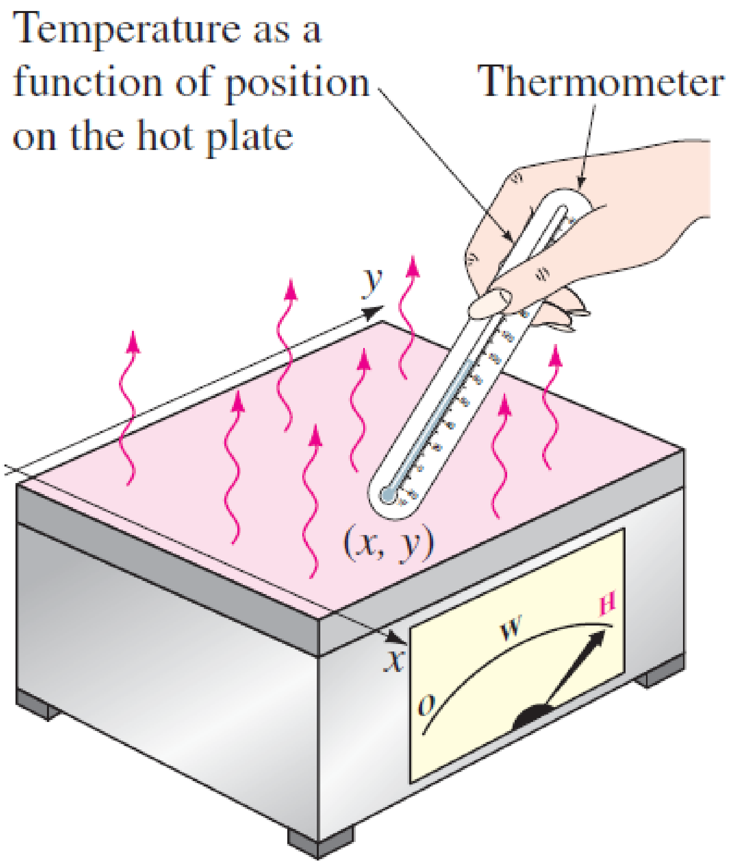
Laplace's Equation

Elliptic-Type Problems



- Laplace's equation usually occurs in *time-independent* problems involving **potentials**.

- Its solution can also be interpreted as a steady-state temperature distribution.

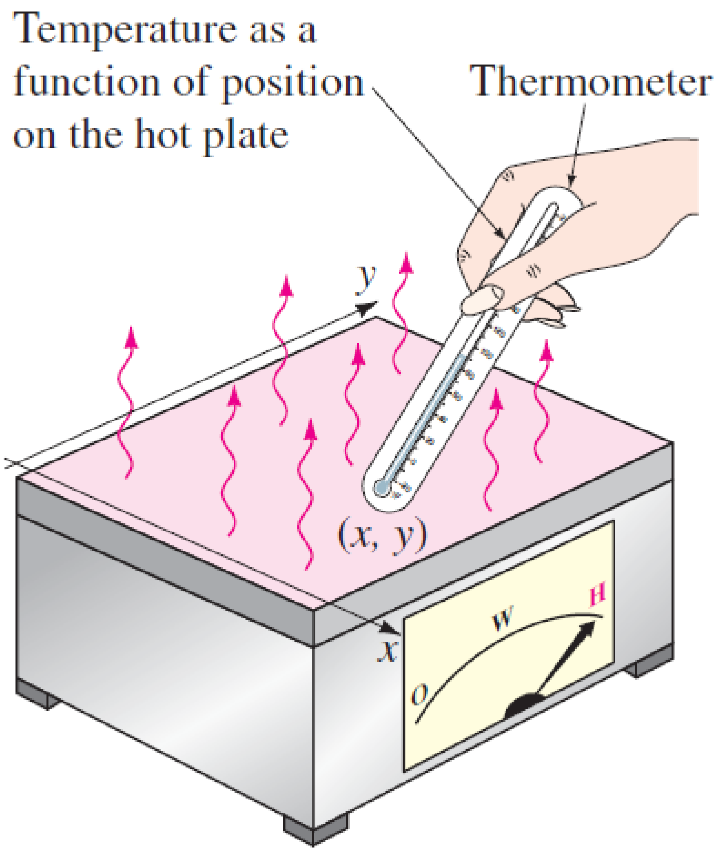
- Two-dimensional Laplace Equation

$$\nabla^2 u := u_{xx} + u_{yy} = 0$$

- Three-dimensional Laplace Equation

$$\nabla^2 u := u_{xx} + u_{yy} + u_{zz} = 0$$

Laplace's Equation: Boundary Conditions



Boundary Conditions:
In the x-direction, at the end points $x = 0$ and $x = a$, give the constraints on u, u_x, or $u_x + hu$.

In the y-direction, at the end points $y = 0$ and $y = b$, give the constraints on u, u_y, or $u_y + hu$.

Examples:

- Both ends in x are insulated

$$u_x(0, y) = 0, \quad u_x(a, y) = 0$$

- Temperatures of two ends in y are held at different distributions

$$u(x, 0) = f(x), \quad u(x, b) = g(x)$$

Example: $u_{xx} + u_{yy} = 0, \quad 0 < x < a, \quad 0 < y < b$ Laplace's equation

$u_x(0, y) = 0, \quad u_x(a, y) = 0, \quad 0 < y < b$ Boundary condition

$u(x, 0) = 0, \quad u(x, b) = f(x), \quad 0 < x < a$ Boundary condition

Solve $u(x, y)$:	$u_{xx} + u_{yy} = 0, \quad 0 < x < a, \quad 0 < y < b$
subject to :	$u_x(0, y) = 0, \quad u_x(a, y) = 0, \quad 0 < y < b$
	$u(x, 0) = 0, \quad u(x, b) = f(x), \quad 0 < x < a$

We focus on solving the above BVP (both ends $x = 0$ and $x = a$ are insulated).

Step 1: Separation of variables:
Assume that the solution $u(x, y) = X(x)\,Y(y)$, $X, Y \neq 0$. Then,

$$u_{xx} + u_{yy} = 0 \implies X''Y + XY'' = 0 \implies \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

$$\implies \begin{cases} X'' + \lambda X &= 0 \\ Y'' - \lambda Y &= 0 \end{cases}$$

The 3 homogeneous boundary conditions become $X'(0) = X'(a) = Y(0) = 0$.

Solve in the x-Dimension and Find λ

$$\boxed{\begin{array}{lll} \text{Solve :} & X'' + \lambda X = 0, & Y'' - \lambda Y = 0 \\ \text{subject to :} & X'(0) = 0, & X'(a) = 0 \\ & Y(0) = 0, & X(x)\,Y(b) = f(x), \quad 0 < x < a \end{array}}$$

Step 2: λ remains to be determined. What values should λ take?

1. $\lambda = 0$: $X(x) = c_1 + c_2 x$. $X'(0) = X'(a) = 0 \implies c_2 = 0$.
2. $\lambda = -\alpha^2 < 0$: $X(x) = c_1 e^{-\alpha x} + c_2 e^{\alpha x}$.
   Plug in $X'(0) = X'(a) = 0$, we get $c_1 = c_2 = 0$.
3. $\lambda = \alpha^2 > 0$: $X(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$.
   Plug in $X'(0) = X'(a) = 0$, we get $c_2 = 0$, and $c_1 \alpha \sin(\alpha a) = 0$.
   Hence, $c_1 \neq 0$ only if $\alpha a = n\pi$.

Since $X \neq 0$, pick $\boxed{\lambda = \dfrac{n^2\pi^2}{a^2}, \; n = 0, 1, 2, \ldots} \implies X(x) = c_1 \cos\left(\frac{n\pi}{a} x\right)$.

Solve in y-Dimension and Superposition

$$\begin{array}{lll} \text{Solve :} & X'' + \lambda X = 0, & Y'' - \lambda Y = 0 \\ \text{subject to :} & X'(0) = 0, & X'(a) = 0 \\ & Y(0) = 0, & X(x)\,Y(b) = f(x), \quad 0 < x < a \end{array}$$

Step 3: Once we fix $\lambda = \frac{n^2\pi^2}{a^2}$, $n = 0, 1, 2, \ldots$, we obtain $X(x) = c_1 \cos\left(\frac{n\pi}{a}x\right)$

$$Y(y) = \begin{cases} \cancel{c_3} + c_4 y, & n = 0 \\ \cancel{c_3 \cosh\left(\frac{n\pi}{a}y\right)} + c_4 \sinh\left(\frac{n\pi}{a}y\right), & n \geq 1 \end{cases} \quad (Y(0) = 0 \implies c_3 = 0)$$

$$\implies u_n(x, y) = \begin{cases} A_0 y, & n = 0 \\ A_n \cos\left(\frac{n\pi}{a}x\right) \sinh\left(\frac{n\pi}{a}y\right), & n \geq 1 \end{cases}, \quad (A_n := c_1 c_4)$$

$$\implies u(x, y) := \sum_{n=0}^{\infty} u_n(x, y) \text{ is a solution, by the superposition principle.}$$

Plug in Initial Condition, Revoke Fourier Series, and Done

$$\boxed{\begin{array}{ll} \text{Solve } u(x,y): & u_{xx}+u_{yy}=0, \quad 0<x<a, \quad 0<y<b \\ \quad\text{subject to}: & u_x(0,y)=0, \quad u_x(a,y)=0, \quad 0<y<b \\ & u(x,0)=0, \quad u(x,b)=f(x), \quad 0<x<a \end{array}}$$

Step 4: Plug in the initial conditions and find $\{A_n \mid n = 1, 2, \ldots\}$.

$$u(x,b) = f(x), \quad u(x,y) = A_0 y + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)$$

$$\Longrightarrow f(x) = A_0 b + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}b\right), \quad 0<x<a$$

From the Fourier cosine series expansion on $(0,a)$, we get

$$2A_0 b = \frac{2}{a}\int_0^a f(x)\,dx, \quad A_n \sinh\left(\frac{n\pi}{a}b\right) = \frac{2}{a}\int_0^a f(x)\cos\left(\frac{n\pi}{a}x\right)\,dx$$

Final Solution

$$\text{Solve } u(x,y): \quad u_{xx}+u_{yy}=0, \quad 0<x<a, \quad 0<y<b$$
$$\text{subject to}: \quad u_x(0,y)=0, \quad u_x(a,y)=0, \quad 0<y<b$$
$$u(x,0)=0, \quad u(x,b)=f(x), \quad 0<x<a$$

