Graphs

Q1 Prove that every graph with n vertices and more than $n^2/4$ edges contains a triangle. Show that for all even $n \geq 2$, there exists a graph with n vertices and $n^2/4$ edges that does not contain a triangle. More generally, show that every graph with n vertices and more than $\frac{(r-2)n^2}{2(r-1)}$ edges contains a complete subgraph with r vertices, for all $r \geq 3$. Also show that if n is a multiple of $r-1$, there exists a graph with n vertices and $\frac{(r-2)n^2}{2(r-1)}$ edges that does not contain a complete subgraph with r vertices. This is known as Turan's theorem.

Q2 The complement G^c of a graph G is the graph with the same set of vertices as G, and two distinct vertices are adjacent in G if and only if they are not adjacent in G^c. Let $k, m \geq 2$ be positive integers. Let $R(k,m)$ denote the smallest positive integer n such that for any graph G with n vertices, either G contains a complete subgraph with k vertices or G^c contains a complete subgraph with m vertices. $R(k,m)$ is called the Ramsey number and $R(k,2) = R(2,k) = k$ for all $k \geq 2$. Prove that for $k, m \geq 3$, $R(k,m) \leq R(k-1,m) + R(k,m-1)$. Prove that $R(3,3) = 6$. Finding the exact values of Ramsey numbers is a very difficult problem, even the value of $R(5,5)$ is not known, only upper and lower bounds. Find $R(3,4)$ and $R(4,4)$.

Q3 Prove that every graph with n vertices and $n+4$ edges must contain two cycles that have no edge in common with each other. Give examples of graphs with n vertices and $n+3$ edges that do not contain two edge-disjoint cycles for all $n \geq 6$. Prove that if $n \geq 6$, any graph with more than $3n-6$ edges must contain two vertex-disjoint cycles. Give an example of a graph with n vertices and $3n-6$ edges that does not contain such cycles, for all $n \geq 6$.

Q4 Prove that the following are all equivalent ways of defining a tree.

1. A graph with n vertices, $n-1$ edges and no cycle.
2. A connected graph with n vertices and $n-1$ edges.
3. A connected graph with no cycles.
4. A graph in which there is a unique path between every pair of vertices.
5. A graph without cycles but adding any edge gives a graph with a cycle.
6. A connected graph such that deleting any edge gives a graph that is not connected.

Q5 Prove that every graph with n vertices and more than $(k-1)n/2$ edges contains a path of length k. A famous conjecture of Erdős and Sós states that any such graph must contain as a subgraph any tree with k edges. This is known to be true for some special kinds of trees and for all trees when n is very large compared to k. Show that there are infinitely many n for which there are graphs with $(k-1)n/2$ edges that do not contain any tree with k edges. Try to find the minimum number of edges that an n vertex graph must have in order to contain a cycle of length at least k.