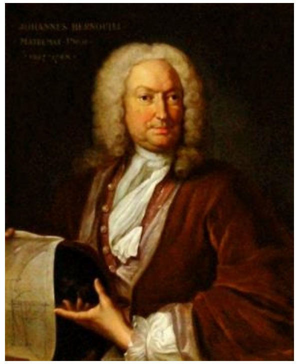
INVENTION OF DIFFERENTIAL EQUATIONS

In mathematics, the history of differential equations traces the development of "differential equations" from calculus, which itself was independently invented by English physicist Isaac Newton and German mathematician Gottfried Leibniz.

The history of the subject of differential equations, in concise form, from a synopsis of the recent article “The History of Differential Equations, 1670- 1950”

Isaac Newton

Gottfried Wilhelm Leibniz



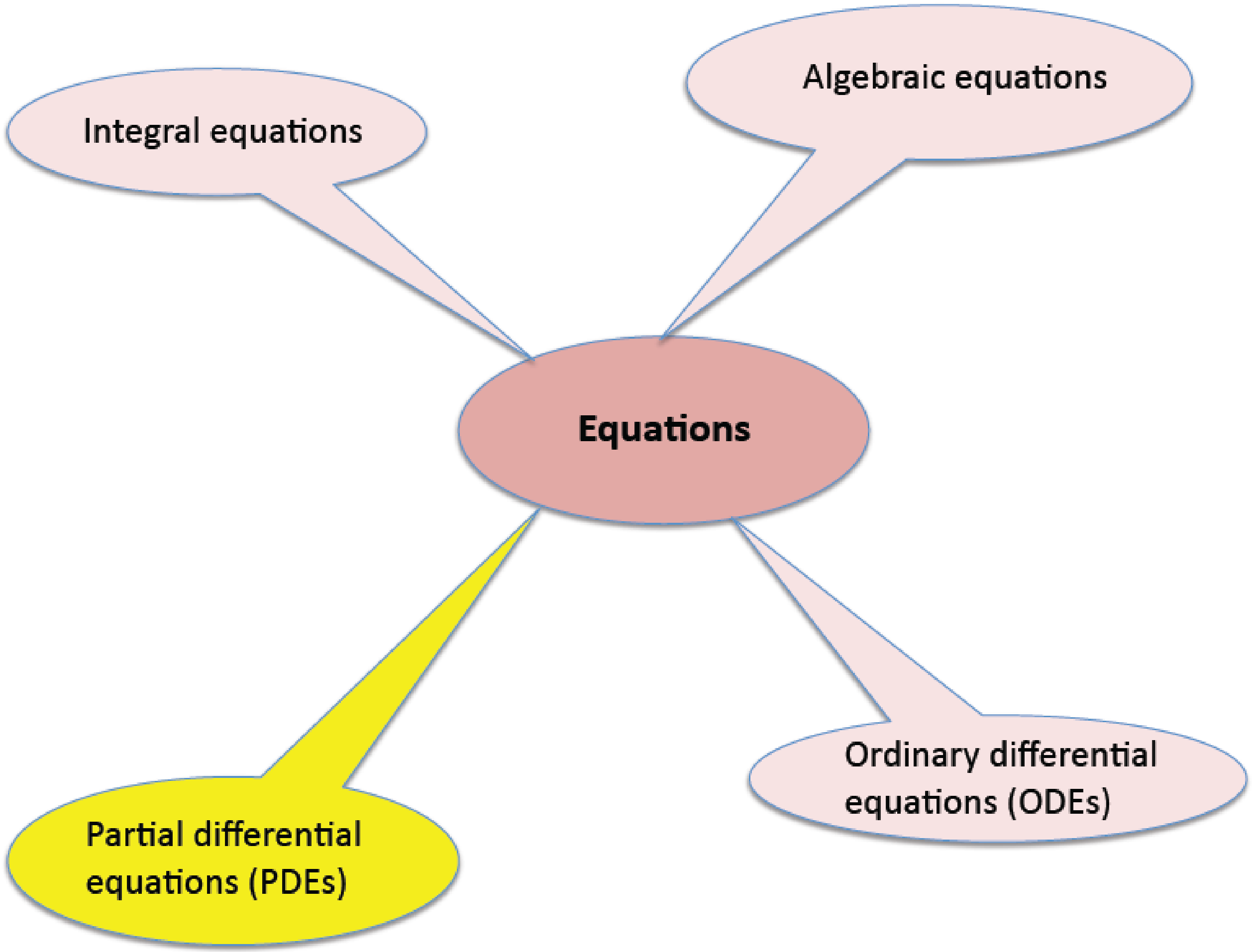
Johann Bernoulli

“Differential equations began with Leibniz, the Bernoulli brothers, and others from the 1680s, not long after Newton’s ‘fluxional equations’ in the 1670s.”

PDE and Boundary-Value Problems

Introduction to Partial Differential Equations

What are PDEs?



- Algebraic equations state relations between unknown number and its power,
  e.g.
  $$x^3 - 7x^2 - 44x = 0.$$
  Three solutions: $x_1 = 0$, $x_2 = -4$, $x_3 = 11$.
- Ordinary differential equations state relations between an unknown function of ONE variable and its derivatives,
  e.g.
  $$u''(t) = 0.$$
  Infinitely many solution: $u(t) = at + b \quad (a, b \in \mathbb{R})$.
- Partial differential equations state relations between an unknown function of SEVERAL variables and its partial derivatives,
  e.g.
  $$\begin{aligned} u_t(x,t) &= u_{xx}(x,t) \quad \text{(heat equation)} \\ u_t(x,y,t) &= u_{xx}(x,y,t) + u_{yy}(x,y,t) \quad \text{(heat equation)} \\ u_{tt}(x,y,z,t) &= u_{xx} + u_{yy} + u_{zz} \quad \text{(wave equation)} \\ u_{tt}(x,t) &= u_{xx} + \alpha u_t + \beta u \quad \text{(telegraph equation)} \end{aligned}$$

Examples of PDEs:

