07/09/23

MEASURE THEORY AND INTEGRATION

* Definition: (X,d) be a metric space $A \subseteq X$. Let $A' = \{$set of all limit points of $A\}$

eg: $(0,1)' = [0,1]$

$\{\frac{1}{n} / n \geq 1, n \in \mathbb{Z}\}' = \{0\}$.

* Defn: Let $A \subseteq X$. The closure of A is defined as the intersection of all closed sets of X containing A. That is, closure of A is the smallest closed set containing A.

That is $\bar{A} = \bigcap\limits_{V \supseteq A} V$

V → closed set in X.

eg: $\overline{(0,1)} = [0,1]$,

* Result: $\bar{A} = A \cup A'$

* Defn: A subset $A \subseteq X$ is called a dense set in X if $\bar{A} = X$

eg: 1) $\mathbb{Q}$ is dense in $\mathbb{R}$

2) $(0,1)$ is dense in $[0,1]$.

* Defn: A subset $A \subseteq X$ of X is called G_δ-set if A can be written as countable intersection of open sets.

This is $A = \bigcap\limits_{i=1}^{\infty} U_i$, where $\{U_i\}_{i \geq 1}$ is family of open sets.

→ Every open set is a G_δ-set. If $U \subseteq X$ is open set then

$U = \bigcap\limits_{i=1}^{\infty} U_i$ where $U_i = U \ \forall i$

→ Every G_δ-set need not be an open set

$[a,b] = \bigcap\limits_{n=1}^{\infty} (a-\frac{1}{n}, b+\frac{1}{n})$.

Defn: A subset $A \subset X$ is called an F_σ set if A can be written as countable union of closed sets. That is

$$A = \bigcup_{i=1}^{\infty} C_i, \text{ for some } \{C_i\}_{i\geq 1}^{\infty} \text{ is a family of closed sets.}$$

1) Any closed set is an F_σ set.

2) Any open interval is an F_σ-set.

Eg: $(a,b) = \bigcup_{n=1}^{\infty} \left[a+\frac{1}{n}, b-\frac{1}{n}\right]$

3) $(a,b) \cup (c,d)$ is an F_σ set.

Defn: Let $A \subseteq X$, then

1) the supremum or the least upper bound of A is defined as

$\sup(A)$ = the smallest upper bound for A

→ An element $b \in \mathbb{R}$ is called an upper bound of $A \subseteq X$, if $x \leq b$ $\forall x \in A$.

→ An element $l \in \mathbb{R}$ is called a supremum of $A \subseteq X$ if given $\varepsilon > 0$, there exist $x \in A$ such that $x \leq l$, & $x > l - \varepsilon$.

→ Recall that a number $l \in \mathbb{R}$ is called a lower bound for $A \subset \mathbb{R}$ if $x \geq l$ $\forall$ $x \in A$.

Defn: A number $l \in \mathbb{R}$ is called infimum or the greatest lower bound of $A \subseteq \mathbb{R}$, if given $\varepsilon > 0$, there exist $x \in A$ such that $l + \varepsilon > x$.

Eg: 1) $\sup((0,1)) = 1$

$\inf((0,1)) = 0$.

2) $\sup([2,5]) = 5$.

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* Reference Books:

1) Measure theory and Integration by Gde Barra.

2) Real Analysis: Measure Theory, Integration and Hilbert space. by Elias M. Stein and R. Shakarchi.

* Motivations

→ length of an interval $[a,b]$ is $b-a$.

→ Also $[a,b] \cup [c,d]$ ⇒ disjoint union, then the length is equal to $(b-a)+(d-c)$.

i.e, length of disjoint intervals is the sum of the lengths of the intervals (additive property)

→ $I \subset \mathbb{R}$ interval, $a \in \mathbb{R}$,

$$a+I = \{a+x \mid x \in I\} \subseteq \mathbb{R}.$$

Then the length of I and $a+I$ are equal. ⇒ The name of this property is Translation invariant.

→ "length of I" ⇝ "?" ← generalise to any subset $E \subset \mathbb{R}$ such that → It is a measure

i) Restrict this to intervals, that coincides with length

ii) Additive

iii) Translation invariant.

→ let $X \neq \phi$ be a set. let $A, B \subseteq X$. Define

$$A \setminus B = \{x \in A \mid x \notin B\} = A \cap B^c$$

$$A \Delta B = (A \setminus B) \cup (B \setminus A) \Rightarrow \text{symmetric difference}$$

$$= (A \cap B^c) \cup (B \cap A^c) = (A \cup B) \setminus (A \cap B)$$

* Properties: For $A, B, C, D \subseteq X$,

1) $A \Delta B = B \Delta A$

2) $(A \Delta B) \Delta C = A \Delta (B \Delta C)$

3) $(A \Delta B) \Delta (C \Delta D) = (A \Delta C) \Delta (B \Delta D)$

4) $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$

<u>Proof</u> 1) $A \Delta B = (A \setminus B) \cup (B \setminus A) = (B \setminus A) \cup (A \setminus B) = B \Delta A$.

2) consider $(A \Delta B)^c = ((A \setminus B) \cup (B \setminus A))^c = ((A \cap B^c) \cup (B \cap A^c))^c$

$$= (A \cap B^c)^c \cap (B \cap A^c)^c = (A^c \cup B) \cap (B^c \cup A)$$

$$= (A^c \cap (B^c \cup A)) \cup (B \cap (B^c \cup A))$$

$$= ((A^c \cap B^c) \cup \underbrace{(A^c \cap A)}_{=\emptyset}) \cup (\underbrace{(B \cap B^c)}_{=\emptyset} \cup (B \cap A))$$

$$= (A^c \cap B^c) \cup (B \cap A)$$

$$\boxed{\therefore (A \Delta B)^c = (A \cap B) \cup (A^c \cap B^c)}$$

→ Now, $(A \Delta B) \Delta C = ((A \Delta B) \setminus C) \cup (C \setminus (A \Delta B))$

$$= ((A \Delta B) \cap C^c) \cup (C \cap (A \Delta B)^c)$$

$$= (((A \cap B^c) \cup (B \cap A^c)) \cap C^c) \cup (C \cap ((A^c \cap B^c) \cup (A \cap B)))$$

$$= (A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C) \cup (A \cap B \cap C)$$

→ By symmetry, this is also equal to

$$A \Delta (B \Delta C) = (\underbrace{B}_{A'} \Delta \underbrace{C}_{B'}) \Delta \underbrace{A}_{C'} \quad \text{(Assume)}$$

Then by above formula, applied to $(A' \Delta B') \Delta C'$, we get.

$$(A \Delta B') \Delta C' = (A' \cap (B')^c \cap (C')^c) \cup ((A')^c \cap B' \cap (C')^c)$$

$$\cup ((A')^c \cap (B')^c \cap C') \cup (A' \cap B' \cap C')$$

$$= (B \cap C^c \cap A^c) \cup (B^c \cap C \cap A^c) \cup (B^c \cap C^c \cap A) \cup (B \cap C \cap A)$$

$$= A \Delta (B \Delta C)$$

3) $(A\Delta B)\Delta(C\Delta D) = ((A\Delta B)\Delta C)\Delta D$ (by ②)

$= (A\Delta(B\Delta C))\Delta D$

$= ((B\Delta C)\Delta A)\Delta D$ (by ①)

$= (B\Delta C)\Delta(A\Delta D)$ (by ②).

$= \cancel{(A\Delta D)\Delta(B\Delta C)}$ $B\Delta(C\Delta A)\Delta D$

$= (B\Delta C\Delta A)\Delta D$

$= ((C\Delta A)\Delta B)\Delta D$ (By ①)

$= (C\Delta A)\Delta(B\Delta D)$

$= (A\Delta C)\Delta(B\Delta D) = RHS.$

4) $A\cap(B\Delta C) = A\cap((B\cap C^c)\cup(C\cap B^c)) = (A\cap B\cap C^c)\cup(A\cap(C\cap B^c))$

$= (A\cap B\cap C^c)\cup(A\cap C\cap B^c) \longrightarrow (*)$.

Now, $(A\cap B)\Delta(A\cap C) = ((A\cap B)\cap(A\cap C)^c)\cup((A\cap C)\cup(A\cap B)^c)$

$= ((A\cap B)\cap(A^c\cup C^c))\cup((A\cap C)\cup(A^c\cup B^c))$

$= (\underbrace{(A\cap B)\cap A^c}_{=\phi})\cup(A\cap B\cap C^c)\cup(A\cap C\cap B^c)\cup(\underbrace{A\cap C\cap A^c}_{=\phi})$

$= (A\cap B\cap C^c)\cup(A\cap C\cap B^c)$

$= A\cap(B\Delta C)$ (from (*)).

$= LHS.$

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* Remark:

→ Let $E_1 \supseteq E_2 \supseteq \dots$. Then,

$\bigcup_{n=1}^{\infty}(E_1\setminus E_n) = E_1\setminus\left(\bigcap_{n=1}^{\infty} E_n\right)$ ($\because$ DeMorgen's Law).

Proof: Exercise.

Equivalence Relation:

→ suppose a set $X \neq \phi$. so an equivalence relation R on X is a subset of $X \times X$ with the following conditions: $(``\sim")$

i) (Reflexive) ⇒ $(x,x) \in R \Rightarrow (x \sim x)$

ii) (symmetric) → if $(x,y) \in R$, then $(y,x) \in R$ $\forall\ x,y \in X$

(if $x \sim y$, then $y \sim x$, $\forall\ x,y \in X$).

iii) (transitive) → if $(x,y) \in R$, $(y,z) \in R$ then $(x,z) \in R$ $\forall\ x,y,z \in X$.

(if $x \sim y$, $y \sim z$, then $x \sim z$ $\forall\ x,y,z \in X$).

→ suppose $\sim$ is an equivalence relation on a non-empty set X then we have "equivalence classes";

$[x] = \{y \in X / x \sim y\}$ is called an "equivalence class of x."

→ Any two equivalence classes are equal or disjoint. i.e,

$[x] = [y]$ (or) $[x] \cap [y] = \phi$ $\forall\ x,y \in X$.

→ $X = \bigcup_{x \in X} [x]$

= disjoint union of certain equivalence classes.

Ex: let $X = \mathbb{Z}$ and $n \in \mathbb{N}$ fixed. Define $\sim$ as follows:

For $x,y \in \mathbb{Z}$, $x \sim y$ if $n / x-y$ i.e, $x-y = n.q$ for any $q \in \mathbb{Z}$

→ check that $\sim$ is an equivalence relation on $\mathbb{Z}$, known as "equivalence modulo n".

→ For $x \in \mathbb{Z}$, $[x] = \{y \in \mathbb{Z} / x \sim y\} = \{y \in \mathbb{Z} / n / x-y\}$

$= \{x + r.n / r \in \mathbb{Z}\}$

$= x + n\mathbb{Z}$; $n\mathbb{Z} = \{nr / r \in \mathbb{Z}\}$

→ $\mathbb{Z} = \bigcup_{x \in \mathbb{Z}} [x] = [0] \cup [1] \cup \cdots \cup [n-1]$ } This is a disjoint union

$x - y = n.q$ for some $q \in \mathbb{Z}$

$y = x - n.q$ for some q

$= x + r.n$ for some r

= finite disjoint union of equivalence classes.

$[0] = n\mathbb{Z}$
$[1] = n\mathbb{Z}+1$
$\vdots$
$[n-1] = n\mathbb{Z}+(n-1)$
$[n] = n\mathbb{Z} = [0]$
$\vdots$

* **Axiom of choice:**

→ suppose $\{E_\alpha\}_{\alpha \in I}$ is a non-empty collection of non-empty disjoint subsets of a set $X \neq \phi$.

Then there exist a set $V \subseteq X$ containing just one element from each set E_α $(\alpha \in I)$. (The set I can be finite, countable infinite or uncountable)

I: index set

$E_\alpha \cap V$ = singleton set $\forall\ \alpha \in I$.

* **Theorem (Heine-Borel theorem):**

→ suppose A is a closed and bounded subset of $\mathbb{R}$. Suppose $\{O_\alpha\}_{\alpha \in I}$ is an open cover of A. Then there exists a finite subcollection of $\{O_\alpha\}_{\alpha \in I}$ say $\{O_{\alpha_1}, O_{\alpha_2}, \ldots O_{\alpha_n}\}$ such that $A \subseteq \bigcup_{i=1}^{n} O_{\alpha_i}$

Recall: A collection of open sets $\{O_\alpha\}_{\alpha \in I}$ in $\mathbb{R}$, is called an open cover of a set $E \subseteq \mathbb{R}$, if $E \subseteq \bigcup_{\alpha \in I} O_\alpha$

* **Theorem (Lindelöf's theorem):**

→ suppose $\mathcal{Y} = \{I_\alpha / \alpha \in \Lambda\}$ (Λ: index set) is a collection of open intervals I_α in $\mathbb{R}$. Then $\exists$ a sub collection of $\mathcal{Y}$, atmost countable in numbers, say $\{I_1, I_2, \ldots\}$ such that

$$\bigcup_{\alpha \in \Lambda} I_\alpha = \bigcup_{n=1}^{\infty} I_n = \bigcup_{n \in \mathbb{N}} I_n$$

→ That is, any arbitrary union of open intervals in $\mathbb{R}$, can be written as countable union of open intervals.

* **Theorem:** Every non-empty open set G in $\mathbb{R}$ is the union of disjoint of open intervals at most countable in numbers.

Proof: let $G \subseteq \mathbb{R}$ be an open set. Define a relation $\sim$ in G as follows.

