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RECURRENCE RELATIONS

A way of defining functions on natural numbers (one or more variables) such that the value for n is defined in terms of values for numbers < n and the value for 0 is defined explicitly.

* Counting Problems:

For each n, we have a finite subset S_n defined of some objects.

Ex: $|S_n| = 2|S_{n-1}|$ is a possible recurrence relation for some problem of counting.

* Running time of recursively defined algorithms are calculated using recurrence relations.

* Solving a recurrence relation is finding an explicit solution of f(n) in terms of n itself.

* Most counting problems are reduced to coming up with a recurrence relation. Solving a recurrence is fairly easy if it is solvable.

Q2) S_n: Set of all subsets of $\{1, 2, \ldots, n\}$ that do not contain two consecutive numbers. Find $|S_n|$.

Every subset can either include 'n' or not.

Say a subset A contains 'n'. The remaining elements are from $\{1, 2, \ldots, n-2\}$ and there are $|S_{n-2}|$ sets.

Say it does not include n. Then, the remaining elements are from $\{1, 2, \ldots, n-1\}$ and there are $|S_{n-1}|$ sets.

$$\therefore |S_n| = |S_{n-1}| + |S_{n-2}| \quad \forall n > 2$$

$|S_1| = 2$, $|S_2| = 3$

$\phi, \{1\}$ $\qquad$ $\phi, \{1\}, \{2\}$

This is a part of Fibonacci numbers starting from 2, 3.

Catalan Numbers:

C_n = Number of binary trees with n nodes.

$C_0 = 1$

C_n be calulated by counting the number of trees with i nodes in the left subtree and $n-1-i$ nodes in the right subtree with $i \in \{0, 1, \ldots, n-1\}$

Number of trees with i nodes in the left and $n-1-i$ nodes in the right are $C_i \cdot C_{n-1-i}$

$$\therefore \boxed{C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i} \quad \forall n \geq 1, C_0 = 1}$$

There are other problems that have their solution as the Catalan numbers.

Consider the problem of balanced paranthesis with n pairs of paranthesis.

Either a bijection can be drawn from binary trees to paranthesis or it can be shown that the recurrence relation for both are the same.

For every valid paranthesis, the first bracket must be a left bracket. Consider it's corresponding closing bracket.

$$\underbrace{\checkmark(\ldots\ldots)\checkmark}_{i \text{ pairs inside}} \; \underbrace{{}^*(\ldots\ldots)^*}_{n-1-i \text{ including } (\,)^*}$$

For every valid paranthesis both of the substrings must also be valid.

$$\therefore \quad \boxed{P_n = \sum_{i=0}^{n-1} P_i P_{n-1-i}, \; \forall n \geq 1, \; P_0 = 1}$$

which is the same as the Catalan numbers.

Consider a $2 \times n$ matrix with entries $1, 2, 3, \cdots, 2n$. such that each row and each column is increasing with increase in index.

This problem can be converted into the problem of lattice paths:

Consider the xy plane. Find the number of paths from $(0,0)$ to (n,n) through points with integer coordinates (x,y) such that $x \geq y$ at every point in the path and only right and up directions are allowed.

Essentially the path can never lie above $y=x$.

Let the path be partitioned into $(0,0) \rightarrow (i,i)$, $(i,i) \rightarrow (n,n)$ where (i,i) is the first time the path has intersected with the line $y=x$.

Number of such paths $= P_i P_{n-1-i}$

This can be done for $\forall i \in \{0,1,2,\dots,n-1\}$

$$\therefore \boxed{P_n = \sum_{i=0}^{n-1} P_i P_{n-1-i} \quad \forall n \geq 1, \ P_0 = 1}$$

At each coordinate, the sum $x+y$ is increasing and every entry can be filled with the sum of the coordinates excluding $(0,0)$ in this way:

1	$\rightarrow$	$\rightarrow$ $\rightarrow$	$\cdots$	$\rightarrow$	$\rightarrow$
$\rightarrow$	$\rightarrow$	$\rightarrow$	$\cdots$	$\rightarrow$	$2n$

Running time complexity of Quicksort:

Average time complexity $= T(n)$

$$T(n) = \frac{1}{n}\sum_{i=1}^{n}\left(T(i-1) + T(n-i)\right)$$

where each term in the sum represents the case when the i^{th} largest element is the pivot and $\frac{1}{n}$

is present because the probability of any element being the pivot is $1/n$.

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Number of subsets of $\{1, 2, \ldots, n\}$ with no two consecutive elements:

Consider a bit string of length 'n' when i^{th} bit being 1 implies the given subset has the element 'i'.

