

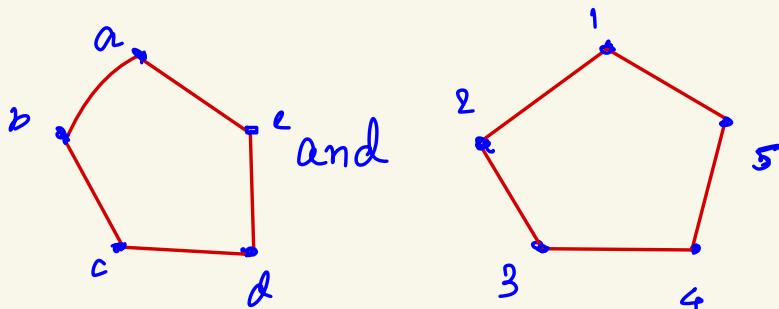
31/10/2023

GRAPHS

- * A graph G consists of a set V of vertices and a set E of edges which are unordered pairs of vertices.
- * Edges define a symmetric and irreflexive relation on the set of vertices. This relation is called an adjacency relation.
i.e. If $\{u, v\}$ is an edge, vertex u is said to be adjacent to v and vice versa.
By definition of the relation (irreflexive), a vertex cannot be adjacent to itself.
- * These graphs are also called undirected graphs.
- * In directed graphs, the relation is not symmetric in general.
- * The edge $\{u, v\}$ is denoted by uv and is said to be incident with vertices u and v .
- * A graph G is said to be isomorphic to a graph H if there exists a bijection from $V(G)$ to $V(H)$. This is like renaming the vertices of the graph G to obtain graph H .
 $uv \in E(G) \iff f(u)f(v) \in E(H)$ for some bijection f .
- * Properties satisfied by a graph G is also satisfied by graphs that are isomorphic to G .

* Isomorphic graphs can be considered to belong to an equivalence class of graphs with respect to some equivalence relation.

* Ex: Number of edges, number of vertices etc. are properties that do not depend on the label.



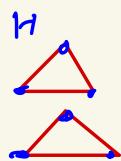
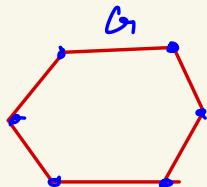
are isomorphic

* Degree of a vertex is the number of vertices adjacent to it.

* The sequence of degrees (σ) degree sequence of a graph is the degrees of vertices arranged in non-increasing order.

Isomorphic graphs have the same degree sequences. However this just this much is not enough to conclude if a pair of graphs are isomorphic to each other or not.

Ex:



Graphs G and H are not isomorphic.

- * There is no simple/efficient algorithm known to determine if two graphs are isomorphic to each other or not.
- * A brute force algorithm would be to test every bijection. This is $O(n!)$ in time where n is the number of vertices.
- * The best known algorithm is of $O(n \log n)$ complexity.
- * For graphs with small degrees, there is a polynomial time algorithm known.
- * A graph H is a **subgraph** of graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph is obtained by deleting edges from G and some vertices and all edges incident at deleted vertices.
- * A **spanning sub-graph** is obtained by only deleting edges from a graph.
- * An **induced sub-graph** is obtained by only deleting vertices. The adjacency relations in the sub-graphs are present in the original graph. $E(I) \subseteq E(G)$ where I is any induced sub-graph.

Some Specific Graphs:

- * P_n : Path with n vertices.

$$V = \{v_1, v_2, \dots, v_n\}$$

$$E = \{ \{v_i, v_{i+1}\} \mid i \in \{1, 2, \dots, n-1\} \}$$

P_n is also called a path of order "n". (number of vertices)

There is only one path with n vertices. Call them as all isomorphic. Number of edges is usually called the size.

* C_n : Cycle with n vertices with $n \geq 3$.

$$V = \{v_1, v_2, \dots, v_n\}$$

$$E = \{ \{v_i, v_{i+1}\} \mid i \in \{1, 2, \dots, n-1\} \} \cup \{v_n, v_1\}$$

* We deal with finite graphs in this course.

Q Every graph with n vertices and at least n edges contain a cycle C_m for some $m \geq 3$.

PROOF:

The statement means, there exists a sub-graph that is isomorphic to C_m for some $m \geq 3$.

Another statement: There exists graphs with n vertices and $n-1$ edges that do not contain any cycle. Such graphs are called trees.

Base Case: $n=1$

This is trivially true since there is no graph with one vertex and at least one edge.

Let G be any graph with n vertices and at least n edges with $n > 1$.

We use induction on ' n ' the number of vertices.
There are 2 cases:

CD Suppose there exists a vertex of degree ≤ 1 in G_i .

Let G'_i be the graph obtained by deleting this vertex.

