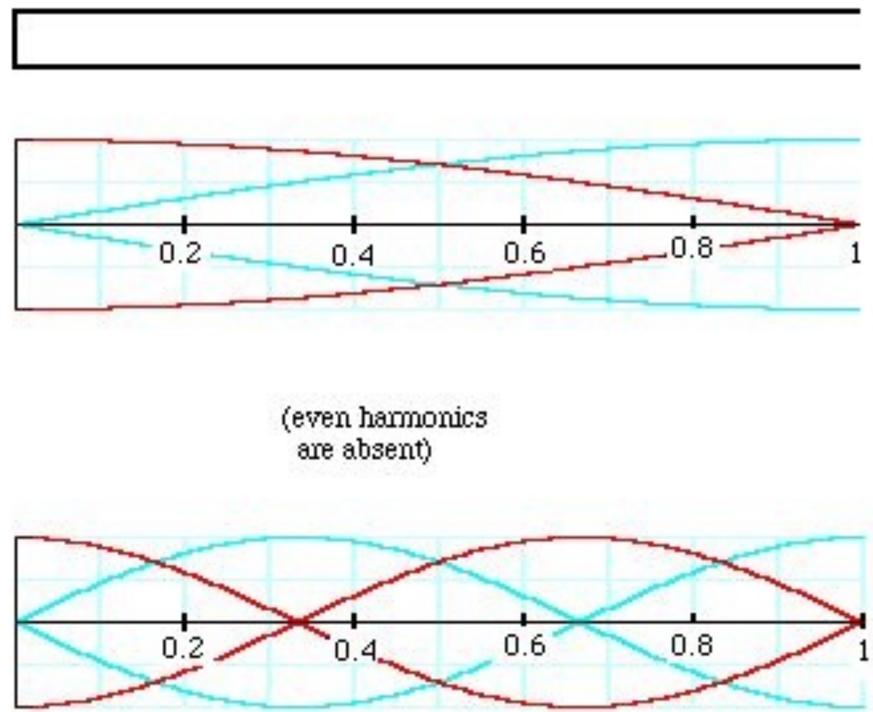
How are the standing waves for a clarinet different from those for a flute?

Flute and Clarinet standing waves and harmonics









Elastic String with Nonzero Initial Displacement. First suppose that the string is disturbed from its equilibrium position and then released at time $t = 0$ with zero velocity to vibrate freely. Then the vertical displacement $u(x,t)$ must satisfy the wave equation

$$a^2 u_{xx} = u_{tt}, \qquad 0 < x < L, \quad t > 0;$$

the boundary conditions

$$u(0,t) = 0, \qquad u(L,t) = 0, \qquad t \geq 0;$$

and the initial conditions

$$u(x,0) = f(x), \qquad u_t(x,0) = 0, \qquad 0 \leq x \leq L, \tag{9}$$

where f is a given function describing the configuration of the string at $t = 0$.

The method of separation of variables can be used to obtain the solution

Assuming that

$$u(x,t) = X(x)T(t) \tag{10}$$

and substituting for u in Eq. (1), we obtain

$$\frac{X''}{X} = \frac{1}{a^2}\frac{T''}{T} = -\lambda, \tag{11}$$

where λ is a separation constant. Thus we find that $X(x)$ and $T(t)$ satisfy the ordinary differential equations

$$X'' + \lambda X = 0, \tag{12}$$

$$T'' + a^2\lambda T = 0. \tag{13}$$

Further, by substituting from Eq. (10) for $u(x,t)$ in the boundary conditions , we find that $X(x)$ must satisfy the boundary conditions

$$X(0) = 0, \qquad X(L) = 0. \tag{14}$$

Finally, by substituting from Eq. (10) into the second of the initial conditions (9), we also find that $T(t)$ must satisfy the initial condition

$$T'(0) = 0. \tag{15}$$

Our next task is to determine $X(x)$, $T(t)$, and λ by solving Eq. (12) subject to the boundary conditions (14) and Eq. (13) subject to the initial condition (15).

The problem of solving the differential equation (12) subject to the boundary conditions (14) is *precisely the same problem* that arose in connection with the heat conduction equation. Thus we can use the results obtained there and at the end of Section the problem (12), (14) has nontrivial solutions if and only if λ is an eigenvalue

$$\lambda = n^2\pi^2/L^2, \qquad n = 1, 2, \ldots, \tag{16}$$

and $X(x)$ is proportional to the corresponding eigenfunction $\sin(n\pi x/L)$.

Using the values of λ given by Eq. (16) in Eq. (13), we obtain

$$T'' + \frac{n^2\pi^2 a^2}{L^2} T = 0. \tag{17}$$

Therefore

$$T(t) = k_1 \cos\frac{n\pi at}{L} + k_2 \sin\frac{n\pi at}{L}, \tag{18}$$

where k_1 and k_2 are arbitrary constants. The initial condition (15) requires that $k_2 = 0$, so $T(t)$ must be proportional to $\cos(n\pi at/L)$.

Thus the functions

$$u_n(x,t) = \sin\frac{n\pi x}{L} \cos\frac{n\pi at}{L}, \qquad n = 1, 2, \ldots \tag{19}$$

satisfy the partial differential equation , the boundary conditions , and the second initial condition (9). These functions are the fundamental solutions for the given problem.

To satisfy the remaining (nonhomogeneous) initial condition (9), we will consider a superposition of the fundamental solutions (19) with properly chosen coefficients. Thus we assume that $u(x,t)$ has the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L} \cos\frac{n\pi at}{L}, \tag{20}$$

where the constants c_n remain to be chosen. The initial condition $u(x,0) = f(x)$ requires that

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L} = f(x). \tag{21}$$

Consequently, the coefficients c_n must be the coefficients in the Fourier sine series of period $2L$ for f; hence

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx, \qquad n = 1, 2, \ldots. \tag{22}$$

Thus the formal solution of the problem is given by Eq. (20) with the coefficients calculated from Eq. (22).