Step 5: The final solution is

$$u(x,y) = A_0 y + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)$$

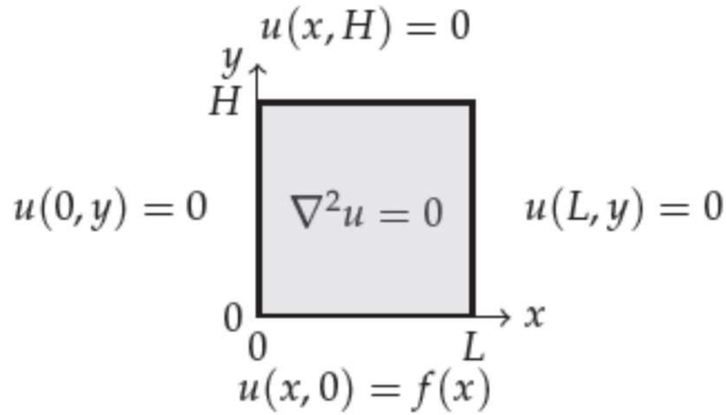
$$A_0 = \frac{1}{ab}\int_0^a f(x)\,dx$$

$$A_n = \frac{2}{a\sinh\left(\frac{n\pi}{a}b\right)}\int_0^a f(x)\cos\left(\frac{n\pi}{a}x\right)\,dx, \quad n \geq 1$$

Example *Equilibrium Temperature Distribution for a Rectangular Plate*

Let's consider Laplace's equation in Cartesian coordinates,

$$u_{xx} + u_{yy} = 0, \quad 0 < x < L, \quad 0 < y < H$$



with the boundary conditions

$$u(0,y) = 0, \quad u(L,y) = 0, \quad u(x,0) = f(x), \quad u(x,H) = 0.$$

The boundary conditions are shown in Figure

As with the heat and wave equations, we can solve this problem using the method of separation of variables. Let $u(x,y) = X(x)Y(y)$. Then, Laplace's equation becomes

$$X''Y + XY'' = 0$$

and we can separate the x and y dependent functions and introduce a separation constant, λ,

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda.$$

Thus, we are led to two differential equations,

$$\begin{aligned} X'' + \lambda X &= 0, \\ Y'' - \lambda Y &= 0. \end{aligned}$$

From the boundary condition $u(0,y) = 0, u(L,y) = 0$, we have $X(0) = 0, X(L) = 0$. So, we have the usual eigenvalue problem for $X(x)$,

$$X'' + \lambda X = 0, \quad X(0) = 0, X(L) = 0.$$

The solutions to this problem are given by

$$X_n(x) = \sin \frac{n\pi x}{L}, \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots.$$

The general solution of the equation for $Y(y)$ is given by

$$Y(y) = c_1 e^{\sqrt{\lambda} y} + c_2 e^{-\sqrt{\lambda} y}.$$

The boundary condition $u(x, H) = 0$ implies $Y(H) = 0$. So, we have

$$c_1 e^{\sqrt{\lambda} H} + c_2 e^{-\sqrt{\lambda} H} = 0.$$

Thus,

$$c_2 = -c_1 e^{2\sqrt{\lambda} H}.$$

Inserting this result into the expression for $Y(y)$, we have

$$\begin{aligned} Y(y) &= c_1 e^{\sqrt{\lambda}y} - c_1 e^{2\sqrt{\lambda}H} e^{-\sqrt{\lambda}y} \\ &= c_1 e^{\sqrt{\lambda}H} \left(e^{-\sqrt{\lambda}H} e^{\sqrt{\lambda}y} - e^{\sqrt{\lambda}H} e^{-\sqrt{\lambda}y} \right) \\ &= c_1 e^{\sqrt{\lambda}H} \left(e^{-\sqrt{\lambda}(H-y)} - e^{\sqrt{\lambda}(H-y)} \right) \\ &= -2c_1 e^{\sqrt{\lambda}H} \sinh \sqrt{\lambda}(H-y). \end{aligned}$$

Since we already know the values of the eigenvalues λ_n from the eigenvalue problem for $X(x)$, we have that the y-dependence is given by

$$Y_n(y) = \sinh \frac{n\pi(H-y)}{L}.$$

So, the product solutions are given by

$$u_n(x,y) = \sin \frac{n\pi x}{L} \sinh \frac{n\pi(H-y)}{L}, \quad n = 1, 2, \ldots.$$

These solutions satisfy Laplace's equation and the three homogeneous boundary conditions and in the problem.

The remaining boundary condition, $u(x,0) = f(x)$, still needs to be satisfied. Inserting $y = 0$ in the product solutions does not satisfy the boundary condition unless $f(x)$ is proportional to one of the eigenfunctions $X_n(x)$. So, we first write down the general solution as a linear combination of the product solutions,

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} \sinh\frac{n\pi(H-y)}{L}.$$

Now we apply the boundary condition, $u(x,0) = f(x)$, to find that

$$f(x) = \sum_{n=1}^{\infty} a_n \sinh\frac{n\pi H}{L} \sin\frac{n\pi x}{L}.$$

Defining $b_n = a_n \sinh\frac{n\pi H}{L}$, this becomes

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}.$$

We see that the determination of the unknown coefficients, b_n, is simply done by recognizing that this is a Fourier sine series.

Potential in a Rectangle

One of the simplest and most important problems in mathematical physics is Dirichlet's problem in a rectangle. To take an easy case, we consider a problem with just two nonzero boundary conditions:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b, \tag{1}$$

$$u(x, 0) = f_1(x), \quad 0 < x < a, \tag{2}$$

$$u(x, b) = f_2(x), \quad 0 < x < a, \tag{3}$$

$$u(0, y) = 0, \quad 0 < y < b, \tag{4}$$

$$u(a, y) = 0, \quad 0 < y < b. \tag{5}$$

It is not immediately clear that separation of variables will work. However, we have a homogeneous partial differential equation and some homogeneous boundary conditions, so we can try the method. If we assume that $u(x, y)$ has a product form $u = X(x)Y(y)$, then Eq. (1) becomes

$$X''(x)Y(y) + X(x)Y''(y) = 0.$$

This equation can be separated by dividing through by XY to yield

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}. \tag{6}$$

The nonhomogeneous conditions Eqs. (2) and (3) will not, in general, become conditions on X or Y, but the homogeneous conditions Eqs. (4) and (5), as usual, require that

$$X(0) = 0, \quad X(a) = 0. \tag{7}$$

Now, both sides of Eq. (6) must be constant, but the sign of the constant is not obvious. If we try a positive constant (say, μ^2), Eq. (6) represents two ordinary equations:

$$X'' - \mu^2 X = 0, \quad Y'' + \mu^2 Y = 0.$$

The solutions of these equations are

$$X(x) = A\cosh(\mu x) + B\sinh(\mu x), \quad Y(y) = C\cos(\mu y) + D\sin(\mu y).$$

In order to make X satisfy the boundary conditions Eq. (7), both A and B must be zero, leading to a solution $u(x, y) \equiv 0$. Thus we try the other possibility for sign, taking both members in Eq. (6) to equal $-\lambda^2$.

Under the new assumption, Eq. (6) separates into

$$X'' + \lambda^2 X = 0, \quad Y'' - \lambda^2 Y = 0. \tag{8}$$

The first of these equations, along with the boundary conditions, is recognizable as an eigenvalue problem, whose solutions are

$$X_n(x) = \sin(\lambda_n x), \quad \lambda_n^2 = \left(\frac{n\pi}{a}\right)^2.$$

The functions Y that accompany the X's are

$$Y_n(y) = a_n\cosh(\lambda_n y) + b_n\sinh(\lambda_n y).$$

The a's and b's are for the moment unknown.

We see that $X_n(x)Y_n(y)$ is a solution of the (homogeneous) potential Eq. (1), which satisfies the homogeneous conditions Eqs. (4) and (5). A sum of these functions should satisfy the same conditions and equation, so u may have the form

$$u(x, y) = \sum_{n=1}^{\infty} \big(a_n \cosh(\lambda_n y) + b_n \sinh(\lambda_n y)\big) \sin(\lambda_n x). \tag{9}$$

The nonhomogeneous boundary conditions Eqs. (2) and (3) are yet to be satisfied. If u is to be of the form of Eq. (9), the boundary condition Eq. (2) becomes

$$u(x, 0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{a}\right) = f_1(x), \quad 0 < x < a. \tag{10}$$

We recognize a problem in Fourier series immediately. The a_n must be the Fourier sine coefficients of $f_1(x)$,

$$a_n = \frac{2}{a} \int_0^a f_1(x) \sin\left(\frac{n\pi x}{a}\right) dx.$$

The second boundary condition reads

$$u(x, b) = \sum_{n=1}^{\infty} \big(a_n \cosh(\lambda_n b) + b_n \sinh(\lambda_n b)\big) \sin\left(\frac{n\pi x}{a}\right)$$
$$= f_2(x), \quad 0 < x < a.$$

This also is a problem in Fourier series, but it is not as neat. The constant

$$a_n \cosh(\lambda_n b) + b_n \sinh(\lambda_n b)$$

must be the nth Fourier sine coefficient of f_2. Since a_n is known, b_n can be determined from the following computations:

$$a_n \cosh(\lambda_n b) + b_n \sinh(\lambda_n b) = \frac{2}{a}\int_0^a f_2(x)\sin(\lambda_n x)\,dx = c_n,$$

$$b_n = \frac{c_n - a_n \cosh(\lambda_n b)}{\sinh(\lambda_n b)}.$$

If we use this last expression for b_n and substitute into Eq. (9), we find the solution

$$u(x,y) = \sum_{n=1}^{\infty}\left\{ c_n \frac{\sinh(\lambda_n y)}{\sinh(\lambda_n b)} + a_n\left[\cosh(\lambda_n y) - \frac{\cosh(\lambda_n b)}{\sinh(\lambda_n b)}\sinh(\lambda_n y)\right]\right\}\sin(\lambda_n x). \qquad (11)$$

Notice that the function multiplying c_n is 0 at $y = 0$ and is 1 at $y = b$. Similarly, the function multiplying a_n is 1 at $y = 0$ and 0 at $y = b$. An easier way to write this latter function is

$$a_n\left[\frac{\sinh(\lambda_n b)\cosh(\lambda_n y) - \cosh(\lambda_n b)\sinh(\lambda_n y)}{\sinh(\lambda_n b)}\right] = a_n\left[\frac{\sinh(\lambda_n (b-y))}{\sinh(\lambda_n b)}\right]$$

as can readily be found from hyperbolic identities.

