The Semicircle Constraint: A Geometric Framework for Quantum-Classical Correlation

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For any normalized qubit state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, the measurement probability $q = |\beta|^2$ and the quantum-classical correlation $C_{qc} = |\alpha||\beta| = \sqrt{q(1-q)}$ satisfy the constraint $(q - \frac{1}{2})^2 + C_{qc}^2 = \frac{1}{4}$. This is a semicircle of radius $\frac{1}{2}$ centered at $(\frac{1}{2}, 0)$ in the (q, C_{qc}) plane, and it follows from the Born rule and normalization alone. Classical states ($q \to 0$ or 1) sit at the endpoints where C_{qc} vanishes, while maximum coherence $C_{qc} = \frac{1}{2}$ is achieved uniquely at $q = \frac{1}{2}$. The Fisher information metric along this curve is constant, giving a total arc length of π. We show that the constraint explains the geometric origin of barren plateaus in variational quantum algorithms: gradient variance scales as $q(1-q) = C_{qc}^2$ and vanishes at the classical endpoints. Hardware validation on IonQ Forte-1 (15 test points, 52 shots each) yields a theory-measurement correlation of $r = 0.943$.

INTRODUCTION

The Born rule $P = |\langle\phi|\psi\rangle|^2$ connects quantum amplitudes to classical probabilities [7], yet the geometric structure implicit in this connection has received surprisingly little attention. Consider a qubit in state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. The normalization $|\alpha|^2 + |\beta|^2 = 1$ forces the measurement probability $q = |\beta|^2$ and the off-diagonal coherence $|\alpha||\beta|$ to be linked. How, exactly?

We show that these two quantities satisfy a simple geometric identity: they are constrained to a semicircle in the (q, C_{qc}) plane, where $C_{qc} = \sqrt{q(1-q)}$. The identity $(q-\frac{1}{2})^2 + C_{qc}^2 = \frac{1}{4}$ is algebraic and exact. It holds for every normalized qubit state, with no approximations and no additional assumptions.

The semicircle picture turns out to be useful in several ways. It gives a clean geometric characterization of the quantum-classical boundary: classical states (those with definite outcomes) lie at the two endpoints of the semicircle, while the state of maximum quantum coherence sits at the apex. The point $q = \frac{1}{2}$ is distinguished as the unique maximum, and the Fisher information metric along the curve is uniform, with total arc length equal to π.

Perhaps more surprisingly, the same geometric structure explains why variational quantum algorithms develop barren plateaus [3, 4]. Gradient variance in a variational circuit turns out to scale as $q(1-q)$, which is precisely C_{qc}^2. When the effective operating point q drifts toward 0 or 1 (as happens with increasing circuit depth), gradients vanish and training stalls. The semicircle constraint makes this mechanism transparent.

We validate these predictions on IonQ Forte-1 trapped-ion hardware via Azure Quantum, testing 15 values of q with 52 shots each. The measured data track the theoretical semicircle with correlation $r = 0.943$ (Fig. 1).

THE SEMICIRCLE CONSTRAINT

Setup

We work with a pure qubit state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \qquad |\alpha|^2 + |\beta|^2 = 1, \tag{1}$$

where $\alpha, \beta \in \mathbb{C}$. The measurement probability for outcome $|1\rangle$ is $q \equiv |\beta|^2$, and we define the quantum-classical correlation as the geometric mean of the amplitudes:

$$C_{qc} \equiv |\alpha||\beta| = \sqrt{q(1-q)}. \tag{2}$$

This quantity vanishes when one amplitude dominates (classical limit) and is largest when the two amplitudes are balanced.