$\Delta u = \sum_{i=1}^{n} u_{x_i x_i} = 0$	Laplace's equation
$u_t - \Delta u = 0$	heat (or diffusion) equation
$u_{tt} - \Delta u = 0$	wave equation
$u_t - \Delta u - \sum_{i=1}^{n} \left(b^i u\right)_{x_i} = 0$	Fokker-Planck equation
$\operatorname{div}\left(\frac{Du}{\sqrt{1+\lvert Du\rvert^2}}\right) = 0$	minimal surface equation

Physical Examples of PDEs

(1) Wave equation, second-order, linear, homogeneous:

$$-\partial_t^2 u + \partial_x^2 u = 0$$

(2) Heat equation, second-order, linear, homogeneous:

$$-\partial_t u + \partial_x^2 u = 0$$

(3) Laplace's equation, second-order, linear, homogeneous:

$$\partial_x^2 u + \partial_y^2 u + \partial_z^2 u = 0$$

(4) Poisson's equation with source function *f*, second-order, linear, inhomogeneous:

$$\partial_x^2 u + \partial_y^2 u + \partial_z^2 u = f(x, y, z)$$

(5) Transport equation, first-order, linear, homogeneous:

$$\partial_t u + \partial_x u = 0$$

(6) Burger's equation, first-order, nonlinear, homogeneous:

$$\partial_t u + u\,\partial_x u = 0$$

(7) Schrödinger equation, second-order, linear, homogeneous:

$$i\,\partial_t u + \partial_x^2 u = 0$$

(8) Maxwell's equations in a vacuum, first-order, linear, homogeneous:

$$\partial_t \mathbf{E} - \nabla \times \mathbf{B} = 0\,, \quad \nabla \cdot \mathbf{E} = 0$$

$$\partial_t \mathbf{B} - \nabla \times \mathbf{E} = 0\,, \quad \nabla \cdot \mathbf{B} = 0$$

Name	2 Vars	3 D
Heat Equation	$u_t = k u_{xx}$	$u_t = k\nabla^2 u$
Wave Equation	$u_{tt} = c^2 u_{xx}$	$u_{tt} = c^2\nabla^2 u$
Laplace's Equation	$u_{xx} + u_{yy} = 0$	$\nabla^2 u = 0$
Poisson's Equation	$u_{xx} + u_{yy} = F(x,y)$	$\nabla^2 u = F(x,y,z)$
Schrödinger's Equation	$iu_t = u_{xx} + F(x,t)u$	$iu_t = \nabla^2 u + F(x,y,z,t)u$

Grad, Div, Curl

Define $\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$ and $u = u(x, y, z)$

$$\operatorname{grad} u = \nabla u = \frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} + \frac{\partial u}{\partial z}\mathbf{k} = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$$

$$\operatorname{div} u = \nabla \cdot u = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$

$$\operatorname{curl} u = \nabla \times u = \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & u & u \end{pmatrix}$$

$$= \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial u}{\partial x}, \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\operatorname{div}(\operatorname{grad} u) = \nabla \cdot (\nabla u) = \nabla^2 u = \Delta u \text{ (Laplacian)}$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

Boundary Conditions for PDEs

Equation (13.2) is a condition on u on the "horizontal" part of the boundary of $\tilde{D}$, but it is not enough to specify u completely; we also need a boundary condition on the "vertical" part of the boundary to tell what happens to the heat when it reaches the boundary surface S of the spatial region D.

(1) One assumption is that S is held at a constant temperature u_0, for example by immersing the body in a bath of ice water. This is called **Dirichlet condition**, where:

$x = u$

$$u(t, \mathbf{x}) - u_0 = 0 \quad \text{for } \mathbf{x} \in S, \quad t > 0 \qquad \textbf{(13.3)}$$

In terms of the heat equation example, Dirichlet conditions correspond to maintaining a fixed temperature at the ends of the rod. The Neumann boundary conditions would correspond to no heat flow across the ends, or insulating conditions, as there would be no temperature gradient at those points. The more general boundary conditions allow for partially insulated boundaries.

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(2) Another assumption is that D is insulated, so that no heat can flow in or out across S. This is called **homogeneous Neumann condition** or **zero-flux condition**. $\Delta j = 0$
Mathematically, this condition amounts to requiring the normal derivative of u along the boundary S to vanish:

$$(\nabla u \cdot \mathbf{n})(t, \mathbf{x}) = 0 \quad \text{for} \quad \mathbf{x} \in S\,, \quad t > 0 \qquad \textbf{(13.4)}$$

(3) **Robin condition** is a condition when the region outside D is held at a constant temperature u_0, and the rate of heat flow across the boundary S is proportional to the difference in temperatures on the two sides:

$$(\nabla u \cdot \mathbf{n})(t, \mathbf{x}) + a(u(t, \mathbf{x}) - u_0) = 0 \qquad \textbf{(13.5)}$$
$$\text{for} \quad \mathbf{x} \in S\,, \quad t > 0$$

Another type of boundary condition that is often encountered is the periodic boundary condition. Consider the heated rod that has been bent to form a circle. Then the two end points are physically the same. So, we would expect that the temperature and the temperature gradient should agree at those points. For this case we write $y(a) = y(b)$ and $y'(a) = y'(b)$. Boundary value problems using these conditions have to be handled differently than the above homogeneous conditions. These conditions leads to different types of eigenfunctions and eigenvalues.

Sturm-Liouville problems

A **Sturm-Liouville problem** consists of

- A Sturm-Liouville equation on an interval:

$$(p(x)y')' + (q(x) + \lambda r(x))y = 0, \quad a < x < b, \tag{1}$$

  together with

- **Boundary conditions**, i.e. specified behavior of y at $x = a$ and $x = b$.

We will assume that p, p', q and r are continuous and $p > 0$ on (at least) the open interval $a < x < b$.

According to the general theory of second order linear ODEs, this guarantees that solutions to (1) exist.

Regularity conditions

A **regular Sturm-Liouville problem** has the form

$$\boxed{\begin{aligned} (p(x)y')' + (q(x) + \lambda r(x))y &= 0, \quad a < x < b, \\ c_1 y(a) + c_2 y'(a) &= 0, \qquad\qquad\qquad (2) \\ d_1 y(b) + d_2 y'(b) &= 0, \qquad\qquad\qquad (3) \end{aligned}}$$

where:

- $(c_1, c_2) \neq (0, 0)$ and $(d_1, d_2) \neq (0, 0)$;
- p, p', q and r are continuous on $[a, b]$;
- p and r are positive on $[a, b]$.

