The Fourier Transform

- The Fourier Series deals with *periodic* signals

$$x(t) = \sum_{n=-\infty}^{\infty} \hat{X}_n e^{jnw_o t}$$

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$$\hat{X}_n = \frac{1}{T}\int_{t_o}^{t_o+T} x(t)e^{-jnw_o t}dt$$

- The Fourier Transform deals with *non-periodic* signals

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$$

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

Fourier Integral and Transform

Fourier integral is a tool used to analyze non-periodic waveforms or non-recurring signals, such as lightning bolts. Fourier integral formula is derived from Fourier series by allowing the period to approach infinity:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(i\omega)\, e^{i\omega t}\, d\omega$$

where the coefficients become a continuous function of the frequency variable ω, as in

$$F(i\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

Fourier Integral and Transform

The function $F(i\omega)$ is called the Fourier integral of $f(t)$. The function $F(i\omega)$ is also called the Fourier transform of $f(t)$. In the same spirit, $f(t)$ is referred to as the inverse Fourier transform of $F(i\omega)$.

The pair allows us to transform back and forth between the time and the frequency domains for a non-periodic signal.

The periodic function $f_T(t)$ can be represented by an exponential Fourier series.

$$f_T(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_o t} \quad \text{where}$$

$$F_n = \frac{1}{T}\int_{-T/2}^{T/2} f_T(t) e^{-jn\omega_o t}\, dt \quad \text{and} \quad \omega_o = 2\pi / T$$

As the magnitude of the Fourier coefficients go to zero when the period is increased, we define

$$\omega_n \equiv n\omega_o \qquad \text{and} \qquad F(\omega_n) \equiv T \cdot F_n$$

The Fourier series pair become

$$f_T(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T} F(\omega_n) e^{j\omega_n t} \quad \text{where} \quad F(\omega_n) = \int_{-T/2}^{T/2} f_T(t) e^{-j\omega_n t}\, dt$$

The spacing between adjacent lines ($\Delta\omega$) in the line spectrum of $f_T(t)$ is

$$\Delta\omega = 2\pi / T$$

$T \infty$

Therefore, we have

$$f_T(t) = \sum_{n=-\infty}^{\infty} F(\omega_n) e^{j\omega_n t} \frac{\Delta\omega}{2\pi}$$

Now as T becomes very large, $\Delta\omega$ becomes smaller and the spectrum becomes denser. In the limit, the discrete lines in the spectrum of $f_T(t)$ merge and the frequency spectrum becomes continuous.

Mathematically, the infinite sum becomes an integral

$$\lim_{T\to\infty} f_T(t) = \lim_{T\to\infty} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} F(\omega_n) e^{j\omega_n t} \Delta\omega$$

$$\Rightarrow \boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega}$$

Inverse Fourier transform of $F(\omega)$

Similarly,

$$F_n = \frac{1}{T}\int_{-T/2}^{T/2} f_T(t) e^{-jn\omega_o t} dt$$

$$\Rightarrow F(\omega_n) = \int_{-T/2}^{T/2} f_T(t) e^{-j\omega_n t} dt \qquad \because \omega_n \equiv n\omega_o \text{ and } F(\omega_n) \equiv TF_n$$

$$\Rightarrow \lim_{T\to\infty} F(\omega_n) = \lim_{T\to\infty} \int_{-T/2}^{T/2} f_T(t) e^{-jn\omega_o t} dt$$

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$$\Rightarrow \boxed{F(\omega) = \int_{-\infty}^{\infty} f_T(t) e^{-jn\omega_o t} dt}$$

Fourier transform of $f(t)$

From Fourier Series to the Fourier Integral

Fourier series of f(x) (period 2L): $f_L(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos w_n x + b_n \sin w_n x), \quad w_n = \frac{n\pi}{L}$

L → ∞, f(x) ?

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)dv + \frac{1}{L}\sum_{n=1}^{\infty}\left[\cos w_n x \int_{-L}^{L} f_L(v)\cos w_n v dv + \sin w_n x \int_{-L}^{L} f_L(v)\sin w_n v dv\right]$$

$$\Delta w = w_{n+1} - w_n = \frac{\pi}{L} \quad \left(\frac{1}{L} = \frac{\Delta w}{\pi}\right)$$

$$\Rightarrow f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)dv + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\begin{array}{l}(\cos w_n x)\Delta w \int_{-L}^{L} f_L(v)\cos w_n v dv \\ + (\sin w_n x)\Delta w \int_{-L}^{L} f_L(v)\sin w_n v dv\end{array}\right]$$

If $f(x) = \lim_{L\to\infty} f_L(x)$ is absolutely integrable, $\int_{-\infty}^{\infty} |f(x)| dx$ exists

L → ∞, then $f(x) = \frac{1}{\pi}\int_{0}^{\infty}\left[\cos wx \int_{-\infty}^{\infty} f(v)\cos wv dv + \sin wx \int_{-\infty}^{\infty} f(v)\sin wv dv\right]dw$

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv dv, \quad B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv dv$$

$$f(x) = \int_{0}^{\infty}[A(w)\cos wx + B(w)\sin wx]dw$$

Fourier Integral

$1 \int^{\infty}$

When $f(x)$ is an even function, $f(u)\sin\omega u$ is an odd function of u,

the **Fourier cosine integral representation** of $f(x)$

$$f(x) = \int_0^\infty A(\omega)\cos\omega x\,d\omega.$$

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(u)\cos\omega u\,du,$$

Similarly, when $f(x)$ is an odd function, $f(u)\cos\omega u$ is an odd function of u,

the **Fourier sine integral representation** of $f(x)$ given by

$$f(x) = \int_0^\infty B(\omega)\sin\omega x\,d\omega.$$

$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(u)\sin\omega u\,du,$$

the **exponential Fourier transform**. where k is an arbitrary scale factor.

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\exp\{-i\omega x\}dx, \qquad f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)\exp\{i\omega x\}d\omega.$$

Fourier transform pair.

$$\mathcal{F}\{f(x)\} = \frac{k}{2\pi}\int_{-\infty}^{\infty} f(x)\exp\{\pm i\omega x\}dx \quad \text{and}$$

$$\mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{k}\int_{-\infty}^{\infty} F(\omega)\exp\{\mp i\omega x\}d\omega,$$

$$\lim_{|\omega|\to\infty} F(\omega) = 0.$$

Properties and Theorems of the Fourier Transform

Linearity

$$a_1 f_1(t) + a_2 f_2(t) + \ldots + a_n f_n(t) \Leftrightarrow a_1 F_1(\omega) + a_2 F_2(\omega) + \ldots + a_n F_n(\omega)$$

Symmetry

$$F(t) \Leftrightarrow 2\pi f(-\omega)$$

Time Scaling

$$f(at) \Leftrightarrow \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

Time Shifting

$$f(t - t_0) \Leftrightarrow F(\omega) e^{-j\omega t_0}$$

Frequency Shifting

$$e^{j\omega_0 t} f(t) \Leftrightarrow F(\omega - \omega_0)$$

$$e^{j\omega_0 t} f(at) \Leftrightarrow \frac{1}{|a|} F\left(\frac{\omega - \omega_0}{a}\right)$$

Time Differentiation

$$\frac{d^n}{dt^n} f(t) \Leftrightarrow (j\omega)^n F(\omega)$$

Frequency Differentiation

$$(-jt)^n f(t) \Leftrightarrow \frac{d^n}{d\omega^n} F(\omega)$$

Time Integration

dc

$$\int_{-\infty}^{t} f(\tau)d\tau \Leftrightarrow \frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega)$$

s

Conjugate Time and Frequency Functions

$$f^*(t) \Leftrightarrow F^*(-\omega)$$

Time Convolution

$$f_1(t)*f_2(t) \Leftrightarrow F_1(\omega)F_2(\omega)$$

Frequency Convolution

$$f_1(t)f_2(t) \Leftrightarrow \frac{1}{2\pi} F_1(\omega)*F_2(\omega)$$

Area Under $f(t)$ If $F(\omega)$ is the Fourier transform of $f(t)$, then,

$$F(0) = \int_{-\infty}^{\infty} f(t)dt$$

that is, the area under a time function $f(t)$ is equal to the value of its Fourier transform evaluated at $\omega = 0$.