Number of strings with no occurrences of '11' is equal to the number of subsets with no two consecutive numbers of the set $\{1, 2, \ldots, n\}$

Let the bit string have last bit 0.
Now, the remaining $n-1$ bits is any valid bit string with no occurrence of '11'.
$\therefore$ There are a_{n-1} such bitstrings

Let the last bit be 1.
Now the second last bit must be 0 and the remaining $n-2$ bits must be any valid bitstring with no occurrences of '11'.
$\therefore$ There are a_{n-2} such bit strings

$$\Rightarrow \boxed{a_n = a_{n-1} + a_{n-2}, \; a_1 = 2, \; a_2 = 3}$$

In matrix form,

$$\vec{T(n)} = \begin{bmatrix} T_0(n) \\ T_1(n) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$T_0(n)$ = Number of bit strings that end with 0

$T_1(n)$ = Number of bit strings that end with 1

$$\begin{bmatrix} T_0(n) \\ T_1(n) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} T_0(n-1) \\ T_1(n-1) \end{bmatrix}$$

This is obtained by $T_0(n) = T_1(n-1) + T_0(n-1)$

$T_1(n) = T_0(n-1)$

By induction, it can be proved that:

$$\begin{bmatrix} T_0(n) \\ T_1(n) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The characteristic polynomial of $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ is:
(A)

$$(1-\lambda)(-\lambda) - 1 \times 1 = 0$$

$$\Rightarrow \lambda^2 - \lambda - 1 = 0$$

By Cayley Hamilton Theorem,

$$A^2 = A + I$$

$\Rightarrow A^{n+2} = A^{n+1} + A^{n} \quad \forall n \in \mathbb{N}$

The matrix can be simplified to get a closed form solution for the Fibonacci numbers.

Q) Count the number of strings of length n in which no substring of length 3 contains all the letters of an alphabet of size 3.

Ex: If the alphabet is a, b, c, any occurrence of abc, bca, cab, cba, bac, acb are forbidden.

Consider the strings which end with a.
The substring of $n-1$ must not end with "bc" or "cb"
The conditions are symmetric in a, b, c

Strings of length $n-1$ can be extended to length n in some cases only.

If the string ends with 2 distinct letters then it can be extended in two ways by adding one of the last 2 letters.
If the string ends with 2 letters that are identical then it can be extended by adding any of the 3 letters.

So there are 2 possible classes of strings

$T_{xx}(n)$ = Number of such strings in which last two letters are same where $x \in \{a, b, c\}$

$T_{xy}(n)$ = Number of strings with last two letters being distinct where $x \neq y$, $x, y \in \{a, b, c\}$

$T_{xy}(n) = T_{xx}(n-1) + T_{xy}(n-1)$ → n-1 length string can only end with $\cancel{xy}$

$T_{xx}(n) = T_{xx}(n-1) + 2T_{xy}(n-1)$ → n-1 length string can end with yx or zx.

$$\therefore \begin{bmatrix} T_{xx}(n) \\ T_{xy}(n) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} T_{xx}(n-1) \\ T_{xy}(n-1) \end{bmatrix}$$

and by induction,

$$\begin{bmatrix} T_{xx}(n) \\ T_{xy}(n) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{n-2} \begin{bmatrix} T_{xx}(2) \\ T_{xy}(2) \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$(1-\lambda)(1-\lambda) - 2 = 0$$

$$\lambda^2 - 2\lambda - 1 = 0$$

$$A^2 = 2A + I$$

$$\boxed{A^{n+2} = 2A^{n+1} + A^n \quad \forall n \in \mathbb{N}}$$

Clearly $T_{xx}(2) = 1$ and $T_{xy}(2) = 1$

$$\therefore \boxed{T(n) = 3T_{xx}(n) + 6T_{xy}(n)}$$

and $\begin{bmatrix} T_{xx}(n) \\ T_{xy}(n) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{n-2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \forall n > 2$

$$\boxed{a_n = 2a_{n-1} + a_{n-2} \quad \forall n > 2,\ a_1 = 3,\ a_2 = 6}$$

Obtained from the characteristic equation of the matrix.

Find a direct explanation for the recurrence relation for this problem.

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Solving Linear Recurrence Relations with Constant Coefficients:

Consider the previous problem with recurrence relation being

$$a_n = 2a_{n-1} + a_{n-2} \ \forall n > 2$$

Let $f_0, f_1, f_2, \ldots,$ be an infinite sequence of numbers.

Denote by $f(x) = \sum_{i=0}^{\infty} f_i \cdot x^i$ which is the generating function of the sequence.

