UNIT 1 RECURRENCE RELATIONS

Structure **Page No.**

1.0 INTRODUCTION

In the previous block, you have learnt to solve various types of combinatorial problems using a varied set of tools. However, there are many problems that have to do with counting which cannot be tackled only with the techniques we have presented so far. To give you one such example, look at the problem of counting the number of binary strings of length n that do not contain two consecutive zeros.

If we denote the number of such binary strings by a_n, then $a_1 = 2$ and $a_2 = 3$. Also, we can show that $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$. This is an example of a **recurrence relation.** This relates the value of a_n, a_{n-1} and a_{n-2}. We will show how to find an explicit expression for a_n using this relation in units 2 and 3. In this unit, we will discuss how to **formulate** such recurrence relations for solving combinational problems. In Sec. 1.2, we will introduce you to recurrence relations through three famous examples, the Fibonacci recurrence, Towers of Hanoi and the number of ways of parenthesising an expression.

In Sec. 1.3, we will discuss some more examples to familiarise you with the process of formulating recurrences.

In Sec. 1.4, we will formally define a recurrence relation and explain some terminology related to recurrences like order and degree of a recurrence relation.

In the Sec. 1.5, we will see how the Divide and Conquer techniques used in the design of algorithms give rise to recurrences in a natural way. Here, we will discuss recurrences associated with algorithms for finding the maximum and minimum elements of a list, fast multiplication of integers etc.

1.1 OBJECTIVES

After going through this unit, you should be able to

- define a recurrence relation;
- give examples of recurrence relations;
- set up recurrence relations;
- write recurrences for divide and conquer algorithms.

1.2 THREE RECURRENT PROBLEMS

Let us begin by exploring three sample problems that will give you an idea of what is to follow. The first two of these problems have been investigated repeatedly. All the three problems have a solution based on the idea of **recurrences**. This means that the solution to each problem depends on the solution to smaller instances of the same problem.

Fibonacci
(1170—1250)

Example:1 (Rabbits and the Fibonacci numbers): Have you heard of the problem of breeding rabbits? This was originally posed by **Leonardo di Pisa,** also known as **Fibonacci,** in 1202 in his book **Liber abaci.** The problem is the following: One pair of rabbits, one male and one female, are left on an island. These rabbits begin breeding at the end of two months and produce a pair of rabbits of opposite sex at the end of each month thereafter. Let f_n denote the number of pairs of rabbits after n months. Then $f_1 = 1$. Note that the rabbits start breeding only after two months and the young ones will be produced one month afterwards. So, young ones are produced only at the end of third month. Therefore the number of pairs of rabbits is still 1 at the end of the second month i.e $f_2 = 1$. At the end of the third month, the pair would have produced one more pair. See Table 1 for details. To find the number of pairs after n months, we must add the number of pairs after n – 1 months to the number of pairs born in the nth month. But the newborns come from pairs at least two months old, i.e. from the pairs that already existed after $n - 2$ months; there are f_{n-2} of these. Therefore the sequence $\{f_n \mid n \geq 1\}$ meets the condition $f_n = f_{n-1} + f_{n-2}$ if $n \geq 3$, and the f_n are called **Fibonacci numbers**.

Table 1: Number of Rabbits on the Island

Months	**Reproducing pairs (at least two months old)**	**Young Pairs (not more than two months old.)**
1		
2		
3		
4		
5		
6		

* * *

So, have we solved the problem? Not quite; but it uniquely defines the sequence we seek, describing its members in terms of some previous members. We can also define f_n as a function of n, as in the following exercise.

E1) Using induction, verify that $\sqrt{5}\, f_n = \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n, n \geq 1.$

We will come back to the Fibonacci sequence later. Now let us consider another important recurrent problem.

Example 2: (The Tower of Hanoi): This problem was invented by the French mathematician **Edouard Lucas** in 1883. We are given a tower of eight discs, initially stacked in decreasing size on one of three pegs.

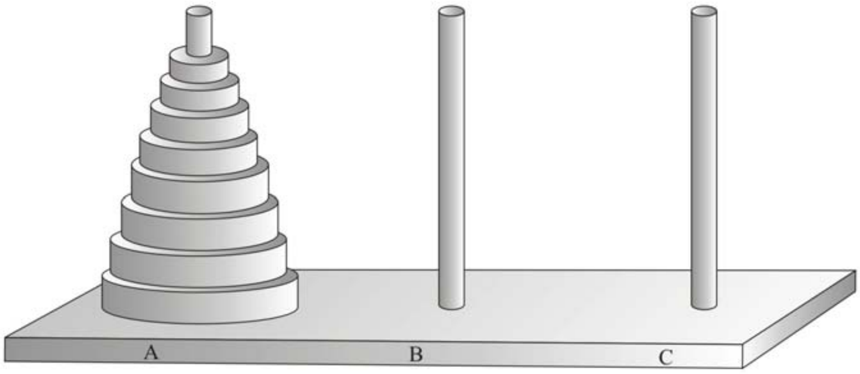
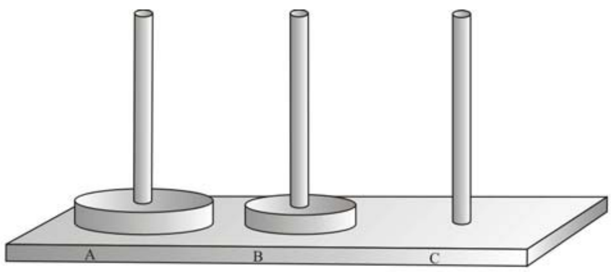


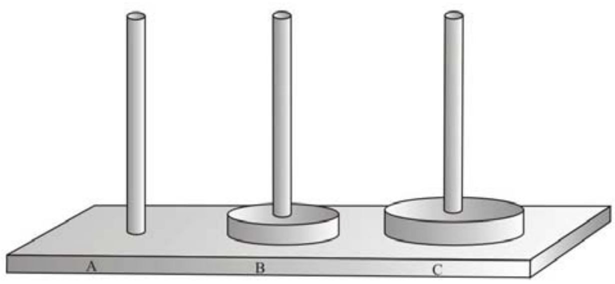
Fig. 2 : Initial position for the towers of Hanoi problem.

The objective is to transfer the entire tower to one of three pegs, moving only one disc at a time without ever moving a larger disc onto a smaller one. Lucas furnished this toy with a legend about a much larger **Tower of Brahma,** which supposedly had 64 discs of pure gold resting on three diamond needles. “At the beginning of time”, he said, “God placed these golden diamond needles”, “God placed these golden discs on the first needle and said that a group of priests should transfer them to a third, according to the rules above. The Tower will crumble and the world will come to an end once the task is finished.”

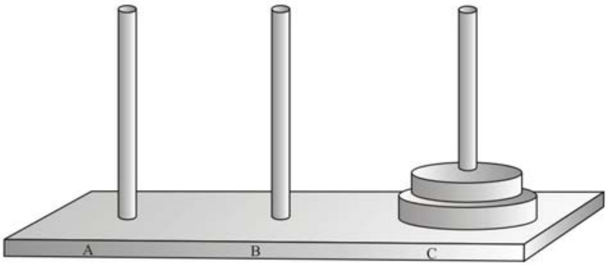
Let us generalise the problem and see what happens if we have n discs instead. Let us say that T_n is the minimum number of moves that will transfer n disks from one peg to another under the rules. Clearly, $T_1 = 1$, and $T_2 = 3$ (see Fig.3). A little bit of experimentation on three disks leads us to the general strategy: We first transfer the $n-1$ smallest disks to B (requiring T_{n-1} moves), then move the largest (requiring one move; remember, it must move!) to C. A is now empty and can be used to transfer the discs on peg B to C. Thus, we can transfer n discs



3a) Transfer disc 2 to peg B



3b) Transfer disc 1 to peg C



3c) Transfer disc 2 to peg C.

Fig. 3 : The moves involved in transferring 2 discs.

(for $n \geq 2$) in **at most** $2T_{n-1} + 1$ moves. So, $T_n \leq 2\,T_{n-1} + 1$, if $n \leq 2$. Why have we used " $\leq$ " instead of " = " here? Our construction proves only that $2T_{n-1} + 1$ moves are enough but can we transfer the disks with lesser number of moves? The answer is "No". At some point, we must move the largest disc. When we do, the $n - 1$ smallest must be on a single peg (why?), and it has taken at least T_{n-1} moves to put them there. After moving the largest disc for the last time, we must transfer the $n - 1$ smallest discs (which must again be on a single peg) back onto the largest; this too requires T_{n-1} moves. Hence, $T_n \geq 2T_{n-1} + 1$ if $n \geq 2$. Both the inequalities, $T_n \geq 2\,T_{n-1} + 1$ and $T_n \leq 2\,T_{n-1} + 1$ can be true only when $T_n = 2T_{n-1} + 1$.

* * *

As with the first example, we shall postpone solving the recurrence relation just obtained to Unit 3. Incidentally, once you have done the following exercise, you will note that the priests will require a minimum of $2^{64} - 1 = 18446\ 744\ 073\ 709\ 551\ 615$ moves to transfer the golden disks. Even at the rate of one move per second, it will take them more than 5×10^{11} years to solve the puzzle, so the doomsday is far away!

Try the next exercise which gives an explicit expression for T_n.

E2) Using induction, show that $T_n = 2^n - 1$, $n \geq 1$.

Now let us consider the third problem we had in mind. This is the number of ways to parenthesise an expression.

Example 3: Let us derive the recurrence relation for the number of ways to parenthesise the expression $x_1 + x_2 + \cdots + x_n$ so that only two terms will be added at a time. For example, the expression $((x_1 + x_2) + x_3)$ is fully parenthesised, but $(x_1 + x_2) + x_3$ is not. Suppose the number of ways of parenthesising the expression $x_1 + x_2 + \cdots + x_n$ is a_n. If we split the expression into $x_1 + x_2 + \cdots + x_{n-1}$ and x_n, $x_1 + x_2 + \cdots + x_{n-1}$ can be parenthesised in a_{n-1} ways and x_n can be parenthesised in a_1 ways. So, in this case, we can parenthesise the expression in $a_{n-1}a_1$ ways. Similarly, if we split up the expression into $x_1 + x_2 + \cdots + x_{n-2}$ and $x_{n-1} + x_{n-2}$, we can parenthesise the expression $x_1 + x_2 + \cdots + x_{n-2}$ in a_{n-2} ways and the expression $x_{n-1} + x_{n-2}$ in a_2 ways. The total number of ways in this case is $a_{n-2}a_2$ ways. In general, if we split up the expression into two sub expressions of size $n - k$ and k, we can parenthesise the expression in $a_{n-k}a_k$ ways. We can get the **total number** of ways of parenthesising the expression by **adding** the different **ways of parenthesising the expression corresponding to different ways of splitting the expression.** This is precisely

$$a_{n-1}a_1 + a_{n-2}a_2 + \cdots + a_{n-k}a_k + \cdots + a_1a_{n-1}.$$

Therefore, the recurrence relation satisfied by a_n is

$$a_n = a_{n-1}a_1 + a_{n-2}a_2 + \cdots + a_{n-k}a_k + \cdots + a_1a_{n-1} \text{for } n \geq 2 \text{ with } a_1 = 1.$$

Using the fact that $a_0 = 0$, we can extend this to

$$a_0 = 0,\ a_1 = 1,\ a_n = a_na_0 + a_{n-1}\, a_1 + \ldots + a_1a_{n-1} + a_{n-1} + a_0a_n\ (n \geq 2).$$

* * *

The numbers a_0, a_1, a_2, … are called Catalan numbers. Catalan numbers arise in several other contexts. We will state two other contexts without proofs.

Example 4: Suppose two candidates A and B poll the same number of votes, n each, in an election. The counting of votes is usually done in some arbitrary order and therefore, during the counting process A may lead for some period and B may lead for some period. The number of ways of counting the votes such that A never trails B is the n^{th} catalan number. We can represent a vote for A by + and a vote for B by –. Then, the n^{th} catalan number is the number of sequences of pluses and minuses such that there are **at least** as many pluses as there are minuses at **any stage** of the sequence. Let us call such a sequence an admissible sequence. Let us take a special case where 8 votes are cast, 4 for A and 4 for B. One voting sequence where A never trails is $(+,-,+,+,-,-,+,-)$. If we drop the last three terms, we get the sequence $(+,-,+,+,-,-)$. Here, there are 3 pluses and 3 minuses i.e. there are at least as many pluses as there are minuses. If we drop the last term, we get $(+,-,+,+,-,-,+)$. Here, there are 4 pluses and 3 minuses, i.e. there are more pluses than minuses.

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Example 5: You would have come across the data structure called **Stack** in MCS-021. Recall that, a stack is a list which can be changed by insertions or deletions at the top of the list. An insertion is called a **push** and a deletion is called a **pop**. A sequence of pushes and pops of length 2n is called **admissible** if there are n pushes and n pops and at each stage of the sequence there are **at least** as many pushes as pops. Suppose we have the string 123…n of the set $N = \{1, 2, 3, \ldots, n\}$ and an admissible sequence of pushes and pops of length 2n. Each push in the sequence transfers the last element in the input string to the stack and every pop transfers the element on the top of the stack to the beginning of the output string. After n pushes and pops have been performed, the output string is a permutation of N called a **stack permutation.** The number of stack permutations of 123…n is the n^{th} Catalan number. Again, we can represent a pop by a + and a push by a –. If you pause and think for a minute, you can convince yourself that every admissible sequence of pops and pushes corresponds to an admissible sequence of pluses and minuses. Let us look at an example. Suppose $n = 4$ and we have the following admissible sequence of pops and pushes: $(+,-,+,+,-,-,+,+,-)$. Let us find the stack permutation corresponding to this. See Table 2.

Table 2 : Finding the slack permutation corresponding to the Admissible sequence

$(+,-,+,+,-,-,+,+,-)$

Sequence	Input String	Stack	Output String
+ (Push 4)	123	[4]	Empty
– (Pop 4)	123	[]	4
+ (Push 3)	12	[3]	4
+ (Push 2)	1	[23]	4
– (Pop 2)	1	[3]	24
– (Pop 3)	1	[]	324
+ (Push 1)	Empty	[1]	324
– (Pop 1)	Empty	[]	1324

So, the permutation obtained is 1324. This is an example of a stack permutation of size 4.

* * *

Try the following exercises now to check if you have understood our discussion so far.

E3) Which of the following sequences are admissible?

i) (+,–,+,–,+,+,–,–)

ii) (+,–,+,–,+,–,+,–)

iii) (+,+,–,–,–,+,–,+)

E4) Is the statement 'An admissible sequence has to start with a plus and end with a minus' true? Explain your answer.

E5) Consider the problem discussed in the introduction. Let a_n be the number of binary sequences of length n that do not contain consecutive zeros. Show that $a_n=a_{n-1} + a_{n-2}$ for $n \geq 3$.

You will have noticed that in each of the three problems, we have been able to express the nth term of a sequence in terms of one or more previous terms and a function of n. This gives you a method to compute the terms of the sequence accurately, given enough time. At times, if the relation between the terms is in a reasonably nice form, we can even "solve" the recurrence, that is, express the nth term as a function of n. You will learn how to solve these three recurrences in the next 2 units.

Let us now consider some more recurrent problems.

1.3 MORE RECURRENCES

You have been exposed to some famous recurrent problems in the previous section. In this section, we shall take another look at setting up recurrence relations for combinational problems of the kind you would have encountered in MCS-013. You will find that in trying to determine recurrence, we are really attempting to describe the counting inductively. In most cases, you will see that the recurrence relation leads to an alternate method of solution, although the methods themselves will be dealt with in Unit 3.

Example 6: Let C_n be the number of comparisons needed to sort a list of n integers. Let us find a recurrence relation for C_n. We first find the minimum of the n elements. This will be the first element of the list. We compare the first two elements and find the minimum among the two. We compare this minimum with the third element, and so on. For finding the minimum of n elements we have to make n–1 comparisons. For example, if we want to find the minimum of the list 3, 2, 4, 1, 5, we will have to make 4 comparisons as shown in Fig.4.

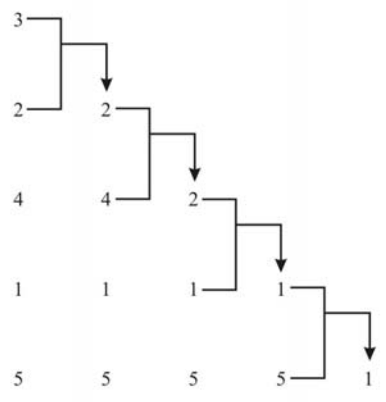

Fig.4 : Comparisons for finding the minimum of 3,2,4,1,5

After making the minimum element the first element of the list we can sort the remaining n–1 elements with C_{n-1} comparisons and append it after first element. So, $n-1 + C_{n-1}$ comparisons are required to sort a list of n elements.

(i.e.) $C_n = C_{n-1} + n-1$

* * *

Try the following exercise which asks you to prove an expression C_n.

E6) In Example 6, using the recurrence relation for C_n, show that

$$C_n = \frac{1}{2} n(n-1), n \geq 1.$$

Example 7: You may recall that the set of all subsets of any non-empty set, S, is called its **power set,** and denoted by *p*(S). Let us determine a recurrence relation satisfied by $s_n = |p(S)|$, where $|S| = n$. Let us take S = {1, 2, …, n}. Now, any subset, A, of S either contains the number n or does not contain the number n. Let us consider these two mutually exclusive cases separately and count the number of such subsets, A. If $n \in A$, then $A = A' \cup \{n\}$, where A′ is a subset of {1, 2, …., n − 1}, there are s_{n-1} such subsets A. So, the number of subsets A that contain n is the same as the number of subsets of $\{1, 2, \ldots, n-1\}$, which is s_{n-1}. On the other hand if $n \notin A$, then, in fact, A is a subset of $\{1, 2, \ldots, n-1\}$, and there are s_{n-1} of these too. Combining these, we see that $s_n = s_{n-1} + s_{n-1} = 2s_{n-1}, n \geq 1$, with $s_0 = 1$.

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We ask you to prove that $s_n = 2^n$ in the next exercise.

E7) In Example 5, using the recurrence relation for s_n, show that $s_n = 2^n$, $n \geq 0$.

Example 8: Recall that a **bijection** is a **one-one, onto** mapping of a set onto itself. It is quite easy to determine directly the number of bijections of an n-set (a set with n elements). We will, however, look at a recurrence relation satisfied by the number of bijections, b_n, of any n-set, say {1,2,….,n}. To begin with, if f is any such bijection, f(n) could be any one of the n elements of the set {1, 2, …, n} ; but now we must map $\{1, 2, \ldots, n-1\}$ bijectively to $\{1, 2, \ldots, n\} \setminus \{f(n)\}$; this can be done in b_{n-1} ways. *f*(n) can be chosen in n ways. In all then, $b_n = nb_{n-1}$, $n \geq 2$, with $b_1 = 1$.

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Again, we ask you to prove an expression for b_n.

E8) In Example 8 using the recurrence relation for b_n, show that $b_n = n!$, $n \geq 1$.

We end this section by looking again at the problem of derangements, discussed in MCS-013.

Example 9: You may recall that the problem is to determine the number of derangements of n objects, d_n, and that we had employed the method of **Inclusion-Exclusion** to solve it. Let us find a recurrence relation for d_n. Recall that d_n counts the number of permutations of n objects that leave **no** object fixed. Any such permutation is called a **derangement**. Let us begin by labeling the objects serially: 1, 2, …, n. In any such derangement of n objects, 1 gets sent to some i, where $i \neq 1$. Two cases arise: For the same derangement, either i gets sent back to 1 or it does not. In the first case, we can leave out 1 and i from the original set and obtain a derangement of n−2 objects; there are d_{n-2} such possibilities. Suppose i is not sent back to 1. Then, we can get a derangement of {2, 3, …,n } as follows. Some number must be mapped back to 1, suppose j is mapped to 1. We can define a derangement of $\{2, 3, \ldots, n\}$ as follows:

See page 56, Block 2, of MCS-013

1) Map j to i
2) Every other element is mapped according to the original derangement

Thus there are d_{n-1} such possibilities to obtain a derangement of n − 1 objects. Therefore, assuming 1 gets sent to i, there is a total of $d_{n-1} + d_{n-2}$ possibilities. Observing that i could have been any number between 2 and n, we conclude that $d_n = (n-1)(d_{n-1} + d_{n-2})$ for $n \ge 3$. To complete the recurrence relation, we note that $d_1 = 0$ and $d_2 = 1$. You will have noticed that to compute d_n one needs to know the values of the two preceding terms. Can we get to compute d_n on the basis of the value of only one preceding term, d_{n-1} ? To explore this, let us write the recurrence in the form

$$d_n - nd_{n-1} = -[d_{n-1} - (n-1)d_{n-2}].$$

We now observe that the expression on the right hand side within the brackets is got from the expression on the left hand side by merely replacing n by n − 1. If we write $D_n = d_n - nd_{n-1}$, we have the simplified expression $D_n = -D_{n-1}$. But then $D_{n-1} = -D_{n-2}$, and so $D_n = D_{n-2}$. Continuing this procedure, we arrive at

$$D_n = (-1)^{n-2} D_2 = (-1)^n [d_2 - 2d_1] = (-1)^n.$$

Therefore, we have $d_n = nd_{n-1} + (-1)^n$ if $n \ge 2$, with $d_1 = 0$.

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In the next exercise, we ask you to prove the expression for d_n we derived in Block 2, Unit 2, but using the recurrence for d_n this time.

E9) Using induction and either of the two recurrence relations for d_n, show that

$$d_n = n!\sum_{i=0}^{n}\frac{(-1)^i}{i!}, n \ge 1.$$

We end this section with a few problems in which you are required to set up the recurrence equations.

E10) Set up a recurrence relation for the determinant of the $n \times n$ matrix with 1 along the main diagonal and with 1 on either side of the main diagonal in each row and zero elsewhere. For example, the 3×3 determinent is $\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}$.

E11) Set up a recurrence relation for the number of n digit sequences on integers {0, 1, 2, 3} having an even number of 0's.

E12) Show that the number of r-permutations of n distinct objects, P(n,r), satisfies the recurrence relation

$$P(n, r) = P(n-1, r) + rP(n-1, r-1), n \ge 1, r \ge 1.$$

E13) Let S_r^n denote the **Stirling numbers of the second kind,** that is, the number of ways to distribute r distinct objects into n nondistinct boxes with no box left empty. Show that S_r^n satisfies the recurrence relation

$$S_{r+1}^n = S_r^{n-1} + nS_r^n, 1 < n < r.$$

In the next section, we give all relevant definitions and introduce the notation for studying recurrence relations.

1.4 DEFINITIONS

We hope you have got a fairly good idea of what a "recurrence relation" is, as well as how to set it up by now. It is time to formalise the procedure and set up a more rigorous mathematical pedestal for it. Let us now define a recurrence relation formally.

Definition: Let $\{a_n : n \geq 0\}$ be a sequence of real or complex numbers. A **recurrence relation (or a recurrence equation)** is an expression of the form

$$a_n = F(a_{n-1}, a_{n-2}, \ldots, n)$$

where F is a function of some of the variables $a_{n-1}, a_{n-2}, \ldots, a_0$, n. Note that all the a_is need not occur in the expression.

In other words, it allows us to compute the n^{th} term of a sequence from one or more of the preceding terms. The symbol "F" merely denotes a (any) function and the variables are (some or all of) the preceding terms in the sequence as also n. For our purposes, we shall only deal with such functions F which are **polynomials and depend** on only **finitely** many variables,

$$a_n = F(a_{n-1}, a_{n-2}, \ldots, a_{n-k}, n)$$

Definition: The **order** of the recurrence relation defined by

$$a_n = F(a_{n-1}, a_{n-2}, \ldots, a_{n-k}, n)$$

is k, where a_n depends on one or more of the previous k terms and k is the **smallest** such integer. We do not define an order for recurrence relations of the form $a_n = F(a_{n-1}, a_{n-2}, \ldots, a_o, n)$ that depend on each of its previous terms.

Therefore, if we can compute the n^{th} term of a sequence from the preceding k terms, but not from the preceding k− 1 terms, we define the order to be k.

Definition: The **degree** of the recurrence relation is the degree of F treated as a polynomial in its variables **excluding** n. If F is not a polynomial in its variables, no degree is assigned to the recurrence relation.

A recurrence relation of degree **one** is also called **linear**, one of degree **two quardratic**, and so on, just like we have in the case of polynomials. After all, the notion of "degree" is tied up with the degree of the defining polynomial F.

Definition: A recurrence relation is called **homogeneous** if it contains no term that depends only on the variable n. A recurrence relation that is not homogeneous is said to be **non-homogeneous** or **inhomogeneous**.

Thus, for a recurrence to be called homogeneous, every term defining the recurrence must contain at least one of the preceding terms of the sequence. Usually, the term **homogeneous** is used for linear recurrences regardless of its order.

Examples:

1) $a_n = 3a_{n-1} + n^2$ is **nonhomogeneous** of order 1 and degree 1.
2) $a_n = na_{n-2} + 2^n$ is **nonhomogeneous** of order 2 and degree 1.
3) $a_n = \sqrt{a_{n-1}} + a_{n-2}^2$ is **homogeneous** of order 2, but has no degree.
4) $a_n = a_{n-1} + a_{n-2} + \ldots + a_0$ is **homogeneous,** has no order, but has degree 1.
5) $a_n = a_{n-1}^2 + a_{n-2}\, a_{n-3}\, a_{n-4}$ is **homogeneous** of order 4 and degree 3.
6) $a_n = \sin a_{n-1} + \cos a_{n-2} + \sin a_{n-3} + \ldots + e^n$ is **nonhomogeneous,** has no order and no degree.
7) $a_n = f_1(n)a_{n-1} + f_2(n)a_{n-2} + \ldots + f_{n-k}(n)\, a_{n-k} + g(n)$ represents the general form of a linear k th order recurrence relation ($f_{n-k}(n) \neq 0$). It is **homogeneous** if g(n) = 0 for each n, and **nonhomogeneous** otherwise.

8) $a_n = a_0a_{n-1} + a_1a_{n-2} + \ldots + a_{n-1}a_0$ $(n \geq 2)$ with $a_0 = 0$, $a_1 = 1$ is a nonlinear recurrence relation.
9) $a_{n,k} = a_{n-1,k} + a_{n-1,k-1}$ is a recurrence relation in two variables n and k. Taking $a_{n,k} = C(n,k)$, the given relation is nothing but Pascal's identity with initial conditions $a_{n,0} = C(n,0) = a_{n,n} = C(n, n) = 1$ for all $n \geq 0$ and $a_{n,k} = 0$, $k \geq n$
10) $a_{n,k} = a_{n-2,k-1} + a_{n-3,k-1} + a_{n-4,k-1}$, with initial conditions $a_{2,1} = a_{3,1} = a_{4,1} = 1$ and $a_{k,1} = 0$ otherwise, is a recurrence relation in two variables (This is the recurrence relation for the ways to distribute n identical balls into k distinct boxes with between two and four balls in each box).
11) $a_n = a_{n/2} + 1$ with $a_1 = 0$ (n a power of 2) is a nonlinear recurrence relation.

Check your understanding of the concepts of order, degree and homogeneity with respect to recurrence relations by trying the following exercises.

E14) Find the order and degree of each of the following recurrences. Also, state whether they are homogeneous or nonhomogeneous.

a) $a_n = a_{n-1} + a_{n-2}$

b) $L_n = L_{n-1} + n$

c) $d_n = nd_{n-1} + (-1)^n$

d) $a_n = a_n a_o + a_{n-1} + \cdots + a_o a_n \, (n \geq 2)$.

You must have observed while looking at the various examples above that the recurrence relation alone will not define for you the terms of the sequence. To be able to do this, one needs to know where to begin the sequence. For example, the n^{th} Fibonacci number and the number of binary sequences of length n that do not contain consecutive zeros satisfy the same recurrence relation. If a_n is defined in terms of a_{n-1} alone, deciding the value for a_0 (or, a_1, or wherever you wish to begin the sequence) uniquely describes the sequence for you. More generally, in case of a kth order recurrence, one needs to know the first k terms, typically $a_0, \ldots, a_{k-1}$ of the sequence in order to uniquely define the sequence. A well-defined linear recurrence relation of degree k consists of a recurrence part and initial conditions for k consecutive values.

Definition: We say that a k^{th} order recurrence relation is a **recurrence relation with initial conditions** provided one or more of the k values $a_0, a_1, \ldots, a_{k-1}$ are known.

Definition: A function $f(n)$ is said to be a **general solution** to the recurrence relation if it satisfies the recurrence equation.
A function g(n) is said to be the **particular solution** to a recurrence relation if it satisfies the recurrence equation, together with the initial conditions.

Please note that there are infinitely many "general solutions" to any recurrence relation without initial condition(s), one for each set of values for the initial terms, but only one "solution" once the first k initial terms are fixed for recurrence relations of order k. You have been verifying that given functions are indeed solutions to the recurrences of the previous two sections. We give a few more examples of a simpler nature. The solution of recurrence relations will be discussed in unit 3.

