CS 215- Data Interpretation and Analysis (Post Midsem)

Fuzzy Sets and Logic, Inverted Pendulum

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Lecture-6

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Recap

Theory of Fuzzy Sets

- Intimate connection between logic and set theory.
- Given any set 'S' and an element 'e', there is a very natural predicate, $\mu_s(e)$ called as the *belongingness predicate*.
- The predicate is such that,

$$\mu_s(e) = 1, \quad \textit{iff } e \in S$$
$$= 0, \quad \textit{otherwise}$$

- For example, $S = \{1, 2, 3, 4\}$, $\mu_s(1) = 1$ and $\mu_s(5) = 0$
- A predicate *P(x)* also defines a set naturally.

  $S = \{x \mid P(x) \text{ is } \textit{true}\}$

  For example, *even(x)* defines $S = \{x \mid x \text{ is even}\}$

Fuzzy Set Theory (contd.)

- Fuzzy set theory starts by questioning the fundamental assumptions of set theory *viz.*, the belongingness predicate, μ*,* value is 0 or 1.
- Instead in Fuzzy theory it is assumed that,

$$\mu_s(e) = [0, 1]$$

- Fuzzy set theory is a generalization of classical set theory *aka* called Crisp Set Theory.
- In real life, *belongingness* is a fuzzy concept.

Example: Let, T = “tallness”

$$\mu_T(\text{height}=6.0\text{ft}) = 1.0$$

$$\mu_T(\text{height}=3.5\text{ft}) = 0.2$$

An individual with height 3.5ft is “tall” with a degree *0.2*

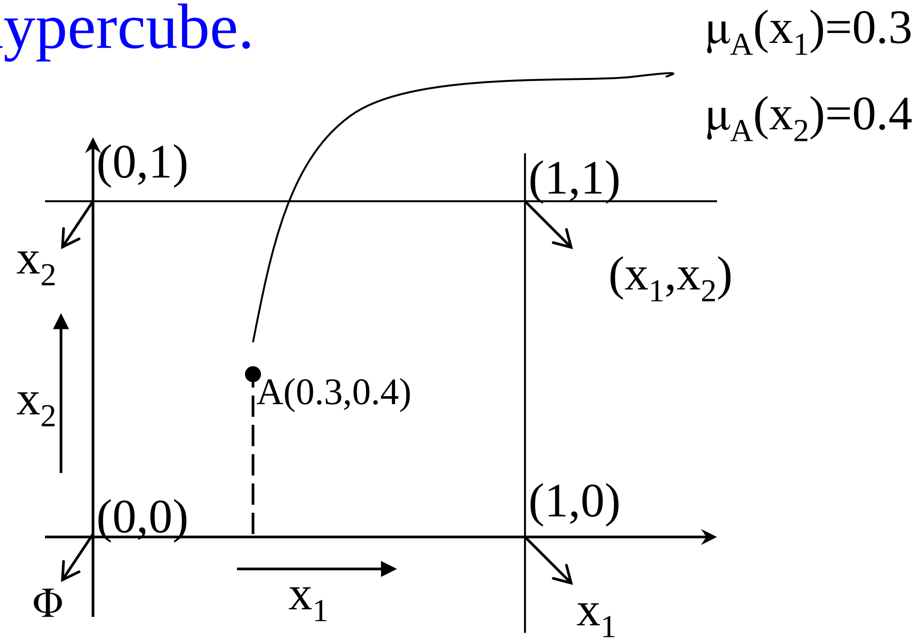
Representation of Fuzzy sets

Let $U = \{x_1, x_2, \ldots.., x_n\}$

$|U| = n$

The various sets composed of elements from U are presented as points on and inside the n-dimensional hypercube. The crisp sets are the corners of the hypercube.

$U=\{x_1,x_2\}$



A fuzzy set A is represented by a point in the n-dimensional space as the point $\{\mu_A(x_1), \mu_A(x_2), \ldots\ldots\mu_A(x_n)\}$

Degree of fuzziness

The centre of the hypercube is the *most fuzzy* set. Fuzziness decreases as one nears the corners

