CS213/293 Data Structure and Algorithms 2023

Lecture 9: Red-Black Trees

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Topic 9.1

Balance and rotation

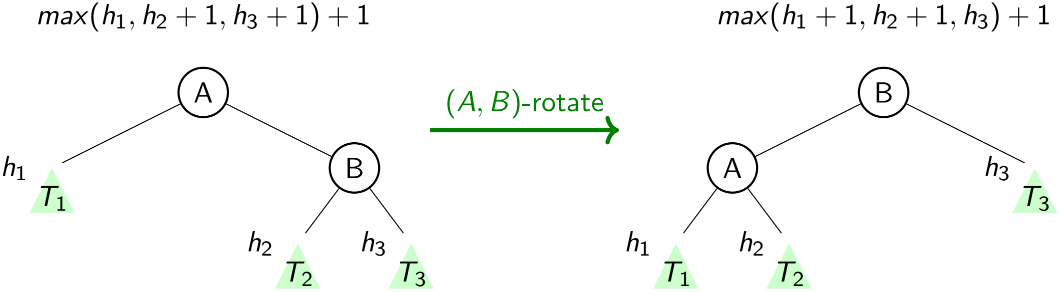
Maintain balance

BST may have a large height.

Height is directly related to branching. More branching implies a shorter height.

We call BST imbalanced when the difference between the left and right subtree height is large.

Balancing height by rotation



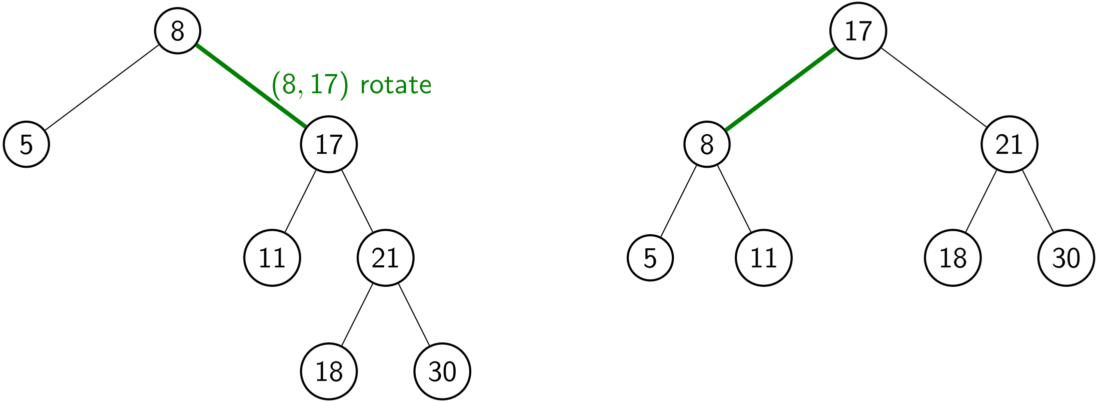
If $h_3 > h_2 = h_1$, if we rotate the BST, we will get a valid more balanced BST with less height.

Note that B could have been the left child. For this situation, we can define symmetric rotation.

Example: rotation

Example 9.1

In the following BST, we can rotate 8-17 *edge.*



When to rotate? Can rotation only fix imbalance?

Rotation is a local operation, which must be guided by non-local measure height.

We need a definition of balance such that rotations operations should be able to achieve the definition.

Design principle:
We minimize the number of rotations while allowing some imbalance.

Topic 9.2

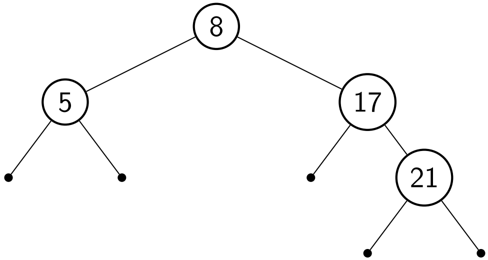
Red-black tree

Null leaves

To describe a red-black tree, we replace the null pointer with absent children by dummy null nodes.

Example 9.2

The following tiny nodes are the dummy null nodes.



Red-black tree

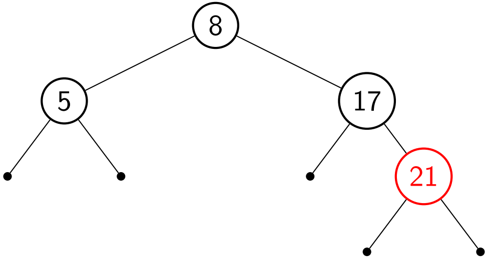
Definition 9.1

A red-black tree T is a binary search tree such that the following holds.