Examples:

1) The general solution to $a_n = a_{n-1}$ is $a_n = c$, where c is any constant, but if in addition $a_0 = 1$, then the solution is $a_n = 1$, $n \geq 0$.
2) The general solution to $a_n = a_{n-1} + 1$ is $a_n = c + n$, where c is any constant; if $a_0 = 0$, then the solution is $a_n = n$, $n \geq 0$.
3) The general solution to $a_n = ka_{n-1}$ is $a_n = ck^n$, where c is any constant; if $a_0 = 1$, then the solution is $a_n = k^n$, $n \geq 0$.
4) The general solution to $a_n = a_{n-1} + a_{n-2}$ is

5) $a_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$, where c_1, c_2 are any constants.

6) If $a_1 = 1$, $a_2 = 3$, then the particular solution is

7) $a_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n, n \geq 1.$

8) The particular solution to $a_n - \frac{n}{n-1} a_{n-1} = n^3$ with $a_1 = 1$, is

9) $a_n = \frac{n^2(n+1)(2n+1)}{6}$

* * *

In the concluding section, we discuss some common types of recurrence relations that result from divide and conquer algorithms.

1.5 DIVIDE AND CONQUER RELATIONS

Often, to solve a problem, we can partition the problem into smaller problems, find solutions to the smaller problems and then combine the solutions to find a solution for the whole. We can repeatedly partition the problem till we reach a stage where we can solve the problem quite easily. This approach is called the **divide-and-conquer** approach. Let us start with an example. **Throughout this section we will assume that** $n = 2^k$.

Example 10: Consider the problem of finding the maximum and minimum of a list of n numbers. Here let us assume that $n = 2^k$ for some k to make things simpler for us. Let a_n be the number of comparisons required for finding the maximum and minimum of a list of n numbers. We can partition the list into two lists of size $\frac{n}{2}$ each. The maximum and minimum of each of the list can be found using $a_{\frac{n}{2}}$ comparisons. We can then compare minimum (resp. maximum) of the lists to get the minimum (resp. maximum) of the list, i.e., we will need two more comparisons. So, $a_n = 2a_{\frac{n}{2}} + 2$ for $n \geq 2$. $a_2 = 2$. We leave it as an exercise for you to check that $a_n = \frac{3}{2}n - 2$, when n is a power of 2.

E15) Check that $a_n = \frac{3}{2}n - 2$ is a solution to the recurrence $a_n = 2a_{\frac{n}{2}} + 2$ where n is a power of 2 and $a_2 = 1$.

Definition: We call a recurrence a divide-and-conquer recurrence if it has the form

$$a_n = ba_{\frac{n}{a}} + d(n)$$

where $a > 1, b > 1$ are integers and d(n) is a function of n.

Such a relation results when we split a problem of size n into b sub problems of size $\frac{n}{a}$. After solving each of the sub problems we may need d(n) steps to get the solution to the original problem of size n. In Example 10, we splitted a problem of size n into 2 sub problems of size $\frac{n}{2}$ each and we needed 2 more steps to get the answer to the original problem. So, a = b = 2 and d(n) = 2 in this example.

Example 11: In a tennis tournament, each entrant plays a match in the first round. Next, all winners from the first round play a second-round match. Winners continue to move on to the next round, until finally only one player is left as the tournament winner. Assuming that tournaments always involve $n = 2^k$ players, for some k, find the recurrence relation for the number rounds in a tournaments of n players.

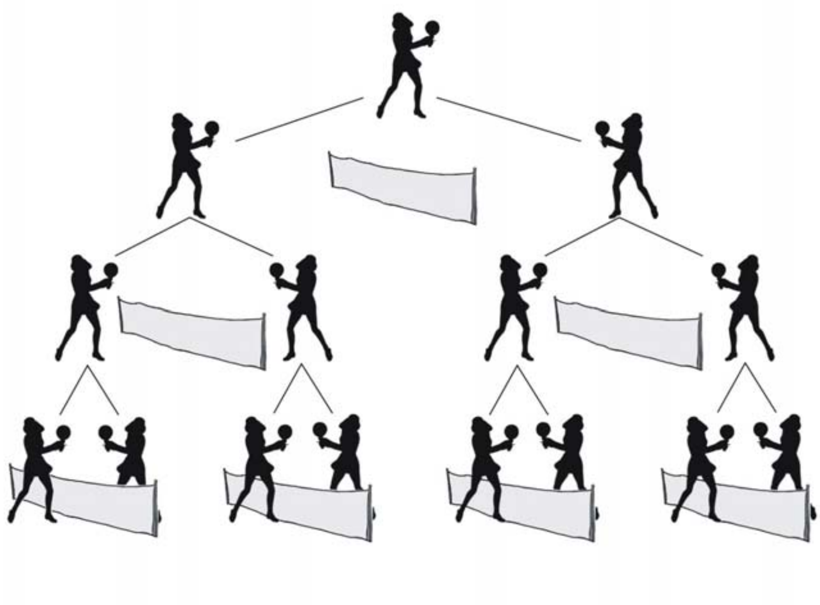

Fig.5: A tournament involving 8 players

Solution: The tournament is subdivided into 2 sub tournaments of n/2 players each. Here the first round of both the tournaments are held together so that they are considered as a single first round. Similarly, all the other rounds are held together. So, after $a_{n/2}$ rounds, one player will be left in both the sub tournaments. The winner of the match between the two is the winner of the tournament. So, $a_n = 2a_{n/2} + 1$. See Fig.5 for a tournament involving 8 players.

* * *

Let us now discuss some more problems where divide and conquer approach can be used.

Example 12: (Binary search) Suppose we have a **sorted** list of $n=2^k$ numbers and we want to check whether a particular number is in the list or not. If we compare the number with all the elements of the list we need n comparisons in the worst case. However, we will describe a better algorithm called the binary search algorithm which uses the divide- and-conquer paradigm. Let us look at a particular example. Suppose we want to check whether 12 is in the list 1,2,3,5,6,7,12,13. We split up the list into two parts. 1,2,3,5 and 6,7,12,13. We compare 12 with the last element of the first list which is 5. We have $5 < 12$ and all the elements in the first list are less than 5. So, we can discard the first list. The second list is again split into two parts, 6,7 and 12,13. Since $7 < 12$, we discard the first part. We split 12,13 into two lists of one element each. The first of these two lists, 12, contains 12. Let a_n denote the number of comparisons required while searching for an element in a sorted list of size n. Then $a_1=1$. In general, we can split up the list into two equal parts of n/2 elements each. We can compare our element with the last element, which is the largest element, of the first list. This will tell us whether we have to search in the first list or in the second list. In either case we have to search in a list of size n/2 only. So, $a_n=a_{n/2}+1$.

* * *

Example 13: (Merge sort) Suppose we want to sort a list of $n=2^k$ elements in ascending order. We can divide the list into parts of size n/2 each, sort them and merge them into a sorted list. Suppose a_n is the number of comparison required to sort a list of n elements then, the two lists can be sorted using $a_{n/2}$ comparisons each. Let us call the lists L_1 and L_2. We will merge them into a new sorted list L. We compare the first (smallest) elements of the list, add the smallest element of the two to the new list L and remove the smallest element from the list from which we selected it. We repeat this till one of the lists, say L_1 is empty. Then we append L_2 to the list L to get a sorted list. See Table 4, where we show how to merge two sorted lists 1,2,5,6 and 3,4.

Step	L_1	L_2	L	Action taken
1	1, 2, 5, 6	3, 4	--	1<3. Remove 1 from L_2 and add it to L.
2	2, 5, 6	3, 4	1	2<3. Remove 2 from L_1 and add it to L.
3	5, 6	3, 4	1, 2	5>3. Remove 3 from L_2 and add it to L.
4	5, 6	4	1, 2, 3	5>4 Remove 4 from L_2 and add it to L.
5	5, 6	--	1, 2, 3, 4	L_2 is empty. No more comparisons required. Add the remaining elements of L_1 to L to get the sorted list 1,2,3,4,5,6

Here L_1 and L_2 are not of the same size. We needed 4+2–1=5 comparisons to sort lists of sizes 4 and 2. In general, to merge two sorted lists of size m and n under this method at most m+n–1 comparisons are required. So, to merge the two lists of size n/2 each, at most n–1 comparisons are required. So, $a_n=2a_{n/2}+n-1$.

E16) Finding the nth power of an integer i, by successive multiplications by i requires n − 1 multiplications. Assuming $n = 2^k$, describe a divide and conquer algorithm such that if a_n is the number of multiplications to find the nth power, then $a_n = a_{n/2} + 1$. Is the algorithm desirable given that the solution is given by $a_n = \log_2(n)$?

E17) Finding the product of a list of n integers by successive multiplication requires n − 1 multiplications. Assuming $n = 2^k$, describe a divide and conquer algorithm for which the number of multiplications a_n satisfies the recurrence relation $a_n = 2a_{n/2} + 1$

E18) To multiply two n-digit numbers, one must do normally n^2 digit-times-digit multiplications. Use a divide and conquer algorithm to do better when n is a power of 2.

With this we have come to the end of this unit. Next two units will deal with the methods of solving recurrence relations. Now let us take a quick look at what we have discussed in this unit.

1.6 SUMMARY

In this Unit **recurrence relations** we:

1) discussed several examples of recurrence relations, drawn from well-known problems and from routine exercises in combinatorics.
2) discussed how to set up recurrence relations after having read this unit.
3) defined the homogeneous, non-homogeneous recurrence relations.
4) defined the order and degree of a recurrence relation.

5) discussed setting up of recurrence relations with the help of divide and conquer paradigm.

1.7 SOLUTIONS/ANSWERS

E1) It is easy to check that $f_1 = 1 = f_2$. With $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$, we observe that $\alpha\beta$ are solutions to the equation $x^2 - x - 1 = 0$. $\therefore \alpha^2 = \alpha + 1$, $\beta^2 = \beta + 1$. If $n \geq 3$,

$$\begin{aligned}\sqrt{5}\,(f_{n-1} + f_{n-2}) &= \left(\alpha^{n-1} - \beta^{n-1}\right) + \left(\alpha^{n-2} - \beta^{n-2}\right)\\ &= \alpha^{n-2}(\alpha+1) - \beta^{n-2}(\beta+1)\\ &= \alpha^{n-2}\alpha^2 - \beta^{n-2}\beta^2\\ &= \alpha^n - \beta^n = \sqrt{5}\, f_n,\end{aligned}$$

as desired.

E2) Observe that $T_1 = 1$. If $n \geq 2$, $2T_{n-1} + 1 = 2(2^{n-1} - 1) + 1 = 2^n - 1 = T_n$, verifying the formula.

E3) 1) and 2) are admissible. If we drop the last three terms from 3), we get $(+,+,-,-,-)$ which has two pluses and three minuses. So, 3) is not admissible.

E4) Yes, this is true. If an admissible sequence starts with a –, the sequence got by removing all the terms except the first one has more minuses (1 minus) than pluses (0 Plus). Suppose an admissible sequence ends in a plus. The sequence we get by removing a plus has more pluses than minuses. If we put back the plus, the sequence will still have more pluses than minuses. But, an admissible sequence must have equal number of pluses and minuses.

E5) Consider any string of length n. The last bit of the sequence is either 0 or 1. If the last bit is 0, the previous bit must be 1. So, it can be obtained by adding '10' to a string of length $n-2$. (Note that 2 consecutive 1s are allowed!)
If the last bit is 1, this can be obtained by adding 1 to a string of length n–1.
$\therefore a_n = a_{n-1} + a_{n-2}$
Note that, a_n satisfies the same recurrence satisfied by the Fibonacci sequence. We have $f_1 = 1$, $f_2 = 1$, but $a_1=2$, $a_2=3$.

E6) It is easy to see that $C_1 = 0$. If $n \geq 2$,

$C_{n-1} + n - 1 = \frac{1}{2}(n-1)(n-2) + n - 1 = \frac{1}{2}n(n-1) = C_n$, as desired.

E7) Observe that $s_0 = 1$. If $n \geq 1$, $2s_{n-1} = 2.2^{n-1} = 2^n = s_n$, verifying the formula.

E8) We note that $b_1 = 1$. If $n \geq 2$, $nb_{n-1} = n.\,(n-1)\,!\; n! = b_n$, as required.

E9) We check that $d_1 = 0$, $d_2 = 1$. To verify the first order recurrence relation, note that if $n \geq 2$,

$$\begin{aligned} nd_{n-1} + (-1)^n &= n\left[(n-1)!\sum_{i=0}^{n-1}\frac{(-1)^i}{i!}\right] + (-1)^n\\ &= n!\left(\sum_{i=0}^{n}\frac{(-1)^i}{i!} - \frac{(-1)^n}{n!}\right) + (-1)^n\end{aligned}$$

$$= n!\sum_{i=0}^{n-1}\frac{(-1)^i}{i!} = d_n$$

as desired.

In case of the second order recurrence relation, if $n \geq 1$,

$$(n-1)(d_{n-1}+d_{n-2}) = (n-1)\left\{\left((n-1)!\sum_{i=0}^{n-1}\frac{(-1)^i}{i!}\right)+\left((n-2)!\sum_{i=0}^{n-2}\frac{(-1)^i}{i!}\right)\right\}$$

$$= n.(n-1)!\sum_{i=0}^{n-1}\frac{(-1)^i}{i!} - (n-1)!\sum_{i=0}^{n-1}\frac{(-1)^i}{i!} + (n-1)!\sum_{i=0}^{n-2}\frac{(-1)^i}{i!}$$

$$= n!\sum_{i=0}^{n-1}\frac{(-1)^i}{i!} - (n-1)!\frac{(-1)^{n-1}}{(n-1)!} = n!\sum_{i=0}^{n-1}\frac{(-1)^i}{i!} + (-1)^n$$

$$= \sum_{i=0}^{n}\frac{(-1)^i}{i!} = d_n,$$

as desired.

E10) Let Δ_n denote the required $n \times n$ determinant. Expanding about the **first** row, we get Δ_{n-1} minus the determinant which when expanded about its first row yields Δ_{n-2}. The corresponding recurrence relation is
$\Delta_n = \Delta_{n-1} - \Delta_{n-2}$, $n \geq 3$, with $\Delta_1 = 1$, $\Delta_2 = 0$.

E11) Let a_n denote the number of n-digit sequences containing an even number of 0′s. Then there are a_{n-1} $(n-1)$-digit sequences that have an even number of 0′s and $4^{n-1} - a_{n-1}$ $(n-1)$-digit sequences that have an odd number of 0′s. To each of the a_{n-1} sequences that have an even number of 0′s, the digit 1,2 or 3 can be appended to yield sequences of length n that contain an even number of 0′s. To each of the $4^{n-1} - a_{n-1}$ sequences that have an odd number of 0′s, the digit 0 must be appended to yield sequences of length n that contain an even number of 0′s. Therefore, for $n \geq 2$, $a_n = 3a_{n-1} + 4^{n-1} - a_{n-1} = 2a_{n-1} + 4^{n-1}$, with $a_1 = 3$.

E12) Of the n distinct objects, pick any **one** object and call it "special". Then, the number of r-permutations in which this "special" object does not appear is $P(n - 1, r)$ because this is the number of r-permutations of the remaining $(n-1)$ objects. On the other hand, if the "special" object does appear, the number of r-permutations is $rP(n-1, r-1)$ because the "special" object could be in any one of r positions between objects or at either end, and we have then to determine the number of $(r-1)$-permutations of $n-1$ objects. Combining the two, we get the required recurrence.

E13) This is somewhat similar to the previous one; choose a "special" object first. The box containing this object either contains no other object or contains at least one more. In the first case, we need to distribute r distinct objects into $n-1$ nondistinct boxes, with no empty box; the number of ways in which to do this is S_r^{n-1}. Otherwise, the special "object" may be placed into any one of the n (nondistinct) boxes (there are n choices), and we still need to distribute r objects into n nondistinct boxes, with no empty box; there are S_r^n such choices for each choice of the box the "special" object is placed in. Combining the two cases gives the recurrence relation.

E14) a) order 2, degree 1
b) order 1, degree 1
c) order 1, degree 1
d) order not defined. Degree is 2.

E15) When n = 2, $\frac{3}{2}n-2=1$. Let us suppose $n = 2^k$. Let us apply induction on k. Suppose it is true for k, say $m = 2^k$, then

$$a_m = 2a_{m/2} + 2$$
$$= 2\left(\frac{2m}{4} - 2\right) + 2$$
$$= \frac{3m}{2} = 4 + 2$$
$$= 3/2\, m - 2$$

Thus it is true for k+1.

E16) Divide n by 2, find $i^{\frac{n}{2}}$, and square it. Thus $a_n = a_{\frac{n}{2}} + 1$. The algorithm is desirable since $\log_2(n)$ grows more slowly than n.

E17) Divide n by 2. Find the product of $\frac{n}{2}$ integers and the product of last $\frac{n}{2}$ integers. Multiply two products obtained. Thus $a_n = 2\,a_{\frac{n}{2}} + 1$.

E18) Let n be a power of 2. Let the two n-digit numbers b A and B. We split each of these numbers into two $\frac{n}{2}$ digit parts:

$A = A_1 10^{\frac{n}{2}} + A_2$ and

$B = B_1 10^{\frac{n}{2}} + B_2$ (like 1235 = 12 × 100 + 35)

Then $A.B = A_1B_1 10^n + A_1B_2\ 10^{\frac{n}{2}} + A_2B_1 10^{\frac{n}{2}} + A_2B_1 10^{\frac{n}{2}} + A_2B_2$

We need only to make three $\frac{n}{2}$ - digit multiplications, $A_1.B_1$, $A_2.B_2$ and $(A_1 + A_2) . (B_1 + B_2)$ to determine A . B since
$A_1 . B_2 + B_2 . A_1 = (A_1 + A_2). (B_1 + B_2) – A_1 . B_1 – A_2 . B_2$
Actually $(A_1 + A_2)$ or $(B_1 + B_2)$ may be $\left(\frac{n}{2}+1\right)$-digit numbers but this slight variation does not effect the general magnitude of our solution (like 1295 = 12 × 10^2 + 95). If a_n represents the number of digit-times multiplications needed to multiply two n-digit numbers by the above procedure, this gives the recurrence relation $a_n = 3a_{\frac{n}{2}}$

a_n is proportional to $n^{\log_2 3} = n^{1.6}$ – a substantial improvement over n^2.

UNIT 2 GENERATING FUNCTIONS

Structure **Page Nos.**

2.0 INTRODUCTION

In your earlier Mathematics courses, you may come across power series expansions of functions like e^x, sin x etc. There, we have to worry about **convergence** questions, i.e for which values of x does the expansion represent the function. Here, we will discuss power series expansions from a different point of view. We will not be interested in convergence questions because we will never substitute numerical values for x; rather we will be interested in the combinatorial properties of the power series. Sequences of numbers that have combinatorial significance appear as the coefficients of power series. We call the power series where coefficients are the terms of a sequence as the **generating function** for the sequence. For example, we will see later in this Unit that the coefficients of the power series of $\frac{z}{\left(1-z-z^2\right)}$ are the Fibonacci numbers. Thus,

$\frac{z}{\left(1-z-z^2\right)}$ is the generating function for the Fibonacci numbers.

In sec.2.2, we shall explain the concept and some elementary uses of generating functions. In Sec.2.3, we shall introduce you to a particular type of generating functions that are used to solve arrangement problems in combinatorics.

In sec.2.4, we shall explore the power of the generating functions as a tool when, for example, it is used to derive some combinatorial identities, solve some combinatorial problems involving general integer equations, find the number of partitions and solve certain recurrence relations.

2.1 OBJECTIVES

After going through this unit, you should be able to

- define and construct generating functions for sequences arising in various types of combinatorial problems;
- use generating functions to find the number of integer solutions to linear equations.
- find the generating function associated with a sequence in closed form in some simple cases
- find the exponential generating function associated with a sequence in closed form in some simple cases;
- solve recurrence relations using generating functions; and
- use generating functions to prove identities involving combinatorial coefficients.

2.2 GENERATING FUNCTIONS

Often, we can relate the solutions of a combinatorial problem to the coefficients of a power series. In the next example we will see how to relate the number of integer solutions to certain linear equations to the coefficients of a power series.

Example 1: Determine the number of integer solutions to linear equation

$$X_1 + X_2 = 3, \text{with } 0 \le X_1 \le 1 \text{ and } 0 \le X_2 \le 2.$$

Solution: By explicit enumeration, the possible values are given below.

X_1	X_2	Sum
0	0	0
0	1	1
0	2	2
1	0	1
1	1	2
1	2	3

Thus, there are two ways to obtain a sum of 1 (also 2) and one way to obtain the sum 3.

Now consider the following product of polynomials:

$$(z^0 + z^1)(z^0 + z^1 + z^2),$$

Where the exponents of symbol z in the first factor corresponded to the possible values of X_1 and in the second factor to the possible values of X_2. On expanding this product, we get

$$\begin{aligned}\left(z^0 + z^1\right)\left(z^0 + z^1 + z^2\right) &= \left(z^0z^0 + z^0z^1 + z^0z^2 + z^1z^0 + z^1z^1 + z^1z^2\right)\\ &= 1 + 2z + 2z^2 + z^3.\end{aligned}$$

Adding the exponents of the symbol z after multiplication corresponds to considering the sum of the values of X_1 and X_2.

We note that the coefficient of z^r , $1 \le r \le 3$, in this expression gives the number of integer solutions to $X_1 + X_2 = r$, with $0 \le X_1 \le 1$ and $0 \le X_2 \le 2$. In particular, because the coefficient of z^3 in the above expression is 1, and there is only one pair of values viz. (1,2), which satisfy the given linear equation.

* * *

Suppose we intend to find non-negative integer solutions to the linear equation

$$X_1 + X_2 + X_3 = 10, \text{with } 0 \le X_1 \le 4, X_2 > 0, \text{and } X_3 \ge 0.$$

Then, by arguments given in the example above, we take the product of the following three polynomials.

$$\left(1 + z + z^2 + z^3 + z^4\right)\left(z + z^2 +\right)\left(1 + z + z^2 + \cdots\right)$$

In the above product, both second and third factors are infinite because there is no upper bound on X_1 and X_2. Also, second factor does not contain the constant term owing to the fact that $X_2 > 0$. Then, as before, the coefficient of z^{10} in the above expression will give us a solution to the linear equation given above.

For finding the coefficients of a power series, we often use the following **results.**

Result 1: (Binomial Theorem)

Let n > 0 then

a) $\left(1+z\right)^{n} = \sum_{r=0}^{\infty} C(n,r)\, z^{r}$

b) $\left(1+z\right)^{-n} = \sum_{r=0}^{\infty} C(n-1+r,r)(-1)^{r} z^{r}$

c) $(1-z)^{-n} = (1+z+z^{2}+\cdots)^{n} = 1+\sum_{r=1}^{\infty} C(n-1+r,r)z^{r}$.

Result 2: $\dfrac{1-z^{n}}{1-z} = 1+z+z^{2}+\cdots+z^{n-1}, z \neq 1.$

Next, we illustrate the technique of identifying the power series associated with a combinatorial problem with the help of following example.

Example 2: Find a power series associated with the problem where we have to find the number of ways to select a **dozen** pieces of fruit from 5 Apples, 10 Bananas and 15 Coconuts.

Solution: To begin with, let us use the letters A, B and C for Apples, Bananas and Coconuts, respectively. So, if we select k Apples, ℓ Bananas and m Coconuts, then we must have k+ ℓ +m =12, with the restriction that $0 \le k \le 5, 0 \le \ell \le 10$ and $0 \le m \le 15$. Let us see what we could do to set up the problem using the symbols A, B and C.

Here we may denote k Apples by A^k, ℓ Bananas by B^{ℓ}, m Coconuts by C^m. Then we have picked the correct number of pieces provided the degree (i.e. the sum k+ ℓ +m) of the term $A^kB^{\ell}C^m$ equals 12. Thus, to find the required number of ways of selecting a dozen pieces of fruit, you simply have to find the number of terms in the expansion.

$$(A^{0}+A^{1}+\cdots+A^{5})(B^{0}+B^{1}+\cdots+B^{10})(C^{0}+C^{1}+\cdots+C^{15}) \qquad (1)$$

whose degree equals 12. This will be the sum of the coefficients of all the terms $A^kB^{\ell}C^m$ in (1) such that k+ ℓ +m =12 i.e. of $A^0B^0C^{12}$, $A^0B^{11}C^{12}$, etc.

At this point it is important to observe that any selection of fruits with the given restriction on the numbers k, ℓ and m correspondes to precisely one term in this product. For instance, if you pick 3 Apples, 4 Bananas and 5 Coconuts, the corresponding term in the product (1) is $A^3B^4C^5$. Conversely, the term AB^2C^9 represents the choice of 1 Apple, 2 Bananas and 9 Coconuts. Thus product (1) when expanded as $\sum_{i,j,k} a_{ijk}A^{i}B^{j}C^{k}$, gives the required (finite) power series for the given problem.

* * *

Now, since our real interest is in the degree of $A^kB^{\ell}C^m$ (i.e. in the sum k+ ℓ +m), we may as well replace each of these symbols in (1) by a common symbol, say z. Then, as before, we are led to determine the coefficient of z^{12} in the following product of polynomials.

$$(1+z+\cdots+z^5)(1+z+\cdots+z^{10})(1+z+\cdots+z^{15}).$$

Now we don't need to look into the possible ways in which A^kB^ℓ and C^m add up to 12 fruits.

Next, let us ask a similar question for the problem given in the following example.

Example 3: How can a power series be associated with the problem in which we have to find the number of selections of fruits if we have Rs.50 with us and it is given that an Apple costs Rs. 5, a Banana Rs.2 and a Coconut Rs.3

Solution: Since here we don't have any restriction on the number of pieces of fruit, the required power series (in terms of money) is of the form

$$(A^0 + A^5 + A^{10} + \cdots)(B^0 + B^2 + B^4 + \cdots)(C^0 + C^3 + C^6 + \cdots),$$

which is the product of three polynomials (infinite because there is no restriction on the number of pieces of fruit). Because an Apple costs Rs.5, so, purchase of k Apples would mean that we have to spend Rs.5k. Similarly, purchase of ℓ Bananas and m Coconuts will amount to spending Rs. (2ℓ+3m). Thus purchase of (k+ℓ+m) fruits correspond to the term $A^{5k}B^{2\ell}C^{3m}$ in the above product of three polynomials. Also because we have Rs.50 only, we must have $5k+2\ell+3m = 50$. On the other hand, each term $A^{5k}B^{2\ell}C^{3m}$ (with $5k+2\ell+3m = 50$) in the above series gives a choice for purchasing k Apples ℓ Bananas and m Coconuts.

Thus, in view of given cost of the Apple, Banana and Coconut, power of symbols A, B and C in the first, second and third polynomials are multiples of 5, 2 and 3, respectively. As before, in this expression we seek the number of terms with degree 50. However, by our discussion following Example 2, if we replace each of these symbols by a common symbol z (say) then the required number is given by the coefficient of z^{50} in the expression.

$$(1+z^5+z^{10}+\cdots)(1+z^2+z^4+\cdots)(1+z^3+z^6+\cdots). \qquad (*)$$

Hence, this product on expansion gives the power series associated with the above problem.

In above example, if we impose some restrictions on our selection of the fruits, then there will be a corresponding change in the associated power series (*).
This is what we want you to see in the following exercise.

E1) Find the power series associated with the problem given in Example 3,
  a) when all our selections are required to have 1 Apple at least;
  b) when each selection has to have at least one fruit of each type.

You have seen above how to associate a power series with a combinatorial problem, such that, the solution of the problem is given by certain coefficients of that series. Certain series can be written in a functional form which we call as **closed form**. For example, it follows from Binomial theorem (see Result 1, a) given above) that $(1-z)^{-1}$ is the closed form (or a functional form) of the power series $\sum_{r=0}^{\infty} z^r$

To distinguish them from exponential generating functions (which we will define in the next section), they are sometimes called **ordinary generating functions.**

Definition: The generating function A (z) (say) for the sequence of real (or complex) numbers, $\{a_0, a_1, \ldots, a_n, \ldots\}$ is given by the powers series

$$A(z) = \sum_{k=0}^{\infty} a_k z^k = a_0 + a_1 z + \ldots + a_n z^n + \ldots$$

Thus, the $(n+1)^{th}$ term a_n of the sequence $\{a_n\}$, $n \geq 0$ is simply the coefficient of z^n in A (z). As said before, the generating function thus serves the purpose of identifying the different terms of a sequence by different powers of the symbol z.

Let G (z) be the generating function of the **geometric progression** $\{ar^n\}$ $n \geq 0$, i.e.

$$G(z) = a + (ar)z + (ar^2)z^2 + \cdots$$

Then,

$$G(z) - a = rz\left[a + (ar)z + \left(ar^2\right)z^2 + \right]$$
$$= rz(G(z))$$

which gives, on simplification, G (z) = a/ $(1 - rz)$

Why don't you try an exercise now?

E2) Verify that

a) The generating function for the finite geometric progression $\{a, ar, ar^2, ..., ar^{k-1}\}$ is a $(1 - r^k z^k)/(1 - rz)$.

b) The generating function for the sequence of Binomial coefficients $\{C(k,0), C(k,1)\,a, C(k,2)\,a^2, ...\}$ is $(1 + az)^k$.

c) the generating function for the sequence of Binomial coefficients $\{C(k-1,0), C(k,l)\,a, C(k+1,2)a^2, ...\}$ is $(1 - az)^{-k}$.