Measure of fuzziness

Called the entropy of a fuzzy set

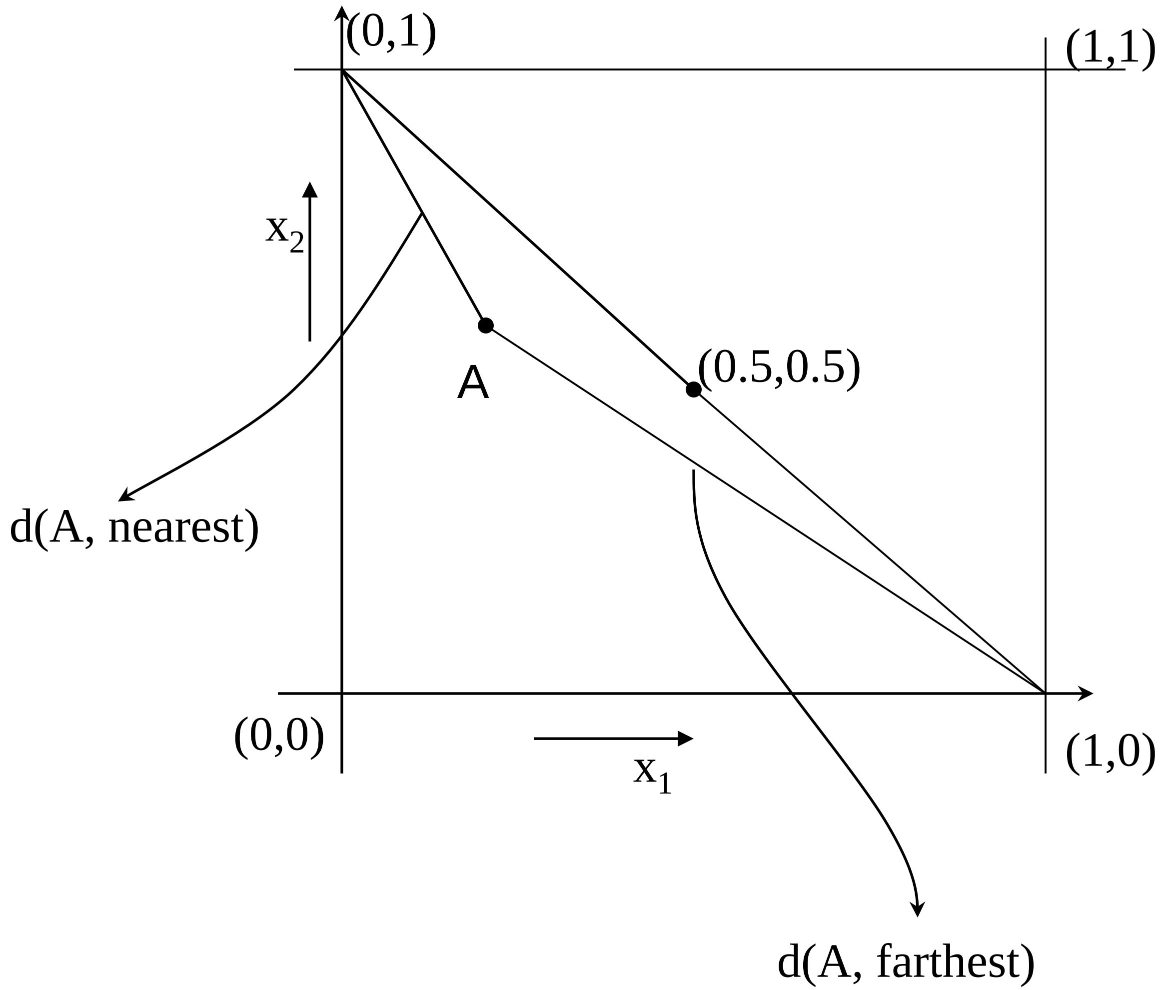
Fuzzy set

Farthest corner

$$E(S) = d(S, nearest) / d(S, farthest)$$

Entropy

Nearest corner


(0,1)
(1,1)
x_2
A
(0.5,0.5)
d(A, nearest)
(0,0)
x_1
(1,0)
d(A, farthest)

Definition

Distance between two fuzzy sets

$$d(S_1, S_2) = \sum_{i=1}^{n} \underbrace{| \mu_{s_1}(x_i) - \mu_{s_2}(x_i) |}_{L_1 \text{ - norm}}$$

Let C = fuzzy set represented by the centre point

d(c,nearest) = |0.5-1.0| + |0.5 – 0.0|

= 1

= d(C,farthest)

=> E(C) = 1

Definition

Cardinality of a fuzzy set

$$m(s) = \sum_{i=1}^{n} \mu_s(x_i)$$ (generalization of cardinality of classical sets)

Union, Intersection, complementation, subset hood

$$\mu_{s_1 \cup s_2}(x) = \max(\mu_{s_1}(x), \mu_{s_2}(x)), \forall x \in U$$

$$\mu_{s_1 \cap s_2}(x) = \min(\mu_{s_1}(x), \mu_{s_2}(x)), \forall x \in U$$

$$\mu_{s^c}(x) = 1 - \mu_s(x)$$

Example of Operations on Fuzzy Set

- Let us define the following:
  - Universe U={X_1 ,X_2 ,X_3}
  - Fuzzy sets
    - A={0.2/X_1 , 0.7/X_2 , 0.6/X_3} and
    - B={0.7/X_1 ,0.3/X_2 ,0.5/X_3}

Then Cardinality of A and B are computed as follows:

Cardinality of A=|A|=0.2+0.7+0.6=1.5

Cardinality of B=|B|=0.7+0.3+0.5=1.5

While distance between A and B

d(A,B)=|0.2-0.7)+|0.7-0.3|+|0.6-0.5|=1.0

What does the cardinality of a fuzzy set mean? In crisp sets it means the number of elements in the set.