- *All nodes are colored either* **red** *or* **black**
- *Null leaves have no color*
- *Root is colored black*
- *All* **red** *nodes have* **black** *children*
- *All paths from the root to null leaves have the same number of* **black** *nodes.*

Example 9.3

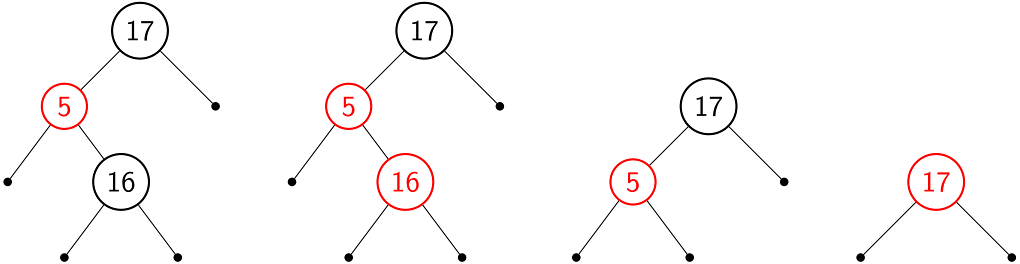
An example of a red-black tree.



Exercise: Identify red-black trees

Exercise 9.1

Which of the following are red-black trees?



Observations:

- Red nodes are not counted in the imbalance. We need them only when there is an imbalance.
- There cannot be too many red nodes. (Why?)
- Red nodes can be at every level except the root.

Black height

Definition 9.2

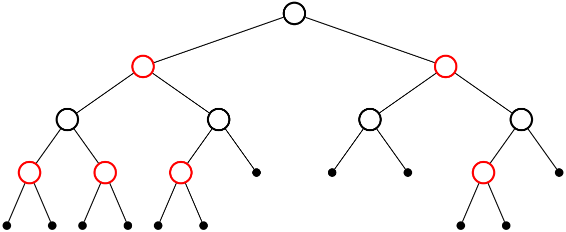
The black height (bh) for each node is defined as follows.

$$bh(n) = \begin{cases} 0 & n \text{ is a null leaf} \\ max(bh(right(n)), bh(left(n))) + 1 & n \text{ is a } \textbf{black} \text{ node} \\ max(bh(right(n)), bh(left(n))) & n \text{ is a red node} \end{cases}$$

Example: black height

Example 9.4

The black height of the following red-black tree is 2.



Exercise 9.2

Can we change the color of some nodes without breaking the conditions of a red-black tree?

Bound on the height of a red-black tree

Let h be the black height of a red-black tree containing n nodes.

- n is the smallest when all nodes are **black**. Therefore, the tree is a complete binary tree. Therefore, $n = 2^h - 1$.
- n is largest when the alternate levels of the tree are red. The height of the tree is $2h$. Therefore, $n = 2^{4h} - 1$.

$$\log_4 n < h < 1 + \log_2 n$$

Search, Maximum, and Successor/Predecessor

We can run search, maximum, and successor/predecessor on the red-black tree as usual.

Their running time will be $O(\log n)$ because $h < 1 + \log_2 n$.

The question is how do we do insertion and deletion on a red-black tree?

Topic 9.3

Insertion in red-black tree

BST insertion in red-black tree

1. Follow the usual procedure of insertion in the BST, which inserts the new node n as a leaf.
   - Note that there are dummy nodes in the red-black tree. n is inserted as the parent of a dummy node.

2. We color n red.

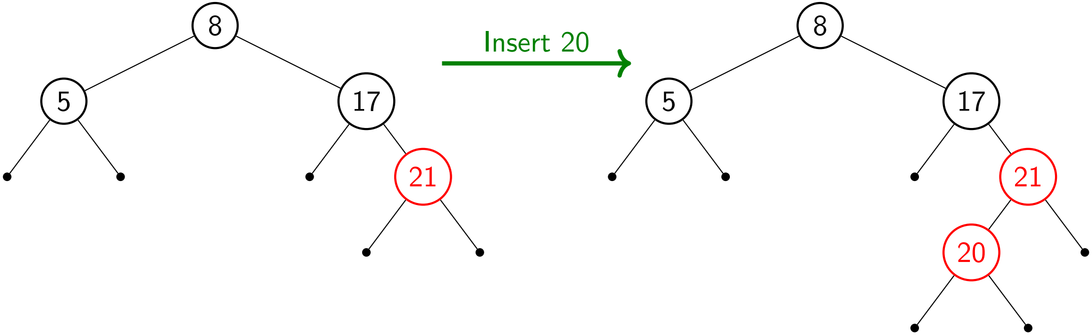
- Good news: No change in the black height of the tree.
- Bad news: n may have a *red* parent.