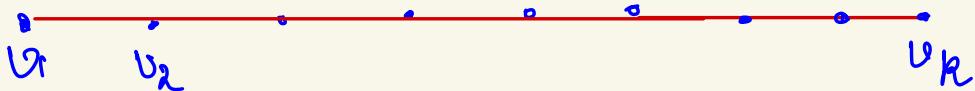
G'_i has $n-1$ vertices and removal of the vertex resulted in the removal of almost one edge.

So G'_i has at least $n-1$ edges.

By Induction, G'_i has a cycle which is also contained in G_i .

(2) Every vertex in G_i has degree at least 2.

Let P be the longest path contained in the graph.



Consider the endpoints v_1, v_k .

Since v_1 has degree 2, it must be connected to some other vertex apart from v_2 .

This vertex v must be from the path itself. If it were not in the path, then the path P can be extended. Therefore there is a cycle.

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TREES

For all $n > 1$, there exists a graph with n vertices and $n-1$ edges that does not contain a cycle.

Proof:

Basis Step: For $n=1$, there exists a graph with a single vertex and 0 edge that satisfies the property.

Inductive Step: For some $n \geq 1$, let there exist a graph satisfying the given property. Let this graph be G_n .

Now add a vertex v to the graph and join it to any vertex in G_n .

The resulting graph G_{n+1} has $n+1$ vertices and n edges. It does not have a cycle because G_n does not have a cycle.

And no new cycle was created by creating by the introduction of v because degree of v is 1.

By Induction, such a graph exists $\forall n > 1$. \square

* These graphs are called 'Trees'.

* This proof confirms the existence of trees -

Property: Every tree with $n \geq 2$ vertices has a vertex of degree 1.

Proof:

Claim: There are no vertices with degree 0.

Proof:

If a vertex has degree 0 then removing it results in a graph with $n-1$ vertices and $n-1$ edges. This graph must contain a cycle.

\Rightarrow The tree contains a cycle

\therefore There is no vertex with degree 0. \square

If every vertex has degree ≥ 2 then there are at least n edges which is again a contradiction.

\therefore There exist at least one vertex with degree 1. \square

* Vertices with degree 1 are called leaf vertices.

Construction of Trees:

Every tree with $n \geq 2$ vertices can be obtained from a tree with $n-1$ vertices by adding a vertex and joining it to some vertex in the tree with $n-1$ vertices.

Proof:

Let T_n denote a tree with n vertices.

T_n has a vertex with degree 1.

Removing this vertex results in a graph with $n-1$ vertices and $n-2$ edges. This is a tree with $n-1$ edges.

\therefore For every tree T_n \exists a tree T_{n-1} . \square

* A weighted graph is a graph in which every vertex and every edge is assigned an integral weight.

* The weight of a sub-graph is the sum of weights of vertices and edges contained in the sub-graph.

Conjecture: For all $k \geq 1$, in any weighted graph with $n \geq k$ vertices and edges $> kn - \frac{k+1}{2}C_2$, there exists a cycle with weight divisible by k .

This has been proven for small values of k but there is no proof known for any value of k in general.

For $k=1$, number of edges $> n-1$. Clearly such a graph has a cycle and every possible weight is divisible by k as $k=1$.

The conjecture is true for $k=1$.

For $k=2$, $n \geq 2$ and number of edges $> 2n-3$.

Let $n \geq 3$ as there are no cycles for $n=2$.

Consider an arbitrary weighted graph with n vertices and at least $2n-2$ edges. If there are more than $2n-2$ edges, then they can be removed until the number of edges = $2n-2$.
Case 1: There is a vertex of degree almost 2.

Deleting this vertex results in a graph with $n-1$ vertices and $2(n-1)-2$ edges.

By induction hypothesis this sub-graph must have a cycle of even weight and we are done.

Case a: Every vertex has degree at least 3.

Claim: There exists at least one vertex of degree 3.

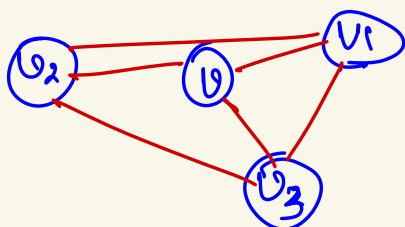
If no vertex had degree 3 then every vertex will have degree at least 4.

$$\Rightarrow \text{Number of edges} \geq \frac{1}{2} \times 4n = 2n > 2n - 2$$

\therefore There is a vertex with degree 3.

Now just removing this vertex will result in a graph that does not satisfy the induction hypothesis.

Case a.1: Every vertex adjacent to the vertex with degree 3 is adjacent to the other 2 vertices.



This is a complete graph K_4 .

Claim: K_4 always has a cycle of even weight.

Proof:

In any combination of 3 vertices, at least 2 of them must have the same parity. For example: In $\{v_1, v_2, v_3\}$, v_1, v_2 might both be even or odd.

WLOG let v_1, v_2 both have the same parity.

Consider 2 possible 3-cycles passing through v_1, v_2 :

$\{v_1, v_1, v_2\}$, $\{v_3, v_1, v_2\}$.

