CS213/293 Data Structure and Algorithms 2023

Lecture 15: Graphs - basics

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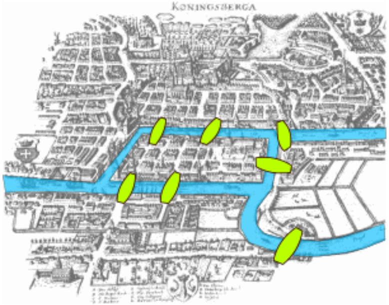
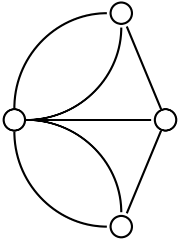
Problem of Konigsberg's bridges

Problem: find a walk through the city that would cross each of those bridges once and only once.



(Source: Wikipedia)

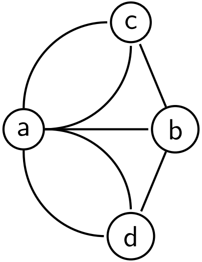


We may view the problem as visiting all nodes without repeating an edge in the above graph.

The first graph theory problem. Euler gave the solution!

Graphs

A graph has vertices (also known as nodes) and vertices are connected via edges.



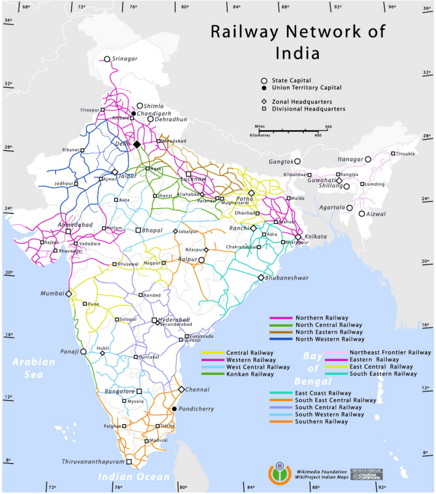
The above is a graph $G = (V, E)$, where

$V = \{a, b, c, d\}$ and

E is a multiset.

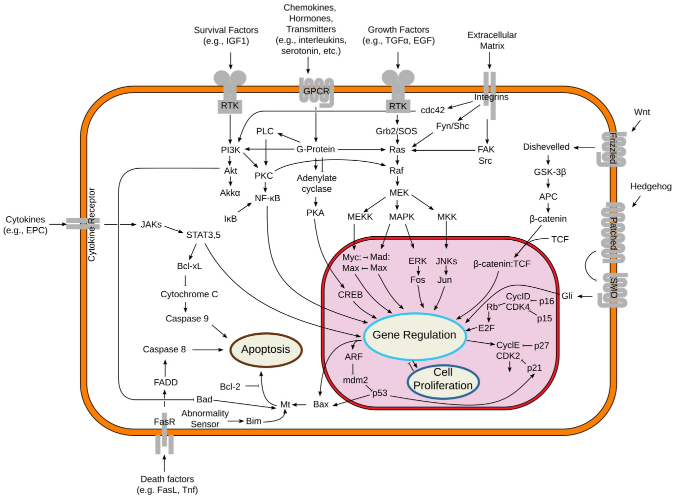
$E = \{\{a, b\}, \{a, c\}, \{a, c\}, \{a, d\}, \{a, d\}, \{b, c\}, \{b, d\}\}$.

Example: graphs are everywhere



(Source: Internet)

Example: graphs are everywhere (2)



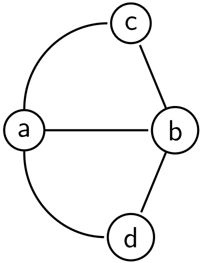
(Source: Wikipedia)

Formal definition

Definition 15.1

A graph $G = (V, E)$ is consists of

- *set of vertices V and*
- *set of edges E is a set of unordered pairs of elements of V.*



Commentary: In the bridge example, E was a multiset and here E is a set. If we want to support multiset, we can define $E \subseteq \mathit{unorderedPairs}(V) \times \mathbb{N}$, which is a natural extension of the above definition. $\mathit{unorderedPairs}(V) = \{\{a, b\} | a, b \in V \wedge a \neq b\}$

Topic 15.1

Basic Terminology

Adjacency and degree

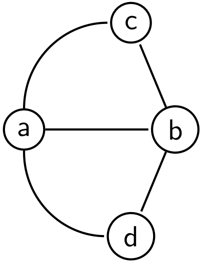
Consider a graph $G = (V, E)$.