Commentary: We have null nodes in this setting. We need to add nulls as children to the new node.

Example: insert in red-black tree

Example 9.5

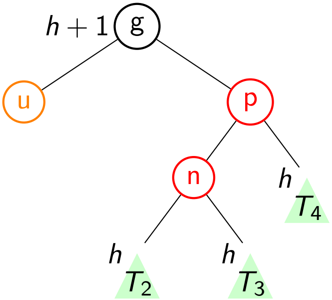
Inserting 20 *in the following tree.*



The insertion results in violation of the conditions of red-black tree that red nodes can only have black children.

Red-red violation

After insertion, we may have a **red**-**red** violation, where a **red** node has a **red** child. Orange color means that we need to consider all possible colors of the nodes.



If n has a **red** parent, we correct the error either by rotation or re-coloring.

We have three cases.

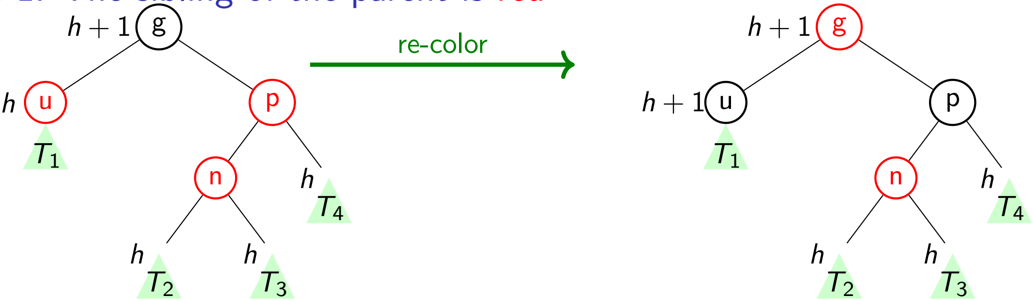
- Case 1: u is **red**
- Case 2: u is not **red** and g to n path is not straight
- Case 3: u is not **red** and g to n path is straight

No transformation should change the black height of g.

Exercise 9.3

Why g must exist and be black?

Case 1: The sibling of the parent is red



In the subtree of g, we have no change in the black height, and in the subtree, there is no red-red error.

Now g is red. We have three possibilities: the parent of g is **black**, the parent of g is **red**, and g is the root.

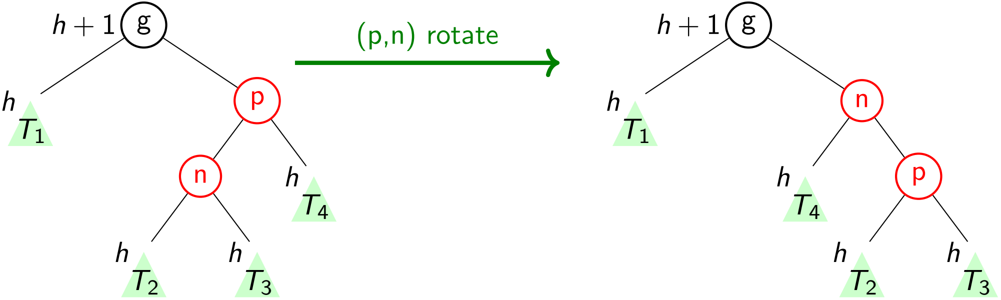
Commentary: Possibility 1: Nothing. Possibility 2: We have a red-red violation a level up and need to apply the transformations there. Possibility 3: turn g back to black.

Exercise 9.4

What do we do in each case?