Area Under $F(\omega)$ If $F(\omega)$ is the Fourier transform of $f(t)$, then,

$$f(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)d\omega$$

that is, the value of the time function $f(t)$, evaluated at $t = 0$, is equal to the area under its Fourier transform $F(\omega)$ times $1/2\pi$.

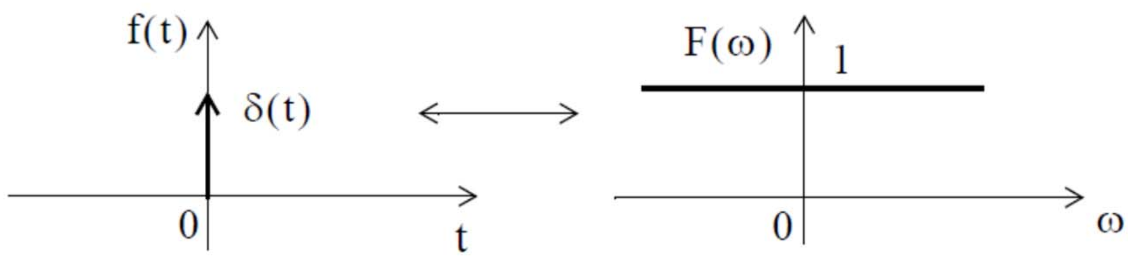
Parseval's Theorem

$$\boxed{\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega}$$

the integral is called *the energy of the signal*.

The Delta Function Pair

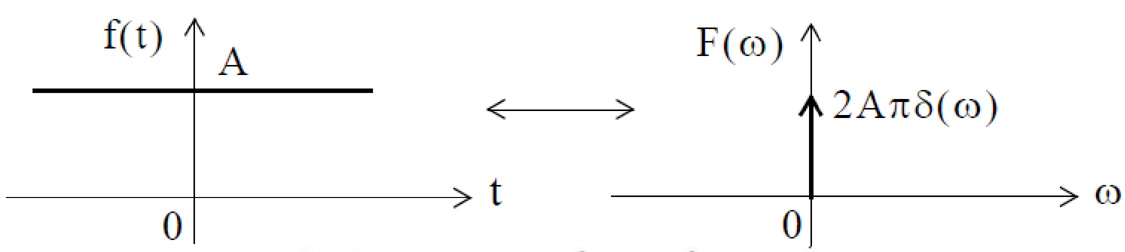
$$\boxed{\delta(t) \Leftrightarrow 1} \qquad \boxed{\delta(t - t_0) \Leftrightarrow e^{-j\omega t_0}}$$



The Fourier transform of the delta function

The Constant Function Pair

$$\boxed{A \Leftrightarrow 2A\pi\delta(\omega)} \qquad \boxed{e^{j\omega_0 t} \Leftrightarrow 2\pi\delta(\omega - \omega_0)}$$



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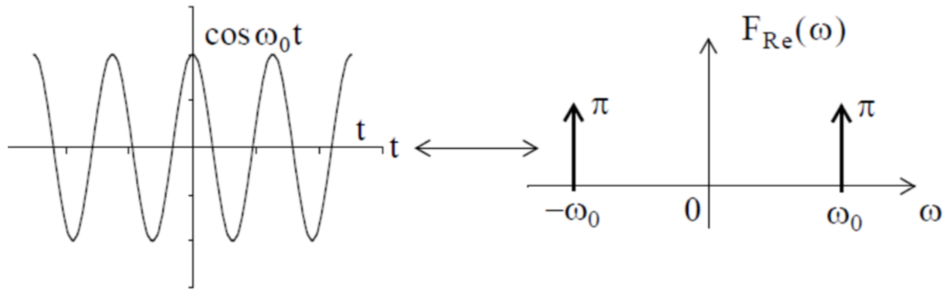
The Fourier transform of constant A

The Cosine Function Pair

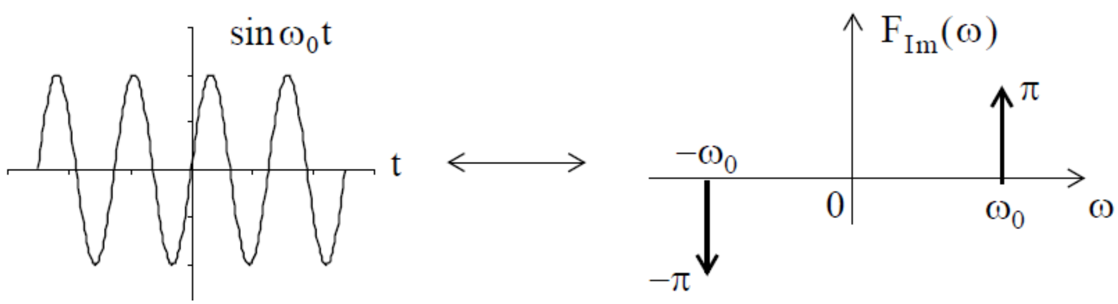
$$\boxed{\cos\omega_0 t = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) \Leftrightarrow \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)}$$



The Fourier transform of $f(t) = \cos\omega_0 t$

The Sine Function Pair

$$\boxed{\sin\omega_0 t = \frac{1}{j2}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right) \Leftrightarrow j\pi\delta(\omega-\omega_0) - j\pi\delta(\omega+\omega_0)}$$

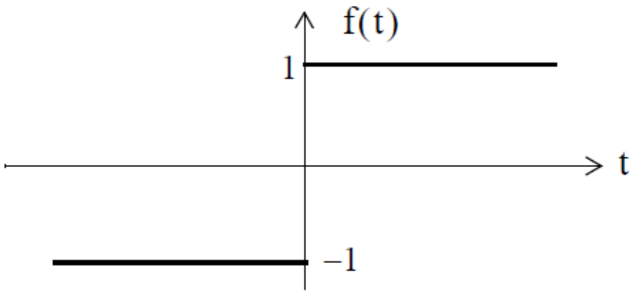


The Fourier transform of $f(t) = \sin\omega_0 t$

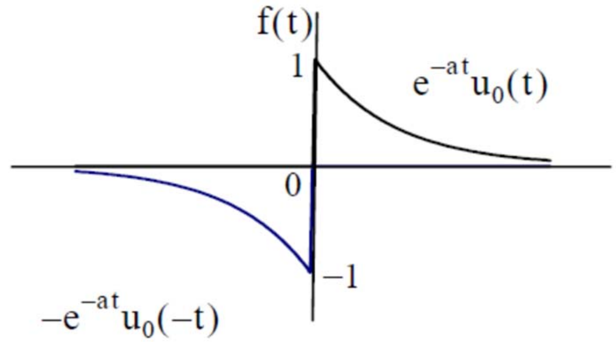
unit step

The Signum Function Pair

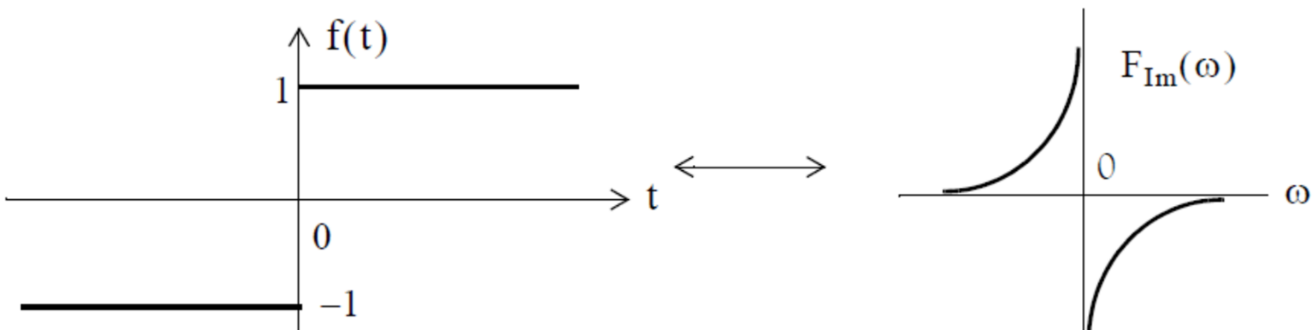
$$\boxed{\operatorname{sgn}(t) = u_0(t) - u_0(-t) \Leftrightarrow \frac{2}{j\omega}}$$



