Department of Computer Science and Engineering
Mid Semester Examination

Course No.: CS 207 Course Name: Discrete Structures
Date: 18/09/2023 Time: 13-30 to 15-30 Marks: 30

Q1 Let n, k be positive integers. Prove that the set of numbers $\{1, 2, 3, \ldots, n\}$ can be partitioned into k parts with equal sums if and only if $n \geq 2k - 1$ and $n(n+1)$ is divisible by $2k$. Hint: Form pairs of numbers with equal sums and use strong induction. (6)

Q2 Let $n > 1$ be a positive integer with the prime factorization $n = p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k}$, where $p_1 < p_2 < \cdots < p_k$ are distinct prime numbers and $e_i > 0$ for $1 \leq i \leq k$. Let $Z_n^* = \{i | 1 \leq i \leq n \text{ and } gcd(i, n) = 1\}$ be the set of numbers between 1 to n that are relatively prime to n. Let $\phi(n) = |Z_n^*|$ be the Euler function. Let $l(n)$ denote the largest positive integer l such that Z_n^* contains l consecutive integers and let $f(n)$ be the number of elements $a \in Z_n^*$ such that $\{a, a+1, a+2, \ldots, a + l(n) - 1\} \subseteq Z_n^*$.

(a) Prove that for all positive integers $m, n > 1$ such that $gcd(m, n) = d$,

$$\phi(m)\phi(n)d = \phi(mn)\phi(d).$$

(4)

(b) Write down an expression for $l(n)$ in terms of the prime factorization of n and prove your answer. (2)

(c) Find the value of $f(4725)$ and explain how you got it. Note that $4725 = 3^3 \times 5^2 \times 7$. If this answer is incorrect, no marks will be given for the (d) part, so take care. (4)

(d) Find an expression for $f(n)$ in terms of the prime factorization of n and prove your answer. Hint: Use the Chinese Remainder Theorem. (6)

Q3 Let N denote the set of natural numbers.

(a) Give an explicit bijection f from $2^N \times 2^N$ to 2^N. In other words, for every pair of subsets $P, Q \subseteq N$, define a subset $f(P, Q) \subseteq N$, such that the function f is a bijection. (3)

(b) Do the same for the sets $(2^N)^N$ and 2^N. The set $(2^N)^N$ is the set of all functions from N to 2^N, that is, all possible infinite sequences $A_0, A_1, \ldots$ where each $A_i \subseteq N$. (5)