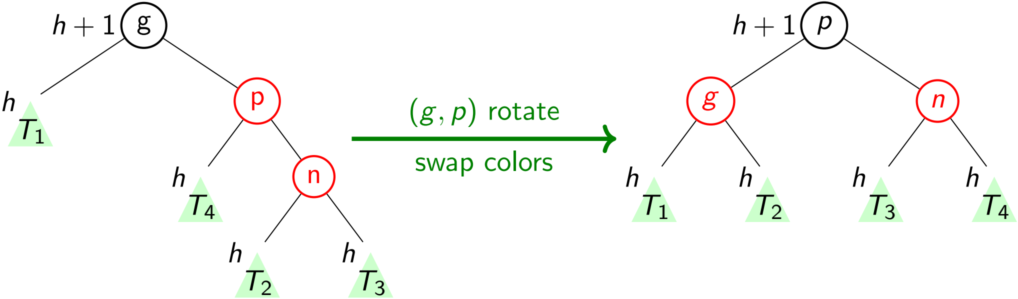
Case 2: The sibling of the parent is not red and the path to the grandparent is not straight

straight means $left^2(parent^2(n)) = n$ or $right^2(parent^2(n)) = n$



This transformation does not resolve the violation but converts the violation to case 3.

Case 3: The sibling of the parent is not red and the path to the grandparent is straight



The transformation removes the red-red violation.

Exercise 9.5

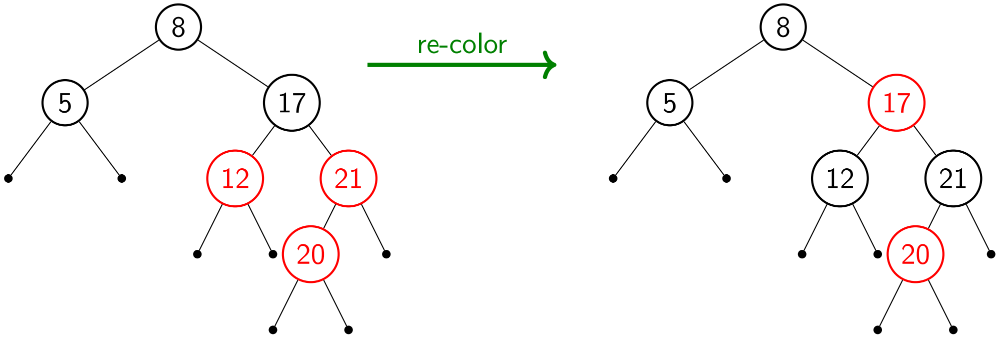
a. *Why roots of T_2, T_3, and T_4 are* **black***?*

b. *Show that if the root of T_1 is red then the above operation does not work.*

Example: red-red correction case 1

Example 9.6

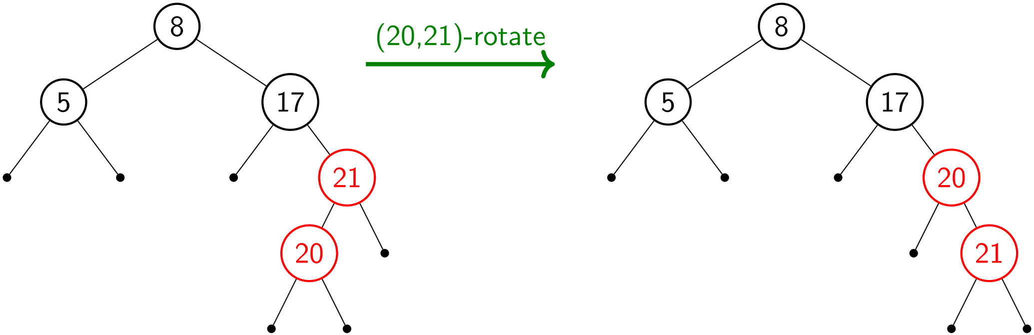
We just inserted 20 *in the following tree. We need to apply case 1 to obtain a red-black tree.*



Example: red-red correction case 2

Example 9.7

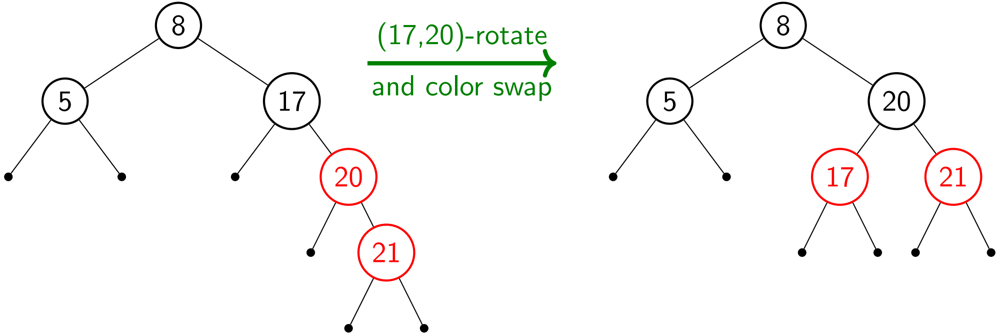
Consider the following example. We are attempting to insert 20. We apply case 2 to move towards a red-black tree.