The signum function



The signum function as an exponential approaching a limit



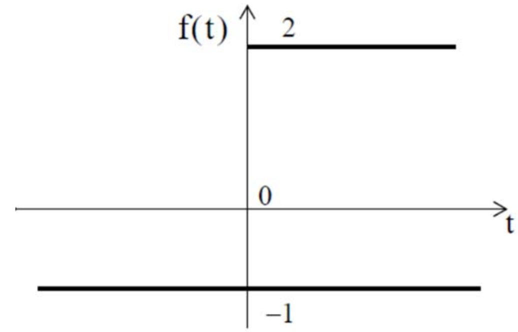
The Fourier transform of $\operatorname{sgn}(t)$

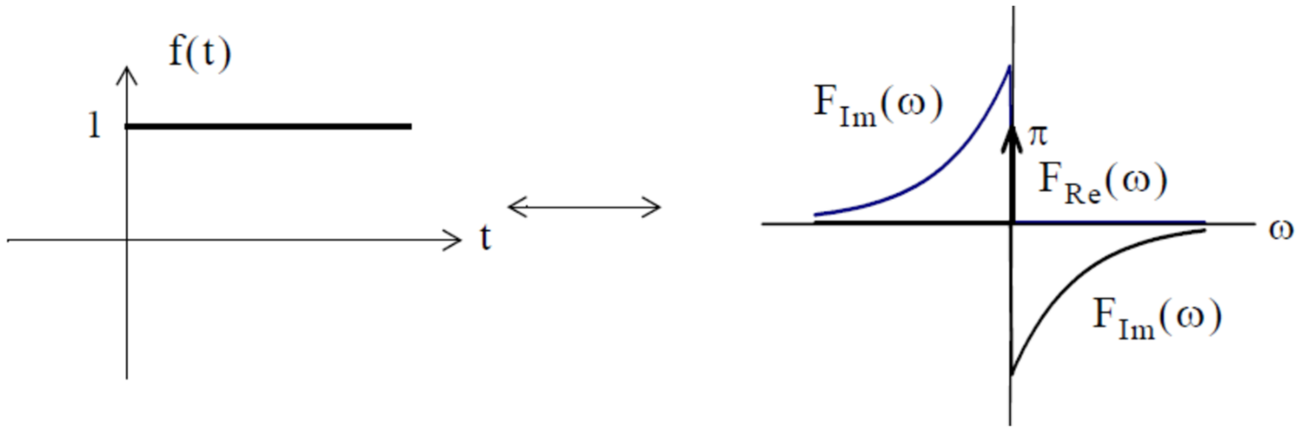
The Unit Step Function Pair

$$\boxed{u_0(t) \Leftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}}$$

$$\boxed{\operatorname{sgn}(t) = 2u_0(t) - 1}$$

Alternate expression for the signum function





The Fourier transform of the unit step function

By the convolution integral, $u_0(t) * f(t) = \int_{-\infty}^{t} f(\tau) u_0(t-\tau) d\tau$

Next, by the time convolution property, $u_0(t) * f(t) \Leftrightarrow U_0(\omega) \cdot F(\omega)$

$$U_0(\omega) \cdot F(\omega) = \left(\pi\delta(\omega) + \frac{1}{j\omega}\right) F(\omega) = \pi\delta(\omega) F(\omega) + \frac{F(\omega)}{j\omega} = \pi F(0)\delta(\omega) + \frac{F(\omega)}{j\omega}$$

The $e^{-j\omega_0 t}u_0(t)$ Function Pair

impulse form DC

$$e^{-j\omega_0 t}u_0(t) \Leftrightarrow \pi\delta(\omega-\omega_0)+\frac{1}{j(\omega-\omega_0)}$$

The $(\cos\omega_0 t)u_0(t)$ Function Pair

$$(\cos\omega_0 t)(u_0 t) \Leftrightarrow \frac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\frac{1}{2j(\omega-\omega_0)}+\frac{1}{2j(\omega+\omega_0)}$$

$$\Leftrightarrow \frac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\frac{j\omega}{\omega_0^2-\omega^2}$$

The $(\sin\omega_0 t)u_0(t)$ Function Pair

$$(\sin\omega_0 t)(u_0 t) \Leftrightarrow \frac{\pi}{j2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\frac{\omega^2}{\omega_0^2-\omega^2}$$

Derivation of the Fourier Transform from the Laplace Transform

If a time function $f(t)$ is zero for $t \leq 0$, we can obtain the Fourier transform of $f(t)$ from the one-sided Laplace transform of $f(t)$ by substitution of s with $j\omega$.

Example

It is known that $\mathcal{L}[e^{-\alpha t}u_0(t)] = \dfrac{1}{s+\alpha}$. Compute $\mathcal{F}\{e^{-\alpha t}u_0(t)\}$

Solution:

$$\mathcal{F}\{e^{-\alpha t}u_0(t)\} = \mathcal{L}[e^{-\alpha t}u_0(t)]\Big|_{s=j\omega} = \frac{1}{s+\alpha}\Big|_{s=j\omega} = \frac{1}{j\omega+\alpha}$$

Thus, we have obtained the following Fourier transform pair.

$$\boxed{e^{-\alpha t}u_0(t) \Leftrightarrow \frac{1}{j\omega+\alpha}}$$

Example

It is known that

$$\mathcal{L}[(e^{-\alpha t}\cos\omega_0 t)u_0(t)] = \frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$$

Compute $\mathcal{F}\{(e^{-\alpha t}\cos\omega_0 t)u_0(t)\}$

Solution:

$$\mathcal{F}\{(e^{-\alpha t}\cos\omega_0 t)\,u_0(t)\} = \mathcal{L}\,[(e^{-\alpha t}\cos\omega_0 t)u_0(t)]\Big|_{s\,=\,j\omega}$$

$$= \left.\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}\right|_{s\,=\,j\omega} = \frac{j\omega+\alpha}{(j\omega+\alpha)^2+\omega_0^2}$$

Thus, we have obtained the following Fourier transform pair.

$$\boxed{(e^{-\alpha t}\cos\omega_0 t)u_0(t) \Leftrightarrow \frac{j\omega+\alpha}{(j\omega+\alpha)^2+\omega_0^2}}$$

We can also find the Fourier transform of a time function $f(t)$ that has non–zero values for $t<0$, and it is zero for all $t>0$. But because the one–sided Laplace transform does not exist for $t<0$, we must first express the negative time function in the $t>0$ domain, and compute the one–sided Laplace transform. Then, the Fourier transform of $f(t)$ can be found by substituting s with $-j\omega$. In other words, when $f(t) = 0$ for $t \ge 0$, and $f(t) \ne 0$ for $t<0$, we use the substitution

$$\boxed{\mathcal{F}\{f(t)\} = \mathcal{L}\,[f(-t)]\Big|_{s\,=\,-j\omega}}$$

Example

Compute the Fourier transform of $f(t) = e^{-a|t|}$

a. using the Fourier transform definition

b. by substitution into the Laplace transform equivalent

Solution:

a. Using the Fourier transform definition, we obtain

$$\mathcal{F}\{e^{-a|t|}\} = \int_{-\infty}^{0} e^{at}e^{-j\omega t}dt + \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt = \int_{-\infty}^{0} e^{(a-j\omega)t}dt + \int_{0}^{\infty} e^{-(a+j\omega)t}dt$$

$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{2}{\omega^2 + a^2}$$

and thus we have the transform pair

$$\boxed{e^{-a|t|} \Leftrightarrow \frac{2}{\omega^2 + a^2}}$$

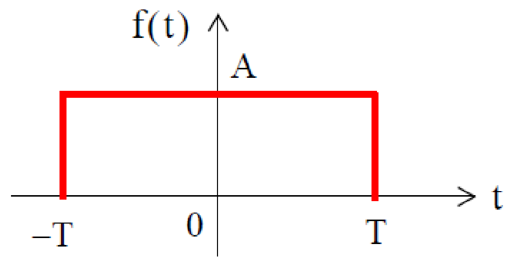
b. By substitution into the Laplace transform equivalent, we obtain

$$\mathcal{F}\{e^{-a|t|}\} = \mathcal{L}[e^{-at}]\Big|_{s=j\omega} + \mathcal{L}[e^{at}]\Big|_{s=-j\omega} = \frac{1}{s+a}\Big|_{s=j\omega} + \frac{1}{s+a}\Big|_{s=-j\omega}$$

$$= \frac{1}{j\omega + a} + \frac{1}{-j\omega + a} = \frac{2}{\omega^2 + a^2}$$

and this result is the same as (8.79). We observe that since $f(t)$ is real and even, $F(\omega)$ is also real and even.