If either $w(v_1, v) + w(v) + w(vv_2) + w(v_1, v_2)$ or $w(v_1, v_3) + w(v_3) + w(v_3v_2) + w(v_1, v_2)$ is even then we get a 3-cycle of even weight and we are done.

If $w(v_1, v) + w(v) + w(vv_2) + w(v_1, v_2)$ and $w(v_1, v_3) + w(v_3) + w(v_3v_2) + w(v_1, v_2)$ are both odd then $w(v_1, v) + w(v) + w(vv_2)$ and $w(v_1, v_3) + w(v_3) + w(v_3v_2)$ must have the same parity.

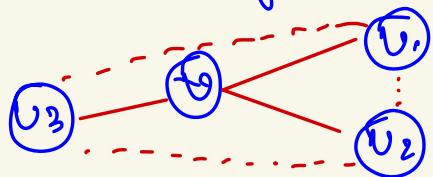
$\Rightarrow w(v_1v_3) + w(v_3) + w(v_3v_2) + w(v_2v) + w(v)$
+ $w(vv_1)$ is even

$\Rightarrow w(v_1v_3) + w(v_3) + w(v_3v_2) + w(v_2) + w(v_2v)$
+ $w(v) + w(vv_1) + w(v_1)$ is even

\Rightarrow A 4-cycle of even weight exists and we are done.

* In fact K_4 is the smallest such graph that has a cycle of even weight and the removal of K_4 in any graph ensures the parity of a cycle of even weight.

Case 2.2: There exists at least a pair of vertices from the set of 3 vertices v, v_1, v_2 that are adjacent to v but are not adjacent to each other.



At most 2 of the dotted lines are edges.

Let v_1 and v_2 be the pair without an edge between them.

Let the weights of v, vv_1 and vv_2 be w_1, w_2 and w_3 respectively.

Delete v and add an edge between v_1 and v_2 with a weight of $w_1 + w_2 + w_3$.

This graph has $n-1$ vertices and $\alpha(n-1) - 2$ edges.

By induction hypothesis it contains an even cycle.

If the cycle does not include v_1, v_2 then it is directly present in the original graph also.

If it includes v_1, v_2 , then in the original graph it passes through v, v_1 and v_2 in that order (or reverse)

□

That concludes the proof.

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Connectivity:

A path from a vertex u to a vertex v in a graph G is a sequence of distinct vertices $u = v_0, v_1, v_2, \dots, v_l = v$, such that $\{v_i, v_{i+1}\}$ is an edge $\forall i, 0 \leq i < l$.

- * Also, a sub-graph with vertex set $\{v_0, v_1, \dots, v_l\}$ and edges $\{v_i, v_{i+1}\} \forall i, 0 \leq i < l$ exists \Leftrightarrow There is a path from v_0 to v_l .
- * If there is a path then there is an induced path.
- * A graph G is **connected** if for every pair of vertices u, v there exists a path from u to v .
- * A **connected component** of a graph is a maximal sub-graph that is connected.
- * If C is a connected component of G then ($\nexists H \supset C \subseteq G$) and (H is a connected component)
 $(\exists \text{ a path from } u \text{ to } v) \wedge (\exists \text{ a path from } v \text{ to } w)$
 $\Rightarrow (\exists \text{ a path from } u \text{ to } w)$

* The existence of a path is an equivalence relation on the set of vertices.

* Proof of Transitivity:

Since \exists a path from u to v \exists vertices $\{v_1, v_2, \dots, v_n\}$ and $\{v_i, v_{i+1}\}$ form an edge and $v_1 = u, \{v_n, v\} \in E(b)$.
and \exists vertices (w_1, w_2, \dots, w_m) such that $\{w_i, w_{i+1}\}$ form an edge and $w_1 = v$ and $\{w_m, w\} \in E(b)$.

$\text{Def } \{v_1, v_2, \dots, v_n\} \cap \{w_1, w_2, \dots, w_m\} = \emptyset$

then $(v_1, v_2, \dots, v_n, w_1, w_2, \dots, w_m)$ are the vertices that form a path from u to w

[2] If there are common vertices, then we find the first vertex v_i that equals w_j for some j .

Now consider the vertices $(v_1, v_2, \dots, v_i, w_{j+1}, \dots, w_m)$ in order. They form a path from u to w . \square

* Connected components of a graph are vertex disjoint and two vertices are in the same component \Rightarrow There exists a path between them.

Property:

* Every connected graph with n vertices contains at least $n-1$ edges.

Proof:

For $n=1$, the statement is trivially true.

Induction hypothesis: Any connected graph with k vertices contains at least $k-1$ edges when $k \leq n$.

Let G be any connected graph with n vertices and the minimum number of edges. Proving the theorem for these graphs \Rightarrow other graphs contain $\geq n-1$ edges.

Deleting an edge gives a graph that is not connected.

(This is because we claimed that G has the minimum number of edges to stay connected. So removing any edge must result in the graph becoming disconnected.)

