SOS - Group Theory Final Report

Anumalasetty Varshith

Mentored by

Arpon Basu

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Over - view

1. **Definition of Isomorphism**
2. **4 steps to prove Isomorphism**
3. **An Example**

Isomorphism

Definition of Group Isomorphism

An isomorphism ϕ from a group G to a group $\overline{G}$ is a one-to-one mapping (or function) from G onto $\overline{G}$ that preserves the group operation.

- Symbolically,
  $$\phi(ab) = \phi(a)\phi(b) \; \forall \, a, b \, in \, G$$
  If there is an isomorphism from G onto $\overline{G}$, we say that G and $\overline{G}$ are isomorphic and write G $\approx \overline{G}$.
- There are four separate steps involved in proving that a group G is isomorphic to a group $\overline{G}$.
  - **Step 1**
    Mapping. Define a candidate for the isomorphism; that is, define a function ϕ from G to $\overline{G}$.

Isomorphism

- **Step 2**
  One - one. Prove that ϕ is one-to-one; that is, assume that $\phi(a) = \phi(b)$ and prove that $a = b$.
- **Step 3**
  Onto. Prove that ϕ is onto; that is, for any element $\overline{g}$ in $\overline{G}$, find an element g in G such that $\phi(g) = \overline{g}$.
- **Step 4**
  O.P. Prove that ϕ is operation-preserving; that is, show that $\phi(ab) = \phi(a)\phi(b)$ for all a and b in G.

An Example

An Example

Prove that the mapping from G to G by ϕ_M(A) = MAM^{-1} for all A in G is an isomorphism, where G = SL(2, $\mathbb{R}$) where SL(2, $\mathbb{R}$) is the group of 2x2 real matrices with determinant 1.
Let G = SL(2, $\mathbb{R}$), the group of 2x2 real matrices with determinant 1. Let M be any 2x2 real matrix with determinant 1. Then we can define a mapping from G to G itself by ϕ_M(A) = MAM^{-1} for all A in G. To verify that ϕ_M is an isomorphism, we carry out the four steps.

- **Step 1**
  ϕ_M is a function from G to G. Here, we must show that ϕ_M(A) is indeed an element of G whenever A is. This follows from properties of determinants:

  $$det(MAM^{-1}) = (det\,M)(det\,A)(det\,M)^{-1} = 1 \cdot 1 \cdot 1^{-1} = 1.$$

  Thus, MAM^{-1} is in G.
- **Step 2**
  ϕ_M is one-to-one. Suppose that ϕ_M(A) = ϕ_M(B). Then MAM^{-1} = MBM^{-1} and, by left and right cancellation, A = B.
- **Step 3**
  ϕ_M is onto. Let B belong to G. We must find a matrix A in G such that ϕ_M(A) = B. How shall we do this? If such a matrix A is to exist, it must have the property that MAM^{-1} = B. But this tells us exactly what A must be! For we can solve for A to obtain A = MBM^{-1} and verify that ϕ_M(A) = MAM^{-1} = M(M^{-1}BM)M^{-1} = B.

An Example

- **Step 4**
  ϕ_M is operation-preserving. Let A and B belong to G. Then,

  $$\phi_M(AB) = M(AB)M^{-1} = MA(M^{-1}M)BM^{-1} = (MAM^{-1})(MBM^{-1}) = \phi_M(A)\phi_M(B)$$

  The mapping ϕ_M is called conjugation by M.

Therefore ϕ mapping from G to G by ϕ_M(A) = MAM^{-1} for all A in G is an **Isomorphism**

Thank you