→ For $a,b \in G$, $a \sim b$ if the closed interval $[a,b]$ or $[b,a]$ lies in G.

(That is $a \sim b$, if $[a,b] \subseteq G$ or $[b,a] \subseteq G$
$\qquad\qquad\qquad\qquad\quad \Downarrow \qquad\qquad\qquad \Downarrow$
$\qquad\qquad\qquad\qquad\; a \leq b \qquad\qquad\quad a \geq b$

→ Check that $\sim$ is an equivalence relation: → $a \sim a$ if $a \in G$

→ suppose $a \sim b$ $(a,b \in G)$ $\qquad\qquad ([a,a] = \{a\} \subseteq G)$

$\Rightarrow [a,b] \subseteq G$ (or) $[b,a] \subseteq G$

$\Rightarrow$ ~~b~a~~ $b \sim a$

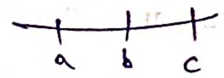
→ for $a,b,c \in G$, $a \sim b$, $b \sim c$

$\Rightarrow [a,b]$ or $[b,a] \subseteq G$, $[b,c]$ (or) $[c,b] \subseteq G$

To show $a \sim c$, i.e., $[a,c] \subseteq G$ (or) $[c,a] \subseteq G$

wlog, $a \leq b \leq c$. Then we have

$[a,b]$, $[b,c] \subseteq G$



Now, $[a,c] = [a,b] \cup [b,c] \subseteq G$

$\Rightarrow a \sim c$

$\therefore \sim$ is an equivalence relation on G.

$\therefore G$ is the union of disjoint equivalence classes.

→ Define $C(a)$ = the equivalence class of a in G

$= \{b \in G \mid b \sim a\}$

To show $C(a)$ is an interval.

suppose $C(a)$ is not an interval $\Rightarrow$ ~~there exists a point~~

~~$c \notin C(a)$ such that for any $b \in C(a)$, either~~

~~c lies between a and b and~~

~~$c \notin C(a)$~~.

→ for any $b \in C(a)$, then $\exists$ $c \notin G$ lies b/w a & b and $c \notin C(a)$.

→ Now $b \in C(a) \Rightarrow [a,b]$ (or) $[b,a] \subseteq G$.

$\Rightarrow$ $[a,c]$ (or) $[b,c] \subseteq G$.

$\Rightarrow c \in C(a)$

$\therefore C(a)$ is an interval.

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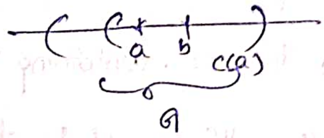
→ continuation of proof:

$C(a) \subseteq G$ → equivalence class

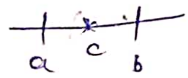


claim: $C(a)$ is an open interval

suppose $a \in C(a)$ is not an interval $\Rightarrow \exists$ a point $b \in C(a)$ and a point $c \notin C(a)$ such that c lies b/w a and b.

$\Rightarrow b \in C(a)$ which implies that

$[a,b]$ (or) $[b,a] \subseteq G$



$\Rightarrow [a,c]$ or $[c,b] \subseteq G$

$\Rightarrow a \sim c$

$\Rightarrow c \in C(a) \Rightarrow \Leftarrow$ contradiction

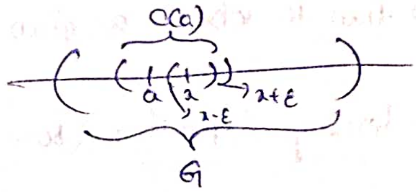
$\therefore C(a)$ is an interval

→ In fact, we show that $C(a)$ is an open interval

let $x \in C(a) \subseteq G$ — open set

$\Rightarrow \exists \ \varepsilon > 0$ such that

$(x-\varepsilon, x+\varepsilon) \subseteq G$



$\Rightarrow (x-\varepsilon, x+\varepsilon) \subseteq C(a)$
(by definition of $\sim$)

$\therefore C(a)$ is open.

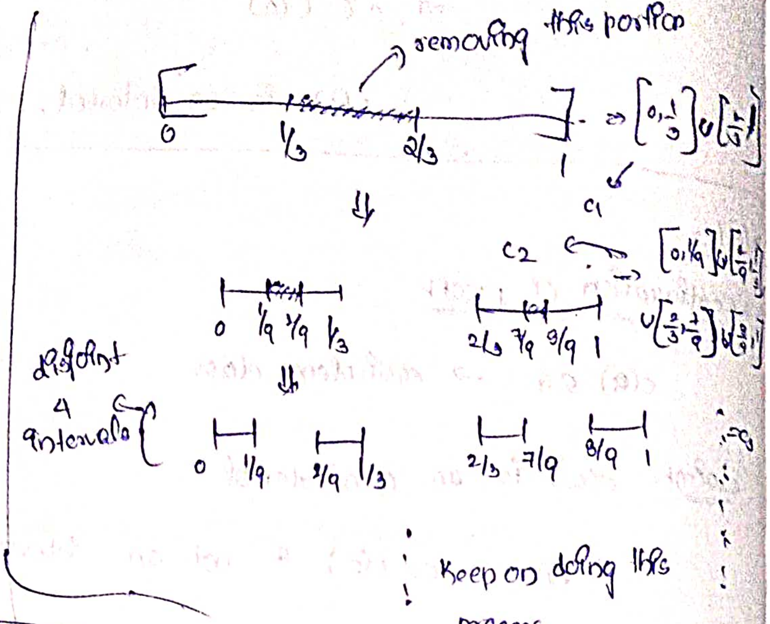
$\therefore G = \bigcup C(a)$ = disjoint union of $C(a)$'s

= disjoint union of open intervals

= disjoint union of at most countable open intervals
(by Lindelöf's theorem)

* **Cantor Set** :-

→ From $[0,1]$, first remove $(1/3, 2/3)$ and $(1/9, 2/9)$, $(7/9, 8/9)$, ... removing at each stage the open interval containing the middle thirds of the closed intervals at previous stage.



→ Then $\boxed{C_n = \frac{1}{3} C_{n-1} \cup \left(\frac{2}{3} + \frac{1}{3} C_{n-1}\right) \quad \forall\, n \geq 1}$

where $\frac{1}{3}A = \left\{ \frac{a}{3} \,\middle|\, a \in A \right\}$

$x + A = \{ x + a \mid a \in A \}$ for any $A \subseteq \mathbb{R}$.

→ Thus we have $C_1, C_2, \ldots$ infinitely many sets.

→ The Cantor set is defined as

$$C = \bigcap_{n=1}^{\infty} C_n$$

→ $C \neq \phi$

→ In fact, the end points of each subinterval in C_n are in C.

→ How to check a given point (efficient way) belongs to Cantor set or not?

* **Ternary Expansion (base 3 instead of base 10)** :-

→ $\frac{1}{3} = 0.\overline{3}$ (base 10) $= 0.1$ (base 3). $= (0.1)_3$

→ For any $c \in (0,1)$ $\overset{c\in A}{}$, in terms of expansion of c is

$$c = \cdots\cdots + c_{-2}\left(\frac{1}{3^2}\right) + c_{-1}\left(\frac{1}{3}\right) + c_0 + c_1(3) + c_2(3)^2 + \cdots\cdots$$

where $c_n \in \{0,1,2\}$ $\forall\, i \in \mathbb{Z}$

→ In fact, for any $c \in (0,1)$,

$$c = \frac{c_1}{3} + \frac{c_2}{3^2} + \frac{c_3}{3^3} + \cdots \quad \text{where } c_n \in \{0,1,2\}$$

→ We denote as $c = (0.c_1c_2c_3\ldots)_3$

Ex:- i) $\frac{1}{4} = 0.25 = (\,?\,)_3$

Follow the below steps

$0.25 \times 3 = 0.75 = 0 + 0.75$ (modulo 3) $\quad c_1 = 0.$

$0.75 \times 3 = 2.25 = 2 + 0.25$ (modulo 3) $\quad c_2 = 2.$

$0.25 \times 3 = 0.75 = 0 + 0.75 \quad c_3 = 0.$

$\vdots$

$$0.25 = (0.c_1c_2\cdots)_3 = (0.0202\ldots)_3$$

$$= (0.\overline{02})_3$$

$$= \frac{c_1}{3} + \frac{c_2}{3^2} + \frac{c_3}{3^3} + \cdots = \frac{0}{3} + \frac{2}{3^2} + 0 + \frac{2}{3^4} + 0 + \cdots$$

$$= 2\left(\sum_{n=1}^{\infty} \frac{1}{3^{2n}}\right) = \frac{2(1/9)}{1-1/9} = 1/4.$$

ii) $\frac{1}{3} = (0.1)_3$

$1/3 \times 3 = (0.333\ldots) \times 3 = 0.9999\ldots = 0 + 0.9999\ldots, \; c_1 = 0$

$0.9999\ldots \times 3 = 2.999\ldots = 2 + 0.999\ldots, \; c_2 = 2$

$c_3 = 2$

$$\therefore \frac{1}{3} = (0.c_1c_2c_3\cdots)_3 = (0.0222\cdots)_3 = (0.0\overline{2})_3$$

$$\therefore \frac{1}{3} = (0.1)_3 = (0.0\overline{2})_3$$

→ Proposition: let $a \in C$, be cantor set, then the ternary expansion of a

$$= \sum_{q=1}^{\infty} \frac{x^q}{3^q} \quad \text{where } x^q \in \{0,2\}$$

→ Conversely, if $x \in [0,1]$ with ternary expansion $x = \sum_{q=1}^{\infty} \frac{x^q}{3^q}$, where $x^q \in \{0,2\}$, then $x \in C$.

Proof: Suppose $x \in [0,1]$ with $x = \sum_{i=1}^{\infty} \frac{x_i}{3^i}$, where $x_i \in \{0,1,2\}$ $\forall i \geq 1$

$= (0.x_1 x_2 x_3 \ldots)$

To show: $x \in C$.

Let y_n be the truncation of x at n places after the decimal point.

i.e) $y_n = (0.x_1 x_2 \ldots x_n)_3$ $\forall n \geq 1$

then $y_n \to x$ as $n \to \infty$

→ In particular, for all $n \geq 1$,

$$y_n \leq x \leq \underbrace{y_n + \frac{1}{3^n}}_{= (0.x_1 x_2 \ldots x_{n-1} \underbrace{(x_n+1)}_{\geq x_n})_3}$$

Ideal:

→ We showed that $y_n \leq x \leq y_n + \frac{1}{3^n}$ $\forall n \geq 1$

→ Observe that the numbers $[0,1]$ whose base-3 expansion goes for exactly n digits after the decimal points & which uses only the digits 0 and 2.

→ These points are precisely the left end points of the intervals whose union is C_n.

$\Rightarrow x \in \left[y_n, y_n + \frac{1}{3^n}\right] \subseteq C_n$ $\forall n$

$\Rightarrow x \in C_n$ $\forall n$

$\Rightarrow x \in \bigcap_{n=1}^{\infty} C_n = C$

→ Conversely, suppose $x \in C$ $\Rightarrow x \in C_n$ $\forall n$

→ Note that the numbers in C are precisely those n^{th} truncation uses only 0 and 2 as digits in base 3.

→ x uses only 0 and 2. $\quad y_n \to x$ as $n \to \infty$, $\to 0.x_1 \cdots x_n$

(II) Cantor set C doesn't contain any sub interval of $[0,1]$.

(3) C is uncountable.

(4) → Look at $I = [a,b)$, $a,b \in \mathbb{R}$, length $l(I) = b-a$.

* <u>Definition</u>: The <u>lebesgue outer measure</u> (or) <u>Outer Measure</u> of $A \subseteq \mathbb{R}$ is defined as

$$m^*(A) = \inf\left(\left\{\sum_{n=1}^{\infty} l(I_n) \,\middle/\, I_n = [a_n,b_n) \text{ and } A \subseteq \bigcup_{n=1}^{\infty} I_n\right\}\right)$$

$$= \inf\left(\sum_{n=1}^{\infty} l(I_n)\right)$$

where infimum is taken over all finite or countable collection of the intervals $\{I_n\}$ of the form, $I_n = [a_n,b_n)$ such that $A \subseteq \bigcup_{n=1}^{\infty} I_n$

→ $m^*(\underbrace{[a,b)}_{A}) = b-a = l([a,b))$.

→ Smallest one is take $I_1 = A$,

$I_2 = I_3 = \cdots = \phi = [a,a)$.

⇒ $\sum_{n=1}^{\infty} l(I_n) = l(A)$. $\quad \therefore m^*(A) = l(A)$.

→ The outer measure of any left closed & right open set is its length.

$m^*(\phi) = m^*([a,a)) = a-a = 0$

∴ ~~outer~~ Outermeasure of empty set $= 0$.

* <u>Properties</u> of m^*: let $A \subseteq \mathbb{R}$.

1) $m^*(A) \geq 0$.

2) $m^*(\phi)=0$

3) If $A\subseteq B\subseteq \mathbb{R}$, then $m^*(A)\le m^*(B)$

4) $m^*(\{x\})=0 \quad \forall x\in \mathbb{R}$

Proof: 1, 2 clear.

3) let $A\subseteq B$.