Example : Find the solution to the one dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi \quad , \quad t > 0$$

$$u(x,0) = \sin(3x) - 4\sin(10x), \quad \frac{\partial u}{\partial t}(x,0) = 2\sin(4x) + \sin(6x) \quad 0 \le x \le \pi$$

$$u(0,t) = u(\pi,t) = 0, \qquad t \ge 0$$

Solution : $$u(x,t) = \sum_{n=1}^{\infty}\left[a_n \cos\left(\left(\frac{n\pi\alpha}{L}\right)t\right) + b_n \sin\left(\left(\frac{n\pi\alpha}{L}\right)t\right)\right]\sin\left(\frac{n\pi x}{L}\right)$$

$\alpha = 2, \ L = \pi$ $$u(x,t) = \sum_{n=1}^{\infty}\left[a_n \cos\left((2n)t\right) + b_n \sin\left((2n)t\right)\right]\sin(nx)$$

when $t = 0$, $$u(x,0) = \sum_{n=1}^{\infty} a_n \sin(nx)$$

when $n = 3$, $\sin(3x) = a_3 \sin(3x)$

$$a_3 = 1$$

when $n = 10$, $-4\sin(10x) = a_{10}\sin(10x)$

$$a_{10} = -4$$

$$\frac{\partial u}{\partial t}(x,0) = 2\sin(4x) + \sin(6x)$$

$$u_t(x,t) = \sum_{n=1}^{\infty}\left[-2na_n \sin((2n)t) + 2nb_n \cos((2n)t)\right]\sin(nx)$$

when $t = 0$, $$u_t(x,0) = \sum_{n=1}^{\infty}[2nb_n]\sin(nx)$$

when $n = 4$, $$8b_4 \sin(4x) = 2\sin(4x) \Rightarrow b_4 = \frac{1}{4}$$

when $n = 6$, $$12b_6 \sin(6x) = \sin(6x) \Rightarrow b_6 = \frac{1}{12}$$

$$u(x,t) = \left[a_n \cos((2n)t)\sin(nx) + b_n \sin((2n)t)\sin(nx)\right]$$

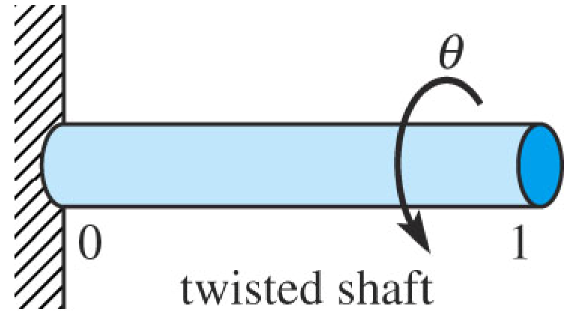
$$u(x,t) = \cos(6t)\sin(3x) + \frac{1}{4}\sin(8t)\sin(4x) + \frac{1}{12}\sin(12t)\sin(6x) - 4\cos(20t)\sin(10x)$$

Example The PDE is described by

$$a^2 \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 \theta}{\partial t^2}, \qquad 0 < x < 1, \qquad t > 0$$

$$\theta(0,t) = 0, \boxed{\frac{\partial \theta}{\partial x} |x = 1} = 0, \qquad t > 0$$

$$\theta(x,0) = x, \boxed{\frac{\partial \theta}{\partial t} |t = 0} = 0, \qquad 0 < x < 1$$



Solution

Similarly we have $\quad X'' + \lambda X = 0 \qquad T'' + a^2 \lambda T = 0$

$$X(0) = 0 \ \text{ and } \quad X'(1) = 0$$

together with the homogeneous boundary conditions,

$$X'' + \lambda X = 0, \qquad x(0) = 0, \qquad X'(1) = 0$$

is a regular Sturm-Liouville problem.

For $\lambda = 0$ and $\lambda = -\alpha^2$, $\alpha > 0$, the only solution is $X = 0$. For $\lambda = \alpha^2$, $\alpha > 0$, applying $X(0) = 0$ and $X'(1) = 0$ to the solution $X = c_1 \cos \alpha x + c_2 \sin \alpha x$ implies $c_1 = 0$, $c_2 \cos \alpha = 0$.

Thus $\alpha_n = (2n - 1)\pi/2$ and the eigenvalues are $\lambda_n = \alpha_n^2 = (2n - 1)^2\pi^2/4$, and the corresponding eigenfunctions are

$$X(x) = c_2 \sin \alpha_n x = c_2 \sin\left(\frac{2n-1}{2}\right)\pi x, n = 1,2,3,\ldots$$

The initial condition $\theta_t(x, 0) = 0$ implies $X(x)T'(0) = 0$ or $T'(0) = 0$. When applied to

$T(t) = c_3 \cos a\alpha_n t + c_4 \sin a\alpha_n t$ of (10) implies $c_4 = 0$, $T(t) = c_3 \cos a\alpha_n t = c_3 \cos a((2n - 1)/2)\pi t$.

Thus

$$\theta_n = XT = A_n \cos a\left(\frac{2n-1}{2}\right)\pi t \sin\left(\frac{2n-1}{2}\right)\pi x$$

$$\theta(x,t) = \sum_{n=1}^{\infty} A_n \cos a\left(\frac{2n-1}{2}\right)\pi t \sin\left(\frac{2n-1}{2}\right)\pi x$$

When $t = 0$, we must have, for $0 < x < 1$,

$$\theta(x,\ 0) = x = \sum_{n=1}^{\infty} A_n \sin\ \left(\frac{2n-1}{2}\right)\ \pi x$$

The set $\{\sin((2n-1)/2)\pi x\}$ is orthogonal w.r.t. the weight function $p(x) = 1$ on $[0, 1]$. We have

$$A_n = \frac{\int_0^1 x\sin\left(\frac{2n-1}{2}\right)\pi x dx}{\int_0^1 \sin^2\left(\frac{2n-1}{2}\right)\pi x dx} = \frac{8(-1)^{n+1}}{(2n-1)^2\pi^2}$$

Finally

$$\theta(x,t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \cos a\left(\frac{2n-1}{2}\right)\pi t \sin\left(\frac{2n-1}{2}\right)\pi x$$

Fixed Ends with a Forcing Term

Example We will solve the problem

$$u_{tt} = c^2 u_{xx} + \boxed{Ae^x} \text{ for } 0 < x < L, t > 0,$$
$$u(0,t) = u(L,t) = 0 \text{ for } 0 < x < L, t > 0,$$
$$u(x,0) = \varphi(x), u_t(x,0) = 0 \text{ for } 0 < x < L.$$

The initial velocity is zero and A is a positive constant.

If $u(x,t) = X(x)T(t)$ is substituted into this wave equation, we obtain

$$XT'' = c^2 X''T + Ae^x,$$

and it is impossible to separate the variables.

