Assignment 9

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Question

- The random variables X_i are i.i.d with density $ce^{-cx}U(x)$. Show that, if $x = x_1 + \cdots + x_n$, then $f_x(x)$ is an Erlang density.

Solution I

Since, we know that,

$$\text{When } f(x) = \gamma x^{b-1} e^{-cx} U(x) \text{ and } \gamma = \frac{c^{b+1}}{\Gamma(b+1)} \tag{1}$$

$$\text{It follows that } \phi(s) = \gamma \int_0^\infty x^{b-1} e^{-(c-s)x} dx = \frac{\gamma \Gamma(b)}{(c-s)^b} = \frac{c^b}{(c-s)^b} \tag{2}$$

Differentiating with respect to s and setting $s = 0$, we obtain

$$\phi^n(0) = \frac{b(b+1)\dots(b+n-1)}{c^n} = E\left(x^n\right) \tag{3}$$

Solution II

With $n = 1$ and $n = 2$, this yields

$$E\left(x\right) = \frac{b}{c} \tag{4}$$

$$E\left(x^2\right) = \frac{b(b+1)}{c^2} \tag{5}$$

$$\sigma^2 = \frac{b}{c^2} \tag{6}$$

The exponential density is a special case obtained with $b = 1, C = \lambda$:

$$f(x) = \lambda e^{-\lambda x} U(x) \tag{7}$$

$$\phi(s) = \frac{\lambda}{\lambda - s} \tag{8}$$

where $E\left(x\right) = \frac{1}{\lambda}$ and $\lambda^2 = \frac{1}{\sigma^2}$

Solution III

Now, It is given that,

$$f_1(x) = ce^{-cx}U(x) \tag{9}$$

$$\text{then } \phi_1(s) = \frac{c}{c-s} \tag{10}$$

$$\phi(s) = \phi_1(s) + \phi_2(s) + \cdots + \phi_n(s) = \frac{c^n}{(c-s)^n} \tag{11}$$

$$\text{Hence, } f(x) = \frac{c^n x^{m-1}}{(n-1)!}e^{-cx}U(x) \tag{12}$$