To show $m^*(A)\le m^*(B)$

let $\{I_n\}_{n\ge 1}$ be a collection of intervals such that $B\subseteq \bigcup_{n=1}^{\infty} I_n$

$$\Rightarrow A\subseteq \bigcup_{n=1}^{\infty} I_n$$

$$m^*(B) = \inf_{A\subseteq B\subseteq \bigcup_{n=1}^{\infty} I_n}\left(\sum_{n=1} l(I_n)\right)$$

(Every covering of B is also a covering of A) $(\mathcal{C}_B \subseteq \mathcal{C}_A)$

$$\ge \inf_{A\subseteq \bigcup_{n=1}^{\infty} I_n'}\left(\sum l(I_n')\right)$$

$$= m^*(A)$$

4) Let $x\in \mathbb{R}$.

consider $I_n = [x, x+\frac{1}{n})$, $\forall n\ge 1$

$x\in I_n \quad \forall n\ge 1$

$\Rightarrow \{x\}\subseteq I_n \quad \forall n$

$\Rightarrow m^*(\{x\}) \le m^*(I_n) \quad$ (By 3)

$= l(I_n)$

$= 1/n$

$\therefore m^*(\{x\}) \le \frac{1}{n} \quad \forall n\ge 1$.

$$\Rightarrow \lim_{n\to\infty} \big(m^*(\underset{\{x\}}{A})\big) \le \lim_{n\to\infty}\left(1/n\right) \Rightarrow m^*(\underset{\{x\}}{A}) \le 0$$

$\therefore m^*(\{x\}) = 0$ (by ①).

One again,

③ Let $A \subseteq B$.

Let $\mathcal{C}_A$ = the collection of all countable open coverings of the form $\{I_n\}$ of A. $[a_n, b_n)$

$$= \left\{ \{I_n\}_{n\geq 1} \;\middle|\; \begin{array}{l} \forall n\geq 1,\ I_n = [a_n, b_n) \\ A \subseteq \bigcup_{n=1}^{\infty} I_n \end{array} \right\}.$$

Hy, $\mathcal{C}_B$.

and $\mathcal{C}_B \subseteq \mathcal{C}_A$

$$m^*(B) = \inf_{\{I_n\}\in\mathcal{C}_B} \left(\sum_n \ell(I_n)\right) \geq \inf_{\{I_n\}\in\mathcal{C}_A} \left(\sum_n \ell(I_n)\right) \quad \left(\begin{array}{l} X\subseteq Y \subseteq \mathbb{R} \\ \inf(X) \geq \inf(Y) \end{array}\right)$$

$$= m^*(A)$$

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\# **Proposition** Let $A \subseteq \mathbb{R}$. Then $m^*(A+x) = m^*(A)$ for any $x \in \mathbb{R}$. i.e.,

$\Rightarrow$ Outer measure is translation invariant.

Proof Let $\varepsilon > 0$. We know that

$$m^*(A) = \inf_{A\subseteq \bigcup_n I_n} \left(\sum_n \ell(I_n)\right)$$

$\Rightarrow$ $\boxed{m^*(A)+\varepsilon \geq \sum_n \ell(I_n)}$ for some $\{I_n\}$ such that $A \subseteq \bigcup_n I_n$ ⊛

We have $A+x \subseteq \bigcup_n (I_n+x)$

$a+x$

where $a \in A \subseteq \bigcup I_n$

$\Rightarrow a \in I_n$ for some n

$\Rightarrow a+x \in I_n+x$

$\Rightarrow a+x \in \bigcup_n (I_n+x)$

Now, $m^*(A+x) \leq m^*\left(\bigcup_n (I_n+x)\right)$ $[,)$

$\leq \sum_n \ell(I_n+x) = \sum_n \ell(I_n) \leq m^*(A)+\varepsilon$ (by ⊛)

$\rightarrow m^*(B) = \inf\limits_{B \subseteq \bigcup_n J_n} \left(\sum_n \ell(J_n)\right)$

$\leq \sum_n \ell(J_n)$, $\forall \{J_n\}$ interval such that $B \subseteq \bigcup_n J_n$

$\rightarrow$ Here the set B, has a covering of intervals $J_n = I_n + a$ $\forall n$

contd.

$\therefore m^*(A+a) \leq m^*(A) + \varepsilon \quad \forall \varepsilon > 0$

$\Rightarrow m^*(A+a) \leq m^*(A)$

consider,

$$m^*(A) = m^*\left(\underbrace{(A+a)}_{B(\text{say})} - \underbrace{a}_{y}\right) = m^*(B+y)$$

$$\leq m^*(A+a) = m^*(B)$$

$\therefore m^*(A) \leq m^*(A+a)$

$\therefore m^*(A+a) = m^*(A).$

Theorem: The outer measure of any interval is equal to its length.

Proof: We already proved that $m^*([a,b]) = b-a = \ell([a,b])$

let $I \subseteq \mathbb{R}$ be any interval.

case-1: $I = [a,b) \quad (a<b)$

To show $m^*(I) = b-a = \ell(I)$

Now, $[a,b] \subseteq [a, b+\frac{1}{n}) \quad \forall n \geq 1$

$$\Rightarrow m^*([a,b]) \leq m^*\left([a, b+\tfrac{1}{n})\right)$$

$$= \ell\left([a, b+\tfrac{1}{n})\right)$$

$$= b + \frac{1}{n} - a$$

$$= (b-a) + 1/n.$$

$\therefore m^*(I) \leq (b-a) + \frac{1}{n} \quad \forall n \geq 1$

$\Rightarrow m^*(I) \leq (b-a)$

consider, $[a,b) \subseteq [a,b]$

$$\Rightarrow m^*([a,b)) \le m^*([a,b])$$
$$\underset{b-a}{\parallel}$$

$$\therefore b-a \le m^*([a,b])$$

$$\therefore m^*([a,b]) = b-a.$$

<u>case-2:</u> let $I = (a,b]$, $a<b$

let $0 < \epsilon < b-a$ and $I' = [a+\epsilon, b] \subseteq (a,b]$

then $I' \subseteq I$

$$\Rightarrow m^*(I') \le m^*(I)$$
$$\parallel \text{(case-I)}$$
$$\ell(I')$$
$$\Rightarrow b-a+\epsilon \le m^*(I)$$
$$\Rightarrow b-a \le m^*(I)$$

$I \subseteq [a,b]$

$$\Rightarrow m^*(I) \le m^*([a,b]) = b-a$$

$$\Rightarrow m^*(I) \le b-a$$

$$\therefore m^*(I) = b-a.$$

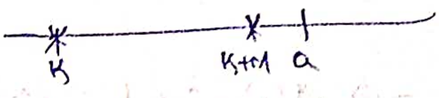
<u>case-3:</u> $I = (a,b)$ (exercise)

<u>case-4:</u> $I = (-\infty, a]$

For any $M>0$, there exists $K \in \mathbb{R}$ such that the interval

$$J_M = [K, K+M) \subseteq I$$

$$\Rightarrow m^*(J_M) \le m^*(I)$$
$$\parallel$$
$$\ell(J_M) = M$$



$(K+M < a)$

$$\Rightarrow m^*(I) \ge M \text{ for any } M \text{ sufficiently large}$$

$$\Rightarrow m^*(I) = \infty$$

→ <u>Other cases:</u> $[a,\infty)$, (exercise)

<u>Example:</u> $m^*([0,3]) = 3$

$$m^*([-1,2] \cup [3,4])$$

* <u>Theorem:</u> Outer measure is countably sub-additive. That is, if

$\{E_n\}_{n\geq 1}$ be a sequence of subsets of $\mathbb{R}$. Then

$$m^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} m^*(E_n)$$

Proof: Let $\varepsilon > 0$.

For each E_n, there exists a sequence of the intervals $\{I_{n,j}\}_{j\geq 1}$

such that $E_n \subseteq \bigcup_{j=1}^{\infty} I_{n,j}$ and

$$m^*(E_n) + \varepsilon/2^n \geq \sum_{j=1}^{\infty} \ell(I_{n,j}) \longrightarrow (*)$$
$\forall\, n \geq 1$

Now, $\bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty}\left(\bigcup_{j=1}^{\infty} I_{n,j}\right)$

$\exists\ B \subseteq \bigcup_{j=1}^{\infty} K_j$

$m^*(B) + \frac{\varepsilon}{2^n} \geq \sum_{j=1}^{\infty} \ell(K_j)$

$\Rightarrow \bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty}\bigcup_{j=1}^{\infty} I_{n,j}$

$$\Rightarrow m^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} \ell(I_{n,j})$$

$$\leq \sum_{n=1}^{\infty}\left(m^*(E_n) + \varepsilon/2^n\right) \quad (\text{By using } (*))$$

$$= \sum_{n=1}^{\infty} m^*(E_n) + \sum_{n=1}^{\infty}\frac{\varepsilon}{2^n}$$

$$\therefore m^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} m^*(E_n) + \varepsilon \quad \forall\, \varepsilon > 0.$$

$$\Rightarrow m^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} m^*(E_n)$$

$\longrightarrow m^*(B) = \inf\left(\sum_n \ell(J_n)\right)$
$B \subseteq \bigcup J_n$

$\varepsilon > 0 \quad (m^*(B) + \varepsilon) > \inf(I \ldots)$

cannot be a lower bound

$$m^*([0,1] \cup [3,5]) \leq m^*([0,1]) + m^*([3,5])$$

$$= 1 + 2 = 3.$$

* Proposition: Let $A \subseteq \mathbb{R}$ be $\varepsilon > 0$. Then there exists an open set $U \subseteq \mathbb{R}$

such that $A \subseteq U$ and $m^*(U) \leq m^*(A) + \varepsilon$

Proof: let $\varepsilon > 0$. There exists $\{I_n\}_{n=1}^{\infty}$, intervals such that $m^*(A) + \varepsilon/2$

$$\geq \sum_{n=1}^{\infty} \ell(I_n) \longrightarrow (*)$$

let $I_n = [a_n, b_n) \quad \forall\, n \geq 1$

set $I_n' = \left(a_n - \frac{\varepsilon}{2^{n+1}}, b_n\right)$ open set $\forall\, n \geq 1$

Note that $I_n \subseteq I_n' \quad \forall\, n \geq 1$

set $U = \bigcup_{n=1}^{\infty} I_n'$ which is an open set.

and $A \subseteq \bigcup_{n=1}^{\infty} I_n \subseteq \bigcup_{n=1}^{\infty} I_n' = U$

and $m^*(U) = m^*\left(\bigcup_{n=1}^{\infty} I_n'\right)$

$$\leq \sum_{n=1}^{\infty} m^*(I_n') \quad \text{(by sub-additive property)}$$

$$= \sum_{n=1}^{\infty} \left(b_n - a_n + \frac{\varepsilon}{2^{n+1}}\right)$$

$$= \sum_{n=1}^{\infty} (b_n - a_n) + \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}}$$

$$= \sum_{n=1}^{\infty} \ell(I_n) + \frac{\varepsilon}{2}$$

$$\leq m^*(A) + \varepsilon/2 + \varepsilon/2 \quad (\text{By } (*))$$

$$= m^*(A) + \varepsilon$$

$$\therefore m^*(U) \leq m^*(A) + \varepsilon$$

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* **Proposition**: Suppose in the definition of outer measure

$$m^*(E) = \inf\left(\left\{\sum_n \ell(I_n) \,\middle/\, \begin{array}{l} E \subseteq \bigcup I_n \\ \{I_n\} \text{ collection of left closed, right open intervals} \end{array}\right\}\right)$$

For $E \subseteq \mathbb{R}$, we stimulate

i) I_n's open intervals, $I_n = (a_n, b_n)$

ii) $J_n = [a_n, b_n)$ $\forall n$ iv) $J_n = [a_n, b_n]$ $\forall n$

iii) $J_n = (a_n, b_n]$ $\forall n$ v) Mixture of all above forms, for different n of the various types of intervals.

→ The same m^* is obtained.

Proof: We define m^* by using case-ii intervals.

let i) $m_0^*(E) = \inf\left(\left\{\sum_n l(J_n) \,/\, E \subseteq \cup J_n, \ J_n\text{'s are open intervals}\right\}\right)$

ii) $m_{oc}^*(E) = \inf\left(\left\{\sum_n l(J_n) \,/\, E \subseteq \cup J_n, \ J_n \text{ are left open right closed intervals}\right\}\right)$

iii) $m_c^*(E) =$ " " $J_n = [a_n, b_n]$

iv) $m_m^*(E) =$ " " J_n's any intervals.