Note that the generating function for a finite sequence is the generating function for a corresponding infinite sequence which can be obtained by setting to zero every term not previously defined. Thus for a finite polynomial $a_0 + a_1 z + a_2 z^2$ we write

$$a_0 + a_1 z + a_2 z^2 + 0.z^3 + 0.z^4 + \cdots$$

Now let us see how the technique of associating a series with a sequence is helpful in solving a combinatorial problem. We try to understand this with the help of following example.

Example 4: Determine the number of subsets of a set of n elements, $n \geq 0$.

Solution: Let s_n denote the number of subsets that a set of n elements can have. In the previous unit, you have seen that the recurrence relation satisfied by the sequence $\{s_n\}$ is given by

$$s_n = 2s_{n-1} \text{ if } n \geq 1 \text{ and } s_0 = 1. \qquad \text{(see Example 7, Unit 1)}$$

Let S (z) stand for the generating function of the sequence $\{s_n\}_{n \geq 0}$. So, we can write

$$S(z) = \sum_{n=0}^{\infty} s_n z^n = 1 + \sum_{n=1}^{\infty} s_n z^n$$

$$= 1 + 2\sum_{n=0}^{\infty} s_{n-1} z^n \text{ (by definition of } S_n, n \geq 1)$$

$$= 1 + 2z\sum_{n=0}^{\infty} s_n z^n = 1 + 2z\,S(z),$$

$$\text{i.e. } S(z) = 1 + 2z\,S(z).$$

Solving last equation for S (z), we get

Two symbolic series $\sum a_n z^n$ and $\sum b_n z^n$ are **equal** iff $\mathbf{a_n} = \mathbf{b_n}$, $\forall$n.

$$S(z) = \frac{1}{1 - 2z} = \sum_{n=0}^{\infty} 2^n z^n. \qquad (\text{by Binomial theorem})$$

Finally, comparing the coefficients of z^n on both sides of above equations, we get $s_n = 2^n$, $n \geq 0$. Thus, the number of subsets of a set of n element is 2^n, $\forall$n.

As you have seen in above example, some (algebraic) operations are needed at the middle stage of the process while writing the general term of a sequence explicitly. These operations on generating functions, which we are defining below, have a crucial role to play in solving combinatorial problems.

Apart from the usual operations of addition, subtraction, multiplication and division of series, we may need to integrate or differentiate a power series. It is important to observe that, while performing last two operations, our aim is to associate with the object $\frac{d}{dz}\left(\sum a_n z^n\right)$ $\left(\text{and} \int\left(\sum a_n z^n\right)dz\right)$ a new power series as given in the right hand side of O_3 (and O_4, respectively).

O_1 (Sum and Difference) $$\sum a_n z^n \pm \sum b_n z^n = \sum (a_n \pm b_n) z^n$$

O_2.(Multiplication) $$\left(\sum a_n z^n\right)\left(\sum b_n z^n\right) = \sum\left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n;$$

O_3.(Differentiation) $$\frac{d}{dz}\left(\sum a_n z^n\right) = \sum (n+1)\, a_{n+1} z^n;$$

O_4.(Integration) $$\int\left(\sum a_n z^n\right)dz = \sum \frac{a_n}{n+1} z^{n+1}.$$

O_5.(Division) $$\left(\sum a_n z^n\right) \Big/ \left(\sum b_n z^n\right) = \sum c_n z^n$$

$$\Leftrightarrow \left(\sum b_n z^n\right)\left(\sum c_n z^n\right) = \sum a_n z^n, \text{i.e.,}\ a_n = \sum_{k=0}^{n} b_k c_{n-k}.$$

The quotient of two power series defined in O_5 above is via the product in the usual manner. In fact, there is no really convenient expression for the quotient.

Next, let us now look at some general results which provide connection between the generating functions of various sequences, terms of which are related in some manner to each other. These results are particularly useful when we know the generating functions of some of these, and want to find the same for others.

If $\{a_n\}$ and $\{b_n\}$ are two sequences, the sequence $\{c_n\}$, where $c_n = \sum_{0}^{n} a_k b_{n-k}$, is called the **convolution** of the sequences. If A(z) and B(z) are the generating functions of the sequences $\{a_n\}$ and $\{b_n\}$, respectively, according to O_2, the generating function of the convolution of $\{a_n\}$ and $\{b_n\}$ is A(z)B(z).
Here is an exercise involving convolution of sequences.

E3) Prove the Binomial identity $\sum_{j=0}^{k} C(m,j)C(n,k-j) = C(m+n,k)$, using Lemma 1. Hence deduce the Binomial identity
$\sum_{j=0}^{k} C(k,j)^2 = C(2k,k)$

We next prove another useful lemma of similar nature.

Lemma 1: Suppose that the sequence $\{a_n\}$ $n \geq 0$, has the generating function

A (z). Then, generating function B(z) (say) for the sequence $\{b_n\} n \geq 0$, where $b_n = a_n - a_{n-1}$ for $n \geq 1$, and $b_0 = a_0$, is given by

$$B(z) = (1 - z)A(z).$$

Proof: By definition, the generating function for the sequence $\{b_n\}$ is

$$\begin{aligned} B(z) &= \sum_{n=0}^{\infty} b_n z^n \\ &= b_0 + \sum_{n=1}^{\infty} b_n z^n \\ &= a_0 + \sum_{n=1}^{\infty} a_n z^n - z\sum_{n=1}^{\infty} a_{n-1} z^{n-1} \text{ (using definition of } b_n) \\ &= a_0 + [A(z) - a_0] - zA(z) \\ &= (1 - z)A(z) \end{aligned}$$

This completes the proof of the lemma.

Try the following exercise now.

E4) a) Use Lemma 1 to find the generating function A(z) (say) for the sequence in arithmetic progression {a, a +d, a+ 2d,…}.

b) Suppose that A(z) is the generating function for the sequence $\{a_n\}, n \geq 0$, Show that the generating function S(z) (say) for the sequence $\{s_n\}$ of its partial sums viz. $s_n = \sum_{k=0}^{n} a_k, (n \geq 0)$ is given by $S(z) = \frac{A(z)}{1-z}$.

c) Use (b) to find the generating function for the sequence {1, 3, 6, …}.

We next look at a problem which you might have solved earlier by different methods. Using generating functions, we shall give you alternative methods of solving them. This is an example involving the sum of k-th power of the first n natural numbers which we denote by σ_n^k

i.e $$\sigma_n^k = 1^k + 2^k + \ldots + n^k = \sum_{i=1}^{n} i^k, \ k \geq 1.$$

You already know how a formula for $\sigma_n^k \ (1 \leq k \leq 3)$ can be verified by induction (see Unit 2 of MCS-013). Let us see how generating function technique makes this task easier. You will see this in operation for the evaluation of $\sigma_n^2 = \sum_{j=1}^{n} j^2$ in the following example.

Example 5: Differentiating the Binomial function $(1 - z)^{-1} = \sum_{j=0}^{\infty} z^j$, we get

$$\sum_{j=1}^{\infty} jz^{j-1} = (1 - z)^{-2} \qquad \text{(see } O_3)$$

Multiplying this by z on both sides, we get

$$\sum_{j=1}^{\infty} jz^j = z(1 - z)^{-2}$$

Repeating this process of first differentiating and then multiplying by z, we get

$$A(z) = \sum_{j=1}^{\infty} j^2 z^j = z(1 + z)(1 - z)^{-3},$$

where we write A (z) for the generating function of the sequence $\{j^2\}_{j \geq 1}$.

Then

$$\sum_{n=1}^{\infty}\sigma_k^2 z^n = \sum_{k=1}^{k}\left(\sum_{j=1}^{\infty} j^2\right)z^n$$
$$= \frac{A(z)}{(1-z)} \text{ (by E4, b)}$$
$$= z(1+z)(1-z)^{-4}$$

Therefore, σ_n^2 is the coefficient of z^n in the series which can be obtained by expanding the function $z(1+z)(1-z)^{-4}$. However, because

$$z(1+z)(1-z)^{-4} = z(1-z)^{-4} + z^2(1-z)^{-4}$$

this is the same as looking for the sum of coefficients of z^{n-1} and z^{n-2} in the expanded form of the Binomial function $(1-z)^{-4}$. Thus, in view of Binomial identity

C (n, k) = C (n – k) we have

$$\sigma_n^2 = C(n+2,3) + C(n+1,3) = n(n+1)(2n+1)/6.$$

Try the following exercise now.

E5) Find the sum σ_n^1 of the first n natural numbers using generating functions.

So far, you learnt how to identify generating functions and use them to solve some simple combinatorial problems. However, there are several combinatorial problems which are hard to crack by using these functions. This is particularly true of problems that involve arrangements (in which **order** plays a crucial role) and distributions of distinct objects (see Block 2 of MCS-013 for more details). In the next section we introduce you to a slightly different kind of generating function which will prove useful for solving these type of problems.

2.3 EXPONENTIAL GENERATING FUNCTIONS

In this section, we shall study a modified form of the series we discussed in the last section. To understand the difference, let us consider the problem of finding the number of three-letter **words** i.e., a string of three letters which can be formed from a two-alphabet set {a, b} (say), with the restriction that not all letters in these **words** are identical.

Thus, we may use either two a's and one b or two b's and one a to form all the three letter **words** out of the two-element set {a, b}. Each of these two possibilities (by our discussion of permutations of objects, not necessarily distinct, in Block 2) give 3!/2!1!=3 distinct **words** viz. aab, aba, baa in the first case, and bba, bab, abb in the second, for a total of six **words.**

An ordered pair (x,y) of positive integers is a solution to the linear equation m+n =3, iff x+y=3.

Now, could we say that the number of distinct possibilities in the problem above is merely the number of positive integer solutions to the linear equation m+ n =3, if we think of this as using m a's and n b's, where m, n ≥ 1? This would have been so if we had not been interested in the position of a and b, in which case aab and aba would mean the same to us. But this is not the case. We are considering the number of three-letter **words** i.e., different strings of three letters. So, the position of the letters is important. Consequently, we would like each integer solution to contribute not 1 but 3 (so total is 3!) to the total number of **words.**

Now, as we wish to count the number of three letter **words**, we should look for the coefficient of z^3 in a series that counts (m+n) !/m!n! each time $z^m z^n = z^3$ appears in that. So, we try the product.

$$\left(\frac{z}{1!}+\frac{z^2}{2!}\right)\left(\frac{z}{1!}+\frac{z^2}{2!}\right)=\frac{z^2}{1!1!}+\frac{z^3}{1!2!}+\frac{z^3}{2!1!}+\frac{z^4}{2!2!}.$$

For r = 1, 2, 3, 4, the coefficient of z^r in this is term of the form 1/m!n!, where m+n = r, m, n $\geq$ 1. We need to multiply this by (m+n)! in order to get the answer we are looking for. Since the coefficient of z^3 in the above expansion is 1, we end up multiplying this by 3! to get a right answer to above problem.
An exponential generating function is precisely the power series of this type. A formal definition is given below.

Definition: The **exponential generating function** A_{exp} (z) (say) for the sequence of real or complex numbers $\{a_0, a_1, \ldots, a_n, \ldots,\}$ is given by the power series.

$$A_{exp}(z)=\sum_{k=0}^{\infty}\frac{a_k}{k!}z^k = a_0+\frac{a_1}{1!}z+\cdots+\frac{a_n}{n!}z^n+\cdots$$

As you can see, the nth term a_n of the given sequence is no longer the coefficient of z^n in $A_{exp}(z)$, rather it is n! times that coefficient.

For example, the exponential generating function for the constant sequence {1, 1, 1, …} is given by

$$\exp(z)=e^z=\sum_{n=0}^{\infty}\frac{1}{n!}z^n=1+z+\frac{z^2}{2!}+\cdots$$

Does it remind you of some function? Of course, it resembles exponential function with which you are familiar but here z is just a symbol and not a variable. It is this resemblance from where these type of generating functions have derived their name.

Try the following exercise now.

E6) Find the exponential generating function of the sequence $\{P(n,k)\}_{n=1}^{k}$ for a fixed $n \in \mathbf{N}$ where P (n, k) denotes the number of k-permutations of n objects.

As before, let us try to identify the exponential generating functions associated with the combinatorial problem given in the following example.

Example 6: Show that the exponential generating function associated with the problem of finding the number of ways to choose some subset of m objects and distribute them into n boxes in such a way that the order within the same box is important, is given by $e^z\,(1-z)^{-n}$.

Solution: First of all, there are n (n+1) … (n+k −1) ways to arrange the objects into n boxes. Let us see why this is so.
Let us first look at an example. Suppose we want to arrange 4 objects, numbered 1, 2, 3, 4 in 3 different boxes labelled a, b, c. Let us first consider the objects to be indistinguishable. Then, the number of ways of distributing 4 indistinguishable objects in 3 distinguishable boxes is C (4+3 −1,4) = C(6,4) = C(6,2) = 15 (See page 68

1 2

a

3 4

b

c

Fig. 1

2 3

a

1 4

b

c

Fig. 2

of Block 2, MCS-013). Let us fix one such distribution, say 2 in the first box, 2 in the second box and none in the third box. One such distribution with objects considered distinct is given in Fig. 1. Let us apply any permutation of 1, 2, 3, 4, say 1 to 2, 2 to 3 and 3 to 1. Then, we will get the arrangement given in Fig. 2. There are 4! such permutations. So, corresponding to each arrangement with objects considered indistinguishable, there are 4! arrangement with objects considered distinguishable. So, the number of arrangements of 4 objects in 3 boxes, when the order inside the boxes matters, is $4!\times C(4+3-1,4)=24\times 15=360$.

Let us now look at the general situation. Let us first assume that the objects are identical and only the number of objects in each box matters. The number of ways of distributing k indistinguishable objects in n distinguishable boxes is $C(n+k-1,k)$. Let us fix any one arrangement and apply all possible permutations of r objects. We will then get r! different arrangements when the order within the boxes are taken into account. So, there are $r!\times C(n+k-1,k) \;= n(n+1)\ldots(n+k-1)$ arrangements. There are C(m,k) ways of choosing k out of m objects. Thus, the total number of ways to choose some subset of m objects and distribute the objects into n boxes in such a way that the order in the same box is matters, are

$$C(m,0)+\sum_{k=1}^{m} n(n+1)...(n+k-1)C(m,k)$$

$$=m!\left[\frac{1}{m!}+\sum_{k=1}^{m}\frac{1}{(m-k)!k!}\times n(n+1)...(n+k-1)\right]$$

Here, we may take n to be fixed, and consider this a sequence in m alone. Therefore, the corresponding exponential generating function for this sequence is

$$\sum_{m=0}^{\infty}\left[\frac{1}{m!}+\sum_{k=1}^{m}\frac{1}{(m-k)!k!}\times n(n+1)...(n+k-1)\right]z^m,$$

which, in turn, is a product of the series

$$\left(\sum_{m=0}^{\infty}\frac{1}{m!}z^m\right)\text{and}\left(1+\sum_{m=1}^{\infty}\frac{n(n+1)...(n+m-1)}{m!}z^m\right).(\text{see}\, O_2)$$

Now the first series equals e^z (by definition), while the second equals $(1-z)^{-n}$, by Binomial theorem. Hence, we have obtained the associated exponential generating function, as claimed.

Let us work out few examples to get a feeling about some elementary uses of the exponential generating functions in solving combinatorial problems.

Example 7: Find the number of bijections on a set of n elements, $n \geq 1$.

Solution: Let b_n denote the number of bijections on a set of n elements, $n \geq 1$. Recall from the previous unit (Example 6) that the recurrence relation satisfied by the sequence $\{b_n\}$ is given by

$$b_n = nb_{n-1} \text{if } n \geq 2 \text{ and } b_1 = 1.$$

Since we do not know b_0, we will ignore this term. The exponential generating function B (z) (say) of the sequence $\{b_n\}$ is given by

$$B(z)=\frac{b_1}{1!}z+\frac{b_2}{2!}z^2+\frac{b_3}{3!}z^3+\cdots+\frac{b_r}{r!}z^r+\cdots$$

Then

$$\begin{aligned}B(z)&=\sum_{n=1}^{\infty}\frac{b_n}{n!}z^n\\ &=z+\sum_{n=2}^{\infty}\frac{nb_{n-1}}{n!}z^n \quad (\text{by definition of } b_n, n\geq 2)\\ &=z+z\sum_{n=1}^{\infty}\frac{b_n}{n!}z^n=z+z.B(z).\end{aligned}$$

Solving for B (z), we get

$$B(z)=z/(1-z)=\sum_{n=1}^{\infty}z^n. \qquad \text{(by Binomial theorem)}$$

So, by comparing coefficients of z^n, we get from the last equality $b_n = n$ for $n\geq 1$

At times, the exponential generating functions are also useful in calculating the sum of an infinite series. Let us see an example of this.

Example 8: Find the sum of the series

$$\sum_{k=0}^{\infty}\frac{(k+1)^2}{k!}=\frac{1^2}{0!}+\frac{2^2}{1!}+\cdots+\frac{(n+1)^2}{n!}+\cdots$$

using exponential generating functions.

Solution: Multiplying by z on the both side of $e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}$, we get

$$ze^z=\sum_{n=0}^{\infty}\frac{z^{n+1}}{n!}$$

This equation when differentiated once, gives

$$(1+z)e^z=\sum_{n=0}^{\infty}\frac{(n+1)z^n}{n!}. \qquad (\text{See}\, O_3)$$

Since we have already got one n+1 term in the numerator we are probably on the right track. We repeat the first two steps viz. multiply each side of the last equation by z and then differentiate, we get.

$$(1+3z+z^2)\,e^z=\sum_{n=0}^{\infty}\frac{(n+1)^2z^n}{n!}.$$

The rest of the job is easy. Put $z=1$ in the last equation to get $5e=\sum_{n=0}^{\infty}(n+1)^2/n!$.

Therefore, the required sum of the given series is 5e.

Why don't you try an exercise now?

E7) Using exponential generating functions, find the number d_n of derangements of n objects. (see Unit 1 and Unit 3, Block 2 of MCS-013 for more details on derangements.)

In the previous two sections, you have seen some elementary use of two type of generating functions. In the next section, we shall give some more applications of generating functions.

2.4 APPLICATIONS

In this section we will see some applications of generating functions. We will see how to derive some combinatorial identities using generating functions. After that we will see how to find the number of integer solutions of linear equations using generating functions. We will also see applications of generating functions to partitions and for solving recurrences.

So let us start by applying generating functions to solve some simple combinatorial identities, particularly those that involve Binomial coefficients.

2.4.1 Combinatorial Identities

By Binomial theorem:

$$(1+z)^n = \sum_{k=0}^{n} C(n,k)z^k, \qquad (2)$$

we know that $(1+z)^n$ is the generating function of the finite sequence $\{C(n,k)\}_{k=0}^{n}$. We shall use this to derive some combinatorial identities given in the following two examples.

Example 9: Prove the Binomial identity

$$C(n,1)+3C(n,3)+5C(n,5)+\cdots = n2^{n-2} = 2C(n,2)+4C(n,4)+6C(n,6)+\cdots$$

Solution: Differentiating both sides of (2) with respect to z, we get

$$n(1+z)^{n-1} = \sum_{k=0}^{n} kC(n,k)z^{k-1}.$$

Now setting $z=1$ and $z=-1$ in the resulting expression, we get

$$\sum_{k=1}^{n} kC(n,k) = n2^{n-1}, \text{and} \qquad (3)$$

$$\sum_{k=1}^{n} (-1)^{k-1} kC(n,k) = 0, \text{ respectively.} \qquad (4)$$

Shifting negative terms to the r.h.s. in (4), we have

$$C(n,1) + 3C(n,3) + 5C(n,5) + \cdots = 2C(n,2) + 4C(n,4) + 6C(n,6) + \cdots$$

Now, on adding terms 2C(n,2), 4C(n,4), 6C(n, 6) ⋯ so on, to both sides of above identity, we get

$$\sum_{n=1}^{\infty} kC(n,k) = 2[2C(n,2) + 4C(n,4) + 6C(n,6) + \cdots]. \quad (5)$$

From this, using (3) it follows that r.h.s of (5) equals $\frac{n2^{n-1}}{2} = n2^{n-2}$. With this we have established the Binomial identity stated above.

Our next application concerns k-permutations of a set of n elements. By E12, of Unit 7, you know that the number of k-permutations of n distinct objects, P (n, k), satisfies the recurrence relation

$$P(n,k) = P(n-1, k) + kP(n-1, k-1), n, k \geq 1. \quad (6)$$

Example 10: For fixed n, find an explicit formula for P (n, k) by making use of its exponential generating function, $P_{exp}(z;n)$ (say) as defined below.

$$P_{exp}(z;n) = \sum_{k=0}^{\infty} (P(n,k)/k!)z^k.$$

Solution: Using (6) and the definition of P_{exp} (z; n) we have

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{P(n,k)}{k!} z^k &= \sum_{k=1}^{\infty} \frac{P(n-1,k)}{k!} z^k + \sum_{k=1}^{\infty} \frac{P(n-1,k-1)}{k!} z^k \\
\text{i.e.} \sum_{k=1}^{\infty} \frac{P(n,k}{k!} z^k &= \sum_{k=1}^{\infty} \frac{P(n-1,k)}{k!} z^k + z\sum_{k=1}^{\infty} \frac{P(n-1,k-1)}{(k-1)!} z^{k-1} \\
\Rightarrow P_{exp}(z;n) - P(n,0) &= \left[P_{exp}(z;n-1) - P(n-1,0)\right] + zP_{exp}(z;n-1) \\
\Rightarrow P_{exp}(z;n) &= (1+z)P_{exo}(z;n-1) (\text{as} P(n,0) = P(n-1,0)) \\
\Rightarrow P_{exp}(z;n) &= (1+z)^n P_{exp}(z;0) = (1+z)^n. \ (\text{by iteration})
\end{aligned}$$

Since the coefficient of z^k in $(1+z)^n$ is C (n, k) (by Binomial theorem), it follows by comparing coefficients, that

$$\frac{P(n,k)}{k!} = C(n,k) \Rightarrow P(n,k) = k!C(n,k) = \frac{n!}{(n-k)!}.$$

Of course, if k > n, C (n, k) = 0, and hence P (n, k) = 0 then. So, we have obtained P(n, k), explicitly.

Try the following exercise now.

E8) Evaluate, using generating function technique, the sum $\sum_{k=1}^{n} k3^k C(n,k)$.

We next consider the application of generating functions to general integer equations.

2.4.2 Linear Equations

Generating functions are also particularly handy when one is looking for non-negative integer solutions to linear equations of the type $a_1+a_2+\cdots+a_k = n$. You may recall that we showed earlier (see Theorem 5 of Unit 2, Block 2 of MCS-013) that this equals $C(n+k-1, k-1)$ by elementary counting techniques. If, on the other hand, each a_j is a positive integer, then the number of such solutions equals $C(n-1, k-1)$.

Generating functions often provide a simpler way to solve such equations. This is illustrated in the following example.

Example 11: Find the number of integer solutions of the linear equation.

$$a_1 + a_2 + \cdots + a_k = n,$$

using generating function techniques when a) $a_i \geq 0$ b) $a_i \geq 1$.

Solution: a) The required number is the coefficient of z^n in the following product of polynomials (see discussion following Example 1)

$$(1 + z + z^2 + \cdots)\cdots(1 + z + z^2 + \cdots) \qquad \text{(k times)}$$

Each term of this product equals $(1-z)^{-1}$ (by Binomial theorem) and the coefficient of z^n in $(1-z)^{-k}$ is

$$C(n+k-1, n) = C(n+k-1, k-1);$$

b) If each $a_j \geq 1$ instead, we seek the coefficient of z^n in the expansion

$$(z+z^2+z^3+\cdots) \ldots (z+z^2+z^3+\cdots). \qquad \text{(k times)}$$

Each term of this product equals $z(1-z)^{-1}$ (by Binomial theorem) and the coefficient of z^n in $z^k(1-z)^{-k}$ is the coefficient of z^{n-k} in $(1-z)^{-k}$. This equals

$C(n-k) + k-1, n-k) = C(n-1, k-1)$.

Of course, this means that there is no solution if $n < k$, as should be the case.

* * *

If, in the example above, we require that one or more of the solutions, a_j are bounded at both ends, and if we allow a_j to be negative, then the number of solutions, even for $k = 2$ or 3 becomes a tedious computation. The method of generating functions is just what you could use for such problems. We illustrate this in the following example.

Example 12: Find the number of integer solutions to $a_1 + a_2 + a_3 = n$, where $-1 \leq a_1 \leq 1$, $1 \leq a_2 \leq 3$ and $a_3 \geq 3$.

Solution: Let us bring this into the situation of Example 11. For this, we put $b_1 = a_1+1$ and $b_3 = a_3 - 3$. Then our problem is same as looking for the number of integer solutions to

$$b_1 + b_2 + b_3 = n - 2, \text{ where } 0 \leq b_1 \leq 2, 1 \leq b_2 \leq 3 \text{ and } b_3 \geq 0.$$

Now in view of these bounds on b_i's, it follows that associated generating function is given by

$$(1+z+z^2)(z+z^2+z^3)(1+z+z^2+\cdots)=\frac{1-z^3}{1-z}\times\frac{z(1-z^3)}{1-z}\times\frac{1}{1-z},$$

by using Binomial theorem and Result 2. As before, we want the coefficient of z^{n-2} in this expansion, which is same as the coefficient of z^{n-3} in

$$(1-z^3)^2(1-z)^{-3}=(1-z)^{-3}-2z^3(1-z)^{-3}+z^6(1-z)^{-3}.$$

Let us assume that C(n,k) = 0 for k > n.

$$(1-z)^{-3}=\sum_{k=0}^{\infty}C(3+k-1,2)z^k$$

$$z^3(1-z)^{-3}=\sum_{k=0}^{\infty}C(3+k-1,2)z^{k+3}$$

$$z^6(1-z)^{-3}=\sum_{k=0}^{\infty}C(3+k-1,2)z^{k+6}$$

So, the coefficient of z^{n-3} in $(1-z)^{-3}(1-z^3)^2$ is

$$C(3+n-3-1,2)-2C(3+n-6-1,2)+C(3+n-9-1,2)$$
$$=C(n-1,2)-2C(n-4,2)+C(n-7,2)$$

Since we have assumed that C(n,k) = 0 if $n<k$ all the terms are non-zero only if $n-7\geq 2$ or $n\geq 9$. If this is the case,

$$C(n-1,2)-2C(n-4,2)+C(n-7,2)$$
$$=\frac{(n-1)(n-2)}{2}-2\frac{(n-4)(n-5)}{2}+\frac{(n-7)(n-8)}{2}$$
$$=\frac{n^2-3n+2-(2n^2-18n+40)+(n^2-15n+56)}{2}=9.$$

If $n-4\geq 2$ and $n-7\leq 1$, $6\leq n\leq 8$. In this case, the answer is

$$\frac{(n-1)(n-2)}{2}-2\frac{(n-4)(n-5)}{2}=\frac{n^2-3n+2-2n^2+18n-40}{2}$$
$$=\frac{-n^2+15n-38}{2}$$

For n = 6, 7, 8, this quantity is 8, 9, 9, respectively.

If $n-4\leq 1$ and $n-1\geq 2$, $3\leq n\leq 5$ and the value is $\frac{(n-1)(n-2)}{2}$. If $n-1<2$ or $n<3$, all the terms are 0, i.e. there are no solutions.

The technique adopted in the example given above is no different if we have more than three summands or if the bounds we had are more general. In principle, therefore, we are in a position to find the number of integer solutions to

$$a_1+a_2+\cdots+a_k=n, \text{with } m_j\leq a_j\leq M_j, m_j, M_j\in Z. \quad (1\leq j\leq k)$$

Why don't you check your understanding of Example 12 by attempting the following exercise?

E9) How many integer solutions are there to $a_1+a_2+a_3+a_4+a_5=28$ with $a_k>k$ for each k, $1\leq k\leq 5$?

Another illustration of the use of generating functions is in the mathematical theory of partitions – historically one of the first problems studied with generating functions. We shall talk about this next.

2.4.3 Partitions

We shall only see one aspect of partitions namely their connection with generating functions. You already had some exposure to them in your earlier Mathematics course. Here we will go a little deeper. For this, we should first define the sequence of partitions, P_n.

Definition: The nth term of the sequence $\{P_n\}$, $n \geq 1$, counts the number of ways in which n can be expressed as a sum of positive integers such that the order of the summands (parts) is not important. We define $P_0 = 1$.

For Example, $P_4 = 5$ since 4 = 3+1=2+2=2+1+1=1+1+1+1. So, partitioning n is the same as distributing n non-distinct objects into n non-distinct boxes, with the empty box allowed (e.g. 4=3+1+0+0). In terms of linear equations discussed above, P_n is the number of non-negative integer solutions to the integer equation.

$$X_1 + X_2 + \cdots + X_k + \cdots = n, X_i = ia_i \; (\forall i),$$

where a_k denotes the number of k's in the partition. Note here, that the number of X_i's is not bounded. It can grow very large according to the size of n.

Let us look at the form that the generating function, P (z) of the sequence $\{P_n\}_{n\geq 0}$ must take.

Note that, in the above linear equation, for each integer $k \geq 1$, we may use none, one or more k's according to the value of $a_k \geq 0$. There is no other restriction on a_k's. Therefore, for each term $X_i = i.a_i$ $(a_i \geq 0)$ the corresponding term in the associated generating function is simply $(1 + z^k + z^{2k} + \cdots)$.

$$P(z) = \prod_{k=1}^{\infty} (1 + z^k + z^{2k} + \cdots) = \prod_{k=1}^{\infty} \frac{1}{1 - z^k}$$

Generating functions of related sequences are not any harder to determine. They play a significant role in proving identities involving partitions. We illustrate this with the help of the following example.

