Partial Orders, Lattices

Q1 Let $(A, \leq)$ be a partial order defined on a finite set A. A total order is a partial order in which any two elements are comparable. Prove that there exists a total order on A that contains the $\leq$ relation. The dimension of a partial order is the minimum number of total orders whose intersection is the given partial order. In other words, $a \leq b$ holds in the partial order if and only if it holds in all the total orders. Prove that the dimension of any partial order on a set with n elements is at most $\lfloor n/2 \rfloor + 1$. Give an example of a partial order on n elements whose dimension is $\lfloor n/2 \rfloor + 1$.

Q2 Let $a_1, a_2, \ldots, a_n$ be a sequence of numbers. Prove that for any number $k \geq 1$, either the sequence can be partitioned into k non-decreasing subsequences, or there exists a decreasing subsequence of $k+1$ numbers (but not both). Using this or otherwise, give an efficient algorithm to find the longest subsequence that can be partitioned into k non-decreasing subsequences. Hint: Define a suitable partial order.

Q3 Let L be a lattice and let $\vee$ and $\wedge$ denote the *lub* and *glb* operations, also called the join and meet. Prove that these operations are commutative, associative and satisfy the absorption laws $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$, for all $a, b \in L$. Conversely, given the two operations satisfying these properties, show that they define a lattice in which $\wedge$ is the *glb* and $\vee$ is the *lub*. Note that it is not necessary to assume the existence of a minimum or maximum element, which may not exist for infinite lattices. The distributive property is not satisfied by many lattices. In particular, show that the lattice of partitions of a set does not satisfy distributivity.

Q4 A permutation p of $[n] = \{1, 2, \ldots, n\}$ is a bijection from $[n]$ to itself, written as a sequence $p(1), \ldots, p(n)$. An inversion pair in a permutation p is an ordered pair (i, j) such that $1 \leq i < j \leq n$ and $p(i) > p(j)$. Let $I(p)$ denote the set of inversion pairs in p. Define a partial order on the set of all permutations by $p \leq q$ if and only if $I(p) \subseteq I(q)$.
(a) If p is the permutation $5, 2, 6, 1, 4, 3$, how many permutations q satisfy $q \leq p$?
(b) Let R be any asymmetric transitive relation on $[n]$ such that $(i, j) \in R \Rightarrow i < j$. Prove that there exists a permutation p of $[n]$ such that $R = I(p)$ if and only if for all $i < j < k$, if $(i, k) \in R$ then either $(i, j) \in R$ or $(j, k) \in R$.
(c) Prove that the $\leq$ relation defines a lattice on the set of permutations by showing that for every pair of permutations of $[n]$ there exists a least upper bound and a greatest lower bound. If q is the permutation $4, 1, 5, 3, 6, 2$, write down the *glb* and *lub* of p and q, where p is the permutation given in part (a). Show how you obtained the answer.
(d) A permutation q is said to cover a permutation p if $p < q$ and there is no permutation r such that $p < r < q$. Here $p < q$ means $p \leq q$ and $p \neq q$. Prove that $\leq$ is a modular lattice that is, p covers $glb(p, q)$ if and only if $lub(p, q)$ covers q for any permutations p, q of $[n]$.