The boundary conditions (2) and (3) are called **separated** boundary conditions.

Properties Theorem 12.5.1 is a list of some of the more important of the many properties of the regular Sturm–Liouville problem. We shall prove only the last property.

Theorem 12.5.1 Properties of the Regular Sturm–Liouville Problem

(a) There exist an infinite number of real eigenvalues that can be arranged in increasing order $\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \cdots$ such that $\lambda_n \to \infty$ as $n \to \infty$.

(b) For each eigenvalue there is only one eigenfunction (except for nonzero constant multiples).

(c) Eigenfunctions corresponding to different eigenvalues are linearly independent.

(d) The set of eigenfunctions corresponding to the set of eigenvalues is orthogonal with respect to the weight function $p(x)$ on the interval $[a, b]$.

In the context of **PDEs**, these questions are answered by **Sturm-Liouville Theory**.

A **Sturm-Liouville equation** is a second order linear differential equation that can be written in the form

$$\boxed{(p(x)y')' + (q(x) + \lambda r(x))y = 0.}$$

Such an equation is said to be in **Sturm-Liouville form.**

- Here p, q and r are specific functions, and λ is a parameter.
- Because λ is a parameter, it is frequently replaced by other variables or expressions.
- Many "familiar" ODEs that occur during separation of variables can be put in Sturm-Liouville form.

Example *Show that $y'' + \lambda y = 0$ is a Sturm-Liouville equation.*

We simply take $p(x) = r(x) = 1$ and $q(x) = 0$

Example Put **the parametric Bessel equation** $x^2y'' + xy' + (\lambda^2x^2 - m^2)y = 0$

in Sturm-Liouville form.

First we divide by x to get

$$\underbrace{xy'' + y'}_{(xy')'} + \left(\lambda^2 x - \frac{m^2}{x}\right) y = 0.$$

This is in Sturm-Liouville form with

$$p(x) = x, \quad q(x) = -\frac{m^2}{x}, \quad r(x) = x,$$

provided we write the parameter as λ^2.

Example Put **Legendre's differential equation** $y'' - \frac{2x}{1-x^2}y' + \frac{\mu}{1-x^2}y = 0$ in Sturm-Liouville form.

First we multiply by $1 - x^2$ to get

$$\underbrace{(1-x^2)y'' - 2xy'}_{((1-x^2)y')'} + \mu y = 0.$$

This is in Sturm-Liouville form with

$$p(x) = 1 - x^2, \quad q(x) = 0, \quad r(x) = 1,$$

provided we write the parameter as μ.

Example Put **Chebyshev's differential equation** $(1-x^2)y'' - xy' + n^2 y = 0$

in Sturm-Liouville form.

First we divide by $\sqrt{1-x^2}$ to get

$$\underbrace{\sqrt{1-x^2}\,y'' - \frac{x}{\sqrt{1-x^2}}\,y'}_{(\sqrt{1-x^2}\,y')'} + \frac{n^2}{\sqrt{1-x^2}}\,y = 0.$$

This is in Sturm-Liouville form with

$$p(x) = \sqrt{1-x^2}, \quad q(x) = 0, \quad r(x) = \frac{1}{\sqrt{1-x^2}},$$

provided we write the parameter as n^2.

Example The boundary value problem

$$\begin{aligned} y'' + \lambda y &= 0, \qquad 0 < x < L, \\ y(0) &= y(L) = 0, \end{aligned}$$

is a regular Sturm-Liouville problem (recall that p(x) = r(x) = 1 and q(x) = 0).

Example The boundary value problem

$$\begin{aligned} ((x^2+1)y')' + (x+\lambda)y &= 0, \qquad -1 < x < 1, \\ y(-1) &= y'(1) = 0, \end{aligned}$$

is a regular Sturm-Liouville problem (here p(x) = x^2 + 1, q(x) = x and r(x) = 1).

Example The boundary value problem

$$\begin{aligned} x^2y'' + xy' + (\lambda^2x^2 - m^2)y &= 0, \qquad 0 < x < a, \\ y(a) &= 0, \end{aligned}$$

is not a regular Sturm-Liouville problem.

Why not? Recall that when put in Sturm-Liouville form we had $p(x) = r(x) = x$ and $q(x) = -m^2/x$. There are several problems:

- p and r are **not positive** when $x = 0$.
- q is **not continuous** when $x = 0$.
- The boundary condition at $x = 0$ is **missing**.

This is an example of a **singular Sturm-Liouville problem.**

TABLE $p(x)$, $q(x)$, $r(x)$ and λ for Some Named Equations

Name	$p(x)$	$q(x)$	$r(x)$	λ
Simple harmonic equation	1	0	1	n^2
Legendre's equation	$1-x^2$	0	1	$\alpha(\alpha+1)$
Bessel's equation	x	$-\nu^2/x$	x	k^2
Bessel's modified equation	x	$-\nu^2/x$	$-x$	k^2
Laguerre equation	xe^{-x}	0	e^{-x}	n
Chebyshev equation	$(1-x^2)^{1/2}$	0	$(1-x^2)^{-1/2}$	n^2
Hermite equation	e^{-x^2}	0	e^{-x^2}	$2n$

Eigenvalues and eigenfunctions

A **nonzero** function y that solves the Sturm-Liouville problem

$$(p(x)y')' + (q(x) + \lambda r(x))y \quad = \quad 0, \qquad a < x < b,$$

(plus boundary conditions),

is called an **eigenfunction**, and the corresponding value of λ is called its **eigenvalue**.

- The **eigenvalues** of a Sturm-Liouville problem are the values of λ for which nonzero solutions exist.
- We can talk about eigenvalues and eigenfunctions for regular or singular problems.

Eigenvalue Problems for y'' + λy = 0

Theorem

The eigenvalue problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0$$

has infinitely many positive eigenvalues $\lambda_n = n^2\pi^2/L^2$, with associated eigenfunctions

$$y_n = \sin\frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots.$$

There are no other eigenvalues.

