Semester project TMA4215

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1 Task

We consider minimization problems of the type

$$\min_{\mathbf{x}\in\mathbb{R}^n} g(\mathbf{x}),\ g(\mathbf{x}) := -\mathbf{b}^T\mathbf{x} + \frac{1}{2}\mathbf{x}^T H\mathbf{x} + \frac{1}{12}\mathbf{x}^T C(\mathbf{x})\mathbf{x},$$

here $\mathbf{b} \in \mathbb{R}^n$ and, H is a $n \times n$ symmetric and positive definite matrix and $C(\mathbf{x})$ is a diagonal matrix with diagonal entries $c_i x_i^2, i = 1, ..., n$. Here $c_i > 0$ are the components of a vector $\mathbf{c} \in \mathbb{R}^{\mathbf{n}}$ and x_i are the components of $\mathbf{x}$.

2 Generation of the data

The data is generated in the function **data**. $\mathbf{b}$ and $\mathbf{c}$ should be of the same dimension and H should be a symmetric matrix which fits to the vectors. Line 6 can be used to check the dimensions of the input data, but this costs a lot of resources, because **data** is often called.

```
1 function [b, H, c] = data
2         b = [1; 0];
3         c = [1; 1];
4         H = [1, 1;
5              1, 1];
6         % b' * H * c;
7 end
```

3 Function, gradient and Hessian of g

```
function [ g ] = ( X )
        [ b, H, c ] = data;
        dim = size(H,1);
        C = zeros ( dim, dim );
        for i = 1 : dim
                C(i,i) = c(i) * X(i) * X(i);
        end
        g = - b' * X + 0.5 * X' * H * X + 1/12 * X' * C * X;
end
```

$$\nabla g = -\mathbf{b} + H\mathbf{x} + \frac{1}{3}C\mathbf{x} \tag{1}$$

```
function [ nablaG ] = grad( X )
        [ b, H, c ] = data;
        dim = size ( H, 1 );
        for i = 1 : dim
            C(i,i) = c(i) * X(i) * X(i);
        end
        nablaG = - b + H * X + 1/3 * C * X;
end
```

$$\nabla^2 g = H + C \tag{2}$$

```
function [ HessG ] = hessian( X )
        [ ~, H, c ] = data;
        dim = size ( H, 1 );
        C = zeros ( dim, dim );
        for i = 1 : dim
                C(i,i) = c(i) * X(i) * X(i);
        end
        HessG = H + C;
end
```

4 Existence of minimum

Let $u \in R^n$ be an arbitrary vector, except $\vec{0}$ then

$$u(\nabla^2 g(x))u = u(H + C)u = uHu + uCu = uHu + \sum_{i=1}^{n} c_i u_i^2 x_i^2.$$

Since H is positive definite, $uHu > 0$ and because $c_i > 0$, $uCu > 0$, so

$$u(\nabla^2 g(x))u > 0$$

That means that the Hessian of g is positive definite.

5 Plot for $n = 2$

6 Steepest decent method

```
function [ iterations, X, Y, Z ] = steepest( Xk, alpha )
        tol = 1e-6;
        maxiterations = 1000;
        [~,H,~] = data;
        dim = size(H, 1);
        X1 = zeros(1, maxiterations);
        X2 = zeros(1, maxiterations);
        Y = zeros(1, maxiterations);
        Z = zeros(1, maxiterations);
        X1(1) = Xk(1);
        X2(1) = Xk(2);
        condition = 1;
        norm_old = norm ( grad ( Xk) );
        Y(1) = norm_old;
        Z(1) = problem(Xk);
        while condition
                maxiterations = maxiterations - 1;
                Xk = Xk - alpha * grad ( Xk );
                if dim == 2
                                X1(1000-maxiterations+1) = Xk(1);
                                X2(1000-maxiterations+1) = Xk(2);
                end
                residual = norm ( grad ( Xk ) ) / norm_old;
                Y(1000-maxiterations+1) = residual;
                Z(1000-maxiterations+1) = problem(Xk);
                condition = (maxiterations > 0) && ( residual > tol);
        end
        X1 = X1(1:1000-maxiterations+1);
        X2 = X2(1:1000-maxiterations+1);
```

```
        X = [X1;X2];
        iterations = (1:1:1000-maxiterations+1);
        Z = Z(1:1000-maxiterations+1);
        Y = Y(1:1000-maxiterations+1);
end
```

7 Equivalence of steepest decent method and forward Euler method

No idea.

8 Improved steepest decent method

To find the optimal α,

$$g\left(\mathbf{x}^{(k+1)}\right) = g\left(\mathbf{x}^{(k)} - \alpha^{(k)}\nabla g(\mathbf{x}^{(k+1)})\right)$$

has to be minimal, so $\frac{\partial}{\partial\alpha} g(\mathbf{x}^{(k+1)})$ has to be zero. This leads to the equation

$$\begin{aligned}\left(\mathbf{b}\nabla g - \mathbf{x}H\mathbf{x} - \frac{1}{3}\mathbf{x}H\nabla g\right) + ((\nabla g)(H+C)\nabla g)\,\alpha \\ + \left(-\sum_{i=1}^{n} c_i x_i (\nabla g)_i^3\right)\alpha^2 + \left(\sum_{i=1}^{n} c_i (\nabla g)_i^4\right)\alpha^3 = 0\end{aligned}$$

9 Newton method

10 Combination of steepest decent methode and Newton method