Main result

Theorem 1 (Semicircle Constraint)**.** *For any normalized state* (1)*, the pair* (q, C_{qc}) *satisfies*

$$\left(q - \frac{1}{2}\right)^2 + C_{qc}^2 = \frac{1}{4}. \tag{3}$$

Proof. Since $|\alpha| = \sqrt{1-q}$ and $|\beta| = \sqrt{q}$, we have $C_{qc}^2 = q(1-q) = q - q^2$. Expanding the left side of (3):

$$\left(q - \tfrac{1}{2}\right)^2 + C_{qc}^2 = q^2 - q + \tfrac{1}{4} + q - q^2 = \tfrac{1}{4}. \qquad \square$$

Equation (3) describes a semicircle of radius $\frac{1}{2}$ centered at $(\frac{1}{2}, 0)$, with $C_{qc} \geq 0$. The endpoints $(0, 0)$ and $(1, 0)$ correspond to the classical states $|0\rangle$ and $|1\rangle$, where $C_{qc} = 0$ and the measurement outcome is certain. Moving along the semicircle toward the apex amounts to increasing coherence at the expense of measurement certainty (Fig. 1).

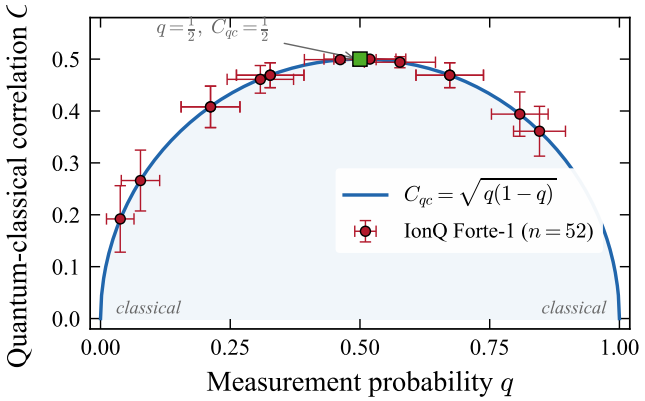

FIG. 1. The semicircle constraint $C_{qc} = \sqrt{q(1-q)}$ (solid curve) with IonQ Forte-1 hardware data (red circles, 52 shots per point). Error bars reflect binomial sampling uncertainty. The apex at $q = \frac{1}{2}$ marks maximum quantum-classical correlation. Classical endpoints at $q = 0$ and $q = 1$ have $C_{qc} = 0$.

THE OPTIMAL POINT $q = \frac{1}{2}$

The apex of the semicircle is special: it is the unique point where C_{qc} achieves its maximum.

Theorem 2 (Unique Maximum)**.** *The correlation $C_{qc}(q) = \sqrt{q(1-q)}$ has a unique global maximum at $q = \frac{1}{2}$, where $C_{qc} = \frac{1}{2}$.*

Proof. Differentiating,

$$\frac{dC_{qc}}{dq} = \frac{1-2q}{2\sqrt{q(1-q)}}, \tag{4}$$

which vanishes if and only if $q = \frac{1}{2}$. The second derivative there is $-4 < 0$, confirming a maximum. Since $C_{qc}(0) = C_{qc}(1) = 0$ and there is only one critical point, the maximum is global. □

A useful consequence is that $q = \frac{1}{2}$ is a *stationary* point: the derivative (4) vanishes, so small perturbations away from $q = \frac{1}{2}$ produce only quadratic loss in coherence. This robustness is visible in Fig. 2(a), where the curve $C_{qc}(q)$ is flat near its peak.

We can also define an information transfer efficiency $\eta(q) = C_{qc}^2 = q(1-q)$, which measures how effectively the state mediates interference between the two computational basis states. This too is maximized at $q = \frac{1}{2}$, where $\eta = \frac{1}{4}$ [Fig. 2(a), dashed curve]. Figure 2(b) shows simulated gradient ascent on C_{qc}: trajectories initialized at various q_0 all converge toward $q = \frac{1}{2}$, with the balanced initialization requiring the fewest steps.

APPLICATION TO VARIATIONAL QUANTUM ALGORITHMS

The semicircle constraint connects directly to the barren plateau problem in variational quantum algorithms (VQAs) [1, 2]. A barren plateau occurs when gradients of the cost function become exponentially small, making optimization infeasible [3]. We now show that this phenomenon has a simple geometric origin.