After deletion there are at least 2 connected components one containing u and another containing v where uv was the deleted edge.

Claim: There are exactly 2 connected components.

If there were more than 2 connected components then they must exist in the original graph also. This is a contradiction since the original graph was connected.

Let n_1 be the number of vertices in the connected component containing u and n_2 be the number of vertices in the component containing v . We know $n_1 + n_2 = n$

By induction, there are at least $n_1 - 1$ and at least $n_2 - 1$ edges in the connected components respectively.

So the original graph contains at least,

$n_1 - 1 + n_2 - 1 + 1$ edges

\Rightarrow There are at least $n - 1$ edges \square

Q. Find the minimum number of n numbers in any array using comparison.

This cannot be done in less than $n - 1$ comparisons for getting correct answers for any input array.

Consider an array a_0, a_1, \dots, a_{n-1} .

Consider a set of vertices with labels a_0, a_1, \dots, a_{n-1} .

If a comparison has been made between a_i, a_j then create an edge between a_i and a_j .

If an algorithm performed less than $n - 1$ comparisons then the graph cannot be connected.

So there are at least 2 connected components and for every possible input, no comparison can be made between two elements belonging to different components.

Consider an input array where the algorithm returns a_i from one of the connected components. Add a quantity ' c ' to every element in this component such that the minimum is now not a_i . The algorithm will still return a_i and hence it is incorrect.

* A similar proof can be done for comparison based sorting algorithms where the time complexity cannot be better than $\Theta(n \log n)$.

* A walk is a sequence of vertices (v_1, v_2, \dots, v_n) such that $\{v_i, v_{i+1}\}$ is an edge $\forall i, 0 \leq i < n$.

* A walk is closed if the first and last vertex v_1 and v_n are the same.

* A walk with no repeated edges is called a trail.

* A trail is Eulerian if every edge of the graph appears exactly once in it.

* A graph is called Eulerian if it has a closed Eulerian trail.

* A closed Eulerian trail is also called an Eulerian circle or Eulerian circuit.

Property:

Every connected graph with n vertices has at least $n-1$ edges.

Proof: (Without using connected components)

Base Case: $n=1$, the statement is trivially true.

Induction Hypothesis: Every connected graph with $n-1$ vertices has at least $n-2$ edges for some $n > a$.

Consider a connected graph with n vertices.

Claim: There exists no vertex of degree 0.

Proof: If it did, then this vertex will not be connected to any other vertex and this is a contradiction as the graph is connected. \square

Case I: There exist no vertex of degree 1.

If this were true then,

$$\begin{aligned}\sum_{v \in V(G)} \text{degree}(v) &\geq 2xn \\ \Rightarrow |E| &\geq 2n \\ \Rightarrow |E| &\geq n\end{aligned}$$

And we are done.

Case II: There exist a vertex of degree 1.

Let this vertex be v .

Now v has to be the last vertex (or first) of a maximal path P in G . Because if v were not at the ends of some maximal path, it would have degree 2 \rightarrow (1)

We remove this vertex v from G . This results in deletion of one edge from G .

The resulting graph G' has $n-1$ vertices and is connected because, there are no vertices in G that are connected by a path containing v (by (1)).

By induction hypothesis, G' has at least $n-2$ edges.

$\Rightarrow G$ has at least $n-1$ edges \square

*The theorem given below is basically the generalised version of the Königsburg bridge problem and is one of the first results in graph theory.

Theorem:

A connected graph has an Eulerian cycle if and only if every vertex has even degree.

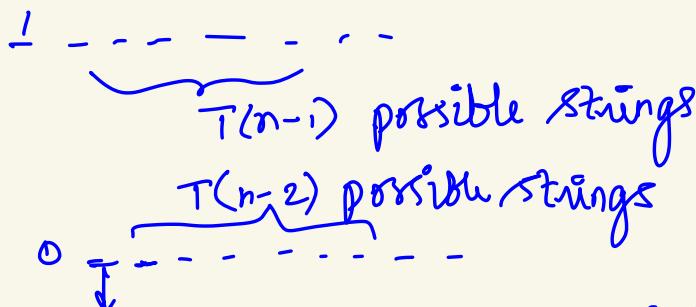
Proof:

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RECURRENCE RELATIONS

Q1 Let $T(n)$ denote the number of bit strings of length n in which every occurrence of 1 is preceded by an even number of occurrences of 0. Derive a recurrence relation for $T(n)$ and show that this number is equal to the number of bit strings of length n that do not contain 11 as a substring. Prove this also by showing an explicit bijection between the two sets.

A string can either start with 1 or 0.