To show: $m^*(E) = m_0^*(E) = m_{oc}^*(E) = m_c^*(E) = m_m^*(E)$

First, to show $m_0^*(E) = m_m^*(E)$

From definition, $m_0^*(E) \geq m_m^*(E) \rightarrow (1)$ $(\because A \subseteq B \Rightarrow \inf(A) \geq \inf(B))$

→ let $\varepsilon > 0$, for any interval J_n, let J_n' be an open interval such that

$l(J_n') = (1+\varepsilon).l(J_n)$ $\forall n \geq 1$ and $J_n \subseteq J_n'$

(For example, $J_n = [a_n, b_n] = [a_n, b_n)$ or $(a_n, b_n]$ or (a_n, b_n),

choose $J_n' = (a_n', b_n')$ where $a_n' = a_n - \frac{(b_n - a_n)}{2}\varepsilon$

$b_n' = b_n + \frac{(b_n - a_n)}{2}\varepsilon$

$l(J_n') = b_n' - a_n' = (b_n - a_n)(1+\varepsilon)$

$= l(J_n) \times (1+\varepsilon)$)

a_n' a_n b_n b_n'

→ suppose $\{J_n\}$ intervals such that $E \subseteq \bigcup_n J_n$ and

$m_m^*(E) + \varepsilon \geq \sum_{n=1}^{\infty} l(J_n) \rightarrow (*)$

We have, $J_n \subseteq J_n'$ $\forall n \geq 1$

$\Rightarrow E \subseteq \bigcup_n I_n \subseteq \bigcup_n I_n'$

$\Rightarrow E \subseteq \underbrace{\bigcup_n I_n'}$ → open intervals

$\Rightarrow m_o^*(E) \leq \sum_{n=1}^{\infty} l(I_n')$ $\quad$ ($\because$ Definition of m_o^*)

$\Rightarrow m_o^*(E) \leq \sum_{n=1}^{\infty} (\epsilon+1) \cdot l(I_n)$

$= (\epsilon+1) \sum_{n=1}^{\infty} l(I_n)$

$\leq (1+\epsilon)(m_m^*(E)+\epsilon)$ $\quad$ (By using (*))

True for any $\epsilon > 0$,

$$m_o^*(E) \leq m_m^*(E) \rightarrow (2)$$

From (1) and (2), $\therefore \boxed{m_o^*(E) = m_m^*(E)}$

Remaining cases: Prove that $m_c^*(E) = m_m^*(E)$, $m_{oc}^*(E) = m_m^*(E)$

(Exercise)

Remark: $m^*(E) = \inf \left(\{ \sum_n l(I_n) / E \subseteq \bigcup_n I_n, I_n\text{'s intervals} \right)$

+

* Definition: let $E \subseteq \mathbb{R}$. We say that E is Lebesgue measurable or measurable, if for each $A \subseteq \mathbb{R}$, we have

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$$

→ Remark: By subadditive property of outermeasure (m^*), we have

$$m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c)$$

→ $m^*(A) = m^*(A \cap \underbrace{(E \cup E^c)}_{\mathbb{R}}) = m^*((A \cap E) \cup (A \cap E^c)) \leq m^*(A \cap E) + m^*(A \cap E^c)$

→ Thus, $E \subseteq \mathbb{R}$ is measurable iff

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c) \quad \forall A \subseteq \mathbb{R}$$

* Proposition: Let $E \subseteq \mathbb{R}$ and $m^*(E) = 0$.

Then E is measurable.

Proof: Given that $m^*(E)=0$

To show: for any $A \subseteq \mathbb{R}$,

$$m^*(A) \geq m^*(A\cap E) + m^*(A\cap E^c)$$

We have $A\cap E \subseteq E$

$$\Rightarrow m^*(A\cap E) \leq m^*(E) = 0$$

$$\& \; 0 \leq$$

$$\Rightarrow m^*(A\cap E) = 0.$$

Also, we have $A\cap E^c \subseteq A$

$$\Rightarrow m^*(A\cap E^c) \leq m^*(A)$$

$$\Rightarrow m^*(A) \geq m^*(A\cap E^c) + 0$$

$$= m^*(A\cap E^c) + m^*(A\cap E)$$

$\therefore E$ is measurable.

Examples:

1) Any finite set is measurable
2) Any countable set is measurable, because it has outer measure zero.

eg: $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$

$\therefore m^*(\mathbb{Q}) = 0$

say $\mathbb{Q} = \{q_1, q_2, \ldots\} = \bigcup_{n=1}^{\infty} \{q_n\}$ $= E_n$ $\neq \emptyset$

$$m^*(\mathbb{Q}) = m^*\left(\bigcup_{n=1}^{\infty} E_n\right)$$

$$\leq \sum_{n=1}^{\infty} m^*(E_n) = \sum_{n=1}^{\infty} 0 = 0.$$

$$\therefore m^*(\mathbb{Q}) = 0.$$

Qn: What are all measurable sets of $\mathbb{R}$?

* **Definition:** A class of subsets $\mathcal{F}$ of an arbitrary space X is said

to be σ-algebra or a σ-field, if it satisfies the following conditions.

i) $X \in \mathcal{F}$

ii) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$

iii) If $E_n \in \mathcal{F}$, $\forall n \geq 1$, then $\bigcup_{n=1}^{\infty} E_n \in \mathcal{F}$

→ Let M = the class of all Lebesgue measurable subsets of $\mathbb{R}$.

→ **Theorem**: M is a σ-algebra.

Proof: From the defn of Lebesgue measurable set, if $E \in M$, then

$E^c \in M$, if $E \in M$

$$\Rightarrow m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c) \quad \forall A \subseteq \mathbb{R}$$

$$= m^*(A \cap \underbrace{E^c}_{F}) + m^*(A \cap \underbrace{(E^c)^c}_{F^c}) \quad \forall A \subseteq \mathbb{R}.$$

$\Rightarrow E^c \in M$

$\mathbb{R} = \phi^c \in M$ ($\because \phi$ is measurable, $m^*(\phi) = 0$.)

$\Rightarrow \phi \in M$.

→ Remains to show: If $\{E_n\}$ is a sequence of measurable sets, then $\bigcup_{n=1}^{\infty} E_n$ is also measurable.

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→ To show: If $\{E_i\}_{i \geq 1}$ is a sequence of measurable sets, then $\bigcup_{i=1}^{\infty} E_i$ is also measurable.

To show for $A \subseteq \mathbb{R}$,

$$m^*(A) \geq m^*\left(A \cap \left(\bigcup_{i=1}^{\infty} E_i\right)\right) + m^*\left(A \cap \left(\bigcup_{i=1}^{\infty} E_i\right)^c\right)$$

→ Given, E_1 is measurable $\Rightarrow m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c)$

Now, E_2 " " so replace E_1 by E_2 and A by $A \cap E_1^c$ in above equality,

we get that

$$m^*(\underbrace{A \cap E_1^c}_{A'}) = m^*(\underbrace{A \cap E_1^c}_{A'} \cap E_2) + m^*(\underbrace{A \cap E_1^c}_{A'} \cap E_2^c)$$

Now, substitute $m^*(A \cap E_1^c)$ in above equality, we get

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c \cap E_2) + m^*(A \cap E_1^c \cap E_2^c)$$

continue in this way,

$$(1) \leftarrow m^*(A) = m^*(A \cap E_1) + \sum_{i=1}^{n} \left(A \cap E_i \cap \left(\bigcap_{j<i} E_j^c\right)\right) + m^*\left(A \cap \left(\bigcap_{j=1}^{n} E_j^c\right)\right) \quad \forall n \geq 1$$

$$\bigcap_{j=1}^{n} E_j^c = \left(\bigcup_{j=1}^{n} E_j\right)^c$$

$$\bigcup_{j=1}^{n} E_j \subset \bigcup_{j=1}^{\infty} E_j$$

$$\Rightarrow \left(\bigcup_{j=1}^{n} E_j\right)^c \cap A \supseteq \left(\bigcup_{j=1}^{\infty} E_j\right)^c \cap A$$

$$\Rightarrow m^*\left(\left(\bigcup_{j=1}^{n} E_j\right)^c \cap A\right) \geq m^*\left(\left(\bigcup_{j=1}^{\infty} E_j\right)^c \cap A\right)$$

From (1), we get that

$$\Rightarrow m^*(A) \geq m^*(A \cap E_1) + \sum_{i=2}^{n} m^*\left(A \cap E_i \cap \left(\bigcap_{j<i} E_j^c\right)\right) + m^*\left(A \cap \left(\bigcup_{j=1}^{\infty} E_j\right)^c\right) \quad \forall n \geq 1$$

$$\Rightarrow m^*(A) \geq m^*(A \cap E_1) + \sum_{i=2}^{\infty} \left(A \cap E_i \cap \left(\bigcap_{j<i} E_j^c\right)\right) + m^*\left(A \cap \left(\bigcap_{j=1}^{\infty} E_j^c\right)\right)$$

$$\geq m^*\left(\bigcup_{i=1}^{\infty} \left((A \cap E_i) \cap \left(\bigcap_{j<i} E_j^c\right)\right)\right) + m^*\left(A \cap \left(\bigcap_{j=1}^{\infty} E_j^c\right)\right)$$

(By sub-Additive property of m^*)

check that $\bigcup_{i=1}^{\infty} \left(A \cap E_i \cap \left(\bigcap_{j<i} E_j^c\right)\right) = \bigcup_{i=1}^{\infty} (E_i \cap A)$ (or) $\left(\bigcup_{i=1}^{\infty} E_i\right) \cap A$

$$\bigcap_{j<i} E_j^c = \left(\bigcup_{j<i} E_j\right)^c$$

Then we have,

$$m^*(A) \geq m^*\left(A \cap \left(\bigcup_{i=1}^{\infty} E_i\right)\right) + m^*\left(A \cap \left(\bigcup_{i=1}^{\infty} E_i\right)^c\right) \quad \forall A \subseteq \mathbb{R}$$

$\Rightarrow \bigcup_{i=1}^{\infty} E_i \in \mathcal{M}$ $\because \mathcal{M}$ is a σ-algebra

Note that

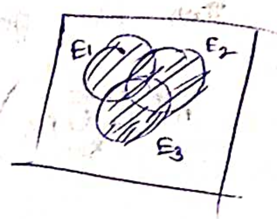
$$\bigcup_{i=1}^{n} \left(E_i \cap \left(\bigcup_{j<i} E_j\right)^c\right) = \bigcup_{i=1}^{n} E_i \quad \forall\, n \geq 1$$

Proof: $n=3$

LHS: $$\bigcup_{i=1}^{3}\left(E_i \cap \left(\bigcup_{j<i} E_j\right)^c\right) = (E_1 \cap \mathbb{R}) \cup (E_2 \cap E_1^c) \cup (E_3 \cap (E_1 \cup E_2)^c)$$

$$= E_1 \cup (E_2 \cap E_1^c) \cup (E_3 \cap (E_1^c \cap E_2^c))$$

$$= E_1 \cup E_2 \cup E_3 = \text{RHS}$$



Proposition:

Suppose $F \in \mathcal{M}$ and $m^*(F \Delta G) = 0$, where $G \subseteq \mathbb{R}$. Then $G \in \mathcal{M}$.

Proof: $F \Delta G = (F \setminus G) \cup (G \setminus F)$

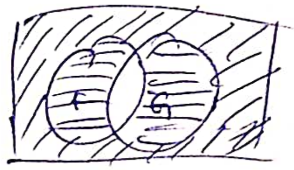
$$0 = m^*(F \Delta G) = m^*((F\setminus G) \cup (G \setminus F))$$

$$\geq m^*(F\setminus G),\ m^*(G\setminus F)$$

$$\Rightarrow m^*(G \setminus F) = 0 = m^*(F \setminus G)$$

$$\Rightarrow G\setminus F,\ F \setminus G \in \mathcal{M}$$

$$\Rightarrow (F\setminus G)^c \in \mathcal{M}$$



and $(F \cap G)^c = \underset{\mathbb{R}-F}{F^c} \cup (F \setminus G) \in \mathcal{M}$

$$\Rightarrow F \cap G \in \mathcal{M}$$

Now $G = \underbrace{(F \cap G)}_{\in \mathcal{M}} \cup \underbrace{(G \setminus F)}_{\in \mathcal{M}} \in \mathcal{M}$

$\therefore G \in \mathcal{M}$

* Theorem: Measurable sets are countably additive.

That is if $\{E_i\}_{i \geq 1}$ is a sequence of disjoint measurable sets in $\mathbb{R}$ $(E_i \cap E_j = \phi \ \forall i \neq j)$ then $m^*\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m^*(E_i)$

Proof: We have, for any $A \subseteq \mathbb{R}$;

$$m^*(A) = m^*(A \cap E_1) + \sum_{i=2}^{\infty} \left(A \cap E_i \cap \left(\bigcap_{j<i} E_j^c\right)\right) + m^*\left(A \cap \left(\bigcup_{j=1}^{\infty} E_j\right)^c\right) \quad \to (1)$$

Take $A = \bigcup_{i=1}^{\infty} E_i$ in above eqn (1),

Then we get that

$$m^*\left(\bigcup_{K=1}^{\infty} E_K\right) = m^*\left(\left(\bigcup_{K=1}^{\infty} E_K\right) \cap E_1\right) + \sum_{i=2}^{\infty} m^*\left(\left(\bigcup_{K=1}^{\infty} E_K\right) \cap E_i \cap \left(\bigcap_{j<i} E_j^c\right)\right)$$

$$+ m^*\underbrace{\left(\left(\bigcup_{K=1}^{\infty} E_K\right) \cap \left(\bigcup_{j=1}^{\infty} E_j\right)^c\right)}_{=\phi}$$

$$\Rightarrow m^*\left(\bigcup_{K=1}^{\infty} E_K\right) = m^*(E_1) + \sum_{i=2}^{\infty} m^*\left(E_i \cap \left(\bigcap_{j<i} E_j^c\right)\right)$$

Consider, $E_i \cap \left(\bigcap_{j<i} E_j^c\right) = E_i \cap (E_1^c \cap E_2^c \cap \cdots \cap E_{i-1}^c) \ \forall i \geq 2$

$= E_i \cap (E_2^c \cap \cdots \cap E_{i-1}^c)$ $\quad (\because E_i \cap E_1 = \phi \ \forall i \neq 1 \Rightarrow E_i \subseteq E_1^c)$

$= E_i$

$$\therefore m^*\left(\bigcup_{K=1}^{\infty} E_K\right) = m^*(E_1) + \sum_{i=2}^{\infty} m^*(E_i)$$

$$\Rightarrow m^*\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{j=1}^{\infty} m^*(E_j) \Rightarrow \text{(Only true for measurable sets)}$$

Theorem: Any interval is measurable.

Proof: Let $I = [a, \infty)$, $a \in \mathbb{R}$

To show I is measurable.

