CS 215- Data Interpretation and Analysis (Post Midsem)

Pushpak Bhattacharyya
Computer Science and Engineering Department
IIT Bombay

Lecture-6

Chi-square distribution, Fitting Distribution to data

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Recap

Type-I and Type-II errors: **Always wrt Null Hypothesis H_0**

as per data / actual	**ACCEPT**	**REJECT**
TRUE	***No Error***	***Type- I error***
FALSE	***Type-II Error***	***No Error***

Significance of level of significance α

- 90% confidence interval, α=0.10
  - ➔Prepared to tolerate 10% Type-I error
  - ➔Probability of wrong rejection of H_0 is 10%
- 95% confidence interval, α=0.05
  - ➔Prepared to tolerate 5% Type-I error
  - ➔Probability of wrong rejection of H_0 is 5%
- 99% confidence interval, α=0.01
  - ➔Prepared to tolerate 1% Type-I error
  - ➔Probability of wrong rejection of H_0 is 1%

Nicotine problem

Problem statement *(Sheldon M. Ross, PSES, 2004)*

All cigarettes presently on the market have an average nicotine content of at least 1.6mg per cigarette. A firm that produces cigarettes claims that it has discovered a new way to cure tobacco leaves that will result in the average nicotine content of a cigarette being less than 1.6 mg. To test this claim, a sample of 20 of the firms cigarettes were analysed. If it is known that the standard deviation of a cigarette's nicotine content is 0.8 mg., what conclusions can be drawn at the 5% level of significance if the average nicotine content of the 20 cigarettes is 1.54?

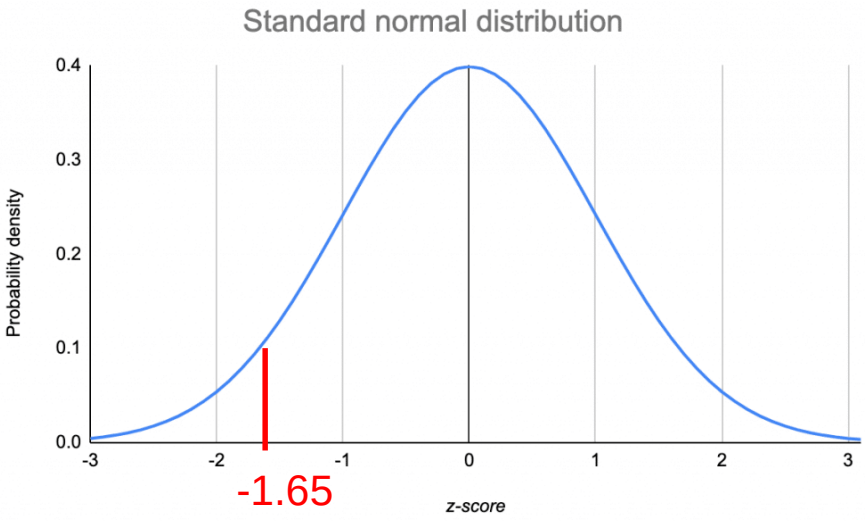
Solution to the Nicotine problem

H_0: μ>=1.6 versus H_1: μ<1.6

With μ=1.6,

$$Z_o = \frac{\sqrt{n}(\overline{X} - \mu)}{\sigma} = \frac{\sqrt{20}(1.54 - 1.6)}{0.8} = -0.33$$

$Z_o > Z_c$, so cannot reject H_0



Conclusion from the nicotine problem

- H_0 cannot be rejected
- Data suggests nicotine content is >=1.6 mg
- Company's claim that the new method of cigarette making ensures < 1.6mg nicotine content is not consistent with the data

End Recap

Chi-Square Distribution

Toss a Die 120 times

- Observe the no. of times each face appears
- Test the hypothesis that the Dice is FAIR

Face	Frequency
1	25
2	17
3	15
4	23
5	24
6	16

Condition for FAIRNESS

- If the dice was fair, we would get 20 times for each face

Face	Frequency	Expected
1	25	20
2	17	20
3	15	20
4	23	20
5	24	20
6	16	20
Total	120	120