Example Non-homogeneous Boundary Conditions

This problem may describe the temperature $u(x, y)$ in a thin plate between insulating sheets.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \qquad 0 < y < b,$$

$$\frac{\partial u}{\partial x}(0, y) = 0, \qquad u(a, y) = Sy, \qquad 0 < y < b,$$

$$\frac{\partial u}{\partial y}(x, 0) = S, \qquad u(x, b) = \frac{Sbx}{a}, \quad 0 < x < a.$$

Since we have nonhomogeneous conditions on adjacent sides, we must split the problem in order to solve by separation of variables. Here are the two problems:

$$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} = 0, \qquad\qquad \frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0,$$

$$\frac{\partial u_1}{\partial x}(0, y) = 0, \quad u_1(a, y) = 0, \qquad \frac{\partial u_2}{\partial x}(0, y) = 0, \quad u_2(a, y) = Sy,$$

$$\frac{\partial u_1}{\partial y}(x, 0) = S, \quad u_1(x, b) = \frac{Sbx}{a}, \qquad \frac{\partial u_2}{\partial y}(x, 0) = 0, \quad u_2(x, b) = 0.$$

The solution of the original problem is the sum $u = u_1 + u_2$. Here is the reasoning in detail.

1. The potential equation is linear and homogeneous. By the Principle of Superposition, the sum of solutions is a solution.

2. At $x = a$ we have $u(a, y) = 0 + Sy$, and at $y = b$ we have $u(x, b) = Sbx/a + 0$. Both conditions are satisfied.

3. From elementary calculus, we know

$$\frac{\partial u}{\partial x} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial x}, \quad \frac{\partial u}{\partial y} = \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial y}.$$

   Then at the left and bottom boundaries, we have

$$\frac{\partial u}{\partial x}(0, y) = 0 + 0, \quad \frac{\partial u}{\partial y}(x, 0) = S + 0.$$

   These are satisfied as well.

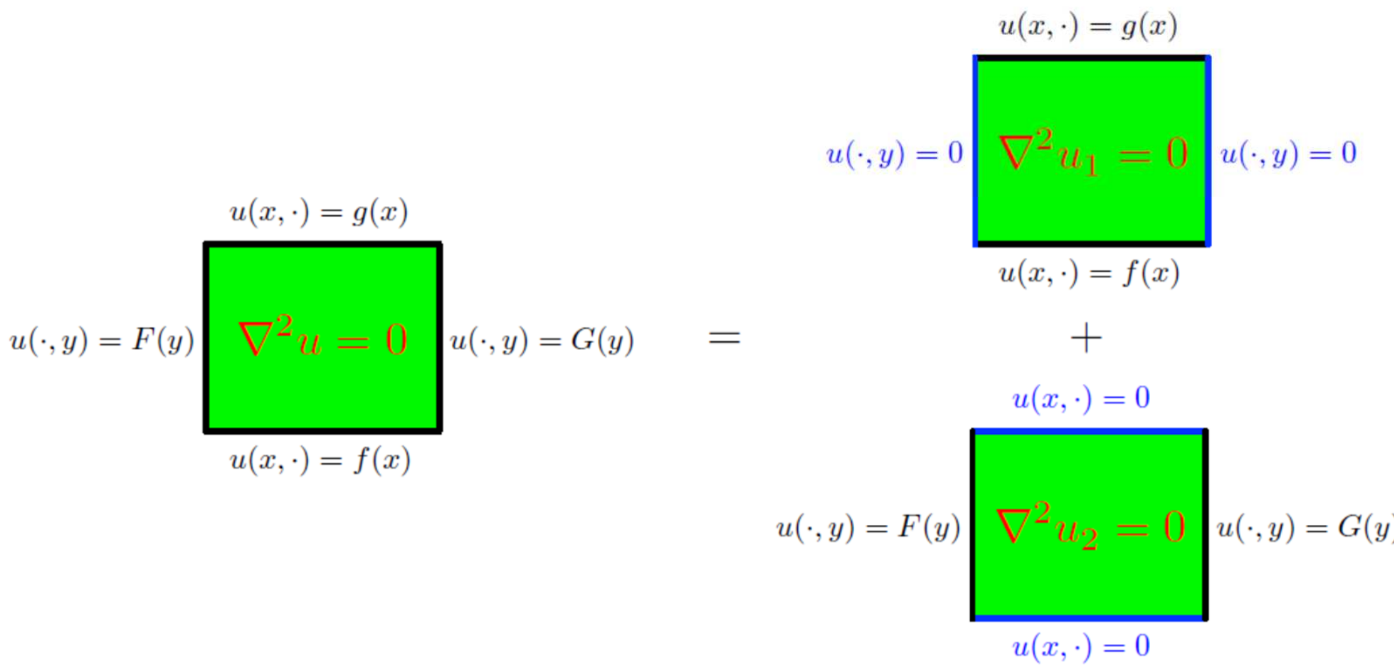
Thus, it remains to solve the two problems for u_1 and u_2.
Here are product solutions. For u_1:

For u_1: $\quad \cos(\lambda_n x)\big(a_n \cosh(\lambda_n y) + b_n \sinh(\lambda_n y)\big), \quad \lambda_n = \left(n - \frac{1}{2}\right)\frac{\pi}{a}, \quad n = 1, 2, \ldots.$

For u_2: $\quad \cos(\mu_n y)\big(A_n \cosh(\mu_n x) + B_n \sinh(\mu_n x)\big), \quad \mu_n = \left(n - \frac{1}{2}\right)\frac{\pi}{b}, \quad n = 1, 2, \ldots.$

Superposition Principle

$$\boxed{\begin{aligned} \text{Solve } u(x,y): &\quad u_{xx} + u_{yy} = 0, \quad 0 < x < a, \quad 0 < y < b \\ \text{subject to}: &\quad u(0,y) = F(y), \quad u(a,y) = G(y), \quad 0 < y < b \\ &\quad u(x,0) = f(x), \quad u(x,b) = g(x), \quad 0 < x < a \end{aligned}}$$



$$\boxed{\begin{array}{ll} \text{Solve } u(x,y): & u_{xx}+u_{yy}=0, \quad 0<x<a, \quad 0<y<b \\ \text{subject to}: & u(0,y)=F(y), \quad u(a,y)=G(y), \quad 0<y<b \\ & u(x,0)=f(x), \quad u(x,b)=g(x), \quad 0<x<a \end{array}}$$

The solution $u(x,y)=u_1(x,y)+u_2(x,y)$, where u_1, u_2 are the solutions of the following 2 BVP's respectively.

$$\boxed{\begin{array}{ll} \text{Solve } u_1(x,y): & u_{xx}+u_{yy}=0, \quad 0<x<a, \quad 0<y<b \\ \text{subject to}: & u(0,y)=0, \quad u(a,y)=0, \quad 0<y<b \\ & u(x,0)=f(x), \quad u(x,b)=g(x), \quad 0<x<a \end{array}}$$

$$\boxed{\begin{array}{ll} \text{Solve } u_2(x,y): & u_{xx}+u_{yy}=0, \quad 0<x<a, \quad 0<y<b \\ \text{subject to}: & u(0,y)=F(y), \quad u(a,y)=G(y), \quad 0<y<b \\ & u(x,0)=0, \quad u(x,b)=0, \quad 0<x<a \end{array}}$$

Problem 1: $$\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial^2 u_1}{\partial y^2}=0, \qquad 0<x<a, \qquad 0<y<b$$
$$u_1(0,y)=0, \qquad u_1(a,y)=0\ , \qquad 0<y<b$$
$$u_1(x,0)=f(x),\ u_1(x,b)=g(x)\ , \qquad 0<x<a$$

Problem 2: $$\frac{\partial^2 u_2}{\partial x^2}+\frac{\partial^2 u_2}{\partial y^2}=0\ , \qquad 0<x<a, \qquad 0<y<b$$
$$u_2(0,y)=F(y),\ \ u_2(a,y)=G(y), \qquad 0<y<b$$
$$u_2(x,0)=0, \qquad u_2(x,b)=0, \qquad 0<x<a$$

Suppose that u_1 and u_2 are solutions of problem 1 and problem 2, respectively. If we define $u = u_1 + u_2$, then

$$u(0,y)=u_1(0,y)+u_2(0,y)=0+F(y)=F(y)$$
$$u(x,b)=u_1(x,b)+u_2(x,b)=g(x)+0=g(x)$$

and so on. The solution of problem 1 and 2 are

$$u_1(x,y)=\sum_{n=1}^{\infty}\left\{A_n\cosh\frac{n\pi}{a}y+B_n\sinh\frac{n\pi}{a}y\right\}\sin\frac{n\pi}{a}x \qquad u_2(x,y)=\sum_{n=1}^{\infty}\left\{A_n\cosh\frac{n\pi}{b}x+B_n\sinh\frac{n\pi}{b}x\right\}\sin\frac{n\pi}{b}y$$

$$A_n=\frac{2}{a}\int_0^a f(x)\sin\frac{n\pi}{a}x\,dx \qquad A_n=\frac{2}{b}\int_0^b F(y)\sin\frac{n\pi}{b}y\,dy$$

$$B_n=\frac{1}{\sinh\frac{n\pi b}{a}}\left(\frac{2}{a}\int_0^a g(x)\sin\frac{n\pi}{a}x\,dx-A_n\cosh\frac{n\pi}{a}b\right) \quad B_n=\frac{1}{\sinh\frac{n\pi a}{b}}\left(\frac{2}{b}\int_0^b G(y)\sin\frac{n\pi}{b}y\,dy-A_n\cosh\frac{n\pi}{b}a\right)$$

Example *Equilibrium Temperature Distribution for a Rectangular Plate for General Boundary Conditions*

A more general problem is to seek solutions to Laplace's equation in Cartesian coordinates,

$$u_{xx} + u_{yy} = 0, \quad 0 < x < L, 0 < y < H$$

with non-zero boundary conditions on more than one side of the domain,

$$u(0,y) = g_1(y), \quad u(L,y) = g_2(y), \quad 0 < y < H,$$

$$u(x,0) = f_1(x), \quad u(x,H) = f_2(x), \quad 0 < x < L.$$

These boundary conditions are shown in Figure 6.9

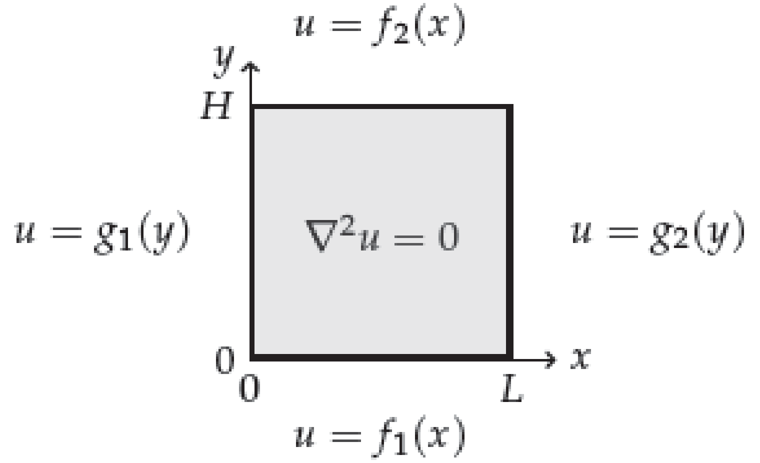


Figure 6.9: In this figure we show the domain and general boundary conditions for the example of determining the equilibrium temperature distribution for a rectangular plate.