Example of Operations on Fuzzy Set (cntd.)

Universe U={X_1 ,X_2 ,X_3}

Fuzzy sets A={0.2/X_1 ,0.7/X_2 ,0.6/X_3} and B={0.7/X_1 ,0.3/X_2 ,0.5/X_3}

A U B= {0.7/X_1, 0.7/X_2, 0.6/X_3}

A ∩ B= {0.2/X1, 0.3/X2, 0.5/X3}

A^c = {0.8/X_1, 0.3/X_2, 0.4/X_3}

Laws of Set Theory

- The laws of Crisp set theory also holds for fuzzy set theory (verify them)
- These laws are listed below:
  - Commutativity: $A \cup B = B \cup A$
  - Associativity: $A \cup (B \cup C)=(A \cup B) \cup C$
  - Distributivity: $A \cup (B \cap C)=(A \cap C) \cup (B \cap C)$
    $A \cap (B \cup C)=(A \cup C) \cap(B \cup C)$
  - De Morgan's Law: $(A \cup B)^{C}= A^{C} \cap B^{C}$
    $(A \cap B)^{C}= A^{C} \cup B^{C}$

Distributivity Property Proof

- Let Universe $U=\{x_1,x_2,\ldots x_n\}$

$p_i = \mu_{A\cup(B\cap C)}(x_i)$

$= \max[\mu_A(x_i), \mu_{(B\cap C)}(x_i)]$

$= \max[\mu_A(x_i), \min(\mu_B(x_i),\mu_C(x_i))]$

$q_i = \mu_{(A\cup B)\cap(A\cup C)}(x_i)$

$= \min[\max(\mu_A(x_i), \mu_B(x_i)), \max(\mu_A(x_i), \mu_C(x_i))]$

Distributivity Property Proof

- Case I: $0<\mu_C<\mu_B<\mu_A<1$

  $p_i = \max[\mu_A(x_i), \min(\mu_B(x_i),\mu_C(x_i))]$

  $= \max[\mu_A(x_i), \mu_C(x_i)]=\mu_A(x_i)$

  $q_i = \min[\max(\mu_A(x_i), \mu_B(x_i)), \max(\mu_A(x_i), \mu_C(x_i))]$

  $= \min[\mu_A(x_i), \mu_A(x_i)]=\mu_A(x_i)$

- Case II: $0<\mu_C<\mu_A<\mu_B<1$

  $p_i = \max[\mu_A(x_i), \min(\mu_B(x_i),\mu_C(x_i))]$

  $= \max[\mu_A(x_i), \mu_C(x_i)]=\mu_A(x_i)$

  $q_i = \min[\max(\mu_A(x_i), \mu_B(x_i)), \max(\mu_A(x_i), \mu_C(x_i))]$

  $= \min[\mu_B(x_i), \mu_A(x_i)]=\mu_A(x_i)$

  Prove it for rest of the 4 cases.