Example13: Show that every nonnegative integer can be written as a unique sum of distinct powers of 2.

Solution: The generating function for the sequence $\{a_n\}$, where a_n denotes the number of ways n can be written as sum of distinct power of 2, is

$$(1+z)\,(1+z^2)\,(1+z^4)\,(1+z^8)\ldots$$

Now we have

$$\begin{aligned}
&(1-z)(1+z)(1+z^2)(1-z^4)(1+z^8)\cdots \\
&= (1-z^2)(1+z^2)(1+z^4)(1+z^8)\cdots \\
&= (1-z^4)(1+z^4)(1+z^8)\cdots \\
&= (1-z^{2n})(1+z^{2n})\cdots \\
&= 1 \quad (\text{assuming } |z| < 1.)
\end{aligned}$$

Thus, in view of operation O_5 and Binomial theorem, it follows that

$$(1+z)(1+z^2)(1+z^8)\cdots = \frac{1}{1-z} = 1 + z + z^2 + \cdots$$

From this, by comparing coefficients, we conclude that the coefficient of z^n in the 1.h.s of the equation is 1. Hence, the number a_n of partitions of n into distinct parts of size 1, 2, 4, 8, 16, …, so on, is 1. In other words, every non-negative integer can be **uniquely** expressed as the sum of distinct powers of 2.

Why don't you try the following exercise now?

E10) Show that the generating function for the sequence of the number of partitions of n with:

a) Parts each of which is at most m is $\prod_{k=1}^{m}(1-z^k)^{-1}$;

b) Unequal parts is $\prod_{k=1}^{\infty}(1+z^k)$;

c) Parts each of which is odd is $\prod_{k=1}^{\infty}(1-z^{2k-1})^{-1}$

E11) Find the generating function for the sequence of the number of partitions of n

i) into primes;
ii) into distinct primes.

Next, we shall discuss one of the most important uses of generating functions, viz., its utility as a tool to solve the recurrence relations.

2.4.4 Recurrence Relations

In unit 1, you have learnt how to set up recurrences for a combinatorial problem. Though we had not talked about how to solve them, we gave you some solutions, which you verified.

For solving a recurrence, we need to know the terms of a sequence explicitly. In other words, for a sequence $\{a_n\}$ that satisfies a given recurrence, we shall use its generating function A (z) (say) to find an explicit formula for a_n in terms of n.

Let us look at an example to see how we can solve recurrences using generating functions.

Example 14: Solve the recurrence $L_n = L_{n-1} + n$ for $n \geq 2$, and $L_1 = 2$.

Solution: The recurrence relation satisfied by the sequence $\{L_n\}$ is $L_n = L_{n-1} + n$ for $n \geq 2$, and $L_1 = 2$. If the same recurrence were to hold for $n \geq 1$ instead, then L_0 must equal 1.

Starting the sequence at L_0, the generating function L (z) (say) of the sequence $\{L_n\}_{n\geq 0}$ is given by

$$L(z) = \sum_{n=0}^{\infty} L_n z^n.$$

Now, by using the recurrence relation, we get

$$
\begin{aligned}
L(z) &= 1+\sum_{n=1}^{\infty}(L_{n-1}+n)z^{n} \\
&= 1+z\sum_{n=1}^{\infty}L_{n-1}z^{n-1}+z\sum_{n=1}^{\infty}nz^{n-1} \\
&= 1+z\sum_{n=0}^{\infty}L_{n}z^{n}+z\sum_{n=1}^{\infty}nz^{n-1} \\
&= 1+z.L(z)+\frac{z}{(1-z)^{2}}.
\end{aligned}
$$

Solving for L (z) in the last equation, we get

$$L(z)=\frac{1}{1-z}+\frac{z}{(1-z)^{3}}.$$

So, using Binomial theorem, we get

$$L(z)=\sum_{n=0}^{\infty}\left\{1+\frac{1}{2}n(n+1)\right\}z^{n}.$$

Finally, equating coefficients of z^n on both sides of the last equation, we get

$$L_n=\frac{1}{2}n(n+1)+1, n\geq 1.$$

We next consider the sequence of Fibonacci numbers $\{f_n\}$ which satisfy the recurrence relation

$$f_n = f_{n-1}+f_{n-2} \text{ if } n \geq 3, \text{ and } f_1=1=f_2.$$

Example 15: Find the generating function associated with the sequence of Fibonacci sequence $\{ f_n \} n \geq 1$. Then deduce a formula for f_n , $n \geq 1$.

Solution: We write F (z) for the associated generating function. Then by definition, we have

$$
\begin{aligned}
F(z) &= \sum_{n=1}^{\infty} f_n z^n \\
&= z+z^2+\sum_{n=3}^{\infty}(f_{n-1}+f_{n-2})z^n \\
&= z+z^2+z\sum_{n=2}^{\infty} f_n z^n+z^2\sum_{n=1}^{\infty} f_n z^n \\
&= z+z^2+z[F(z)-z]+z^2.F(z).
\end{aligned}
$$

Then $(1-z-z^2)$ F (z) = z. Therefore, with $\alpha=(1+\sqrt{5})/2$ and $\beta=\left(1-\sqrt{5}\right)/2$,

$$\beta = (1-\sqrt{5})/2,$$

$$\begin{aligned}
F(z) &= \frac{z}{(1-\alpha z)(1-\beta z)} \qquad \text{(by solving equation } z^2 + z - 1 = 0) \\
&= \frac{1}{\alpha-\beta}\left(\frac{1}{1-\alpha z} - \frac{1}{1-\beta z}\right) \\
&= \frac{1}{\alpha-\beta}\sum_{n=0}^{\infty}(\alpha^n - \beta^n) z^n. \text{ (by binomial theorem)}
\end{aligned}$$

Comparing coefficients of z^n now gives $f_n = (\alpha^n - \beta^n)/(\alpha-\beta)$, for all $n \geq 1$.
Try the following exercise now.

E12) Solve the recurrence relation $T_n = 2T_{n-1}+1$ if $n \geq 2$ and $T_1 = 1$, using generating functions technique. (See Tower of Hanoi problem in Unit 1).

If you have understood the steps that we followed in solving the recurrence relation involving Fibonacci sequence in previous example, then it should not be difficult for you to understand the proof of the following general result.

Theorem 1: The generating function, denoted by U(z), for a general linear, homogeneous recurrence relation with constant coefficients, of order k,

$u_n = a_1 u_{n-1} + a_2 u_{n-2} + \cdots + a_k u_{n-k}$, $n \geq k$, with $u_0 = c_0, \ldots, u_{k-1} = c_{k-1}$ satisfies the equation

$$\left(1 - a_1 z - a_2 z^2 - \cdots - a_k z^k\right) U(z) = c_0 + \sum_{n=1}^{k-1}(c_n - a_1 c_{n-1} - \cdots - a_n c_0) z^n.$$

Proof: We have, by definition,

$$\begin{aligned}
U(z) &= \sum_{n=0}^{\infty} u_n z^n \\
&= (u_0 + u_1 z + \ldots + u_{k-1} z^{k-1}) + \sum_{n=k}^{\infty} u_n z^n
\end{aligned}$$

$$\begin{aligned}
\sum_{n=k}^{\infty} u_n z^n &= \sum_{n=k}^{\infty}\left(a_1 u_{n-1} + a_2 u_{n-2} + \cdots + a_k u_{n-k}\right) Z^n \\
&= a_1 z \sum_{n=k}^{\infty} u_{n-1} z^{n-1} + a_2 z^2 \sum_{n=k}^{\infty} u_{n-2} z^{n-2} + \cdots + a_k z^k \sum_{n=k}^{\infty} u_n z^{n-k} \\
&= a_1 z \sum_{n=k-1}^{\infty} u_n z^n + a_2 z^2 \sum_{n=k-2}^{\infty} u_n z^n + \cdots + a_k z^k \sum_{n=0}^{\infty} u_n z^n \\
&= a_1 z\left(U(z) - \left(u_0 + u_1 z + u_2 z^2 + \cdots + u_{k-2} z^{k-2}\right)\right) \\
&\quad + a_2 z^2\left(U(z) - \left(u_0 + u_1 z + u_2 z^2 + \cdots + u_{k-3} z^{k-3}\right)\right) \\
&\quad + \cdots + a_{k-1} z^{k-1}\left(U(z) - u_0\right) + a_k z^k U(z)
\end{aligned}$$

Using $u_i = c_i$ for $0 \leq i \leq k$,

$$\begin{aligned}\sum_{n=k}^{\infty} u_n z^n &= \left(a_1 z + a_2 z^2 + \cdots + a_k z^k\right)U(z) - a_1 z\left(c_0 + c_1 z + c_2 z^2 + \cdots + c_{k-2} z^{k-2}\right) \\ &\qquad - a_2 z^2\left(c_0 + c_1 z + c_2 z^2 + \cdots + c_{k-2} z^{k-2}\right) \\ &\qquad \cdots \\ &\qquad - a_{k-2} z^{k-2}\left(c_0 + c_1 z\right) - a_{k-1} c_0 z^{k-1} \\ &= \left(a_1 z + a_2 z^2 + \cdots + a_k z^k\right)U(z) - \left(a_1 c_0 z + (a_1 c_1 + a_2 c_0) z^2\right. \\ &\quad + \cdots + \left(a_1 c_{k-3} + a_2 c_{k-4} + \cdots + a_{k-3} c_1\right) z^{k-2} \\ &\qquad \left. + \left(a_1 c_{k-2} + a_2 c_{k-3} + \cdots + c_0 a_{k-1}\right) z^{k-1}\right) \\ \therefore U(z) &= c_0 + c_1 z + c_2 z^2 + \cdots + c_{k-1} z^{k-1} - \sum_{n=k}^{\infty} u_n z^n \\ &= \left(a_1 z + a_2 z^2 + \cdots + a_k\right)\left(U(z)\right) + c_0 + \left(c_1 - a_1 c_0\right) z + \left(c_2 - a_1 c_1 - a_2 c_0\right) z^2 \\ &\quad + \cdots + \left(c_{k-1} - a_1 c_{k-2} - a_2 c_{k-3} - \cdots - c_0 a_{k-1}\right) z^k\end{aligned}$$

$$\left(1 - \left(a_1 z + a_2 z^2 + \cdots + a_k z^k\right)\right)U(z) = c_0 + \sum_{i=1}^{k-1}\left(c_i - c_{i-1} a_1 - c_{i-2} a_2 + \cdots + c_0 a_i\right) z^i$$

$$\therefore U(z) = \frac{Q_k(z)}{P_{k-1}(z)}$$

where

$$Q_k(z) = 1 - \left(a_1 z + a_2 z^2 + \cdots + a_k z^k\right)$$

and $P_{k-1}(z) = c_0 + \sum\limits_{n=0}^{k-1} (c_n - a_1 c_{n-1} - \cdots - a_n c_0) z^n$

This completes the proof of the theorem.

A first conclusion that you can easily deduce from the theorem above, is given in the following result.

Corollary 1: The generating function of linear, homogeneous recurrence relations with constant coefficients given in Theorem 1 is a rational function, p(z) / q (z), with the numerator, p(z), a polynomial of degree at most one less than the order of the recurrence.

Also observe that 1+q(z) is equal to the polynomial obtained from the R. H. S. of the given recurrence relation given in Theorem 1 by replacing u_{n-i} with z^i $(1 \le i \le k)$. While applying this corollary, you need to pay careful attention to the form of q (z). You should not try to memorize p (z) at all. After all, once you know q (z), p (z) can be obtained by multiplying q (z) by the generating series $\sum\limits_{n=0}^{\infty} u_n z^n$.

Let us employ Theorem 1 and Corollary 1 to solve the following recurrence relation.

Example 16: Solve the third-order recurrence

$$u_n - 9u_{n-1} + 26u_{n-2} - 24u_{n-3} = 0, n \ge 3,$$

with the initial conditions $u_0 = 6$, $u_1 = 17$ and $u_2 = 53$.

Solution: We denote by U (z) the generating function for the sequence $\{u_n\}$. Then, by Theorem 1, we know that $(1 - 9z + 26z^2 - 24z^3)\, U(z) = P(z)$ is a polynomial of degree 4 in z.. Now, a little more calculations will lead you to conclude that

$$(1 - 9z + 26z^2 - 24^3)\, U(z) = (1 - 2z)(1 - 3z)(1 - 4z)\, U(z)$$

$$= \sum_{n=0}^{\infty} u_n z^n - 9\sum_{n=0}^{\infty} u_n z^{n+1} + 26\sum_{n=0}^{\infty} u_n z^{n+2} - 24\sum_{n=0}^{\infty} u_n z^{n+3}$$

$$= u_0 + (u_1 - 9u_0)z + (u_2 - 9u_1 + 26u_0)z^2$$

$$+ \sum_{n=3}^{\infty}(u_n - 9u_{n-1} + 26u_{n-2} - 24u_{n-3})z^n$$

$= 6 - 37z + 56z^2$, by using the given recurrence relation and substituting the values $u_0 = 6$, $u_1 = 17$ and $u_2 = 53$.

Therefore,

$$U(z) = (6 - 37z + 56z^2) / (1 - 2z)(1 - 3z)(1 - 4z).$$

Decomposing the R.H.S. into partial fractions, we then get

$$U(z) = 3(1-2z)^{-1} + (1-3z)^{-1} + 2(1-4z)^{-1}.$$

Expanding the R.H.S. using Binomial theorem and comparing the co-efficient of z^n both sides, we get

$$u_n = 3.2^n + 3^n + 2.4^n, n \geq 0.$$

Try the following exercise now.

E13) Determine the generating function for the sequence $\{t_n\}_{n=0}^{\infty}$ given by the recurrence relation $(n \geq 3)$

$$t_n = \begin{cases} t_{n-3} & \text{if n is even} \\ t_{n-3} + \dfrac{n + (-1)^{(n+1)/2}}{4} & \text{if n is odd} \end{cases}$$

You may take $t_0 = t_1 = t_2 = 0$.

In yet another situation, let us next consider the case of nonhomogenous recurrences viz. when the nonhomogenous term (s) are either of the types r^n $(r \in C)$ or n^k $(k \in N \cup \{0\})$. Below we consider the case when it is of the form r^n. The method of generating functions, and in particular Theorem 1, can still be of use to good effect as the following example shows.

Example 17: Solve the third-order nonhomogeneous linear recurrence with constant coefficients viz. $u_n - 3u_{n-2} - 2u_{n-3} = an + b.2^n$ in terms of the initial conditions u_0, u_1 and u_2.

Solution: Write U (z) for the generating function of the sequence $\{u_n\}_{n \geq 0,}$ then

$$(1 - 3z^2 - 2z^3)U(z) = (1+z)^2(1-2z)U(z)$$

$$= \sum_{n=0}^{\infty} u_n z^n - 3\sum_{n=0}^{\infty} u_n z^{n+2} - 2\sum_{n=0}^{\infty} u_n z^{n+3}$$

$$= u_0 + u_1 z + (u_2 - 3u_0)z^2 + \sum_{n=3}^{\infty}(u_n - 3u_{n-2} - 2u_{n-3})z^n$$

$$= u_0 + u_1 z + (u_2 - 3u_0)z^2 + az\sum_{n=3}^{\infty} nz^{n-1} + b\sum_{n=3}^{\infty}(2z)^n$$

$$= (u_0 - b) + (u_1 - a - 2b)z + (u_2 - 3u_0 - 2a - 4b)z^2$$

$$-\frac{a}{(1-z)^2} - \frac{a}{1-z} + \frac{b}{1-2z}.$$

The rest of the calculation is tedious, but routine. We employ partial fractions, to get U (z) in the form

$A(1-z)^{-1} + B(1-z)^{-2} + C(1+z)^{-1} + D(1+z)^{-2} + E(1-2z)^{-1} + F(1-2z)^{-2}$ for some choice of A, … , F. In terms of these constants.

$$u_n = A + B(n+1) + C(-1)^n + D(-1)^n(n+1) + E.2^n + F.2^n(n+1), n \geq 0.$$

Try the following exercise now.

E14) Use Theorem 1 to solve the recurrence
$a_n - 3a_{n-1} - 10a_{n-2} = 28 \times 5^n$ for $n \geq 2$, with $a_0 = 25$ and $a_1 = 120$.

It is sometimes possible to solve even non-linear recurrences with the help of generating functions. We illustrate this by solving a recurrence about which you have read before in Unit 1.

Example 18: Solve the recurrence relation for Catalan numbers given by

$a_n = a_{n-1}a_1 + a_{n-2}a_2 + \cdots + a_2 a_{n-2} + a_1 a_{n-1}, n \geq 2$, with $a_n \geq 0\,(\forall n)$ and $a_1 = 1$.

Solution: In order to extend the validity of the given recurrence to $n \geq 1$. we define $a_0 = 0$. If we denote its generating function by A (z), we get

$$\sum_{n=2}^{\infty} a_n z^n = \sum_{n=2}^{\infty}(a_{n-1}a_1 + a_{n-2}a_2 + \cdots + a_2 a_{n-2} + a_1 a_{n-1})z^n$$

$$\Rightarrow A(z) - a_1 z - a_0 = \{A(z)\}^2 - (a_1 a_0 + a_0 a_1)z - a_0^{\,2} \quad (\text{by } O_2)$$

$$\Rightarrow \{A(z)\}^2 - A(z) + z = 0$$

$$A(z) = \frac{1 \pm \sqrt{1-4z}}{2}.$$

Now, using Binomial Theorem, the coefficient of z^n in $(1-4z)^{1/2}$ is equal to

$$\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-n+1\right)}{n!}(-4)^n,$$

which you can easily simplify to $-\frac{2}{n}C(2n-2, n-1)$.

We choose the solution $A(z) = (1-\sqrt{1-4z})/2$, so that the terms a are non negative for $n \geq 1$, we thus have

$$a_n = \frac{1}{n}C(2n-2, n-1) = \frac{(2n-2)!}{(n-1)!\,n!}.$$

E15) Using Theorem 1, find the n^{th} term, L_n of the Lucas sequence given by $L_n = L_{n-1} + L_{n-2}, n \geq 3, \text{with } L_1 = 1, L_2 = 3.$

So far we have discussed the use of generating functions in various areas. Regarding linear recurrence relations, we have seen how useful they are for finding solutions of such equations. There are several others methods for solving equations of this kind. We shall discuss them in the next unit. For now let us summarise what we have covered in this limit.

2.5 SUMMARY

In this unit we have seen how to:

1) construct generating functions for sequences arising from combinatorial problems.
2) find the number of integer solutions to linear equations.
3) find the generating functions associated with sequences in closed form in certain simple cases.
4) find the exponential generating functions associated with sequences in closed form in certain special cases.
5) solve recurrence relations in certain special cases.
6) use generating functions to prove identities involving combinatorial co-efficients.

2.6 SOLUTIONS / ANSWERS

Note: In all the following solutions, we will skip some steps and you are encouraged to work out the individual steps to ensure understanding of the computational procedure. In most cases, second block of MCS-013 will be helpful.

E1) a) The associated power series is
$(z^5 + z^{10} + \cdots)(1 + z^2 + z^4 + \cdots)(1 + z^3 + z^6 + \cdots).$
Here the first polynomial does not contain the constant term because of the given condition.

b) Since each k, ℓ and m are positive by given condition, and so, for a choice of (k+ℓ+m) fruits (with 5k+2ℓ+3m=50) , the associated power series is $(z^5 + z^{10} + \cdots)(z^2 + z^4 + \cdots)(z^3 + z^6 + \cdots)$.

E2) a) The generating function for the finite geometric progression is
$$\sum_{n=0}^{k-1} ar^n z^n = a \sum_{n=0}^{k-1} (rz)^n = a(1 - r^k z^k)/(1 - rz), \text{ by result } \mathbf{R2}.$$

b) Replacing z by az in the Binomial theorem, it follows that $(1 + az)^k$ is the generating function for the sequence $\{C(k,n)a^n\}_{n=0}^{\infty}$, if k is negative. This gives solution of (b).

c) Replacing z by az in Binomial theorem result IC, we get the result.

E3) For positive m and n, since $(1+z)^m$ is the generating function for the sequence $\{C(m,k)\}_{k=0}^{\infty}$, and $(1+z)^n$ is the generating function for $\{C(n,k)\}_{k=0}^{\infty}$, the function $(1+z)^m (1+z)^n$ is the generating function for the sequence with kth term $\sum_{j=0}^{k} C(m,j)C(n,k-j)$. However, $(1+z)^{m+n}$ is the generating function for

$\{C(m+n,k)\}_{k=0}^{\infty}$. Hence the first identity.The second identity follows from the first by taking m = n = k and using the identity C (n, k) = C (n, n-k).

E4) a) Write $a_n = a+nd, n \geq 0$. Then $a_n - a_{n-1} = d, \forall n \geq 1$. and $a_0 = a$. Let $\{b_n\}$ denote the sequence, where $b_0 = a$ and $b_n = d, \forall n \geq 1$.
By definition, B (z) = a+dz+dz^2+ … = a+zd [1+z+z^2+…] = a+dz $(1-z)^{-1}$, which is the generating function for the sequence $\{b_n\}_{n\geq 1}$. Thus, by Lemma 2, B (z) = (1–z) A (z)
$\Rightarrow \quad A(z) = a(1-z)^{-1} + z\,d(1-z)^{-2} = \{a+(d-a)z\}(1-z)^{-2}$.

b) Since $a_n = s_n - s_{n-1}$, for $n \geq 1$, and $a_0 = s_0$, so, we have
$(1-z)S(z) = A(z)$. (by Lemma 1)
Finally, proof is complete by using the definition O_5 of quotients of series.

c) The n th term of the given sequence is the n th partial sum of the sequence {1, 2, 3, …,} whose generating function A (z) (say) is $(1-z)^{-2}$ by (a). Hence, by (b), the generating function for the sequence {1, 3, 6,…} equals $(1-z)^{-3}$.

E5) Differentiating the Binomial function $(1-z)^{-1} = \sum_{j=0}^{\infty} z^j$, we get

$$\sum_{j=0}^{\infty} jz^{j-1} = (1-z)^{-2} \qquad (\text{see } O_3).$$

On multiplying this by z both sides, we get

$$A(z) = \sum_{j=1}^{\infty} jz^j = z(1-z)^{-2},$$

where we write A (z) for the generating function of the sequence $\{j\}_{j\geq 1}$. Then

$$\sum_{k=1}^{\infty} \sigma_k^1 z^k = \sum_{k=1}^{\infty}\left(\sum_{j=1}^{k} j\right) z^k$$
$$= \frac{A(z)}{(1-z)} \qquad (\text{by E4 b})$$
$$= z(1-z)^{-3}.$$

Therefore, σ_n^1 is the coefficient of z_n in the series which can be obtained by expanding the function z $(1-z)^{-3}$. However, this is the same as looking for the coefficients of z^{n-1}. In the expanded form of the Binomial function $(1-z)^{-3}$. Thus, in view of Binomial identity C (n, k) = C (n, n– k), we have

$$\sigma_n^1 = C(n+1, n-1) = C(n+1, 2) = n(n+1)/2.$$

E6) By definition, exponential generating function of the sequence $\{P(n,k)\}_{k=1}^{n}$ is

$$\sum_{k=0}^{\infty} \frac{P(n,k)}{k!} z^k = \sum_{k=0}^{\infty} C(n,k) z^k = (1+z)^n.$$

E7) A first-order recurrence equation that the sequence $\{d_n\}$ satisfies is given by
$d_n = nd_{n-1} + (-1)^n, n \geq 2, \text{with } d_1 = 0, d_2 = 1$. (see Problem 7 of Unit 1).
In order that the recurrence also holds for n =1, we define d_0=1.Then, with

$$D_{exp}(z)=\sum_{n=0}^{\infty}(d_n/n!)z^n, \text{we have}$$

$$\sum_{n=1}^{\infty}\frac{d_n}{n!}z^n=\sum_{n=1}^{\infty}\frac{nd_{n-1}}{n!}z^n+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}z^n$$

$$\Rightarrow D_{exp}(z)-d_0=zD_{exp}(z)+(e^{-z}-1)$$

$$\Rightarrow D_{exp}(z)=\frac{e^{-z}}{1-z}.$$

Now the coefficient of z^n in the expansion of e^{-z} equals $\frac{(-1)^n}{n!}$ and so, the coefficient of z^n in the expansion of D_{exp} (z) is $\sum_{k=0}^{n}(-1)^k/k!$ (see E3 (b)). It then follows that $d_n = n! \sum_{k=0}^{n}(-1)^k/k!\ \ \forall n,$ by comparing coefficients of z^n.

E8) Differentiating and then multiplying by z on both sides of the identity $\sum_{k=0}^{n}C(n,k)z^k=(1+z)^n$, We get $\sum_{k=1}^{n}kC(n,k)z^k=nz(1+z)^{n-1}$.
Putting z = 3 yields
$\sum_{k=1}^{n}k3^kC(n,k)=3\times 4^{n-1}n.$

E9) Since the required generating function is
$$(z^2+z^3+z^4+\cdots)(z^3+z^4+z^5+\cdots)(z^4+z^5+z^6+\cdots)$$
$$\times(z^5+z^6+z^7+\cdots)(z^6+z^7+z^8+\cdots).$$
$=z^{20}(1+z+z^2+\cdots)^5$, the number of integer solutions is the coefficient of z^8 in $(1-z)^{-5}$, which is C(12,4)=495.

E10) a) The contribution to the generating function from a part k is $(1+z^k+z^{2k}+...)$. Since $1\le k\le m$, the required generating function is
$$\prod_{k=1}^{m}\left(1+z^k+z^{2k}+\cdots\right)=\prod_{k=1}^{m}\left(1-z^k\right)^{-1}.$$

b) If we use unequal parts, no part k may be repeated. The corresponding term in the generating function is $(1+z^k)$, so that k may be used at most once. Therefore, the generating function is $\prod_{k=1}^{\infty}(1+z^k)$.

c) The contribution from the odd part, 2k–1, is $(1+z^{2k-1}+z^{2(2k-1)}+\cdots)$. Thus, the required generating function is
$$\prod_{k=1}^{\infty}(1+z^{2k-1}+z^{2(2k-1)}+\cdots)=\prod_{k=1}^{\infty}(1-z^{2k-1})^{-1}.$$

E11) i) By above discussion, the required generating function is

$(1+z^{p_1}+z^{2p_1}+\cdots)(1+z^{p_2}+z^{2p_2}+\cdots)$ where $p_1, p_2, \ldots$ are the prime numbers.

ii) Similarly, here generating function will be

$$\left(1+z^{p_1}\right)\left(1+z^{p_1}\right)\cdots$$

E12) Defining $T_0 = 0$, so that the recurrence is valid for $n\ge 1$, and writing T (z) for the generating function of $\{T_n\}_{n=0}^{\infty}$, we have

$$T(z) = \sum_{n=0}^{\infty} T_n z^n = T_0 + 2\sum_{n=1}^{\infty} T_{n-1} z^n + 2\sum_{n=1}^{\infty} T_{n-1} z^n + \sum_{n=1}^{\infty} z^n$$

$$= 2z.T(z) + z/(1-z)$$

Therefore, $T(z) = z/(1-z)(1-2z) = (1-2z)^{-1} - (1-z)^{-1}$ and hence $T_n = 2^n - 1$, $n \geq 0$, by comparing coefficients after applying Binomial theorem on r.h.s. of the last equality.

E13) Let $T(z) = \sum_{n=0}^{\infty} t_n z^n$. Then,

$$T(z) = (t_0 + t_1 z + t_2 z^2) + \sum_{n=3}^{\infty} t_{n-3} z^n + \sum_{n=1}^{\infty} \frac{2n+1+(-1)^{n+1}}{4} z^{2n+1}$$

$$= z^3.T(z) + \frac{z}{4}\sum_{n=1}^{\infty}(2n+1)z^2 + \frac{z^3}{4}\sum_{n=0}^{\infty}(-1)^n z^{2n}$$

$$\Rightarrow (1-z^3)T(z) = \frac{z}{4}\frac{d}{dz}\sum_{n=1}^{\infty} z^{2n+1} + \frac{z^3}{4(1-z^2)}$$

$$= \frac{z}{4}\frac{d}{dz}\left(z\sum_{n=1}^{\infty} z^{2n}\right) + \frac{2^3}{4(1-z^{2)}}$$

$$= \frac{z}{4}\frac{d}{dz}\left(\frac{z}{(1-z^2)}\right) + \frac{z^3}{4(1-z^2)}$$

$$= \frac{3z^3}{4(1-z^3)^2} + \frac{z^3}{4(1-z^2)}$$

$$\Rightarrow T(z) = \frac{z^3(4-3z^2-2z^3+z^6)}{4(1-z^3)^3(1-z^2)}.$$

E14) Let $A(z) = \sum_{n=0}^{\infty} a_n z^n$. Then,

$$(1-3z-10z^2)A(z) = a_0 + (a_1 - 3a_0)z + \sum_{n=2}^{\infty}(a_n - 3a_{n-1} - 10a_{n-2})z^n$$

$$= 25 + 45z + 28\sum_{n=2}^{\infty}(5z)^n = 25 + 45z + 28\left\{\frac{1}{1-5z} - (1+5z)\right\}$$

$$= (25 - 80z + 475z^2)/(1-5z).$$

Using partial fractions, we get

$$A(z) = (25 - 80z + 475z^2)/(1+2z)(1-5z)^2 =$$

$15(1+2z)^{-1} - 10(1-5z)^{-1} + 20(1-5z)^{-2}$. Equating coefficients of z^n, we get

$$a_n = 15(-2)^n - 10.5^n + 20(n+1)5^n = 15(-2)^n + (10+20n)5^n, n \geq 0.$$

E15) We set $L_0 = L_2 - L_1 = 2$, so that the recurrence is valid for $n \geq 2$. By Theorem 1, $(1-z-z^2)\,L(z) = L_0 + (L_1 - L_0)z = 2 - z$. Therefore, $L(z) = (1-\alpha z)^{-1} + (1+\beta z)^{-1}$, where $\alpha + \beta = 1 = -\alpha\beta$.