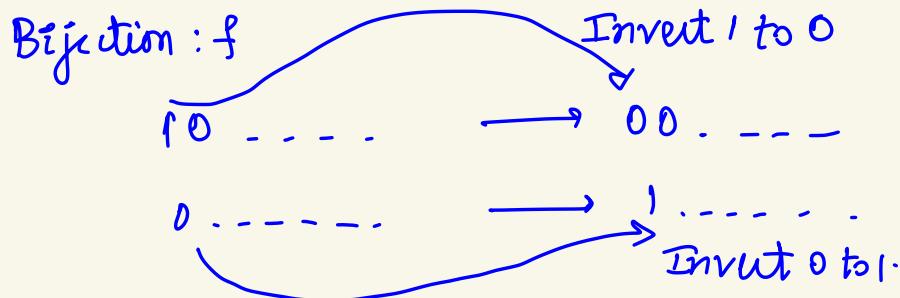
This digit has to 0. Because if it were 1,
the property would be violated

$$\therefore T(n) = T(n-1) + T(n-2)$$

$$T(0) = 1, T(1) = 2$$

This recurrence is the same as that for n-bit strings not containing '11'.

If string starts with 1, then there are $F(n-2)$ strings
and if it starts with 0, then there are $F(n-1)$ strings
(For the '11' strings)



If the string starts with 1, then change 1 to 0 and retain the next 0. Now recursively apply f on the $n-2$ sized string.

If the string starts with 0, then invert it to 1 and apply f recursively to the $n-1$ sized string.

Another Method:

Now $T(n) = T_E(n) + T_O(n)$ where $T_E(n)$ is the number of bit strings with even number of 0s and $T_O(n)$ is the number of bit strings with odd number of 0s.

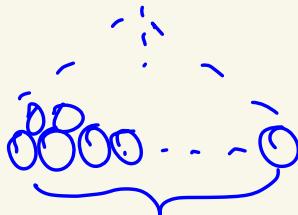
? I am not very sure if this is how Sir partitioned the cases.

He gave a solution with the matrix method. That matrix must satisfy the characteristic equation

$$A^2 = A + I$$

If anyone can share the solution I will add it.

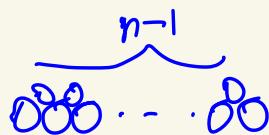
Q2 Consider an arrangement of balls in layers such that the balls in each layer are placed consecutively touching each other, and except for the bottom layer, each ball is placed between two balls in the layer below. If there are n balls in the bottom layer, how many distinct arrangements are possible? For $n = 1$, there is only one, for $n = 2$, there are two, one in which there is only one layer and the other with a single ball in layer 2. For $n = 3$, there are five, with distributions (3), (3, 1), (3, 1), (3, 2) or (3, 2, 1) of balls in layers. Prove that the number of possible arrangements is the Fibonacci number F_{2n-2} . Prove it using a recurrence and also by showing a bijection with the set of bit strings of length $2n - 3$ not containing 11, for $n > 1$.



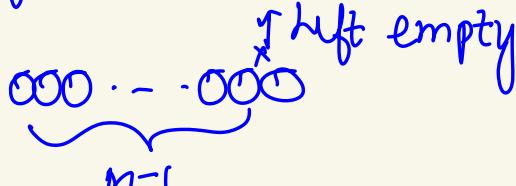
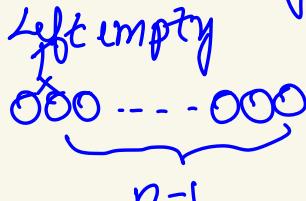
n balls are present at the bottom.

Claim: Number of such arrangements = Number of bitstrings of length $2n-3$ with no occurrences of '11'.

A possibility is, every gap is filled in the first level from the bottom.



Now the remaining arrangements are $T(n-1)$ in number



There are $2T(n-1)$ arrangements here

But this kind of split counts all arrangements with both corners empty twice

There are $T(n-2)$ such arrangements

$$\Rightarrow T(n) = T(n-1) + 2T(n-1) - T(n-2)$$

$$\Rightarrow T(n) = 3T(n-1) - T(n-2), T(1) = 1, T(2) = 2$$

$\forall n \geq 3$

$$F(n) = F(n-1) + F(n-2)$$

$$= 2F(n-2) + F(n-3)$$

$$= 2F(n-2) + F(n-2) - F(n-4)$$

$$F(n) = 3F(n-2) - F(n-4)$$

$$F(2n-2) = 3F(2n-4) - F(2n-6)$$

$\underbrace{\hspace{10em}}$

Similar to the recurrence for $T(n)$

$$F(0) = 1, F(2) = 2,$$

$$F(2n-2) = 3F(2n-4) - F(2n-6), \forall n \geq 3$$

$$\therefore \boxed{T(n) = F(2n-2) \quad \forall n \geq 1, F(0) = 1, F(2) = 2}$$

Now we derive a bijection between such arrangements and bit strings of length $2n-3$ not containing '11'.

$$\frac{1}{0} \frac{0}{1} \frac{1}{2} \frac{0}{3} \frac{0}{4} \overset{0}{5} \overset{1}{6} \overset{0}{7} \overset{1}{8} \overset{0}{9} \overset{1}{10} \overset{0}{11} \overset{1}{12} \overset{0}{13} \overset{1}{14}$$

$$dn-3=15$$

Encoded as $dn-3$
Bit string $n=9$

Denotes that last two spaces are empty

Denotes that first 2 spaces are empty.