Try putting

$$u(x,t) = v(x,t) + f(x),$$

as we did with the heat equation when the ends of the bar were kept at different temperatures. Now

$$u_{tt} = v_{tt} = c^2 u_{xx} = c^2(v_{xx} + f''(x)) + Ae^x.$$

This will be a standard wave equation $v_{tt} = c^2 v_{xx}$ for $v(x,t)$ if we choose $f(x)$ so that

$$c^2 f''(x) + Ae^x = 0.$$

We need

$$f''(x) = -\frac{A}{c^2}e^x.$$

Integrate twice to get

$$f(x) = -\frac{A}{c^2}e^x + K_1x + K_2,$$

in which K_1 and K_2 have to be chosen to make this method work.

Now look at the boundary conditions. First,

$$u(0,t) = 0 = v(0,t) + f(0) = v(0,t) - \frac{A}{c^2} + K_2.$$

This will allow us to set $v(0,t) = 0$ if we choose

$$K_2 = \frac{A}{c^2}.$$

So far

$$f(x) = -\frac{A}{c^2}e^x + K_1x + \frac{A}{c^2}.$$

Next

$$\begin{aligned} u(L,t) &= v(L,t) + f(L) \\ &= v(L,t) - \frac{A}{c^2}e^L + K_1L + \frac{A}{c^2} = 0 \end{aligned}$$

and we can set $v(L,t)=0$ if

$$-\frac{A}{c^2}e^L + K_1 L + \frac{A}{c^2} = 0.$$

This leads us to set

$$K_1 = \frac{A}{Lc^2}e^L - \frac{A}{Lc^2} = \frac{A}{Lc^2}\left(e^L - 1\right).$$

We now have

$$f(x) = -\frac{A}{c^2}e^x + \frac{A}{Lc^2}\left(e^L - 1\right)x + \frac{A}{c^2}.$$

Finally, look at the initial conditions. First,

$$u(x,0) = v(x,0) + f(x) = \varphi(x),$$

so

$$v(x,0) = \varphi(x) - f(x).$$

And

$$u_t(x,0) = 0 = v_t(x,0).$$

The problem for $v(x,t)$ is, therefore,

$$\begin{aligned} v_{tt} &= c^2 v_{xx} \text{ for } 0 < x < L, t > 0, \\ v(0,t) &= v(L,t) = 0 \text{ for } t > 0, \\ v(x,0) &= \varphi(x) - f(x), v_t(x,0) = 0. \end{aligned}$$

This problem has the solution

$$v(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right),$$

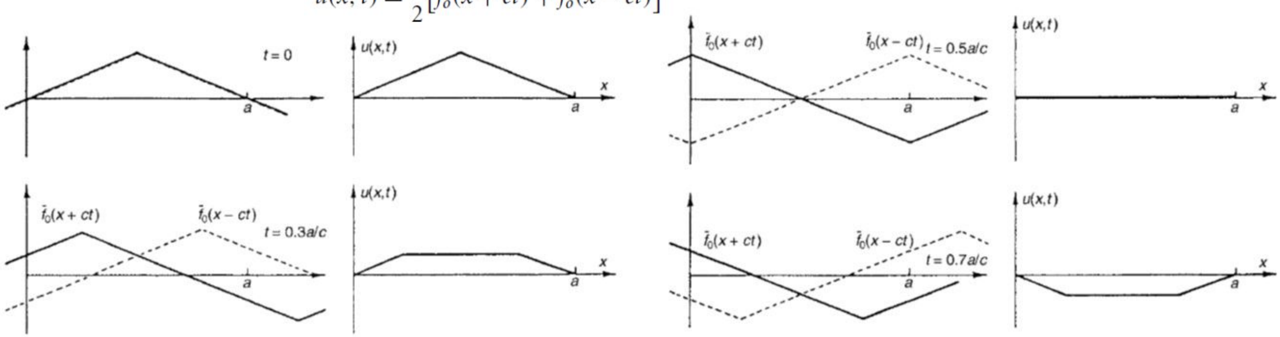
with

$$c_n = \frac{2}{L} \int_0^L (\varphi(\xi) - f(\xi)) \sin\left(\frac{n\pi x}{L}\right) d\xi.$$

The solution of the forced problem for $u(x,t)$ is

$$u(x,t) = v(x,t) - \frac{A}{c^2} e^x + \frac{A}{Lc^2}\left(e^L - 1\right) x + \frac{A}{c^2}.$$

$$u(x,t) = \frac{1}{2}[\bar{f}_o(x+ct) + \bar{f}_o(x-ct)]$$



d'Alembert's Solution

The first problem we will solve using this observation is called the *Cauchy problem for the wave equation on the real line*:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \text{ for } -\infty < x < \infty, t > 0, \\ u(x,0) &= \varphi(x), u_t(x,0) = \psi(x) \text{ for } -\infty < x < \infty. \end{aligned} \tag{3.7}$$

This problem is posed on the entire real line, so there are no end points and no boundary conditions. We will write a solution for this problem in terms of the initial position and velocity functions. Begin with the fact that $u(x,t)$ must look like

$$u(x,t) = F(x-ct) + G(x+ct).$$

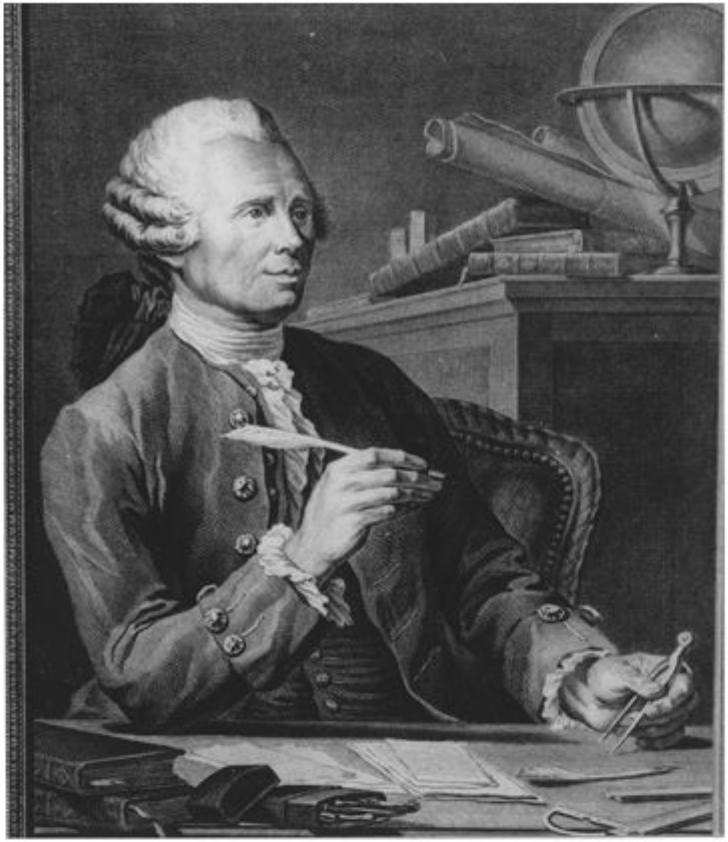
The idea is to use φ and ψ to choose F and G to produce a solution satisfying given initial conditions. First,