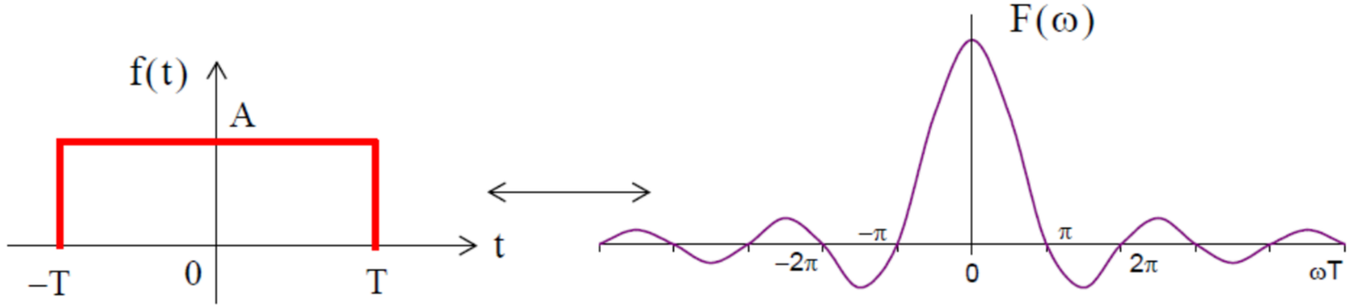
The Transform of $f(t) = A[u_0(t+T) - u_0(t-T)]$



Rectangular pulse waveform $f(t) = A[u_0(t+T) - u_0(t-T)]$

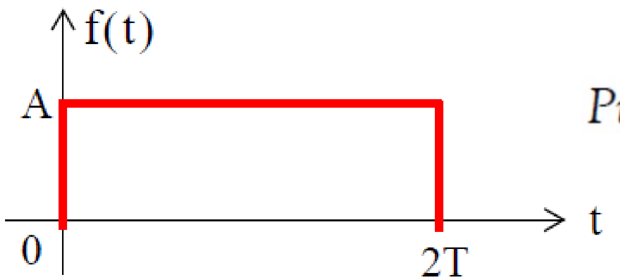
Thus, we have the waveform pair

$$\boxed{A[u_0(t+T) - u_0(t-T)] \Leftrightarrow 2AT\frac{\sin\omega T}{\omega T}}$$



The waveform $f(t) = A[u_0(t+T) - u_0(t-T)]$ *and its Fourier transform*

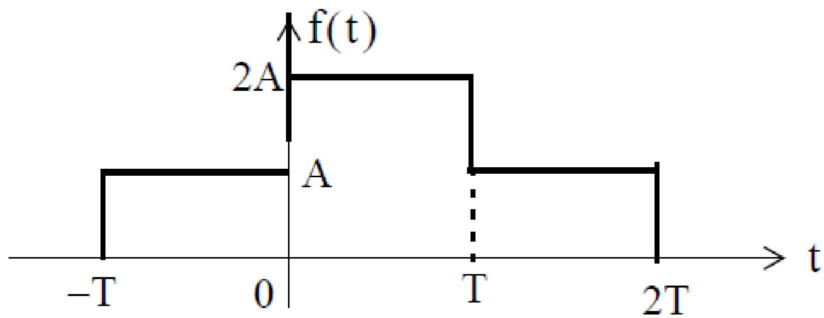
The Transform of $f(t) = A[u_0(t) - u_0(t-2T)]$



Pulse for $f(t) = A[u_0(t) - u_0(t-2T)]$

$f(t-t_0) \Leftrightarrow F(\omega)e^{-j\omega t_0}$ Thus, multiplying $2AT\dfrac{\sin\omega T}{\omega T}$ by $e^{-j\omega T}$

The Transform of $f(t) = A[u_0(t+T) + u_0(t) - u_0(t-T) - u_0(t-2T)]$



Waveform for $f(t) = A[u_0(t+T) + u_0(t) - u_0(t-T) - u_0(t-2T)]$

$$F(\omega) = F_1(\omega) + F_2(\omega) = 2AT\frac{\sin\omega T}{\omega T} + 2ATe^{-j\omega T}\left(\frac{\sin\omega T}{\omega T}\right)$$

$$F(\omega) = 4ATe^{-j\frac{\omega T}{2}}\cos\left(\frac{\omega T}{2}\right)\frac{\sin\omega T}{\omega T}$$

The Transform of $f(t) = A\cos\omega_0 t[u_0(t+T) - u_0(t-T)]$

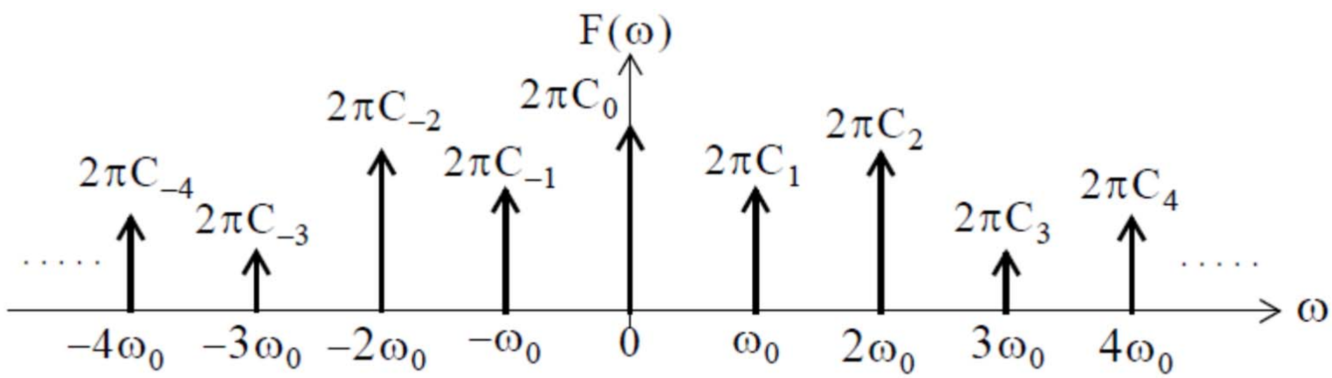
$$\boxed{A\cos\omega_0 t[u_0(t+T) - u_0(t-T)] \Leftrightarrow AT\left[\frac{\sin[(\omega-\omega_0)T]}{(\omega-\omega_0)T} + \frac{\sin[(\omega+\omega_0)T]}{(\omega+\omega_0)T}\right]}$$

The Transform of a Periodic Time Function with Period T

the exponential Fourier series $\quad f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$

we obtain

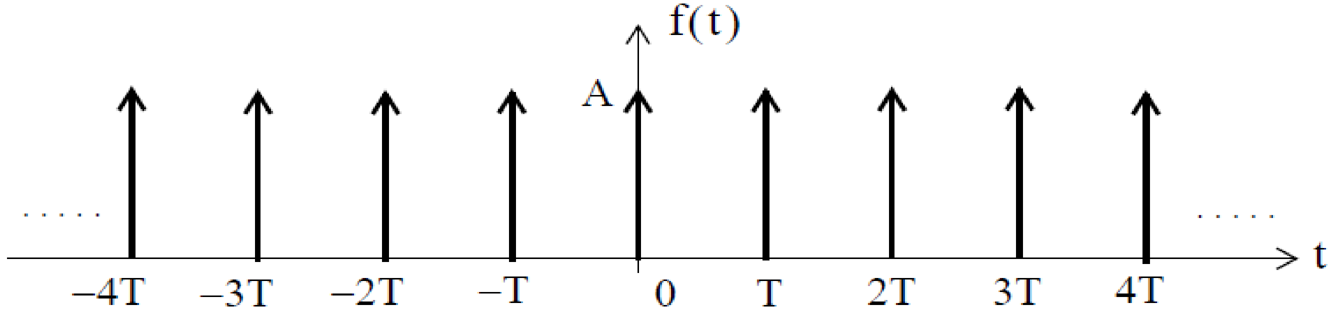
$$\mathcal{F}\{f(t)\} = \mathcal{F}\left\{\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}\right\} = \sum_{n=-\infty}^{\infty} C_n \mathcal{F}\{e^{jn\omega_0 t}\} = 2\pi \sum_{n=-\infty}^{\infty} C_n \delta(\omega - n\omega_0)$$



Line spectrum

The Transform of the Periodic Time Function $f(t) = A\sum_{n=-\infty}^{\infty}\delta(t-nT)$

The periodic time function $\quad f(t) = A\sum_{n=-\infty}^{\infty}\delta(t-nT)$



Waveform for $f(t) = A\sum_{n=-\infty}^{\infty}\delta(t-nT)$

$F(\omega) = 2\pi\sum_{n=-\infty}^{\infty}C_n\delta(\omega-n\omega_0)$ where $C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}dt$

By substitution $\quad C_n = \frac{1}{T}\int_{-T/2}^{T/2}\delta(t)e^{-jn\omega_0 t}dt = \frac{1}{T}$

we obtain $\quad F(\omega) = \frac{2\pi}{T}\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)$

THEOREM

Fourier transform of $x^n f(x)$ Let $f(x)$ be a continuous and differentiable function with an n times differentiable Fourier transform $F(\omega)$. Then

(a) $$\mathcal{F}\{xf(x)\} = i\frac{d}{d\omega}[F(\omega)]$$

and

(b) $$\mathcal{F}\{x^n f(x)\} = i^n\frac{d^n}{d\omega^n}[F(\omega)],$$

for all n such that $\lim_{|\omega|\to\infty} F^{(n)}(\omega) = 0$.