The problem with this example is that none of the boundary conditions are homogeneous. This means that the corresponding eigenvalue problems will not have the homogeneous boundary conditions which Sturm-Liouville theory in Section 4 needs. However, we can express this problem in terms of four different problems with nonhomogeneous boundary conditions on only one side of the rectangle.

In Figure 6.10 we show how the problem can be broken up into four separate problems for functions $u_i(x,y)$, $i = 1, \ldots, 4$. Since the boundary conditions and Laplace's equation are linear, the solution to the general problem is simply the sum of the solutions to these four problems,

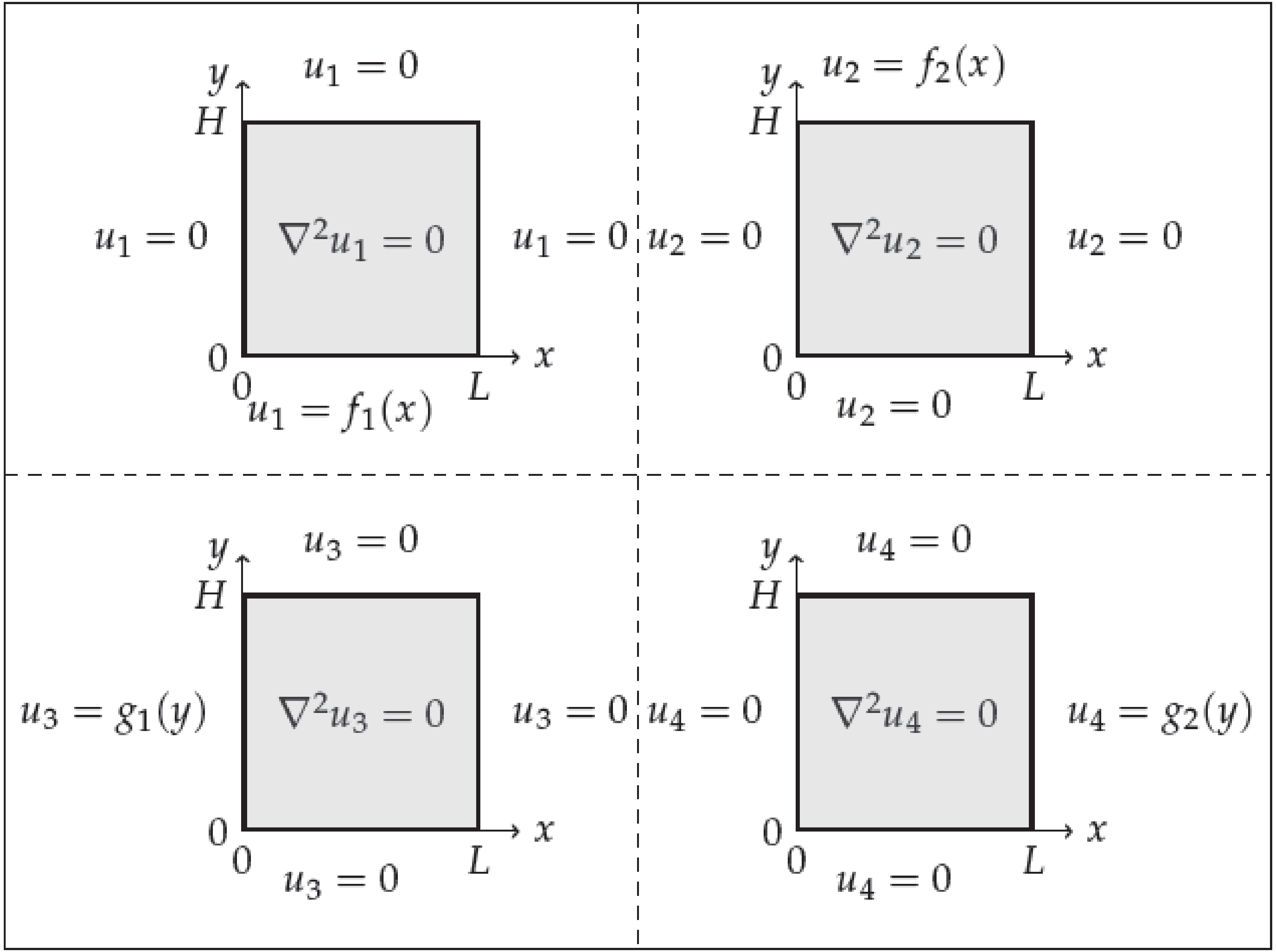


Figure 6.10: The general boundary value problem for a rectangular plate can be written as the sum of these four separate problems.

$$u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y).$$

Then, this solution satisfies Laplace's equation,

$$\nabla^2 u(x,y) = \nabla^2 u_1(x,y) + \nabla^2 u_2(x,y) + \nabla^2 u_3(x,y) + \nabla^2 u_4(x,y) = 0,$$

and the boundary conditions. For example, using the boundary conditions defined in Figure 6.10, we have for $y = 0$,

$$u(x,0) = u_1(x,0) + u_2(x,0) + u_3(x,0) + u_4(x,0) = f_1(x).$$

The other boundary conditions can also be shown to hold.

We can solve each of the problems in Figure 6.10 quickly based on the solution we obtained in the last example. The solution for $u_1(x,y)$, *which satisfies the boundary conditions*

$$u_1(0,y) = 0, \quad u_1(L,y) = 0, \quad 0 < y < H,$$

$$u_1(x,0) = f_1(x), \quad u_1(x,H) = 0, \quad 0 < x < L,$$

is the easiest to write down. It is given by

$$u_1(x,y) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi(H-y)}{L}.$$

where

$$a_n = \frac{2}{L \sinh \frac{n\pi H}{L}} \int_0^L f_1(x) \sin \frac{n\pi x}{L} \, dx.$$

For the boundary conditions

$$u_2(0, y) = 0, \quad u_2(L, y) = 0, \quad 0 < y < H,$$

$$u_2(x, 0) = 0, \quad u_2(x, H) = f_2(x), \quad 0 < x < L.$$

the boundary conditions for $X(x)$ *are* $X(0) = 0$ *and* $X(L) = 0$*. So, we get the same form for the eigenvalues and eigenfunctions as before:*

$$X_n(x) = \sin \frac{n\pi x}{L}, \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1, 2, \ldots.$$

The remaining homogeneous boundary condition is now $Y(0) = 0$*. Recalling that the equation satisfied by* $Y(y)$ *is*

$$Y'' - \lambda Y = 0,$$

we can write the general solution as

$$Y(y) = c_1 \cosh \sqrt{\lambda} y + c_2 \sinh \sqrt{\lambda} y.$$

Requiring $Y(0) = 0$, we have $c_1 = 0$, or

$$Y(y) = c_2 \sinh \sqrt{\lambda} y.$$

Then, the general solution is

$$u_2(x, y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi y}{L}.$$

We now force the nonhomogeneous boundary condition, $u_2(x, H) = f_2(x)$,

$$f_2(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi H}{L}.$$

Once again we have a Fourier sine series. The Fourier coefficients are given by

$$b_n = \frac{2}{L \sinh \frac{n\pi H}{L}} \int_0^L f_2(x) \sin \frac{n\pi x}{L}\, dx.$$

Next we turn to the problem with the boundary conditions

$$u_3(0, y) = g_1(y), \quad u_3(L, y) = 0, \quad 0 < y < H,$$

$$u_3(x, 0) = 0, \quad u_3(x, H) = 0, \quad 0 < x < L.$$

In this case the pair of homogeneous boundary conditions $u_3(x,0) = 0, \quad u_3(x,H) = 0$ *lead to solutions*

$$Y_n(y) = \sin\frac{n\pi y}{H}, \quad \lambda_n = -\left(\frac{n\pi}{H}\right)^2, \quad n = 1,2\ldots.$$

The condition $u_3(L,0) = 0$ *gives* $X(x) = \sinh\frac{n\pi(L-x)}{H}$.