$$u(x,0) = F(x) + G(x) = \varphi(x) \tag{3.8}$$

and, by chain rule differentiations,

$$u_t(x,0) = -cF'(x) + cG'(x) = \psi(x). \tag{3.9}$$

Integrate equation 3.9 and rearrange terms to obtain



Although largely self-educated in mathematics, Jean Le Rond d'Alembert (1717–1783) gained equal fame as a mathematician and *philosophe* of the continental Enlightenment. By the middle of the eighteenth century, he stood with such leading European mathematicians and mathematical physicists as Clairaut, D. Bernoulli, and Euler. Today we best remember him for his work in fluid dynamics and applying partial differential equations to problems in physics. (Portrait courtesy of the Archives de l'Académie des sciences, Paris.)

$$-F(x) + G(x) = \frac{1}{c}\int_0^x \psi(s)\,ds - F(0) + G(0).$$

Add this to equation 3.8 to obtain

$$2G(x) = \varphi(x) + \frac{1}{c}\int_0^x \psi(s)\,ds - F(0) + G(0).$$

Solve this for $G(x)$:

$$G(x) = \frac{1}{2}\varphi(x) + \frac{1}{2c}\int_0^x \psi(s)\,ds - \frac{1}{2}F(0) + \frac{1}{2}G(0). \tag{3.10}$$

But then, from equation 3.8,

$$\begin{aligned} F(x) &= \varphi(x) - G(x) \\ &= \frac{1}{2}\varphi(x) - \frac{1}{2c}\int_0^x \psi(s)\,ds + \frac{1}{2}F(0) - \frac{1}{2}G(0). \end{aligned} \tag{3.11}$$

Finally, use equations 3.10 and 3.11 to write

$$u(x,t) = F(x-ct) + G(x+ct)$$

$$\boxed{\begin{aligned} u(x,t) &= \frac{1}{2}\varphi(x-ct) - \frac{1}{2c}\int_0^{x-ct}\psi(s)\,ds + \frac{1}{2}(F(0)-G(0)) \\ &+ \frac{1}{2}\varphi(x+ct) + \frac{1}{2c}\int_0^{x+ct}\psi(s)\,ds - \frac{1}{2}(F(0)-G(0)). \end{aligned}}$$

Upon making cancelations and combining the two integrals, we obtain *d'Alembert's solution of the Cauchy problem for the wave equation on the real line*:

$$\boxed{u(x,t) = \frac{1}{2}(\varphi(x-ct)+\varphi(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)\,ds.} \qquad (3.12)$$

Example We will solve the Cauchy problem

$$u_{tt} = 9u_{xx} \text{ for } -\infty < x < \infty, t > 0,$$
$$u(x,0) = \cos(x), u_t(x,0) = \sin(2x) \text{ for } -\infty < x < \infty.$$

With $\varphi(x) = \cos(x)$ and $\psi(x) = \sin(2x)$, and $c = 3$, the solution 3.12 is

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}(\cos(x-3t)+\cos(x+3t)) \\
&\quad + \frac{1}{6}\int_{x-3t}^{x+3t} \sin(2s)\,ds \\
&= \frac{1}{2}(\cos(x-3t)+\cos(x+3t)) \\
&\quad - \frac{1}{12}(\cos(2(x+3t))-\cos(2(x-3t))).
\end{aligned}
$$

$$
\boxed{\begin{aligned}
\cos A + \cos B &= 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} \\
\cos A - \cos B &= -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}
\end{aligned}}
$$

Using trigonometric identities, this solution can also be written

$$
u(x,t) = \cos(x)\cos(3t) + \frac{1}{6}\sin(2x)\sin(6t).
$$

From the d'Alembert solution, we know that the Cauchy problem has a solution, and that this solution is unique. We will also show that the solution depends continuously on the initial data, so this Cauchy problem is well posed.

The solution of the nonhomogeneous wave equation

nonhomogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + f(x,t)$$

is given by

$$u(x,t) = \frac{h(x-ct)+h(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} k(\sigma)d\sigma + \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\sigma,\tau)d\sigma d\tau.$$

An important and useful result can be derived directly from the general solution of the wave equation, and the fact that its characteristics are $x - ct$ = constant and $x + ct$ = constant.

Example 3.10 We will solve the problem

$$u_{tt} = 4u_{xx} + x\cos(t) \text{ for } -\infty < x < \infty, t > 0,$$
$$u(x,0) = e^{-x}, u_t(x,0) = \sin(x) \text{ for } -\infty < x < \infty.$$

The solution is first evaluated by the double integral with c = 2.

$$
\begin{aligned}
\iint_{\Delta} X\cos(T)\,dX\,dT &= \int_0^t \left(\int_{x-2t+2T}^{x+2t-2T} X\,dX\right)\cos(T)\,dT \\
&= \int_0^t \left[\frac{1}{2}X^2\right]_{x-2t+2T}^{x+2t-2T} \cos(T)\,dT \\
&= \int_0^t \frac{1}{2}\left((x+2t-2T)^2-(x-2t+2T)^2\right)\cos(T)\,dT \\
&= \int_0^t 4x(t-T)\cos(T)\,dT \\
&= 4x(1-\cos(t)).
\end{aligned}
$$

The solution is

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}\left(e^{-(x-2t)}+e^{-(x+2t)}\right)+\frac{1}{4}\int_{x-2t}^{x+2t}\sin(s)\,ds + x(1-\cos(t)) \\
&= e^{-x}\cosh(t)-\frac{1}{4}(\cos(x+2t)-\cos(x-2t))+x(1-\cos(t)).
\end{aligned}
$$

With some manipulation, we can also write

$$
u(x,t) = e^{-x}\cosh(t)+\frac{1}{2}\sin(x)\sin(2t)+x(1-\cos(x)).
$$

Example Solve the problem

$$\begin{array}{ll} u_{tt} - u_{xx} = t^7 & -\infty < x < \infty,\ t > 0, \\ u(x,0) = 2x + \sin x & -\infty < x < \infty, \\ u_t(x,0) = 0 & -\infty < x < \infty. \end{array}$$

