Counting with Symmetries

Q1. Consider the problem of counting necklaces taking into account rotational symmetry. This can also be considered as counting the number of distinct ways of colouring the corners of a regular polygon with n sides, where each corner is assigned one of k colours. Suppose $k = 2$ and the colours are black and white. Find the polynomial $N_n(b)$ in one variable b such that the coefficient of b^k is the number of necklaces with exactly k corners coloured black. If rotational symmetry is not considered, this polynomial is just $(1 + b)^n$. Do the same if colourings obtained by reflection are also considered to be equivalent.

Q2. Consider the 3-dimensional unit cube whose corners are the points (x, y, z) where each of x, y, z is either 0 or 1. How many different ways of colouring the corners are there with at most k colours, if colourings that can be obtained by applying any rotations of the cube are considered to be equivalent? Do the same if reflections are also allowed. Also find the number of distinct ways of colouring the 12 edges and the 6 faces of the cube. In general, many different groups of bijections arise by considering symmetries of different geometrical objects.

Q3 Let A be a set with 3 elements and R_1 and R_2 relations defined on A. Suppose R_1 is equivalent to R_2 iff there exists a bijection f from A to A such that $(a_1, a_2) \in R_1$ if and only if $(f(a_1), f(a_2)) \in R_2$. In other words, the relation R_2 can be obtained from R_1 by simply relabeling the elements of A. Find the number of distinct relations on A. Do the same for relations from A to B where both A and B have 3 elements each. In this case R_1 is equivalent to R_2 iff there are bijections f from A to A and g from B to B such that $(a, b) \in R_1$ iff $(f(a), g(b)) \in R_2$. Try to extend to sets with n elements. The second problem is the same as finding the number of distinct $n \times n$ matrices with $0, 1$ entries, where two matrices are considered to be the same if one can be obtained from the other by permuting the rows and columns.

Q4 Consider the set of strings of length $2n$ over an alphabet that contains k letters. A string $a_0a_1 \ldots a_{2n-1}$ is equivalent to the string $b_0b_1 \ldots b_{2n-1}$ iff for all $0 \leq i < n$, either $a_{2i} = b_{2i}$ and $a_{2i+1} = b_{2i}$ or $a_{2i} = b_{2i+1}$ and $a_{2i+1} = b_{2i}$. In other words, two strings are equivalent if one can be obtained from the other by swapping a letter in an even position with the letter in the next position. Find the number of equivalence classes of this relation. Find the number of inequivalent strings in which a particular letter a appears exactly m times. Can you find this answer without using Burnside's Lemma?