Compute $\chi^2_{observed}$

- Take $(O-E)^2/E$ for each observation
- Sum them; that gives $\chi^2_{observed}$

Face	Frequency (O)	Expected (E)	O-E	(O-E)^2	(O-E)^2/E
1	25	20	5	25	1.25
2	17	20	-3	9	0.45
3	15	20	-5	25	1.25
4	23	20	3	9	0.45
5	24	20	4	16	0.8
6	16	20	-4	16	0.8
Total	120	120		$\chi^2_{observed}$	5

Find $\chi^2_{critical}$

DoF=6-1=5; Significance level $\alpha=0.05$; $\chi^2_{critical}=11.1$

Critical values of the Chi-square distribution with *d* degrees of freedom

	Probability of exceeding the critical value						
d	0.05	0.01	0.001	*d*	0.05	0.01	0.001
1	3.841	6.635	10.828	11	19.675	24.725	31.264
2	5.991	9.210	13.816	12	21.026	26.217	32.910
3	7.815	11.345	16.266	13	22.362	27.688	34.528
4	9.488	13.277	18.467	14	23.685	29.141	36.123
5	11.070	15.086	20.515	15	24.996	30.578	37.697
6	12.592	16.812	22.458	16	26.296	32.000	39.252
7	14.067	18.475	24.322	17	27.587	33.409	40.790
8	15.507	20.090	26.125	18	28.869	34.805	42.312
9	16.919	21.666	27.877	19	30.144	36.191	43.820
10	18.307	23.209	29.588	20	31.410	37.566	45.315

***INTRODUCTION TO POPULATION GENETICS*, Table D.1**

Compare $\chi^2_{observed}$ and $\chi^2_{critical}$

- $\chi^2_{observed} < \chi^2_{critical}$
- So cannot reject NULL Hypothesis
- H_0: the dice is FAIR

Another Problem

We have fixed a GPS tracker on our pet bird. The tracker gives the location of the bird with some error. The error is normally distributed with mean 0 and standard deviation 2, i.e., $\sim N(0,2)$. We write

$$\xi \sim N(0,2)$$

We want to know what probability is there of getting our bird's location wrong by 3 meters.

This probability turns out to be 0.522 which is pretty high chance. Time to buy a better tracker!!

Technique and Knowledge Needed

Coordinate Geometry

$$d^2 = (x - x_t)^2 + (y - y_t)^2 + (z - z_t)^2$$

Transmitted location from GPS
<x,y,z>

Actual Location,
$<x_t, y_t, z_t>$

Find Probability P(d>3m)

Distance between transmitted and actual location is expressed in terms of "error"

$\xi_x = x - x_t,\ \xi_y = y - y_t,\ \xi_z = z - z_t$

$$d^2 = \xi_x^{\,2} + \xi_y^{\,2} + \xi_z^{\,2}\ ,\ \ \xi \sim \mathrm{N}(0,2)$$

We need to find the PDF of 'd' to get to our goal P(d>3)

Invoke standard normal distribution

$Z_x = \xi_x/2$

$Z_y = \xi_y/2$

$Z_z = \xi_z/2$

$$Z_{x/y/z} \sim \mathrm{N}(0,1)$$

$$d^2 = \xi_x^{\,2} + \xi_y^{\,2} + \xi_z^{\,2}\ ,\ \xi \sim \mathrm{N}(0,2)$$

$$d^2 = 4(Z_x^{\,2} + Z_y^{\,2} + Z_z^{\,2})$$

Ch-Square Distribution Definition

Sheldon Edition 3, Pp 185

Statement of Chi-Square distribution

$$X = Z_1^{\,2} + Z_2^{\,2} + Z_3^{\,2} + \ldots Z_n^{\,2}$$

Each Z_i is a standard normal variable ($Z_i \sim N(0,1)$)