Example . Solve the equation

$$y''(x) + \lambda y(x) = 0 \tag{4.20}$$

on the interval $[0, l]$ with boundary conditions

$$y(0) = 0 \quad \text{and} \quad y(l) = 0. \tag{4.21}$$

Solution. First, comparing Equation (4.20) with Sturm-Liouville problem , it is clear that we have a Sturm-Liouville problem with linear operator $L = -d^2/dx^2$, i.e. functions $q(x) = 0$ and $p(x) = r(x) = 1$. As a *Reading Exercise*, verify that L is Hermitian.

Let us discuss the cases $\lambda = 0$, $\lambda < 0$ and $\lambda > 0$ separately. If $\lambda = 0$, then a general solution to Equation (4.20) is

$$y(x) = C_1 x + C_2$$

and from boundary conditions (4.21) we have $C_1 = C_2 = 0$, i.e. there exists only the trivial solution $y(x) = 0$. If $\lambda < 0$, then

$$y(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}$$

and the boundary conditions (4.21) again give $C_1 = C_2 = 0$ and therefore the trivial solution $y(x) = 0$. Thus we have only the possibility$\lambda > 0$, in which case we write $\lambda = \mu^2$ with μ real and we have a general solution of Equation (4.20) given by

$$y(x) = C_1 \sin \mu x + C_2 \cos \mu x.$$

The boundary condition $y(0) = 0$ requires that $C_2 = 0$ and the boundary condition $y(l) = 0$ gives $C_1 \sin \mu l = 0$. From this we must have $\sin \mu l = 0$ and $\mu_n = \frac{n\pi}{l}$ since the choice $C_1 = 0$ again gives the trivial solution. Thus, the eigenvalues are

$$\lambda_n = \mu_n^2 = \left(\frac{n\pi}{l}\right)^2, \quad n = 1, 2, \ldots$$

and the eigenfunctions are $y_n(x) = C_n \sin \frac{n\pi x}{l}$, where for $n = 0$ we have the trivial solution $y_0(x) = 0$. It is obvious that we can restrict ourselves to positive values of n since negative values do not give new solutions. These eigenfunctions are orthogonal over the interval $[0, l]$

Thus, we have shown that the boundary value problem consisting of Equations (4.20) and (4.21) has eigenfunctions that are *sine* functions. It means that *the expansion in eigenfunctions of the Sturm-Liouville problem for solutions to Equations* (4.20) *and* (4.21) *is equivalent to the trigonometric Fourier sine series.*

Theorem

The eigenvalue problem

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(L) = 0$$

has infinitely many positive eigenvalues $\lambda_n = (2n-1)^2\pi^2/4L^2$, with associated eigenfunctions

$$y_n = \cos\frac{(2n-1)\pi x}{2L}, \quad n = 1, 2, 3, \ldots.$$

There are no other eigenvalues.

Example Determine the eigenvalues and corresponding eigenfunctions for the Sturm-Liouville problem

$$y''(x) + \lambda y(x) = 0, \tag{4.25}$$

$$y'(0) = 0, \quad y(l) = 0. \tag{4.26}$$

Solution. As in the previous example, the reader may check as a *Reading Exercise* that the parameter λ must be positive in order to have nontrivial solutions. Thus, we may write $\lambda = \mu^2$, so that we have oscillating solutions given by

$$y(x) = C_1 \sin \mu x + C_2 \cos \mu x.$$

The boundary condition $y'(0) = 0$ gives $C_1 = 0$ and the boundary condition $y(l) = 0$ gives $C_2 \cos \mu l = 0$. If $C_2 = 0$ we have a trivial solution; otherwise we have $\mu_n = (2n+1)\pi/2l$, for $n = 0, 1, 2, \ldots$. Therefore, the eigenvalues are

$$\lambda_n = \mu_n^2 = \left[\frac{(2n-1)\pi}{2l}\right]^2, \quad n = 1, 2, \ldots \tag{4.27}$$

and the eigenfunctions are

$$y_n(x) = C_n \cos \frac{(2n-1)\pi x}{2l}, \quad n = 1, 2, \ldots, \quad n = 1, 2, \ldots. \tag{4.28}$$

Theorem

The eigenvalue problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(L) = 0$$

has infinitely many positive eigenvalues $\lambda_n = (2n-1)^2\pi^2/4L^2$, with associated eigenfunctions

$$y_n = \sin\frac{(2n-1)\pi x}{2L}, \quad n = 1, 2, 3, \ldots.$$

There are no other eigenvalues.

Example Solve the eigenvalue problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(L) = 0.$$

Solution:

Any eigenvalues of Equation must be positive. If y satisfies Equation with $\lambda > 0$, then

$$y = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x,$$

where c_1 and c_2 are constants. The boundary condition $y(0) = 0$ implies that $c_1 = 0$. Therefore $y = c_2 \sin\sqrt{\lambda}\, x$. Hence, $y' = c_2\sqrt{\lambda}\cos\sqrt{\lambda}\, x$ and the boundary condition $y'(L) = 0$ implies that $c_2 \cos\sqrt{\lambda}\, L = 0$. To make $c_2 \cos\sqrt{\lambda}\, L = 0$ with $c_2 \neq 0$ we must choose

$$\sqrt{\lambda} = \frac{(2n-1)\pi}{2L},$$

where n is a positive integer. Then $\lambda_n = (2n-1)^2\pi^2/4L^2$ is an eigenvalue and

$$y_n = \sin\frac{(2n-1)\pi x}{2L}$$

is an associated eigenfunction.

Theorem

The eigenvalue problem

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(L) = 0$$

has the eigenvalue $\lambda_0 = 0$, with associated eigenfunction $y_0 = 1$, and infinitely many positive eigenvalues $\lambda_n = n^2\pi^2/L^2$, with associated eigenfunctions

$$y_n = \cos\frac{n\pi x}{L}, n = 1, 2, 3 \ldots.$$

There are no other eigenvalues.