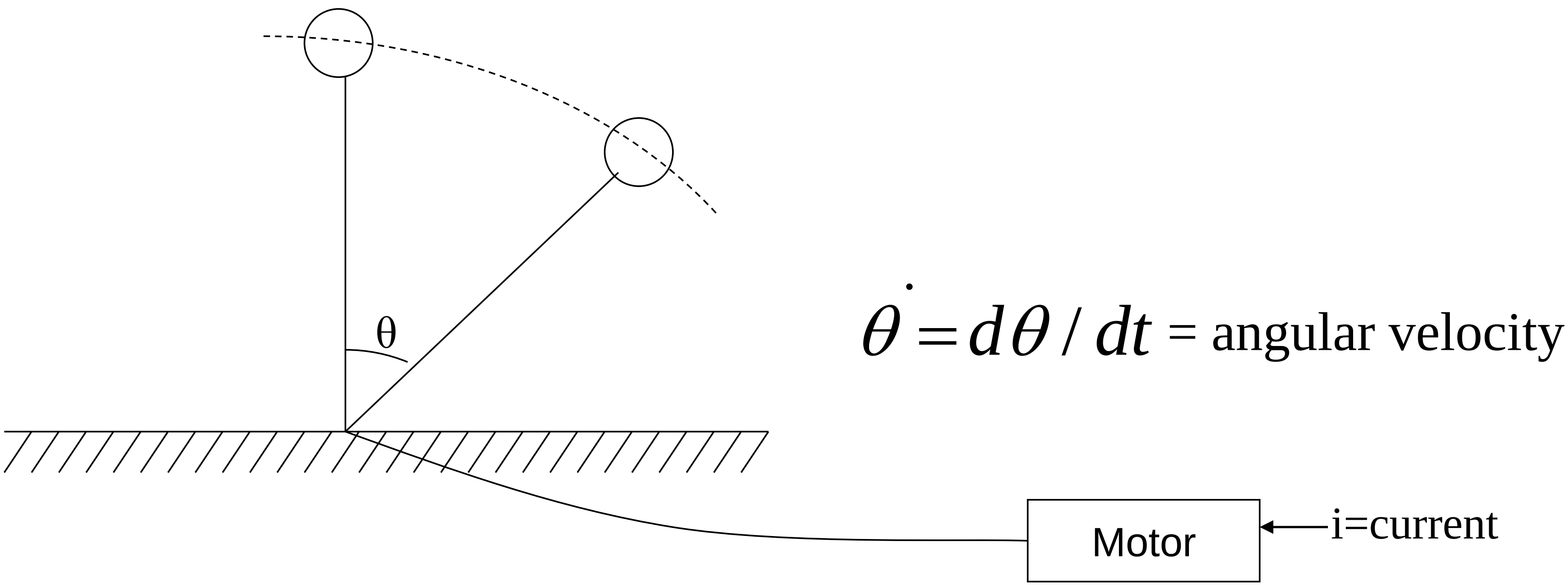
End Recap

Application of Fuzzy Logic

An example

An Example

Controlling an inverted pendulum:



$$\dot{\theta} = d\theta / dt$$ = angular velocity

The goal: To keep the pendulum in vertical position ($\theta=0$) in dynamic equilibrium. Whenever the pendulum departs from vertical, a torque is produced by sending a current 'i'

Controlling factors for appropriate current

Angle θ, Angular velocity $\dot{\theta}$

<u>Some intuitive rules</u>

If θ is +ve small and $\dot{\theta}$ is –ve small

then current is zero

If θ is +ve small and $\dot{\theta}$ is +ve small

then current is –ve medium

Control Matrix

$\dot{\theta}$ \ θ	-ve med	-ve small	Zero	+ve small	+ve med
-ve med					
-ve small		+ve med	+ve small	Zero	
Zero		+ve small	Zero	-ve small	
+ve small		Zero	-ve small	-ve med	
+ve med					

