Recurrence Relations

Q1 Let $T(n)$ denote the number of bit strings of length n in which every occurrence of 1 is preceded by an even number of occurrences of 0. Derive a recurrence relation for $T(n)$ and show that this number is equal to the number of bit strings of length n that do not contain 11 as a substring. Prove this also by showing an explicit bijection between the two sets.

Q2 Consider an arrangement of balls in layers such that the balls in each layer are placed consecutively touching each other, and except for the bottom layer, each ball is placed between two balls in the layer below. If there are n balls in the bottom layer, how many distinct arrangements are possible? For $n = 1$, there is only one, for $n = 2$, there are two, one in which there is only one layer and the other with a single ball in layer 2. For $n = 3$, there are five, with distributions (3), $(3, 1)$, $(3, 1)$, $(3, 2)$ or $(3, 2, 1)$ of balls in layers. Prove that the number of possible arrangements is the Fibonacci number F_{2n-2}. Prove it using a recurrence and also by showing a bijection with the set of bit strings of length $2n - 3$ not containing 11, for $n > 1$.

Q3. Prove that the number of bit strings of length n that do not contain any occurrence of 010 or 101 is $2F_n$. Prove it using a recurrence relation and also by giving a 1 to 2 function from the set of such bit strings to the set of bit strings of length $n - 1$ not containing any occurrence of 11.

Q4. Suppose a fair coin is tossed n times. Given that two consecutive heads did not occur in the n trials, what is the expected number of heads that have occurred? Let S be the set of sequences in which each entry is either +1 or -1 and the sum of entries in any prefix of the sequence is at least -2 and at most 2. How many such sequences of length n are possible? If each entry in the sequence is generated randomly with equal probability, and the sequence terminates as soon as the prefix condition is violated, what is the expected length of the sequence?

Q5. The number of binary trees with n nodes T_n satisfies the recurrence $T_0 = 1$ and $T_n = \sum_{i=0}^{n-1} T_i T_{n-1-i}$. These numbers are called Catalan numbers. Prove that T_n is odd if and only if $n = 2^k - 1$ for some $k \geq 0$. One way of doing it is by showing that binary trees can be paired up so that exactly one is left unpaired when $n = 2^k - 1$ and otherwise all are paired up. Describe such a pairing of binary trees. Two binary trees are considered to be isomorphic if one can be obtained from the other by exchanging the left and right subtrees of some nodes. Write down a recurrence relation to compute the number of non-isomorphic binary trees with n nodes.