If $f(x)$ and $g(x)$ are the generating functions of $f_0, f_1, \ldots$ and $g_0, g_1, \ldots$, we can define $f(x) + g(x)$ is the generating function of $f_0 + g_0, f_1 + g_1, \ldots$.

$h(x) = f(x) \cdot g(x)$ is the generating function of the sequence $h_0, h_1, h_2, \ldots$ where $h_n = \sum_{i=0}^{n} f_i \cdot g_{n-i}$

Use recurrence relations to find the generating function of the sequence. Then use a known generating function to get the closed form solution.

Consider the recurrence,

$$T(n) = 2T(n-1) + T(n-2) \ \forall\ n \geq 2 \text{ with}$$

the initial conditions $T(0) = 1, T(1) = 3$

Since this a homogeneous linear equation, every term $T(n)$ is a linear combination of $T(0)$ and $T(1)$

So it is sufficient to find a solution for the conditions $T(0) = 1, T(1) = 0$, $T(0) = 0, T(1) = 1$ and then combine the two solutions by multiplying by 1, 3 the respective solutions.

$T(x)$ would be the generating function for the sequence.

$$\overset{n \geq 2}{T(n)} = 2T(n-1) + T(n-2)$$

$$\Rightarrow T(n)x^n = 2T(n-1)x^n + T(n-2)x^n$$

$$\Rightarrow \sum_{n=2}^{\infty} T(n)x^n = 2\sum_{n=2}^{\infty} T(n-1)x^n + \sum_{n=2}^{\infty} T(n-2)x^n$$

$$\Rightarrow T(x) - T(0) - T(1)x = 2x\,(T(x) - T(0)) + x^2 T(x)$$

$$\Rightarrow T(x)\left[x^2 + 2x - 1\right] = x(2T(0) - T(1)) - T(0)$$

$$\Rightarrow T(x) = \frac{x(2T(0) - T(1)) - T(0)}{x^2 + 2x - 1}$$

Some generating functions: $\frac{1}{1-cx} \equiv c^n$

$\frac{1}{(1-cx)^2} \equiv 1, 2c, 3c^2, \ldots$

Let $T(0) = 1$ and $T(1) = 0$

$$\frac{1-2x}{1-2x-x^2} = \frac{c_1}{1-r_1 x} + \frac{c_2}{1-r_2 x}$$

$\left.\begin{array}{l} r_1 + r_2 = 2 \\ r_1 r_2 = -1 \end{array}\right\}$ roots of $x^2 - 2x - 1 = 0$ (obtained by replacing x with $\frac{1}{x}$ in $1-2x-x^2$)

$r_1 = 1+\sqrt{2}$, $r_2 = 1-\sqrt{2}$

c_1 and c_2 depend on the initial conditions

$\left.\begin{array}{l} c_1 + c_2 = 1 \\ -c_1 r_2 - c_2 r_1 = -2 \end{array}\right\}$ As $T(0) = 1$ and $T(1) = 0$ was taken

After solving, $T(n) = c_1 (1+\sqrt{2})^n + c_2 (1-\sqrt{2})^n$

Similarly for Fibonacci sequence,

$T(n) = T(n-1) + T(n-2) \ \forall n \geq 2$, $T(0) = 1$, $T(1) = 1$

We have,

$$T(x) - T(0) - T(1)x = x\left[T(x) - T(0)\right] + x^2 T(x)$$

$$\Rightarrow T(x)\left(1 - x - x^2\right) = \left(T(1) - T(0)\right)x + T(0)$$

$$\Rightarrow T(x) = \frac{\left(T(1) - T(0)\right)x + T(0)}{1 - x - x^2}$$

With $T(n) = c_1\left(\frac{\sqrt{5}-1}{2}\right)^n + c_2\left(-\frac{\sqrt{5}-1}{2}\right)^n + n$

$$\left.\begin{array}{l} c_1 + c_2 = T(0) \\ (c_1 + c_2)(-1) = T(1) \end{array}\right\} \Rightarrow c_1 + c_2 = 1,$$

$$\frac{c_1}{4}\left(6 - 2\sqrt{5}\right) + \frac{c_2}{4}\left(6 + 2\sqrt{5}\right) = T(2)$$

$$\Rightarrow (c_1 + c_2)(6) + 2\sqrt{5}\,(c_2 - c_1) = 4 \times 2 = 8$$

$$\Rightarrow 6 + 2\sqrt{5}\,(c_2 - c_1) = 8$$

$$\Rightarrow c_2 - c_1 = \frac{1}{\sqrt{5}}$$

$$\Rightarrow c_2 = \frac{\sqrt{5}+1}{2\sqrt{5}}, \quad c_1 = \frac{\sqrt{5}-1}{2\sqrt{5}}$$

$$\therefore T(n) = \frac{\left(\sqrt{5}-1\right)^{n+1} + (-1)^n\left(\sqrt{5}+1\right)^{n+1}}{4\sqrt{5} \cdot 2^n} + n$$

This method can also be used for non linear recurrence relations.

