Linear Transformation

- Theorem 1:

Let V & W be two vector space over field F. A function $T: V \to W$ is linear if and only if

$$T(\lambda \cdot v_1 + v_2) = \lambda \cdot T(v_1) + T(v_2)$$
$$\forall \lambda \in F \text{ \& } v_1, v_2 \in V$$

- Theorem 2:

If $T: V \to W$ is a linear map then $T(\vec{0}_V) = \vec{0}_W$.

I. Problems:

Verify whether the given transformation are linear

1) $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_0, x_1) = (x_0, 2x_0 + x_1)$ $\forall (x_0, x_1) \in \mathbb{R}^2$.

Soln Let $\lambda \in \mathbb{R}$,

$(x_0, x_1), (y_0, y_1) \in \mathbb{R}^2$

Consider,

$$\begin{aligned}
T(\lambda(x_0, x_1) + (y_0, y_1)) &= T(\lambda x_0, \lambda x_1) + (y_0, y_1) \\
&= T(\lambda \cdot x_0 + y_0, \lambda x_1 + y_1) \\
&= (\lambda x_0 + y_0, 2(\lambda x_0 + y_0) + (\lambda x_1 + y_1)) \\
&= (\lambda x_0 + y_0, \lambda 2x_0 + 2y_0 + \lambda x_1 + y_1) \\
&= (\lambda x_0 + y_0, \lambda(2x_0 + x_1) + 2y_0 + y_1) \quad \text{--- (1)}
\end{aligned}$$

Consider

$$\begin{aligned}
\lambda \cdot T(x_0, x_1) + T(y_0, y_1) &= \lambda(x_0, 2x_0 + x_1) + (y_0, 2y_0 + y_1) \\
&= (\lambda x_0, \lambda(2x_0 + x_1)) + (y_0, 2y_0 + y_1) \\
&= (\lambda x_0 + y_0, \lambda(2x_0 + x_1) + (2y_0 + y_1)) \quad \text{--- (2)}
\end{aligned}$$

Teacher's Signature

From ① & ②, we have

$T(\lambda_1(x_0, x_1) + (y_0, y_1)) = \lambda \cdot T(x_0, x_1) + T(y_0, y_1)$

$\therefore$ T is linear

2) $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T(x_0, x_1, x_2) = (x_0 - x_1, 2x_2, 1)$

soln Here $T(0, 0, 0) = (0-0, 2\cdot 0, 1)$

$= (0, 0, 1)$

$T(0, 0, 0) \neq (0, 0, 1)$

$\therefore$ By Theorem 2, T is not linear

⊙ Theorem 3.1:

Let $T: V \rightarrow W$ be a linear map

Then $R(T)$ is a subspace of W & $N(T)$ is a subspace of V.

⊙ Theorem ~~3.2~~:

The Rank - Nullity Theorem

Definition - Let V & W be two vector space over a field F and $T: V \rightarrow W$ be a linear map.

i) The range space of T, denoted by $R(T)$ is defined to be the set $\{T(v) \mid v \in V\}$

$(R(T) \subseteq W)$

ii) The nullspace of T, denoted by $N(T)$, is defined to be the set $\{v \in V \mid T(v) = \vec{0}_W\}$

It is also called kernel (T)

$(N(T) \subseteq V)$

Definition - Let $T: V \to W$ be a linear map. If $R(T)$ & $(N(T))$ are finite dimensional then the $\dim(R(T))$ is called the rank (T) and the $\dim(N(T))$ is called the nullity (T).

⊙ Theorem 4:

Let V & W be vector space over a field F & $T: V \to W$ be a linear map. If V is finite dimensional, then

$$\text{rank}(T) + \text{nullity}(T) = \dim(V)$$

i.e

$$\dim(R(T)) + \dim(N(T)) = \dim(V)$$

PROBLEMS:

