Propositions, Predicates, Numbers

Q1. Assume the predicate $prime(n)$ is defined, which is true if and only if n is a prime number. Express the following statements in predicate logic.

1. There exist infinitely many prime numbers.
2. There exist arbitrarily long sequences of consecutive numbers such that none of the numbers is a prime number.
3. For all positive numbers n, there exists a prime number p such that $n \leq p \leq 2n$.

Q2. These are some famous problems involving primes, some of which are still unsolved. Express them using predicate logic. Do NOT attempt to prove them or read the proofs, where available.

1. There exist infinitely many prime numbers p such that $p + 2$ is also a prime. This is also called the twin primes conjecture.
2. Every even number > 2 can be written as a sum of two (not necessarily distinct) prime numbers. This is called the Goldbach conjecture.
3. There exist arbitrarily long arithmetic progressions of prime numbers. This was proved in the early 2000's by Ben Green and Terrence Tao.
4. There exists a constant c such that there are infinitely many pairs of distinct prime numbers that differ by at most c. This was proved by Yitan Zhang in 2013 for c around 70 million. The value has now been reduced considerably.

Q3 Define the addition and multiplication operation of numbers using the basic assumptions. Prove that they are commutative and associative and that multiplication distributes over addition.

Q4 Prove that for any natural number $m > 0$, for every number n there exist unique natural numbers q and r such that $n = qm + r$ and $m > r \geq 0$. This is called the division property of numbers and is the starting point for many basic results in number theory.