That is, to show that for any $A \subseteq \mathbb{R}$

$$m^*(A) \geq m^*(A \cap I) + m^*(A \cap I^c)$$

let $A \subseteq \mathbb{R}$, $A_1 = A \cap I^c = A \cap (-\infty, a)$

$A_2 = A \cap I = A \cap [a, \infty)$.

let $\varepsilon > 0$, Then $\exists$ intervals $\{I_n\}$ $\ni$ $A \subseteq \bigcup_{n=1}^{\infty} I_n$ and $m^*(A) + \varepsilon$

$\geq \sum_{n=1}^{\infty} \ell(I_n)$

Write $I_n' = I_n \cap I^c = I_n \cap (-\infty, a)$, $I_n'' = I_n \cap I = I_n \cap [a, \infty)$ $\forall\ n \geq 1$

Then $I_n = I_n' \cup I_n''$

and $\ell(I_n) = \ell(I_n') + \ell(I_n'')$

Now, $A_1 = A \cap (-\infty, a) \subseteq \bigcup_{n=1}^{\infty} I_n \cap (-\infty, a) = \bigcup_{n=1}^{\infty} I_n'$

Ily, $A_2 = A \cap [a, \infty) \subseteq \bigcup_{n=1}^{\infty} I_n \cap [a, \infty) = \bigcup_{n=1}^{\infty} I_n''$

Now, $m^*(A_1) + m^*(A_2) \leq m^*\left(\bigcup_{n=1}^{\infty} I_n'\right) + m^*\left(\bigcup_{n=1}^{\infty} I_n''\right)$

$\leq \sum_{n=1}^{\infty} m^*(I_n') + \sum_{n=1}^{\infty} m^*(I_n'')$

$= \sum_{n=1}^{\infty} \left(\ell(I_n') + \ell(I_n'')\right)$

$= \sum_{n=1}^{\infty} \ell(I_n)$

$\leq m^*(A) + \varepsilon$

This is true from any $\varepsilon > 0$,

therefore $m^*(A_1) + m^*(A_2) \leq m^*(A)$

$\Rightarrow$ $m^*(A) \geq m^*(A \cap I^c) + m^*(A \cap I)$

as required

$I \in \mathcal{M}$

$I = [a, \infty) \in \mathcal{M}$

$I^c = (-\infty, a) \in \mathcal{M}$

$(a,\infty) = \bigcup_{n=1}^{\infty} \underbrace{[a+1/n, \infty)}_{\in \mathcal{M}} \in \mathcal{M}$

$(a,b) = \underbrace{(-\infty,b)}_{\in \mathcal{M}} \cap \underbrace{(a,\infty)}_{\in \mathcal{M}} \in \mathcal{M}$

$[a,b] = \underbrace{(-\infty,b]}_{\in \mathcal{M}} \cap \underbrace{[a,\infty)}_{\in \mathcal{M}} \in \mathcal{M}$

$[a,b) \in \mathcal{M}$, $(a,b] \in \mathcal{M}$

$\therefore$ Any interval is measurable.

→ complement of measurable set is measurable
→ countable union of measurable sets is measurable

* Proposition: countable intersection of measurable sets is measurable.

Proof: suppose $\{E_i\}_{i\geq 1}$ is a sequence of measurable sets.

To show: $\bigcap_{i=1}^{\infty} E_i \in \mathcal{M}$

consider $\left(\bigcap_{i=1}^{\infty} E_i\right)^c = \bigcup_{i=1}^{\infty} \underbrace{E_i^c}_{\in \mathcal{M}} \in \mathcal{M}$

$\Rightarrow \bigcap_{i=1}^{\infty} E_i \in \mathcal{M}$.

* Definition: A map $m : \mathcal{M} \to \mathbb{R} \cup \{+\infty\}$ defined as $m(A) = m^*(A)$ is called lebesgue measure and $m(A)$ is called Lebesgue measure of A or simply measure of A.

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→ The "Lebesgue measure" is the map $m : \mathcal{M} \to \mathbb{R} \cup \{+\infty\}$ defined as $m(E) = m^*(E)$ where

$\mathcal{M}$ = The σ-algebra of all measurable sets of $\mathbb{R}$.

→ $m(E+x) = m(E)$, $\forall x \in \mathbb{R}$. (Translation invariant)

$\rightarrow m(I) = l(I)$ for any interval $I \subset \mathbb{R}$

$\rightarrow m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m(E_i)$ $\forall \{E_i\} \subseteq \mathcal{M}$ (countably additive)

& disjoint

*** Theorem:** Let $\mathcal{A}$ be a class of subsets of subsets of metric space (X,d). Then there exists a smallest σ-algebra $\mathcal{S}$ containing $\mathcal{A}$.

$\rightarrow$ We say that $\mathcal{S}$ is the σ-algebra generated by $\mathcal{A}$.

Proof: Let $\{\mathcal{S}_\alpha\}$ be any collection of σ-algebras of subsets of X. Then,

$\bigcap_\alpha \mathcal{S}_\alpha$ is also a σ-algebra.

(arbitrary intersection)

Now take $\mathcal{S}$ is the intersection of all σ-algebras of X containing $\mathcal{A}$.

That is, $\mathcal{S} = \bigcap \mathcal{S}_\alpha$ ($\because \mathcal{S}_\alpha$ is a σ-algebra and $\mathcal{A} \subseteq \mathcal{S}_\alpha$)

Then, $\mathcal{S}$ is the smallest σ-algebra containing $\mathcal{A}$.

*** Definition:** The σ-algebra generated by the class of all the ~~subsets~~ intervals of the form $[a,b)$, $a,b \in \mathbb{R}$ is called Borel σ-algebra and is denoted by $\mathcal{B}$.

$\rightarrow$ The members of $\mathcal{B}$ are called Borel sets of $\mathbb{R}$.

*** Theorem:** 1) Every Borel set is Lebesgue measurable i.e., $\mathcal{B} \subseteq \mathcal{M}$

2) $\mathcal{B}$ is the σ-algebra generated by each of the following class:

$\rightarrow$ The open intervals (or)

$\rightarrow$ The open sets (or)

$\rightarrow$ G_δ - sets

$\rightarrow$ F_σ - sets.

Proof: 1) We have $[a,b) \in \mathcal{M}$ $\forall$ $a,b \in \mathbb{R}$.

and $\mathcal{M}$ is a σ-algebra

$\rightarrow$ Let $A = \{[a,b) / a,b \in \mathbb{R}\}$

Then $A \subseteq \mathcal{M}$

and we know $\mathcal{M}$ is a σ-algebra.

$\Rightarrow \mathcal{B} \subseteq \mathcal{M}$. (By defn of Borel-$\sigma$-algebra)

(Since $\mathcal{B}$ is the smallest σ-algebra containing A)

2) Let $\mathcal{B}_1$ = the σ-algebra generated by the open intervals.

To show that $\mathcal{B}_1 = \mathcal{B}$

We have any open interval as a countable union of the intervals of the form $[a,b)$.

That is, $(a,b) = \bigcup_{n=1}^{\infty} [a+\frac{1}{n}, b) \in \mathcal{B}$

$\Rightarrow$ any open interval $(a,b) \in \mathcal{B}$ $\forall a,b \in \mathbb{R}$.

$\Rightarrow \mathcal{B}_1 \subseteq \mathcal{B}$. ($\because \mathcal{B}_1$ is the smallest σ-algebra containing open intervals and $\mathcal{B}$ is a σ-algebra).

$\rightarrow$ Now, $[a,b) = \bigcap_{n=1}^{\infty} \underbrace{(a-\frac{1}{n}, b)}_{\in \mathcal{B}_1}$ $\Rightarrow [a,b) \in \mathcal{B}_1$

$\Rightarrow [a,b) \in \mathcal{B}_1$, $\forall a,b \in \mathbb{R}$.

$\Rightarrow \mathcal{B} \subseteq \mathcal{B}_1$.

($\because \mathcal{B}$ is the smallest σ-algebra containing $[a,b)$'s and $\mathcal{B}_1$ is a σ-algebra)

$\therefore \boxed{\mathcal{B} = \mathcal{B}_1}$

$\rightarrow$ Remaining proofs exercises.

$\rightarrow$ Question: Does $\mathcal{B} = \mathcal{M}$?

Ans: No (why?) i.e., $\mathcal{B} \subsetneq \mathcal{M}$.

(We will see that there exists a measurable set which isn't a Borel set).

→ Question: $\mathcal{M} = \underbrace{P(\mathbb{R})}_{\text{power set of } \mathbb{R}}$?

Ans: No. (why?) i.e, $\mathcal{M} \subsetneq P(\mathbb{R})$

(we will see that there exists a non-measurable set in $\mathbb{R}$).

That is, $\mathcal{B} \subsetneq \mathcal{M} \subsetneq P(\mathbb{R})$.

* Proposition: for any $A \subseteq \mathbb{R}$, there exists a measurable set $E \subseteq \mathbb{R}$ such that $A \subseteq E$ and $m^*(A) = m^*(E)$.

Proof: let $\varepsilon = \frac{1}{n} > 0$

There exists an open set $U_n \subseteq \mathbb{R}$ such that $A \subseteq U_n$ and

$m^*(A) + \varepsilon \geq m^*(U_n) \quad \forall n \geq 1$

let $E = \bigcap_{n=1}^{\infty} U_n$,

need not be open set

but is G_δ-set

$E \in \mathcal{B} \subseteq \mathcal{M}$

$\therefore E$ is a measurable set

Now, $m^*(E) = m^*\left(\bigcap_{n=1}^{\infty} U_n\right) \leq m^*(U_n)$

$\Rightarrow m^*(E) \leq m^*(U_n) \quad \forall n \geq 1$

$\leq m^*(A) + \frac{1}{n} \quad \forall n \geq 1$

$\Rightarrow m^*(E) \leq m^*(A)$.

Also, we have $A \subseteq U_n \quad \forall n \geq 1$

$\Rightarrow A \subseteq \bigcap_{n=1}^{\infty} U_n = E$

$\Rightarrow A \subseteq E$

$\Rightarrow m^*(A) \leq m^*(E)$

$\therefore \boxed{m^*(A) = m^*(E)}$

* Definition: For any sequence of sets $\{E_i\}_{i=1}^{\infty}$, and

$\limsup_i (E_i) = \bigcap_{n=1}^{\infty}\left(\bigcup_{i \geq n} E_i\right)$

$$= \left(\bigcup_{i\geq 1} E_i\right) \cap \left(\bigcup_{i\geq 2} E_i\right) \cap \left(\bigcup_{i\geq 3} E_i\right) \cap \cdots\cdots$$

(Looks like an empty set but not empty)

And,

$$\liminf_i (E_i) = \bigcup_{n=1}^{\infty} \left(\bigcap_{i\geq n} E_i\right)$$

$$= \left(\bigcap_{i\geq 1} E_i\right) \cup \left(\bigcap_{i\geq 2} E_i\right) \cup \left(\bigcap_{i\geq 3} E_i\right) \cup \cdots$$

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* **Proposition** for $\{E_i\}$ any sequence of sets,
$$\liminf_i (E_i) \subseteq \limsup_i (E_i)$$

Proof let $a \in \liminf_i (E_i) = \bigcup_{n=1}^{\infty} \left(\bigcap_{i\geq n} E_i\right)$

$\Rightarrow a \in \bigcap_{i\geq n_0} E_i$ for some $n_0 \geq 1$.

$\Rightarrow a \in E_i \quad \forall\, i \geq n_0$.

$\Rightarrow a \in E_1 \cup E_2 \cup \cdots E_{n_0} \cup \cdots$

$\Rightarrow a \in E_2 \cup \cdots E_{n_0} \cup \cdots$

$\Rightarrow$ ⋮

$$\Rightarrow a \in \left(\bigcup_{i=1}^{\infty} E_i\right) \cap \left(\bigcup_{i=2}^{\infty} E_i\right) \cap \left(\bigcup_{i=3}^{\infty} E_i\right) \cap \cdots$$

$$= \left(\bigcup_{i\geq 1} E_i\right) \cap \left(\bigcup_{i\geq 2} E_i\right) \cap \cdots$$

$$= \bigcap_{n=1}^{\infty} \left(\bigcup_{i\geq n} E_i\right)$$

$$= \limsup_i (E_i)$$

* **Remark** $\limsup_i (E_i)$ = the set of points belonging to infinitely many of the sets E_i

and $\liminf_i (E_i)$ = the set of points belonging to all but finitely many of the sets E_i.

