CS 215- Data Interpretation and Analysis (Post Midsem)

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Lecture-3

Towards Hypothesis Testing

12oct23

Recap

A Practical Problem

- A bridge is being built. The weight it can tolerate has a distribution with $\mu=400$ and $\sigma=40$. A car that goes on the bridge has weight distribution given by $\mu=3$ and $\sigma=0.3$. We want the probability of damage to the bridge to be less than 0.1. How many cars can we allow to go on the bridge?

Another Problem

- We have to estimate the percentage of sand grains in a pile of sand resulting from the fragmentation of a mineral compound which fall in a particular range.

Integration of Normal PDF cumbersome: Use Standard Normal Form Table

$$P(Z < V) = \int_{-\infty}^{V} \frac{1}{\sqrt{2\pi}} \exp\left(-y^2/2\right) dy$$

- Difficult to find an algebraic closed form expression
- Values found numerically
- Tabulated in standard normal form tables
- Z-scores

Z-score table

Standard Normal Probabilities

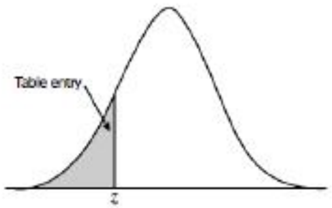


Table entry for z is the area under the standard normal curve to the left of z.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
−0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

Standard Normal Probabilities

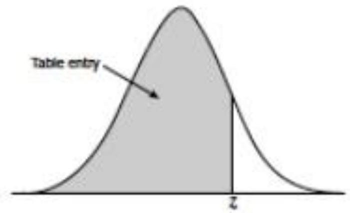


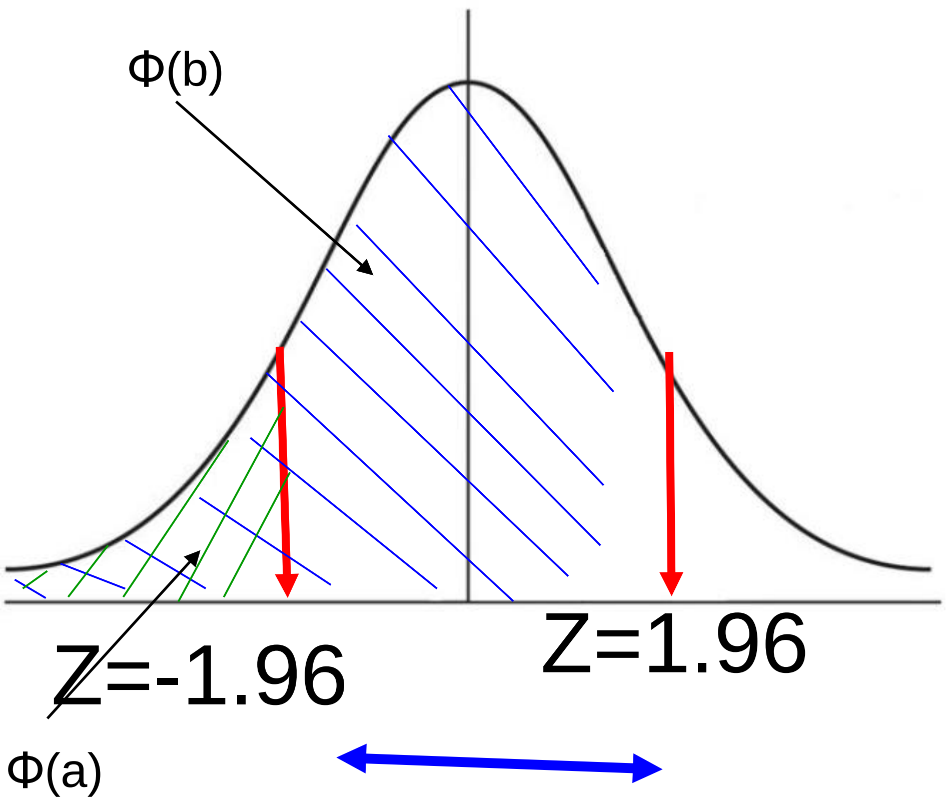
Table entry for z is the area under the standard normal curve to the left of z.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

How to use Z-score table

- Read off values, adding row heading and column heading
- Lets verify that 95% of the area under the normal curve lies within +-1.96σ, **plus and minus 1.96 times standard deviation**
- The so called 95% confidence interval
- Often used in “Hypothesis Testing”

$\Phi(1.96)-\Phi(-1.96)=0.95$



-1.96 <= Z <= +1.96

- Area under the curve from *-1.96* to +*1.96* is the probability of Z being in interval

$$P\left(-1.96 <= Z <= +1\right.$$

The importance of 95% confidence interval is supreme !