Comparing the coefficients of z^n, we get $L_n = \alpha^n + \beta^n, n \geq 0$.

UNIT 1 BASIC PROPERTIES OF GRAPHS

Structure **Page Nos.**

1.0 INTRODUCTION

In our everyday life, we come across various problems requiring us to look at structures of objects and some family of subsets of those objects. For example, we may need to put up an electric network where different electrical gadgets are the objects, and they are to be connected by electric wires. The lengths of these wires may not be important, but it is important to know how the wires are connected. This means that it is important to know which gadgets are connected to the endpoints of the wires.

Another example is that of the public transport system in a city. Various places are the objects here, the bus routes are the connections, and we need to know the places connected to the railway station, say. Yet another problem could be that of establishing communication links between different centres.

All these problems can be represented pictorially with a set of dots called vertices and a set of edges connecting various pairs of dots. Such representations are called graphs. The solutions to the given problems can be obtained by analysing their graphs. Ideas given by various mathematicians to solve such problems gave birth to a branch of mathematics called **graph theory.**

In this unit we shall begin with defining a graph and study some of its basic properties. In Sec.1.2 and Sec.1.3 we have defined various types of graphs. Throughout the sections, these graphs and their properties are illustrated with the help of examples.

Next, Sec.1.4 is devoted to the study of subgraphs.

In the following units of this block you would notice how these simple basic ideas help us to solve many tough problems of day-to-day life. We can have graphs with vertices representing points in space, people, animal species, sports teams etc., and edges might represent roads, telephone lines, communication channels etc.

Before getting into the subject matter, let us take a look at the broad objectives of this unit.

1.1 OBJECTIVES

After studying this unit, you should be able to

- identify different ways of representing a graph;
- identify complete graphs, paths, cycles;
- obtain the complement of a graph;
- write the degree sequence of a graph and obtain the number of edges of a graph using the degrees of vertices;

- identify graphs isomorphic to a given graph;
- obtain a subgraph of G induced by a subset of V(G);
- draw a regular graph on p vertices having degree r, where p and r are integers with r < p, such that at least one of them is even.

1.2 GRAPHS

You must have used the term 'graph' while studying the calculus of real-valued functions of a real variable. It is a set of the form {(x, f(x)): x is in the domain of the function f}. Such a set helps us to analyse the function f. The graphs that we will define presently are a little different. Before giving a formal definition of a graph let us look at some simple examples.

Example 1: Take two points x_1, x_2 in the plane and join them by any line. This line may be a straight line or an arc. There are many ways of joining these points. In Fig.1 we have shown three different ways.

Fig.1

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Similarly in Examples 2 and 3 we have shown different ways of joining 4 points.

Example 2: Take four points x_1, x_2, x_3, x_4 in the plane. Join x_i to x_{i+1} by a line for $1 \le i \le 3$. Then join x_4 to x_1. In Fig.2 we have given two different ways of doing this.

Fig.2

As far as our study in this block is concerned these drawings represent the same object.

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Example 3: Take four points x_1, x_2, x_3, x_4 in the plane. Join x_1 to the three other points by lines.

Fig.3

Again in Fig.3 above, the three drawings represent the same object.

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From the examples above, you may have observed the points are important as objects but their positions are not important. Similarly, it is important to know which pairs are joined to each other, but the lines or the curves joining them are not important.

In the diagrams above, each point is called a **vertex,** and the curve joining any pair is called an **edge**. So, for instance, in Example 1 we have two **vertices** (plural of 'vertex') x_1 and x_2, and one edge.

In this way, to each drawing corresponds two sets — one consisting of vertices, say V, and one of edges**,** say E.

Now, note that, to identify an edge, we need to know which vertices it joins. So, we denote an edge joining x_1 and x_2 by (x_1, x_2). So, any edge is given by a pair of points from V. Now, we are in a position to define the objects we wanted to.

Definition : An **undirected graph** G is a finite non-empty set V together with a set E consisting of pairs of points of V. The set V is called the **vertex set** of G, the set E is called the **edge set** of G. To show the relationship between V, E and G, we write G = (V(G), E(G)).

If $|V| = p$ and $|E| = q$, then G is a **(p,q)-graph**.

Now, suppose the edges have a direction, then the edge going from a vertex x_1 to a vertex x_2 is not the same as the one going from x_2 to x_1. So, $(x_1, x_2) \neq (x_2, x_1)$. In this case, the edges are represented by **ordered** pairs, that is, elements of $V \times V$, where V is the vertex set. This leads us to the following definition.

Definition: A **directed graph,** or **digraph,** G consists of a finite non-empty set V together with a subset E of the Cartesian product set $V \times V$. We call V the **vertex set** of G and E the **edge set** of G, and we write G = (V(G), E(G)).

In Fig.4 we have shown directed as well as undirected graphs. **Note that**, in an undirected graph, if (u,v) is an edge, so is (v, u). So, the relation E(G) is a symmetric binary relation on V(G). However, in a digraph E(G) need not be symmetric.

Sometimes, it may happen that in a graph there is a **loop**, i.e., an edge joins a vertex to itself as in Fig.4 (c), where (u, u) shows a loop. It may also happen that there are two
or more edges joining the same vertices, as in Fig.4 (c), where there are two edges joining y to x. Such edges are called parallel or multiple edges, and a graph with any parallel edges is called a **multigraph**.

(a) **(b)** **(c)**

Fig.4

An undirected graph is either a simple graph or a multigraph.

Definition: An undirected graph without loops or parallel edges is called a **simple graph**.

Why don't you try an exercise now ?

E1) There are four basic blood types: A,B, AB and O. Type O can donate to any of the four types. A and B can donate to AB as well as to their own types, but type AB can only donate to AB. Draw a graph that presents this information.

So far, we have seen several kinds of graphs. However, in this block, **we shall only discuss simple graphs**, and **shall just refer to them as graphs**. Also, whenever there is no confusion, **we shall write V and E in place of V(G) and E(G)**.

Now, let us introduce some terminology.

- If e is an edge joining the vertices u and v of a graph G=(V,E), we will **denote it as uv**. In this case, u and v are called **adjacent vertices** (or **neighbours**), and are the **endpoints** of e. We also say that e is **incident** with u and v.

- If distinct edges e_1 and e_2 of G have at least one vertex in common, then e_1 and e_2 are called **adjacent edges**.

For an example of these concepts, consider the graph G_1 of Fig.4. G_1 = (V, E), where V = {u, v, x, y} and E = {uv,ux,uy,vx,xy}. So, G_1 is a (4, 5)-graph. The only non-adjacent vertices of G_1 are v and y. The edges uv and vx are adjacent, since both are incident with the vertex v. The edges uv and xy are non-adjacent. Two other ways of representing the graph G_1 are shown in Fig.5. Thus, **there is no unique way of drawing a graph**; the relative placing of the points and curves have no special significance.

Fig.5

Remark : Since a diagram of a graph, as in Figures 1-5, completely describe the graph, **we often refer to the diagram of a graph G as G itself**.

It is interesting to know that the structure of molecules can also be represented by graphs (see Fig.6). Various atoms are represented by the vertices and the structural bonds are represented by the edges. For example, butane as well as isobutane are both the hydrocarbons $C_4 H_{10}$. But the manner in which the bonds are present between the carbon and hydrogen atoms makes the difference. In both the compounds each carbon atom is attached to four other atoms. Unlike isobutane, in butane there is no carbon atom which is attached to all the other carbon atoms.

Fig.6

You may now try some exercises.

E2) Take three vertices x, y, z and draw all possible (3,2)-graphs on these vertices.

E3) Write down the vertex set V and edge set E of each graph in Examples 1, 2 and 3.

We now define some graphs which are used for data communication and parallel processing.

A) **A complete graph** is a graph in which any two vertices are adjacent, i.e., each vertex is joined to every other vertex by an edge. We denote the complete graph on n vertices by $\mathbf{K}_n$.

In Fig.7, we have shown K_n for various n. K_1 is just a single vertex; K_2 consists of two vertices and an edge; K_3 is often called a triangle. The last two figures in Fig.7 show two ways of representing K_4.

Fig.7

B) **Star topology**: Various computers, printers and plotters on a campus can be connected using a local area network. Some of these networks are based on graphs like the one given in Fig.8, called a **star topology**. In this graph n vertices are adjacent to one central vertex. This represents n devices connected to a central control device from which messages are sent to them.

y

Fig.8 : Star topology

C) **Cycles:** A **cycle $\mathbf{C_n}$** is a graph on n vertices $\{x_1, \ldots, x_n\}$ where $E(C_n) = \{x_i x_{i+1} : 1 \le i \le n-1\} \cup \{x_n x_1\}$. For instance, C_{16} is shown in Fig.9.

Fig.9 : C_{16}, a cycle on 16 vertices

You may now try the following exercise.

E4) How many edges do C_n and K_n have, $n \geq 3$?

We now take up some definitions related to certain algorithms that you have studied in MCS-031.

Definition: Let G = (V, E) be a (p, q)-graph. By its **complement** $\overline{G}$, we mean the graph with $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{xy : xy \notin E(G), x, y \in V(G)\}$.

Note that $\overline{G}$ is a (p, $\overline{q}$)-graph, where $\overline{q}$ = (number of pairs of elements of V) – q.

Since in a set V with p elements, there can be C(p, 2) = $\frac{p(p-1)}{2}$ such pairs of elements, $\overline{q} = \frac{p(p-1)}{2} - q$.

Remark : The complement of $\overline{G}$ is G. Can you prove this ?

Let us consider some examples, Fig. 10 shows C_5 and its complement. Check whether $\overline{C_5}$ is also a cycle or not.

Fig.10

Two graphs G and H are **disjoint** if $V(G) \cap V(H) = \phi$

For another example, consider the graph shown in Fig.11 (a). Its complement breaks into two disjoint graphs. One is K_3 and the other is K_4 (see Fig.11 (b)).

(a) **(b)**

Fig.11

Consider the complements of all the graphs you have seen so far.

Notice that C_5 is a (5, 5) graph, and $\overline{C_5}$ has 5 edges. Also, in Fig.11, G is a (7, 12)-graph and $\overline{G}$ has 9 edges. Do you see any relation between the number of vertices of G and number of edges of $\overline{G}$? You may try the following exercises and look for the answer.

E5) Three graphs G_1, G_2 and G_3 are listed below

$G_1 = (\{u_1, u_2, u_3, u_4, u_5, u_6\}, \{u_1u_2, u_1u_5, u_1 u_6, u_2u_3, u_2u_5, u_3u_4, u_4u_5\})$

$G_2 = (\{u_1,u_2, u_3, u_4,u_5\}, \{u_1u_2,u_1u_3, u_1u_4,u_1u_5, u_2u_4, u_2u_5, u_3u_4, u_3u_5\})$

$G_3 = (\{u_1, u_2, u_3, u_4, u_5, u_6\}, \{u_1u_2, u_1u_4, u_1u_5, u_2u_3, u_3u_4, u_3 u_6, u_5u_6\})$

Find $\overline{G_1}$, $\overline{G_2}$ and $\overline{G_3}$.

E6) If G is a (p, q)-graph, then how many edges can $\overline{G}$ have ?

So far we haven't really looked at the ways in which graphs are related to the real-life situations given in Sec.1.0. Let us consider some graphs that will help us see this connection.

1.3 DEGREE, REGULARITY AND ISOMORPHISM

You may recall that in the beginning we defined two vertices of a graph G to be adjacent if they are joined by an edge. Such vertices are also called **neighbours**. The set of all neighbours of a fixed vertex x of G is called the **neighbourhood set of x**, denoted by $\mathbf{N_G(x)}$. Since our graphs are simple, there is a one-one correspondence between N_G (x) and the set of all edges of G incident with the vertex x. Related to this set, we get the following number.

Definition: By the **degree** of a vertex x in G, we mean the number of edges incident with x. We denote the degree of x by $\mathbf{d_G(x)}$.

By definition, $d_G(x) = |N_G(x)|$, where $|N_G(x)|$ denotes the number of elements of the set $N_G(x)$.

Since in a (p, q)-graph G the maximum number of edges incident with a vertex x can be (p –1), we have

$\mathbf{0 \le d_G(x) \le (p-1)}$ for every vertex x in G.

Whenever there is no possibility of confusion, we will simply write d(x) instead of $d_G(x)$.

Also a vertex x of a graph G is called an **even vertex** if d_G (x) is even; otherwise it is called an **odd vertex**. A vertex with degree 0 is called an **isolated vertex**.

Now let us look at the following example.

Example 4 : Consider the graph G shown in Fig. 12. First consider the vertex x_1. Clearly, three edges are incident with it, so that $d(x_1) = 3$. Likewise you may observe that $d(x_2) = 4$, $d(x_3) = 5$, $d(x_4) = 6$ and $d(x_5) = 7$.

We can write these observations as $d(x_i) = i + 2$ for $1 \le i \le 5$.

Fig .12

In the same way, we can write $d(y_j) = 1$ for $1 \le j \le 15$.

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You may now try the following exercises.

E7) Write down the degrees of all the vertices for the graphs given in Figures 5 to 9.

E8) If G is a (p, q)-graph and x is a vertex in G, show that the degree of x in $\overline{G}$ is $p - 1 - d_G(x)$.

Note that in Example 4

$$\begin{aligned} d(x_1) + d(x_2) + \ldots + d(x_5) + d(y_1) + \ldots + d(y_{15}) &= 40 \\ &= 2 \times 20 \\ &= 2 \times \text{(number of edges in G)} \end{aligned}$$

Does a similar relationship hold for the graphs in Fig.5 to Fig.9 ? In fact, it should, because of the following theorem.

Theorem 1 (Handshaking Theorem) : If G is a (p, q)-graph with $V(G) = \{v_1, \ldots, v_p\}$, and if $d_i = d_G(v_i)$, $1 \le i \le p$, then $2q = \sum_{i=1}^{p} d_i$.

That is, **the sum of the degrees of the vertices of G is twice the number of edges**.

Proof: Consider the set $S = \{(x, e): x \in V(G), e \in E(G), x \text{ is an endpoint of } e\}$. Choose a vertex $v_i \in V$. This can be done in p ways. Now, since $d_i = d(v_i)$, there are precisely d_i edges incident with this vertex v_i. These edges give d_i elements of the set S. Adding over all the vertices of G, we get

$$|S| = \sum_{i=1}^{p} d_i. \qquad (1)$$

Now choose an edge e in E (G). This can be done in q ways. This edge has precisely two endpoints, and they give two elements of S. Summing over every edge $e \in E(G)$, we get

$$|S| = 2q \qquad (2)$$

This is because every edge is counted twice, once for each vertex it contains. Equating (1) and (2) we get the required result.

The next result immediately follows from Theorem 1.

Corollary 1 : The sum of the degrees of all the vertices of any graph is even.

You have already verified Theorem 1 for the graphs in Fig.5 to Fig.9. Now let us look at a useful application of Theorem 1.

So far, in the discussion, you must have noticed that for a simple (p, q)-graph G the edge set E(G) is a subset of the set of all subsets of size 2 of elements of V (G). This means $q \le \frac{p(p-1)}{2}$. But then you may wonder : is it always possible to go the other way round? That is, for any pair of positive integers (p, q) with $q \le \frac{p(p-1)}{2}$, is it always possible to find a (p, q)-graph?

Theorem 1 gives us a necessary condition on p and q under which a (p, q)-graph exists. It helps us to see that **there does not always exist a graph with vertices having given degrees**.

For instance, can we construct a graph on 12 vertices with 2 of them having degree1, three having degree 3, and the remaining seven having degree 10 ? This is not possible. Why? If such a graph existed, the sum of the degrees of all its vertices would be 1+1+3+3+3+10+10+10+10+10+10+10=81, which is not even.
So, the condition of Theorem 1 would not be satisfied.

Theorem 1 can also be used to obtain another result which we discuss now.
Corollary 2 : Any graph can only have an even number of odd vertices.

Proof : Let G be a (p, q)- graph and let $\{x_1,\ldots,x_t\}$ be the set of its odd vertices and $\{x_{t+1},\ldots,x_p\}$ be the set of its even vertices. Let $d_G(x_i) = 2c_i+1$, $1 \le i \le t$ and $d_G(x_i) = 2r_i$, $t+1 \le i \le p$.

Then Theorem 1 says that $2q = \sum_{1}^{p} d_G(x_i)$

$$\Rightarrow 2q = \sum_{1}^{t}(2c_i+1) + \sum_{t+1}^{p}(2r_i) = 2(c_1+c_2+\ldots+c_t)+t+2(r_{t+1}+\ldots+r_p),$$

which shows that t is even.

What this result says is that **if a graph has any odd vertices**, then their number has to be even. So, for instance, we can't have a graph with only 1 odd vertex. In some graphs, like K_{10}, all the ten vertices are odd vertices. On the other hand, in K_{11} all the vertices have degree 10, that is, none of the vertices are odd.

We now give the reason for the name given to Theorem 1.

Corollary 3 : At any party, the number of people who shake the hands of an odd number of people is even.

You can see this from Theorem 1 if you consider a handshake between two hands as an edge between two adjacent vertices.

Let us now consider some numbers that help us judge the type of graph we are dealing with.

Definitions: If G = (V, E) is a (p, q)-graph, then
$\delta(G) = \min\{d_G(x) : x \in V(G)\}$ is called the **minimum vertex degree of G,** and
$\Delta(G) = \max\{d_G(x) : x \in V(G)\}$ is called the **maximum vertex degree** of G.
Clearly, $\delta(G)$ and $\Delta(G)$ are non-negative integers.

We can, in fact, re-number the vertices of V (G) as $\{v_1, \ldots, v_p\}$ with $d_i = d(v_i)$, $1 \le i \le p$ such that $d_1 \ge d_2 \ge \ldots \ge d_p$, that is, place the vertices in decreasing order of their degrees. This is called the **degree sequence** of the graph G.

For instance, the degree sequence of the graph G in Fig.11 is 7, 6, 5, 4, 3, $\underbrace{1,1,\ldots,1}_{15\text{times}}$.

And now some exercises for you.

E9) Write down $\delta(G)$ and $\Delta(G)$ for all the graphs in Examples 1,2, 3.

E10) For each of the number sequences given below, give an example of a graph having this as a degree sequence, if possible. Otherwise explain why such a graph does not exist.

(i) (3,2,2,2,1) (ii) (3,2,2,2,1,1) (iii) (4,3,2,1,0)
(iv) (4,4,3,3,2,2) (v) (5,5,5,4,4,3,3)

E11) Let G be a (p, q)-graph, each of whose vertices has degree k or k+1. If G has m vertices of degree k and r vertices of degree k+1, then show that
m = (k+1) p – 2q.

Let us now consider graphs that have a constant degree sequence, that is, each of their vertices has the same degree. For example, the degree sequence of C_5 and its complement is 2,2,2,2,2, that is, it is a constant 2. Such graphs have a special status in graph theory, and we name them as follows.

Definition: A (p, q)-graph G is said to be **regular, with degree of regularity r,** if $d_G(x) = r$ for every vertex $x \in V(G)$. In this case we also say that G is an **r-regular graph**. Of course, $0 \le r \le (p-1)$.

You have seen that K_3 is regular. What about K_n for n>3? In fact, it is (n–1)-regular.

As you will see in the next unit, 3-regular graphs, called **cubic graphs**, are important. A well known example of a cubic graph is the **Petersen graph**. This is named after the Danish mathematician J.P.C. Petersen. He worked in several areas of pure and applied mathematics. Two representations of the Petersen graph are shown in Fig.13. **Note** that it is a (10,15)-graph.

Fig.13

Fig

Fig. 13 : Julius Petersen (1839-1910)

Let us now consider the following example of a regular graph.

Example 5 (Hypercube Q_n) : Let the vertex set consist of all n-tuples with entries 0,1 only. The edge set is given by

$E(Q_n) = \{$ **ab** : **a** and **b** differ exactly at one coordinate$\}$.

Here, by **a** we mean an n-tuple $(a_1, \ldots, a_n)$, where a_i = 0 or 1, for $1 \le i \le n$. In Fig.14, we show Q_2 and Q_3.

(a) **(b)**

Fig.15

Any vertex **a** is adjacent to precisely n other vertices. For example, (0,0, …,0) is adjacent to (1, 0, 0, …, 0), (0, 1, 0, 0, …0), … , (0, 0, …, 1). Hence, the hypercube Q_n is n-regular. You should check that Q_n has 2^n vertices and $n2^{n-1}$ edges.

***

You have seen some examples of regular graphs, and you can think of some more. You also know that if G is an r-regular graph on p vertices, then by Theorem 1, 2q = pr. So, pr is even. Therefore, at least one of p or r is even. **Is the converse true** ? That is, given a pair of integers p, r, $0 \le r \le (p-1)$, where pr is even, can we always construct an r-regular graph on p vertices? This is true. The proof is by construction on the same lines as the examples we give below. We consider two cases — when r is even, and when r is odd.

Example 6 : We construct a 4-regular graph G with 12 vertices. Let $V(G)=\{x_1,\ldots,x_{12}\}$. Place the vertices in a circular manner. Join x_i to x_{i+1} by an edge for every i, $1 \le i \le 11$. Join x_{12} to x_1 also by an edge. Now all the vertices have acquired
degree 2. Now, join each x_i to x_{i+2} for every i = 1,…10. Finally, join x_{11} to x_1 and x_{12} to x_2, as in Fig.16. You can see that the resulting graph is 4-regular.

Fig.16

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Example 7 : We shall construct a 5-regular graph on 12 vertices now. We first construct a graph on 12 vertices and with regularity (r–1), i.e., 5 – 1 = 4 — in fact, the graph constructed in Example 6.
Now join x_i to x_{i+6} for every i, $1 \le i \le 6$.

We choose 6 because $\frac{p}{2} = 6$. Notice that the resulting graph is 5-regular (see Fig.17).

Fig. 17

***

The constructions above can be generalized to obtain a regular graph on p vertices with degree of regularity r, where at least one of p and r is even. In the following exercise, we give you an opportunity to do this.

E12) Construct a 5-regular graph on 10 vertices.

Often, when we look at two graphs, they appear different when they may essentially be the same. For instance, the complement of C_5 is essentially C_5, though they don't look the same. To see why we say this, let us look at the two graphs in Fig.10. Now, consider the function f: $V(C_5) \to V(\overline{C_5})$ defined by
$f(x_1) = x_1$, $f(x_2) = x_3$, $f(x_3) = x_5$, $f(x_4) = x_2$, $f(x_5) = x_4$.
With this definition, note that whenever $x_i x_j \in E(C_5), f(x_i)f(x_j) \in E(\overline{C_5})$. In other words, $x_i x_j$ is an edge of C_5 if and only if $f(x_i)f(x_j)$ is an edge of $\overline{C_5}$. In such a situation we say that f is an isomorphism between C_5 and $\overline{C_5}$.

This leads us to the following definition.

Definition : Let G = (V(G), E(G)), H = (V(H), E(H)) be two graphs. By an **isomorphism** f from the graph G to the graph H, we mean a map f: V(G) $\to$ V(H) such that

(i) f is one-one and onto; and
(ii) $x y \in E(G)$ if and only if $f(x) f(y) \in E(H)$.

In this case we say that G and H are **isomorphic.** Otherwise they are called **non-isomorphic**.

Note that two graphs G and H are isomorphic **if and only if** there is a one-one correspondence between V(G) and V(H) that preserves adjacencies and non-adjacencies. In this case we say that **the map f preserves the structure of G**.

Let us consider one more example.

Example 8 : Consider the two graphs G and H shown in Fig.18.

Fig.18

Define a map $f : V(G) \rightarrow V(H)$ as follows:
$f(x_1) = a$, $f(x_2) = b$, $f(x_3) = c$, $f(y_1) = d$, $f(y_2) = e$, $f(y_3) = g$, $f(y_4) = h$.
Observe that $uv \in E(G)$ **if and only if** $f(u) f(v) \in E(H)$.
The two graphs shown in Fig.18 are isomorphic under this correspondence.

***

As you can see, if G and H are isomorphic, many properties of a vertex in V(G) are shared by its image in V(H). For example, you can check that $d_G(u) = d_H(f(u))$, for every $u \in V(G)$.

Here's an exercise about isomorphic graphs.

E13) Show that the graphs G and H given in Fig.19 are isomorphic.

Fig.19

As you have seen, in order to show that two graphs are isomorphic, it is enough to produce one isomorphism from one of them to the other. In some cases, as in Fig.20, it is clear that the 2 graphs are not isomorphic. However, if we are given two similar looking graphs, it is not easy to show that there **does not exist** any isomorphism between them. The six properties given below are of help in this matter. We shall state them, without proof.

Theorem 3 : Let f be an isomorphism from a graph G to a graph H. Then the following hold :

i) If G is a (p, q)-graph, then H is also a (p, q)-graph.

ii) The inverse map f^{-1} is an isomorphism from the graph H to the graph G.

iii) If g is an isomorphism from the graph H to a graph K, then the composite map $g \circ f$ is an isomorphism from the graph G to the graph K.

iv) f induces a bijective map $\tilde{f}$: E (G) $\rightarrow$ E (H), given by $\tilde{f}$ (x y) = f (x) f(y).

v) For every x $\in$ V (G), a vertex y belongs to N_G (x) if and only if f (y) belongs to N_H (f (x)). (This means that d_G(x) = d_H(f(x)), for every x $\in$ V (G).) Thus, the degree sequence of the graph G is the same as the degree sequence of the graph H.

vi) If G has a set of vertices $\{x_1, \ldots, x_n\}$ such that $x_n x_1$ and $x_i x_{i+1}$ are in E (G) $\forall$i, $1 \le i \le (n-1)$, then the vertices $\{f(x_1), \ldots, f(x_n)\}$ in V (H) are such that f (x_n) f (x_1) as well as f (x_i) f (x_{i+1}) are in E(H) $\forall$i, $1 \le i \le (n-1)$. Thus, for every positive integer n $\ge$ 3, the number of copies of C_n in G is equal to the number copies of C_n in H.

Let us now consider the following examples where we have used these properties to show non-isomorphism of the two graphs.

Example 9 : Consider the two graphs shown in Fig.21. Both are (8, 8)-graphs and

Fig.21

have a copy of C_6 inside them. Are they isomorphic ? The degree sequence of the graph G is 4, 2, 2, 2, 2, 2, 1, 1 and of the graph H is 3, 3, 2, 2, 2, 2, 1, 1. This contradicts (v) of Theorem 3. Therefore, G and H are not isomorphic.

***

Example 10 : Consider the graphs G and H in Fig.22.

Fig.22

Both are (8, 12)-graphs and have a copy of C_8 inside them. Moreover, both have degree sequences 3, 3, 3, 3, 3, 3, 3, 3. They are still not isomorphic. This can be seen by observing that the graph G has no copy of a triangle inside it and the graph H has two triangles $\{x_1, x_2, x_8\}$ and $\{x_4, x_5, x_6\}$, which contradicts (vi) of Theorem 3.

***

Example 11: Consider the graphs G and H shown in Fig. 23.

Fig.23

Both are (6, 6)-graphs having 3, 3, 2, 2, 1, 1 as their degree sequence. However, they are not isomorphic. In the graph G the two vertices x_3, x_5 having degree 3 are adjacent. Under any isomorphism (if it exists) they should be mapped to two adjacent vertices of degree 3. We observe that in the graph H the two vertices of degree 3 are not adjacent.

***

Notice that the two graphs shown in Fig. 6, corresponding to butane and isobutane, are not isomorphic. Unlike isobutane, no carbon atom is attached to all the other carbon atoms of butane.

And now the following exercises for you to try.

E14) Draw at least 3 non-isomorphic graphs on four vertices.

E15) A graph G is said to be **self complementary** if it is isomorphic to its complement $\overline{G}$. Show that for a self complementary (p, q)-graph G, either p or (p – 1) is divisible by 4.

It is often the case that a graph under study is contained within some larger graph also being investigated. When we talk of an electric circuit, it is often described in terms of various sub-circuits. Transport in a country is always divided into various sections, for example, the railway transport in India is divided into Central Railway, Western Railway, …etc. That is, whenever we study any system, it is important to study its subsystems. Likewise here in the next section we study subgraphs.

1.4 SUBGRAPHS

Let us start with considering the graph G = (V(G), E(G)) shown in Fig.24.

Fig.24

What if we just take a part of this graph G? Would this be a graph? Yes, it would. For example, consider the following.