THEOREM

Fourier transform of $x^m f^{(n)}(x)$ Let $f(x)$ be a continuous n times differentiable function. Furthermore, let $x^m f^{(r)}(x)$ for $r = 1, 2, \ldots, n$ satisfy Dirichlet conditions and be absolutely integrable over $(-\infty, \infty)$, and let $\omega^n F(\omega)$ possess an m times differentiable inverse Fourier transform. Then, provided $\lim_{|x|\to\infty} f^{(n-1)}(x) = 0$,

$$\mathcal{F}\left\{x^m f^{(n)}(x)\right\} = (i)^{m+n}\frac{d^m}{d\omega^m}[\omega^n F(\omega)].$$

EXAMPLE

Find the Fourier transforms of

$$\text{(a)}\quad f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a, \end{cases} \qquad \text{(b)}\quad g(x) = \begin{cases} 1, & 0 < x < a \\ 0, & \text{otherwise,} \end{cases} \qquad \text{(c)}\quad p(x) = \frac{1}{x^2 + a^2}$$

by making use of the standard integral $\int_{-\infty}^{\infty} \frac{\cos \omega x}{x^2+a^2} dx = \frac{\pi}{a} e^{-|\omega| a}$ $(a > 0)$ and (d) $q(x) = \begin{cases} e^{iax}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$. In each case confirm that the Fourier transform vanishes as $\omega \to \pm\infty$.

Solution

(a)

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-i\omega x} dx = \frac{1}{\omega\sqrt{2\pi}} \left[\frac{e^{i\omega a} - e^{-i\omega a}}{i} \right]$$

$$= \frac{1}{\omega} \sqrt{\frac{2}{\pi}} \left[\frac{e^{i\omega a} - e^{-i\omega a}}{2i} \right] = \sqrt{\frac{2}{\pi}} \frac{\sin \omega a}{\omega}.$$

As $\sin \omega a$ is bounded, it follows directly that $\lim_{|\omega| \to \infty} F(\omega) = 0$.

(b)

$$G(\omega) = \frac{1}{\sqrt{2\pi}} \int_{0}^{a} e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \left(\frac{1 - e^{-i\omega a}}{i\omega} \right).$$

As the numerator of $G(\omega)$ is bounded, it follows that $\lim_{|\omega| \to \infty} G(\omega) = 0$. This example shows that although $f(x)$ may be real, its Fourier transform can be complex.

(c) $$P(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{-i\omega x}}{x^2+a^2}dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\cos\omega x}{x^2+a^2}dx-\frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\sin\omega x}{x^2+a^2}dx.$$

The integrand of the second integral is odd, so the value of the integral is zero. Using the standard result

$$\int_{-\infty}^{\infty}\frac{\cos\omega x}{x^2+a^2}dx=\frac{\pi}{a}e^{-|\omega|a}$$

in the remaining integral on the right, we find that

$$P(\omega)=\sqrt{\frac{\pi}{2}}\frac{e^{-|\omega|a}}{a}\quad(a>0).$$

In this case the factor $e^{-|\omega|a}$ ensures that $\lim_{|\omega|\to\infty}P(\omega)=0$.

(d) $$Q(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}q(x)e^{-i\omega x}dx=\frac{1}{\sqrt{2\pi}}\int_{0}^{1}e^{-i(\omega-a)x}dx$$
$$=\frac{i}{\sqrt{2\pi}}\left(\frac{1-e^{-i(\omega-a)}}{a-\omega}\right).$$

As the numerator of the Fourier transform is bounded, the denominator causes the transform to vanish as $|\omega|\to\infty$. This example shows that a complex function can also have a Fourier transform and, in general, that the transform will be complex.

EXAMPLE Find the Fourier transform of $f(x) = \exp(-a^2x^2)(a > 0)$.

Solution The function $f(x)$ is continuous and differentiable for all x and

$$\int_{-\infty}^{\infty} |\exp(-a^2x^2)|dx = \int_{-\infty}^{\infty} \exp(-a^2x^2)dx = \frac{1}{a}\int_{-\infty}^{\infty} \exp(-u^2)du = \frac{\sqrt{\pi}}{a},$$

where we have made use of the standard integral $\int_{-\infty}^{\infty} \exp(-u^2)du = \sqrt{\pi}$. This shows that $f(x)$ is absolutely integrable over the interval $(-\infty, \infty)$, and so $f(x)$ has a Fourier transform. A straightforward calculation establishes that $f(x)$ satisfies the differential equation

$$f' + 2a^2xf = 0.$$

Taking the Fourier transform of this equation using Theorem 10.2 gives

$$\mathcal{F}\{f'(x)\} + 2a^2\mathcal{F}\{xf(x)\} = 0.$$

Applying Theorem to the first term and Theorem to the second term and cancelling a factor i reduces this to the variables separable equation for $F(\omega)$,

$$2a^2F' + \omega F = 0, \quad \text{where } F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp(-a^2x^2)e^{-i\omega x}dx.$$

When variables are separated, the equation becomes

$$\int \frac{F'}{F} d\omega = -\frac{1}{2a^2} \int \omega d\omega,$$

so

$$\ln F(\omega) = -\frac{\omega^2}{4a^2} + \ln A, \quad \text{or} \quad F(\omega) = A \exp\left[-\frac{\omega^2}{4a^2}\right],$$

where, for convenience, the arbitrary integration constant has been written in the form $\ln A$. To determine A we use the fact that $A = F(0)$, but

$$F(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-a^2x^2)dx = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\pi}}{a} = \frac{1}{a\sqrt{2}},$$

and so

$$\mathcal{F}\{\exp(-a^2x^2)\} = F(\omega) = \frac{1}{a\sqrt{2}} \exp\left\{-\frac{\omega^2}{4a^2}\right\} \quad (a > 0).$$

When working with Fourier integrals, as with the Laplace transform, it is useful to introduce the convolution operation to establish the relationship between the functions $f(x)$ and $g(x)$ and their respective Fourier transforms $F(\omega)$ and $G(\omega)$.

The **convolution** of functions $f(x)$ and $g(x)$ denoted by $f * g$ is a function of x, and if the dependence on a variable x in the convolution is to be emphasized, it is then denoted by $(f * g)(x)$. The convolution of $f(x)$ and $g(x)$ is defined as

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt = \int_{-\infty}^{\infty} f(x-t)g(t)dt.$$

A slightly different definition of the convolution operation for the Fourier transform is also to be found in the literature, where it is defined as

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)g(x-t)dt.$$

THEOREM

The convolution theorem for Fourier transforms Let the functions $f(x)$ and $g(x)$ be piecewise continuous, bounded, and absolutely integrable over $(-\infty, \infty)$ with the respective Fourier transforms $F(\omega)$ and $G(\omega)$. Then

(a) $$\mathcal{F}\{(f * g)(x)\} = 2\pi\, \mathcal{F}\{f(x)\}\mathcal{F}\{g(x)\}, \text{ or } \mathcal{F}\{f * g\} = 2\pi\, F(\omega)G(\omega)$$

and, conversely,

(b) $$(f * g)(x) = \sqrt{2\pi} \int_{-\infty}^{\infty} F(\omega)G(\omega)e^{i\omega x} d\omega.$$

THEOREM

The Parseval relation for the Fourier transform If $f(x)$ has the Fourier transform $F(\omega)$, then

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega.$$

THEOREM

Fourier transforms involving scaling x by a, shifting x by a, and shifting ω by λ If $f(x)$ has a Fourier transform $F(\omega)$, then

(i) $$\mathcal{F}\{f(ax)\} = \frac{1}{a}F(\omega/a) \quad (a > 0)$$

(ii) $$\mathcal{F}\{f(x-a)\} = e^{-i\omega a}F(\omega)$$

(iii) $$\mathcal{F}\{e^{i\lambda x} f(x)\} = F(\omega - \lambda)$$

EXAMPLE Using the function $f(x)$ and its Fourier transform $F(\omega) = \sqrt{\frac{2}{\pi}}\frac{\sin \omega a}{\omega}$. (a) $\mathcal{F}\{f(2x)\}$, (b) $\mathcal{F}\{f(x-\pi)\}$, and (c) $\mathcal{F}\{e^{ix} f(x)\}$.