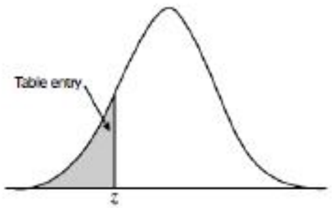
- Often used backward
- We know that Z=+-1.96 in Z=z, that gives me 95% confidence interval

$$Z = \frac{X - \mu}{\sigma}$$

- But
- This gives me values of X, mean, standard deviation if 2 out of 3 quantities are known

Z-score table

Standard Normal Probabilities



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−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
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−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
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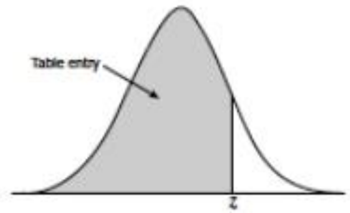
Standard Normal Probabilities



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0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
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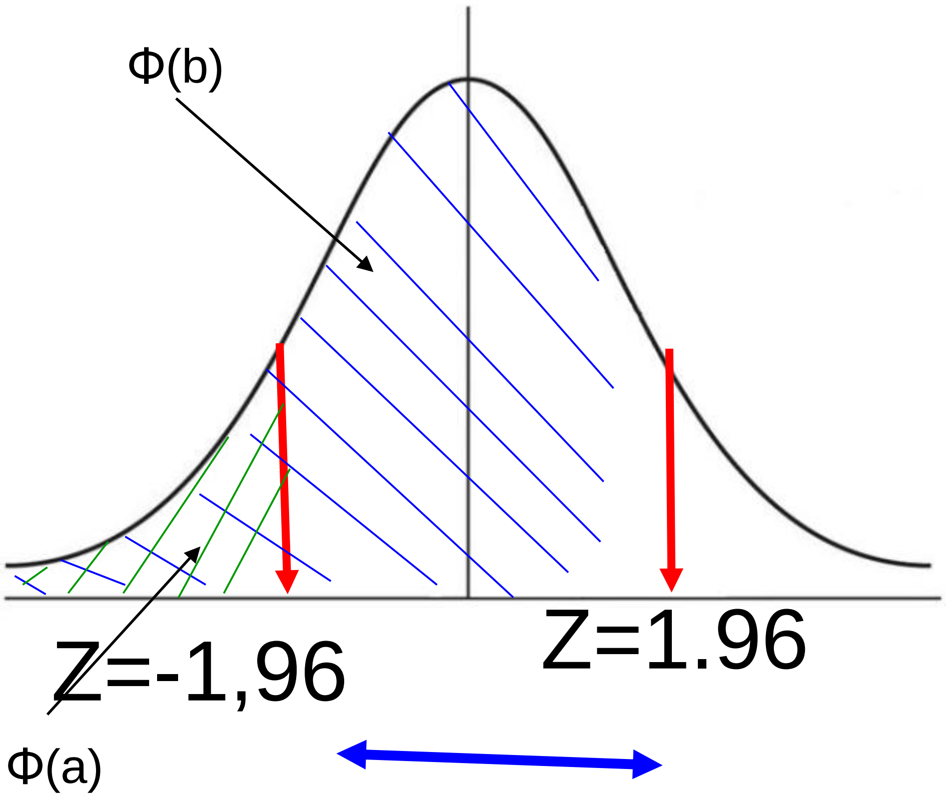
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

The 95% confidence interval



-1.96 <= Z <= +1.96

- $\Phi(1.96)=0.9750$
- By symmetry

$\Phi(-1.96)= 1-\Phi(1.96)$

$\rightarrow \Phi(1.96)-\Phi(-1.96)= 2.\Phi(1.96)-1= 2 \text{ X } 0.975-1.0=1.95-1.0=0.95$