Let V (G_1) = $\{x_1, x_2, x_3, x_4\}$, E (G_1) = $\{x_i\, x_{i+1}: 1 \le i \le 3\} \cup \{x_4 x_1\}$.
Note that G_1 is **isomorphic** to C_4.
If V (G_2) = $\{x_8, x_9\}$, E (G_2) = $\{x_8x_9\}$, then G_2 is **isomorphic** to K_2.
Also the graph G_3 is **isomorphic** to C_7, where
V (G_3) = $\{x_9, \ldots, x_{15}\}$, E (G_3) = $\{x_{15}\, x_9\} \cup \{x_i x_{i+1} : 9 \le i \le 14\}$

Note that all these graphs have one thing in common. Their vertex sets are subsets of V (G) and edge sets are subsets of E (G). In this sense, all these graphs are 'portions' of the graph G. Formally, we have the following definition.

Definition : Let G = (V (G), E (G)) be a graph. A **subgraph** H of the graph G is a graph, such that every vertex of H is a vertex of G, and every edge of H is an edge of G also, that is, V (H) $\subseteq$ V (G) and E (H) $\subseteq$ E (G).
Further, if H is a **subgraph** of a graph G, such that V (H) = V (G) and E (H) $\subseteq$ E (G), that is, H and G have exactly the same vertex set, then H is called a **spanning subgraph** of G.

So, G_1, G_2 and G_3 given above are subgraphs of G given in Fig.23. However, the graph H, with $V(H) = V(G_3), E(H) = E(G_3) \cup \{x_9 x_{12}\}$ is **not** a subgraph of the graph G, since the edge x_9x_{12} is not in E (G).

Note : You should write down the reasons why the following statements are true.

1) Every graph G is a subgraph of itself, i.e., **G is a subgraph of G**.
2) **For any v $\in$ V (G), {v} is a subgraph of G**.

Now for an example of a spanning subgraph.

Example 12: Consider G = K_4 on four vertices x_1, x_2, x_3, x_4 as shown in Fig.25. From the figure you can see G_1, G_2, G_3 are subgraphs of G, with $V(G_1)=V(G_2)=V(G_3)=V(G)$. So, G_1, G_2 and G_3 are spanning subgraphs of the graph G.

Fig.25

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Example 13: Consider the Petersen graph G, with the vertex set $\{x_i : 1 \le i \le 5\} \cup \{y_j : 1 \le 1 \le 5\}$ shown in Fig.26. Consider the graph G_1, where $V(G_1) = \{y_j : 1 \le j \le 5\}, E(G_1) = \{y_1 y_3, y_3 y_5, y_5 y_2, y_2 y_4\}$.

Fig.26 : The Petersen Graph

Here every edge of G_1 is an edge in G. On the other hand, $y_4\, y_1$ is an edge in G but not an edge in G_1. Thus, G_1 is a subgraph of G. So, is the graph G_2 shown in Fig.27. However, there is a difference in the two subgraphs. Though y_1 and y_4 lie in $V(G_1)$, $y_1 y_4$ is in E(G) but not in $E(G_1)$. But, whenever two vertices of G_2 are joined by an edge in G, that edge belongs to E (G_2)

Fig.27

The property of the subgraph G_2 that we have just mentioned leads us to the following definition.

Definition: Let G be a graph and let S $\subseteq$ V (G). By **the subgraph of the graph G**, **induced by the set S**, we mean the subgraph H with V(H)=S and the edge set consisting of those edges of G which are joining the vertices in S. That is, $E(H) = \{x\,y : x \ne y, x \in S, y \in S, x\,y \in E(G)\}$. We denote H by **$<S>_G$**.

Note that two points of S are adjacent in $<S>_G$ if and only if they are adjacent in G.

For example, the subgraph G_2 in Example 13 is an induced subgraph of the graph G, induced by $\{x_1, x_2, x_3, x_4, x_5, y_1, y_3\}$, whereas the subgraph G_1 is not induced.

Note that for a vertex v $\in$ V (G), by **G – v we mean the subgraph $< V(G) - \{v\} >_G$**, which means a subgraph of G consisting of all points of G except v, and all edges of G except for the edges incident with v.
For a subset S of V (G), the subgraph **$< V(G) - S >_G$ is often written as G–S**.

We now illustrate various types of subgraphs, relating their minimum and maximum vertex degrees.

Example 14 : Consider the graph G shown in Fig.28 (a). Observe that Fig.28 (b)

Fig.28

Fig.28 (contd.)

shows a subgraph H_1, Fig.28 (c) gives a vertex induced subgraph H_2 with $V(H_2) = V(H_1)$, Fig.28 (d) shows $H_3 = G - v_4$, and Fig.28 (e) gives the spanning subgraph H_4.

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Now, a few exercises for you.

E16) Show that for a subgraph H of a graph G, $\Delta(H) \leq \Delta(G)$.

E17) Give an example of a subgraph H of a graph G with $\delta(G) < \delta(H)$ and $\Delta(H) < \Delta(G)$.

E18) Let G be a graph with n vertices and m edges, and let v be a vertex of G of degree k. How many vertices and edges does $G - v$ have?

E19) Is every subgraph of a regular graph regular ? Give reasons for your answer.

We now end this unit by giving a summary of what we have covered here.

1.5 SUMMARY

1) A simple graph G consists of a finite non-empty set V of points together with a prescribed set E of 2 element subsets of V.

2) The complete graph K_n is a graph with n vertices such that every vertex is joined to every other vertex by an edge.

3) The path P_n is a graph on n vertices $\{x_1, x_2, \ldots, x_n\}$ in which any two consecutive edges are adjacent and where no edge and no vertex is repeated.

4) A cycle is a circuit in which the only repeated vertex is the first vertex, which is the same as the last vertex.

5) The complement of the (p, q)-graph G is a (p, $\bar{q}$)-graph $\bar{G}$ where $\bar{q}$ = (number of pairs of elements of V) – q.

6) The number of edges incident with a vertex in a graph G gives the degree of the vertex. A graph having the same degree of all its vertices is regular.

7) In any graph the sum of the degrees of all its vertices is even.

8) There always exists an r-regular graph on p vertices, where p, r are non-negative integers and at least one of them is even.

9) For a graph G = (V (G), E (G)), a graph H = (V (H), E (H)) is a subgraph of G whenever V (H) $\subseteq$ V (G) and E (H) $\subseteq$ E (G).

10) A subgraph of a regular graph may or may not be regular.

11) A subgraph H of a graph G is a spanning subgraph of G if V(H) = V(G).

12) For any $S \subseteq V(G)$, the subgraph of G induced by S is $H = \langle S \rangle_G$, where $V(H) = S$ and $E(H) = \{xy : x \neq y, x \in S, y \in S, xy \in E(G)\}$.

1.6 SOLUTIONS / ANSWERS

E1)

Fig.29

E2)

Fig.30

E3) Example 1, $V = \{x_1, x_2\}$, $E = \{x_1x_2\}$
Example 2, $V = \{x_1, x_2, x_3, x_4\}$ $E = \{x_1x_2, x_2x_3, x_3x_4, x_4x_1\}$
Example 3, $V = \{x_1, x_2, x_3, x_4\}$, $E = \{x_1x_2, x_1x_3, x_1x_4\}$

E4) C_n has n edges and K_n has $\frac{n(n-1)}{2}$ edges.

E5) $E(\overline{G}_1) = \{u_1u_3, u_1u_4, u_2u_4, u_2u_6\ u_3u_5, u_3u_6, u_4u_6, u_5u_6\}$
$E(\overline{G}_2) = \{u_2u_3, u_4u_5\}$
$E(\overline{G}_3) = \{u_1u_3, u_1u_6, u_2u_4, u_2u_5, u_2u_6, u_3u_5, u_4u_5, u_4u_6\}$

E6) $\overline{G}$ can have $\frac{p(p-1)}{2} - q$ edges.

E7) For Fig.5, $d(x_i) = 4$, $1 \le i \le 5, d(y_i) = 2, 1 \le i \le 7$.
For Fig.8, $d(x_i) = 2$, $1 \le i \le 5$.
Do the others similarly.

E8) $d_{\overline{G}}(x) = |N_{\overline{G}}(x)| = |\{y \in V(G) : xy \notin E(G)\}| = |V(G)| - 1 - |N_G(x)|$
$= p - 1 - d_G(x)$.

E9) (1) 1, 1 (2) 2, 2 (3) 1, 3

E10) ii) The graph has 3 vertices of odd degree, contradicting Corollary 1 of Theorem 1.
v) The sum of the degrees of all the vertices of a graph is odd, contradicting Corollary 1.
For the rest you can draw graphs whose degree sequence is the given one.

E11) $km + (k+1)r = 2q$ (Using Theorem 1)
Also, $m+r = p$
Therefore, $km+(k+1)(p-r) = 2q$
$E \Rightarrow m = (k+1)p - 2q$

E12) Here $p = 10$, $r = 5$. So $\frac{r-1}{2}$ is an integer. Take 10 vertices $\{x_1, x_2, \ldots, x_{10}\}$. Join x_i to x_{i+1} for $1 \le i \le 9$. Join x_{10} to x_1. Now all the vertices have acquired degree $\frac{r-1}{2} = 2$. Join x_i to x_{i+2} for i=1,..,8. Join x_9 to x_1 and x_{10} to x_2.We now have a 4-regular graph. Here $\frac{p}{2} = n = 5$.
Thus, to obtain a 5-regular graph join x_i to x_{i+5} for $1 \le i \le 5$ (see Fig.31).

Fig.31

E13) Consider the function
$\phi : V(G) \to V(H) : \phi(u) = \lambda, \phi(v) = m, \phi(w) = n, \phi(x) = p, \phi(y) = q, \phi(z) = r.$
You can check that ϕ is an isomorphism.

E14) If p = 4, then q $\leq$ C(4, 2) = 6. So we want (4, q)-graphs, with $0 \leq q \leq 6$. We are giving several in Fig. 32.

Fig.32

E15) Suppose G is a (p, q)-graph. Then
E (G) $\cup E(\overline{G})$ = {the set of all pairs of vertices in V (G)}. Thus,

$$q + \overline{q} = \frac{p(p-1)}{2}.$$

If the graph G is self complementary, then q = $\overline{q}$. Thus, p (p –1) = 2q + 2$\overline{q}$ = 4q, that is 4 divides p (p –1). Since only one of p or (p –1) is even, this means either p or (p – 1) is divisible by 4.

E16) Let x $\in V(H)$ such that d_H (x) = $\Delta(H)$. Then, N_H (x) $\subseteq N_G$ (x). Thus,

$$\Delta(H) = |N_H(x)| \leq |N_G(x)| \leq \Delta(G).$$

E17) $\delta(G) = 1 < 2 = \delta(H)$
$\Delta(H) = 2 < 3 = \Delta(G)$

Fig.33

E18) G–v will have (n – 1) vertices and m – k edges

E19) No, for example, any cycle is regular. However, if you remove one of its edges, you get a subgraph which is not regular.

UNIT 2 CONNECTEDNESS

Structure **Page Nos.**

2.0 INTRODUCTION

In the last unit you saw that graphs are often used to represent (that is, model) communication or transportation networks and several other systems such as representation of a molecule in a chemical compound. In a transportation network, it is necessary to know which destinations are connected by a direct route. For example, if air travel is abolished then the people without any seaport cannot go to any other country unless their neighbours provide the initial road passage through their territory. When we use a graph to model this situation, we need to see which vertices are connected. We also need to ensure that there is an edge between any two vertices. Such graphs are called connected graphs. In Sec.2.2 we will define connected graphs and we will show that any graph can be partitioned into connected graphs.

In Sec.2.3, we will familiarise you with a type of graph which is useful in electronics and other areas. These graphs are called bipartite graphs. Such graphs are also very useful in studying neural networks.

In Sec. 2.4 we have considered another type of graph, a type, which also represents the chemical compounds butane and isobutane. We call such graphs trees. Here we will show that a tree has got several interesting properties, which are used in studying some real-life situations and various chemical compounds.

2.1 OBJECTIVES

After studying this unit, you should be able to

- distinguish between walks, paths, circuits and cycles in a graph;
- identify the components of various graphs;
- define, and recognize, bipartite graphs, and trees.

2.2 CONNECTED GRAPHS

From Unit 1, you know that graphs model different real-life situations, especially situations involving routes — the vertices represent towns or junctions and each edge represents a road or some other form of communication link. This kind of a picture is very helpful in understanding connected graphs that we introduce in this section. To understand such graphs we need some definitions which describe ways of “going from one vertex to another". We shall first give these definitions in the following subsection.

2.2.1 Paths, Circuits and Cycles

Consider the graph in Fig.1. Imagine yourself walking along its edges, going from vertex to vertex.

Fig.1

Suppose we want to start at the vertex x_1 and reach the vertex x_{12}. Is this possible? One possible way is to start from vertex x_1, walk along the edge x_1x_2, reach x_2, walk along the edge x_2x_3, reach x_3, walk along x_3x_4, reach x_4, and continue this till we reach x_{12}. Suppose we denote the edge joining x_{i-1} and x_i as $x_{i-1}x_i$. Then we can describe this 'walk' in an alternating sequence of vertices and edges as x_1, x_1x_2, x_2, x_2x_3, x_3, x_3x_4, x_4, x_4x_5, x_5, x_5x_6, x_6, x_6x_9, x_9, x_9x_{10}, x_{10}, $x_{10}x_{11}$, x_{11}, $x_{11}x_{10}$, x_{10}, $x_{10}x_{13}$, x_{13}, $x_{13}x_{12}$, x_{12}. This is by no means the shortest way to reach x_{12} from x_1. We could have gone from x_1 to x_5 directly. Moreover, we passed through the vertex x_{10} twice. This is not necessary. So the walk above can be described as a leisurely walk. If we have more time at our disposal, we can trace and retrace more edges. For example, we could have gone from x_6 to x_9, and again back to x_6.

So what are we doing when choosing a walk? We are, in fact, choosing a sequence whose elements are vertices and edges, alternately. Let us formally define a walk.

Definition: A **walk** in a graph G is a finite sequence W = $\{v_0, e_1, v_1, e_2, \ldots, e_k, v_k\}$, where $v_0, v_1, \ldots, v_k$ are vertices of G and $e_1, e_2, \ldots, e_k$ are edges joining the vertices v_{i-1} and v_i, $1 \le i \le k$. (Note that all the v_is or e_is may not be distinct. There may be repetition.)

In this case we say that **W is a walk from v_0 to v_k,** or **W is a v_0-v_k walk**, or **W is a walk joining v_0 and v_k.** The vertex v_0 is called the **initial vertex** and the vertex v_k is called the **end vertex** of the walk W. The **number of edges contained in a walk,** i.e., k, is called **the length** of the walk **W**, and is denoted by ℓ **(W)**. Since the vertices as well as the edges can be repeated, the length can very well be greater than the number of edges of the graph G.

Note: As you have seen, in a walk the vertices as well as edges can be repeated. So we cannot view this as a subgraph unless all the vertices as well as the edges in the walk are distinct.

Let's consider an example.

Example1 : Consider the graph on 5 vertices and 7 edges given in Fig.2. Find x_1-x_5 walks of length 8 and length 4, respectively.

Fig. 2

Solution: Consider the walk
$W = \{x_1, x_1\, x_2, x_2, x_2\, x_3, x_3, x_3\, x_4, x_4, x_4\, x_2, x_2, x_2\, x_5, x_5, x_5\, x_3, x_3, x_3\, x_4, x_4, x_4\, x_5, x_5\}$.
Then W is an x_1-x_5 walk of length 8.

Again, $W' = \{ x_1, x_1x_2, x_2, x_2x_4, x_4, x_4x_3, x_3, x_3x_5, x_5\}$ is an x_1-x_5 walk of length 4.

***

Why don't you try an exercise now?

E1) For the graph given in Fig.3, find a u-v walk of length 7.

Fig.3

Since we are considering only simple graphs, we often write a walk W as $\{v_0, v_1, \ldots, v_k\}$. While doing so, we assume that two consecutive vertices in the walk are joined by an edge in the graph, and that edge is included in the walk. For example, the x_1-x_{12} walk corresponding to Fig.1 that we discussed can be written as
$W = \{x_1, x_2, x_3, x_4, x_5, x_6, x_9, x_{10}, x_{11}, x_{10}, x_{13}, x_{12}\}$.

Let us now consider some particular kinds of walks, that we use in studying computer science.

Definitions :

1) A walk W is called a **path** if all its vertices are distinct, and hence, all its edges are distinct.

2) A u-v walk is **closed** if u = v, and **open** if $u \neq v$.

3) A walk in which all the edges are distinct and the only repeated vertex is the first vertex, this being the same as the last vertex, is called a **cycle**. (Remember, we had introduced you to cycles in Unit 1.)

Let us consider an example.

Example 2 : In the graph in Fig.4, find the following:

i) a closed walk which is not a cycle;
ii) a walk which is not a path;
iii) a cycle.

Solution: i) There are several closed walks in it which are not cycles. For instance, $W = \{x_5, x_6, x_7, x_8, x_5, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{11}, x_5\}$ is a closed walk. Since the edge $x_5\, x_{11}$ is repeated, it is not a cycle.

Fig.4

ii) $W_0 = \{x_5, x_6, x_7, x_8, x_5, x_9, x_{10}, x_5\}$ is a walk. Here the vertex x_5 is repeated three times. Thus, this is not a path.

iii) $W_1 = \{x_5, x_6, x_7, x_8, x_5\}$ is a cycle. So is $W_2 = \{x_{11}, x_{12}, x_{13}, x_{14}, x_{15\,2}{}^{x}{}_{11}\}$.

***

Try these exercises now.

E2) If all edges are distinct, then all vertices are distinct. True or false ? Why ?

E3) Let $G = (V, E)$ be a graph, where $V = \{t, u, v, w, x, y, z\}$ and $E = \{tu, tv, tw, ux, vw, vy, uz, wx, wz, xy, xz\}$. In G, find
i) a u-v walk that is not a path,
ii) a (u-u) walk that is not a cycle,
iii) a (u-u) cycle of minimum length.

E4) Let G be a graph such that $\delta(G) \geq k$. Use the principle of induction to show that G has a path of length k starting at any given vertex.
(Recall that $\delta(G) = \min \{d_G(x) : x \in V(G)\}$.)

Now, we know that a walk need not be a path. However, we shall now prove that in any u-v walk, we can always find a path from u to v.

Theorem 1: If W is a u-v walk joining two distinct vertices u and v, then there is a path joining u and v contained in the walk.

Proof : We will prove this using the principle of mathematical induction (see Unit 2, Block 1, MCS-013) on the length of the walk.

Let P (k) denote the statement 'If W is a u-v walk of length k, then there exists a path joining u and v contained in W.'

If k = 1, then P (1) is true since every walk of length 1 is a path.

Now, let us assume that the statement P (k−1) is true. In other words, we assume that given any x-y walk of length $\leq k-1$, there exists a path joining x and y contained in the walk. Then we want to show that the statement P (k) is true.

So, consider the u-v walk $W = \{u = u_0, e_1, u_1, \ldots, e_k, u_k = v\}$ of length k.
If W is already a path, we are done. Otherwise, there is at least one vertex which is repeated. Suppose j is the smallest integer such that the vertex u_j is repeated. Then

there is an integer $t > j$ such that $u_j = u_t$. Now consider the walk W_1 obtained by removing the part $\{e_{j+1}, \ldots, e_t\}$ in W, that is,
$W_1 = \{u = u_0, e_1, \ldots, u_j = u_t, e_{t+1}, \ldots e_k, u_k = v\}$. Clearly, W_1 is a u-v walk contained in the walk W, and its length is $k–t+j<k$, since $j < t$. Hence, by induction, we can get a path P joining u and v contained in W_1. Since P is contained in W_1 and W_1 is contained in W, the path P is contained in the walk W.
So, the result is true for any walk of length k, i.e., P(k) is true.

Therefore, by induction, P(n) is true for all n. Hence the result.

Note that the theorem above does not say that there is a walk joining any two vertices in a graph. In fact, in many practical situations it is very important to know which vertices in a graph can be joined by a walk, **and hence** by a path. For instance, in the graph in Fig.5, there is no a-x walk. Hence there is no path from a to x.

Try an exercise now.

Fig.5

E5) Consider the walk $\{x_3, x_5, x_2, x_4, x_3, x_2, x_1\}$ in the graph given in Fig.2. Obtain two distinct x_3-x_1 paths contained in this walk.

While studying networks, we often need to know whether two vertices in a graph are joined by a walk or not. This leads us to the definition of a connected graph, which we will introduce now.

2.2.2 Components

As you have noticed, almost all the graphs we have discussed so far have been ‘in one piece’. Some, like the one in Fig.5, are not. We can formalise this difference by introducing the concept of connectedness.

Definition: A graph G = (V, E) is called **connected** if for any two vertices $u, v \in V$, there exists a u-v walk in G. If G is not connected, then it is called **disconnected**.

This means that in a connected graph any two distinct vertices are joined by a path (by Theorem 1). For instance, K_n is connected $\forall n \geq 1$. However, a **null graph**, i.e., a graph whose edge set is empty, is totally disconnected (see Fig.6).

Fig.6 : A null graph

Here are some related exercises for you.

E6) Can a graph with one vertex be connected? Give reasons for your answer.

E7) Which of the graphs given in Fig.7 are connected?

Fig.7

E8) “If a graph G is connected, then all its subgraphs are connected.” Prove or disprove this statement.

While solving E8, you would have realised that subgraphs of connected graphs need not be connected. Similarly, some subgraphs of disconnected graphs are connected. Let us discuss such subgraphs now.

Definition: Let G = (V, E) be a graph. A subgraph H of G is called a **component** of G if H is connected and it is not a subgraph of any other connected subgraph of G. Thus, a component of G is, in a sense, a 'maximal' connected subgraph of G.

The number of components of G is denoted by **c(G)**. For instance, the graph in Fig.5 has two components, the graph in Fig.6 has 6 components, and the graph in Fig.1 has one component.

Let us consider another example.

Example 3: Find all the components of the graph G given in Fig.8.

Fig.8

Solution: G has three components, given in Figures 9 (a), (b) and (c).

Fig.9

* * *

You can now try this exercise.

E9) Consider the graph G given by Fig. 10. Find
- i) all the connected subgraphs of G;
- ii) all the components of G. Are they disjoint? Give reasons for your answer.

Fig.10

Consider all the graphs you have seen so far in this unit. In each case, it can be written as a disjoint union of its components. This phenomenon generalizes to any graph, as you will see in the following theorem, which we shall only state.

Theorem 2: Every graph can be partitioned into its components.

You may ask how this result can be of help. Knowing the number of components, for instance, helps us to find a bound for the number of edges of a graph. Here is a result about this, again without proof.

Theorem 3: If G is a graph with n vertices and k components, then G can have at least n–k edges, and at most $\frac{1}{2}(n-k)(n-k+1)$ edges.

A very useful result that immediately follows from this is :

Corollary 1: If G is a connected (n, m)-graph, then $n-1 \leq m \leq \frac{1}{2}n(n-1)$.

Try some exercises now.

E10) Give an example of a graph with
- i) 4 components, each of which is complete;
- ii) 3 components, where no two components are isomorphic.

E11) Can a graph have more components than vertices ? Give reasons for your answer.

Another approach used in the study of connected graphs is to ask ‘how connected’ a connected graph is. One possible interpretation of this question is to ask how many edges or vertices must be removed from the graph in order to disconnect it. We shall discuss this in the next subsection.

2.2.3 Connectivity

Let us now consider a graph showing an electric circuit (see Fig.11). This graph is connected. Suppose we cut the wire connecting d and e in the electric circuit. This means that in the graph showing the circuit, we are actually removing the edge de. When we cut the wire, the circuit becomes disconnected. Correspondingly, the removal of the edge de in the graph makes the graph disconnected.

Removing the edge de does not mean that we remove the vertices d and e.

Fig.11

We just saw a situation in which the removal of one edge disconnects the graph. This lead us to the following definition.

Definition: An edge e of a connected graph G is called a **bridge** in G if the removal of e disconnects G. When we remove an edge e from the graph G, we denote the resulting graph by **G–e**.

Not every edge is a bridge. For instance, if we remove the edge ab in Fig. 11, the resulting graph is not disconnected. This also holds for the graph given in Fig. 12, which represents the roads connecting the main towns in a state.

Fig.12

In this case no edge is a bridge since there always exist alternative connections.

Here are some related exercises for you.

E12) Find the bridges in each of the graphs in Fig.13.

Fig.13

E.13) How many bridges do C_n and K_n have, where $n \geq 3$?

Let us consider the graph given in Fig.11 again. This graph is connected. Here, if we remove the edge de, then the resulting graph gets disconnected, the number of components becoming 2. On the other hand, if we remove the edge dc, then the graph does not get disconnected. Note that the edge dc belongs to the cycle {a, b, c, d, a}, but the edge de does not belong to any such cycle. The cycle seems to provide an alternative connection between the vertices c and d.

In fact, it follows from the definition of a bridge that **an edge e of a graph G is a bridge if and only if e does not belong to any cycle of G**.

While doing E13, you have obtained many graphs which do not have a bridge. To disconnect such a graph we need to remove more than one edge. Therefore, given a graph, it is natural to ask how many edges need to be removed before it gets disconnected. This leads us to the following definition.

Definition: The **edge-connectivity, λ (G),** of a connected graph G is the least number of edges that need to be removed for G to become disconnected.

For example, the edge-connectivity of any graph with a bridge is 1. Let us consider another kind of example.

Fig.14

Example 4: Find the edge-connectivity of the graph G given in Fig.14.

Solution: First note that this graph does not have any bridges. Therefore its edge-connectivity is more than 1. Now, if we remove the edges xz, zy, then the graph gets disconnected. Similarly, there are other sets of two edges, namely, {xv, vu}and {uw, wy}, the removal of which disconnects G. Therefore, the edge connectivity of G is 2.

***

Note : In the **context of computer networks**, the edge-connectivity of a graph representing such a network gives the number of link failures that can be tolerated before the network becomes disconnected.

Why don't you try some exercises now?

E14) Find the edge-connectivity of C_n and K_n for $n \geq 3$.

E15) Find λ (G), where G is the Petersen graph.

Let us now look at a particular type of set of edges of a connected graph.

Definition : A **cutset** of a connected graph G is a set S of edges with the following properties:

i) the removal of all the edges in S disconnects G;
ii) the removal of any proper subset of S will not disconnect G.

For example, consider the graph given in Fig.15

Fig.15

The set {uw, ux, vx} and {uw, wx, xz} are cutsets for this graph. However, the set {uw, wx, xz, yz} is not a cutset since this set has a subset {uw, wx, xz}, the removal of which disconnects G.

Note : 1) **Two cutsets of a graph need not have the same number of edges**. For example, the sets {uw, ux, vx} and {wy, xz} are both cutsets of the graph in Fig.15.

2) **The edge-connectivity of a graph G is the size of the smallest cutset of G**.

Try this exercise now.

E16) Which of the following sets of edges are cutsets of the graph given in Fig.16, and what is the edge-connectivity of the graph ?
i) {su, sv}, ii) {uv, wx, yz}, iii) {ux, vx, wx, yz},
iv) {yt}, v) {wx, xz, yz}, vi) {uw, wx, wy}

Fig.16

We can also think of connectivity in terms of the minimum number of vertices which need to be removed in order to disconnect a graph. Remember that, when we remove an edge, we do not remove its end vertices. However, **when we remove a**

vertex, then any edge incident with that vertex also gets removed.

Analogous to the notion of a bridge, we define a cut-vertex.

Definition : A **cut-vertex** of a connected graph G is a vertex v of G such that G−v is disconnected.

For instance, in Fig.11, both d and e are cut-vertices.

Now we can define vertex-connectivity and a vertex-cutset on similar lines as we have done for edges. Why don't you try it for yourself (see E 17)?

E17) Define vertex-connectivity and vertex-cutset of a graph.

E18) Find the vertex-connectivity and a vertex-cutset for the graph given in Fig.16.

Once again, **if a graph represents a computer network, then its vertex connectivity gives the number of node failures that the network can tolerate**.

We shall now introduce you to another type of graph which underlies many computer, and other, applications.

2.3 BIPARTITE GRAPHS

In this section we shall define bipartite graphs and explain their importance through various problems. Let us first start with the following problem.

Four persons x_1, x_2, x_3 and x_4 are available to fill five jobs y_1, y_2, y_3, y_4 and y_5. x_1 is qualified for the jobs y_1 and y_2; x_2 is qualified for the jobs y_1 and y_3; x_3 is qualified for the job y_4; and x_4 is qualified for the jobs y_2, y_3 and y_5. The assignment problem is concerned with the following questions:

i) *Can each person be assigned to a single job for which she is qualified?*
ii) *If so, how should the assignment be made?*
iii) *If assigning to a single job is not possible, at most how many jobs should be assigned to each person ?*

The problem of the kind stated above is known as an assignment problem. To solve this problem it is convenient to consider the following graph-theoretic model of the situation (see Fig.17).

Fig.17

The graph G, representing the problem, has an edge joining x_i and y_j if x_i is qualified for the job y_j. Then the problem of assigning people to jobs for which they are qualified is equivalent to the problem of selecting a subset of the set of edges such that each x will be connected to exactly one y by one of these edges.

Now, if you look at the graph given in Fig.17, you will see that the set of its vertices can be divided into two disjoint subsets such that no two vertices in a subset are adjacent. Let us formally define such graphs.