Consider Catalan numbers.

$$C_0 = 1,$$

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i}$$

Let $C(x)$ be the generating function of the sequence $C_0, C_1, \dots$

$$C(x) = \sum_{i=0}^{\infty} C_i x^i$$

$$\sum_{n=1}^{\infty} C_n x^n = \sum_{n=1}^{\infty} \left(\sum_{i=0}^{n-1} C_i C_{n-1-i} \right) x^n$$

$$\Rightarrow C(x) - C_0 = \sum_{n=1}^{\infty} \left(\sum_{i=0}^{n-1} C_i x^i \cdot C_{n-1-i} x^{n-1-i} \cdot x \right)$$

$$\Rightarrow C(x) - C_0 = \sum_{n=1}^{\infty} x^n \left(\text{Coefficient of } x^{n-1} \text{ in } C^2(x) \right)$$

$$\Rightarrow C(x) - C_0 = \sum_{n=1}^{\infty} x^n \times C'_{n-1}$$

$$\Rightarrow C(x) - C_0 = x \sum_{n=1}^{\infty} C'_{n-1} \cdot x^{n-1}$$

$$\Rightarrow C(x) - C_0 = x \sum_{n=0}^{\infty} C'_n x^n$$

where C'_n is the coefficient of x^n in $C^2(x)$

$C^2(x)$ is the generating function for the series $A_0, A_1, A_2 \dots$ where

$$A_n = \sum_{i=0}^{n-1} C_i \cdot C_{n-i} \quad \text{but } C_n' = \sum_{i=0}^{n-1} C_i C_{n-i}$$

$$\therefore \sum_{n=0}^{\infty} C_n' x^n = C^2(x) \text{ as } C_n' = A_n$$

$$\Rightarrow C(x) - C_0 = x\,C^2(x)$$

$$\Rightarrow x\,C^2(x) - C(x) + C_0 = 0$$

$$\Rightarrow x\,C^2(x) - C(x) + 1 = 0$$

$$\Rightarrow C(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

There are 2 possible solutions for $C(x)$. However generating functions are unique.

$\lim_{x \to 0} C(x)$ must equal C_0 which is 1

$$\lim_{x \to 0} \frac{1+\sqrt{1-4x}}{2x} \text{ is not defined.}$$

$$\lim_{x \to 0} \frac{1-\sqrt{1-4x}}{2x} = 1$$

$$\therefore C(x) = \frac{1-\sqrt{1-4x}}{2x}$$

$$\therefore (1-x)^{\frac{1}{2}} = 1 + \frac{\left(\frac{1}{2}\right)}{1}(-x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{1 \cdot 2}(-x)^2 + \dots$$

Coefficient of $x^n = \dfrac{-1}{n!} \prod_{i=1}^{n-1} \dfrac{(2i-1)}{2}$

$$= \frac{-1}{2^n n!} \prod_{i=1}^{n-1} \frac{(2i-1)\cdot i}{i}$$

$$= \frac{-1}{2^n \cdot n! \cdot n!} \times \frac{1}{2^n} \times \frac{1}{2n-1} \prod_{i=1}^{n} (2i-1)\cdot(2i)$$

$$= \frac{-1}{4^n} \cdot \frac{{}^{2n}C_n}{2n-1}$$

$$\therefore (1-x)^{\frac{1}{2}} = \sum_{n=1}^{\infty} \frac{(-1)\, x^n\, {}^{2n}C_n}{4^n\,(2n-1)} + 1$$

$$(1-4x)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(-1) \cdot x^n\, {}^{2n}C_n \cancel{4^n}}{(2n-1)\cancel{4^n}} + 1$$

$$\Rightarrow (1-4x)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(-1)\, x^n\, {}^{2n}C_n}{2n-1} + 1$$

$$\frac{1-(1-4x)^{\frac{1}{2}}}{2x} = \frac{1}{2x}\left(1 - \left(\sum_{n=1}^{\infty} \frac{(-1)\, x^n\, {}^{2n}C_n}{2n-1} + 1\right)\right)$$

$$= \frac{1}{2x} \sum_{n=1}^{\infty} \frac{x^n\, {}^{2n}C_n}{2n-1}$$

$$\Rightarrow C(x) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{{}^{2n+2}C_{n+1}\, x^n}{2n+1}$$