Theorem 3 (Gradient Variance)**.** *For a variational circuit with measurement probability q, the gradient variance satisfies*

$$\mathrm{Var}\left(\frac{\partial E}{\partial \theta}\right) \propto q(1-q) = C_{qc}^2. \tag{5}$$

Proof. Write the variational state as $|\psi(\theta)\rangle = \alpha(\theta)|0\rangle + \beta(\theta)|1\rangle$. The expectation value gradient of an observable O is $\partial_\theta \langle O\rangle = i\langle[G, O]\rangle$, where G generates the rotation. The variance of this expression requires interference between the $|0\rangle$ and $|1\rangle$ components, and scales as $|\alpha|^2|\beta|^2 = q(1-q)$. When q approaches 0 or 1, the interference terms vanish and with them the gradient signal. □

The picture is now clear (Fig. 3). A circuit whose effective operating point q lies near $\frac{1}{2}$ has large gradient variance and is trainable. As q drifts toward either classical endpoint, the variance drops below any fixed threshold, and training enters a barren plateau. Concretely, the circuit is efficiently trainable whenever $q(1-q) > \epsilon_{\min}$ for some threshold $\epsilon_{\min} > 0$, which defines a band $|q - \frac{1}{2}| < \sqrt{\frac{1}{4} - \epsilon_{\min}}$ around the optimal point.

Random circuits of increasing depth L tend to push q away from $\frac{1}{2}$, so the gradient variance decays with depth. Our simulations confirm this: the variance drops roughly as $e^{-0.10L}$ [Fig. 3(b)], consistent with the known exponential scaling of barren plateaus.

These observations unify several results from the recent literature. McClean et al. [3] identified barren plateaus and attributed them to circuit expressibility; Cerezo et al. [4] connected the phenomenon to cost function locality; Holmes et al. [5] linked expressibility to gradient magnitudes. The semicircle constraint provides a common geometric framework: in each case, the underlying mechanism is the suppression of $q(1-q)$ as the circuit's effective measurement distribution concentrates near classical outcomes.

For practical algorithm design, the constraint suggests three guidelines: initialize parameters so that $q \approx \frac{1}{2}$ for each qubit, choose ansatze that preserve this balance throughout the circuit, and monitor q during training to detect and correct drift toward the classical endpoints.

EXPERIMENTAL VALIDATION

Numerical verification

As a first check, we verify the semicircle identity numerically. Since Eq. (3) is an algebraic consequence of