$$X \sim \chi^2$$

X is said to have a chi-square distribution with ‘n’ degrees of freedom

Back to bird tracker

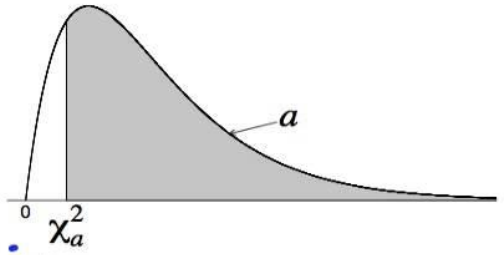
Use Chi-Square distribution

$$d^2 = 4(Z_x^{\ 2} + Z_y^{\ 2} + Z_z^{\ 2})$$

$d^2/4$ is chi-square distributed

$(d>3) \rightarrow (d^2>9) \rightarrow (d^2/4>9/4=2.25)$

Chi-Square table



df	$\chi^2_{0.9995}$	$\chi^2_{0.999}$	$\chi^2_{0.995}$	$\chi^2_{0.990}$	$\chi^2_{0.975}$	$\chi^2_{0.95}$	$\chi^2_{0.90}$	$\chi^2_{0.85}$	$\chi^2_{0.80}$
1	0.000	0.000	0.000	0.000	0.001	0.004	0.016	0.036	0.064
2	0.001	0.002	0.010	0.020	0.051	0.103	0.211	0.325	0.446
3	0.015	0.024	0.072	0.115	0.216	0.352	0.584	0.798	1.005
4	0.064	0.091	0.207	0.297	0.484	0.711	1.064	1.366	1.649
5	0.158	0.210	0.412	0.554	0.831	1.145	1.610	1.994	2.343
6	0.299	0.381	0.676	0.872	1.237	1.635	2.204	2.661	3.070
7	0.485	0.598	0.989	1.239	1.690	2.167	2.833	3.358	3.822
8	0.710	0.857	1.344	1.646	2.180	2.733	3.490	4.078	4.594
9	0.972	1.152	1.735	2.088	2.700	3.325	4.168	4.817	5.380
10	1.265	1.479	2.156	2.558	3.247	3.940	4.865	5.570	6.179
11	1.587	1.834	2.603	3.053	3.816	4.575	5.578	6.336	6.989
12	1.934	2.214	3.074	3.571	4.404	5.226	6.304	7.114	7.807
13	2.305	2.617	3.565	4.107	5.009	5.892	7.042	7.901	8.634
14	2.697	3.041	4.075	4.660	5.629	6.571	7.790	8.696	9.467

$D^2/4$ is chi-square distributed; *$(d>3)$* ➔ *$(d^2>9)$* ➔*$(d^2/4>9/4=2.25)$*

$P(d^2/4>9/4=2.25)=0.522$

Motivation Slide

- Scarcity of data
- Have to use transfer learning
- Transfer learning is essentially borrowing the distribution of another domain and/or data for the purpose at hand
- For example, sentiment analysis in the movie domain- in case data is absent- can be attempted through data in book domain (sentiment about the book ***that*** was picturized will have bearing on the sentiment on the picture)

Transfer Learning = Distribution Adaptation

- Distribution adaptation needs distribution fitting
- Distribution fitting needs testing the goodness of fit
- A well established classical area
- HYPOTHESIS TESTING: Distribution ‘D’ fits the Data ‘d’

coin

Fitting Distribution to Data

Fitting Poisson Distribution: Proverb Data

- The table below shows the number of times proverbs occur in a set of 50 documents.

X (num proverbs)	F(num docs)
0	21
1	18
2	7
3	3
4	1
	50

Poisson Formula

$$P(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

P(X=x) is the probability of the random variable X taking the value x.