Definition 15.2
Let $\textit{adjacent}(v) = \{v' | \{v, v'\} \in E\}$.

Definition 15.3
Let $\textit{degree}(v) = |\textit{adjacent}(v)|$.

Exercise 15.1
a. What is $\sum_{v \in V} \textit{degree}(v)$*?*
b. Is $\{v, v\} \in E$ *possible?*

Example 15.1



$\textit{adjacent}(a) = \{c, b, d\}$ and $\textit{adjacent}(d) = \{a, b\}$.

$\textit{degree}(a) = 3$ and $\textit{degree}(d) = 2$.

Commentary: $\sum_{v \in V} \textit{degree}(v) = 2|E|$

Paths, simple paths, and cycles

Consider a graph $G = (V, E)$.

Definition 15.4

A path is a sequence of vertices $v_1, ..., v_n$ such that $v_i, v_{i+1} \in E$ for each $i \in [1, n)$.

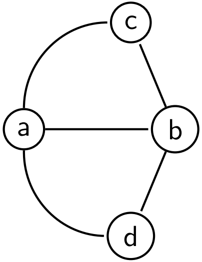
Definition 15.5

A simple path is a path $v_1, ..., v_n$ such that $v_i \neq v_j$ for each $i < j \in [1, n]$.

Definition 15.6

A a cycle is a path $v_1, ..., v_n$ such that $v_1, ..., v_{n-1}$ is a simple path and $v_1 = v_n$.

Example 15.2



abcad is a path but not a simple path.

abd is a simple path.

abda is a cycle.

Subgraph

Consider a graph $G = (V, E)$.

Definition 15.7

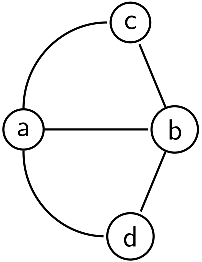
A graph $G' = (V', E')$ is a subgraph of G if $V' \subseteq V$ and $E' \subseteq E$.

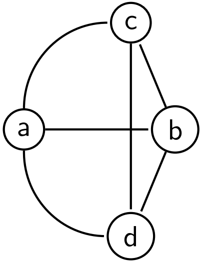
Definition 15.8

For a set of vertices V', let $G - V'$ be $(V - V', \{e|e \in E \wedge e \subseteq V - V'\})$.

Example 15.3

The left graph is a subgraph of the right graph.





Connected graph

Consider a graph $G = (V, E)$.

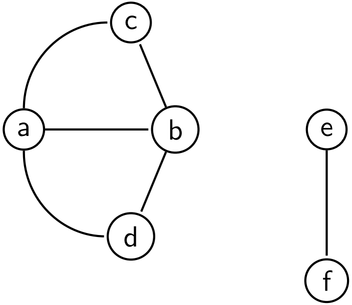
Definition 15.9

G is connected if for each $v, v' \in V$ there is a path $v, ..., v'$ in E.

Definition 15.10

A graph G' is a connected component of G if G' is a maximal connected subgraph of G.

Example 15.4



The above is not a connected graph.

The above has two connected components.

Complete graph

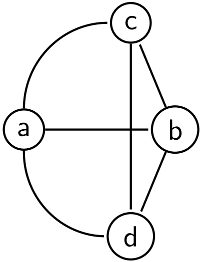
Consider a graph $G = (V, E)$.