The above is not a red-black tree. We need to further apply case 3 to finally obtain the red-black tree.

Example: red-red correction case 3 (continued)

We apply case three as follows.



Summary of insertion

1. Insert like BST and assign **red** color to the new node.
2. While we have case 1, re-color nodes and move up the **red**-**red** violation.
3. If we find case 2 or 3, we rotate and the violation is finished.
4. If the root becomes **red** in the process, then turn it back to black.

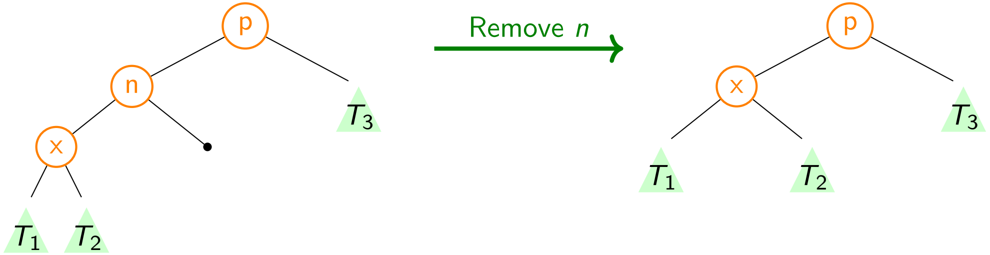
We may never find case 2/3.

Topic 9.4

Deletion red-black tree

BST deletion in red-black tree

- Delete a node as if it is a binary search tree.
- Recall: In the BST deletion we always delete a node n that has at most one non-null child.



x can be either a null or non-null node.

What can go wrong with a red-black tree?

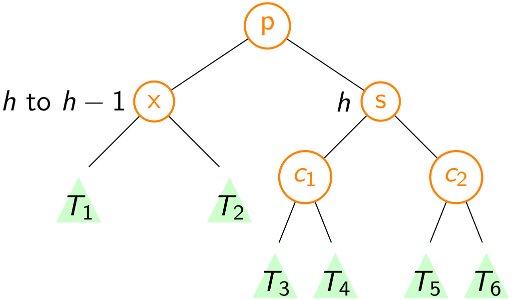
Since a child x of n takes the role of the node, we need to check if x can replace n.

- If n was **red**, no violations occur. (Why?)
- if n was **black**, $\underbrace{bh(x) = bh(n) - 1}_{\text{black height violation}}$, or it is possible that $\underbrace{\text{both } x \text{ and } parent(x) \text{ are}}_{\text{red-red violation}}$ **red**.

The leaves of the subtree rooted at x will have one less black depth.

Violation pattern

After deletion, we may need to consider the following five nodes around x.



We correct the violation either by rotation or re-coloring.

There are five cases

1. x is **red**
2. x is **black** and s is **red**
3. x, s, c_1, and c_2 are **black**
4. x, s, and c_2 are **black** and c_1 is **red**
5. x and s are **black** and c_2 is **red**

Exercise 9.6

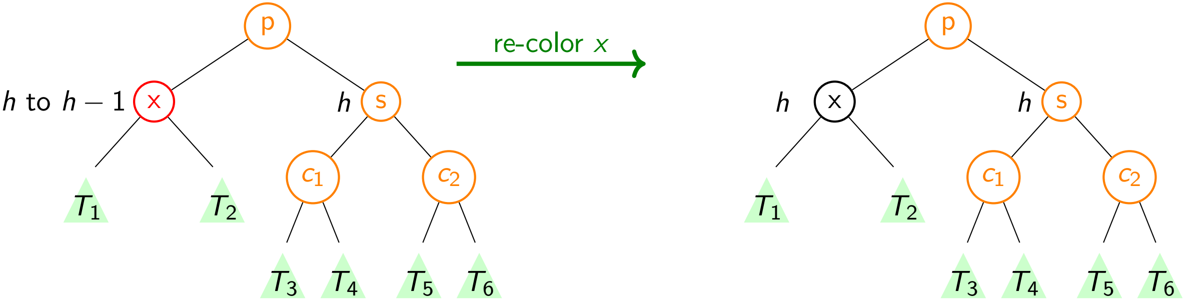
a. If p does not exist, what should we do?

b. How many cases to consider? Is it 32?

