Problem:-

Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x,y,z) = (x-y+z, 2x+3y-\frac{1}{2}z, x+y-2z)$ w.r.t the bases $B_1 = \{(-1,1,0), (5,-1,2), (1,2,1)\}$ and $B_2 = \{(1,1,0), (0,0,1), (1,5,2)\}$. Find $[T]_{B_1}^{B_2}$. Verify the answer using python code.

Step 1: Consider

$$T(-1,1,0) = (-1-1+0,\ 2(-1)+3(1)-\tfrac{1}{2}(0),\ -1+1-2(0))$$
$$= (-2,1,0)$$

Suppose c_1, c_2, c_3 are scalar s/t

$$(-2,1,0) = c_1(1,1,0) + c_2(0,0,1) + c_3(1,5,2)$$
$$(-2,1,0) = (c_1+c_3,\ c_1+5c_3,\ c_2+2c_3)$$

$$c_1 + c_3 = -2$$
$$c_1 + 5c_3 = 1$$
$$c_2 + 2c_3 = 0$$

$$\left(\begin{array}{ccc:c} 1 & 0 & 1 & -2 \\ 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & 0 \end{array}\right)$$

$R_2 \to R_2 - R_1$

$$\left(\begin{array}{ccc:c} 1 & 0 & 1 & -2 \\ 0 & 0 & 4 & 3 \\ 0 & 1 & 2 & 0 \end{array}\right)$$

$R_2 \leftrightarrow R_3$

$$\left(\begin{array}{ccc:c} 1 & 0 & 1 & -2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 4 & 3 \end{array}\right)$$

Here $\rho(A) = \rho(A:b) = 3$

So the system has unique solution

from the last row we get $4c_3 = 3$, $c_3 = 3/4$

from the second row we get $c_2 + 2c_3 = 0 \Rightarrow c_2 = -2/3 = -2(3/4)$

$\therefore c_2 = -3/2$

from the first row we get $c_1 + c_3 = -2$, $c_1 = -2(-c_3)$

$c_1 = -2 - 3/2 = -11/4$

$\therefore T(-1,1,0) = -11/4\,(1,1,0) + (-3/2)(0,0,1) + 3/4\,(1,5,2)$ — ①

Step 2:-

$T(5,-1,2) = (5-(-1)+2,\ 2(5)+3(-1)-1/2(2),\ 5+(-1)-2(2))$

$= (8,6,0)$

Suppose c_1, c_2, c_3 are scalars s/t

$(8,6,0) = c_1(1,1,0) + c_2(0,0,1) + c_3(1,5,2)$

$(8,6,0) = (c_1 + c_3,\ c_1 + 5c_3,\ c_2 + 2c_3)$

$c_1 + c_3 = 8$, $c_1 + 5c_3 = 6$, $c_2 + 2c_3 = 0$

$$\left(\begin{array}{ccc:c} 1 & 0 & 1 & 8 \\ 1 & 0 & 5 & 6 \\ 0 & 1 & 2 & 0 \end{array}\right)$$

$R_3 \to R_2 - R_1$

$$\left(\begin{array}{ccc:c} 1 & 0 & 1 & 8 \\ 0 & 0 & 4 & -2 \\ 0 & 1 & 2 & 0 \end{array}\right)$$

$R_2 \leftrightarrow R_3$

$$\left(\begin{array}{ccc:c} 1 & 0 & 1 & 8 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 4 & -2 \end{array}\right)$$

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$\rho(A) = \rho(A:b) = 3$

So the system has unique solution

from last row $2c_3 = -2$ $\quad c_3 = -1/2$

from the second row $c_2 + 2c_3 = 0$ $\quad c_2 = -2c_3 = -2(-1/2)$

$c_2 = 1$

from third row $c_1 + c_3 = 8 \Rightarrow c_1 = 8 - c_3 \Rightarrow c_1 = 8 - (-1/2) = 17/2$

$T(5,-1,2) = 17/2\,(1,1,0) + 1(0,0,1) - \frac{1}{2}(1,5,2)$ — ①

Step ③

$T(1,2,1) = (1-2+1,\ 2(1) + 3(2) - \frac{1}{2}(1),\ 1+2-2(1))$

$= (0, 15/2, 1)$

Suppose c_1, c_2, c_3 are scalar s/t

$(0, 15/2, 1) = c_1(1,1,0) + c_2(0,0,1) + c_3(1,5,2)$

$\Rightarrow c_1 + c_3 = 0,\ c_1 + 5c_3 = 15/2,\ c_2 + 2c_3 = 1$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 0 & 5 & 15/2 \\ 0 & 1 & 2 & 1 \end{array}\right)$$