EXAMPLE A Regular Sturm–Liouville Problem

Solve the boundary-value problem

$$y'' + \lambda y = 0, \qquad y(0) = 0, \qquad y(1) + y'(1) = 0. \tag{10}$$

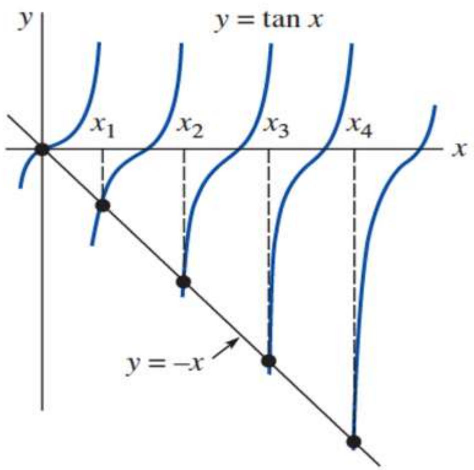
SOLUTION You should verify that for $\lambda = 0$ and for $\lambda = -\alpha^2 < 0$, where $\alpha > 0$, the BVP in (10) possesses only the trivial solution $y = 0$. For $\lambda = \alpha^2 > 0$, $\alpha > 0$, the general solution of the differential equation $y'' + \alpha^2 y = 0$ is $y = c_1 \cos \alpha x + c_2 \sin \alpha x$. Now the condition $y(0) = 0$ implies $c_1 = 0$ in this solution and so we are left with $y = c_2 \sin \alpha x$. The second boundary condition $y(1) + y'(1) = 0$ is satisfied if

$$c_2 \sin \alpha + c_2 \alpha \cos \alpha = 0.$$

Choosing $c_2 \neq 0$, we see that the last equation is equivalent to

$$\tan \alpha = -\alpha. \tag{11}$$

If we let $x = \alpha$ in (11), then **FIGURE** shows the plausibility that there exists an infinite number of roots of the equation $\tan x = -x$, namely, the x-coordinates of the points where the graph of $y = -x$ intersects the branches of the graph of $y = \tan x$.



FIGURE

Positive roots of –x = tan x

Theorem

The eigenvalue problem

$$y'' + \lambda y = 0, \quad y(-L) = y(L), \quad y'(-L) = y'(L),$$

has the eigenvalue $\lambda_0 = 0$, with associated eigenfunction $y_0 = 1$ and infinitely many positive eigenvalues $\lambda_n = n^2\pi^2/L^2$, with associated eigenfunctions

$$y_{1n} = \cos\frac{n\pi x}{L} \quad \text{and} \quad y_{2n} = \sin\frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots.$$

There are no other eigenvalues.

Example Solve the eigenvalue problem

$$y'' + \lambda y = 0, \quad y(-L) = y(L), \quad y'(-L) = y'(L).$$

Solution

Any eigenvalues of Equation must be positive. If y satisfies Equation with $\lambda > 0$, then

$$y = c_1 \cos \sqrt{\lambda}\, x + c_2 \sin \sqrt{\lambda}\, x, \tag{11.1.6}$$

where c_1 and c_2 are constants. The boundary condition $y(-L) = y(L)$ implies that

$$c_1 \cos(-\sqrt{\lambda}\, L) + c_2 \sin(-\sqrt{\lambda}\, L) = c_1 \cos \sqrt{\lambda}\, L + c_2 \sin \sqrt{\lambda}\, L. \tag{11.1.7}$$

Since

$$\cos(-\sqrt{\lambda}\, L) = \cos \sqrt{\lambda}\, L \quad \text{and} \quad \sin(-\sqrt{\lambda}\, L) = -\sin \sqrt{\lambda}\, L, \tag{11.1.8}$$

Equation 11.1.7 implies that

$$c_2 \sin \sqrt{\lambda}\, L = 0. \tag{11.1.9}$$

Differentiating Equation 11.1.6 yields

$$y' = \sqrt{\lambda} \left(-c_1 \sin \sqrt{\lambda} x + c_2 \cos \sqrt{\lambda} x \right).$$

The boundary condition $y'(-L) = y'(L)$ implies that

$$-c_1 \sin(-\sqrt{\lambda}\, L) + c_2 \cos(-\sqrt{\lambda}\, L) = -c_1 \sin\sqrt{\lambda}\, L + c_2 \cos\sqrt{\lambda}\, L,$$

and Equation 11.1.8 implies that

$$c_1 \sin\sqrt{\lambda}\, L = 0. \tag{11.1.10}$$

Eqns. Equation 11.1.9 and Equation 11.1.10 imply that $c_1 = c_2 = 0$ unless $\sqrt{\lambda} = n\pi/L$, where n is a positive integer. In this case Equation 11.1.9 and Equation 11.1.10 both hold for arbitrary c_1 and c_2. The eigenvalue determined in this way is $\lambda_n = n^2\pi^2/L^2$, and each such eigenvalue has the linearly independent associated eigenfunctions

$$\cos\frac{n\pi x}{L} \quad \text{and} \quad \sin\frac{n\pi x}{L}.$$

In this lecture, we focus on solving some *classical* partial differential equations in **boundary-value problems**.

Instead of solving the general solutions, we are only interested in finding *useful* particular solutions.

We focus on linear second order PDE: ($A, \cdots, G$: functions of x, y)

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G.$$

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$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G.$$

notation: for example, $u_{xy} := \frac{\partial^2 u}{\partial x \partial y}$.

Method: Separation of variables – convert a PDE into two ODE's

Types of Equations:

- Heat Equation
- Wave Equation
- Laplace Equation

- Parabolic equations describe heat flow and diffusion processes.
- Hyperbolic equations describe vibrating systems and wave motion.
- Elliptic equations describe steady-state phenomena.

Classification of Linear Second Order PDE

$$Au_{xx} + \underline{Bu_{xy}} + Cu_{yy} + Du_x + Eu_y + Fu = G.$$

notation: for example, $u_{xy} := \frac{\partial^2 u}{\partial x \partial y}$.