The general solution satisfying the homogeneous conditions is

$$u_3(x,y) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi y}{H}\sinh\frac{n\pi(L-x)}{H}.$$

Applying the nonhomogeneous boundary condition, $u_3(0,y) = g_1(y)$, *we obtain the Fourier sine series*

$$g_1(y) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi y}{H}\sinh\frac{n\pi L}{H}.$$

The Fourier coefficients are found as

$$c_n = \frac{2}{H\sinh\frac{n\pi L}{H}}\int_0^H g_1(y)\sin\frac{n\pi y}{H}\,dy.$$

Finally, we can find the solution $\quad u_4(0,y) = 0, \quad u_4(L,y) = g_2(y), \quad 0 < y < H,$

$$u_4(x,0) = 0, \quad u_4(x,H) = 0, \quad 0 < x < L.$$

Following the above analysis, we find the general solution

$$u_4(x,y) = \sum_{n=1}^{\infty} d_n \sin\frac{n\pi y}{H} \sinh\frac{n\pi x}{H}.$$

The nonhomogeneous boundary condition, $u(L,y) = g_2(y)$, is satisfied if

$$g_2(y) = \sum_{n=1}^{\infty} d_n \sin\frac{n\pi y}{H} \sinh\frac{n\pi L}{H}.$$

The Fourier coefficients, d_n, are given by

$$d_n = \frac{2}{H \sinh\frac{n\pi L}{H}} \int_0^H g_1(y) \sin\frac{n\pi y}{H}\, dy.$$

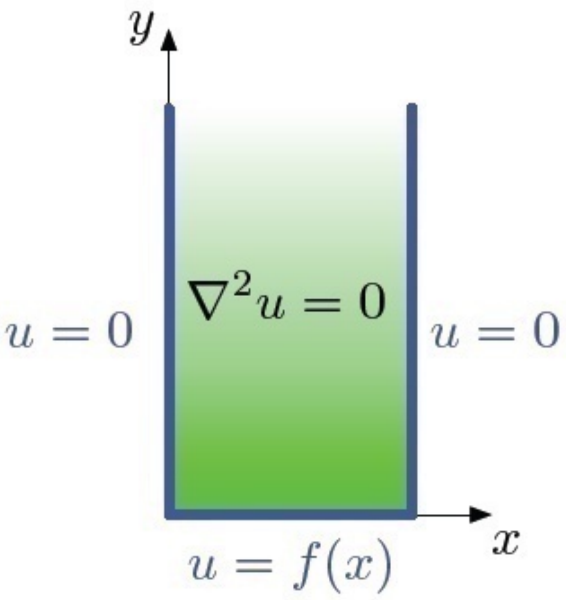
The solution to the general problem is given by the sum of these four solutions.

$$\begin{aligned} u(x,y) \quad = \quad & \sum_{n=1}^{\infty} \left[\left(a_n \sinh\frac{n\pi(H-y)}{L} + b_n \sinh\frac{n\pi y}{L} \right) \sin\frac{n\pi x}{L} \right. \\ & \left. + \left(c_n \sinh\frac{n\pi(L-x)}{H} + d_n \sinh\frac{n\pi x}{H} \right) \sin\frac{n\pi y}{H} \right], \end{aligned}$$

where the coefficients are given by the above Fourier integrals.

Semi-finite plate

$$\begin{array}{ll} \text{Solve } u(x,y): & u_{xx} + u_{yy} = 0, \quad 0 < x < a, \quad y > 0 \\ \text{subject to}: & u(0,y) = 0, \quad u(a,y) = 0, \quad y > 0 \\ & u(x,0) = f(x), \quad |u(x,\infty)| < \infty, \quad 0 < x < a \end{array}$$



Following the same steps as before (setting $u(x,y) = X(x)\,Y(y)$), we can convert the original problem into

$$\begin{array}{ll} \text{Solve } u(x,y): & X'' + \lambda X = 0, \quad Y'' - \lambda Y = 0 \\ \text{subject to}: & X(0) = 0, \quad X(a) = 0, \quad y > 0 \\ & X(x)\,Y(0) = f(x), \quad |\,Y(\infty)| < \infty, \quad 0 < x < a \end{array}$$

Step 1: First we solve $X(x) = c_2 \sin\left(\frac{n\pi}{a}x\right)$ and find the possible $\lambda = \frac{n^2\pi^2}{a^2}$, $n = 1, 2, \ldots$.

Step 2: Next we solve $Y(y) = c_3 e^{\frac{n\pi}{a} y} + c_4 \quad e^{-\frac{n\pi}{a} y}$.

By the condition $|Y(\infty)| < \infty$, we have $c_3 = 0$.

Hence,

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a} x\right) e^{-\frac{n\pi}{a} y}.$$

Final Solution:

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a} x\right) e^{-\frac{n\pi}{a} y}.$$

By the condition $u(x, 0) = f(x), 0 < x < a$, we have

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a} x\right) \implies A_n = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi}{a} x\right) \, dx.$$

EXAMPLE Find the steady state temperature distribution $T(x, y)$ in the uniform slab of metal shown in Fig. 18.33, given that no heat sources are present in the slab and the temperatures on the boundaries are

$$T(x,0) = T(x,a) = 0 \quad \text{for } 0 < x < \infty, \text{ and } T(0, y) = f(y),$$

where $f(y)$ is a bounded function. State any additional condition that must be imposed on $T(x, y)$ for the solution to be physically possible.

Solution As the metal is uniform and there are no heat sources present, it follows that the steady state temperature must be a solution of the Laplace equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0.$$

The sides of the slab are parallel to the coordinate axes, and the equation is homogeneous, so we may separate variables by setting

$$T(x) = X(x)Y(y).$$

Substituting this expression into Laplace's equation and proceeding in the normal manner, we arrive at the separated form of the equation

$$\frac{Y''}{Y} = -\frac{X''}{X} = -\lambda,$$

where $\lambda > 0$ is a separation constant.

This last result separates Laplace's equation into the two differential equations

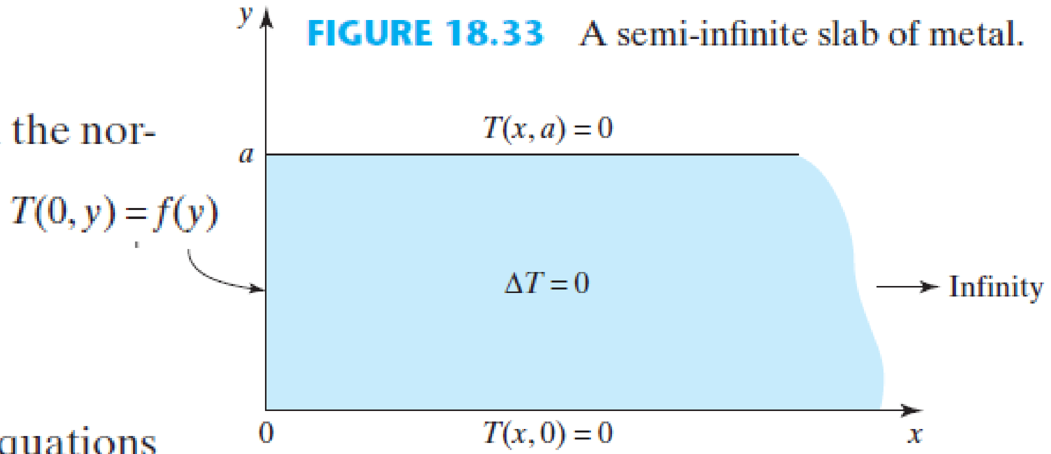


FIGURE 18.33 A semi-infinite slab of metal.

$$Y'' + \lambda Y = 0 \quad \text{and} \quad X'' - \lambda X = 0,$$

where the boundary conditions for $Y(y)$ are easily seen to be $Y(0) = Y(a) = 0$. Thus, we have arrived at the following Sturm–Liouville problem for $Y(y)$:

$$Y'' + \lambda Y = 0 \quad \text{with} \quad Y(0) = Y(a) = 0.$$

The general solution for $Y(y)$ is

$$Y(y) = A\cos\left(\sqrt{\lambda}y\right) + B\sin\left(\sqrt{\lambda}y\right).$$

Imposing these boundary conditions on the general solution for $Y(y)$ shows that the eigenvalues are $\lambda_n = n^2\pi^2/a^2$ and the corresponding eigenfunctions are $Y_n(y) = \sin(n\pi y/a)$, for $n = 1, 2, \ldots$.

Setting $\lambda = \lambda_n$ in the equation for $X(x)$ and integrating gives

$$X_n(x) = C_n \exp(-n\pi x/a) + \cancel{D_n \exp(n\pi x/a)}.$$

To make further progress it is now necessary to recognize that when no sources are present in the metal, and a finite temperature is imposed along the boundary $x = 0$, $0 < y < a$, a physically possible temperature distribution is one that must be bounded throughout the metal. This being so, we must set the coefficients $D_n = 0$ to remove the terms $\exp(n\pi x/a)$ that would otherwise become infinite as $x \to \infty$, thereby causing the functions $X_n(x)$ to simplify to $X_n(x) = \exp(-n\pi x/a)$. Notice that for convenience we have set all scale factors $C_n = 1$, since in what is to follow they will be absorbed into the new arbitrary constants d_n.

Writing $T_n(x, y) = X_n(x)Y_n(y) = \exp(-n\pi x/a)\sin(n\pi y/a)$, we now seek a solution of the form

$$T(x, y) = \sum_{n=1}^{\infty} d_n X_n(x)Y_n(y) = \sum_{n=1}^{\infty} d_n \exp(-n\pi x/a)\sin(n\pi y/a).$$

If we set $x = 0$ in this summation and use the boundary condition $T(0, y) = f(y)$, this reduces to

$$f(y) = \sum_{n=1}^{\infty} d_n \sin(n\pi y/a),$$

from which it follows in the usual manner that

$$d_n = \frac{2}{a}\int_0^a f(y)\sin\left(\frac{n\pi y}{a}\right) dy, \text{ for } n = 1, 2, \ldots .$$

The solution has been found by imposing the extra condition that $T(x, y)$ remains *bounded* in the (open) semi-infinite strip, which compensates for the normal requirement for elliptic equations that the region is closed ■

Potential in Unbounded Regions

The potential equation, as well as the heat and wave equations, can be solved in unbounded regions. Consider the following problem, in which the region involved is half a vertical strip, or a slot:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y, \tag{1}$$

$$u(x, 0) = f(x), \quad 0 < x < a, \tag{2}$$

$$u(0, y) = g_1(y), \quad 0 < y, \tag{3}$$

$$u(a, y) = g_2(y), \quad 0 < y. \tag{4}$$

As usual, we required that $u(x, y)$ remain bounded as $y \to \infty$.

In order to make the separation of variables work, we must break this up into two problems. Following the model   we set $u(x, y) = u_1(x, y) + u_2(x, y)$ and require that the parts satisfy these two solvable problems:

$$
\begin{array}{llll}
\nabla^2 u_1 = 0, & \nabla^2 u_2 = 0, & 0 < x < a, & 0 < y, \\
u_1(x,0) = f(x), & u_2(x,0) = 0, & 0 < x < a, & \\
u_1(0,y) = 0, & u_2(0,y) = g_1(y), & 0 < y, & \\
u_1(a,y) = 0, & u_2(a,y) = g_2(y), & 0 < y. &
\end{array}
$$

We attack the problem for u_1 by assuming the product form and separating variables:

$$u_1(x,y) = X(x)Y(y), \quad \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda^2.$$

The sign of the constant $-\lambda^2$ is determined by the boundary conditions at $x = 0$ and $x = a$, which become homogeneous conditions on the factor $X(x)$:

$$X(0) = 0, \quad X(a) = 0. \tag{5}$$

(We also can see that the condition to be satisfied along $y = 0$ demands functions of x that permit a representation of an arbitrary function.)