Definition: A graph G is said to be **bipartite** if V (G) = X $\cup$ Y, where X and Y are non-empty sets such that X $\cap$ Y = ϕ and every edge in E (G) has one end vertex in the set X and the other end vertex in the set Y. The sets X and Y form a **partition** of the set V (G), and we often say that X $\cup$ Y is a **bipartition** of the graph G. We also denote such a graph by **G(X,Y)**.

An alternative way of thinking of a bipartite graph is in terms of colouring its vertices with two colours, say red and blue — a graph is bipartite if we can colour each vertex red or blue in such a way that every edge has a red end and a blue end.

Bipartite graphs are useful in studying various real-life problems, including neural networks. One model that emulates the essential working of the network using graph theory is given in Fig.18. As you can see, this is a bipartite graph, so that the properties of bipartite graphs are useful for studying this model.

Fig.18

Let us consider some other examples of bipartite graphs.

Example 5 : Show that C_6 is bipartite and K_3 is not bipartite.

Solution : In Fig.19 we show C_6. Since $V(C_6)$ can be partitioned into {a, c, e} and {b, d, f}, C_6 is bipartite.

In K_3 each vertex is adjacent to every other vertex. Therefore, no bipartition is possible.

***

Note that in a bipartite graph G(X,Y), it is not necessary that each vertex of X is joined to each vertex of Y. For instance, in C_6 in Fig.19, a is not joined to d. This leads us to the following definition.

Fig.19 : C_6 is bipartite

Definition : A **complete bipartite graph** is a bipartite graph G(X,Y) in which each $x \in X$ is joined to every $y \in Y$, i.e., G is also a complete graph. If $|X| = r$ and $|Y| = s$, we denote G(X,Y) by $\mathbf{K_{r,s}}$. In Fig.20 we have shown a few of these graphs.

Fig 20 : Some complete bipartite graphs

Now, given a bipartite graph, you may wonder if the bipartition is unique. The following example will give you an answer to this question.

Example 5: Find two different bipartitions of the graph given in Fig.21.

Fig.21

Solution: The vertex set is $\{x_1, x_2, y_1, y_2, y_3, z_1, z_2\}$.
One way of partitioning this can be by taking $X = \{x_1, x_2, z_2\}$, $Y = \{z_1, y_1, y_2, y_3\}$.
Another way can be $X_1 = \{x_1, x_2, y_3\}$, $Y_1 = \{z_2, z_1, y_1, y_2\}$.
Both these partitions make G bipartite.

***

Note that the graph in Fig.21 is not connected. Had it been connected it would not have been possible to find more than one bipartition of G. In fact, we have the following theorem.

Theorem 4 : A connected bipartite graph has a unique bipartition.

We shall now state a theorem, which gives a characterisation for bipartite graphs.

Theorem 5: A graph G is bipartite if and only if G does not contain any cycle of odd length as a subgraph.

This result is very useful. For instance, using it we know that **C_n is not bipartite whenever n is odd**.

You can try some exercises now.

E19) Check whether the hypercube Q_3 and the star graphs are bipartite.

E20) For which values of m and n is $K_{m,n}$ regular ?

E21) i) Is the subgraph of a bipartite graph bipartite ?
ii) Is the complement of a bipartite graph bipartite ?
Give reasons for your answers.

E22) Show that if $G_1, \ldots, G_n$ are bipartite, then $\bigcup_{i=1}^{n} G_i$ is bipartite.

Let us now go back to the assignment problem. In that problem we are interested in finding those special subgraphs of the associated bipartite graph which give a solution to the problem. We have defined such subgraphs below.

Definition: A **matching** in a bipartite graph G is a set of edges such that no two edges have a common end vertex. In other words, a matching in G (X,Y) defines a one-to-one correspondence between the vertices in a subset of X and the vertices in a subset of Y.

For example, Fig.22 shows a bipartite graph and one of its matchings. Can you find any other matching? We leave this as an exercise for you to check (see E 23).

(a) **(b)**

Fig.22 : a) G is bipartite; b) A matching in G

Related to the concept of matching, we have another concept.

Definition: A matching of X into Y is called a **complete matching** of X and Y if there is an edge incident with every vertex in X. In other words, a matching is complete if a one-to-one correspondence is defined between all the vertices in X and the vertices in a subset of Y.

Is the matching given in Fig.22 (b) a complete matching? No, because in this matching, the vertex 4 is not included.

In graph-theoretic terminology, the **assignment problem can be stated in the following way**: if G = G (X, Y) is a bipartite graph, when does there exist a complete matching from X to Y in G? So, for a given bipartite graph, we want to know whether there is a complete matching of the set of vertices in X into the set of vertices in Y. The following theorem gives a **necessary and sufficient condition** for the existence of such a matching. As before we shall only state the theorem, omitting the proof.

Theorem 6: Let G = G (X, Y) be a bipartite graph. A complete matching of X into Y exists in G **if and only if** $|A| \leq |R(A)|$ for every subset A of X, where $|A|$ denotes the number of elements in A (also called cardinality of A) and R(A) denotes the set of vertices in Y that are adjacent to the vertices in A.

Let us apply this theorem to the assignment problem in the following example.

Example 6 : Verify the conditions of Theorem 6 for the assignment problem given at the beginning of this section (see Fig.17).

Solution : To check the theorem we have to consider all subsets of the vertex set X = $\{x_1,x_2,x_3,x_4\}$, their cardinality, the corresponding sets R(A), and their cardinality. Table 1 gives a list of all the possibilities.

Table 1

A	$\lvert A\rvert$	$R(A)$	$\lvert R(A)\rvert$
ϕ	0	ϕ	0
$\{x_1\}$	1	$\{y_1,y_2\}$	2
$\{x_2\}$	1	$\{y_2,y_3\}$	2
$\{x_3\}$	1	$\{y_4\}$	1
$\{x_4\}$	1	$\{y_2,y_3,y_5\}$	3
$\{x_1,x_2\}$	2	$\{y_1,y_2,y_3\}$	2
$\{x_2,x_3\}$	2	$\{y_1,y_3,y_4\}$	2
$\{x_3,x_4\}$	2	$\{y_2,y_3,y_4,y_5\}$	4
$\{x_1,x_4\}$	2	$\{y_1,y_2,y_3,y_5\}$	4
$\{x_2,x_4\}$	2	$\{y_1,y_2,y_3,y_5\}$	4
$\{x_1,x_3\}$	2	$\{y_1,y_2,y_4\}$	3
$\{x_1,x_2,x_3\}$	3	$\{y_1,y_2,y_3,y_4\}$	4
$\{x_2,x_3,x_4\}$	3	$\{y_1,y_2,y_3,y_4,y_5\}$	5
$\{x_1,x_3,x_4\}$	3	$\{y_1,y_2,y_3,y_4\}$	4
$\{x_1,x_2,x_4\}$	3	$\{y_1,y_2,y_3,y_5\}$	4
$\{x_1,x_2,x_3,x_4\}$	4	$\{y_1,y_2,y_3,y_4,y_5\}$	5

It shows that the condition $\lvert A\rvert \le \lvert R(A)\rvert$ is satisfied for all subsets A of X. Hence the condition of Theorem 6 is satisfied. This shows that there exists a complete matching from X into Y for the assignment problem. Therefore, the assignment problem is solved.

***

You can now try an exercise.

E 23) For the bipartite graph given in Fig.22, find a matching, apart from the given one. Does the graph have a complete matching ? Give reasons for your answer.

Let us now see another type of graph which has come into prominence because of its applications to electrical networks.

2.4 TREES

We are all familiar with the idea of a family tree. The concept of a tree in graph theory first arose in connection with work of a mathematician G. Kirchoff on electric networks in the 1840s, and with the work of another mathematician Cayley on the enumeration of chemical molecules in the 1870s. More recently, trees are used in many areas, ranging from linguistics to computing. For instance, trees are used to study the following problems :

- How should items in a list be stored so that an item can be easily located ?
- How should a set of characters be efficiently coded by bit strings ?

So, let us begin our study of trees by considering the following graphs.

G **H**

Fig.23

Can you find any difference in their structures ? You might have noticed that G is disconnected, and has no cycles. On the other hand, H is connected and has no cycles. From the following definition you will see that H is an example of a tree.

Definition : A **tree** is a connected graph with no cycles. A **forest** is a graph, each of whose components is a tree.

Fig.24 shows a graph with four components, each of which is a tree. Hence, the graph is a forest.

Fig.24

A tree has several defining properties which we shall list in the following theorem.

Theorem 7 : Let G be a graph with n vertices. Then the following statements are equivalent.

i) G is a tree.
ii) G has no cycles and has $(n - 1)$ edges.
iii) G is connected and has $(n - 1)$ edges.
iv) G is connected and every edge is a bridge.
v) Any two vertices of G are connected by exactly one path.

Proof : If $n = 1$, all the five results are trivial. We shall, therefore, assume that $n \geq 2$. Now, from your study of mathematical logic you know that if we prove (i)$\Rightarrow$(ii), (ii) $\Rightarrow$(iii), (iii) $\Rightarrow$(iv), (iv) $\Rightarrow$(v) and (v) $\Rightarrow$(i), then all the statements will be proved to be equivalent. So, let us prove the implications one by one.

(i) $\Rightarrow$(ii) : By definition, G does not have any cycles. We shall show that G has $(n - 1)$ edges, by induction.
If $n = 2$, then the number of edges is 1. Therefore, the result is true for $n = 2$.
So, now let us assume that every tree on p vertices has $(p - 1)$ edges for any positive integer p such that $2 \leq p < n$. Then we have to show that every tree on n vertices has $(n - 1)$ edges.
Now suppose we remove any edge. Since G has no cycles, the removal of any edge disconnects G into two graphs G_1 and G_2, such that G_1 and G_2 are connected and have no cycles. Therefore, G_1 and G_2 are trees and each has less than n vertices.
Let n_1 and n_2 be the vertices in G_1 and G_2. Then $n_1 + n_2 = n$.
Since n_1 and n_2 are less than n, by our induction assumption, the number of edges in G_1 and G_2 are $n_1 - 1$ and $n_2 - 1$, respectively. Therefore, the total number of edges in both the graphs is $n_1 + n_2 - 2 = n - 2$. These edges, together with the edge which is removed, will give the total number of edges in the original graph. Therefore, the total number of edges in G is $n - 1$.

Thus, we have shown that every tree on n vertices has $n - 1$ edges. By induction, this is true for all n.

(ii) $\Rightarrow$(iii) : Suppose that G is disconnected. Let $c(G) = t > 1$. Let $G_1, G_2, \ldots, G_t$ be the components of G such that the number of vertices in each G_i is p_i for $i = 1,2,\ldots,t$, and the number of edges in each G_i is q_i, for $i = 1,2,\ldots,t$. Then
$p = p_1 + p_2 + \ldots + p_t$, $q = q_1 + \ldots + q_t$.

Now, since every G_i is connected and without cycles, G_i is a tree for i = 1,2,…,t. Therefore, by what we have shown while proving (i) $\Rightarrow$ (ii), $q_i = p_i - 1 \le i \le t$. Then
$p - 1 = q = q_1 + \ldots + q_t = p - t$.
That is, t = 1, which contradicts our assumption that t > 1. Therefore, G is connected.

(iii) $\Rightarrow$ **(iv)** : Suppose there is an edge which is not a bridge. Then the removal of that edge will result in a graph with n vertices and (n–2) edges. This is not possible when G is connected, by Corollary 1 to Theorem 3. Therefore, every edge is a bridge.

(iv) $\Rightarrow$ **(v)** : Since T is connected, each pair of vertices is connected by at least one path. If a given pair of vertices is connected by two paths, then they form a cycle, which contradicts the fact that every edge is a bridge. Therefore, there is a unique path joining any two vertices.

(v) $\Rightarrow$ **(i)** : We are assuming that any two vertices are connected by a unique path. So, the graph G is connected. Now, suppose G contains a cycle $C = \{x_0, x_1, \ldots, x_n = x_0\}$. Then we can find two distinct paths $P_1 = \{x_0, x_1\}$ and $P_2 = \{x_0, x_{n-1}, \ldots, x_2, x_1\}$ connecting the vertices x_0 and x_1, which contradicts our assumption. Therefore, G does not contain any cycle, and hence, is a tree.

Why don't you try some exercises now ?

E24) For which values of m and n is $K_{m,n}$ a tree ?

E25) Which of the following graphs are trees, and why ?

Fig.25

The theorem above tells us that a tree has got several nice properties which a general graph does not have. In fact, the importance of trees in graph theory is that every connected graph contains a tree which has all the vertices of the original graph, as you will now see.

Let us consider a connected graph G. Consider a cycle in it and remove one of its edges such that the resulting graph stays connected. We repeat this procedure with one of the remaining cycles, continuing until there are no cycles left. The graph which remains is a connected subgraph of G which does not have any cycle. Therefore, it is a tree. Note that this tree has all the vertices of G. Such a graph is called a spanning tree, as you will realize from the following definition.

Definition : A **spanning tree** for a graph G is a subgraph of G which contains all the vertices of G and is a tree.

This concept is useful for finding, for example, the minimum number of roads to be kept open to maintain connections in a given transport network.

Now, the question is whether every graph has a spanning tree. The following theorem, the proof of which is omitted, tells us about this.

Theorem 7 : A graph G is connected **if and only** if it has a spanning tree.

The theorem above tells us that in a graph with k components, each component will have a spanning tree. Because of this result and because of the special structure of trees, in trying to prove a general result in graph theory, it is sometimes convenient to try to prove the corresponding result for a tree. The general result would, then, follow.

Note : A **spanning tree is not unique**. For instance, Fig. 26 shows a connected graph G and two of its spanning trees, T_1 and T_2.

Fig.26

You can try some exercises now.

E26) Draw three spanning trees of the following graph.

Fig.27

E27) Is a tree a bipartite graph ? Give reasons for your answer.

Spanning trees are important in data networking, particularly in multicasting over Internet Protocol (IP) networks. To send data from a source computer to multiple receiving computers, each of which is a subnetwork, data could be sent separately to each computer. This type of networking, called unicasting, is inefficient, since many copies of the same data are transmitted over the network. To make the transmission of data to multiple receiving computers more efficient, IP multicasting is used. With IP multicasting, a computer sends a single copy of data over the network, and as data reaches intermediate routers the data are forwarded to one or more other routers so that ultimately all receiving computers in their various subnetworks receive these data.

For data to reach receiving computers as quickly as possible, there should be no loops (which in graph theory terminology are circuits or cycles) in the path that data take through the network. That is, once data have reached a particular router, data should never return to this router. To avoid loops, the multicast routers use network algorithms to construct a spanning tree in the graph that has the multicast source, the routers, and the subnetworks containing receiving computers as vertices, with edges representing the links between computers and/or routers.

So far we have seen three types of graphs : connected graphs, bipartite graphs and trees. You will see several other types of graphs in the following units. Let us now summarise what we have covered in this unit.

2.5 SUMMARY

In this unit we have discussed the following points.

1) The definition and use of the terms 'walk', 'path' and 'cycle' in a graph. We have also proved and used the fact that in every u-v walk there is a u-v path.

2) Properties of connected graphs, how to find components of a graph, the effect of removal of a vertex or an edge on the number c(G) of the components of a graph G.

3) The application of the fact that if G is a (p,q)-graph with k components, then $p - k \le q \le \frac{1}{2}(p-k)(p-k+1)$.

4) The definition and properties of bipartite graphs, and a characterisation of such graphs in terms of not containing odd cycles.

5) What a matching is, and when a complete matching exists.

6) Explanation of trees and spanning trees, particularly the importance of such graphs among the class of all connected graphs.

7) The proof and application of the statement that a graph G with n vertices is a tree **iff** it has no cycles and has (n–1) edges **iff** it is connected and has (n–1) edges.

2.6 SOLUTIONS/ANSWERS

E1) {u, uv, ux, x, xy, y, yv, v, vb, b, bu, u, uv, v} is a walk of length 7. This is not the only one. Think of some others too.

E2) False. For instance, in a cycle all the edges are distinct, not all the vertices.

E3) i) It is easy to find examples if you draw the walk. One example is {u,x,w,z,y,x,w,v}. There are other examples.

ii) W = {u,v,y,z,w,x,z,u} is a walk in which the vertex z is repeated. Therefore, W is not a cycle.

iii) W_0 = {u,t,w,x,u} is a cycle such that all other cycles have length greater than $\ell(W_0)$.

E4) We use induction on k. If k = 1, then every vertex has at least one neighbour. Thus, there exists a path of length 1 starting at any vertex.
Now, by induction, assume that in every graph H with $\delta(H) \ge (k-1)$, there is a path of length (k–1) starting at any given vertex.

Let G be a path with $\delta(G) \ge k\,(>1)$. Let x_0 be any vertex in G. Choose any edge e_1 incident on x_0. Consider G–e_1. Removal of one edge reduces only the degree of its end vertices by one. Thus, $\delta(G - e_1) \ge (k-1)$. Thus, by induction, there is a path $\{x_1,e_2,\ldots,e_k,x_k\}$ of length (k–1) in G_1. Moreover, since the degree of x_{k-1} is at least k, we can choose x_k different from $x_0,x_1,\ldots,x_{k-2}$. Also $\{x_0,e_1,x_1,e_2,\ldots,x_k\}$ is a path of length k in G starting from x_0. Therefore, there exists a path of length k starting at any vertex in G.

Since this is true for all k, the result follows.

E5) $\{x_3, x_5, x_2, x_1\}$ and $\{x_3, x_2, x_1\}$.

E6) It is connected. Because if it is disconnected then there exists two distinct vertices which are not joined by a path, which is not possible since the graph does not have two distinct vertices.

E7) (a) and (b) are connected, (c) is disconnected.

E8) The statement is false. For example, consider the graph K_3 given in Fig.28(a). The subgraph of this graph obtained by deleting the edges v_1v_3 and v_2v_3, given in Fig.28(b), is not connected.

Fig.28

E9) i) The graphs having single vertices a,b,c,d,e,f, and the graphs having the following vertices and edges

i)	V = {a,b},	E = {ab}
ii)	V = {a,c},	E = {ac}
iii)	V = {c,d},	E = {cd}
iv)	V = {c,f},	E = {cf}
v)	V = {d,c,a},	E = {dc,ca}
vi)	V = {b,a,c},	E = {ba,ac}
vii)	V = {a,b,c,d},	E = {ab,ac,cd}

ii) Two components are the graph formed by the vertices a,b,c and d, and the graph formed by the vertices e and f. They are disjoint, for if they have a vertex in common the two component graphs would be connected.

E10) An example each is given below. You can think of many others.

Fig.29 : $K_3 \cup K_4 \cup K_1 \cup K_5$

Fig.30

E11) No, since each component must contain at least one vertex.

E12) In (a) there is only one bridge given by bc. In (b) there are several bridges, e.g., u_1u_3, u_3u_6, u_5u_6.

E13) None.

E14) $\lambda(C_n) = 2, \lambda(K_n) = n-1$.

E15) $\lambda(G) = 3$.

E16) The sets given in (i), (iii), (iv) and (vi) are cutsets. The set given in (ii) is not a cutset, since its removal does not disconnect the graph; the set given in (v) is also not a cutset, since we can disconnect the graph by removing just xz and yz.

E17) The **vertex-connectivity** of a connected graph G is the smallest number of vertices whose removal disconnects G.

A **vertex-cutset** of a connected graph G is a set H of vertices with the following properties :

i) the removal of all vertices in H disconnects G;.
ii) the removal of any proper subset of H will not disconnect G.

E18) The vertex-connectivity is 1, and the vertex-cutset is $\{w\}$.

E19) Q_3 does not contain any odd cycle. Therefore, by Theorem 5, it is bipartite. The star network with n+1 vertices is $K_{1,n}$, and hence, is bipartite.

E20) Only for m=n, is $K_{m,n}$ regular.

E21) i) Yes. Let G be a bipartite graph, with a bipartition $X \cup Y$. Let H be a subgraph of G. If V(H) is disjoint from either X or Y, then E(H) = ϕ. And then, any partition of V(H) into two subsets will serve as a bipartition. In the other situtation, $(V(H) \cap X) \cup (V(H) \cap Y)$ is a bipartition of V(H).

ii) No, e.g., the complement of G given in Fig.21 contains C_7. So, by Theorem 5, G is not bipartite.

E22) Let $G_i, 1 \le i \le n$, be bipartite graphs with the bipartitions $V(G_i) = X_i \cup Y_i$, respectively. Let $G = \bigcup_{i=1}^{n} G_i$. Then E(G) is the disjoint union $\bigcup_{i=1}^{n} E(G_i)$.

$V(G) = A \cup B$, where $A = \bigcup_{i=1}^{n} X_i$ and $B = \bigcup_{i=1}^{n} Y_i$, is a bipartition of V(G). This can be seen as follows :
Let e be an edge in E(G). Since E(G) is a disjoint union of $E(G_1),\ldots,E(G_n)$, the edge e belongs to only one of them. Without loss of generality, suppose $e \in E(G_r)$. Since G_r is bipartite with a bipartition $X_r \cup Y_r$, this means e has one end vertex in X_r and the other in Y_r, that is, e has one end vertex in A and the other in B. Thus, G is bipartite with a bipartition $A \cup B$.

E23) Fig.31 gives another matching, which is in fact a complete matching.

Fig.31

E24) For m=1, $n \geq 1$ or $n = 1$ and $m \geq 1$ only. For $m \geq 2, n \geq 2, K_{m,n}$ will contain a cycle.

E25) The ones in (a) and (c) are trees by Condition (iii), Theorem 7. The one in (b) is not, for the same reason. The one in (d) is not because it contains C_5.

E26)

Fig.32

E27) Yes. Since a tree does not have any cycles, by Theorem 5 it is a bipartite graph.

UNIT 3 EULERIAN AND HAMILTONIAN GRAPHS

Structure **Page Nos.**

3.0 INTRODUCTION

Suppose you go to a new city as a salesperson. You would naturally like to familiarise yourself with all the important routes. One way to do this is to buy a map of the city and go around the city. If you do this without proper planning, you may pass through some of the streets more than once. To avoid this, you would need to sit down and plan your route. The most efficient route would involve traversing every street in it only once. But is it possible to find such a route?

This question is so natural that you may not be surprised to know that a similar question was raised more than 250 years ago. Königsberg was a city in what was known as Prussia those days. The Pregel river flowed through this city forming two islands (see B and C in Fig.1).

Fig. 1: A schematic diagram of Königsberg

The two islands and the rest of the city were connected to each other by seven bridges. Some of the citizens used to amuse themselves with the following question: Is it possible to go around the city using each bridge exactly once?

Fig. 2: Leonhard Euler (1707-1783)

In 1736, the great Swiss mathematician Leonhard Euler (pronounced as 'oiler') answered this question by converting this into a problem in graph theory. We will see this problem in Section 3.2 (Sec.3.2 in brief), while discussing graphs named after Euler.

There is one more question similar to the Königsberg problem in recreational mathematics — which figures can be drawn without lifting the pen from the paper and without going over any of the lines twice? This question is also answered in Sec.3.2. A mathematical puzzle invented by Hamilton involves finding a cycle containing all the vertices of a certain graph. Motivated by this, we will discuss conditions for a graph to contain a cycle containing all the vertices of the graph. Such as graph is called a Hamiltonian graph, in honour of Hamilton. In Sec. 3.3 we will give some necessary and sufficient conditions for a graph to be Hamiltonian.

Finally, in Sec.3.4 we discuss a related question, the travelling salesperson problem.

3.1 OBJECTIVES

After studying this unit, you should be able to

- check whether a given graph is Eulerian or not;
- check whether a given graph satisfies certain necessary conditions for a Hamiltonian graph;
- check whether a given graph satisfies certain sufficient conditions for a Hamiltonian graph;
- apply the 1-exchange algorithm to reduce the weight of a Hamiltonian cycle.

3.2 EULERIAN GRAPHS

As we mentioned in the introduction, Euler solved the Königsberg problem by converting it into a problem in graph theory. He represented each land area by a vertex and each bridge by an edge (see Fig.3(a)).

(a) **(b)**

Fig. 3

You might have noticed that the graph in Fig.3(a) is a **multigraph**. Here A and C are connected by two edges; So are C and D. Let us break up one of the edges connecting C and D by adding a new vertex E. Similarly, break up one of the edges joining A and C by adding a new vertex F. Then, we get the simple graph in Fig.3 (b). If we can find a way of going around the graph in Fig.3(b) using each edge only once, then we can do so in the graph in Fig.3 (a) also, and vice-versa. This process of subdividing the edges can be carried out for any multigraph.

To understand the problem, let us introduce some terms first.

Definitions :

i) A **trail** is a walk in which no edge is repeated.
ii) A **circuit** is a trail whose starting vertex and end vertex are the same.
iii) A trail which is not a circuit is sometimes called an **open trail**.
iv) A circuit (resp. trail) in a graph G containing all the edges of G, is called an **Eulerian circuit** (resp. **Eulerian trail**).
v) A **graph is Eulerian** if it contains an Eulerian circuit.

So, we can rephrase the Königsberg bridge problem in the following way:
Is the graph in Fig. 3(b) Eulerian?

Before going further, we give a clarification of our definition of Eulerian graphs in the form of a remark.

Remark: You may have noticed that we made connectedness a part of the definition of Eulerian graphs. This is to avoid examples like the one given in Fig. 4. Here the graph has a circuit which contains all the edges of the graph. However, there is no edge through which we can reach the isolated vertex. Unless there is a very special reason, we will not bother about a place to which there is no access! So, such isolated vertices are of no interest to us. By making connectedness a part of the definition, such situations can be avoided.

Fig.4

Now, let us consider some examples of Eulerian graphs. The simplest class of example is a cycle, for example, C_6 in Fig. 5(a). We can get another example by adding a cycle of length 3 to the graph in Fig. 5(a) at v_1 (see Fig. 5(b)).

(a) **(b)** **(c)**

Fig. 5

This is also Eulerian because we can start at the vertex v_1, traverse the inner triangle, come back to v_1 and traverse the outer cycle. We get yet another Eulerian graph by incorporating a cycle of length 6 at v_1 to Fig.5 (a) (see Fig. 5(c)).

Now you may like to verify whether you have understood the definition of an Eulerian circuit by attempting the following exercise.

E1) Prove that the graph given in Fig.5(c) is Eulerian by producing an Eulerian circuit in it.

E2) What is the difference between an Eulerian graph and an Eulerian circuit?

You probably found E1 easy. In a simple example like this, you can easily prove that a graph is Eulerian by producing an Eulerian circuit by trial and error. This may not be possible in more complicated cases. It is **impossible** to prove that a graph is **not** Eulerian by trial and error — we may miss some clever way of tracing an Eulerian circuit. So, we need a necessary and sufficient condition for a graph to be Eulerian. The condition should also be easy to apply. The next theorem gives such a condition. Euler's proof of the necessary part of the theorem appeared in *Solutio problematis geometriam situs pertinentis* (The solution of a Problem relating to the Geometry of Position). Hierholzer proved the sufficiency part.

Theorem 1: A connected graph G is Eulerian **if and only if** the degree of each of its vertices is even.

Proof: We shall first assume that the graph G is Eulerian and prove that all its vertices have even degree. So, let T be an Eulerian circuit in G. Every time the circuit passes through a vertex, it uses two edges, one to reach the vertex v and one to leave it. What about the vertex v from which we start tracing the circuit? The edge with which we start the circuit is paired with the edge with which we end the circuit. Apart from this, every time we pass through v in the intermediate stages, we will use two edges incident at the vertex as before. Also, we traverse each edge only once. So, all the vertices of the graph have even degree.

To **prove the converse**, consider a connected graph in which each vertex has even degree. We will now prove that G contains an Eulerian circuit, by induction on the number of edges in G.

Suppose that the number of edges is 0. Since we have assumed that the graph is connected, it consists of a single isolated point. Since the edge set is empty the statement that there is an Eulerian circuit containing all the edges is vacuously true.

Next, assume that all the graphs with fewer edges than G contain an Eulerian circuit. All the vertices of G have even degree and G has no vertex of degree 0 (isolated vertex) since it is connected. So, all the vertices have degree at least 2. We can start from an arbitrary point $u = u_0$ and trace a circuit C as follows:

We choose any edge u_0u_1 incident at u_0. Since u_1 has degree at least two, there is another edge incident at u_1, say u_1u_2. We go on tracing a circuit like this, always making sure that we enter and leave any vertex by different edges. During the course of tracing C, we may pass through u_0 several times. The process ends when we reach u_0 and find that there is no unused edge to leave u_0. If the circuit we have obtained contains all the edges, we are done. Otherwise, we remove this circuit from G and call the resulting (possibly disconnected) graph H. All the vertices in each of the components of H have even degree and all the components have fewer edges than G.

So all the components are Eulerian. We now get an Eulerian circuit in G as follows: We start from any vertex v on the circuit C and traverse the edges of C till we come to a vertex that lies on one of the components of H. We then traverse the Eulerian circuit in that component, eventually returning to the circuit C. We continue along C in this fashion, taking Eulerian circuits of components of H as we come to them, finally returning to the vertex v we started with. We would have used each of the edges only once, that is, we have obtained an Eulerian circuit.

Note that, by connectedness of G, each component of H must contain a point of C.

Hence, by induction, the result is true for all graphs satisfying the condition of the converse.

Let us now see if we can solve the Königsberg bridge problem using Theorem 1.