Once the empty places are denoted by the number of '10' or '01', we place a 0 at either ends to signify that between these 0's the balls are filled:

Now in between the 0's at 4 and 9, we fill the string recursively for $n=4$ (there are 3 places in the string $(2(4)-3)$)

These strings can never have any occurrence of '11'.

\therefore There is a one-one function from the arrangement of balls to the bit strings

Now we need to prove that this is a bijection ie show that given a string we know how the string is filled.

Given the string, we find the number of occurrences of '10' and '01' and let the numbers be k_1 and k_2 .

If the string is of length m , we find $n = \frac{m+3}{2}$.

Now we construct an arrangement:

$\underset{k_1}{\textcircled{O}\textcircled{O}\dots\textcircled{O}} \underset{x}{\textcircled{O}\textcircled{O}\textcircled{O}\dots\textcircled{O}} \underset{k_2}{\textcircled{O}\textcircled{O}\dots\textcircled{O}}$

After this we recursively decode the string between the two bounding 0s and do the same for

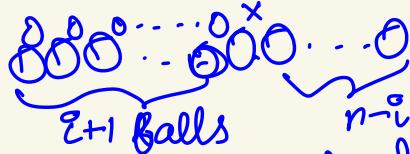
$$m' = m - 2(R_1 + R_2 + i) \text{ and } n' = n - 1 - R_1 - R_2 \text{ with}$$

n', m' satisfying, $m' = 2n' - 3$

\therefore For every string there exists a valid arrangement \square

By removing the condition for balls of any layer to be in contact with neighbours, the problem becomes similar to finding the number of binary trees. So the answer would be a Catalan number.

Place ~~of~~ first absence



i balls have been placed
on top of the first layer with each ball
in contact. \therefore there are $i+1$ balls in the bottom

now

Number of such arrangements = $T(i) \cdot T(n-1-i)$
This is for the second last layer.

$$\therefore T(n) = \sum_{i=0}^{n-1} T(i) T(n-1-i), T(0) = 1, T(1) = 1$$

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Q4. Suppose a fair coin is tossed n times. Given that two consecutive heads did not occur in the n trials, what is the expected number of heads that have occurred? Let S be the set of sequences in which each entry is either +1 or -1 and the sum of entries in any prefix of the sequence is at least -2 and at most 2. How many such sequences of length n are possible? If each entry in the sequence is generated randomly with equal probability, and the sequence terminates as soon as the prefix condition is violated, what is the expected length of the sequence?

The number of valid strings is F_n where $F_n = F_{n-1} + F_{n-2}$
and $F_1 = 2, F_2 = 3$

S_n = Sum of number of 1's in every string of the sample space.

$$\text{Expectation} = \frac{S_n}{F_n}$$

(We consider every string to be equally probable)

$$\begin{aligned} \text{We know } F_n &= \underbrace{F_{n-1} + F_{n-2}}_{\Rightarrow S_{n-1} + S_{n-2} + F_{n-2}} \rightarrow \langle _ _ _ _ _ \dots 0 _ _ \rangle \\ \Rightarrow S_n &= S_{n-1} + S_{n-2} + F_{n-2} \quad \langle _ _ _ _ _ \dots 0 _ _ \rangle \\ S_0 &= 0, S_1 = 1 \quad (\text{The extra 1}) \end{aligned}$$

The generating function for F_n is $F(x) = \frac{1+x}{1-x-x^2}$

$$\begin{aligned} S_n &= S_{n-1} + S_{n-2} + F_{n-2} \\ \Rightarrow S(x) - x &= xS(x) + x^2 S(x) + x^2 F(x) \\ \Rightarrow S(x)(1-x-x^2) &= x + \frac{x^2(1+x)}{1-x-x^2} \\ \Rightarrow S(x) &= \frac{x}{1-x-x^2} + \frac{x^2(1+x)}{(1-x-x^2)^2} \\ &= \frac{x}{1-x-x^2} \cdot \frac{(1-x-x^2+x+x^2)}{(1-x-x^2)} \end{aligned}$$

$$\Rightarrow S(x) = \frac{x}{(1-x-x^2)^2}$$

$$\approx \frac{C_1}{\left(1 - \frac{x}{\frac{\sqrt{5}+1}{2}}\right)^2} \quad (\text{This is the dominant term for large } n)$$

$$\therefore S_n \approx C_1 \cdot \left(\frac{\sqrt{5}+1}{2}\right)^n \cdot n \text{ for large } n.$$

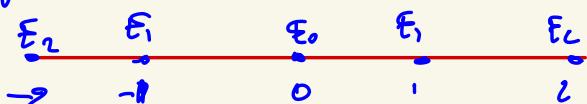
$$F_n \approx \left(\frac{\sqrt{5}+1}{2}\right)^n \text{ for large } n$$

C_1 comes out to be $\frac{5-\sqrt{5}}{10}$

$$\therefore E_n = \frac{S_n}{F_n} \approx \left(\frac{5-\sqrt{5}}{10}\right)^n < \frac{1}{2}$$

Consider a particle constrained to move on the x -axis. It starts at the origin and it is supposed to move only by one unit at a time and its displacement from origin can never be more than +2 or less than -2. It ceases to exist if the displacement violates the given condition.