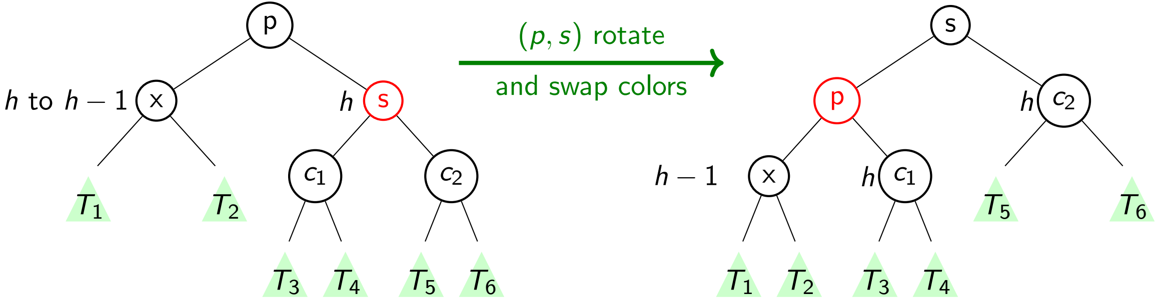
The goal is to restore the black height of p.

Commentary: If p does not exist, x is the root. If x is red, we change x to black. We are done.

Case 1: x is red



Case 2: x is **black** and the sibling of x is **red**

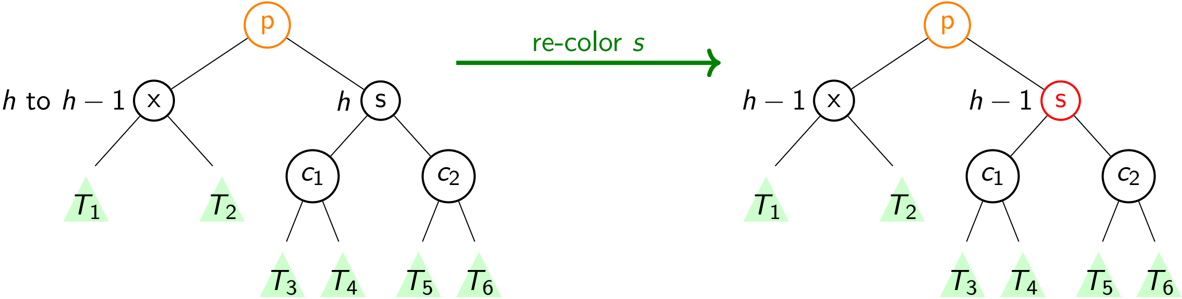


The transformation does not solve the height violation at parent of x but changes the sibling of x from **red** to **black**.

Exercise 9.7

Why p, c_1, and c_2 must be black?

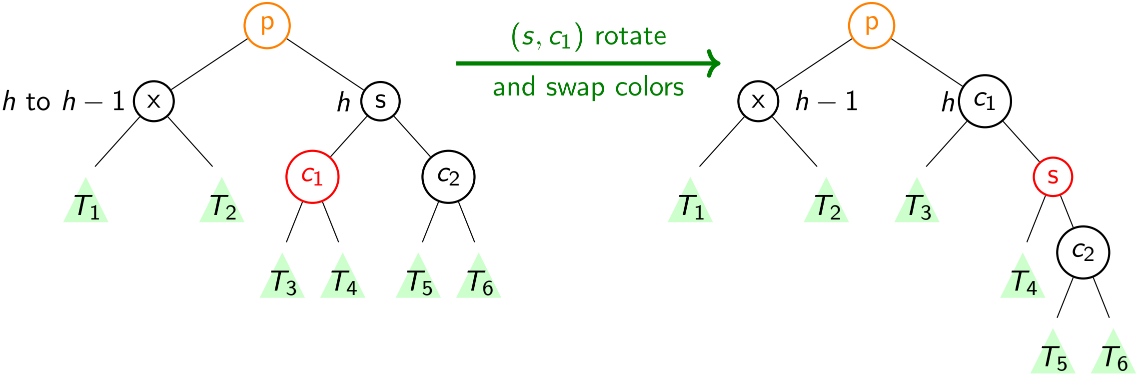
Case 3: x and the sibling of x are **black** and both children of the sibling are **black**



The above transformation reduces $bh(p)$ by 1 and the violation is moved to the lower level.