Solution Using the results of Theorem

(a) $\mathcal{F}\{f(2x)\} = \frac{1}{2}\sqrt{\frac{2}{\pi}}\left(\frac{\sin(\omega a/2)}{(\omega/2)}\right) = \sqrt{\frac{2}{\pi}}\left(\frac{\sin(\omega a/2)}{\omega}\right)$

(b) $\mathcal{F}\{f(x-\pi)\} = e^{-i\pi\omega}\sqrt{\frac{2}{\pi}}\left(\frac{\sin \omega a}{w}\right)$

(c) $\mathcal{F}\{e^{ix} f(x)\} = \sqrt{\frac{2}{\pi}}\left(\frac{\sin(\omega-1)a}{\omega-1}\right)$

The **Dirac delta function** $\delta(x)$ was introduced in connection with the Laplace transform, where it was recognized that it is not a function in the usual sense, but an *operation* that only has meaning when it appears in the integrand of a definite integral. Because of its many uses in connection with physical problems described by differential equations, we now extend its definition in a way that is suitable for use with Fourier transforms. This is accomplished by defining $\delta(x-a)$ in a symmetrical manner about $x = a$ in terms of the integrals

$$\int_{-\infty}^{\infty} \delta(x-a) f(x) dx = \int_{-\infty}^{\infty} \delta(a-x) f(x) dx = f(a),$$

where a is any real number.

This definition allows the Fourier transform of $\delta(x-a)$ to be represented as

$$\mathcal{F}\{\delta(x-a)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \delta(x-a)e^{-i\omega x}\,dx = \frac{1}{\sqrt{2\pi}}e^{-i\omega a}.$$

EXAMPLE Find the Fourier transform of $f(x) = \delta(x-a)\exp[-b^2x^2]$ $(b > 0)$.

Solution By definition

$$\mathcal{F}\{\delta(x-a)\exp[-b^2x^2]\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \delta(x-a)\exp[-b^2x^2]e^{-i\omega x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\exp[-(a^2b^2 + i\omega a)].$$

Fourier Transforms of Partial Derivatives with Respect to *x* of a Function *f*(*x*, *t*)

The Fourier transform with respect to x of a function $f(x,t)$ of two independent variables x and t, denoted by $F(\omega, t)$, is defined as

$$_x\mathcal{F}\{f(x,t)\} = F(\omega,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x,t)e^{-i\omega x}\,dx,$$

where the prefix suffix x shows the variable that is being transformed.

and the Fourier transforms of the partial derivatives of $f(x,t)$ with respect to x are given by

$$ {}_x\mathcal{F}\left\{\frac{\partial^n}{\partial x^n}[f(x,t)]\right\} = (i\omega)^n F(\omega,t) $$

$$ {}_x\mathcal{F}\{x^n f(x,t)\} = i^n \frac{\partial^n}{\partial \omega^n}[F(\omega,t)] $$

$$ {}_x\mathcal{F}\left\{x^m \frac{\partial^n}{\partial x^n}[f(x,t)]\right\} = i^{m+n} \frac{\partial^m}{\partial \omega^m}[\omega^n F(\omega,t)]. $$

Fourier Cosine and Sine Transforms

Fourier cosine transform

$$ F_C(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos \omega x dx. $$

$$ F_C(\omega) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos \omega x dx, $$

Fourier cosine inversion integral

$$ f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_C(\omega) \cos \omega x d\omega. $$

Fourier sine transform

$$F_S(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin \omega x dx,$$

Fourier cosine inversion integral

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_S(\omega) \sin \omega x d\omega.$$

EXAMPLE

Find $\mathcal{F}_C\{e^{-ax}\}$ and $\mathcal{F}_S\{e^{-ax}\}$ when $a > 0$, and use the results with the Fourier cosine and sine inversion integrals and an interchange of variables to show that

$$\mathcal{F}_C\left\{\frac{1}{x^2 + a^2}\right\} = \sqrt{\frac{\pi}{2}} \frac{e^{-a\omega}}{a} \quad \text{and} \quad \mathcal{F}_S\left\{\frac{x}{x^2 + a^2}\right\} = \sqrt{\frac{\pi}{2}} e^{-a\omega}.$$

Solution By definition

$$\mathcal{F}_C\{e^{-ax}\} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \cos \omega x dx$$

$$= \mathrm{Re} \sqrt{\frac{2}{\pi}} \left\{ \int_0^{\infty} e^{-ax} e^{i\omega x} dx \right\} = \sqrt{\frac{2}{\pi}} \mathrm{Re} \left\{ \frac{1}{a - i\omega} \right\} = \sqrt{\frac{2}{\pi}} \left(\frac{a}{\omega^2 + a^2} \right).$$

Similarly,

$$\begin{aligned}\mathcal{F}_S\{e^{-ax}\} &= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\ \sin\omega x dx \\ &= \operatorname{Im}\sqrt{\frac{2}{\pi}}\left\{\int_0^\infty e^{-ax}e^{i\omega x}dx\right\} = \sqrt{\frac{2}{\pi}}\left(\frac{\omega}{\omega^2+a^2}\right).\end{aligned}$$

Using these results in the Fourier cosine and sine inversion integrals gives

$$e^{-ax} = \frac{2a}{\pi}\int_0^\infty \frac{\cos\omega x}{\omega^2+a^2}d\omega = \frac{2}{\pi}\int_0^\infty \frac{\omega\sin\omega x}{\omega^2+a^2}d\omega,\ \text{for } a>0,$$

so after x and ω are interchanged, these results become

$$e^{-a\omega} = \frac{2a}{\pi}\int_0^\infty \frac{\cos\omega x}{x^2+a^2}dx = \frac{2}{\pi}\int_0^\infty \frac{x\cos\omega x}{x^2+a^2}dx.$$

However,

$$\mathcal{F}_C\left\{\frac{1}{x^2+a^2}\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\cos\omega x}{x^2+a^2}dx \quad \text{and} \quad \mathcal{F}_S\left\{\frac{x}{x^2+a^2}\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{x\sin\omega x}{x^2+a^2}dx,$$

so combining results gives

$$\mathcal{F}_C\left\{\frac{1}{x^2+a^2}\right\} = \sqrt{\frac{\pi}{2}}\frac{e^{-a\omega}}{a} \quad \text{and} \quad \mathcal{F}_S\left\{\frac{x}{x^2+a^2}\right\} = \sqrt{\frac{\pi}{2}}e^{-a\omega}..$$

THEOREM

Fourier cosine and sine transforms of derivatives Let $f(x)$ be continuous and absolutely integrable over $[0, \infty)$ and such that $\lim_{x\to\infty} f(x) = 0$. Then if $f'(x)$ and $f''(x)$ are piecewise continuous on each finite subinterval of $[0, \infty)$,

(i) $$\mathcal{F}_C\{f'(x)\} = \omega \mathcal{F}_S\{f(x)\} - \sqrt{\frac{2}{\pi}} f(0)$$

(ii) $$\mathcal{F}_S\{f'(x)\} = -\omega \mathcal{F}_C\{f(x)\}$$

(iii) $$\mathcal{F}_C\{f''(x)\} = -\omega^2 \mathcal{F}_C\{f(x)\} - \sqrt{\frac{2}{\pi}} f'(0)$$