E_q = Expected number of steps when starting from $|x|=2$ before ceasing to exist.



$$E_2 = \frac{1}{2}(1) + \frac{1}{2}(1+E_1) \rightarrow \underset{\text{leads to } \pm 1 \text{ with probability } \frac{1}{2}}{1 + \frac{1}{2}E_1}$$

Already at $|x|=2$ and goes outside range with probability $\frac{1}{2}$.

$$\varepsilon_1 = \frac{1}{2}(1+\varepsilon_2) + \frac{1}{2}(1+\varepsilon_0)$$

$$E_0 = 1 + E_1$$

Solving the three equations gives, $E_0=9$, $\varepsilon_1=8$, $\varepsilon_2=5$

This is true for $n=2$

$$\text{Now for } n=1, \quad \begin{aligned} E_0 &= 1 + \varepsilon_1 \\ \varepsilon_1 &= \frac{1}{2}(1 + E_0) \end{aligned} \quad \left. \begin{array}{l} \varepsilon_0 = 3, \varepsilon_1 = 4 \end{array} \right\}$$

For a general n , finding E_0 is the question when the bounds are $\pm n$

9/11/2023

GRAPHS

Q1 Prove that every graph with n vertices and more than $n^2/4$ edges contains a triangle.
 Show that for all even $n \geq 2$, there exists a graph with n vertices and $n^2/4$ edges that does not contain a triangle. More generally, show that every graph with n vertices and more than $\frac{(r-2)n^2}{2(r-1)}$ edges contains a complete subgraph with r vertices, for all $r \geq 3$. Also show that if n is a multiple of $r-1$, there exists a graph with n vertices and $\frac{(r-2)n^2}{2(r-1)}$ edges that does not contain a complete subgraph with r vertices. This is known as Turan's theorem.

First we need to show that any graph with n vertices and more than $\frac{n^2}{4}$ edges contains a triangle.

The largest graph not containing a triangle has $\frac{n^2}{4}$ edges (when n is even) and is a complete Bipartite graph $K_{\frac{n}{2}, \frac{n}{2}}$.

If n is odd, then the largest such graph is a complete Bipartite graph $K_{\frac{n-1}{2}, \frac{n+1}{2}}$.

Let $n > 2$. Since the graph has $> \frac{n^2}{4}$ edge, it must contain at least one edge.

Let the edge be uv .

If any vertex w is adjacent to both u, v then it is a triangle.

Thus we consider the remaining vertex to be connected to at most one of them.

Deleting u and v will result in the number of edges being $> \frac{n^2}{4} - (n-2)$

$$\begin{aligned}&= \frac{n^2 - 4n + 8}{4} \\&= \frac{(n-2)^2}{4}\end{aligned}$$

By induction hypothesis this graph has a triangle.

\therefore The original graph has a triangle. \square

For general case if the graph has more than $\frac{n-2}{2(n-1)} n^2$ edges and it is not a complete graph (i.e. not K_n) then, $x < n$.

So we only need to care about cases where $x \leq n$

$P(k) : \forall n \geq k \geq 3$, a graph with n vertices and more than $\frac{k-2}{2(k-1)} n^2$ edges contains a sub-graph isomorphic to K_k .

We have proven that $P(3)$ is true in the previous part of the question. $\rightarrow (1)$

Inductive hypothesis: For some k , $P(m)$ is true $\forall m < k \rightarrow (2)$

Consider a graph with n vertices and more than $\frac{k-2}{2(k-1)} n^2$ edges.

$$\frac{k-2}{2(k-1)} > \frac{k-3}{2(k-2)}$$

$$\Rightarrow \frac{k-2}{2(k-1)} n^2 > \frac{k-1-2}{2(k-1-1)} n^2$$

$$\Rightarrow \text{Number of edges} > \frac{(k-1-2)}{2(k-1-1)} n^2$$

\therefore This graph contains K_{k-1} . (By induction hypothesis)

Now let the vertices v_1, v_2, \dots, v_{k-1} be the vertex set of K_{k-1} .

Consider the remaining $n - (k-1)$ vertices. Let their set be V' .

Case 1: There exists a vertex $v \in V'$ such vv_i is an edge $\forall i \in \{1, 2, \dots, k-1\}$.

$\Rightarrow v, v_1, v_2, \dots, v_{k-1}$ form a complete graph K_k and we are done.