Definition 15.11

G is a complete graph if for all pairs $v_1, v_2 \in V$

- *if $v_1 \neq v_2$, $v_1 \in adjacent(v_2)$, and*
- *if $v_1 = v_2$, $v_1 \notin adjacent(v_1)$.*

Example 15.5



Exercise 15.2

If $|V| = n$, how many edges does a complete graph have?

Topic 15.2

Tree (a new non-recursive definition of tree)

Tree

Consider a graph $G = (V, E)$.

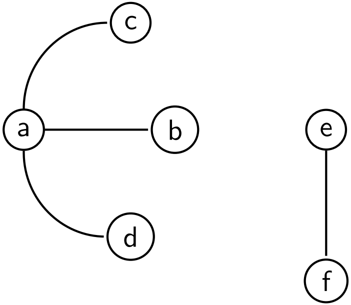
Definition 15.12
G is a tree if G is connected and has no cycles.

Definition 15.13
G is a forest if G is a disjoint union of trees.

Definition 15.14
$G = (V, E, v)$ is a rooted tree if (V, E) is a tree and $v \in V$ is called root.

The trees in the earlier lectures are rooted tree.

Example 15.6



The above is a forest containing two trees.

Exercise 15.3
Which nodes of a tree can be selected for root?

Every tree has a leaf

Theorem 15.1

For a finite tree $G = (V, E)$ and $|V| > 1$, there is $v \in V$ such that $degree(v) = 1$.

Proof.

Since there are no cycles in G, there is a path $v_1, ..., v_n$ of G that cannot be extended at either ends (assuming finite graph).

Therefore, there must be two nodes such that $degree(v) = 1$.

□

Number of edges in a tree

Theorem 15.2
For a finite tree $G = (V, E)$, $|E| = |V| - 1$.

Proof.
Base case:
Let $|V| = 2$. We have $|E| = 1$.

Induction step:
Let $|V| = n + 1$.
Consider a leaf $v \in V$ and $\{v, v'\} \in E$.
Since $|adjacent(v) = 1|$ in G, $G - \{v\}$ is a tree.
Due to the induction hypothesis, $G - \{v\}$ has $|V| - 2$ edges.
Hence proved. □

Number of edges in a tree

Theorem 15.3

Let $G = (V, E)$ be a finite graph. If $|E| < |V| - 1$, G is not connected.

Proof.

Let us suppose there are cycles in the graph.

If we remove an edge from a cycle, it does not change the connectedness of any pairs of vertices. (Why?)

We keep removing such edges until no more cycles left.

Since $|E| < |V| - 1$, the remaining graph is not a tree. Therefore, G was not connected. $\square$

Commentary: The definition of

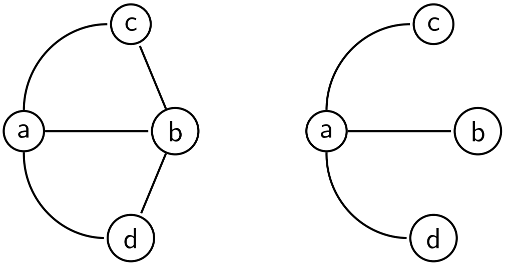
Spanning tree

Consider a graph $G = (V, E)$.

Definition 15.15

A spanning tree of G is a subgraph of G that is a tree and contains all vertices of G.

Example 15.7

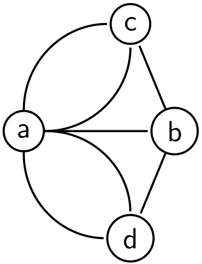


The right graph is the spanning tree of the left graph.

