Department of Computer Science and Engineering
End Semester Examination

Course No.: CS 207 Course Name: Discrete Structures
Date: 17/11/2023 Time: 8-30 to 11-30 Marks: 50

Q1 Let $r_1 = p_1/q_1$ and $r_2 = p_2/q_2$ be positive rational numbers such that p_1, p_2, q_1, q_2 are positive integers with $gcd(p_1, q_1) = gcd(p_2, q_2) = 1$. Prove that there exists a positive rational number r_3 such that r_1/r_3 and r_2/r_3 are integers and for any rational number r such that r_1/r and r_2/r are integers, r_3/r is also an integer. Give an expression for r_3 in terms of p_1, p_2, q_1, q_2. (5)

Q2 Let f be any function from a non-empty finite set A to itself. Prove that there exists a non-empty subset $B \subseteq A$ such that f restricted to B, that is $f \cap (B \times B)$, is a bijection from B to B. Give an example to show that this may not hold for infinite sets A. (5)

Q3 Let A be a finite subset of points in the plane and let P be the partial order on A defined by $((x_1, y_1), (x_2, y_2)) \in P$ if and only if $x_1 \leq x_2$ and $y_1 \leq y_2$, where $(x_1, y_1), (x_2, y_2)$ are the coordinates of the two points. Prove that there exists a partial order Q defined on the same set of points such that any two distinct points in A are comparable in P if and only if they are not comparable in Q. Conversely, suppose P and Q are partial orders defined on an arbitrary finite set A, such that two distinct elements in A are comparable in P if and only if they are not comparable in Q. Prove that it is possible to assign integer coordinates (x_a, y_a) to each element $a \in A$ such that $(a, b) \in P$ if and only if $x_a \leq x_b$ and $y_a \leq y_b$. (10)

Q4 Let G_n be the graph with $2n$ vertices $V_n = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and the edges $u_i v_i$ for $1 \leq i \leq n$, $u_i u_{i+1}$ and $v_i v_{i+1}$ for $1 \leq i < n$. Let $S(n)$ denote the number of distinct non-empty subsets X of V_n such that the subgraph of G_n induced by X is connected. Then $S(0) = 0, S(1) = 3, S(2) = 13$ and for $n \geq 3$, $S(n) = a_1 S(n-1) + a_2 S(n-2) + a_3 S(n-3) + a_4$ for some constants a_1, a_2, a_3, a_4. Find the values of a_1, a_2, a_3, a_4 with explanation. (10)

Q5 (a) How many different graphs G are there with vertex set $\{1, 2, \ldots, n\}$ such that G is the only graph with the same vertex set that is isomorphic to G? Prove your answer. (3)

(b) Prove that for all $n \geq 7$, there exists a graph with vertex set $\{1, 2, \ldots, n\}$ which is isomorphic to $n!$ different graphs with the same vertex set. (4)

(c) Let G be the graph with vertex set $\{1, 2, 3, 4, 5, 6\}$ in which the vertices $\{1, 2, 3\}$ induce a triangle as do the vertices $\{4, 5, 6\}$. There are 3 other edges $\{1, 4\}, \{2, 5\}, \{3, 6\}$. How many different graphs with the same vertex set are isomorphic to G? (3)

Q6 Let G be a graph with n vertices that does not contain any cycle of length at least 4 but for every pair of non-adjacent vertices u, v in G, the graph obtained by adding the edge uv contains a cycle of length at least 4. Find the minimum possible number of edges in G, as a function of n, and prove your answer. (10)