End recap

Interval Estimate

Sample Mean and Population Mean

- $X_1, X_2, X_3, \ldots, X_n$ is a sample from a normal distribution having unknown mean μ and known variance σ^2.
- Maximum likelihood point estimator of μ is

$$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$$

$\bar{X}$

- We know that $\bar{X}$ is normally distributed with mean μ and known standard deviation $\sigma/\sqrt{n}$
- So the following is standard normal distribution:

$$\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}$$

95% confidence interval

$$P\left[-1.96 < \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} < 1.96\right] = 0.95$$

$$\Rightarrow P\left[-1.96\frac{\sigma}{\sqrt{n}} < \bar{X}-\mu < 1.96\frac{\sigma}{\sqrt{n}}\right] = 0.95$$

$$\Rightarrow P\left[\bar{X}-1.96\frac{\sigma}{\sqrt{n}} < \mu < \bar{X}+1.96\frac{\sigma}{\sqrt{n}}\right] = 0.95$$

A Manufacturing situation

Suppose that a machine part manufacturer has made parts with the dimensions as given below:

(a) 9 pieces of the machine part

(b) dimensions are respectively

5, 8.5, 12, 15, 7, 9, 7.5, 6.5, 10.5

Suppose somehow it is known that IF the parts could be measured on the whole population sample by sample, the variance of the measurement of dimensions would be 4

(*artificial? Yes, but useful for concept building*)

95% confidence interval for μ

5+8.5+12+15+7+9+7.5+6.5+10.5=81

$\bar{X}$=81/9=9

It follows that under the assumption that the values are independent, a 95% confidence interval for μ is

[9-1.96.(2/3), 9+1.96.(2/3)]

=**(7.6, 10.31)**

Interpretation of the observation

Based on the

(a) observation (9 samples)

(b) knowledge obtained somehow that variance is 4

We reach the 95% confidence interval as (7.69, 10.31)

Qualitatively

- If the manufacturer says, “I can ensure a part dim of 15”, we cannot trust!!

- If the maker says, “I assure dim of 10”, we CAN trust

- Trust with 95% confidence

Two sided and one sided confidence intervals

- What we saw is 2 sided confidence interval
- Similarly, one sided upper and lower confidence intervals

95% one sided intervals

- Upper

$$\left(\bar{X} - 1.645\frac{\sigma}{\sqrt{n}}, \infty\right)$$

- Lower

$$\left(-\infty, \bar{X} + 1.645\frac{\sigma}{\sqrt{n}}\right)$$

Towards test of hypothesis

Problem statement *(Sheldon M. Ross, PSES, 2004)*

All cigarettes presently on the market have an average nicotine content of at least 1.6mg per cigarette. A firm that produces cigarettes claims that it has discovered a new way to cure tobacco leaves that will result in the average nicotine content of a cigarette being less than 1.6 mg. To test this claim, a sample of 20 of the firms cigarettes were analysed. If it is known that the standard deviation of a cigarette’s nicotine content is 0.8 mg., what conclusions can be drawn at the 5% level of significance if the average nicotine content of the 20 cigarettes is 1.54?

95% one sided intervals

- Upper

$$\left(\bar{X}-1.645\frac{\sigma}{\sqrt{n}},\infty\right)=\left(1.54-1.645\frac{0.8}{\sqrt{20}},\infty\right)=\left(1.24,\infty\right)$$

- Lower

$$\left(-\infty,\bar{X}+1.645\frac{\sigma}{\sqrt{n}}\right)=\left(-\infty,1.54+1.645\frac{0.8}{\sqrt{20}}\right)=\left(-\infty,1.84\right)$$

Conclusion from the analysis

1. There is 95% probability that the nicotine content will lie in the range $(-\infty, 1.84)$

2. There is 95% probability that the nicotine content will lie in the range $(1.24, +\infty)$

Both these observations shake our confidence in the claim of the manufacturer that the nicotine content is less that 1.6mg/cigarette.