Topic 15.3

Multi-graph

Multi graph



Definition 15.16

A graph $G = (V, E)$ is consists of

- *set of vertices V and*
- *set of edges E is a* multiset *of unordered pairs of elements of V.*

The above is a graph $G = (V, E)$, where

$V = \{a, b, c, d\}$ and

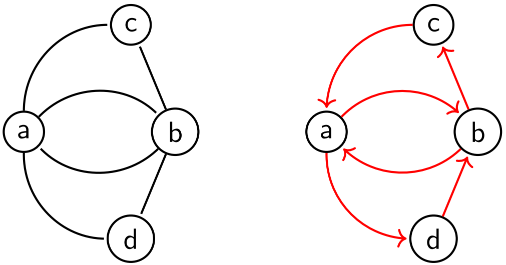
$E = \{\{a, b\}, \{a, c\}, \{a, c\}, \{a, d\}, \{a, d\}, \{b, c\}, \{b, d\}\}$.

Eulerian tour

Consider a graph $G = (V, E)$.

Definition 15.17
For a (multi)graph G, an Eulerian tour is a path that traverses every edge exactly once and returns to the same node.

Example 15.8



Eulerean path: cadbabc

Exercise 15.4
Why an Eulerian tour is not a cycle?

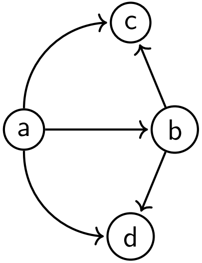
Theorem 15.4
A graph has an Eulerian tour if and only if all vertices have even degrees.

Proof.
Hint: Replace edges $\{v_1, v_2\}$ and $\{v_2, v_3\}$ by $\{v_1, v_3\}$. □

Topic 15.4

Directed graph

Directed graph



Definition 15.18

A graph $G = (V, E)$ is consists of

- *set of vertices V and*
- *set of edges $E \subseteq V \times V$.*

The above is a directed graph $G = (V, E)$, where

$V = \{a, b, c, d\}$ and

$E = \{(a, b), (a, c), (a, d), (b, c), (b, d)\}$.

Definition 15.19

A path *is a sequence of vertices $v_1, ..., v_n$ such that $(v_i, v_{i+1}) \in E$ for each $i \in [1, n)$.*

There is a path from a to d, but not d to a.

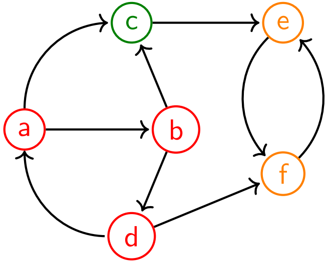
Strongly connected component (SCC)

Consider a directed graph $G = (V, E)$.

Definition 15.20
G is strongly connected if for each $v, v' \in V$ there is a path $v, ..., v'$ in E.

Definition 15.21
A graph G' is a strongly connected component (SCC) of G if G' is a maximal strongly connected subgraph of G.

Example 15.9



abd, c, and ef are SCCs.

SCC-Graph

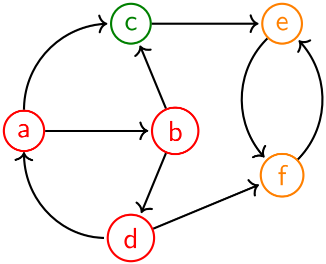
Let G be a directed graph.

Definition 15.22

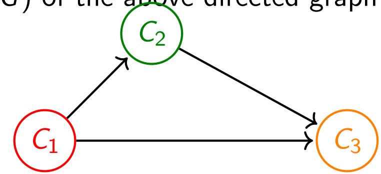
SCC-graph $SCC(G)$ is defined as follows.

- *Let $C_1, ..., C_n$ be SCCs of G.*
- *For each C_i, create a vertex v_i in $SCC(G)$.*
- *Add an edge (v_i, v_j) to $SCC(G)$, if there are two vertices u_i and u_j in G with $u_i \in C_i, u_j \in C_j$ and $(u_i, u_j) \in E$.*

Example 15.10



$SCC(G)$ of the above directed graph G is



SCC(G) is acyclic

Theorem 15.5

For any directed graph $G = (V, E)$, $SCC(G)$ is acyclic.

Proof.

Let us suppose there is a cycle in $SCC(G) = (V', E')$.

There must be $u, u' \in V'$ such that there are paths from u to u' and in the reverse direction.