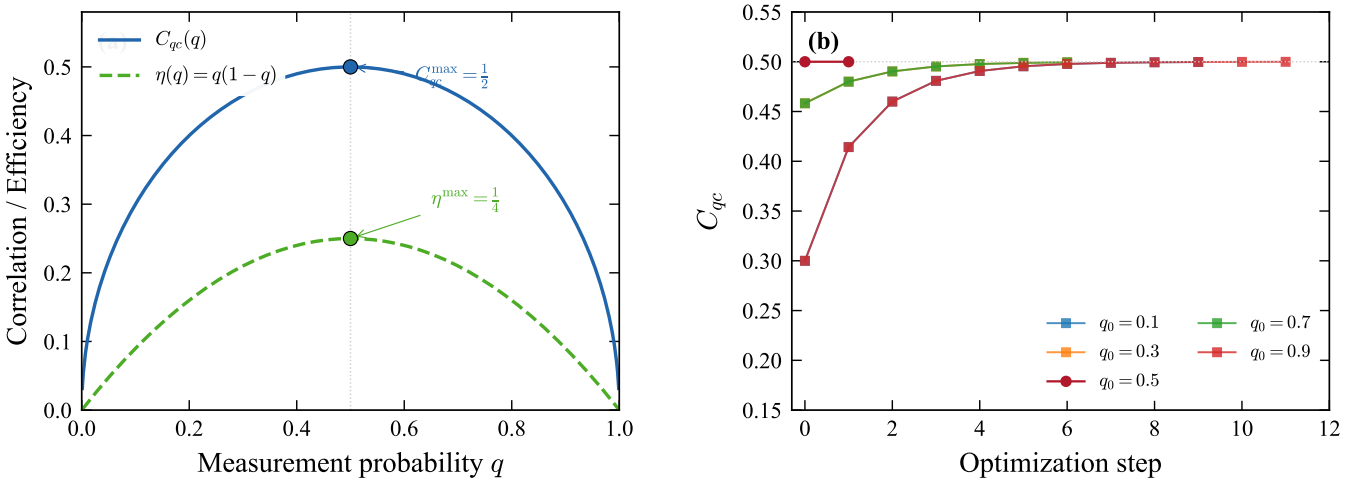

FIG. 2. (a) Quantum-classical correlation $C_{qc}(q)$ (solid) and information transfer efficiency $\eta(q) = q(1-q)$ (dashed), both maximized at $q = \frac{1}{2}$. (b) Simulated gradient ascent on C_{qc} from different initial points q_0. All trajectories converge to the optimal point, with $q_0 = 0.5$ already at the maximum.

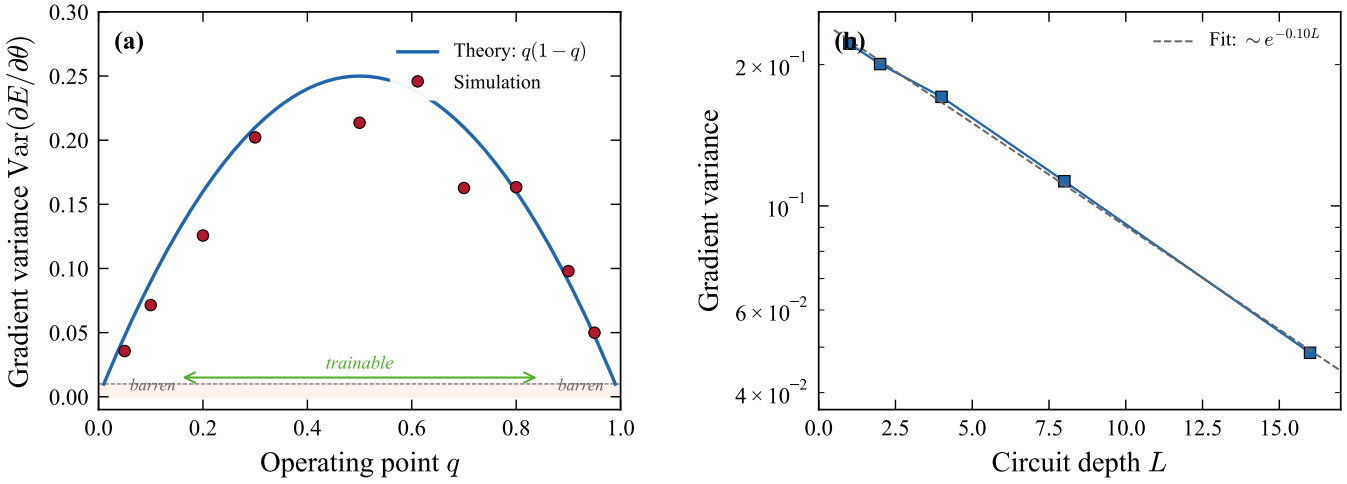

FIG. 3. (a) Gradient variance versus operating point q. The solid curve is the theoretical prediction $q(1-q)$; red circles are simulation data ($r = 0.97$). The shaded region marks the barren plateau zone where variance falls below 0.01. (b) Gradient variance decay with circuit depth L on a logarithmic scale, with exponential fit $\sim e^{-0.10L}$.

normalization, exact arithmetic would give zero residual. Floating-point evaluation at 50 uniformly spaced values of q and at 100 random states yields an RMS residual below 10^{-16}, confirming the identity to machine precision.