Case 2: No vertex in V' is adjacent to all vertices in

$\{v_1, v_2, \dots, v_{n-1}\} \Rightarrow$ Every vertex in V' can be adjacent to at most $n-2$ vertices in $\{v_1, v_2, \dots, v_{n-1}\}$.
We delete the set of vertices $\{v_1, v_2, \dots, v_{n-1}\}$.

This results in a loss of $\frac{(n-1)(n-2)}{2}$ edges among themselves
and at most $(n-(n-1)) \frac{n}{2}(n-2)$ edges among this set and
vertices from V' .

\Rightarrow The resulting graph has $n-(n-1)$ vertices and more

than $\frac{n^2(n-2)}{2(n-1)} - \frac{(n-1)(n-2)}{2} = (n-(n-1))(n-2)$

$$= \frac{n-2}{2(n-1)} [n^2 - (n-1)^2 - 2(n-1)(n-(n-1))]$$

$$= \frac{n-2}{2(n-1)} [n^2 - (n-1)^2 - 2n(n-1) + 2(n-1)^2]$$

$$= \frac{n-2}{2(n-1)} [n^2 - 2n(n-1) + (n-1)^2]$$

$$= \frac{n-2}{2(n-1)} (n-(n-1))^2$$

We are now left with a graph containing $n-(n-1)$ vertices and
more than $\frac{n-2}{2(n-1)} (n-(n-1))^2$ edges.

Now we induct on n for a fixed λ .

$Q(n)$: $\forall n$, for some $\lambda \geq 3$, a graph having n vertices and more than $\frac{\lambda-2}{2(\lambda-1)} n^2$ edges, there exists a sub-graph isomorphic to K_λ .

Let $Q(l)$ be true $\forall l < n$ for some $n \rightarrow (2)$

Since $n-(\lambda-1) < n$, $Q(n-(\lambda-1))$ is true

\Rightarrow a graph with $n-(\lambda-1)$ vertices and more than

$\frac{(n-(\lambda-1))^2(\lambda-2)}{2(\lambda-1)}$ edges contains a sub-graph

isomorphic to K_λ .

$\Rightarrow Q(n)$ is true since this graph was obtained by removing some vertices and edges. $\rightarrow (4)$

(3) \Rightarrow (4)

\therefore By induction $Q(n)$ is true $\forall n$.

$P(n)$ is true $\rightarrow (5)$

(1) \wedge (2) \Rightarrow (5)

\therefore By induction $P(n)$ is true $\forall n$. \square

Now we prove Turan's theorem:

If n is a multiple of $k-1$, then there exists a graph with n vertices and $\frac{(k-2)n^2}{2(k-1)}$ edges that does not contain a subgraph isomorphic to K_k .

For $k=3$, this can be proven quite easily.

Let n be divisible by 2. The graph has $\frac{n^2}{4}$ edges.

We partition the vertex set into 2 equal sized sets X and Y such that

$$xy \in E(G) \iff (x \in X \wedge y \in Y) \vee (x \in Y \wedge y \in X)$$

i.e two vertices are adjacent if and only if they belong to different sets. Such a graph is called a bipartite graph and the sets X and Y are called the partite sets of G .

Now say G contains a triangle. Let v_1, v_2, v_3 be the vertices forming the triangle.

By pigeon hole principle, at least 2 of these vertices must belong to the same partite set.

\Rightarrow There cannot be an edge between at least 2 of them

$\Rightarrow v_3$ cannot exist.

A contradiction.

$\therefore G$ cannot contain a triangle.

This concludes the proof for $n=3$.

Now we prove the theorem for a general n .

We have graph with n vertices and $\frac{(n-2)n^2}{2(n-1)}$ edges such that $n-1$ divides n .

$n-1 \mid n \Rightarrow$ The vertex set can be partitioned into $n-1$ sets given by V_1, V_2, \dots, V_{n-1} each of size $\frac{n}{n-1}$ and the property that $uv \in E(G) \Leftrightarrow i \in V_i \wedge v \in V_j^c, i \neq j$

\Rightarrow Every vertex in V_i^c is connected to every other vertex in $V_j^c \forall j \neq i$.

\Rightarrow degree of every vertex $v = \frac{(n-2)n}{n-1}$

$$\begin{aligned}\Rightarrow \text{Number of edges} &= \frac{1}{2} \sum_{i=1}^{n-1} \frac{n(n-2)}{n-1} \\ &= \frac{n^2(n-2)}{2(n-1)}\end{aligned}$$

Thus we have constructed a graph satisfying the sufficient conditions.

Now say G has a subgraph isomorphic to K_n .

Let v_1, v_2, \dots, v_n be the vertices that are the part of K_n .

By Pigeonhole Principle, at least two of them must belong to V_i^c for some i .

But then they cannot have an edge between them.

A contradiction.

\therefore There is no subgraph isomorphic to K_n

□