1. Homogeneous vs. Nonhomogeneous

Homogeneous	$\iff$	$G = 0$
Nonhomogeneous	$\iff$	$G \neq 0.$

2. Hyperbolic, Parabolic, and Elliptic: $A, B, C, \cdots, G$: constants,

wave eqn

Wave	Hyperbolic	$\iff$	$B^2 - 4AC > 0$
Heat	Parabolic	$\iff$	$B^2 - 4AC = 0$
Laplace	Elliptic	$\iff$	$B^2 - 4AC < 0$

Superposition Principle

Theorem

If $u_1(x,y), u_2(x,y), \ldots, u_k(x,y)$ are solutions of a homogeneous linear PDE, then a linear combination

$$u(x,y) := \sum_{n=1}^{k} c_n u_n(x,y)$$

is also a solution.

Note: We shall assume without rigorous argument that the linear combination can be an infinite series

$$u(x,y) := \sum_{n=1}^{\infty} c_n u_n(x,y)$$

How do you solve a PDE?

The most important methods are those that change PDEs into ODEs. The useful techniques are:

- *Separation of Variables*. This technique reduce a PDE in n variables to n ODEs. Homo
- *Integral Transforms*. This procedure reduces a PDE in n independent variables to one in $n-1$ variables; hence, a PDE in two variables could be changed to an ODE.
- *Eigenfunction Expansion*. This method attemts to find the solution of a PDE as an infinite sum of *eigenfunctions*. These eigenfunctions are found by solving what is known as an eigenvalue problem. แก้ได้ยาก
- *Numerical Methods*.

Method of separation of variables dates back to the time of Joseph Fourier (in fact, it's occasionally called *Fourier's method*).

- Jean Baptiste Joseph Fourier (1768-1830)

Field:	Mathematician, physisist and historian
Alma mater:	École Normale
Advisor:	Joseph Lagrange
Doctoral students:	Gustav Dirichlet, Claude-Louis Navier
Known for:	Fourier series, Fourier transform, Fourier's law of conduction

Separation of Variables

To find a particular solution of an PDE, one method is **separation of variables**, that is, assume that the solution $u(x, y)$ takes the form of a product of a x-function and a y-function:

$$u(x, y) = X(x)\,Y(y).$$

Then, with the following, *sometimes* the PDE can be converted into an ODE of X and an ODE of Y:

$$u_x = \frac{dX}{dx}Y = X'Y, \qquad u_y = X\frac{dY}{dy} = XY'$$

$$u_{xx} = \frac{d^2X}{dx^2}Y = X''Y, \qquad u_{yy} = X\frac{d^2Y}{dy^2} = XY'', \qquad u_{xy} = X'Y'$$

Note: Derivatives are with respect to different independent variables. For example, $X' := \dfrac{dX}{dx}$.

Some Remarks

1. The method of separation of variables can only solve for *some* linear second order PDE's, not all of them.

2. For the PDE's considered in this lecture, the method works.

3. The method may work for both homogeneous ($G = 0$) and nonhomogeneous ($G \neq 0$) PDE's

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G.$$

Overview of Separation of Variables

Separation of variables looks for simple-type solutions to the PDE of the form

$$u(x,t) = X(x)T(t)$$

where $X(x)$ is some function of x and $T(t)$ is some of t.

The solutions are simple because any temperature $u(x,t)$ of this form will retain its basic „shape“ for different values of time t.

- The general idea is that it is possible to find an infinite number of these solutions to the PDE (which, at the same time, also satisfy the BCs).
- These simple functions

  $$u_n(x,t) = X_n(x)T_n(t)$$

  (called fundamental solutions) are the building blocks of our problem.
- We are looking for the solution $u(x,t)$ of our IBVP as the resulting sum

$$u(x,t)=\sum_{n=1}^{\infty} A_n X_n(x)T_n(t)$$

which satisfies the initial condition.

Problem

To find the function $u(x,t)$ that satisfies

$$\text{PDE:}\quad u_t=\alpha^2 u_{xx},\qquad 0<x<1,\quad 0<t<\infty$$

Not [illegible]

$$\text{BCs:}\quad \begin{cases} u(0,t)=0 \\ u(1,t)=0 \end{cases},\qquad 0<t<\infty$$

Homo

$$\text{IC:}\quad u(x,0)=\phi(x),\qquad 0\leqslant x\leqslant 1$$

Step 1 (Finding elementary solutions to the PDE)

- We look for solutions of the form $u(x,t)=X(x)T(t)$ by substituting $X(x)T(t)$ into the PDE and solving for $X(x)T(t)$. As a result we get

$$\boxed{X(x)T'(t)=\alpha^2 X''(x)T(t).}\qquad (5.1)$$

- If we divide each side of (5.1) by $\alpha^2 X(x)T(t)$, we have

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} \tag{5.2}$$

and obtain what is called separated variables, that is, the left-hand side of (5.2) depends only on t and the right-hand side of (5.2), only on x.

- Since x and t are independent of each other, each side must be a fixed constant (say k); hence, we can write

$$T'(t) - k\alpha^2 T(t) = 0$$
$$X''(x) - kX(x) = 0.$$

- Now we can solve each of these two ODEs, multiply them together to get a solution to the PDE (note that we have changed a second-order PDE to two ODEs).
- We change $k := -\lambda^2$. Otherwise, $T(t)$ factor doesn't go to zero as $t \to \infty$. As a result we get

$$\begin{aligned} T'(t) + \lambda^2\alpha^2 T(t) &= 0 \\ X''(x) + \lambda^2 X(x) &= 0. \end{aligned} \tag{5.3}$$

- Both of equations in (5.3) are standard-type ODEs and have solutions

$$T(t) = C_1 e^{-\lambda^2 \alpha^2 t} \quad (C_1 \text{ an arbitrary constant})$$
$$X(x) = C_2 \sin(\lambda x) + C_3 \cos(\lambda x) \quad (C_2, C_3 \text{ arbitrary}).$$

- Hence all functions

$$u(x,t) = e^{-\lambda^2 \alpha^2 t}(A \sin(\lambda x) + B \cos(\lambda x))$$

(with A, B and λ arbitrary) satisfies the PDE $u_t = \alpha^2 u_{xx}$.