The boundary conditions, Eq. (5), together with the differential equation

$$X'' + \lambda^2 X = 0 \tag{6}$$

that comes from the separation of variables, constitute a familiar eigenvalue problem, whose solution is

$$X_n(x) = \sin\left(\frac{n\pi x}{a}\right), \quad \lambda_n^2 = \left(\frac{n\pi}{a}\right)^2, \quad n = 1, 2, 3, \ldots.$$

The equation for Y is

$$Y'' - \lambda^2 Y = 0, \quad 0 < y.$$

In addition to satisfying this differential equation, Y must remain bounded as $y \to \infty$. The solutions of the equation are $e^{\lambda y}$ and $e^{-\lambda y}$. Of these, the first is unbounded, so

$$Y_n(y) = \exp(-\lambda_n y).$$

Finally, we can write the solution of the first problem as

$$u_1(x, y) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{a}\right) \exp\left(\frac{-n\pi y}{a}\right). \tag{7}$$

The constants a_n are to be determined from the condition at $y = 0$.

The solution of the second problem is somewhat different. Again we seek solutions in the product form $u_2(x, y) = X(x)Y(y)$. The homogeneous boundary condition at $y = 0$ and the boundedness condition become conditions on $Y(y)$:

$$Y(0) = 0, \quad Y(y) \text{ bounded as } y \to \infty.$$

Then the potential equation becomes

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0, \tag{8}$$

and both ratios must be constant. If Y''/Y is positive, the auxiliary conditions force Y to be identically 0. Thus, we take $Y''/Y = -\mu^2$, or $Y'' + \mu^2 Y = 0$, and find that the solution that satisfies the auxiliary conditions is

$$Y(y) = \sin(\mu y),$$

for any $\mu > 0$. Then the general solution of the equation $X''/X = \mu^2$ is

$$X(x) = A\frac{\sinh(\mu x)}{\sinh(\mu a)} + B\frac{\sinh(\mu(a-x))}{\sinh(\mu a)}.$$

from p.16 (11)

We have chosen this special form on the basis of our experience in solving the potential equation in the rectangle.

Since μ is a continuous parameter, we combine our product solutions by means of an integral, finding

$$u_2(x,y) = \int_0^\infty \left[A(\mu)\frac{\sinh(\mu x)}{\sinh(\mu a)} + B(\mu)\frac{\sinh(\mu(a-x))}{\sinh(\mu a)}\right]\sin(\mu y)\, d\mu. \quad (9)$$

The nonhomogeneous boundary conditions at $x = 0$ and $x = a$ are satisfied if

$$u_2(0,y) = \int_0^\infty B(\mu)\sin(\mu y)\, d\mu = g_1(y), \quad 0 < y,$$

$$u_2(a,y) = \int_0^\infty A(\mu)\sin(\mu y)\, d\mu = g_2(y), \quad 0 < y.$$

Obviously these two equations are Fourier integral problems, so we know how to determine the coefficients $A(\mu)$ and $B(\mu)$.

Polar Coordinates

Dirichlet Problem for a Circle. Consider the problem of solving Laplace's equation in a circular region $r < a$ subject to the boundary condition

$$u(a,\theta) = f(\theta), \tag{18}$$

where f is a given function on $0 \le \theta < 2\pi$ In polar coordinates Laplace's equation has the form

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0. \tag{19}$$

To complete the statement of the problem, we note that for $u(r,\theta)$ to be single-valued, it is necessary that u be periodic in θ with period 2π. Moreover, we state explicitly that $u(r,\theta)$ must be bounded for $r \le a$, since this will become important later.

To apply the method of separation of variables to this problem, we assume that

$$u(r,\theta) = R(r)\Theta(\theta), \tag{20}$$

and substitute for u in the differential equation (19). This yields

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0,$$

or

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda, \tag{21}$$

where λ is the separation constant. Thus we obtain the two ordinary differential equations

$$r^2R'' + rR' - \lambda R = 0, \tag{22}$$

$$\Theta'' + \lambda\Theta = 0. \tag{23}$$

In this problem there are no homogeneous boundary conditions; recall, however, that solutions must be bounded and also periodic in θ with period 2π. It is possible to show that the periodicity condition requires λ to be real. We will consider in turn the cases in which λ is negative, zero, and positive.

If $\lambda < 0$, let $\lambda = -\mu^2$, where $\mu > 0$. Then Eq. (23) becomes $\Theta'' - \mu^2\Theta = 0$, and consequently,

$$\Theta(\theta) = c_1e^{\mu\theta} + c_2e^{-\mu\theta}. \tag{24}$$

Thus $\Theta(\theta)$ can be periodic only if $c_1 = c_2 = 0$, and we conclude that λ cannot be negative.

If $\lambda = 0$, then Eq. (23) becomes $\Theta'' = 0$, and thus

$$\Theta(\theta) = c_1 + c_2\theta. \tag{25}$$

For $\Theta(\theta)$ to be periodic we must have $c_2 = 0$, so that $\Theta(\theta)$ is a constant. Further, for $\lambda = 0$, Eq. (22) becomes

$$r^2R'' + rR' = 0. \tag{26}$$

This equation is of the Euler type and has the solution

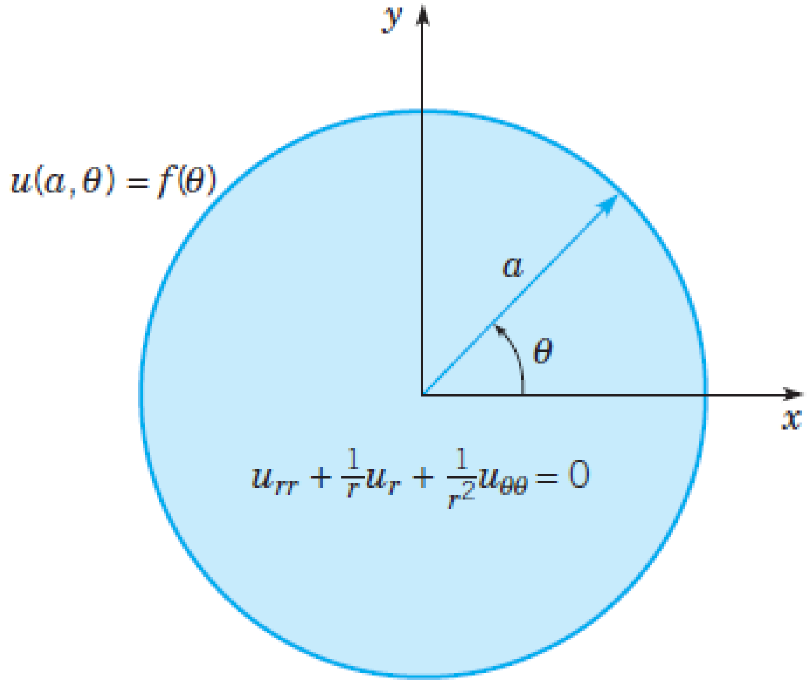


FIGURE 10.8.4 Dirichlet problem for a circle.

$$R(r) = k_1 + k_2 \ln r. \qquad (e^{-\infty} \cong 0) \tag{27}$$

The logarithmic term cannot be accepted if $u(r,\theta)$ is to remain bounded as $r \to 0$; hence $k_2 = 0$. Thus, corresponding to $\lambda = 0$, we conclude that $u(r,\theta)$ must be a constant—that is, proportional to the solution

$$u_0(r,\theta) = 1. \tag{28}$$

Finally, if $\lambda > 0$, we let $\lambda = \mu^2$, where $\mu > 0$. Then Eqs. (22) and (23) become

$$r^2R'' + rR' - \mu^2 R = 0 \tag{29}$$

and

$$\Theta'' + \mu^2\Theta = 0, \tag{30}$$

respectively. Equation (29) is an Euler equation and has the solution

$$R(r) = k_1 r^\mu + k_2 r^{-\mu}, \qquad (k_2 \frac{1}{r^\mu}) \tag{31}$$

while Eq. (30) has the solution

$$\Theta(\theta) = c_1 \sin \mu\theta + c_2 \cos \mu\theta. \tag{32}$$

In order for Θ to be periodic with period 2π, it is necessary for μ to be a positive integer n. With $\mu = n$ it follows that the solution $r^{-\mu}$ in Eq. (31) must be discarded since it becomes unbounded as $r \to 0$. Consequently, $k_2 = 0$ and the appropriate solutions of Eq. (19) are

$$u_n(r,\theta) = r^n \cos n\theta, \qquad v_n(r,\theta) = r^n \sin n\theta, \qquad n = 1, 2, \ldots. \tag{33}$$

These functions, together with $u_0(r,\theta) = 1$, form a set of fundamental solutions for the present problem.

In the usual way we now assume that u can be expressed as a linear combination of the fundamental solutions; that is,

$$u(r,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} r^n (c_n \cos n\theta + k_n \sin n\theta). \tag{34}$$

The boundary condition (18) then requires that

$$u(a,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} a^n (c_n \cos n\theta + k_n \sin n\theta) = f(\theta) \tag{35}$$

for $0 \leq \theta < 2\pi$. The function f may be extended outside this interval so that it is periodic with period 2π and therefore has a Fourier series of the form (35). Since the extended function has period 2π, we may compute its Fourier coefficients by integrating over any period of the function. In particular, it is convenient to use the original interval $(0, 2\pi)$; then

$$a^n c_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta, \qquad n = 0, 1, 2, \ldots; \tag{36}$$

$$a^n k_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta, \qquad n = 1, 2, \ldots. \tag{37}$$

With this choice of the coefficients, Eq. (34) represents the solution of the boundary value problem of Eqs. (18) and (19). Note that in this problem we needed both sine and cosine terms in the solution. This is because the boundary data were given on $0 \leq \theta < 2\pi$ and have period 2π. As a consequence, the full Fourier series is required, rather than sine or cosine terms alone.

EXAMPLE Find the steady state temperature distribution in the semicircular region of radius ρ lying in the upper half-plane and centered on the origin, as shown in Fig. 18.28. The temperature on the straight boundary is $u = 0$, and that on the semicircular boundary is $u = u_0\theta(\pi - \theta)$.