Region of interest

Each cell is a rule of the form

If θ is <> and $\dot{\theta}$ is <>

then i is <>

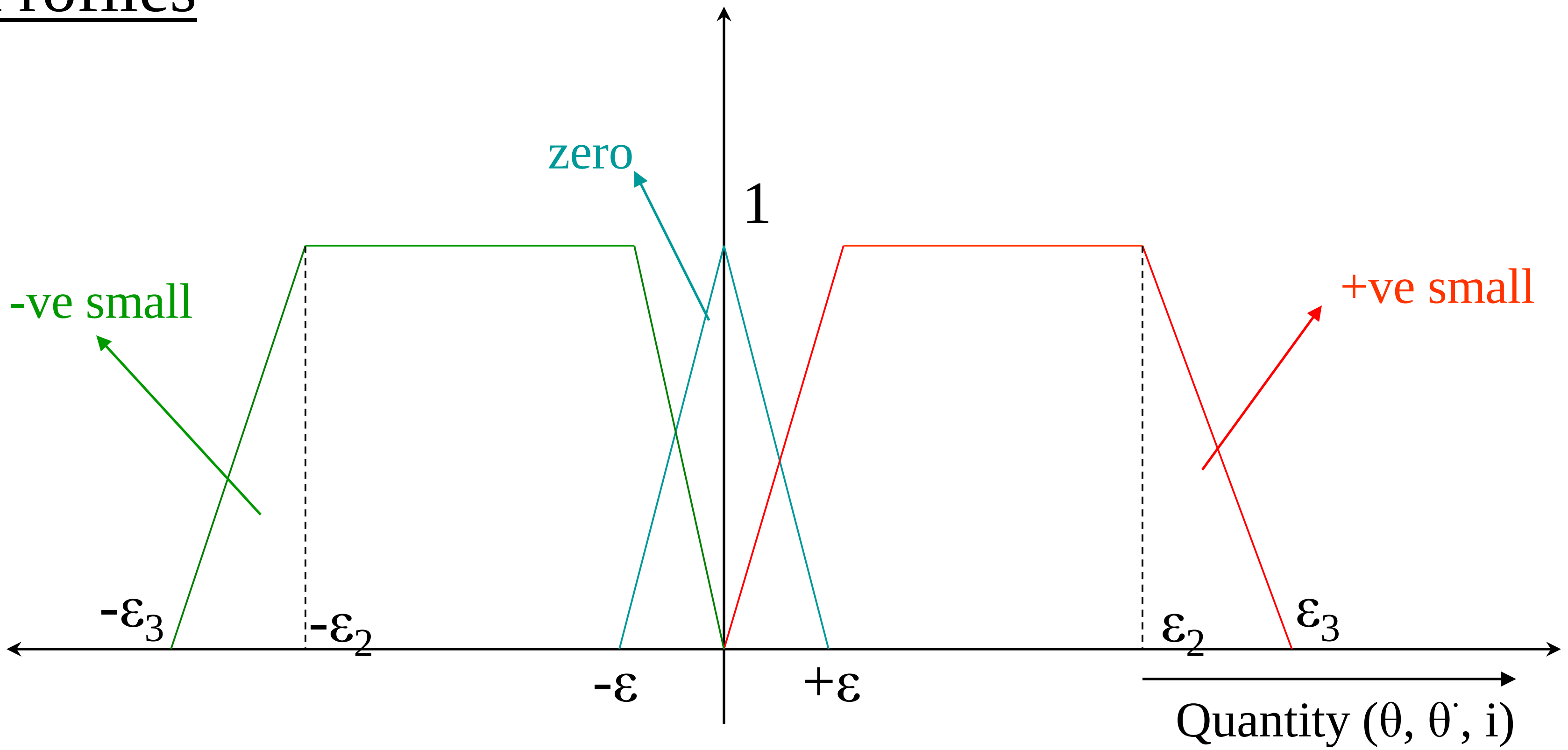
<u>4 “Centre rules”</u>

1. if θ = = Zero and $\dot{\theta}$ = = Zero then i = Zero
2. if θ is +ve small and $\dot{\theta}$ = = Zero then i is –ve small
3. if θ is –ve small and $\dot{\theta}$ = = Zero then i is +ve small
4. if θ = = Zero and $\dot{\theta}$ is +ve small then i is –ve small
5. if θ = = Zero and $\dot{\theta}$ is –ve small then i is +ve small