* **Definition:** If $\liminf_q (E_q) = \limsup_q (E_q)$, then we define

$$\lim_q (E_q) = \liminf_q (E_q) = \limsup_q (E_q)$$

Examples: 1) Suppose $E_1 \subseteq E_2 \subseteq \cdots$, then,

$$\liminf_q (E_q) = \bigcup_{n=1}^{\infty} \left(\bigcap_{q \geq n} E_q \right) = \bigcup_{n=1}^{\infty} (E_n) \quad (\because \text{From given cond}^n).$$

$$\limsup_q (E_q) = \bigcap_{n=1}^{\infty} \left(\bigcup_{q \geq n} E_q \right)$$

$$= \left(\bigcup_{q \geq 1} E_q \right) \cap \left(\bigcup_{q \geq 2} E_q \right) \cap \left(\bigcup_{q \geq 3} E_q \right) \cap \cdots$$

$$\therefore \limsup_q (E_q) = \bigcup_{n=1}^{\infty} E_n = \liminf_q (E_q)$$

$$\therefore \lim_q (E_q) = \bigcup_{q=1}^{\infty} E_q.$$

→ Suppose $E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots$, then,

$$\limsup_q (E_q) = \liminf_q (E_q) = \bigcap_{q=1}^{\infty} E_q \quad (\text{exercise})$$

* **Theorem:** Let $\{E_q\}_{q \geq 1}$ be a sequence of measurable sets in $\mathbb{R}$. Then,

i) If $E_1 \subseteq E_2 \subseteq \cdots$, then

$$m\left(\lim_q (E_q) \right) = \lim_{q \to \infty} (m(E_q))$$

ii) If $E_1 \supseteq E_2 \supseteq \cdots$, and $m(E_1) < \infty$ $\forall q \geq 1$. Then

$$m\left(\lim_q (E_q) \right) = \lim_{q \to \infty} (m(E_q))$$

Proof: i) Note that $\lim_q (E_q) = \bigcup_{q=1}^{\infty} E_q$

Let $F_1 = E_1$,

$F_2 = E_2 \setminus E_1 = E_2 \cap E_1^c \in \mathcal{M}$

$F_3 = E_3 \setminus E_2 = E_3 \cap E_2^c \in \mathcal{M}$

$\vdots$

$F_q = E_q \setminus E_{q-1} = E_q \cap E_{q-1}^c \in \mathcal{M}$

$\vdots$

Then, $\bigcup_{q=1}^{\infty} F_q = \bigcup_{q=1}^{\infty} E_q$ and each $F_q \in \mathcal{M}$.

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = m\left(\bigcup_{i=1}^{\infty} F_i\right)$$

$$= \sum_{i=1}^{\infty} m(F_i) \quad \text{(By countably additive property of measurable sets)}$$

$$= \lim_{n\to\infty}\left(\sum_{i=1}^{n} m(F_i)\right)$$

$$= \lim_{n\to\infty}\left(m\left(\bigcup_{i=1}^{n} F_i\right)\right) \quad (\text{''})$$

$$= \lim_{n\to\infty}\left(m\left(\bigcup_{i=1}^{n} E_i\right)\right)$$

$$= \lim_{n\to\infty}\left(m(E_n)\right) \quad \left(\because \bigcup_{i=1}^{n} E_i = E_n\right)$$

$$\boxed{\therefore m\left(\lim_i (E_i)\right) = \lim_{n\to\infty} m(E_n)}$$

ii) Assume $E_1 \supseteq E_2 \supseteq \cdots$

$\Rightarrow \phi = E_1 \setminus E_1 \subseteq E_1 \setminus E_2 \subseteq E_1 \setminus E_3 \subseteq E_1 \setminus E_4 \cdots$

(Ascending chain of measurable sets)

By (i),

$$m\left(\bigcup_{i=1}^{\infty} (E_1 \setminus E_i)\right) = \lim_{n\to\infty} m(E_1 \setminus E_n)$$

Now, $m(E_1) = m(E_1 \cup E_i) \quad \forall i \geq 1$

$$= m\left(\underbrace{(E_1 \setminus E_i)}_{\in \mathcal{M}} \cup \underbrace{E_i}_{\in \mathcal{M}}\right) \quad (\because E_1 \supseteq E_i \ \forall i \geq 1)$$

$$= m(E_1 \setminus E_i) + m(E_i)$$

$\Rightarrow m(E_1 \setminus E_i) = m(E_1) - m(E_i) \ \forall i \geq 1 \quad (\because m(E_i) < \infty \ \forall i \geq 1)$ (?)

$$\Rightarrow \lim_{i\to\infty} m(E_1 \setminus E_i) = \lim_{i\to\infty} \left(m(E_1) - m(E_i)\right)$$

$$\Rightarrow \boxed{\lim_{i\to\infty} \left(m(E_1 \setminus E_i)\right) = m(E_1) - \lim_{i\to\infty} (m(E_i))} \text{ — (*)}$$

consider,

$$\lim_i (E_1 \setminus E_i) = \bigcup_{i=1}^{\infty} (E_1 \setminus E_i)$$

$$= \left(E_1 \setminus \left(\bigcap_{q=1}^{\infty} E_q\right)\right)$$

$$= E_1 \setminus \left(\lim_q (E_q)\right)$$

$$\Rightarrow m\left(\lim_q (E_1 \setminus E_q)\right) = m\left(E_1 \setminus \lim_q (E_q)\right)$$

$$\Rightarrow \lim_{q\to\infty} m(E_1 \setminus E_q) = m\left(E_1 \setminus \lim_q (E_q)\right)$$

$$\Rightarrow (\text{By } (*)) \quad m(E_1) - \lim_{q\to\infty} m(E_q) = m(E_1) - m\left(\lim_q (E_q)\right)$$

$$\Rightarrow \boxed{\lim_{q\to\infty} m(E_q) = m\left(\lim_q (E_q)\right)}$$

* Propositions:-

i) Every non-empty open set has positive measure. that is if $U \subseteq \mathbb{R}$ is open, then $m(U) > 0$.

ii) Let $Q = \{q_1, q_2, q_3, \ldots\}$ and $G = \bigcup_{n=1}^{\infty} \left(q_n - \frac{1}{n^2}, q_n + \frac{1}{n^2}\right)$. then for any closed set $F \subseteq \mathbb{R}$, $m(G \Delta F) > 0$.

Proof:-

i) We know any open set in $\mathbb{R}$, can be written as countable union of disjoint open intervals.

let $U = \bigcup_{n=1}^{\infty} I_n$, each I_n is an open interval.

$$m(U) = m\left(\bigcup_{n=1}^{\infty} I_n\right) = \sum_{n=1}^{\infty} m(I_n)$$

(By countable additive property of measurable sets) (CAPMS)

Note that all I_n are not empty.

$\Rightarrow$ there exists one $I_{n_0} \neq \phi$

$$\Rightarrow m(I_{n_0}) = l(I_{n_0}) > 0.$$

$$\Rightarrow m(U) = \sum_{n=1}^{\infty} m(I_n) \geq m(I_{n_0}) > 0$$

$$\Rightarrow m(U) > 0$$

ii) Note that $Q \subseteq G_{\in \mathcal{U}}$. G is an open set.

By above (i), $m(G) > 0$

<u>To show</u> $m(G \Delta F) > 0$ for any closed set $F \subseteq \mathbb{R}$.

Let $F \subseteq \mathbb{R}$ be a closed set,

$$G \Delta F = \underbrace{(G \setminus F)}_{\in \mathcal{M}} \cup \underbrace{(F \setminus G)}_{\in \mathcal{M}} \in \mathcal{M}$$

$$\therefore m(G \Delta F) = m(G \setminus F) + m(F \setminus G) \quad \text{(by CAPMS)}$$

If $m(G \setminus F) > 0$, then $m(G \Delta F) > 0$, as required.

Assume $m(G \setminus F) = 0$,

We know $G \setminus F = G \cap F^c$ is an open set.

$\therefore$ By (ii), $G \setminus F = \emptyset \Rightarrow G \subseteq F$

$\mathbb{Q} \subseteq G \subseteq F$

Take closure, then, $\bar{\mathbb{Q}} = \mathbb{R}$ and $\bar{F} = F$

so, $\mathbb{R} \subseteq \bar{G} \subseteq \bar{F} = F \subseteq \mathbb{R}$.

$$\Rightarrow m(F) = m(\mathbb{R}) = +\infty.$$

Also, $m(G) = m\left(\bigcup_{n=1}^{\infty} \left(q_n - \frac{1}{n^2}, q_n + \frac{1}{n^2}\right)\right)$

$$\leq \sum_{n=1}^{\infty} m\left(\left(q_n - \frac{1}{n^2}, q_n + \frac{1}{n^2}\right)\right) \quad \text{(by sub-additive prop)}$$

$$= \sum_{n=1}^{\infty} \frac{2}{n^2} < \infty.$$

$$\Rightarrow m(G \setminus F) = m(G) - m(F) = +\infty$$

This is a contradiction bcoz, $m(G \setminus F) = 0$.

Thus, $m(G \setminus F) > 0$.

$$\Rightarrow m(G \Delta F) > 0.$$

* <u>Proposition</u> The cantor set has measure zero.

Thus, there exists an uncountable set of measure zero.

Does uncountable sets can have measure zero?
Yes

Proof: The cantor set $= C = \bigcap_{n=1}^{\infty} C_n$, where $C_n = \bigcup_{r=1}^{2^{n-1}} J_{n,r}$

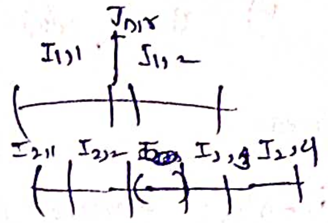
$\Rightarrow C \in B \in \mathcal{M}$. and $\ell(J_{n,r}) = \frac{1}{3^n}$

<u>To show</u> $m(C) = 0$

Now $C^c = [0,1] \setminus C$

$$= [0,1] \setminus \left(\bigcap_{n=1}^{\infty} C_n\right) = \bigcup_{n=1}^{\infty} \left([0,1] \setminus C_n\right) \quad (\because \text{DeMorgen's Law})$$

$$= \bigcup_{n=1}^{\infty} \underbrace{\left([0,1] \setminus \left(\bigcup_{r=1}^{2^{n-1}} J_{n,r}\right)\right)}_{\bigcup_{r=1}^{2^{n-1}} J_{n,r}}$$



$$= \underbrace{\bigcup_{n=1}^{\infty} \left(\bigcup_{r=1}^{2^{n-1}} J_{n,r}\right)}$$

$$\therefore m([0,1] \setminus C) = m\left(\bigcup\right)$$

$$= \sum_{n=1}^{\infty} \sum_{r=1}^{2^{n-1}} \ell(J_{n,r})$$

$$= \sum_{n=1}^{\infty} \sum_{r=1}^{2^{n-1}} \frac{1}{3^n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{3^n} \cdot \left(\sum_{r=1}^{2^{n-1}} 1\right) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = 1$$

$$\therefore m([0,1] \setminus C) = 1$$

$$\Rightarrow m([0,1]) - m(C) = 1$$

$$\Rightarrow m(C) = 0.$$

<u>02/02/23</u>

→ The following result states that the measurable sets are those which can be approximated closely in terms of m^* by open or closed sets.

Theorem: Let $E \subseteq \mathbb{R}$. Then the following conditions are equivalent:-

i) E is measurable

ii) Given $\varepsilon > 0$, there exists an open set $U \subseteq \mathbb{R}$ such that $E \subseteq U$ and $m^*(U \setminus E) \leq \varepsilon$

iii) There exists a G_δ-set $G \subseteq \mathbb{R}$ such that $E \subseteq G$ and $m^*(G \setminus E) = 0$

iv) Given $\varepsilon > 0$, there exists a closed set $F \subseteq \mathbb{R}$ such that $F \subseteq E$ and $m^*(E \setminus F) \leq \varepsilon$

v) There exists an F_σ-set $F \subseteq \mathbb{R}$ such that $F \subseteq E$ and $m^*(E \setminus F) = 0$.

Proof:-

* (i) ⟹ (ii) :- Assume (i), that is E is a measurable set

let $\varepsilon > 0$.

To show there exists an open set $U \subseteq \mathbb{R}$ such that $E \subseteq U$ and $m^*(U \setminus E) \leq \varepsilon$

suppose $m(E) < \infty$

We already proved that, there exists open set $U \subseteq \mathbb{R}$ such that

$E \subseteq U$ and $m^*(U) \leq m^*(E) + \varepsilon$

If $m(E) < \infty$, then $m^*(U \setminus E) = m^*(U) - m^*(E) \leq \varepsilon$ as required. (?)

Assume $m(E) = \infty$

We have $\mathbb{R} = \overset{\infty}{\underset{n=1}{\dot{\bigcup}}} I_n$, a disjoint union of finite open intervals ~~disjoint countable union~~

Then $E = E \cap \mathbb{R} = E \cap \left(\overset{\infty}{\underset{n=1}{\dot{\bigcup}}} I_n \right) = \overset{\infty}{\underset{n=1}{\dot{\bigcup}}} \underbrace{(E \cap I_n)}_{= E_n} = \overset{\infty}{\underset{n=1}{\dot{\bigcup}}} E_n$

and $m(E_n) < \infty \ \forall\, n \geq 1$

Also E_n's are measurable sets.

→ Then, there exists an open set $U_n \subseteq \mathbb{R}$ such that $E_n \subseteq U_n$ and

$m^*(U_n \setminus E_n) \leq \frac{\varepsilon}{2^n} \quad (\forall\, n \geq 1)$

let $U = \bigcup_{n=1}^{\infty} U_n$ ⟹ Then U is an open set (∵ Union of open sets)

and $E = \bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} U_n = U$ (∵ $E_n \subseteq U_n \ \forall n$)

Now, $U \setminus E = \left(\bigcup_{n=1}^{\infty} U_n\right) \setminus \left(\bigcup_{n=1}^{\infty} E_n\right) \subseteq \bigcup_{n=1}^{\infty} (U_n \setminus E_n)$ (check it!)

by taking one element of it & see it belongs to this

then $m^*(U \setminus E) \le m^*\left(\bigcup_{n=1}^{\infty} (U_n \setminus E_n)\right)$

$$\le \sum_{n=1}^{\infty} m^*(U_n \setminus E_n)$$

$$= \sum_{n=1}^{\infty} \varepsilon/2^n = \varepsilon$$

$\therefore m^*(U \setminus E) \le \varepsilon$, U open set, $E \subseteq U$. This proves (ii).

→(ii) → (iii): Assume (ii).