Statement and proof of CLT

First we state/prove some lemma

Expectation and Variance of sum of random variables

- Let $X_1, X_2, X_3, \ldots, X_n$ be n independent random variables
- Let

$$S_n = X_1 + X_2 + X_3, \ldots + X_n$$

- Then,

$$E(S_n) = E(X_1) + E(X_2) + E(X_3) \ldots + E(X_n)$$

- *and,*

$$Var(S_n) = Var(X_1) + Var(X_2) + Var(X_3) \ldots + Var(X_n)$$

Proofs related to Variance

$$\sigma^2 = E[X - E(X)]^2$$

$$= E(X - \mu]^2$$

$$= E[X^2 - 2.X.\mu + \mu^2]$$

$$= E(X^2) - E(2.X.\mu) + E(\mu^2)$$

$$= E(X^2) - 2.\mu.E(X) + \mu^2$$

$$= E(X^2) - 2.\mu.\mu + \mu^2$$

$$= E(X^2) - \mu^2$$

$$= E(X^2) - [E(X)]^2$$

Proof wrt Variance of sum

$Variance = E[X - E(X)]^2$

$$Var(X_1+X_2)$$
$$=E[(X_1+X_2)^2] - [E(X_1+X_2)]^2$$
$$=E[X_1^2+2.X_1.X_2+X_2^2] - [E(X_1)+E(X_2)]^2$$
$$=E[X_1^2]+2.E[X_1].\ E[X_2]+E[X_2^2] - [E(X_1)^2+2.E(X_1).E(X_2)+E(X_2)^2]$$
$$=[E(X_1^2)-E(X_1)^2] + [E(X_2^2)-E(X_2)^2]$$
$$=Var(X_1) + Var(X_2)$$

Variance of subtraction of R.V.s

$$Var(X_1-X_2)$$

$$=E[(X_1-X_2)^2] - [E(X_1-X_2)]^2$$

$$=E[X_1^2-2.X_1.X_2+X_2^2]-[E(X1)-E(X_2)]^2$$

$$=E[X_1^2]-2.E[X_1].\ E[X_2]+E[X_2^2]$$

$$-[E(X_1)^2-2.E(X_1).E(X_2)+E(X_2)^2]$$

$$=[E(X_1^2)-E(X_1)^2] + [E(X_2^2)-E(X_2)^2]$$

$$=Var(X_1) + Var(X_2)$$

Statement of Central Limit Theorem

- Let $X_1, X_2, X_3, \ldots, X_n$ be n independent random variables, each with mean μ and variance σ^2
- Also let

  $$S_n = X_1 + X_2 + X_3, \ldots + X_n$$

- Then,

  the following is **standard normal**

$$S_n^* = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

Mathematical adjustment

$$S_n^* = \frac{S_n - n\mu}{\sigma\sqrt{n}}, \ gives$$

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} = \frac{\frac{S_n}{n} - \mu}{\frac{\sigma\sqrt{n}}{n}} = \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

Hence, another equivalent statement of CLT

Let $X_1, X_2, X_3, \ldots, X_n$ be n independent random variables forming a sample from a population with mean μ and variance σ^2.

Then the sample mean is normally distributed with mean μ and variance σ^2/n.

MGF

Moment Generating Function

$M_X(t)=E(e^{tX})$,

X is a random Variable and

$f(x_j)=P(X=x_j)$

$$M_X(t)=\sum_{j=1}^{n} e^{tx_j} f(x_j)$$

for discrete distribution

$$M_X(t)=\int_{-\infty}^{+\infty} e^{tx} f(x)dx$$

for continuous distribution

Significance of MGF (1/2)

- The n^{th} derivatives of the MGF at $t=0$ gives the nth moment of the distribution of the random variable

- Thus the 1^{st} derivative at $t=0$ gives the mean of the distribution

- The 2^{nd} derivative at $t=0$ minus the 1^{st} derivative at $t=0$ gives the variance

Significance of MGF (2/2)

- Similarly the 3rd derivative at $t=0$ along with the lower derivatives (combined with proper operators) at $t=0$ gives the skewness, i.e., how symmetric is the about the mean

- 4th derivative at $t=0$ similarly can lead to the kurtosis, i.e., how heavily the tails of a distribution differ from the tails of a normal distribution

Proof regarding n^{th} derivative and n^{th} moment

$$
\begin{aligned}
M_X{}'(t) &= \frac{d}{dt}E(e^{tX})\\
&= E\left[\frac{d}{dt}(e^{tX})\right]\\
&= E[Xe^{tX}]\\
&= E(X)\\
&= M_X{}'(0)
\end{aligned}
$$

$$
\begin{aligned}
M_X{}''(t) &= \frac{d}{dt}M_X{}'(t)\\
&= \frac{d}{dt}E(Xe^{tX})\\
&= E\left[\frac{d}{dt}(Xe^{tX})\right]\\
&= E[X^2e^{tX}]\\
&= E(X^2);\quad at \quad t=0
\end{aligned}
$$

$$\therefore \operatorname{var}(X) = M_X{}''(0) - [M_X{}'(0)]^2$$

Uniqueness Theorem

- Suppose X and Y are random variables having moment generating functions $M_X(t)$ and $M_Y(t)$ respectively.
- Then X and Y have the same probability distribution if and only if $M_X(t)=M_Y(t)$ identically.