(iv) $$\mathcal{F}_S\{f''(x)\} = -\omega^2 \mathcal{F}_S\{f(x)\} + \sqrt{\frac{2}{\pi}} \omega f(0).$$

The Parseval relation for the Fourier cosine and sine transforms Let $f(x)$ have the respective Fourier cosine and sine transforms $F_C(\omega)$ and $F_S(\omega)$. Then the Parseval relation for the Fourier cosine transform is

$$\int_0^\infty |F_C(\omega)|^2 d\omega = \int_0^\infty |f(x)|^2 dx,$$

and the Parseval relation for the Fourier sine transform is

$$\int_0^\infty |F_S(\omega)|^2 d\omega = \int_0^\infty |f(x)|^2 dx.$$

THEOREM

Shifting ω and scaling x in Fourier cosine and sine transforms Let $f(x)$ have the respective Fourier cosine and sine transforms $F_C(\omega)$ and $F_S(\omega)$. Then

(i) $$\mathcal{F}_C\{\cos(ax)f(x)\} = \tfrac{1}{2}\{F_C(\omega + a) + F_C(\omega - a)\}$$

(ii) $$\mathcal{F}_C\{\sin(ax)f(x)\} = \tfrac{1}{2}\{F_S(a + \omega) + F_S(a - \omega)\}$$

(iii) $$\mathcal{F}_S\{\cos(ax)f(x)\} = \tfrac{1}{2}\{F_S(\omega + a) + F_S(\omega - a)\}$$

(iv) $$\mathcal{F}_S\{\sin(ax)f(x)\} = \tfrac{1}{2}\{F_C(\omega - a) - F_C(\omega + a)\}$$

(v) $$\mathcal{F}_C\{f(ax)\} = \frac{1}{a}F_C(\omega/a) \quad (a > 0)$$

(vi) $$\mathcal{F}_S\{f(ax)\} = \frac{1}{a}F_S(\omega/a) \quad (a > 0).$$

Fourier cosine and sine transforms of partial derivatives of a function f(x, y)

$$_x\mathcal{F}_C\{f'(x,t)\} = \omega F_S(\omega,t) - \sqrt{\frac{2}{\pi}}f(0,t)$$

$$_x\mathcal{F}_S\{f'(x,t)\} = -\omega F_C(\omega,t)$$

$$_x\mathcal{F}_C\{f''(x,t)\} = -\omega^2 F_S(\omega,t) - \sqrt{\frac{2}{\pi}}f'(0,t)$$

$$_x\mathcal{F}_S\{f''(x,t)\} = -\omega^2 F_S(\omega,t) + \sqrt{\frac{2}{\pi}}\omega f(0,t)$$

EXAMPLE

Given $f(x) = e^{-ax}$ with $a > 0$, use the results of Theorem find (a) $\mathcal{F}_C\{\cos bx f(x)\}$ and (b) $\mathcal{F}_S\{f(bx)\}$, when $b > 0$.

Solution

(a) Using Theorem (i) with

$$\mathcal{F}_C\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\left(\frac{a}{\omega^2 + a^2}\right),$$

gives

$$\mathcal{F}_C\{\cos bx e^{-ax}\} = \frac{1}{2}\sqrt{\frac{2}{\pi}}\left(\frac{a}{(\omega+b)^2+a^2}\right) + \frac{1}{2}\sqrt{\frac{2}{\pi}}\left(\frac{a}{(\omega-b)^2+a^2}\right)$$
$$= \sqrt{\frac{2}{\pi}}\frac{a(\omega^2+a^2+b^2)}{[(\omega+b)^2+a^2][(\omega-b)^2+a^2]}.$$

(b) Using Theorem (vi) with

$$\mathcal{F}_S\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\left(\frac{\omega}{\omega^2+a^2}\right)$$

gives

$$\mathcal{F}_S\{f(bx)\} = \mathcal{F}_S\{e^{-abx}\} = \frac{1}{b}\sqrt{\frac{2}{\pi}}\left(\frac{\omega/b}{(\omega/b)^2+a^2}\right) = \sqrt{\frac{2}{\pi}}\left(\frac{\omega}{\omega^2+a^2b^2}\right).$$

TABLE Fourier Transform Pairs

$f(x)$	$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$
1. $af(x) + bg(x)$	$aF(\omega) + bG(\omega)$
2. $f^{(n)}(x)$	$(i\omega)^n F(\omega)$
3. $x^n f(x)$	$(i)^n \frac{d^n}{d\omega^n}[F(\omega)]$
4. $x^m f^{(n)}(x)$	$(i)^{m+n} \frac{d^m}{d\omega^m}[\omega^n F(\omega)]$
5. $f(ax)(a > 0)$	$\frac{1}{a} F(\omega/a)$
6. $f(x - a)$	$e^{-i\omega a} F(\omega)$
7. $e^{i\lambda x} f(x)$	$F(\omega - \lambda)$
8. $(f * g)(x)$	$\sqrt{2\pi} F(\omega) G(\omega)$ (convolution theorem)

9. $\int_{-\infty}^{\infty} \lvert f(x)\rvert^2 dx$	$\int_{-\infty}^{\infty} \lvert F(\omega)\rvert^2 d\omega$ (Parseval relation)
10. $\begin{cases} 1, & \lvert x\rvert < a \\ 0, & \lvert x\rvert > a \end{cases} \quad (a > 0)$	$\sqrt{\frac{2}{\pi}}\left(\frac{\sin a\omega}{\omega}\right)$
11. $\frac{\sin ax}{x} \quad (a > 0)$	$\begin{cases} \sqrt{\frac{\pi}{2}}, & \lvert\omega\rvert < a \\ 0, \lvert\omega\rvert > a \end{cases}$
12. $\begin{cases} 1, & a < x < b \\ 0, & \text{otherwise} \end{cases} \quad (0 < a < b)$	$\frac{1}{\sqrt{2\pi}}\left(\frac{e^{-ia\omega} - e^{-ib\omega}}{i\omega}\right)$
13. $\begin{cases} a - \lvert x\rvert, & \lvert x\rvert < a \\ 0, & \lvert x\rvert > a \end{cases}$	$\sqrt{\frac{2}{\pi}}\left(\frac{1 - \cos \omega a}{\omega^2}\right)$
14. $\frac{1}{a^2 + x^2} \; (a > 0)$	$\sqrt{\frac{\pi}{2}} \frac{e^{-a\lvert\omega\rvert}}{a}$
15. $\begin{cases} e^{-ax}, & x > 0 \\ 0, & x < 0 \end{cases} \; (a > 0)$	$\frac{1}{\sqrt{2\pi}}\left(\frac{1}{a + i\omega}\right)$
16. $\begin{cases} e^{ax}, & b < x < c \\ 0, & \text{otherwise} \end{cases} \; (a > 0)$	$\frac{1}{\sqrt{2\pi}}\left[\frac{e^{(a-i\omega)c} - e^{(a-i\omega)b}}{a - i\omega}\right]$

$f(x)$	$F(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$
17. $e^{-a\|x\|} \quad (a > 0)$	$\sqrt{\dfrac{2}{\pi}} \left(\dfrac{a}{a^2 + \omega^2} \right)$
18. $xe^{-a\|x\|} \quad (a > 0)$	$-\sqrt{\dfrac{2}{\pi}} \dfrac{2ia\omega}{(a^2 + \omega^2)^2}$
19. $\begin{cases} e^{iax}, & \|x\| < b \\ 0, & \|x\| > b \end{cases}$	$\sqrt{\dfrac{2}{\pi}} \left(\dfrac{\sin b(\omega - a)}{\omega - a} \right)$
20. $\exp(-a^2x^2) \quad (a > 0)$	$\dfrac{1}{a\sqrt{2}} \exp\left\{ -\dfrac{\omega^2}{4a^2} \right\}$
21. $\begin{cases} e^{-x}x^a, & x > 0 \\ 0, & x \le 0 \end{cases}$	$\dfrac{\Gamma(a)}{\sqrt{2\pi}(1 + i\omega)^a}$
22. $J_0(ax) \quad (a > 0)$	$\sqrt{\dfrac{2}{\pi}} \dfrac{H(a - \|\omega\|)}{(a^2 - \omega^2)^{1/2}}$
23. $\delta(x - a) \quad (a \text{ real})$	$\dfrac{1}{\sqrt{2\pi}} e^{-ia\omega}$