To show: there exists a G_δ-set $G \subseteq \mathbb{R}$ such that $E \subseteq G$ and $m^*(G \setminus E) = 0$

let $\varepsilon = 1/n > 0$

Then by (ii), $\exists$ an open set $U_n \subseteq \mathbb{R}$ such that $E \subseteq U_n$ and $m^*(U_n \setminus E) \le \frac{1}{n}$ $(\forall n \ge 1)$

let $G = \bigcap_{n=1}^{\infty} U_n$, then G is a G_δ-set

Then, $E \subseteq \bigcap_{n=1}^{\infty} U_n = G$ $(\because E \subseteq U_n \ \forall n \ge 1)$

Now, $(G \setminus E) \subseteq U_n \setminus E \quad \forall n \ge 1$

$G = \bigcap_{n=1}^{\infty} U_n \subseteq U_n \quad \forall n \ge 1$

$\Rightarrow G \setminus E \subseteq U_n \setminus E \quad \forall n \ge 1$

$\Rightarrow m^*(G \setminus E) \le m^*(U_n \setminus E) \le \frac{1}{n} \quad \forall n \ge 1$

$\Rightarrow m^*(G \setminus E) \le 0$

$\Rightarrow m^*(G \setminus E) = 0$ as required. This proves (iii).

→(iii) → (i): Assume (iii). $(E \subseteq G, m^*(G \setminus E) = 0)$

To show: E is measurable

We have $E = G \setminus (G \setminus E)$

Since G is a G_δ-set, therefore G is a Borel set.

and hence G is measurable.

and $m^*(G \setminus E) = 0$ which gives that $G \setminus E$ is a measurable set.

Then, $E = \left(\underbrace{G}_{\in \mathcal{M}} \setminus \underbrace{(G \setminus E)}_{\in \mathcal{M}}\right) \in \mathcal{M}$

$= G \cap (G \setminus E)^c$

$\therefore E$ is measurable.

This proves (i).

→ (i) ⇒ (iv) : Assume (i). That is assume E is measurable

(Means (ii), (iii) are also true)

let $\varepsilon > 0$.

To show: There exists a closed set $F \subseteq \mathbb{R}$, $F \subseteq E$, $m^*(E \setminus F) \leq \varepsilon$

Since E is measurable, therefore $E^c \in \mathcal{M}$

Then by (ii), there exists an open set $U \subseteq \mathbb{R}$ such that

$E^c \subseteq U$ and $m^*(U \setminus E^c) \leq \varepsilon$

Now, $U \setminus E^c = U \cap (E^c)^c = U \cap E = E \cap (U^c)^c = E \setminus U^c$

let $F = U^c$, then F is a closed set and

since $E^c \subseteq U$, $(E^c)^c \supseteq U^c$, which implies that

$$E \supseteq U^c = F$$

$$F \subseteq E$$

Now, $m^*(E \setminus F) = m^*(E \setminus U^c)$

$= m^*(U \setminus E^c)$

$\leq \varepsilon$

→ (iv) ⇒ (v) : Assume (iv).

To show: There exists an F_σ-set $F \subseteq \mathbb{R}$ such that $F \subseteq E$

and $m^*(E \setminus F) = 0$

let $\varepsilon = 1/n > 0$

Then by (iv), there exists a closed $F_n \subseteq \mathbb{R}$ such that

$F_n \subseteq E$ and $m^*(E \setminus F_n) \leq \frac{1}{n}$ $(\forall\, n \geq 1)$

let $F = \bigcup_{n=1}^{\infty} F_n$. Then F is an F_σ-set.

We have each $F_n \subseteq E \quad \forall n \geq 1$

$$\Rightarrow F = \bigcup_{n=1}^{\infty} F_n \subseteq E$$

$\therefore F \subseteq E$

And also $E \setminus F \subseteq E \setminus F_n \quad \forall n \geq 1$ $\quad (\because F_n \subseteq F \ \forall n$, $\Rightarrow E \setminus F_n \supseteq E \setminus F \ \forall n \geq 1)$

$$\Rightarrow m^*(E \setminus F) \leq \underbrace{m^*(E \setminus F_n)}_{\leq \frac{1}{n}} \quad \forall n \geq 1.$$

$$\Rightarrow m^*(E \setminus F) \leq 0$$

$$\Rightarrow m^*(E \setminus F) = 0$$

this proves (v).

→ $\underline{(v) \Rightarrow (i)}$: Assume (v).

$\underline{\text{To show}}$ E is measurable

By (v), we have an F_σ-set $F \subseteq \mathbb{R}$ such that $F \subseteq E$ and $m^*(E \setminus F) = 0$

We know F is a Borel set and hence measurable.

and $E = \underbrace{F}_{\in \mathcal{M}} \cup \underbrace{(E \setminus F)}_{\in \mathcal{M}} \in \mathcal{M}$ $\quad (\because m^*(E \setminus F) = 0)$

$\therefore E \in \mathcal{M}$

$$\begin{array}{ccc} (i) & \Rightarrow & (ii) \\ \swarrow \quad \uparrow & \nwarrow & \Downarrow \\ (iv) & & (iii) \\ \Rightarrow (v) & & \end{array}$$

* $\underline{\text{Measurable functions}}$

* $\underline{\text{Def}^n}$: Let $E \subseteq \mathbb{R}$ be a measurable set.

An extended real valued f^n

$$f : E \to \mathbb{R} \cup \{\pm\infty\}$$

is said to be "$\underline{\text{measurable } f^n}$"

or measurable, if for each $\alpha \in \mathbb{R}$, the set

$$\{x \in E / f(x) > \alpha\} = f^{-1}((\alpha, \infty))$$

is measurable $\quad$ ↳ Notation of pre-image of f

$(\because$ All continuous f^ns are measurable f^ns$)$

Examples:

i) Let $E \subseteq \mathbb{R}$. The characteristic function of E,

$$\chi_E : \mathbb{R} \to \mathbb{R}, \quad \chi_E(x) = \begin{cases} 0 & \text{if } x \notin E \\ 1 & \text{if } x \in E \end{cases}$$

$\rightarrow$ For $\alpha \in \mathbb{R}$,

$$\chi_E^{-1}((\alpha, \infty)) = \{x \in \mathbb{R} / \chi_E(x) > \alpha\}$$

$$= \begin{cases} \phi \in \mathcal{M} & \text{if } \alpha \geq 1 \\ E & \text{if } 0 \leq \alpha < 1 \\ \mathbb{R} \in \mathcal{M} & \text{if } \alpha < 0 \end{cases}$$

$\therefore \chi_E$ is measurable iff $E \in \mathcal{M}$

* **Proposition:** Any continuous function defined on a measurable set is measurable.

Proof: Let $E \subseteq \mathbb{R}$ be measurable and $f: E \to \mathbb{R}$, a continuous fn.

To show: f is a measurable function,

Let $\alpha \in \mathbb{R}$,

Since f is continuous, $f^{-1}(\underbrace{(\alpha, \infty)}_{\text{open set}})$ is an open set.

Therefore, $f^{-1}((\alpha, \infty))$ is a Borel set and hence measurable.

* **Theorem:** Let $E \subseteq \mathbb{R}$ be a measurable set and $f: E \to \mathbb{R} \cup \{+\infty\}$ be any function then the following statements are equivalent.

i) f is measurable.

ii) $\forall\, \alpha \in \mathbb{R}$, $\{x \in E / f(x) \geq \alpha\}$ is measurable

iii) $\forall\, \alpha \in \mathbb{R}$, $\{x \in E \,/\, f(x) < \alpha\}$ is measurable

iv) $\forall\, \alpha \in \mathbb{R}$, $\{x \in E \,/\, f(x) \leq \alpha\}$ is measurable.

Proof: (i) ⇒ (ii):- Assume f is measurable.

⇒ $\forall\, \alpha \in \mathbb{R}$, $\{x \in E \,/\, f(x) > \alpha\}$ is measurable

Now, for any $\alpha \in \mathbb{R}$,

$$\{x \in E \,/\, f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \underbrace{\{x \in E \,/\, f(x) > \alpha - \tfrac{1}{n}\}}_{\in \mathcal{M}} \in \mathcal{M}$$

⇓

countable intersection of measurable

(ii) ⇒ (iii):-

Assume (ii).

Let $\alpha \in \mathbb{R}$.

$\{x \in E \,/\, f(x) < \alpha\} = \{x \in E \,/\, f(x) \geq \alpha\}^c \in \mathcal{M}$ (complement of measurable is measurable)

(iii) ⇒ (iv):- Assume (iii).

Let $\alpha \in \mathbb{R}$.

$$\{x \in E \,/\, f(x) \leq \alpha\} = \bigcap_{n=1}^{\infty} \underbrace{\{x \in E \,/\, f(x) < \alpha + \tfrac{1}{n}\}}_{\in \mathcal{M}} \in \mathcal{M}.$$

(iv) ⇒ (i):- Assume (iv).

To show: f is measurable

Let $\alpha \in \mathbb{R}$, $\{x \in E \,/\, f(x) > \alpha\} = \underbrace{\{x \in E \,/\, f(x) \leq \alpha\}}_{\in \mathcal{M}}^c \in \mathcal{M}$

∴ f is measurable

→ f is measurable ⇒ $\forall \alpha \in \mathbb{R}$, $\{x \in E \,/\, f(x) = \alpha\}$ is also measurable but

converse is not true.

* ~~Def~~ Proposition:- Suppose $f : E \to \mathbb{R} \cup \{\pm\infty\}$ is measurable then

$\{x \in E \,/\, f(x) = \alpha\}$ is measurable, for any $\alpha \in \mathbb{R}$.

Proof: let $\alpha \in \mathbb{R}$,

Now $\{x \in E \mid f(x) = \alpha\} = \underbrace{\{x \in E \mid f(x) \leq \alpha\}}_{\in \mathcal{M}} \cap \underbrace{\{x \in E \mid f(x) \geq \alpha\}}_{\in \mathcal{M}}$

$\in \mathcal{M}$

*** Theorem:** let f, g ~~ER~~ $: E \to \mathbb{R}$ be measurable f^n defined on a measurable set $E \subseteq \mathbb{R}$ and $c \in \mathbb{R}$.

Then $f+c$, cf, $f+g$, $f-g$, fg, f^2 are measurable f^ns.

Proof: let $\alpha \in \mathbb{R}$.

consider, $\{x \in E \mid (f+c)(x) > \alpha\}$

$= \{x \in E \mid f(x) + c > \alpha\}$

$= \{x \in E \mid f(x) > \alpha - c\}$

$= f^{-1}((\alpha - c, \infty)) \in \mathcal{M}$

Then $f+c$ is measurable f^n,

If $c = 0$, then $cf \equiv 0$ which is measurable.

Assume $c \neq 0$, Now

$$(cf)^{-1}((\alpha, \infty)) = \{x \in E \mid cf(x) > \alpha\}$$

$$= \begin{cases} \{x \in E \mid f(x) > \alpha/c\} \in \mathcal{M} & \text{if } c > 0 \\ \{x \in E \mid f(x) < \frac{\alpha}{c}\} \in \mathcal{M} & \text{if } c < 0. \end{cases}$$

$\therefore cf$ is measurable.

To show: $f+g$ is measurable.

Let $\mathbb{Q} = \{r_1, r_2, r_3, \cdots \infty\}$

Let $\alpha \in \mathbb{R}$,

$$(f+g)^{-1}((\alpha,\infty)) = \{x \in E \,/\, (f+g)(x) > \alpha\}$$
$$= \{x \in E \,/\, f(x) + g(x) > \alpha\}$$
$$= \{x \in E \,/\, f(x) > \alpha - g(x)\}$$
$$= A \text{ (say)}$$

Suppose $x \in A$, $\Rightarrow f(x) > \alpha - g(x)$

$\Rightarrow$ there exists $r_q \in \mathbb{Q}$ such that

$$f(x) > r_q > \alpha - g(x)$$

$\Rightarrow f(x) > r_q$ and $r_q > \alpha - g(x) \Rightarrow g(x) > \alpha - r_q$

$$\Rightarrow x \in \{x \in E \,/\, f(x) > r_q\} \cap \{x \in E \,/\, g(x) > \alpha - r_q\}$$

$$\subseteq \bigcup_{q=1}^{\infty} \left(\{x \in E \,/\, f(x) > r_q\} \cap \{x \in E \,/\, g(x) > (\alpha - r_q)\} \right) = B \text{ (say)}$$

$\therefore A \subseteq B$

Note that $B \in \mathcal{M}$

check that $B \subseteq A$ (Exercise)

$\therefore A = B \in \mathcal{M}$

$\Rightarrow A \in \mathcal{M}$

$\therefore f+g$ is measurable.

$\rightarrow f-g = \underset{\in \mathcal{M}}{f} + \underbrace{(-g)}_{\in \mathcal{M}}$ (As $(-1).g \in \mathcal{M}$, $cg \in \mathcal{M}$). $\Rightarrow f-g \in \mathcal{M}$.