In our case X is the r.v denoting the #proverbs in a document

λ is the parameter of the distribution, equal to the mean and standard deviation (can be shown by MGF)

Mean of Poisson for the example

Mean=λ=
(0*21+1*18+2*7+3*3+4*1)/50=45/50
= 0.9

X (num proverbs)	F(num docs)
0	21
1	18
2	7
3	3
4	1
	50

Calculate the Expected No. of proverbs

$$P(X=0)=\frac{e^{-0.9}\lambda^{0}}{0!}=0.4$$

Similarly find P(X=1), P(X=2), P(X=3), P(X=4)

P(X=1)= 0.37,

P(X=2)=0.17,

P(X=3)=0.05,

P(X=4)=0.01

Get expected values, P(X=x)*50

X (#Proverbs)	F(#docs)	Exp #docs, after rounding off
0	21	20
1	18	18
2	7	8
3	3	2
4	1	1

$(Exp-obs)^2/Exp$

X (#Proverbs)	F(Observed #docs)	Exp #docs	(obs-exp)^2/exp
0	21	20	0.05
1	18	18	0
2	7	8	0.125
3	3	2	0.5
4	1	1	0

The fit looks good at the first impression!

Get ChiSquare Observed

$$\chi^2_{obs} = \sum_{i \in categories} \frac{(\exp_i - obs_i)^2}{\exp}$$

$$= 0.675$$

and Compare with ChiSq Critical

$$\chi^2_{critical,\ dof=5-1=4,\ \alpha=0.05}$$

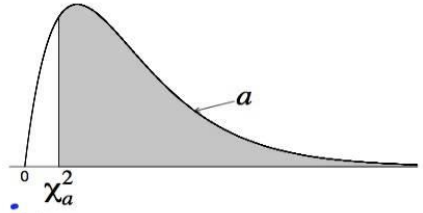
$$= 9.48$$

$ChiSq_{obseved} < ChiSq_{critical}$

No reason to reject null hypothesis

H_0= Data follows Poisson distribution

Chi-Square table



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13	2.305	2.617	3.565	4.107	5.009	5.892	7.042	7.901	8.634
14	2.697	3.041	4.075	4.660	5.629	6.571	7.790	8.696	9.467

Fitting Binomial Distribution

Die Tossing: χ^2 Test

Toss a Die 120 times (already done)

- Observe the no. of times each face appears
- Test the hypothesis that the Dice is FAIR

Face	Frequency
1	25
2	17
3	15
4	23
5	24
6	16

Fitting Normal Distribution

Cricket Score problem

Range	Midpoint (MP)	#innings (I)
0-20	10	10
21-40	30	20
41-60	50	40
61-80	70	20
81-100	90	10
		100

Compute Mean

Range	Midpoint (MP)	#innings (I)	MP X I
0-20	10	10	100
21-40	30	20	600
41-60	50	40	2000
61-80	70	20	1400
81-100	90	10	900
		100	5000
		AV	50

Compute Standard Deviation

Midpoint (MP), Xi	#innings (I)	MP X I		(Xi-av)	(Xi-av)^2	sqr X I
10	10	100		-40	1600	16000
30	20	600		-20	400	8000
50	40	2000		0	0	0
70	20	1400		20	400	8000
90	10	900		40	1600	16000
	100	5000			4000	48000
	AV	50				
	var	484.848				
	std	22.0193				