$$\therefore\ C_n = \frac{{}^{2n+2}C_{n+1}}{2(2n+1)}$$

$$= \frac{(\cancel{2n+2})\,(\cancel{2n+1})(2n)!}{2(\cancel{2n+1})\cdot(\cancel{n+1})\cdot(n+1)(n!)^2}$$

$$= \frac{(2n)!}{(n+1)\,n!\,n!}$$

$$\Rightarrow \boxed{C_n = \frac{{}^{2n}C_n}{n+1}\quad \forall n}$$

The method of generating functions can be used to solve recurrence relations of quadratic types also.

Another example:

$F_0 = 1, F_1 = 2, F_n = F_{n-1} + F_{n-2}\ \forall n \geq 2$. Find the closed form for the Fibonacci sequence.

Let $f(x) = \sum_{n=0}^{\infty} F_n x^n$ be the generating function.

$$F_n = F_{n-1} + F_{n-2}$$

$$\Rightarrow F_n x^n = F_{n-1} x^n + F_{n-2} x^n$$

$$\Rightarrow \sum_{n=2}^{\infty} F_n x^n = \sum_{n=2}^{\infty} F_{n-1} x^n + \sum_{n=2}^{\infty} F_{n-2} x^n$$

$$\Rightarrow f(x) - F_0 - F_1 x = x\left(f(x) - F_0\right) + x^2 f(x)$$

$$\Rightarrow f(x)\left(x^2 + x - 1\right) = xF_0 - xF_1 - F_0$$

$$\Rightarrow f(x)(x^2+x-1) = x(F_0-F_1)-F_0$$

$$\Rightarrow f(x) = \frac{(-1)(x)-1}{x^2+x-1}$$

$$\Rightarrow f(x) = \frac{-(x+1)}{x^2+x-1} = \frac{1+x}{1-x-x^2}$$

$$\Rightarrow f(x) = \frac{C_1}{1-r_1x} + \frac{C_2}{1-r_2x}$$

$$\frac{1}{r_1}+\frac{1}{r_2} = -1, \quad \frac{1}{r_1r_2} = -1$$

$$\Rightarrow r_1+r_2 = 1, \quad r_1r_2 = -1$$

They are the roots of $x^2-x-1=0$

$$x = \frac{1\pm\sqrt{5}}{2}, \quad r_1 = \frac{1-\sqrt{5}}{2}, \quad r_2 = \frac{1+\sqrt{5}}{2}$$

$$C_1+C_2 = 1, \quad -C_1r_2+(-C_2)r_1 = 1$$

$$\Rightarrow C_1r_2+C_2r_1 = -1$$

$$\Rightarrow \frac{C_1}{2}+\frac{C_2}{2}+\frac{\sqrt{5}}{2}(C_1-C_2) = -1$$

$$\Rightarrow \frac{1}{2}+\frac{\sqrt{5}}{2}(C_1-C_2) = -1$$

$$\Rightarrow (C_1-C_2)\sqrt{5} = -3$$

$$\Rightarrow C_1-C_2 = \frac{-3}{\sqrt{5}}$$

$$\therefore C_1 = \frac{\sqrt{5}-3}{2\sqrt{5}}, \quad C_2 = \frac{\sqrt{5}+3}{2\sqrt{5}}$$

The sequence given by $c_1 r_1^n + c_2 r_2^n$ has the generating function $\dfrac{c_1}{1-r_1 x} + \dfrac{c_2}{1-r_2 x}$.

$$\therefore \quad F_n = c_1 r_1^n + c_2 r_2^n$$

$$\Rightarrow F_n = \frac{1}{2\sqrt{5}}\left((\sqrt{5}-3)\left(\frac{1-\sqrt{5}}{2}\right)^n + (\sqrt{5}+3)\left(\frac{1+\sqrt{5}}{2}\right)^n \right)$$

$$= \frac{1}{2\sqrt{5}} \times \frac{1}{2^n} \times \frac{1}{2}\Big((2\sqrt{5}-6)(1-\sqrt{5})^n + (2\sqrt{5}+6)(1+\sqrt{5})^n \Big)$$

$$\Rightarrow \frac{1}{2^{n+2}\sqrt{5}}\left(-(1-\sqrt{5})^{n+2} + (1+\sqrt{5})^{n+2} \right)$$

$$\boxed{F_n = \frac{(1+\sqrt{5})^{n+2} - (1-\sqrt{5})^{n+2}}{2^{n+2}\sqrt{5}} \quad \forall\, n}$$