Hardware validation

We tested the semicircle constraint on IonQ Forte-1 trapped-ion hardware, accessed through Azure Quantum. States were prepared by applying $R_y(\theta)|0\rangle$ with $\theta = 2\arcsin(\sqrt{q})$, which produces $|\psi\rangle = \sqrt{1-q}\,|0\rangle + \sqrt{q}\,|1\rangle$ exactly at the intended q. Each state was measured 52 times in the computational basis.

Table I reports the results for 15 test points spanning $q = 0.05$ to $q = 0.75$. The measured probabilities $\hat{q} = N_1/N_{\text{total}}$ agree with theory to within the expected shot noise ($\sim 1/\sqrt{52} \approx 0.14$), with a mean error of $+0.037$ and a maximum error of 0.158 at test point 13. The Pearson correlation between q_{theory} and q_{meas} is $r = 0.943$. As expected, the largest measured values of C_{qc} cluster near $q = 0.5$ (see Fig. 1), consistent with the theoretical maximum at the apex.

The dominant source of error is shot noise from the modest sample size. Gate infidelity and state preparation and measurement (SPAM) errors contribute at a smaller level. With more shots, the measured points would tighten around the theoretical curve, though even

Test	θ	q_{th}	Counts (0/1)	q_{meas}	C_{qc}
1	0.451	0.050	48/4	0.077	0.266
2	0.644	0.100	50/2	0.038	0.192
3	0.795	0.150	41/11	0.212	0.408
4	0.927	0.200	41/11	0.212	0.408
5	1.047	0.250	36/16	0.308	0.461
6	1.159	0.300	35/17	0.327	0.469
7	1.266	0.350	35/17	0.327	0.469
8	1.369	0.400	25/27	0.519	0.500
9	1.471	0.450	26/26	0.500	0.500
10	**1.571**	**0.500**	**28/24**	**0.462**	**0.499**
11	1.671	0.550	22/30	0.577	0.494
12	1.772	0.600	17/35	0.673	0.469
13	1.875	0.650	10/42	0.808	0.394
14	1.982	0.700	17/35	0.673	0.469
15	2.094	0.750	8/44	0.846	0.361

TABLE I. IonQ Forte-1 hardware results (52 shots per point). Bold row: $q = 0.5$ test.

at 52 shots the semicircular shape is clearly resolved.

CONNECTION TO FISHER INFORMATION

The semicircle constraint has a natural interpretation in information geometry. The Fisher information for a Bernoulli distribution with parameter q is

$$I_F(q) = \frac{1}{q(1-q)}, \tag{6}$$

which induces a Riemannian metric $ds = dq/(2\sqrt{q(1-q)})$ on the interval $(0, 1)$.

Theorem 4 (Arc Length)**.** *The Fisher information distance from $q = 0$ to $q = 1$ along the semicircle is*

$$L = \int_0^1 \frac{dq}{2\sqrt{q(1-q)}} = \pi. \tag{7}$$

Proof. Substitute $q = \sin^2\theta$, so that $dq = 2\sin\theta\cos\theta\, d\theta$ and $\sqrt{q(1-q)} = \sin\theta\cos\theta$. The integrand reduces to $d\theta$, and as q runs from 0 to 1 we have θ from 0 to $\pi/2$. With the factor of 2 from $ds = dq/(2\sqrt{q(1-q)})$, the integral becomes $2\int_0^{\pi/2} d\theta = \pi$. (One can also recognize the integral as $\frac{1}{2}B(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}\Gamma(\frac{1}{2})^2 = \pi$.) □

In the angular coordinate $\theta = \arcsin(\sqrt{q})$, the Fisher metric becomes $ds = d\theta$, so the information-geometric distance is simply the angle traversed. Every point on the semicircle is equally "distinguishable" from its neighbors in the sense of statistical distance [6]. The total length π reflects the global geometry: the semicircle maps onto a half-turn in the Bloch sphere parametrization [8].