Linguistic variables

1. Zero
2. +ve small
3. -ve small

Profiles

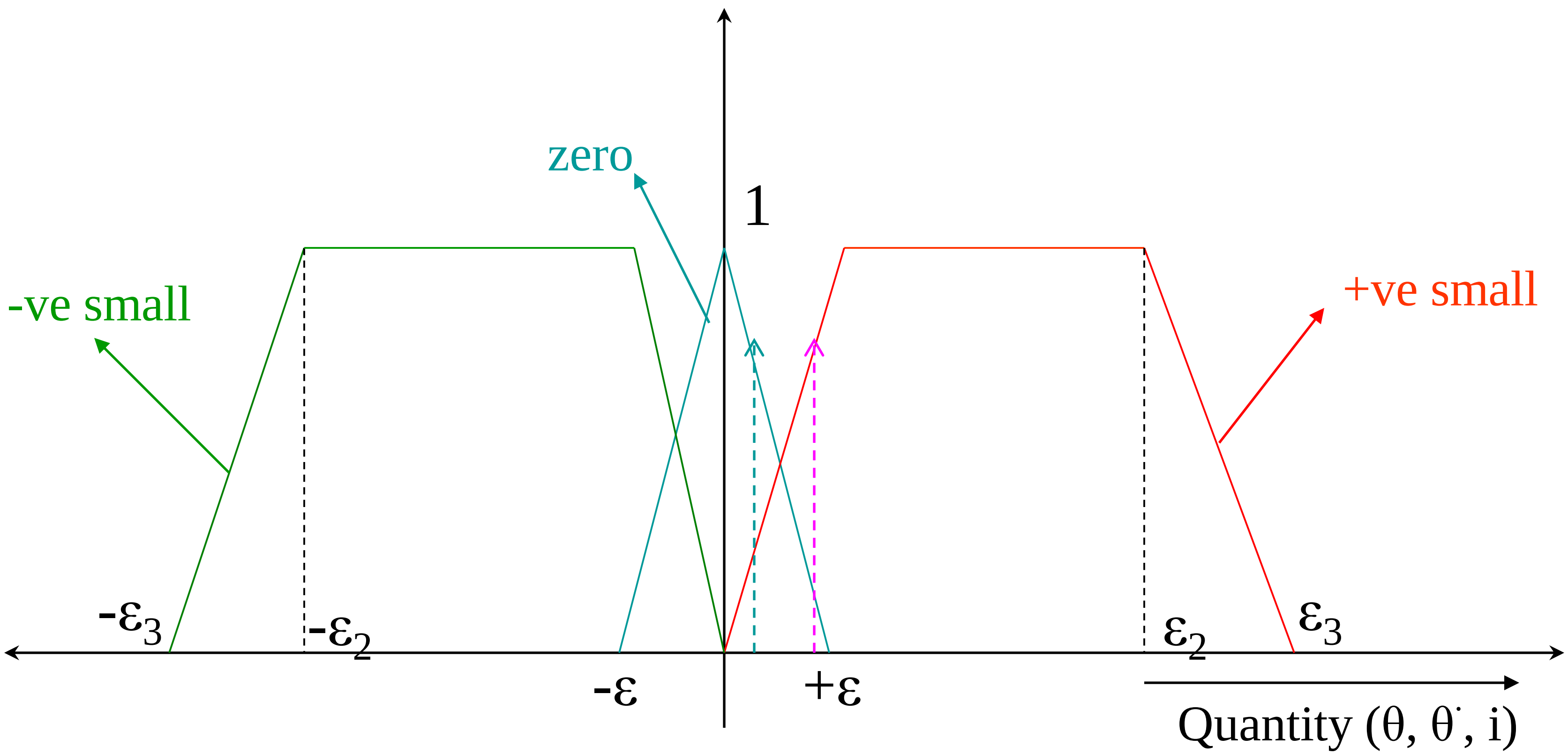


Inference procedure

1. Read actual numerical values of θ and $\dot{\theta}$
2. Get the corresponding μ values μ_{Zero}, $\mu_{(+ve\ small)}$, $\mu_{(-ve\ small)}$. This is called FUZZIFICATION
3. For different rules, get the fuzzy I-values from the R.H.S of the rules.
4. "Collate" by some method and get ONE current value. This is called DEFUZZIFICATION
5. Result is one numerical value of 'i'.

Rules Involved

if θ is Zero and dθ/dt is Zero then i is Zero
if θ is Zero and dθ/dt is +ve small then i is –ve small
if θ is +ve small and dθ/dt is Zero then i is –ve small
if θ +ve small and dθ/dt is +ve small then i is -ve medium



Fuzzification

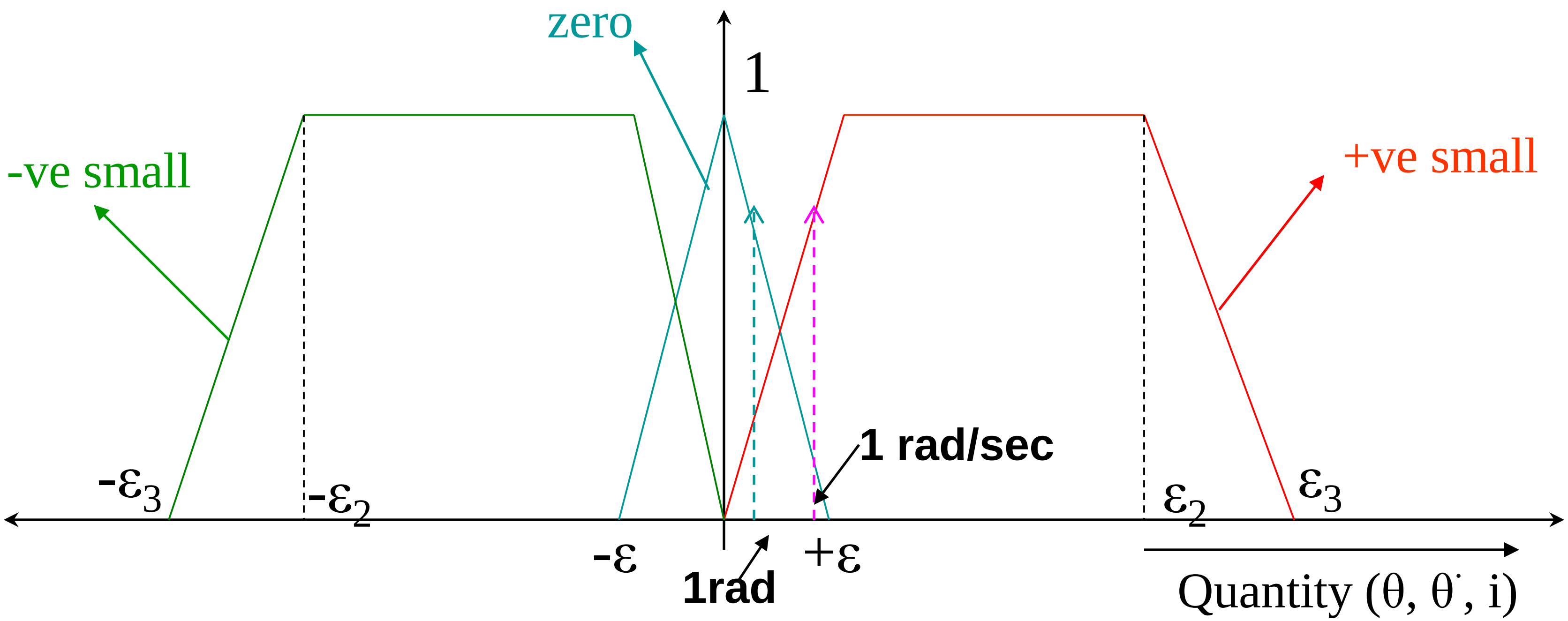
Suppose θ is 1 radian and dθ/dt is 1 rad/sec