Because of the special form of the nonhomogeneous equation, we look for a particular solution of the form $v = v(t)$. Indeed it can be easily verified that $v(x,t) = \frac{1}{72}t^9$ is such a solution. Consequently, we need to solve the homogeneous problem

$$\begin{array}{ll} w_{tt} - w_{xx} = 0 & -\infty < x < \infty,\ t > 0, \\ w(x,0) = f(x) - v(x,0) = 2x + \sin x & -\infty < x < \infty, \\ w_t(x,0) = g(x) - v_t(x,0) = 0 & -\infty < x < \infty. \end{array}$$

Using d'Alembert's formula for the homogeneous equation, we have

$$w(x,t) = 2x + \tfrac{1}{2}\sin(x+t) + \tfrac{1}{2}\sin(x-t),$$

and the solution of the original problem is given by $\boxed{\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}$

$$u(x,t) = 2x + \sin x\cos(t) + \frac{t^9}{72}.$$

Using nonhomogeneous formula

$$u(x,t)=\frac{h(x-ct)+h(x+ct)}{2}+\frac{1}{2c}\int_{x-ct}^{x+ct}k(\sigma)d\sigma$$
$$+\frac{1}{2c}\int_0^t\int_{x-c(t-\tau)}^{x+c(t-\tau)}f(\sigma,\tau)d\sigma d\tau.$$

The 1st and 2nd solution are still the same so finding the 3rd solution by formula instead.

3rd term $=\frac{1}{2}\int_0^t\int_{x-(t-T)}^{x+(t-T)}T^7dXdT=\frac{1}{2}\int_0^t\left(XT^7\Big|_{x-(t-T)}^{x+(t-T)}\right)dT=\frac{1}{2}\int_0^t T^7[x+(t-T)-x+(t-T)]dT$

$$=\frac{1}{2}\int_0^t T^7[x+(t-T)-x+(t-T)]dT=\frac{1}{2}\int_0^t 2(t-T)T^7dT=\frac{1}{2}\int_0^t 2(tT^7-T^8)dT$$

$$=\frac{1}{2}\int_0^t 2(tT^7-T^8)dT=\int_0^t\frac{tT^8}{8}-\frac{T^9}{9}dT=\frac{tT^8}{8}-\frac{T^9}{9}\Big|_0^t=\frac{9t^9-8t^9}{72}=\frac{t^9}{72}$$

$$u(x,t)=2x+\sin(x)\cos(t)+\frac{t^9}{72}$$

Solution of the Semi-infinite String via the D'Alembert Formula

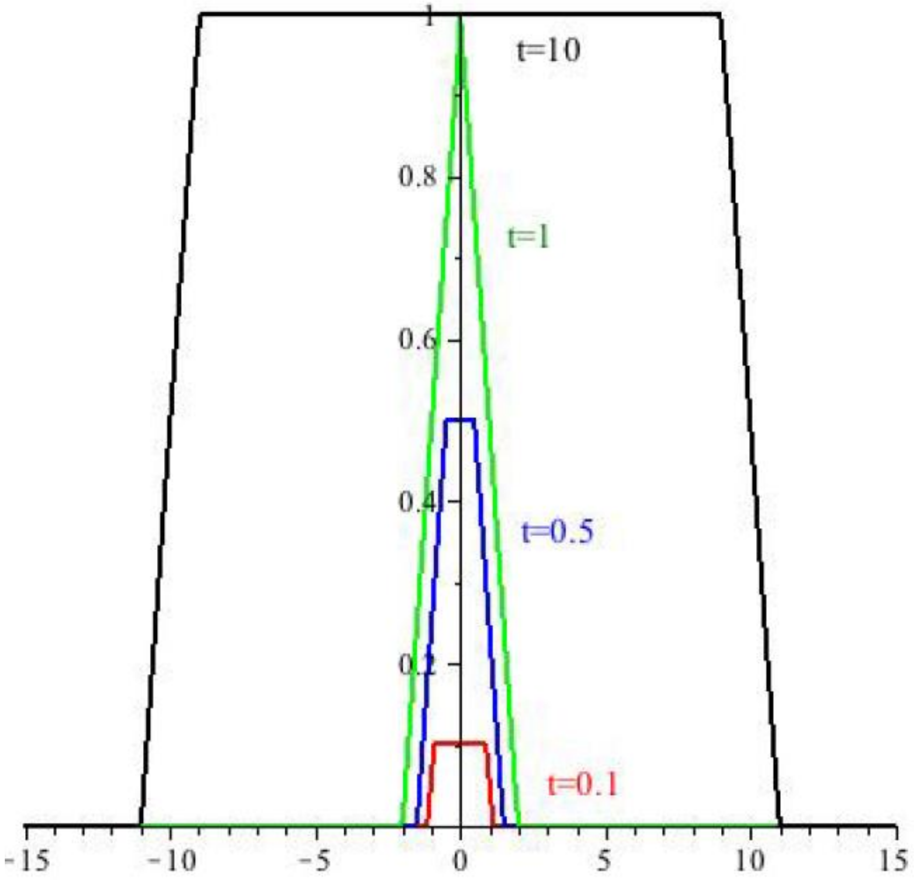


Fig. 13.2a Solution of problem for various values of time

We will solve the IBVP for the semi-infinite string

Problem

To find the function $u(x,t)$ that satisfies

$$\text{PDE:} \quad u_{tt} = c^2 u_{xx}, \qquad 0 < x < \infty, \quad 0 < t < \infty$$

$$\text{BC:} \quad u(0,t) = 0, \qquad 0 < t < \infty$$

$$\text{ICs:} \quad \begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases} \qquad 0 < x < \infty$$

by modifying the D'Alembert formula. To find the solution of problem , we proceed in a manner similar to that used with the infinite string.

- We find the general solution to the PDE

$$u(x,t) = \phi(x - ct) + \psi(x + ct).$$

- Substituting this general solution into ICs we arrive at

$$
\begin{aligned}
\phi(x-ct) &= \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_{x_0}^{x-ct} g(s)ds \\
\psi(x+ct) &= \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_{x_0}^{x+ct} g(s)ds
\end{aligned} \tag{13.1}
$$

We now have a problem that we didn't encounter when dealing with the infinite string.

- Since we are looking for the solution $u(x,t)$ for $x > 0$ and $t > 0$, it is obvious that we must find

$$
\begin{aligned}
\phi(x-ct) \quad &\forall \quad -\infty < x-ct < \infty \\
\psi(x+ct) \quad &\forall \quad 0 < x+ct < \infty.
\end{aligned}
$$

- Unfortunately, the first equation in (13.1) only gives us $\phi(x-ct)$ for $x-ct \geqslant 0$, since our initial data $f(x)$ and $g(x)$ are only known for positive arguments.