$R_2 \to R_2 - R_1$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 4 & 15/2 \\ 0 & 1 & 2 & 1 \end{array}\right)$$

$R_2 \leftrightarrow R_3$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 4 & 15/2 \end{array}\right)$$

Here $\rho(A) = \rho(A:b) = 3$.

So the System has unique solution
from the last row, $4c_3 = 15/2$ $c_3 = 15/8$
from the second row, $c_2 + 2c_3 = 1$ $c_2 = 1 - 2c_3 = 1 - 2(15/8)$
$c_2 = \frac{4-15}{4} = \frac{-11}{4}$

from the first row, $c_1 + c_3 = 0 \Rightarrow c_1 = -c_3 \Rightarrow c_1 = -15/8$

$$T(1,2,1) = \frac{-15}{8}(1,1,0) - \frac{11}{4}(0,0,1) + \frac{15}{8}(4,5,2) \quad -③$$

$$[T]_{B_1}^{B_2} = \begin{bmatrix} -11/4 & 17/2 & -15/8 \\ -3/2 & 1 & -11/4 \\ 3/4 & -1/2 & 15/8 \end{bmatrix}$$

$$= \begin{bmatrix} -2.75 & 8.5 & -1.875 \\ -1.5 & 1 & -2.75 \\ 0.75 & -0.5 & 1.875 \end{bmatrix}$$

II Find the linear transformation when the matrix & basis are specified

x

1. [Find the linear transformation when the matrix and basis are specified]

1. Find the linear transformation corresponding to the matrix $\begin{pmatrix} 2 & 3 \\ 4 & -5 \end{pmatrix}$ w.r.t the bases

$B_1 = \{(1,-1),(1,1)\}$ and $B_2 = \{(1,0),(0,1)\}$

Soln/ Given $[T]_{B_1}^{B_2} = \begin{pmatrix} 2 & 3 \\ 4 & -5 \end{pmatrix}$

$B_1 = \{(1,-1), (1,1)\}$ & $B_2 = \{(1,0), (0,1)\}$

Here $T(1,-1) = 2(1,0) + 4(0,1) = (2,4)$

$T(1,1) = 3(1,0) + (-5)(0,1) = (3,-5)$

we want to find (x,y) for any $(x,y) \in \mathbb{R}^2$

Suppose $c_1, c_2 \in \mathbb{R}$ s/t

$(x,y) = c_1(1,-1) + c_2(1,1)$

Then

$(x,y) = (c_1 + c_2, -c_1 + c_2)$

$c_1 + c_2 = x$

$-c_1 + c_2 = y$

$2c_2 = x+y$

$c_1 = \frac{x+y}{2}$

$c_1 + \left(\frac{x+y}{2}\right) = x$

$c_1 = x - \frac{x+y}{2}$

$c_1 = \frac{x-y}{2}.$

$\therefore (x,y) = \left(\frac{x-y}{2}\right)(1,-1) + \left(\frac{x+y}{2}\right)(1,1)$

$T(x,y) = T\left(\left(\frac{x-y}{2}\right)(1,-1) + \left(\frac{x+y}{2}\right)(1,1)\right)$

$= \left(\frac{x-y}{2}\right) T(1,-1) + \left(\frac{x+y}{2}\right) T(1,1)$

$= \left(\frac{x-y}{2}\right)(2,4) + \left(\frac{x+y}{2}\right)(3,-5)$ $\because T$ is linear

$= \left(\left(\frac{x-y}{2}\right)2, \left(\frac{x-y}{2}\right)4\right) + \left(3\left(\frac{x+y}{2}\right), (-5)\left(\frac{x+y}{2}\right)\right)$

$= (x-y), 2x-2y) + \left(\frac{3x+3y}{2}, \frac{-5x-5y}{2}\right)$

$= x-y + \frac{3x+3y}{2}, 2x-2y + \left(\frac{-5x-5y}{2}\right)$