Making the ranges continuous, Computing low and high values

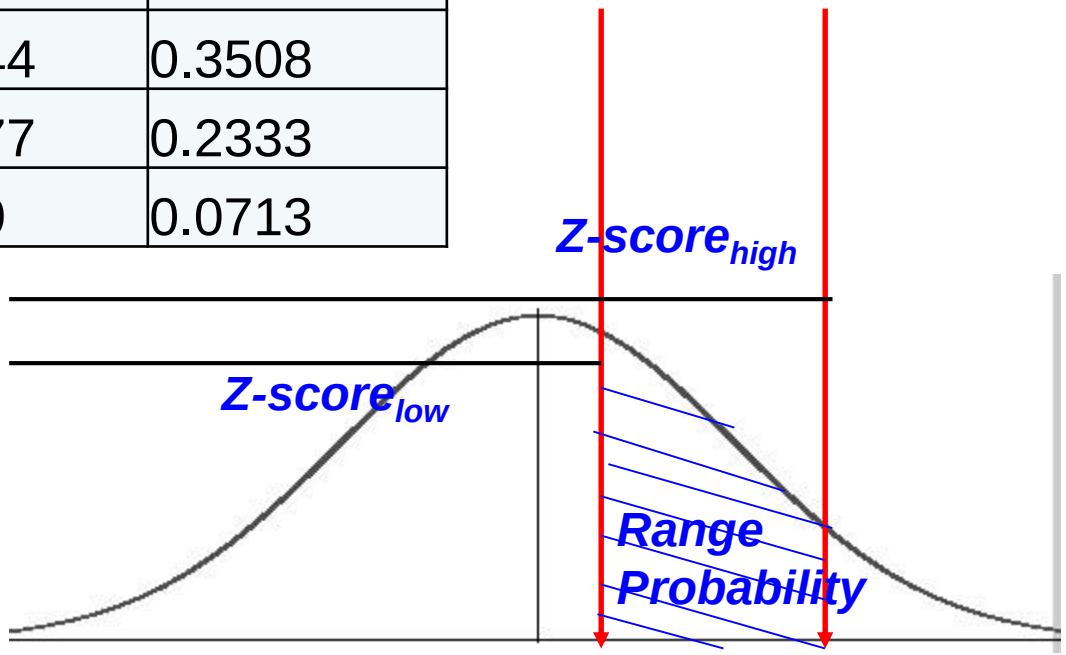
Range	Low range	Hi range	X_low-mu	X_high-mu
0-20	-0.5	20.5	-50.5	-29.5
21-40	20.5	40.5	-29.5	-9.5
41-60	40.5	60.5	-9.5	10.5
61-80	60.5	80.5	10.5	30.5
81-100	80.5	100.5	30.5	50.5
	Mean=50			

Z_{low} [=(X_{low}-μ)/σ] and Z_{high} [=(X_{high}-μ)/σ)

Low range	Hi range	X_low-mu	X_high-mu	Zlow	Zhigh
-0.5	20.5	-50.5	-29.5	-2.29337	-1.33969
20.5	40.5	-29.5	-9.5	-1.33969	-0.43143
40.5	60.5	-9.5	10.5	-0.43143	0.47684
60.5	80.5	10.5	30.5	0.47684	1.3851
80.5	100.5	30.5	50.5	1.3851	2.29337
				Mean=50	
				std 22.02	

Compute *Z-score* and the range probability

Z-score,low	Zscore, high	Range Probability (P)
0.011	0.0901	0.0791
0.0901	0.3336	0.2435
0.3336	0.6844	0.3508
0.6844	0.9177	0.2333
0.9177	0.989	0.0713



Compute Expected Frequency, Range Probability X Midpoint

Range	Midpoint (MP)	Range Probability (P)	Expected freq (MP X P)
0-20	10	0.0791	7.91
21-40	30	0.2435	24.35
41-60	50	0.3508	35.08
61-80	70	0.2333	23.33
81-100	90	0.0713	9.33
			100

Compare Observed and Expected

Range	Midpoint (MP)	Observed Frequency #innings (I)	Expected freq (MP X P)
0-20	10	10	7.91
21-40	30	20	24.35
41-60	50	40	35.08
61-80	70	20	23.33
81-100	90	10	9.33
		100	100

Seems like from Normal Distribution!

Compute $X^2_{observed}$

Observed Frequency #innings (I)	Expected freq (MP X P)	obs-expected	(obs-exp)^2	(obs-exp)^2/exp
10	7.91	2.09	4.3681	0.552225032
20	24.35	-4.35	18.9225	0.777104723
40	35.08	4.92	24.2064	0.690034208
20	23.33	-3.33	11.0889	0.475306472
10	9.33	0.67	0.4489	0.048113612
100	100			

$X^2_{observed}$=2.54 (sum of last col)

Compare $X^2_{observed}$ and $X^2_{critical}$

- $X^2_{observed} = 2.54$
- $X^2_{critical} = 9.48$ (DoF: 4, $\alpha=0.05$)

- Cannot reject the null hypothesis

H_0: The data comes from a normal distribution with $\mu=50$ and $\sigma=22.01$