I Find the range space, rank, nullspace & nullity for the linear Transformation

$T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$T(C_1) = (1, -1, 0)$

$T(C_2) = (2, 0, 1)$

$T(C_3) = (1, 1, 1)$

where $C_1 = (1, 0, 0)$

$C_2 = (0, 1, 0)$

$C_3 = (0, 0, 1)$

And hence verify Rank-Nullity Theorem

Also, write a python program to verify the answers obtained

Soln Here,

$T(x,y,z) = T(x\cdot(1,0,0) + y(0,1,0) + z(0,0,1))$
$= x\cdot T(1,0,0) + y\cdot T(0,1,0) + z(0,0,1)$
$= x\cdot(1,-1,0) + y(2,0,1) + z(1,1,1)$
$= (x+2y+z, -x+z, y+z)$

To find N(T):

w.k.T

$N(T) = \{(x,y,z) \mid T(x,y,z) = (0,0,0)\}$
$= \{(x,y,z) \mid (x+2y+z, -x+z, y+z) = (0,0,0)\}$

$$= \left\{ (x,y,z) \;\middle|\; \begin{array}{l} x+2y+z=0 \\ -x+z=0 \\ y+z=0 \end{array} \right\}$$

Consider the co-efficient matrix of the above system

$$\begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$R_2 \to R_2 + 1 \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$

$$R_3 \to R_3 - \left(\tfrac{1}{2}\right)R_2 \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$x+2y+z=0$

$2y+z=0$

$\boxed{y=-z}$

$$x = -2y - z$$
$$= -2(-z) - z$$
$$= 2z - z$$
$$= z$$

$\boxed{\therefore x = z}$ & $\boxed{y = -z}$

z is arbitrary

$\therefore$ The soln is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ -z \\ z \end{pmatrix} = z\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, $\forall \in \mathbb{R}$

$\therefore N(T) = \{z(1,-1,1) \mid z \in \mathbb{R}\}$

So, nullity $(T) = 1$

To Find Rank (T):

Fact: If B is a basis for a v.s, v, then

$\text{Span}(T(B)) = R(T)$

In this Problem,

$\text{Span}(T(e_1), T(e_2), T(e_3)) = R(T)$

i.e $\text{Span}((1,-1,0), (2,0,1), (1,1,1)) = R(T)$

Let $A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - R_1$

$$\begin{vmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{vmatrix}$$

Rank $= 2$

Rank + Nullity $= 2 + 1 = 3$

$= \dim(V)$

$R_3 \to R_2 - R_3$

$$\begin{vmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 1 \end{vmatrix}$$

Teacher's Signature

Program:

I
```
from sympy import symbols, expand
import numpy as np
def T(x):
    x0 = x[0]
    x1 = x[1]
    return np.array([x0, 2*x0, x1])
def check_linear(T, no_co_ordinates_input):
    x= np.array(symbols(f'x:{no_co_ordinates_input}'))
    y=np.array(symbols(f'y:{no_co_ordinates_input}'))
    c=symbols('c')
    lhs=T(c*x+y)
    rhs= c*T(x) + T(y)
    lhs=[expand(i) for i in lhs]
    print('T(cx+y) is :', lhs)
    rhs=[expand(i) for i in rhs]
    print('cT(x) + T(y) is :', rhs)
    if lhs=rhs
        print('And, T(x+y)=cT(x)+T(y). So, the given
                    T is linear')
    else:-
        print('And, T(x+y) is not equal to cT(x)+T(y).
                 So, the given T is not linear')
```

1)
```
check_linear(T,2)
def T(x):
  x0 =x[0]
  x1 = x[1]
  x2 = x[2]
```

```
return np.array([x0-x1, 2*x2, 1])
c) check_linear (T, 3)
```

```
II from sympy import Matrix
dim_V = 3
A = Matrix([[1,-1,0],[2,0,1],[4,1,1]])
r = A.rank()
B = A.rref()
print('Range space of T is spanned by first', r 'rows of : V', B[0])
A_Nullspace = A.transpose().nullspace()
print("Nullspace of A is generated by the colum of:",
                                        A_Nullspace)

lhs = r + len(A_Nullspace)
rhs = dim_V
print('Rank of T:', r)
print('Nullity of T:', len(A_Nullspace))
print('Dimension of the domain of T:', dim_V)
if lhs==rhs;
    print('Rank and Nullity Theorem is verified!')
else
    print('There is a mistake!')
```

M 4/5/22