Step 2 (Finding solutions to the PDE and the BCs)

- The next step is to choose a certain subset of solutions

$$e^{-\lambda^2 \alpha^2 t}(A \sin(\lambda x) + B \cos(\lambda x)) \tag{5.4}$$

that satisfy the boundary conditions

$$u(0,t) = 0, \qquad u(1,t) = 0 \qquad \forall t > 0.$$

- To do this, we substitute solutions (5.4) into BCs, getting

$$u(0,t) = Be^{-\lambda^2\alpha^2 t} = 0 \quad \Rightarrow \quad B = 0$$

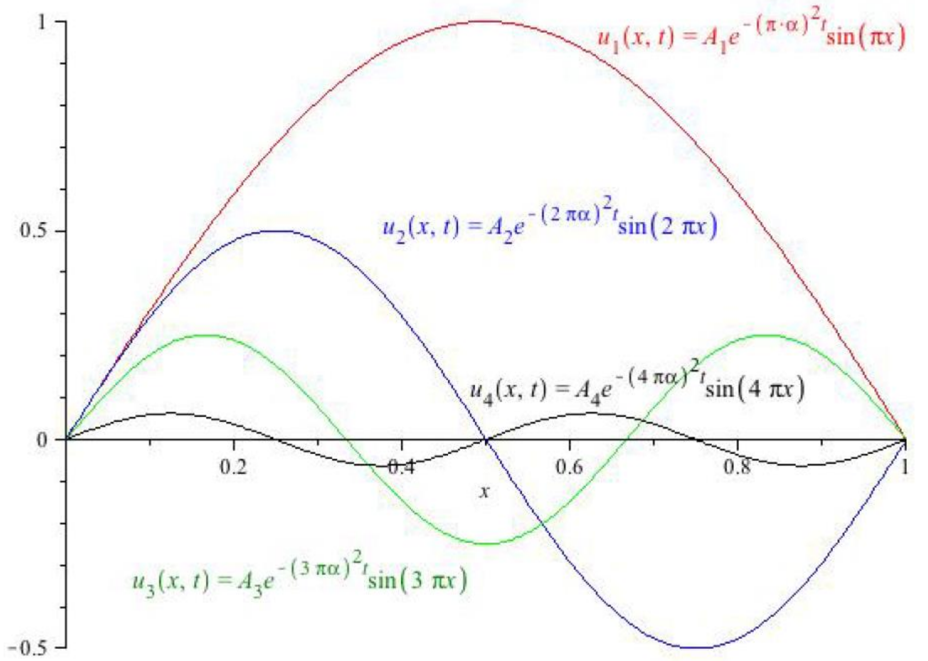
$$u(1,t) = Ae^{-\lambda^2\alpha^2 t}\sin\lambda = 0 \;\Rightarrow\; \sin\lambda = 0 \;\Rightarrow\; \lambda_n = \pm n\pi.$$

- Case $A = B = 0$ is not interesting.
- We have now finished the second step; we have found an infinite number of functions

$$\boxed{u_n(x,t) = A_n e^{-(n\pi\alpha)^2 t}\sin(n\pi x) \qquad n = 1, 2, 3, \ldots}$$

each one satisfying the PDE and BCs.

- These functions u_n are the building blocks (fundamental solutions) of the problem, and our desired solution will be a certain sum of these simple functions; the specific sum will depend on the initial conditions.



Fundamental solutions

Step 3 (Finding solutions to the PDE, BCs, and the IC)

- The last step is to add the fundamental solutions

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi\alpha)^2 t}\sin(n\pi x) \qquad (5.5)$$

in such a way (pick the coefficients A_n) that the initial condition $u(x, 0) = \phi(x)$ is satisfied.

- Substituting (5.5) into the IC gives

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x).$$

Question:

Is it possible to expand the initial temperature $\phi(x)$ as the sum of the elementary function as follows:

$$A_1 \sin(\pi x) + A_2 \sin(2\pi x) + A_3 \sin(3\pi x) + \ldots ?$$

Answer:

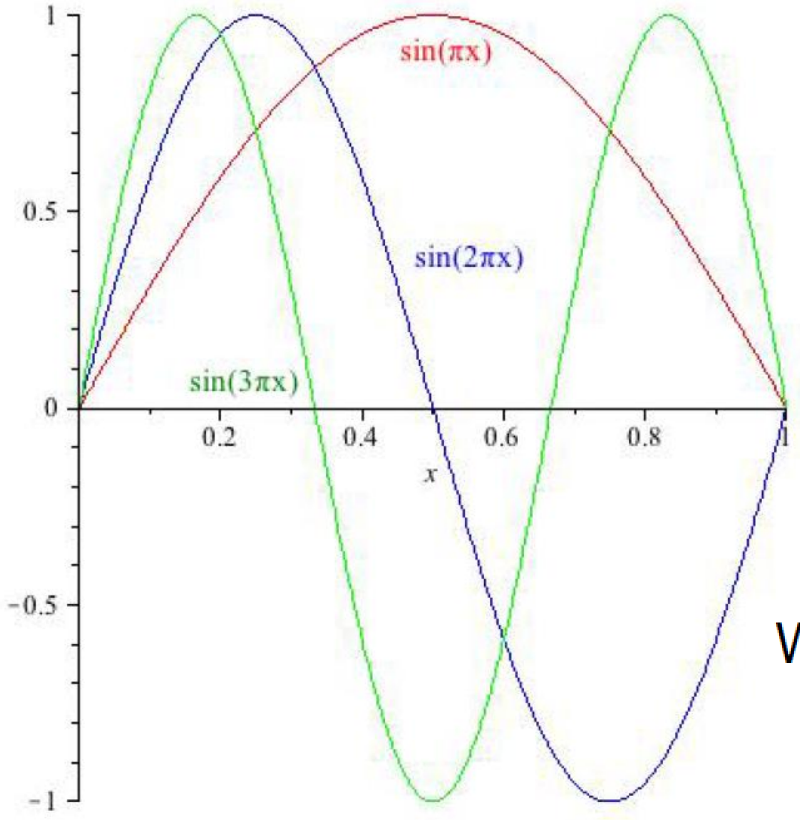
The answer to this question is YES provided $\phi(x)$ is continuous.

Hence, the question now becomes how to find the coefficients A_n.

