Sets, Relations, Functions

Q1. Prove that if A is any finite set, then any function f from A to A is 1-1 if and only if it is onto. Use induction on the number of elements in A. Give examples of a 1-1 function from A to A that is not onto, and also an onto function from A to A that is not 1-1, for some set A. Prove that every set is either finite or there exists a 1-1 function f from N to A.

Q2. Prove that the set of all finite subsets of N has the same cardinality as N. Find an explicit bijection between these sets. In other words, given a finite subset of natural numbers, show how to compute a natural number from it, so that the original set can be recovered uniquely from the computed number. Do the same for all finite sequences of natural numbers. Note that there is no fixed bound on the size of the finite subset or the length of the finite sequence. All that is known is that it is some natural number.

Q3. Let $A_1, A_2, \ldots$ be an infinite sequence of sets such that the intersection of any finite number of sets in the sequence is not empty. Is it true that there exists an element x such that $x \in A_i$ for all i? If so prove it, else give an example for which it is false. Suppose $a_1, a_2, \ldots$ is an infinite sequence of numbers such that the *gcd* of any finite subsequence of numbers in the sequence is greater than 1. Prove that there exists a prime p such that p divides a_i for all i.

Q4. Let R be relation from a set A to itself. Let R^k be defined inductively by $R^0 = I$ and $R^k = R \cdot R^{k-1}$. Suppose there exists a number $k > 0$ such that R^k is the identity relation. Prove that R must be a bijection. Prove that if R is a bijection and A is finite, there exists a $k > 0$ such that $R^k = I$. Give an example to show that this may not be true if A is infinite. For a finite set A with n elements, what is the smallest number k such that for every bijection R from A to A, $R^k = I$. Prove your answer.

Q5. Let R_1, R_2, R_3 be relations on a set A. Prove or disprove the following statements.

(i) $(R_1 \cdot R_2)^{-1} = R_2^{-1} \cdot R_1^{-1}$.

(ii) $(R_1 \cup R_2) \cdot R_3 = (R_1 \cdot R_3) \cup (R_2 \cdot R_3)$.

(iii) $(R_1 \cap R_2) \cdot R_3 = (R_1 \cdot R_3) \cap (R_2 \cdot R_3)$.

(iv) $R_1 \cdot R_2$ is a function if and only if both R_1 and R_2 are functions.