Standard Normal Distribution, *N(0,1)* and its PDF

Normal:

$$P(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(y-\mu)^2}{2\sigma} \right)$$

Standard normal:

$$P(Z = y) = \frac{1}{\sqrt{2\pi}} \exp\left(-y^2 / 2 \right)$$

MGF of *N(0,1)*

$$MGF = \int_{-\infty}^{+\infty} e^{ty}\,\frac{1}{\sqrt{2\pi}}e^{\left(-y^2/2\right)}dy$$

$$= \int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{\left(-y^2/2+ty\right)}dy$$

$$= \int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(y^2-2yt+t^2)}.e^{\frac{t^2}{2}}dy$$

$$= e^{\frac{t^2}{2}}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(y-t)^2}dy$$

$$= e^{\frac{t^2}{2}}$$

Proof of CLT

- To prove that

$$S_n^* = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

- Is standard normal, we will show that

$$M_{S_n^*}(t) = M_Z(t)$$

- i.e., the moment generating function of S_n* is equal to the moment generating function of standard normal r.v.

Proof: MGF

$$E(e^{tS_n^*}) = E[e^{t(S_n - n\mu)/\sigma\sqrt{n})}]$$

$$= E[e^{t(\sum_{i=1}^{n} X_i - n\mu)/\sigma\sqrt{n})}]$$

$$= E[e^{\sum_{i=1}^{n} t(X_i - \mu)/\sigma\sqrt{n})}]$$

$$= E[\prod_{i=1}^{n} (e^{t(X_i - \mu)/\sigma\sqrt{n})})]$$

$$= \prod_{i=1}^{n} E[e^{t(X_i - \mu)/\sigma\sqrt{n})}]$$

$$= \{E[e^{t(X_i - \mu)/\sigma\sqrt{n})}]\}^n$$

Proof: cntd.

$$\{E[e^{t(X_i-\mu)/\sigma\sqrt{n})}]\}^n$$

now,

$$e^{t(X_i-\mu)/\sigma\sqrt{n})}=[1+\frac{t(X_i-\mu)}{\sigma\sqrt{n}}+\frac{t^2(X_i-\mu)^2}{2\sigma^2 n}+\ldots],$$

by Taylor series exp*ansion*

Proof: working with *E*

$$E[1+\frac{t(X_i-\mu)}{\sigma\sqrt{n}}+\frac{t^2(X_i-\mu)^2}{\sigma^2 n}+...],$$

$$=E(1)+\frac{tE(X_i-\mu)}{\sigma\sqrt{n}}+\frac{t^2E(X_i-\mu)^2}{2\sigma^2 n}+...]$$

$$=1+0+\frac{t^2}{2n}+...$$

As n tends to infinity…

$$E(e^{tS_n^*}) = (1 + \frac{t^2}{2n} + \ldots)^n$$

$$Study\;\; L_n = (1 + \frac{t^2}{2n} + \ldots)^n,\; as\; n -> \infty$$

$$\log L_n = n \log(1 + \frac{t^2}{2n} + \ldots)$$

$$= \frac{\log(1 + \frac{t^2}{2n} + \ldots)}{1/n}$$

$$Both\;\; num\; and\;\; denom \to 0, \quad as\; n -> \infty$$

As n tends to infinity…

$$take\ derivative\ of\ \ numerator\ \ and$$

$$numerator\ \ as\ \ per\ \ L'Hospital\ \ rule$$

$$= \frac{\dfrac{(-\dfrac{t^2}{2n^2})}{(1+\dfrac{t^2}{2n}+\ldots)}}{-1/n^2} = \frac{\dfrac{t^2}{2}}{(1+\dfrac{t^2}{2n}+\ldots)}$$

$$= \frac{t^2}{2},\ as\ \ n \to \infty$$

$$same\ as\ the\ mgf\ \ of\ \ Z$$