Assignment 13

Varshini Jonnala - CS21BTECH11024

June 16, 2022

Question

Let $P_n(z)$ represent the Levinson polynomial of the first kind such that:

(a) If one of the roots of $P_n(z)$ lie on the unit circle, then show that all other roots of $P_n(z)$ are simple and lie on the unit circle.

(b) If the reflection coefficient $s_k \neq 0$, then show that $P_k(z)$ and $P_{k+1}(z)$ have no roots in common.

Solution (a) - i

(a) Let $z = e^{j\theta_1}$ represent one of the roots of the Levinson Polynomial $P_n(z)$ that lie on the unit circle. In that case,

$$P_n(e^{j\theta_1}) = 0 \tag{1}$$

and substituting this into the recursion equation, we get

$$|s_n| = \left| \frac{P_{n-1}(e^{j\theta_1})}{\tilde{P}_{n-1}(e^{j\theta_1})} \right| \tag{2}$$

$$= 1 \tag{3}$$

$$\text{so that } s_n = e^{j\alpha} \tag{4}$$

Let

$$P_{n-1}(e^{j\theta}) = R(\theta)e^{j\chi(\theta)} \tag{5}$$

Solution (a) - ii

and since $P_{n-1}(z)$ is free of zeros in $|z| \leq 1$, we have $R(\theta) > 0, 0 < \theta < 2\pi$, and once again substituting these into recursion equation, we obtain

$$\sqrt{1 - s_n^2}P_n(e^{j\theta}) = R(\theta)e^{j\chi(\theta)} - e^{j(\theta+\alpha)}e^{j(n-1)\theta}R(\theta)e^{-j\chi(\theta)} \tag{6}$$

$$= R(\theta)\left[e^{j\chi(\theta)} - e^{j(n\theta+\alpha)}e^{-j\chi(\theta)}\right] \tag{7}$$

$$= 2jR(\theta)e^{j(n\theta+\alpha)/2} sin\left(\chi(\theta) - \frac{n\theta}{2} - \frac{\alpha}{2}\right) \tag{8}$$

Solution (a) - iii

We observe that,

(i) As θ varies from 0 to 2π, there is no net increment in the phase term $\chi(\theta)$, and the entire argument of the sine term above increases by $n\pi$.

(ii) Consequently, $P_n(e^{j\theta})$ equals 0 atleast at n distinct points $\theta_1, \theta_2, \ldots, \theta_n, 0 < \theta_i < 2\pi$.

(iii) However, $P_n(z)$ is a polynomial of degree n in z and can have atmost n zeros. Thus, all the above zeroes are simple and they all lie on the unit circle.

Solution - b

(b) Suppose $P_n(z)$ and $P_{n-1}(z)$ has a common root at $z = z_0$. Then, $|z_0| > 1$ and from the recursion equation, we get

$$z_0 s_n \tilde{P}_{n-1}(z_0) = 0 \tag{9}$$

which gives $s_n = 0$, since $\tilde{P}_{n-1}(z_0) \neq 0$ as it has all its zeroes in $|z| < 1$. Thus, $s_n \neq 0$ implies $P_n(z)$ and $P_{n-1}(z)$ do not have any common roots. Hence, proved.