μ_{zero}(θ =1)=0.8 (say)

$M_{+ve\text{-}small}$(θ =1)=0.4 (say)

μ_{zero}(dθ/dt =1)=0.3 (say)

$\mu_{+ve\text{-}small}$(dθ/dt =1)=0.7 (say)



Fuzzification

Suppose θ is 1 radian and dθ/dt is 1 rad/sec

$\mu_{zero}(\theta = 1) = 0.8$ (say)

$\mu_{+ve\text{-}small}(\theta = 1) = 0.4$ (say)

$\mu_{zero}(d\theta/dt = 1) = 0.3$ (say)

$\mu_{+ve\text{-}small}(d\theta/dt = 1) = 0.7$ (say)

if θ is Zero and dθ/dt is Zero then i is Zero

min(0.8, 0.3)=0.3

hence ***$\mu_{zero}(i)=0.3$***

if θ is Zero and dθ/dt is +ve small then i is –ve small

min(0.8, 0.7)=0.7

hence ***$\mu_{-ve\text{-}small}(i)=0.7$***

if θ is +ve small and dθ/dt is Zero then i is –ve small

min(0.4, 0.3)=0.3

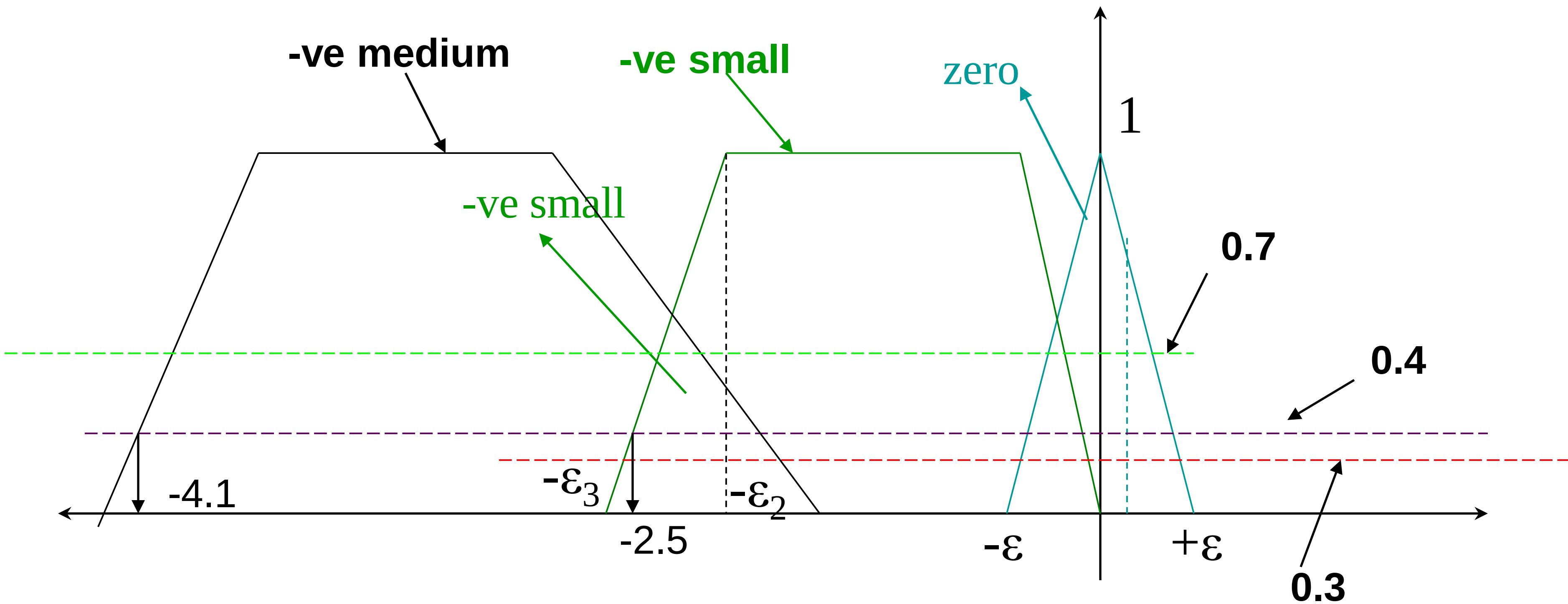
hence ***μ-ve-small(i)=0.3***

if θ +ve small and dθ/dt is +ve small then i is -ve medium

min(0.4, 0.7)=0.4

hence ***$\mu_{-ve\text{-}medium}(i)=0.4$***

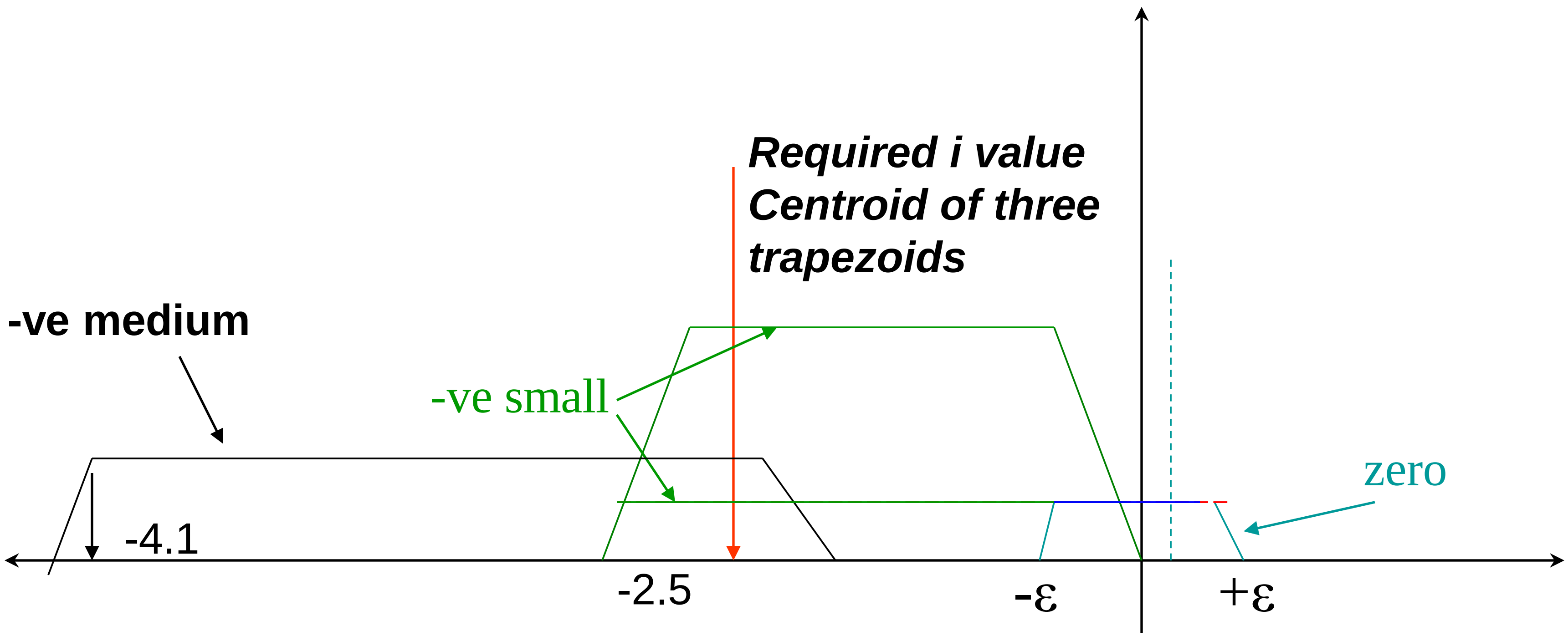
Finding *i*



Possible candidates:

i=0.5 and -0.5 from the "zero" profile and μ=0.3
i=-0.1 and -2.5 from the "-ve-small" profile and μ=0.3
i=-1.7 and -4.1 from the "-ve-small" profile and μ=0.3

Defuzzification: Finding *i* by the *centroid* method



Possible candidates:

i is the x-coord of the centroid of the areas given by the blue trapezium, the green trapeziums and the black trapezium