Remark

The functions

$$\sin(n\pi x), \quad n = 1, 2, \ldots$$



Orthogonal sequence of functions

are orthogonal to each other in the sense

$$\int_0^1 \sin(m\pi x)\sin(n\pi x)dx = \begin{cases} 0, & m \neq n \\ 1/2, & m = n \end{cases}$$

This property can be illustrated by looking at the graph of these functions.

We are now in position to solve for the coefficients in the expression

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x). \tag{5.6}$$

- We multiply each side of (5.6) by $\sin(m\pi x)$ (m is an arbitrary integer) and integrate from zero to one. As a result we get

$$\int_0^1 \phi(x)\sin(m\pi x)dx = A_m \int_0^1 sin^2(m\pi x)dx = \frac{1}{2}A_m \quad \Rightarrow \quad A_m = 2\int_0^1 \phi(x)\sin(m\pi x)dx.$$

We are done; the solution is

$$\boxed{u(x,t) = 2\sum_{n=1}^{\infty}\left[\int_0^1 \phi(s)\sin(n\pi s)ds\right] e^{-(n\pi\alpha)^2 t}\sin(n\pi x)}.$$

Remark

The terms in the solution

$$u(x,t) = A_1 e^{-(\pi\alpha)^2 t}\sin(\pi x) + A_2 e^{-(2\pi\alpha)^2 t}\sin(2\pi x) + \dots$$

become small very fast due to the factor

$$e^{-(n\pi\alpha)^2 t}.$$

Hence, for long time periods, the solution is approximately equal to the first term

$$u(x,t) \cong A_1 e^{-(\pi\alpha)^2 t}\sin(\pi x).$$

Example Find product solution of $\dfrac{\partial^2 u}{\partial x^2} = 4\dfrac{\partial u}{\partial y}$.

Solution Let $u = X(x)Y(y)$ and then We introduce a real separation constant as $-\lambda$.

$$X''Y = 4XY', \frac{X''}{4X} = \frac{Y'}{Y}$$

$$\frac{X''}{4X} = \frac{Y'}{Y} = -\lambda$$

Thus $$X'' + 4\lambda X = 0, Y' + \lambda Y = 0$$

For the three cases:

$\lambda = 0$: $X'' = 0,\ Y' = 0$

$\lambda = -\alpha^2 > 0, \alpha > 0$ $X'' - 4\alpha^2 X = 0,\ Y' - \alpha^2 Y = 0$

$\lambda = \alpha^2 > 0, \alpha > 0$ $X'' + 4\alpha^2 X = 0,\ Y' + \alpha^2 Y = 0$

Case I: ($\lambda = 0$) The solutions are $X = c_1 + c_2x$ and $Y = c_3$. Thus

$$u = XY = (c_1 + c_2x)c_3 = A_1 + B_1x$$

where $A_1 = c_1c_3$, $B_1 = c_2c_3$.

Case II: ($\lambda = -\alpha^2$) The solutions of are $X = c_4 \cosh 2\alpha x + c_5 \sinh 2\alpha x$

and $Y = c_6e^{\alpha^2y}$. Thus $\quad u = XY = (c_4 \cosh 2\,\alpha x$

or $\quad u = A_2e^{\alpha^2y} \cosh 2\,\alpha x + B_2e^{\alpha^2y} \sinh 2\,\alpha x$

where $A_2 = c_4c_6$, $B_2 = c_5c_6$.

Case III: ($\lambda = \alpha^2$) The solutions are $X = c_7 \cos 2\alpha x + c_8 \sin 2\alpha x$ and $Y = c_9e^{-\alpha^2y}$.

Thus $\quad u = A_3e^{-\alpha^2y} \cos 2\,\alpha x + B_3e^{-\alpha^2y} \sin 2\,\alpha x$

where $A_3 = c_7c_9$, $B_3 = c_8c_9$.

Example Classify the following equations:

$$(a)\quad 3\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} \qquad (\mathrm{b})\quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} \qquad (\mathrm{c})\quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Solution

(*a*) $$3\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = 0\,;\, A = 3, B = 0, C = 0;$$

$$B^2 - 4AC = 0 : parabolic$$

$$(b)\quad \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0\,;\, A = 1, B = 0, C = -1;$$

$$B^2 - 4AC > 0 : hyperbolic$$

$$(c)\quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,;\, A = 1, B = 0, C = 1;$$

$$B^2 - 4AC < 0 : elliptic$$

Three Classical PDE's

In this lecture we focus on solving boundary-value problems of the following three classical PDE's that arise frequently in physics, mechanics, and engineering:

1. (One-Dimensional) Heat Equation/Diffusion Equation

$$\boxed{ku_{xx} = u_t,\ \ k > 0}$$

2. (One-Dimensional) Wave Equation/Telegraph Equation

$$\boxed{a^2 u_{xx} = u_{tt}}$$

3. (Two-Dimensional) Laplace Equation

$$\boxed{u_{xx} + u_{yy} = 0}$$

Introduction

Typical second-order PDEs: $k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$, $k > 0$

$$a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad\qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

They are known as one-dimensional **heat equation**, one-dimensional **wave equation**, and **Laplace's equations** in two dimensions, respectively.

- Laplace's equation is abbreviated $\nabla^2 u = 0$, where $\nabla^2 u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$

  is called the two-dimensional Laplacian of u. In three dimension the Laplacian of u is

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

Boundary-Value Problems

Solve: $$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \quad , \quad 0 < x < L \quad , \quad t > 0$$

Subject to:

(BC) $$u(0,\ t) = 0 \quad , \quad u(L,\ t) = 0 \quad , \quad t > 0$$

(IC) $$u(x,\ 0) = f(x) \quad , \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) \quad , \quad 0 < x < L$$

and Solve: $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad , \quad 0 < x < a \quad , \quad 0 < y < b$$

Subject to:

(BC) $$\begin{cases} \left.\dfrac{\partial u}{\partial x}\right|_{x=0} = 0 \,, & \left.\dfrac{\partial u}{\partial x}\right|_{x=a} = 0 \,, \quad 0 < y < b \\ u(x,\ 0) = 0 \,, & u(x,\ b) = f(x) \,, \quad 0 < x < a \end{cases}$$