Let C and C' be the SSCs in G corresponding to u and u' respectively.

There must be a path from nodes in C to nodes in C' and in the reverse direction.

C and C' cannot be SSCs of G. Contradiction. □

Topic 15.5

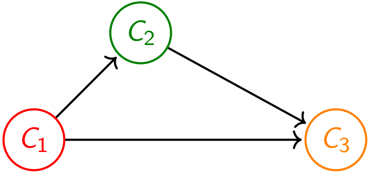
Directed acyclic graph (DAG)

Directed acyclic graph (DAG)

Consider a directed graph $G = (V, E)$.

Definition 15.23

G is a directed acyclic graph (DAG) if G has no cycles.



The above is a directed acyclic graph.

Exercise 15.5

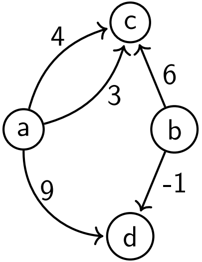
Define a tree from DAG.

Commentary: We may view that DAG $SCC(G)$ is embedded in graph G.

Topic 15.6

Labeled graph

Directed labeled graph



Definition 15.24

A graph $G = (V, E)$ is consists of

- *set of vertices V and*
- *set of edges $E \subseteq V \times L \times V$,*

where L is the set of labels.

The above is a labelled directed graph $G = (V, E)$, where

$L = \mathbb{N}$, $V = \{a, b, c, d\}$ and

$E = \{(a, 3, c), (a, 4, c), (a, 9, d), (b, 6, c), (b, -1, d)\}$.

Topic 15.7

Representation of graph

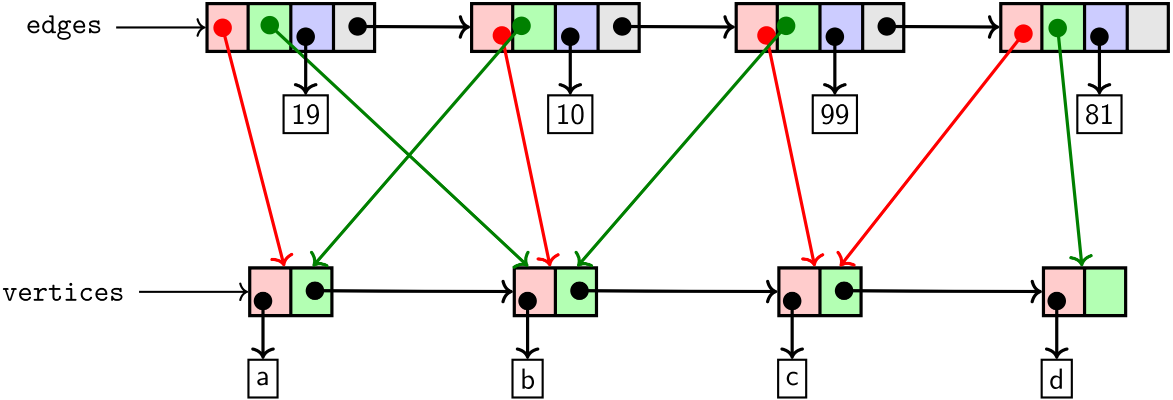
Representations of graph

- Edge list
- Adjacency list
- Matrix

Edge list

- Store vertices as a sequence (array/list)
- Store edges as a sequence with pointers to vertices

Example: edge list



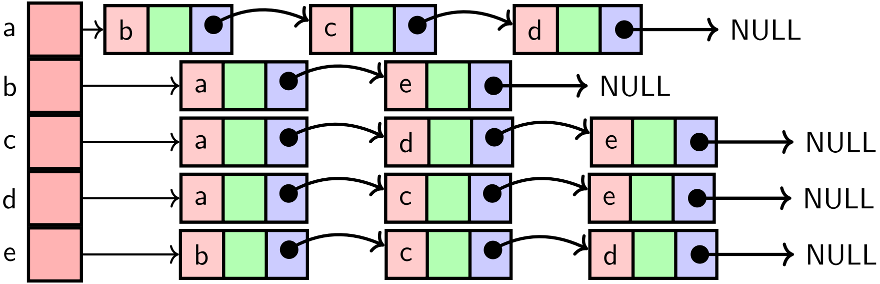
Exercise 15.6

a. *What is the cost of computing adjacent(v)?*

b. *What is the cost of insertion of an edge?*

Adjacency list

- Each vertex maintains the list of adjacent nodes.