Solution The geometry of the problem suggests that the Laplace equation for the steady state temperature distribution u should be expressed in terms of the polar coordinates r and θ. In terms of these variables the Laplace equation $\Delta u = 0$ becomes

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

To separate the variables we now set $u(r, \theta) = R(r)\Theta(\theta)$ and substitute into the equation. After dividing by $R\Theta$ and rearranging terms, we find that

$$r^2\frac{R''}{R} + r\frac{R'}{r} = -\frac{\Theta''}{\Theta},$$

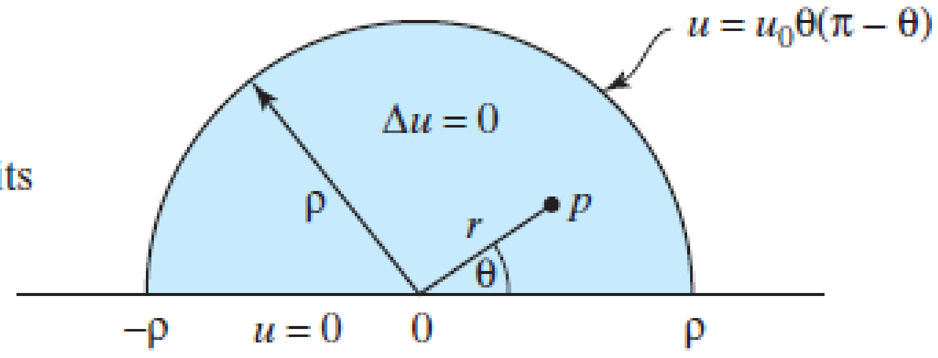


FIGURE 18.28 The semicircular domain and its boundary conditions.

but as the expression on the left is a function of only r and the one on the right is a function of only θ, both must be equal to a separation constant k, so we have

$$r^2 R'' + rR' - kR = 0 \quad \text{and} \quad \Theta'' + k\Theta = 0.$$

The sign of k is determined by the fact that only when $k > 0$ will the θ variation be periodic in nature, as would be expected, because increasing θ by a multiple of 2π will simply reproduce the original problem. If we set $k = \lambda^2$, the functions R and Θ are seen to satisfy the two equations

$$r^2 R'' + rR' - \lambda^2 R = 0 \quad \text{and} \quad \Theta'' + \lambda^2\Theta = 0.$$

The first of these equations is a **Cauchy–Euler equation**, which was seen to have the general solution

$$R(r) = \widetilde{A}r^{\lambda} + \widetilde{B}\frac{1}{r^{\lambda}}.$$

As the solution must be *finite* at the origin, we must set $\widetilde{B} = 0$, so $R(r)$ must be of the form $R(r) = \widetilde{A}r^{\lambda}$. Now, as $u(r, \theta) = R(r)\Theta(\theta)$ and $u(r, 0) = u(r, \pi) = 0$ (in polar coordinates these two conditions represent the boundary condition on the straight line boundary), it follows that the boundary conditions for Θ are $\Theta(0) = \Theta(\pi) = 0$.

The general solution for Θ is

$$\Theta(\theta) = \widetilde{C}\cos\lambda\theta + \widetilde{D}\sin\lambda\theta,$$

so imposing the first of the boundary conditions gives $\widetilde{C} = 0$, and when the second one is imposed we find that λ must satisfy

$$0 = \widetilde{D} \sin \lambda\pi,$$

so the eigenvalues λ_n are

$$\lambda_n = n, \quad \text{for } n = 1, 2, \ldots.$$

The eigenfunctions $R_n(r)$ become

$$R_n(x) = A_n r^n, \quad \text{for } n = 1, 2, \ldots,$$

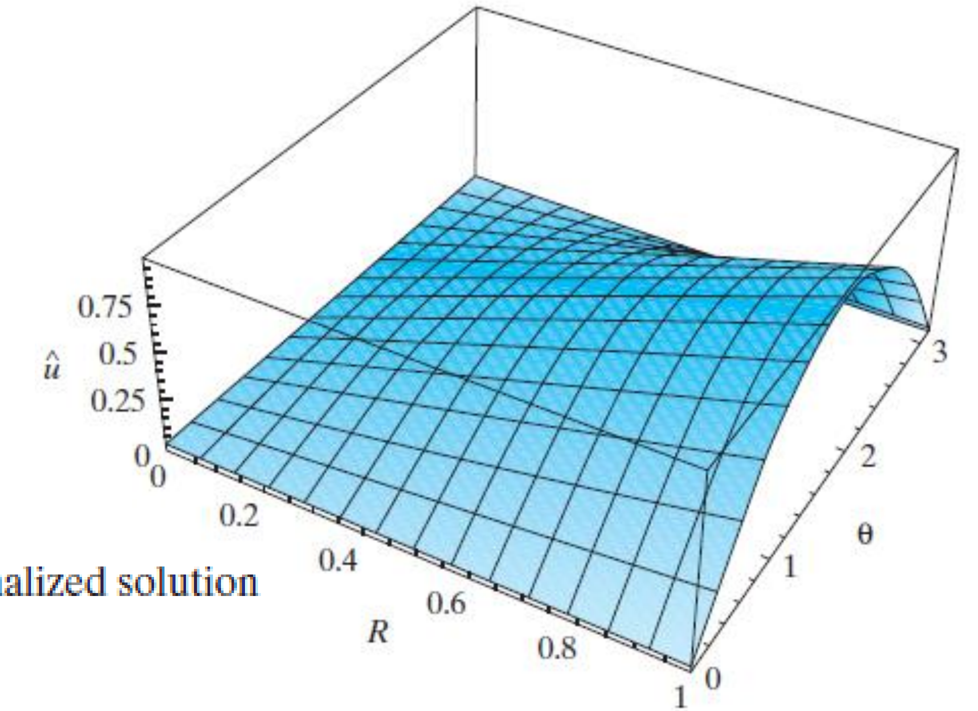


FIGURE 18.29 A plot of the normalized solution $\hat{u} = (\pi/8u_0)u(r, \theta)$.

and the eigensolutions $u_n(r, \theta) = A_n r^n \sin n\theta$, where the product of the arbitrary constants $\widetilde{A}\,\widetilde{D}$, each of which depends on n, has been denoted by A_n.

We now seek a solution in the form of the linear combination of the eigensolutions

$$u(r, \theta) = \sum_{n=1}^{\infty} u_n(r, \theta) = \sum_{n=1}^{\infty} A_n r^n \sin n\theta.$$

Substituting the boundary condition $u(\rho,\theta)=u_0\theta(\pi-\theta)$ on the left of this series and setting $r=\rho$ in the expression on the right gives

$$u_0\theta(\pi-\theta)=\sum_{n=1}^{\infty}A_n\rho^n\sin n\theta.$$

Definite integral:

$$\int_0^{\pi} x(\pi-x)\sin(n x)\,dx=\frac{-\cancel{\pi n\sin(\pi n)}-2\cos(\pi n)+2}{n^3}$$

The coefficients A_n now follow from the orthogonality properties of the sine function over the interval $0\le\theta\le\pi$. Multiplying the last result by $\sin m\theta$ and integrating over the interval $0\le\theta\le\pi$, we find that

$$2u_0\left(\frac{1-(-1)^n}{n^3}\right)=\frac{1}{2}A_n\rho^n\pi\quad\text{and so}\quad A_n=\frac{4u_0}{\pi}\frac{(1-(-1)^n)}{n^3\rho^n}.$$

Substituting these coefficients into the series now gives the required solution,

$$u(r,\theta)=\frac{8u_0}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{\rho}\right)^{2n-1}\frac{\sin(2n-1)\theta}{(2n-1)^3}.$$

Figure 18.29 shows a plot of $\hat{u}=(\pi/8u_0)u(r,\theta)$ as a function of $R=r/\rho$ for $0\le R\le 1$ and $0\le\theta\le\pi$ using 10 terms of the series.

Fourier Transform

Sine transforms

$$\begin{cases} \mathcal{F}_s[f] = F(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\sin(\omega t)dt & \text{(Fourier sine transform)} \\ \mathcal{F}_s^{-1}[F] = f(t) = \int_0^\infty F(\omega)\sin(\omega t)d\omega & \text{(inverse sine transform)} \end{cases}$$

Cosine transforms

$$\begin{cases} \mathcal{F}_c[f] = F(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\cos(\omega t)dt & \text{(Fourier cosine transform)} \\ \mathcal{F}_c^{-1}[F] = f(t) = \int_0^\infty F(\omega)\cos(\omega t)d\omega & \text{(inverse cosine transform)} \end{cases}$$

Sine and Cosine transforms of derivatives

$$\mathscr{F}_s\{f'(x)\} = -\alpha\mathscr{F}_c\{f(x)\}$$

$$\mathscr{F}_c\{f'(x)\} = \alpha\mathscr{F}_s\{f(x)\} - f(0).$$

$$\mathscr{F}_s\{f''(x)\} = -\alpha^2F(\alpha) + \alpha f(0).$$

$$\boxed{\mathscr{F}_c\{f''(x)\} = -\alpha^2F(\alpha) - f'(0).}$$

$$\mathscr{F}\{f'(x)\} = -i\alpha F(\alpha).$$

$$\mathscr{F}\{f''(x)\} = (-i\alpha)^2\,\mathscr{F}\{f(x)\} = -\alpha^2F(\alpha).$$

$$\mathscr{F}\{f^{(n)}(x)\} = (-i\alpha)^n\,\mathscr{F}\{f(x)\} = (-i\alpha)^nF(\alpha),$$

$$\mathcal{F}[f] \equiv F(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[f(x)e^{-i\xi x}dx\right] \quad \text{(Fourier transfor}$$

$$\mathcal{F}^{-1}[F] \equiv f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[F(\xi)e^{i\xi x}d\xi\right] \quad \text{(inverse FT)}$$

- Transformation of partial derivatives:

$$\mathcal{F}[u_x] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u_x(x,t)e^{-i\xi x}dx = i\xi\mathcal{F}[u]$$

$$\mathcal{F}[u_{xx}] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u_{xx}(x,t)e^{-i\xi x}dx = -\xi^2\mathcal{F}[u]$$

$$\mathcal{F}[u_t] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u_t(x,t)e^{-i\xi x}dx = \frac{\partial}{\partial t}\mathcal{F}[u]$$

$$\mathcal{F}[u_{tt}] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u_{tt}(x,t)e^{-i\xi x}dx = \frac{\partial^2}{\partial t^2}\mathcal{F}[u]$$

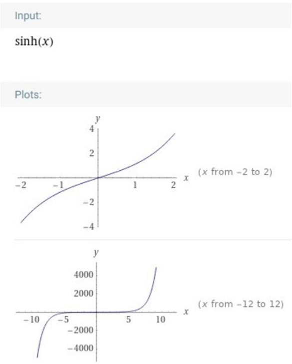
EXAMPLE : Using the Cosine Transform

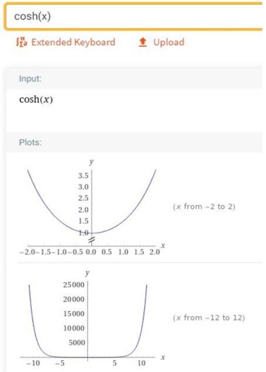
The steady-state temperature in a semi-infinite plate is determined from

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < \pi, \quad \boxed{y > 0}$$

$$u(0, y) = 0, \quad u(\pi, y) = e^{-y}, \quad y > 0$$

$$\boxed{\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0,} \quad 0 < x < \pi.$$





Solve for $u(x, y)$.