- As long as $x - ct \geqslant 0$, we have

$$
\begin{aligned}
u(x,t) &= \phi(x-ct) + \psi(x+ct) \\
&= \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds.
\end{aligned}
$$

- The question is, what to do when $x < ct$?
- When $x < ct$, we use our BC. Substituting the general solution u into the BC $u(0,t) = 0$ gives

$$\phi(-ct) = -\psi(ct)$$

- Hence, by functional substitution

$$\phi(x - ct) = -\psi(ct - x) = -\frac{1}{2} f(ct - x) - \frac{1}{2c} \int_{x_0}^{ct-x} g(s)ds$$

- Substituting this value of ϕ into the general solution gives

$$u(x,t) = \frac{1}{2}[f(x+ct) - f(ct-x)] + \frac{1}{2c} \int_{ct-x}^{x+ct} g(s)ds \quad 0 < x < ct.$$

- Combining the solutions for $x < ct$ and $x > ct$ we have our result

$$\boxed{u(x,t) = \begin{cases} \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)ds, & x \geqslant ct \\ \frac{1}{2}[f(x+ct) - f(ct-x)] + \frac{1}{2c} \int_{ct-x}^{x+ct} g(s)ds, & x < ct \end{cases}}$$

Example Solve

$$u_{tt} = u_{xx} \text{ for } x > 0, t > 0,$$
$$u(0,t) = 0 \text{ for } t > 0,$$
$$u(x,0) = \varphi(x) = 1 - e^{-x}, u_t(x,0) = \psi(x) = \cos(x) \text{ for } x \geq 0.$$

Equation gives the solution in terms of the odd extensions of $\varphi(x)$ and $\psi(x)$. These extensions are

$$\Phi(x) = \begin{cases} \varphi(x) = 1 - e^{-x} & \text{if } x \geq 0, \\ -\varphi(-x) = -1 + e^{x} & \text{if } x < 0. \end{cases}$$

and

$$\Psi(x) = \begin{cases} \psi(x) = \cos(x) & \text{if } x \geq 0, \\ -\psi(-x) = -\cos(x) & \text{if } x < 0. \end{cases}$$

The solution is

$$u(x,t) = \frac{1}{2}\left(\Phi(x-t) + \Phi(x+t)\right) + \frac{1}{2}\int_{x-t}^{x+t} \Psi(s)\, ds.$$

We will write $u(x,t)$ explicitly in terms of x and t. This requires two cases.

Case 1. If $x - t \geq 0$, then

$$\begin{aligned}
u(x,t) &= \frac{1}{2}(\boxed{\Phi(x-ct)} + \Phi(x+ct)) + \frac{1}{2}\int_{x-ct}^{x+ct} \Psi(s)\,ds \\
&= \frac{1}{2}\left(\boxed{1 - e^{-(x-t)}} + 1 - e^{-(x+t)}\right) + \frac{1}{2}\int_{x-t}^{x+t} \cos(s)\,ds \\
&= 1 - \frac{1}{2}e^{-x}\left(e^{t} + e^{-t}\right) + \frac{1}{2}(\sin(x+t) - \sin(x-t)) \\
&= 1 - e^{-x}\cosh(t) + \cos(t)\sin(t).
\end{aligned}$$

Case 2. If $x - t < 0$, then

$$\begin{aligned}
u(x,t) &= \frac{1}{2}(\boxed{\Phi(x-t)} + \Phi(x+t)) + \frac{1}{2}\int_{x-t}^{x+t} \Psi(s)\,ds \\
&= \frac{1}{2}\left(\boxed{-1 + e^{x-t}} + 1 - e^{-(x+t)}\right) \\
&\quad \boxed{+ \frac{1}{2}\int_{x-t}^{0} -\cos(s)\,ds} + \int_{0}^{x+t} \cos(s)\,ds \\
&= \frac{1}{2}\left(e^{x}e^{-t} - e^{-x}e^{-t}\right) - \left[\frac{1}{2}\sin(s)\right]_{x-t}^{0} + \left[\frac{1}{2}\sin(s)\right]_{0}^{x+t}.
\end{aligned}$$

After some manipulation, this yields

$$u(x,t) = e^{-t}\sinh(x) + \sin(x)\cos(t).$$

In summary,

$$u(x,t) = \begin{cases} 1 - e^{-x}\cosh(t) + \cos(x)\sin(t) & \text{for } x - t \geq 0, \\ e^{-t}\sinh(x) + \sin(x)\cos(t) & \text{for } x - t < 0. \end{cases}$$

EXAMPLE : Using the Laplace Transform to Solve a BVP

Solve $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < 1,\ t > 0$$

subject to $$u(0, t) = 0, \quad u(1, t) = 0,\ t > 0$$

$$u(x, 0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \sin \pi x,\ 0 < x < 1.$$

SOLUTION The partial differential equation is recognized as the wave equation with $a = 1$. From (4) and the given initial conditions, the transformed equation is

$$\mathscr{L}\left\{\frac{\partial^2 u}{\partial t^2}\right\} = s^2U(x, s) - su(x, 0) - u_t(x, 0).$$

$$\frac{d^2U}{dx^2} - s^2U = -\sin \pi x, \tag{5}$$

where $U(x, s) = \mathscr{L}\{u(x, t)\}$. Since the boundary conditions are functions of t, we must also find their Laplace transforms:

$$\mathscr{L}\{u(0, t)\} = U(0, s) = 0 \quad \text{and} \quad \mathscr{L}\{u(1, t)\} = U(1, s) = 0. \tag{6}$$

The results in (6) are boundary conditions for the ordinary differential equation (5). Since (5) is defined over a finite interval, its complementary function is

$$U_c(x, s) = c_1 \cosh sx + c_2 \sinh sx.$$

The method of undetermined coefficients yields a particular solution

$$U_p(x, s) = \frac{1}{s^2 + \pi^2} \sin \pi x.$$

Hence $$U(x, s) = c_1 \cosh sx + c_2 \sinh sx + \frac{1}{s^2 + \pi^2} \sin \pi x.$$

But the conditions $U(0, s) = 0$ and $U(1, s) = 0$ yield, in turn, $c_1 = 0$ and $c_2 = 0$. We conclude that

$$U(x, s) = \frac{1}{s^2 + \pi^2} \sin \pi x$$

$$u(x, t) = \mathscr{L}^{-1}\left\{\frac{1}{s^2 + \pi^2} \sin \pi x\right\} = \frac{1}{\pi} \sin \pi x \, \mathscr{L}^{-1}\left\{\frac{\pi}{s^2 + \pi^2}\right\}.$$

Therefore $$u(x, t) = \frac{1}{\pi} \sin \pi x \sin \pi t.$$ ≡

EXAMPLE : Using the Laplace Transform to Solve a BVP

A very long string is initially at rest on the nonnegative x-axis. The string is secured at $x = 0$, and its distant right end slides down a frictionless vertical support. The string is set in motion by letting it fall under its own weight. Find the displacement $u(x, t)$.