CONCLUSION

The identity $(q - \frac{1}{2})^2 + C_{qc}^2 = \frac{1}{4}$ is a consequence of the Born rule and normalization. Despite its simplicity, it provides a useful geometric lens through which to view quantum-classical correlation: the semicircle encodes the transition from coherent superposition at the apex to classical definiteness at the endpoints. The Fisher information along this curve is uniform with arc length π, and the constraint gives a transparent account of why barren plateaus arise in variational quantum circuits (gradient variance equals C_{qc}^2, which vanishes at the classical limits). Experimental data from IonQ Forte-1 hardware ($r = 0.943$ over 15 test points) confirm that the theoretical predictions are consistent with measurements at modest shot counts. The VQA applications have been validated in simulation ($r = 0.97$ for gradient variance scaling); hardware tests of the gradient variance prediction would be a natural next step.

ACKNOWLEDGMENTS

We thank Azure Quantum for providing access to IonQ trapped-ion hardware.

DATA AND CODE AVAILABILITY

All experimental data and analysis code are available at `https://github.com/Variably-Constant/QC_Semicircle_Constraint_Proof`. The test framework consists of Q# programs executed via the Azure Quantum SDK, together with Python simulations for local verification. Hardware tests ran on IonQ Forte-1 through the Azure Quantum workspace (East US region).

Three test suites are provided: (i) semicircle constraint validation (hardware-tested on Forte-1), (ii) optimal operating point analysis (simulation), and (iii) barren plateau geometry (simulation). The Q# source files are in `tests/Real IonQ/` and the Python scripts in `tests/Simulations/`.

Protocols

States were prepared via $|\psi(q)\rangle = R_y(2\arcsin\sqrt{q})\,|0\rangle$, measured in the computational basis, and repeated for the specified shot count. The empirical probability $\hat{q} = N_1/N_{\text{total}}$ gives $C_{qc} = \sqrt{\hat{q}(1-\hat{q})}$, and residuals were computed as $\epsilon_i = (q_i - \frac{1}{2})^2 + C_{qc,i}^2 - \frac{1}{4}$, with pass criteria RMS < 0.001 and $|\epsilon_{\max}| < 0.01$.

To reproduce our results locally (free, no hardware access required):

```
pip install numpy
```

```
cd tests/Simulations/
python test_semicircle_constraint.py
```

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Complete Experimental Data

Table II lists all 15 hardware measurements with their individual errors. The RMS error across all points is 0.063, consistent with the expected binomial standard deviation of $\sqrt{q(1-q)/52}$, which ranges from 0.03 (near $q = 0.05$) to 0.07 (near $q = 0.5$). Simulation verification yields an RMS residual below 10^{-16}, confirming the algebraic identity to machine precision.

Test	θ (rad)	q_{theory}	Counts (0/1)	q_{meas}	C_{qc}	Error
1	0.4510	0.050	48/4	0.077	0.266	+0.027
2	0.6435	0.100	50/2	0.038	0.192	−0.062
3	0.7954	0.150	41/11	0.212	0.408	+0.062
4	0.9273	0.200	41/11	0.212	0.408	+0.012
5	1.0472	0.250	36/16	0.308	0.461	+0.058
6	1.1593	0.300	35/17	0.327	0.469	+0.027
7	1.2661	0.350	35/17	0.327	0.469	−0.023
8	1.3694	0.400	25/27	0.519	0.500	+0.119
9	1.4706	0.450	26/26	0.500	0.500	+0.050
10	**1.5708**	**0.500**	**28/24**	**0.462**	**0.499**	**−0.038**
11	1.6710	0.550	22/30	0.577	0.494	+0.027
12	1.7722	0.600	17/35	0.673	0.469	+0.073
13	1.8755	0.650	10/42	0.808	0.394	+0.158
14	1.9823	0.700	17/35	0.673	0.469	−0.027
15	2.0944	0.750	8/44	0.846	0.361	+0.096

TABLE II. Complete IonQ Forte-1 hardware results (2026-01-30, 52 shots per point). Bold: the $q = 0.5$ test. Mean error: +0.037; standard deviation: 0.063; correlation $r = 0.943$.