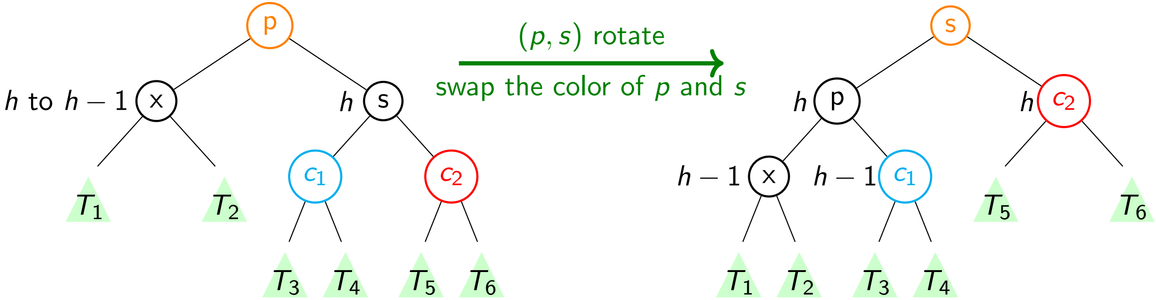
The only case that kicks the can upstairs!!

Case 4: x and the sibling of x are **black** and the left and right child of the sibling are **red** and **black** respectively



The above transformation does not solve the height violation. It changes the right child of the sibling from **red** to **black**, which is the case 5.

Case 5: x and sibling of x are **black** and the right child of the sibling is **red**



The above transformation solves the black height violation. All cases are covered.

Exercise 9.8

Why case 5 transformation cannot be applied to case 4?

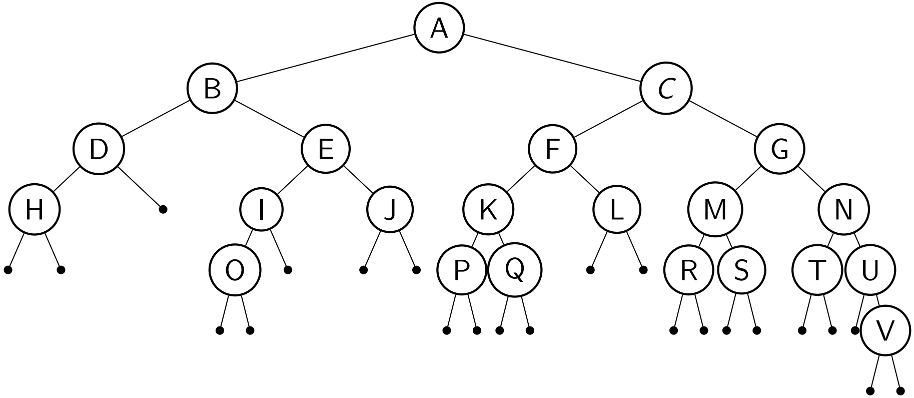
Topic 9.5

Problems

Insert and delete

Exercise 9.9

Consider the tree below. Can it be colored and turned into a red-black tree? If we wish to store the set 1, . . . , 22, label each node with the correct number. Now add 23 to the set and then delete 1. Also do the same in the reverse order. Are the answers the same? When will the answers be the same?



Topic 9.6

Extra slides: AVL trees

AVL (Adelson, Velsky, and Landis) tree

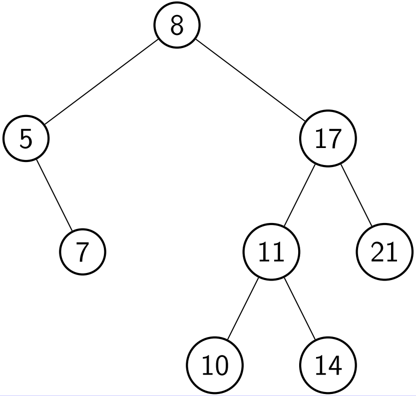
Definition 9.3

An AVL tree is a binary search tree such that for each node n

$$|height(right(n)) - height(left(n))| \leq 1.$$

Example 9.8

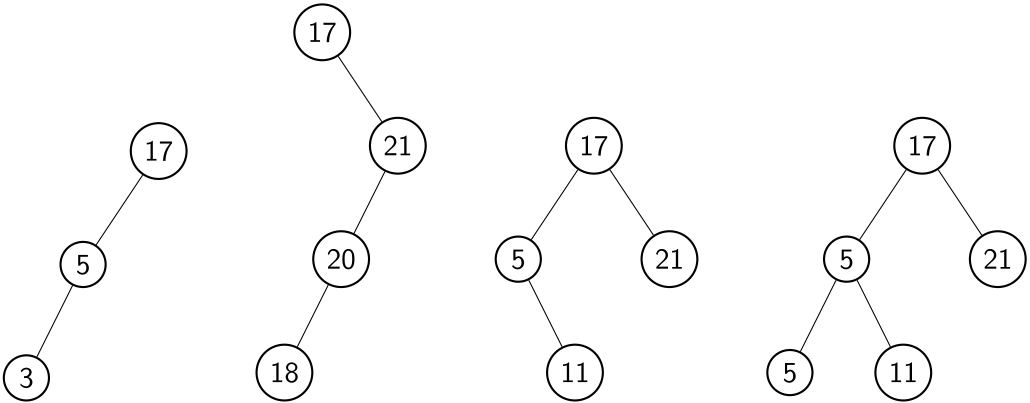
An example of an AVL tree.