TABLE Fourier Cosine Transform Pairs

$f(x)$	$F_C(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \omega x dx$
1. $af(x) + bg(x)$	$aF(\omega) + bG(\omega)$
2. $\cos(ax) f(x)$	$\frac{1}{2}\{F_C(\omega + a) + F_C(\omega - a)\}$
3. $\sin(ax) f(x)$	$\frac{1}{2}\{F_S(a + \omega) + F_S(a - \omega)\}$
4. $f(ax)$	$\frac{1}{a} F_C\left(\frac{\omega}{a}\right)(a > 0)$
5. $f'(x)$	$\omega F_S(\omega) - \sqrt{\frac{2}{\pi}} f(0)$
6. $f''(x)$	$-\omega^2 F_C(\omega) - \sqrt{\frac{2}{\pi}} f'(0)$
7. $\int_0^\infty \lvert f(x)\rvert^2 dx$	$\int_0^\infty \lvert F(\omega)\rvert^2 d\omega$ (Parseval relation)
8. $\int_0^\infty f(x) g(x) dx$	$\int_0^\infty F_C(\omega) G_C(\omega) d\omega$

9.	$\begin{cases} 1, & 0 < x < a \\ 0, & \text{otherwise} \end{cases}$	$\sqrt{\dfrac{2}{\pi}}\left(\dfrac{\sin a\omega}{\omega}\right)$
10.	$\begin{cases} 1, & a < x < b \\ 0, & \text{otherwise} \end{cases}$	$\sqrt{\dfrac{2}{\pi}}\left(\dfrac{\sin b\omega - \sin a\omega}{\omega}\right)$
11.	$x^{\alpha-1}(0 < \alpha < 1)$	$\sqrt{\dfrac{2}{\pi}}\dfrac{\Gamma(\alpha)}{\omega^{\alpha}}\cos\dfrac{\alpha\pi}{2}$
12.	$\begin{cases} x, & a < x < b \\ 0, & \text{otherwise} \end{cases}$	$\sqrt{\dfrac{2}{\pi}}\left(\dfrac{\cos b\omega + b\omega\sin b\omega - \cos a\omega - a\omega\sin a\omega}{\omega^2}\right)$
13.	$e^{-ax}(a > 0)$	$\sqrt{\dfrac{2}{\pi}}\left(\dfrac{a}{\omega^2 + a^2}\right)$
14.	$xe^{-ax}(a > 0)$	$\sqrt{\dfrac{2}{\pi}}\dfrac{(a^2 - \omega^2)}{(a^2 + \omega^2)^2}$
15.	$\exp\{-ax^2\}\ (a > 0)$	$\dfrac{1}{\sqrt{2a}}\exp\left\{-\dfrac{\omega^2}{4a}\right\}$
16.	$\dfrac{1}{x^2 + a^2}(a > 0)$	$\sqrt{\dfrac{\pi}{2}}\dfrac{e^{-a\omega}}{a}$

$f(x)$	$F_C(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos \omega x dx$
17. $J_0(ax)(a > 0)$	$\sqrt{\frac{2}{\pi}} \frac{H(a-\omega)}{(a^2-\omega^2)^{1/2}}$
18. $\frac{\sin ax}{x}(a > 0)$	$\sqrt{\frac{2}{\pi}} H(a-\omega)$

TABLE Fourier Sine Transform Pairs

$f(x)$	$F_S(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin \omega x dx$
1. $af(x) + bg(x)$	$aF(\omega) + bG(\omega)$
2. $\cos(ax) f(x)$	$\frac{1}{2}\{F_S(\omega+a) + F_S(\omega-a)\}$
3. $\sin(ax) f(x)$	$\frac{1}{2}\{F_C(\omega-a) - F_C(\omega+a)\}$
4. $f(ax)$	$\frac{1}{a} F_S\left(\frac{\omega}{a}\right) \quad (a > 0)$

TABLE Fourier Sine Transform Pairs

$f(x)$	$F_S(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin \omega x dx$
5. $f'(x)$	$-\omega F_C(\omega)$
6. $f''(x)$	$-\omega^2 F_S(\omega) + \sqrt{\frac{2}{\pi}} \omega f'(0)$
7. $\int_0^{\infty} \lvert f(x)\rvert^2 dx$	$\int_0^{\infty} \lvert F(\omega)\rvert^2 d\omega$ (Parseval relation)
8. $\int_0^{\infty} f(x)g(x)dx$	$\int_0^{\infty} F_S(\omega)G_S(\omega)d\omega$
9. $\begin{cases} 1, & 0 < x < a \\ 0, & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}} \left(\frac{1 - \cos a\omega}{\omega} \right)$
10. $\begin{cases} 1, & a < x < b \\ 0, & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}} \left(\frac{\cos a\omega - \cos b\omega}{\omega} \right)$

$f(x)$	$F_S(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x dx$
11. $x^{\alpha-1}$ $(0 < \alpha < 1)$	$\sqrt{\frac{2}{\pi}} \frac{\Gamma(\alpha)}{\omega^\alpha} \sin \frac{\alpha\pi}{2}$
12. e^{-ax} $(a > 0)$	$\sqrt{\frac{2}{\pi}} \frac{\omega}{(\omega^2 + a^2)}$
13. xe^{-ax} $(a > 0)$	$\sqrt{\frac{2}{\pi}} \frac{2a\omega}{(\omega^2 + a^2)^2}$
14. $x \exp\{-ax^2\}$ $(a > 0)$	$\frac{\omega}{(2a)^{3/2}} \exp\left\{-\frac{\omega^2}{4a}\right\}$
15. $\frac{x}{x^2 + a^2}$ $(a > 0)$	$\sqrt{\frac{\pi}{2}} e^{-a\omega}$
16. $\frac{\cos ax}{x}$ $(a > 0)$	$\sqrt{\frac{\pi}{2}} H(\omega - a)$
17. $\operatorname{erfc}\left\{\frac{x}{2a}\right\}$ $(a > 0)$	$\sqrt{\frac{2}{\pi}} \left\{\frac{1 - \exp(-a^2\omega^2)}{\omega}\right\}$

Fourier Transform Table (1)

No.	$x(t)$	$X(\omega)$	
1	$e^{-at}u(t)$	$\dfrac{1}{a+j\omega}$	$a>0$
2	$e^{at}u(-t)$	$\dfrac{1}{a-j\omega}$	$a>0$
3	$e^{-a\|t\|}$	$\dfrac{2a}{a^2+\omega^2}$	$a>0$
4	$te^{-at}u(t)$	$\dfrac{1}{(a+j\omega)^2}$	$a>0$
5	$t^n e^{-at}u(t)$	$\dfrac{n!}{(a+j\omega)^{n+1}}$	$a>0$
6	$\delta(t)$	1	
7	1	$2\pi\delta(\omega)$	
8	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	
9	$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	
10	$\sin\omega_0 t$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$	
11	$u(t)$	$\pi\delta(\omega)+\dfrac{1}{j\omega}$	
12	$\operatorname{sgn} t$	$\dfrac{2}{j\omega}$	

Fourier Transform Table (2)

No.	$x(t)$	$X(\omega)$	
13	$\cos \omega_0 t\, u(t)$	$\frac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\frac{j\omega}{\omega_0^2-\omega^2}$	
14	$\sin \omega_0 t\, u(t)$	$\frac{\pi}{2j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]+\frac{\omega_0}{\omega_0^2-\omega^2}$	
15	$e^{-at}\sin \omega_0 t\, u(t)$	$\frac{\omega_0}{(a+j\omega)^2+\omega_0^2}$	$a>0$
16	$e^{-at}\cos \omega_0 t\, u(t)$	$\frac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$	$a>0$
17	$\operatorname{rect}\left(\frac{t}{\tau}\right)$	$\tau \operatorname{sinc}\left(\frac{\omega\tau}{2}\right)$	
18	$\frac{W}{\pi}\operatorname{sinc}(Wt)$	$\operatorname{rect}\left(\frac{\omega}{2W}\right)$	
19	$\Delta\left(\frac{t}{\tau}\right)$	$\frac{\tau}{2}\operatorname{sinc}^2\left(\frac{\omega\tau}{4}\right)$	
20	$\frac{W}{2\pi}\operatorname{sinc}^2\left(\frac{Wt}{2}\right)$	$\Delta\left(\frac{\omega}{2W}\right)$	
21	$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\omega_0\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)$	$\omega_0=\frac{2\pi}{T}$
22	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}e^{-\sigma^2\omega^2/2}$	