SOLUTION The domain of the variable y and the prescribed condition at $y = 0$ indicate that the Fourier cosine transform is suitable for the problem. We define

$$\mathscr{F}_c\{u(x, y)\} = \int_0^\infty u(x, y) \cos \alpha y \, dy = U(x, \alpha).$$

$$\mathscr{F}_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} + \mathscr{F}_c\left\{\frac{\partial^2 u}{\partial y^2}\right\} = \mathscr{F}_c\{0\}$$

becomes

$$\frac{d^2U}{dx^2} - \alpha^2 U(x, \alpha) - u_y(x, 0) = 0 \qquad \text{or} \qquad \frac{d^2U}{dx^2} - \alpha^2 U = 0.$$

Since the domain of x is a finite interval, we choose to write the solution of the ordinary differential equation as

$$U(x, \alpha) = c_1 \cosh \alpha x + c_2 \sinh \alpha x.$$

Now $\mathscr{F}_c\{u(0, y)\} = \mathscr{F}_c\{0\}$ and $\mathscr{F}_c\{u(\pi, y)\} = \mathscr{F}_c\{e^{-y}\}$ are in turn equivalent to

$$U(0, \alpha) = 0 \qquad \text{and} \qquad U(\pi, \alpha) = \frac{1}{1 + \alpha^2}.$$

13. $e^{-ax} (a > 0)$	$\sqrt{\frac{2}{\pi}}\left(\frac{a}{\omega^2 + a^2}\right)$

When we apply the two conditions in to the solution we obtain $c_1 = 0$ and $c_2 = 1/[(1 + \alpha^2) \sinh \alpha]$. Therefore,

π

$$U(x, \alpha) = \frac{\sinh \alpha x}{(1 + \alpha^2) \sinh \alpha\pi},$$

and we arrive at

$$u(x, y) = \frac{2}{\pi} \int_0^\infty \frac{\sinh \alpha x}{(1 + \alpha^2) \sinh \alpha\pi} \cos \alpha y \, d\alpha. \qquad \equiv$$

EXAMPLE

Use an integral transform to find the solution of the two-dimensional Laplace equation $u_{xx} + u_{yy} = 0$ in the infinite strip $0 \le y \le a$, given that $u(x,0) = 0$ and $u(x,a) = f(x)$, and interpret the result in terms of two different physical problems.

Solution As $-\infty < x < \infty$, it is necessary to use the Fourier transform with respect to x, so transforming the equation we find that

$$(i\omega)^2 U(\omega, y) + \frac{d^2U(\omega, y)}{dy^2} = 0.$$

The solution of this ODE for the Fourier transform $U(\omega, y)$ of the solution $u(x, y)$ is

$$U(\omega, y) = A(\omega)e^{\omega y} + B(\omega)e^{-\omega y},$$

where the functions $A(\omega)$ and $B(\omega)$ of ω are to be determined. Assuming that $f(x)$ has the Fourier transform $F(\omega)$, the Fourier transform of the boundary conditions becomes

$$_x\mathcal{F}\{u(x,0)\} = U(\omega,0) = 0 \quad \text{and} \quad {}_x\mathcal{F}\{u(x,a)\} = U(\omega,a) = F(\omega).$$

The transform $U(\omega, y)$ is required to satisfy these two-point boundary conditions, and a routine calculation shows that

$$U(\omega, y) = F(\omega)\frac{\sinh(\omega y)}{\sinh(\omega a)}.$$

$$A(\omega) = -B(\omega)$$

$$F(\omega) = 2A(\omega)\left(\frac{e^{a\omega} - e^{-a\omega}}{2}\right)$$

$$= 2A(\omega)\sinh a\omega$$

$$A(\omega) = \frac{F(\omega)}{2\sinh a\omega}$$

$$U(\omega, y) = \frac{F(\omega)}{\sinh a\omega}\left(\frac{e^{\omega y} - e^{-\omega y}}{2}\right)$$

$$= F(\omega)\frac{\sinh \omega y}{\sinh a\omega}$$

Applying the Fourier inversion integral to $U(\omega, y)$ gives

$$u(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} U(\omega, y) e^{i\omega x} d\omega.$$

If $G(\omega, y)$ is defined as

$$G(\omega, y) = \frac{\sinh(\omega y)}{\sinh(\omega a)},$$

we can write

$$U(\omega, y) = F(\omega) G(\omega, y),$$

and so
$$u(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) G(\omega, y) e^{i\omega x} d\omega.$$

If $g(x, y) = {}_x\mathcal{F}^{-1}\{G(\omega, y)\}$, an application of the Fourier convolution theorem to the expression on the right gives

$$u(x, y) = \frac{1}{\sqrt{2\pi}} (f * g).$$

By definition
$$g(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\sinh(\omega y)}{\sinh(\omega a)} e^{i\omega x} d\omega,$$

so after expansion of the factor $e^{i\omega x}$ this becomes

$$g(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\sinh(\omega y)}{\sinh(\omega a)} \cos(\omega x) d\omega + \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\sinh(\omega y)}{\sinh(\omega a)} \sin(\omega x) d\omega.$$

The last integral is zero because its integrand is an odd function of ω, but the integrand of the first integral is an even function of ω, so

$$g(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\sinh(\omega y)}{\sinh(\omega a)} \cos(\omega x) d\omega = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \frac{\sinh(\omega y)}{\sinh(\omega a)} \cos(\omega x) d\omega.$$

Using these results in the convolution theorem now gives

$$u(x, y) = \frac{1}{\sqrt{2\pi}} (f * g) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} \int_{\omega=0}^{\infty} \int_{-\infty}^{\infty} f(\tau) \frac{\sinh(\omega y)}{\sinh(\omega a)} \cos[(\omega - \tau)x] d\tau d\omega,$$

and so

$$u(x, y) = \frac{1}{\pi} \int_{\omega=0}^{\infty} \int_{-\infty}^{\infty} f(\tau) \frac{\sinh(\omega y)}{\sinh(\omega a)} \cos[(\omega - \tau)x] d\tau d\omega.$$

One physical interpretation of this problem is that it provides the steady state temperature distribution in a slab of metal of thickness a when the lower face is maintained at a temperature $u(x, 0) = 0$ and the upper face is maintained at the temperature $u(x, a) = f(x)$. Another interpretation is that it provides the potential distribution in air between two parallel conducting plates a distance a apart, when the lower plate is maintained at zero potential and the upper one is maintained at the potential $u(x, a) = f(x)$. ■

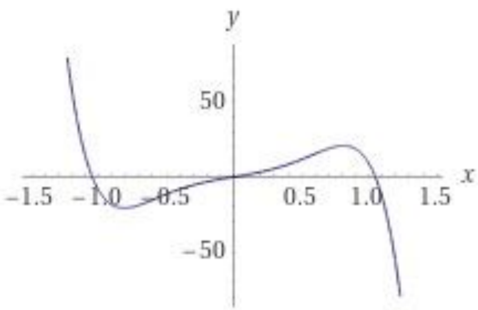
Input:

$$\sinh(5\,x) \times \frac{\sin(3\,x)}{\sinh(x)}$$

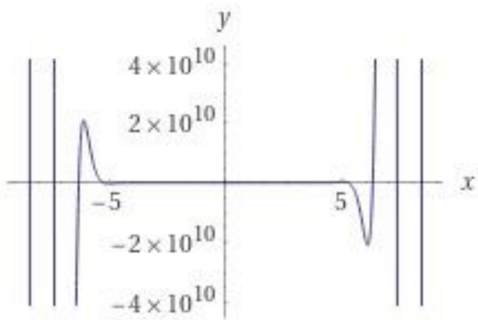
Result:

$$\sin(3\,x)\sinh(5\,x)\operatorname{csch}(x)$$

Plots:



(x from −1.5 to 1.5)



(x from −9 to 9)

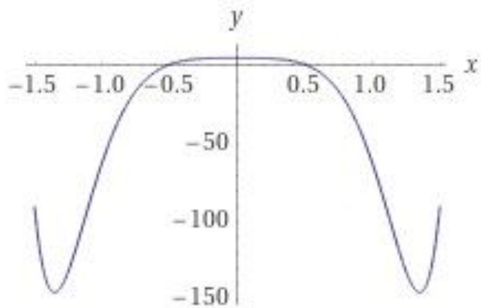
Input:

$$\sinh(5\,x) \times \frac{\cos(3\,x)}{\sinh(x)}$$

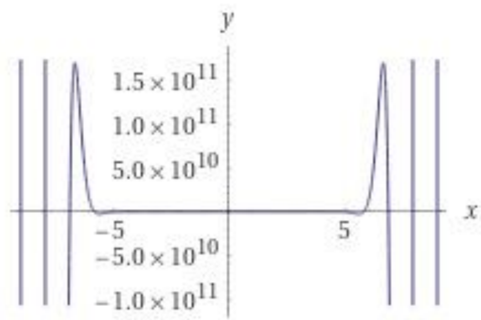
Result:

$$\cos(3\,x)\sinh(5\,x)\operatorname{csch}(x)$$

Plots:



(x from −1.5 to 1.5)



(x from −9 to 9)