Space: $O(|V| + \sum degree(v)) = O(|V| + |E|)$

Exercise 15.7

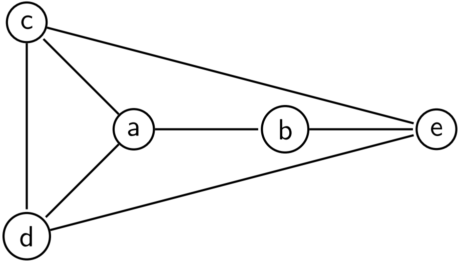
a. Draw the graph for the above data structure.

b. What is the cost of adjacent(v), and find vertices of an edge given by edge number?

c. How can we mix the edge list and adjacency list to make the above operations efficient?

Adjacency Matrix

Store adjacency relation on a matrix.



$$\begin{array}{c|c} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Space: $O(|V|^2)$

Exercise 15.8

a, What is the cost of adding a node? $O(n^2)$

b. What is the cost of adjacent(v)*?* $O(n)$

c. What is the cost of finding vertices of an edge which is given as a pair of positions? $O(1)$

d. How can we mix edge list and adjacency matrix?

Topic 15.8

Tutorial problems

Exercise: modeling COVID

Exercise 15.9

The graph is an extremely useful modeling tool. Here is how a Covid tracing tool might work. Let V be the set of all persons. We say (p,q) is an edge (i) in E1 if their names appear on the same webpage, and (ii) in E2 if they have been together in a common location for more than 20 minutes. What significance do the connected components in these graphs and what does the BFS do? Does the second graph have epidemiological significance? If so, what? If not, how would you improve the graph structure to get a sharper epidemiological meaning?

Commentary: Source: Milind notes.

Exercise: Bipartite graphs

Definition 15.25

A graph $G = (V, E)$ is bipartite if $V = V_1 \uplus V_2$ and for all $e \in E$ $e \notin V_1$ and $e \notin V_2$.

Exercise 15.10

Show that a bipartite graph does not contain cycles of odd length.

Exercise: Planer graphs

Exercise 15.11

Let us take a plane paper and draw circles and infinite lines to divide the plane into various pieces. There is an edge (p,q) between two pieces if they share a common boundary of intersection (which is more than a point). Is this graph bipartite? Under what conditions is it bipartite?

Commentary: Source: Milind notes.

Exercise: Die hard puzzle

Exercise 15.12

There are three containers A, B, and C, with capacities of 5,3, and 2 liters respectively. We begin with A has 5 liters of milk and B and C are empty. There are no other measuring instruments. A buyer wants 4 liters of milk. Can you dispense this? Model this as a graph problem with the vertex set V as the set of configurations c=(c1,c2,c3) and an edge from c to d if d is reachable from c. Begin with (5,0,0). Is this graph directed or undirected? Is it adequate to model the question: How to dispense 4 liters?

Commentary: Source: Milind notes.

Topic 15.9

Problems

Exercise: Modeling call center

Exercise 15.13

Suppose that there are M workers in a call center for a travel service that gives travel directions within a city. It provides services for N cities - $C1,\ldots,CN$. Not all workers are familiar with all cities. The numbers of requests from cities per hour are $R1,\ldots,RN$. A worker can handle K calls per hour. Is the number of workers sufficient to address the demand? How would you model this problem? Assume that $R1,\ldots,RN$, and K are small numbers.

Commentary: Source: Milind notes.

End of Lecture 15