$\rightarrow fg = \frac{1}{4}\left((f+g)^2 - (f-g)^2\right)$

Enough to show f^2 is measurable.

consider,

$$(f^2)^{-1}((\alpha,\infty)) = \{x\in E \mid (f(x))^2 > \alpha\}$$

$$= \begin{cases} E & \text{if } \alpha < 0 \\ \underbrace{\{x\in E \mid f(x) > \sqrt{\alpha}\}}_{\in \mathcal{M}} \cup \underbrace{\{x\in E \mid f(x) < -\sqrt{\alpha}\}}_{\in \mathcal{M}} \end{cases}$$

$$\Rightarrow (f^2)^{-1}((\alpha,\infty)) \in \mathcal{M}.$$

07/02/23

* **Definitions**: let $\{f_i\}_{i\geq 1}$ be a sequence of fns defined on a set $E\subseteq \mathbb{R}$
→ (these can be defined for extended value fns)

i) $\sup\limits_{1\leq i\leq n}(f_i) = \max\{f_1, f_2, \dots, f_n\}$

$\sup\limits_{1\leq i\leq n}(f_i) : E \to \mathbb{R}$, $\sup\limits_{1\leq i\leq n}(f_i)(x) = \sup\limits_{1\leq i\leq n}(f_i(x))$

$= \max\{f_1(x), f_2(x), \dots, f_n(x)\}$

ii) $\inf\limits_{1\leq i\leq n}(f_i) = \min\{f_1, f_2, \dots, f_n\}$

$\inf\limits_{1\leq i\leq n}(f_i) : E \to \mathbb{R}$, $\inf\limits_{1\leq i\leq n}(f_i)(x) = \min\{f_1(x), f_2(x), \dots, f_n(x)\}$

iii) $\sup\limits_{n}(f_n) : E \to \mathbb{R}$, defined as

$n \in \{1,2,\dots,\infty\}$

$$\sup_n(f_n)(x) = \sup(\{f_1(x), f_2(x), \dots\})$$
$$= \sup_n(f_n(x))$$

iv) $\inf\limits_{n}(f_n) : E \to \mathbb{R}$, defined as

$$\inf_n(f_n) = \inf(\{f_1(x), f_2(x), \dots\})$$

v) $\limsup\limits_{n}(f_n) : E \to \mathbb{R}$, defined as

$$\limsup_n(f_n)(x) = \inf_n\left(\sup_{i\geq n}(f_i(x))\right)$$

~~vi) $\liminf_n (f_n) : E \to \mathbb{R}$, def.~~

$$= \inf_n \left(\left\{ \sup_{i \geq 1} f_i(x), \sup_{i \geq 2} f_i(x), \ldots \right\} \right)$$

vi) $\liminf_n (f_n) : E \to \mathbb{R}$ defined as

$$\liminf_n (f_n)(x) = \sup_n \left(\inf_{i \geq n} (f_i(x)) \right)$$

Remark: Let $S \subseteq \mathbb{R}$

$\sup(S)$ = the least upper bound of S

$= \alpha$ (say)

$\Rightarrow$ $x \leq \alpha$ $\forall x \in S$ and α is least such number

$\Rightarrow$ $-x \geq -\alpha$ $\forall x \in S$ and $-\alpha$ is the greatest such number

$\Rightarrow$ $-\alpha = \inf(-S)$

$\Rightarrow$ $-\sup(S) = \inf(-S)$

$\Rightarrow$ $\inf(-S) = -\sup(S)$

$\Rightarrow$ $\boxed{\inf(S) = -\sup(-S)}$

Theorem: Let $\{f_n\}_{n \geq 1}$ be a sequence of measurable fs defined on a measurable set $E \subseteq \mathbb{R}$. Then

i) $\sup_{1 \leq i \leq n} (f_i)$ is measurable

ii) $\inf_{1 \leq i \leq n} (f_i)$ " "

iii) $\sup_n (f_n)$ " "

iv) $\inf_n (f_n)$ " "

v) $\limsup_n (f_n)$ " "

vi) $\liminf_n (f_n)$ " "

Proof: i) To show $\sup_{1\le i\le n}(f_i)$ is measurable.

Let $\alpha\in\mathbb{R}$, consider,

$$\left(\sup_{1\le i\le n}(f_i)\right)^{-1}((\alpha,\infty)) = \{x\in E \mid \sup_{1\le i\le n}(f_i)(x) > \alpha\}$$

$$= \{x\in E \mid \sup_{1\le i\le n}(f_i(x)) > \alpha\}$$

$$= \{x\in E \mid \max\{f_1(x), f_2(x), \dots, f_n(x)\} > \alpha\}$$

$$= \bigcup_{i=1}^{n} \{x\in E \mid f_i(x) > \alpha\}$$

$$= \bigcup_{i=1}^{n} \underbrace{f_i^{-1}((\alpha,\infty))}_{\in \mathcal{M}} \in \mathcal{M}.$$

$\therefore \sup_{1\le i\le n}(f_i)$ is measurable.

ii) $\inf_{1\le i\le n}(f_i) = -\sup_{1\le i\le n}(-f_i)$ $\quad (\because f_i\in\mathcal{M} \Rightarrow -f_i\in\mathcal{M})$

is measurable by (i).

iii) To show $\sup_n(f_n)$ is measurable

Let $\alpha\in\mathbb{R}$, consider,

$$\left(\sup_n(f_n)\right)^{-1}((\alpha,\infty)) = \{x\in E \mid (\sup_n(f_n))(x) > \alpha\}$$

$$= \{x\in E \mid \sup_n(f_n(x)) > \alpha\}$$

$$= \bigcup_{n=1}^{\infty} \underbrace{\{x\in E \mid f_n(x) > \alpha\}}_{\in\mathcal{M}} \in \mathcal{M}$$

$\therefore \sup_n(f_n)$ is measurable

iv) $\inf_n(f_n) = -\sup_n(-f_n)$ is measurable by iii), To show,

v) $\limsup_n (f_n)$ is measurable.

$$\limsup_n (f_n) = \inf_n \underbrace{\Big(\underbrace{\sup_{i\geq n}(f_i)}_{\in \mathcal{M}}\Big)}_{\in \mathcal{M}}$$

vi) $\liminf_n (f_n) = -\limsup_n (-f_n)$ (check it!)

which is measurable by (v).

Definition: Let $E \subseteq \mathbb{R}$ be a (Borel) measurable set. A f^n $f: E \to \mathbb{R} \cup \{\pm\infty\}$ is said to be a Borel-measurable f^n, if for each $\alpha \in \mathbb{R}$,

$$f^{-1}((\alpha,\infty)) = \{x \in E \mid f(x) > \alpha\}$$

is a Borel measurable set

That is, $f^{-1}((\alpha,\infty)) \in \mathcal{B}$, the Borel σ-algebra

→ **Examples:** Any continuous function defined on a Borel measurable set is Borel measurable

Exercises

Theorem: Let $f: E \to \mathbb{R}$ be a Borel Measurable f^n. Then the following statements are equivalent:

i) f is Borel-measurable

ii) $\forall \alpha \in \mathbb{R}$, $\{x \in E \mid f(x) \geq \alpha\}$ is a Borel set

iii) $\forall \alpha \in \mathbb{R}$, $\{x \in E \mid f(x) < \alpha\}$ " " " "

iv) " $\{x \in E \mid f(x) \leq \alpha\}$ " " " "

Exercises

Theorem: Suppose $f, g: E \to \mathbb{R}$ are Borel measurable f^ns and $c \in \mathbb{R}$. Then, $f+c$, cf, $f+g$, $f-g$, fg, f^2 are Borel measurable f^ns!

Exercise

* **Theorem**: let $f_n : E \to \mathbb{R}$ be a measurable for each $n \geq 1$. then

i) $\sup\limits_{1 \leq i \leq n} (f_i)$ is Borel measurable.

ii) $\inf\limits_{1 \leq i \leq n} (f_i)$ " " "

iii) $\sup\limits_{n} (f_n)$ " " "

iv) $\inf\limits_{n} (f_n)$ " " "

v) $\limsup\limits_{n} (f_n)$ " " "

vi) $\liminf\limits_{n} (f_n)$ " " "

* **Definition**: We say that a property P holds <u>almost everywhere</u>, (a.e)

if P holds except on a set of measure zero.

→ <u>Examples</u>

1) If $f : E \to \mathbb{R}$ continuous a.e, means that

$$m(\{a \in E \mid f \text{ is not continuous at } x = a\}) = 0.$$

i.e., $\lim\limits_{x \to a} f(x) = f(a)$?? why it is measurable.

2) $\{f_n\}$, a sequence of measurable f's defined on E

$f_n \to f$, as $n \to \infty$ a.e.

(pointwise convergence)

That is, $\{x \in E \mid f_n(x) \not\to f(x) \text{ as } n \to \infty\}$ has measure zero.

* **Theorem**: let $f : E \to \mathbb{R}$ be a measurable f^n and let $g : E \to \mathbb{R}$

be any function. suppose $f=g$ a.e. Then g is measurable.

Proof: Given f is measurable. For $\alpha \in \mathbb{R}$,

$\{x \in E \mid f(x) > \alpha\} \in \mathcal{M}$.

Given $f=g$ a.e

$\Rightarrow m(\{x \in E \mid f(x) \neq g(x)\}) = 0$

let $A = \{a \in E \mid f(a) > \alpha\}$, $B = \{a \in E \mid g(a) > \alpha\}$

look at $A \Delta B = (A \setminus B) \cup (B \setminus A)$

$= \{a \in E \mid f(a) > \alpha \text{ and } g(a) \leq \alpha\} \cup$

$\{a \in E \mid g(a) > \alpha \text{ and } f(a) \leq \alpha\}$

$\Rightarrow A \Delta B \subseteq \{a \in E \mid f(a) \neq g(a)\}$

$m^*(A \Delta B) \leq m^*(\{a \in E \mid f(a) \neq g(a)\}) = 0$

$m^*(A \Delta B) = 0$. Note that $A \in \mathcal{M}$.

$\therefore B \in \mathcal{M}$ (by using a proposition proved earlier)

$\Rightarrow g^{-1}((\alpha, \infty)) \in \mathcal{M}$

$\therefore g$ is a measurable fn

* Proposition: let $f_n : E \to \mathbb{R}$, $\forall\, n \geq 1$ be measurable. suppose $f_n \to f$ pointwise a.e as $n \to \infty$. Then f is measurable.

Proof: Given $f_n \to f$ a.e

That is, $m(\{x \in E \mid f_n(x) \not\to f(x)\}) = 0$

If $f_n(x) \to f(x)$, then

$f(x) = \lim_{n \to \infty} f_n(x)$

$= \limsup_n (f_n(x))$ (check it!)

$= \liminf_n (f_n(x))$

Look at, $\{x \in E \,/\, f(x) = \limsup_n (f_n(x))\}$

$= \{x \in E \,/\, (f - \limsup_n (f_n))(x) = 0\}$

We have $\limsup_n (f_n)$ is measurable and

$f = \limsup_n (f_n)$ a.e

$\therefore f$ is measurable.

* Proposition: Let $f_n : E \to \mathbb{R}$ be measurable $\forall n \geq 1$. Then

$\{x \in E \,/\, \{f_n(x)\}$ converges as $n \to \infty\} \in \mathcal{M}$

Proof: $\{x \in E \,/\, f_n(x)$ converges as $n \to \infty\} = \{x \in E \,/\, \limsup_n (f_n(x)) = \liminf_n (f_n(x))\}$

$= \{x \in E \,/\, \limsup_n (f_n(x)) - \liminf_n (f_n(x)) = 0\}$

$= \{x \in E \,/\, \underbrace{(\limsup_n (f_n) - \liminf_n (f_n))}_{h} (x) = 0\}$

$\in \mathcal{M}$ $\left(h \text{ m} \atop \{x \in E \,/\, h(x) = a\} \in \mathcal{M} \right)$

* Def^n: Let $f : E \to \mathbb{R} \cup \{\pm \infty\}$ be a measurable f^n. Then the essential supremum of f is defined as

$\text{esssup}(f) := \inf (\{\alpha \in \mathbb{R} \,/\, f \leq \alpha \text{ a.e on } E\})$

$= \inf(U_f)$ (say)

$U_f \neq \phi$

where

$U_f = \{\alpha \in \mathbb{R} / m(f^{-1}(\alpha, \infty)) = 0\}$

$f \leq \alpha$ a.e on E

$m(\{x \in E \mid f(x) \not\leq \alpha\}) = 0.$

Since $f \leq \alpha$ a.e on E $\Leftrightarrow$ $m(\{a \in E \mid f(a) \not\leq \alpha\}) = 0$

$\Leftrightarrow$ $m(\{a \in E \mid f(a) > \alpha\}) = 0$

$\Leftrightarrow$ $m(f^{-1}((\alpha, \infty))) = 0$

$\sup(\phi) = -\infty$

$\inf(\phi) = +\infty.$

and $\operatorname{essup}(f)$ is defined to be $+\infty$, if $U_f \neq \phi$

Example 1) let $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} -1 & \text{if } x = 1 \\ 30 & \text{if } x = 0 \\ 10 & \text{otherwise} \end{cases}$$

$\sup(f) = 30$, $\inf(f) = -1$.

$\operatorname{essup}(f) = ?$ Ans: 10. $f \leq 10$ a.e $\{0,1\}$

2) $$f(x) = \begin{cases} x^5 & \text{if } x \in \mathbb{Q} \\ \tan^{-1}(x) & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

$f = \tan^{-1}(x)$ a.e. $\operatorname{essen inf}(f) = -\pi/2$

$\operatorname{esssup}(f) = \inf(\{\alpha \in \mathbb{R} / f \leq \alpha \text{ a.e}\})$

$= \inf(\{\alpha \in \mathbb{R} \mid \tan^{-1} x \leq \alpha \text{ a.e}\})$

$= \pi/2$

* Defn: let $f: E \to \mathbb{R} \cup \{\pm\infty\}$ be a measurable fn. Then the essential infimum of is defined as

$\operatorname{essen inf}(f) = \sup(\{\alpha \in \mathbb{R} \mid f \geq \alpha \text{ a.e on } E\})$