SOLUTION Since the force of gravity is taken into consideration, it can be shown that the wave equation has the form

$$a^2 \frac{\partial^2 u}{\partial x^2} - g = \frac{\partial^2 u}{\partial t^2}, \quad x > 0, \ t > 0,$$

where g is the acceleration due to gravity. The boundary and initial conditions are, respectively,

$$u(0, t) = 0, \quad \lim_{x \to \infty} \frac{\partial u}{\partial x} = 0, \ t > 0$$

$$u(x, 0) = 0, \quad \left. \frac{\partial u}{\partial t} \right|_{t=0} = 0, \ x > 0.$$

The second boundary condition $\lim_{x\to\infty} \partial u/\partial x = 0$ indicates that the string is horizontal at a great distance from the left end. Now

$$\mathscr{L}\left\{a^2 \frac{\partial^2 u}{\partial x^2}\right\} - \mathscr{L}\{g\} = \mathscr{L}\left\{\frac{\partial^2 u}{\partial t^2}\right\}$$

becomes $$a^2 \frac{d^2 U}{dx^2} - \frac{g}{s} = s^2 U - su(x, 0) - u_t(x, 0)$$

or, in view of the initial conditions,

$$\frac{d^2 U}{dx^2} - \frac{s^2}{a^2} U = \frac{g}{a^2 s}.$$

The transforms of the boundary conditions are

$$\mathscr{L}\{u(0, t)\} = U(0, s) = 0 \qquad \text{and} \qquad \mathscr{L}\left\{\lim_{x\to\infty} \frac{\partial u}{\partial x}\right\} = \lim_{x\to\infty} \frac{dU}{dx} = 0.$$

With the aid of undetermined coefficients, the general solution of the transformed equation is found to be

$$U(x, s) = c_1 e^{-(x/a)s} + c_2 e^{(x/a)s} - \frac{g}{s^3}.$$

The boundary condition $\lim_{x\to\infty} dU/dx = 0$ implies $c_2 = 0$, and $U(0, s) = 0$ gives $c_1 = g/s^3$. Therefore

$$U(x, s) = \frac{g}{s^3} e^{-(x/a)s} - \frac{g}{s^3}.$$

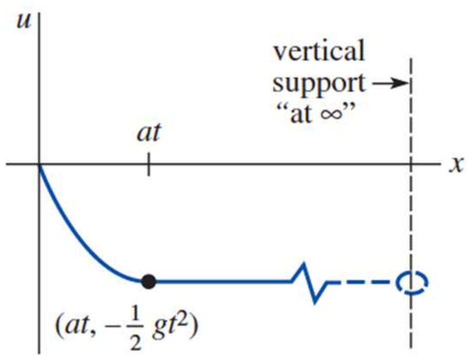
$t^n, \quad n = 1, 2, 3, \ldots$	$\dfrac{n!}{s^{n+1}}$
$u_c(t) f(t-c)$	$\mathbf{e}^{-cs} F(s)$

Now by the second translation theorem we have

$$u(x, t) = \mathscr{L}^{-1}\left\{\frac{g}{s^3} e^{-(x/a)s} - \frac{g}{s^3}\right\} = \frac{1}{2} g\left(t - \frac{x}{a}\right)^2 \mathcal{U}\left(t - \frac{x}{a}\right) - \frac{1}{2} g t^2$$

or

$$u(x, t) = \begin{cases} -\dfrac{1}{2} g t^2, & 0 \le t < \dfrac{x}{a} \\ -\dfrac{g}{2a^2}(2axt - x^2), & t \ge \dfrac{x}{a}. \end{cases}$$



To interpret the solution, let us suppose $t > 0$ is fixed. For $0 \le x \le at$, the string is the shape of a parabola passing through the points (0, 0) and $(at, -\frac{1}{2} gt^2)$. For $x > at$, the string is described by the horizontal line $u = -\frac{1}{2} gt^2$. See **FIGURE 15.2.1**. ≡

Example

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, && 0 < x < 1, \quad 0 < t,\\
&u(0, t) = 0, \quad u(1, t) = 0, && 0 < t,\\
&u(x, 0) = \sin(\pi x), && 0 < x < 1,\\
&\frac{\partial u}{\partial t}(x, 0) = -\sin(\pi x), && 0 < x < 1.
\end{aligned}
$$

Under transformation the problem becomes

$$\mathcal{L}\left\{\frac{\partial^2 u}{\partial t^2}\right\} = s^2 U(x, s) - su(x, 0) - u_t(x, 0).$$

$$
\begin{aligned}
&\frac{d^2 U}{dx^2} = s^2 U - s\sin(\pi x) + \sin(\pi x), \quad 0 < x < 1,\\
&U(0, s) = 0, \quad U(1, s) = 0.
\end{aligned}
$$

The function U is found to be $\quad U(x, s) = \dfrac{s-1}{s^2 + \pi^2}\sin(\pi x),$

from which we find the solution,

$$u(x, t) = \left(\cos(\pi t) - \frac{1}{\pi}\sin(\pi t)\right)\sin(\pi x).$$

The Finite Fourier Transforms (Sine and Cosine Transforms)

Remarks

- Earlier, we learned about the Fourier and Laplace transforms and their applications for problems in free space (no boundaries).
- Now, we show how to solve BVPs (with boundaries) by transforming the bounded variables.