Exercise: Identify the AVL trees

Exercise 9.10

Which of the following are AVL trees?



Topic 9.7

Height of AVL tree

AVL tree height

Theorem 9.1

The height of an AVL tree T having n nodes is $O(\log n)$.

Proof.

Let $n(h)$ be the minimum number of nodes for height h.

Base case:

$n(1) = 1$ and $n(2) = 2$.

Induction step:

Consider an AVL tree with height $h \geq 3$. In the minimum case, one child will have height $n - 1$ and the other $n - 2$. (Why?)

Therefore, $n(h) = 1 + n(h-1) + n(h-2)$. ...

Commentary: We need to show that $n(h) > n(h-1)$ is monotonous. Ideally, $n(h) = 1 + n(h-1) + min(n(h-2), n(h-1))$. This proves that $n(h) > n(h-1)$.

AVL tree height(2)

Proof(continued.)

Since $n(h-1) > n(h-2)$,

$$n(h) > 2n(h-2).$$

Therefore,

$$n(h) > 2^i n(h-2i).$$

For $i = h/2 - 1$(Why?),

$$n(h) > 2^{h/2-1} n(2) = 2^{h/2}.$$

Commentary: Here is the explanation of the last step. Consider an AVL tree with m nodes and h height. By definition, $h(n) \leq m$. Since $h < 2\log n(h)$, $h < 2\log m$. Therefore, h is $O(\log m)$.

Therefore,

$$h < 2\log n(h).$$

Therefore, the height of an AVL tree is $O(\log n)$. (Why?) □

Closest leaf

Theorem 9.2

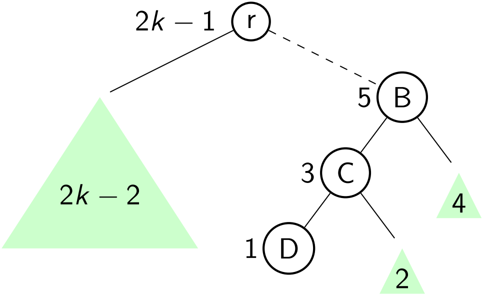
Let T be an AVL tree. Let the level of the closest leaf to the root of T is k.

$$height(T) \leq 2k - 1$$

Proof.

Let D be the closest leaf of the tree.

- The height of $right(C)$ cannot be more than 2. (Why?)
- Therefore, the maximum height of C is 3.
- Therefore, the maximum height of $right(B)$ is 4.
- Therefore, the maximum height of B is 5.
- Continuing the argument, the maximum height of root r is $2k - 1$.



□

A part of AVL is a complete tree

Theorem 9.3

Let T be an AVL tree. Let the level of the closest leaf to the root of T is k. Upto level $k-2$ all nodes have two children.

Proof.

A node at level $k-2-i$ cannot be a leaf. (Why?)

Let us assume that a node n at level $k-2-i$ has a single child n'.

The height of n' cannot be more than 1. (Why?)

Therefore, n' is a leaf. Contradiction. □

Exercise 9.11

Show T has at least 2^{k-1} nodes.

Another proof of tree height bound

Let T have n nodes and the height of T be h.

We know the following from the previous theorems.

- $n \geq 2^{k-1}$, and
- $2k - 1 \geq h$.

Therefore,

$$n \geq 2^{k-1} \geq 2^{(h-1)/2}$$

Exercise 9.12

What is the maximum number of nodes given height h?

Problem: A sharper bound for the AVL tree

Exercise 9.13

a. Find largest c such that $c^{k-2} + c^{k-1} \geq c^k$

b. Recall $n(h) = 1 + n(h-1) + n(h-2)$. Let c_0 be the largest c. Show that $n(h) \geq c^{h-1}$.

c. Prove that the above bound is a sharper bound than our earlier proof.

Topic 9.8

Insertion and deletion

End of Lecture 9