The finite sine and cosine transforms are defined by

$$\begin{cases} S[f] = S_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin(n\pi x/L)dx, \quad \text{(finite sine transform)} \\ \qquad\qquad n = 1, 2, \ldots \\ f(x) = \displaystyle\sum_{n=1}^{\infty} S_n \sin(n\pi x/L) \quad \text{(inverse sine transform)} \end{cases}$$

$$\begin{cases} C[f] = C_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos(n\pi x/L)dx, \quad \text{(finite cosine transform)} \\ \qquad\qquad n = 0, 1, 2, \ldots \\ f(x) = \dfrac{C_0}{2} + \displaystyle\sum_{n=1}^{\infty} C_n \cos(n\pi x/L) \quad \text{(inverse cosine transform)} \end{cases}$$

Properties of the Transforms

- If $u(x,t)$ is a function of two variables, then (note we're transforming the x-variable)

$$S[u] = S_n(t) = \frac{2}{L}\int_0^L u(x,t)\sin(n\pi x/L)dx$$

$$C[u] = C_n(t) = \frac{2}{L}\int_0^L u(x,t)\cos(n\pi x/L)dx$$

Finite Sine Transform

	$f(x) = \sum_{n=1}^{\infty} S_n \sin(nx)$ $0 \leqslant x \leqslant \pi$	$S_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx$ $n = 1, 2, \ldots$
1.	$\sin(mx)$	$\begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$
2.	$\sum_{n=1}^{\infty} a_n \sin(nx)$	a_n
3.	$\pi - x$	$\frac{2}{n}$
4.	x	$\frac{2}{n}(-1)^{n+1}$
5.	1	$\frac{2}{n\pi}[1-(-1)^n]$
6.	$\begin{cases} -x, & x \leqslant a \\ \pi - x, & x > a \end{cases}$	$\frac{2}{n}\cos(na), \quad 0 < a < \pi$
7.	$\begin{cases} (\pi - a)x, & x \leqslant a \\ (\pi - x)a, & x > a \end{cases}$	$\frac{2}{n^2}\sin(na), \quad 0 < a < \pi$
8.	$\frac{\pi}{2}e^{ax}$	$\frac{n}{n^2+a^2}[1-(-1)^n e^{a\pi}]$
9.	$\frac{\sinh a(\pi - x)}{\sinh a\pi}$	$\frac{2n}{\pi(n^2+a^2)}$

Finite Cosine Transform

	$f(x) = \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n \cos(nx)$ $0 \leqslant x \leqslant \pi$	$C_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)dx$ $n = 0, 1, 2, \ldots$
1.	$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$	a_n
2.	$f(\pi - x)$	$(-1)^n \frac{2}{\pi} C_n$
3.	1	$\begin{cases} 2, & n = 0 \\ 0, & n = 1, 2, \ldots \end{cases}$
4.	$\cos(mx), \quad m = 1, 2, \ldots$	$\begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$

	$f(x) = \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n \cos(nx)$ $0 \leqslant x \leqslant \pi$	$C_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)dx$ $n = 0, 1, 2, \ldots$
5.	x	$\begin{cases} \pi, & n = 0 \\ \frac{2}{\pi n^2}[(-1)^n - 1], & n = 1, 2, \ldots \end{cases}$
6.	x^2	$\begin{cases} 2\pi^2/3, & n = 0 \\ \frac{4}{n^2}(-1)^n, & n = 1, 2, \ldots \end{cases}$
7.	$-\log(2\sin(x/2))$	$\begin{cases} 0, & n = 0 \\ \frac{1}{n}, & n = 1, 2, \ldots \end{cases}$
8.	$\frac{1}{a}e^{ax}$	$\frac{2}{\pi}\left[\frac{(-1)^n e^{a\pi} - 1}{n^2 + a^2}\right]$
9.	$\begin{cases} 1, & 0 < x < a \\ -1, & a < x < \pi \end{cases}$	$\begin{cases} \frac{2}{\pi}(2a - \pi), & n = 0 \\ \frac{4}{n\pi}\sin(na), & n = 1, 2, \ldots \end{cases}$

1. $S[u_t] = \dfrac{dS[u]}{dt}$
2. $S[u_{tt}] = \dfrac{d^2 S[u]}{dt^2}$
3. $S[u_{xx}] = -[n\pi/L]^2 S[u] + \dfrac{2n\pi}{L^2}\left[u(0,t) + (-1)^{n+1} u(L,t)\right]$
4. $C[u_{xx}] = -[n\pi/L]^2 C[u] - \dfrac{2}{L}\left[u_x(0,t) + (-1)^{n+1} u_x(L,t)\right]$

EXAMPLE **The wave equation for an infinite string**

Solve the boundary value problem

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2\frac{\partial^2 u}{\partial x^2} && (-\infty < x < \infty, t > 0), \\ u(x,0) &= f(x) && \text{(initial displacement)}, \\ \frac{\partial}{\partial t}u(x,0) &= g(x) && \text{(initial velocity)}. \end{aligned}$$

Assume that both f and g have Fourier transforms, and give the answer as an inverse Fourier transform.

Solution We fix t and take the Fourier transform of both sides of the differential equation and the initial conditions. Using $n = 2$, we get

$\mathscr{F}\{f''(x)\} = (-i\alpha)^2\,\mathscr{F}\{f(x)\} = -\alpha^2 F(\alpha).$

$$\begin{aligned} &(6) & \frac{d^2}{dt^2}\widehat{u}(\omega,t) &= -c^2\omega^2\widehat{u}(\omega,t), \\ &(7) & \widehat{u}(\omega,0) &= \widehat{f}(\omega), \\ &(8) & \frac{d}{dt}\widehat{u}(\omega,0) &= \widehat{g}(\omega). \end{aligned}$$

It is clear that (6) is an ordinary differential equation in $\widehat{u}(\omega,t)$, where t is the variable. Let us write (6) in the standard form

$$\frac{d^2}{dt^2}\,\widehat{u}(\omega,t) + c^2\omega^2\widehat{u}(\omega,t) = 0\,.$$

The general solution of this equation is

$$\widehat{u}(\omega, t) = A(\omega)\cos c\omega t + B(\omega)\sin c\omega t,$$

where $A(\omega)$ and $B(\omega)$ are constant in t. (You should note that while A and B are constant in t, they can depend on ω, which explains writing $A(\omega)$ and $B(\omega)$.) We determine $A(\omega)$ and $B(\omega)$ from the initial conditions (7) and (8) as follows:

$$\begin{aligned}\widehat{u}(\omega, 0) = A(\omega) &= \widehat{f}(\omega), \\ \frac{d}{dt}\widehat{u}(\omega, 0) &= c\,\omega B(\omega) = \widehat{g}(\omega).\end{aligned}$$

So

$$\widehat{u}(\omega, t) = \widehat{f}(\omega)\cos c\omega t + \frac{1}{c\omega}\,\widehat{g}(\omega)\sin c\omega t.$$

To obtain the solution we use the inverse Fourier transform and get

$$u(x, t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\widehat{f}(\omega)\cos c\omega t + \frac{1}{c\omega}\,\widehat{g}(\omega)\sin c\omega t\right]e^{i\omega x}\,d\omega. \tag{9}$$

■