Example 1: Check whether the Königsbergians can go round the city using each bridge only once.

Solution: You may recall that we have reduced the Königsberg bridge problem to finding an Eulerian circuit in Fig.3(b). According to the necessary part of the theorem, if a graph has an Eulerian circuit, it has no edges of odd degree. But, as you can see, all the vertices, except E and F, have odd degree. So, this graph does not have an Eulerian circuit. So, the Königsbergians cannot go around the city using each vertex only once.

* * *

Now, here are some exercises to test your understanding.

E3) After Euler proved his theorem, much water has flown under the bridges in Königsberg. In 1875, an extra bridge was built in Königsberg, joining the land areas A and D (see Fig.6). Is it possible now for the Königsbergians to go round the city, using each bridge only once?

Fig. 6

E4) By writing the degree sequences of the following graphs, check whether they are Eulerian. For the graphs that are Eulerian, write down an Eulerian circuit.

G_1 **G_2**

Fig.7

E5) a) For which values of n is K_n Eulerian?

b) For which values of n and m is $K_{n,m}$ Eulerian?

E6) Check whether Q_3 and Q_4 are Eulerian.

E7) Show that, in a connected Eulerian graph, an Eulerian circuit can be traced starting from any vertex.

Suppose now that the people of Königsberg will be happy if they can go around the city, still using all the bridges only once, but they do not mind ending their tour at a point different from their starting point. Is this possible? Let us now examine this question. We will convert this to a problem in graph theory. But, before that, we need a definition that will be helpful in formulating our problem.

Definition: A graph G is **edge traceable** if G contains an open trail that contains all the edges of G.

For instance, the graph in Fig.8. is edge traceable because it contains the open trail $\{v_5, v_1, v_2, v_5, v_4, v_3, v_2, v_4\}$. This contains all the seven edges of the graph and the end vertices are distinct.

In view of the definition of an edge traceable graph, citizens of Königsberg will have to check whether the graph in Fig.3(b) is edge traceable. As an immediate consequence of Theorem 1 , we get the following characterisation of edge traceable graphs.

Theorem 2: A connected graph G with two or more vertices is edge traceable **if and only if** it has exactly two vertices of odd degree.

Proof: Suppose G is an edge traceable graph. Then, there is an open trail T containing all the edges of G. Suppose x and y are the first and the last vertices of T. We now add a new vertex a and join this to x and y. Let us call the new graph we obtain G'. This is illustrated for a particular case in Fig.9 below:

Fig.8

G **G′**

Fig.9

In the graph G' we get an Eulerian circuit as follows:
We start at a, trace the edge ax, trace the open trail T, and trace the edge ya, So, by Theorem 1 all the edges of G' have even degree. Except for x and y, the degrees of all the vertices are unaffected by the addition of the edges ax and ay. So, all of them must have even degree, considered as vertices in G. In the case of vertices x and y, their degrees have become even after the edges ax and ay are added, i.e., after their degrees are increased by one. So, before the addition of the edges, their degrees must have been odd.

Conversely, suppose that exactly two vertices x and y have odd degree. Then, by adding a new vertex a and two new edges ax and ay, the degrees of all the vertices become even. So, we can find an Eulerian circuit starting at a. Let this Eulerian circuit be $\{v_0 = a, v_1, \ldots.., v_n = a\}$. Since x and y are the only vertices to which a is adjacent, either $v_1 = x$ or $v_{n-1} = x$. If $v_1 = x$, we must have $v_{n-1} = y$ and $\{v_1 = x, v_2, \ldots., v_{n-1} = y\}$ is an open Eulerian trail. Similarly, if $v_1 = y$, we must have $v_{n-1} = x$, and $\{v_1 = y, v_2, \ldots\ldots, v_{n-1} = x\}$ is an open Eulerian trail. Hence, the theorem is proved.

Let us now look at the question that motivated us to prove the theorem above.
Example 2: Check whether it is possible for the Königsbergians to go around the city, still using each bridge only once, but ending the trip at a point different from the starting point (see Fig.3 (b)).

Solution: Referring to Fig.3(b), as we observed before, all the vertices except E and F have odd degree, i.e., there are four vertices of odd degree. So, it is not possible for Königsbergians to tour the city using each bridge only once, even if they are allowed to start and end the tour at two different points.

* * *

Here are some related exercises for you to try.

E8) Consider the situation after the addition of a new bridge in 1875 (see Fig.6). Is it possible to tour the city using each bridge only once, if starting and ending the tour at two different points is permitted?

E9) By writing down the degree sequence, find out which of the following graphs are edge traceable.

(a) **(b)**

Fig. 10

We considered one more problem that we mentioned in the introduction to this unit. This asks for a method for determining whether a given figure can be drawn without lifting the pencil from the paper and without going over any of the lines twice. There is such a method, which we shall now illustrate.

Example 3: Check whether the graph in Fig.11(a) can be drawn without lifting the pencil from the paper and without going over any of the lines twice.

Fig.11(a)

Solution: The method involves 4 steps.

Step 1: **(Add vertices at the junctions where two or more lines meet, and at the ends of line segments.)** In Fig.11 (a) there are three such junctions A, B and C. So, add vertices at A, B and C to get the multigraph with a loop in Fig.11 (b). Note that the curve joining A and B in Fig.11 (a) is replaced by a straight edge in Fig.11 (b). Similarly, the curve joining A and C is represented by the edge AC.

Fig.11(b)

Step 2: **(If there are no loops, go to Step 3. If there are loops, eliminate the loops by adding two vertices of degree two.)** If we add two vertices D and E of degree 2 to the earlier loop at A, we get the figure in Fig.11(c).

Step 3: **(If there are no multiple edges go to Step 4. Otherwise, eliminate the multiple edges by adding vertices of degree 2.)** In Fig.11(c), B and C are connected by two edges. We eliminate one of the multiple edges by adding a vertex F to it.

Fig.11(c)

Step 4: **(Count the number of edges of odd degree in the resulting graph. If there are two vertices of odd degree, the graph is edge traceable. If there is no vertex of odd degree, the graph is Eulerian . So, the graph can be drawn without lifting pen from paper. Therefore, the figure we started with can be traced without lifting pen from paper.)** As you can see from Fig.11(d) there are exactly two edges, B and C, of odd degree. So, the figure can be traced without lifting the pencil from the paper.

Fig.11(d)

* * *

If you go through the example above carefully, you may realize that there is a much easier method for deciding whether a figure can be drawn without lifting the pencil form the paper and without going over any of the lines twice. In analogy with graphs, let us call the number of lines that meet in a junction, the degree of the junction for convenience. Note that, only those junctions where more than two lines meet can give rise to vertices of odd degree. All the other vertices that we added are of even degree. In view of this observation, we have the following result.

Theorem 3: A figure can be drawn without lifting the pencil from the paper and without going over any of the lines twice **if and only if** the number of junctions whose degree is odd, and at least 3, is either 2 or 0.

Here is an opportunity for you to apply the method described above.

E10) Which of the following figures can be drawn without lifting pen from paper and without covering any line segment more than once? (Note that the end points of the vertical line in G_1 are vertices of degree 1.)

G_1 **G_2** **G_3**

Fig.12

E11) Construct, if possible, Eulerian graphs with the following number of vertices and edges. When it is not possible, explain why you think so.

	a	b	c
Number of vertices	5	6	7
Number of edges	10	10	6

So far, we have seen that if all the vertices of a graph have even degree, it is Eulerian. However, there are situations where we know that a graph is Eulerian, but we still may not be able to find an Eulerian circuit in it. There is an algorithm due to Fleury that gives a method of finding an Eulerian circuit in an Eulerian graph. You will study the algorithm in MCS-031.

In this section we were interested in finding circuits in which all the edges of the graph occur exactly once. In the next section we are interested in finding cycles in which all the vertices occur exactly once.

3.3 HAMILTONIAN GRAPHS

Suppose a transport company operates bus services between 10 different places. There are places with no direct bus service between them, but there is always a route between any two places that go through the other places. In this situation, the company wants to offer a round trip that passes through each of the cities exactly once. Is this possible?

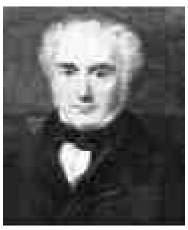

Fig. 13: Hamilton
(1805-1865)

Let us formulate this question as a problem in graph theory. Let us represent the places by vertices. Two vertices are adjacent if a direct bus connects the corresponding places. Since it is possible to go from each place to another, the graph we get is a connected graph. So, the transport company's problem is :
Is there a cycle in the graph in which each vertex occurs precisely once?

A similar question was a basis of the mathematical game described by William Rowan Hamilton. He called this game 'Traveller's Dodecahedron', and also 'A Voyage Round the World'. In this two-player game, we take a regular dodecahedron, each of its 20 vertices representing a city of the world. One player inserts a pin in a vertex.

The other player is supposed to find a 'world tour' starting from this vertex, touching the remaining 19 cities once and returning to the starting vertex. This amounts to finding a cycle covering all the vertices of the regular dodecahedron. Fig.14 gives such a cycle.

Fig.14

Such a cycle is, aptly, named after Hamilton.

Definition : A cycle C in a graph G is called a **Hamiltonian cycle** if it contains all the vertices of G. A graph is called **Hamiltonian** if it contains a Hamiltonian cycle. A graph is called **non-Hamiltonian** if it is not Hamiltonian.

Can you think of examples of Hamiltonian graphs other than the one given in Fig.14? For instance, is any cycle a Hamiltonian graph? Is any graph obtained by adding edges to a Hamiltonian graph also Hamiltonian? The answer to both these questions is 'yes'. For example, the graphs in Fig.15 are Hamiltonian.

Fig.15

Are there any non-Hamiltonian graphs? Trees are obvious examples of non-Hamiltonian graphs. Since they don't have any cycles, they cannot have a cycle containing all the vertices!

Note that, by definition, a Hamiltonian graph contains a cycle containing all the vertices. So, a Hamiltonian graph cannot have cut vertices or pendant vertices. (Recall that **a pendant vertex** is a vertex of degree 1.) This gives a simple method for constructing examples of non-Hamiltonian graphs. For example, the graph in Fig.16 is non-Hamiltonian because it has a cut vertex, namely, the vertex common to both the triangles.

Fig.16

Here are some exercises to help you test your understanding of the discussion above.

E12) Construct a non-Hamiltonian graph on 5 vertices.

E13) i) Is a Hamiltonian graph Eulerian ?

ii) Is an Eulerian graph Hamiltonian ?
Give reasons for your answers.

E14) Check whether the hypercube Q_3 is Hamiltonian.

We have used the existence of a cut vertex to prove that the graph in Fig.16 is not Hamiltonian. However, this does not give us a foolproof method of identifying non-Hamiltonian graphs. For example, $K_{m,n}$, m, n $\geq$ 2, has no cut vertices or pendant vertices, and it is not Hamiltonian when m + n is odd, as we shall now show.

Example 4: Show that $K_{m,n}$ is not Hamiltonian when m + n is odd.

Solution: Since $K_{m,n}$ is bipartite, it does not have cycles of odd length. On the other hand, it has an odd number of vertices. So, a Hamiltonian cycle in this graph, if it exists, must be of odd length. Therefore, $K_{m,n}$ is not Hamiltonian when m + n is odd.

* * *

From the previous example it is clear that to identify non-Hamiltonian graphs we need some conditions which do not depend on the existence of a cut vertex or pendant vertex. The following theorem gives necessary condition for a graph to be Hamiltonian. We will omit the proof of this theorem in this course.

Recall that c(G) denotes the number of components of G.

Theorem 4: If G is a Hamiltonian graph, then for every proper subset S of V(G), we must have $c(G - S) \leq |S|$.

Let us now look at an example to illustrate the use of Theorem 4.

Example 5: Show that $K_{m,n}$ is not Hamiltonian if m < n.

Solution: Recall that the vertex set of $K_{m,n}$ can be partitioned into two disjoint subsets X and Y of cardinality m and n, respectively, in such a way that no two edges in the same subset are adjacent and every vertex in X is adjacent to every vertex in Y. Let us take X to be the set S in Theorem 4. So, $|S| = m$ in this case. If we delete all the vertices in X, the graph becomes totally disconnected. So, there are n components in G – S, one corresponding to each vertex of Y. So, $c(G–S) = n > m = |S|$ in this case. Therefore, by Theorem 4, $K_{m,n}$ is non-Hamiltonian.

***

Remark : If the condition given in Theorem 4 is not satisfied, the graph is non-Hamiltonian. However, if the condition is satisfied, it does not mean that the graph is Hamiltonian. For example, consider the Petersen graph in Fig. 17. You can check that for each subset S of V(G), $c(G - S) \leq |S|$. But G is non-Hamiltonian (which is not easy to check!).

Fig.17

Now for some exercises to check your understanding of Theorem 4.

E15) Show that the following graph is non-Hamiltonian.
(**Hint:** Find a set $S \subset V(G)$ such that $c(G - S) > |S|$.)

Fig.18

E16) Check whether the following graphs are Hamiltonian.

Fig.19

So far, we have seen some necessary conditions for a graph to be Hamiltonian. They are helpful if we want to show that a given graph is non-Hamiltonian. They are of no use if we want to show that a given graph is Hamiltonian. We need some sufficient conditions for this purpose. Since we are looking for a cycle covering all the vertices, it is reasonable to expect success whenever, at every vertex, there are enough choices of edges. This is confirmed by the following theorems. Theorem 5 was proved by Gabriel Dirac in 1952. This was generalized to Theorem 6 by Oystein Ore in 1960.

Theorem 5 (Dirac's criterion) : If G is a simple graph on p vertices, $p \geq 3$, and if $\delta(G) \geq \frac{p}{2}$, then G is Hamiltonian.

$\delta(G) = \min \{\deg_G (x) \mid x \in V(G)\}$

Theorem 6 (Ore's criterion) : Let G be a simple graph on p vertices, $p \geq 3$, satisfying the condition that $d(u) + d(v) \geq p$ for any two non-adjacent vertices u and v in G. Then G is Hamiltonian.

Can you see that Dirac's theorem follows from Ore's theorem? This is because if $\delta(G) \geq \frac{p}{2}$, then for **any** two vertices u and v, we have
$d(u) + d(v) \geq 2\delta(G) \geq p$.

So, the conditions of Ore's theorem are satisfied whenever the conditions of Dirac's Theorem are satisfied. So, if we prove Ore's criterion, we will have also proved Dirac's criterion.

Proof of Theorem 6 : We shall prove this result **by contradiction**. Suppose the theorem is false. Then, there are non-Hamiltonian graphs with p vertices satisfying Ore's criterion. So, the following set is non-empty:

$F = \{G \mid |V(G)| = p$, G is non-Hamiltonian and satisfies Ore's condition$\}$
Choose a graph in F with the maximum number of edges among all such graphs. (Such a graph must exist, because there are only finitely many graphs in F.) Let us denote this graph by G_M.

As G_M is non-Hamiltonian, it cannot be complete. So, there are two vertices, call them u and v, which are not adjacent. So, adding the edge e = uv to G_M, we get a new graph G'_M. The number of vertices in G'_M is still p because we haven't removed any vertex. Since we haven't removed any edge, the degrees of each of the vertices has not decreased. So, Ore's condition holds for any two vertices in G'_M also. But then, G'_M must be Hamiltonian. If it is not, it will be in F. This is not possible because $|E(G'_M)| = |E(G_M)| + 1$, and G_M was chosen to be a graph in F with the maximum possible edges.

Now, since G'_M is Hamiltonian, we can choose a Hamiltonian cycle C in G'_M. Since G is non-Hamiltonian, the edge uv must lie on C. (Why?) Removing this edge, we get a path in G containing all the vertices. Let
$P = \{u = u_1, u_2, \ldots, u_p = v\}$ be this path. Define
$S = \{u_j : uu_{j+1} \in E(G_M)\}$, $T = \{u_j : u_jv \in E(G_M)\}$.

Clearly, $u_p = v \notin S \cup T$. (Why?) Hence, $|S \cup T| < p$. Now, if possible, suppose $S \cap T \neq \phi$. Then, let $u_r \in S \cap T$. $\{u_1, \ldots, u_r, u_p, u_{p-1}, \ldots, u_{r+1}, u_1\}$ is a Hamiltonian cycle in the graph G (see Fig.20). This contradicts the assumption that G is non-Hamiltonian.

Fig. 20

Hence, $S \cap T = \phi$, that is, $|S \cap T| = 0$.
But then, $p \le d_{G_M}(u) + d_{G_M}(v) = |S| + |T| = |S \cup T| < p$, i.e., $p < p$.
This is a contradiction. Thus, our assumption that the theorem is false, is wrong. In other words, every graph G on $p \ge 3$ vertices, satisfying Ore's condition, is Hamiltonian.

Remark: Note that Theorem 5 and Theorem 6 are sufficient conditions. They are not at all necessary. For example, C_n, $n > 4$, is always Hamiltonian, but C_n is a 2-regular graph, and therefore, $d(u) + d(v) = 4 < n$ always.

Here is an example to illustrate the use of the theorems.

Example 6 : To which of the graphs in Fig.21 does Dirac's criterion apply? To which does Ore's criterion apply?

Fig.21

Solution: For the graph in Fig.21(a), p = 6 and d(v) = 3 for each vertex v. So, $\delta(G) = 3$. Thus, Dirac's criterion is satisfied for this graph.

For the graph in Fig.21(b), p = 5, but d(x) = 2. So, Dirac's criterion is not satisfied by this graph. However, $d(u) + d(v) \geq 5$ for all pairs of non-adjacent vertices u and v (in fact, for all pairs u and v). So, Ore's criterion applies in this case.

* * *

Try the following exercise now to test your understanding of the example above.

E17) To which of the following graphs does Ore's criterion apply? To which of these does Dirac's criterion apply?

(a) **(b)** **(c)** **(d)**

Fig.22

So far, we have seen a few necessary conditions and some sufficient conditions for a graph to be Hamiltonian. Are there any conditions that are both necessary and sufficient for a graph to be Hamiltonian? So far no such conditions have been found.

Now, we have come to the end of our discussion on the problem stated in the beginning of the section. In the next section we consider a related, but slightly different problem where we assume that any two places are directly connected by a bus route. We are interested in finding a way of going around all the places, visiting each place only once, and doing so in the shortest possible time.

3.4 TRAVELLING SALESPERSON PROBLEM

A travelling salesperson wants to visit a number of towns and return to her base. The travelling time between any two towns is known. How should she plan her journey so that she spends as short a time as possible but visits each town precisely once? This is known as the **travelling salesperson problem.** Here, one assumes that a direct route connects any two towns without passing through any of the other towns on the list. If we try to represent the towns by vertices and the direct route by edges, then we simply get a complete graph. How should we represent the time required to go from one town to the other? This question leads to the concept of a weighted graph.

Definition: A **weighted graph** is a pair (G, f), where G is a graph and f is a real-valued function on the set E(G).

In simple language, we associate some real number f(e) with each edge e of the graph G. In the case of the travelling salesperson problem, f(e) is simply the time required to travel from one end vertex of e to the other end vertex.

Related to this we have another definition.

Definition: Let W be a walk in a weighted graph G. By the **weight of the walk W,** we mean the sum of the weights of all the edges in W.

So, our traveller's problem reduces to finding a Hamiltonian cycle of minimum weight in a weighted complete graph. One possible approach is to find a Hamiltonian cycle first and then search for edges having smaller weight and modify the cycle using them. The modifications can be made as below:
Let $C = \{v_1, \ldots, v_p, v_1\}$ be a Hamiltonian cycle in a weighted **complete graph**. For a fixed i, first check whether there is a j such that

$$f(v_iv_j) + f(v_{i+1}v_{j+1}) < f(v_iv_{i+1}) + f(v_jv_{j+1}).$$

If this inequality holds, then replace the cycle C by

$$C_{i,j} = \{v_1, \ldots, v_i, v_j, v_{j-1}, \ldots, v_{i+1}, v_{j+1}, v_{j+2}, \ldots, v_p, v_1\}.$$

In Fig.23, we have shown this diagrammatically.

(a) **(b)**

Fig.23

The algorithm for reducing the weight of the cycle is called the **1-exchange heuristic**, and was proposed by Lin and Kernighan**.**

Clearly, the weight of the cycle $C_{i,j}$ is strictly less than that of the cycle C. After performing a sequence of such modifications, one is left with a cycle whose weight cannot be reduced further by this process. Of course, **there is no guarantee that the resulting cycle will have the least possible weight**. There may be other cycles with lower weight. But it will often be fairly good. In fact, finding the minimum weight cycle is an NP-hard problem (ref. the course MCS-031.)

Let us consider an example of how the 1-exchange heuristic is applied.

Example 7: Consider the copy of a weighted K_6 given in Fig.24. Starting with the cycle {L, M, N, O, P, T, L}, modify it to a cycle of lesser weight. The number on the edge e indicates the weight f(e) assigned to it.

Solution: You can check that f(LO) + f(MP) = 80 < f(LM) + f(OP) = 107.
So, we modify the cycle to {L, O, N, M, P, T, L}(see Fig.25(a)).
Now, f(MT) + f(PL) = 121 < f(MP) + f(TL) = 138 (see Fig.25(b)).
So, again we modify the cycle to {L, O, N, M, T, P, L}.
Again, f(OP) + f(NL) = 86 < f(ON) + f(PL) = 87 (see Fig.25(c)).
Hence, modify the cycle to {L, O, P, T, M, N, L} (see Fig.25 (d)).
You can check that we can't decrease the weight of the cycle in the graph further.

Fig.24

(a) **(b)** **(c)** **(d)**

Fig.25

Hence, by this method we have reduced a cycle of weight 237 to a cycle of weight 192.

* * *

Here is a related exercise for you to try !

E18) Start with the cycle $\{v_1, v_2, v_3, v_4, v_5, v_1\}$ in the following weighted copy of K_5. Carry out the reduction step once to get a cycle of lesser weight.

Fig.26

We have now reached the end of our unit. Let us briefly summarise what we have studied in this unit.

3.5 SUMMARY

In this unit we defined the following terms:

i) Eulerian circuit: A circuit in a graph is called Eulerian if each edge of the graph occurs exactly once in the circuit.

ii) Eulerian graph: A connected graph is Eulerian if it contains an Eulerian circuit.

iii) Open trail: A trail is open if the initial and end vertices of the trail are distinct.

iv) Edge traceable graphs: A connected graph is edge traceable if it has an open trail.

v) Hamiltonian cycle: A cycle is Hamiltonian if each vertex of the graph occurs exactly once in the cycle.

vi) Hamiltonian graphs: A graph is called Hamiltonian if it contains a Hamiltonian cycle.

We also discussed the following points in the unit.

1) The proof and application of the statement : A connected graph is Eulerian **iff** the degree of each of its vertices is even.

2) The proof and use of the statement : A connected graph with two or more vertices is edge traceable **iff** it has exactly two vertices of odd degree.

3) The application of an algorithm for checking if a figure can be drawn without lifting pen from paper and without going over any of the lines twice.

4) The application of the fact that if G is Hamiltonian, then $c(G-S) \leq |S| \; \forall \, S \subseteq V(G)$.
We also gave an example to show that this condition is not sufficient.

5) The application of the Dirac and Ore criteria for a graph to be Hamiltonian. According to these criteria, for a simple graph G on p vertices, $p \geq 3$,
   i) if $\delta(G) \geq \frac{p}{2}$, G is Hamiltonian (Dirac)
   ii) if $d(u)+d(v) \geq p$ for any two non-adjacent vertices u and v of G, then G is Hamiltonian (Ore).

6) The travelling salesperson problem, namely, applying the 1-exchange heuristic to obtain a Hamiltonian cycle of smaller weight than that of a given cycle in a complete weighted graph.

3.6 SOLUTIONS/ANSWERS

E1) Here is an Eulerian circuit:
$\{v_1, v_2, v_3, v_4, v_5, v_6, v_1, v_7, v_8, v_4, v_{10}, v_9, v_1\}$
Of course, there are many different Eulerian circuits in the graph, and you may have come up with a different one.

E2) The graph G is the pair (V(G), E(G)), and the circuit is a finite sequence consisting of elements of V and E alternately such that every element of E exists once in this sequence.

E3) The situation will be as in Fig.27. After the addition of the new edge, both the vertices A and D have become even degree vertices. However, B and C still have odd degree. So, it is still not possible for the Königsbergians to go around the city using each bridge exactly once.

Fig.27

E4) The degree sequence of G_1 is {8, 4, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2}. All the vertices are even, and hence the graph is Eulerian. You can check that the following gives an Eulerian circuit in it.
$\{x_1, x_2, x_3, x_4, x_1, x_5, x_6, x_3, x_7, x_8, x_1, x_9, x_{10}, x_{11}, x_1, x_{12}, x_{13}, x_{14}, x_{15}, x_1\}$.
The degree sequence of G_2 is {8, 4, 4, 4, 4, 4, 2, 2, 2, 2}. Since all the degrees are even, it is Eulerian. An Eulerian circuit in G_2 is
$\{x_1, x_2, x_3, x_4, x_5, x_1, x_3, x_5, x_2, x_4, x_1, x_6, x_7, x_8, x_1, x_9, x_7, x_{10}, x_1\}$.

E5) a) K_n is an $(n-1)$-regular graph. So, it is Eulerian when $n-1$ is even, i.e., n is odd.

b) $K_{n,m}$ has n vertices of degree $m-1$ and m vertices of degree $n-1$. So, it is Eulerian when n, m are odd.

E6) In Q_3, every vertex has degree 3, and hence it is a non-Eulerian graph. On the other hand, all the vertices of Q_4, have degree 4. Hence, Q_4 is Eulerian.

E7) Suppose G is an Eulerian graph and $\{v_0, v_1, \ldots, v_n = v_0\}$ is an Eulerian circuit in it. Let $x = v_i$ be any vertex in G. Then, the following is an Eulerian trail starting and ending at x:
$\{x = v_i, v_{i+1}, \ldots, v_n = v_0, v_1, \ldots, v_{i-1}\}$

E8) Refer to Fig.27. After the construction of the new bridge all the vertices except B and C are even, i.e., there are two vertices of odd degree. So, it is possible to go round the city using each bridge only once, starting and ending the trip at two different points.

E9) a) Let us write down the degree sequence of the graph. It is {4, 4, 4, 3, 3, 3, 3, 3, 3, 3, 3}. It has eight vertices of odd degree. So, the graph in Fig.10(a) is not edge traceable.

b) The degree sequence of the graph in Fig.10(b) is {4, 3, 3, 2, 2, 2}. So, it has exactly two vertices of odd degree. So, the graph is edge traceable.

E10) Since G_1 has exactly two vertices of odd degree, it can be drawn without lifting pen from paper and without going over any of the vertices twice.
Since G_2 has 6 vertices of odd degree (degree 3), it cannot be traced without lifting pen from paper.
Since G_3 has precisely two vertices of odd degree, this can also be traced without lifting pen from paper.

E11) The solutions for (a) and (b) are given below.

(a) **(b)**

Fig.28

c) Recall that any Eulerian graph is connected. Here, the number of vertices is one more than the number of edges. So, such a graph is a tree, and therefore, does not contain any cycle. Thus, there is no Eulerian graph with the given number of vertices and edges.

E12) For example, consider the graph in Fig.29. This is non-Hamiltonian because the vertex x is a cut vertex.

Fig.29

E13) i) See Fig.30. This has a Hamiltonian cycle $\{v_1, v_2, v_3, v_4, v_5, v_6, v_1\}$. But, it is not Eulerian because the vertices v_2 and v_5 have odd degrees.

ii) The graph given in Fig.29 is Eulerian because all its vertices have even degree. As, we have seen already, it is not Hamiltonian.

E14) A Hamiltonian cycle in Q_3 is
{000, 100, 110, 010, 011, 111, 101, 001, 000}.

E15) If you remove the vertices marked x, y and z in Fig.31, you will get four connected components, namely, one inner triangle and three isolated outer vertices.

Fig.30

Fig.31

Hence, by Theorem 4, the given graph is non-Hamiltonian.

E16) A Hamiltonian cycle in the graph G_1 is $\{x_7, x_3, x_4, x_2, x_6, x_5, x_1, x_7\}$.
The following cycle in the graph G_2 is Hamiltonian :
$\{x_{12}, x_{14}, x_8, x_9, x_{10}, x_{11}, x_{13}, x_7, x_6, x_5, x_4, x_3, x_2, x_1, x_{12}\}$.

E17) a) This is a 4-regular graph. So, $\delta(G) = 4$. Here $p = 6$, and therefore, the condition $\delta(G) \geq \frac{p}{2}$ is satisfied. So, Dirac's criterion (and therefore, Ore's criterion) applies here.

b) Here $p = 7$. The vertices v_6 and v_7 have degree $3 < \frac{7}{2}$. Therefore, Dirac's criterion does not apply. However, the only pairs of non-adjacent vertices in this graph are (v_6, v_4), (v_6, v_5), (v_6, v_3), (v_7, v_4), (v_7, v_5), (v_7, v_1).
Ore's condition is satisfied for these pairs of vertices. So, this graph is Hamiltonian.

c) Here $p = 8$ and the graph is 4-regular. So, Dirac's criterion is satisfied.

d) Here $p = 8$, but the vertices v_8 and v_4 have degree 3 which is less than $\frac{p}{2} = 4$. So, Dirac's criterion is not satisfied. The only pairs of non adjacent vertices are (v_7, v_3), (v_7, v_4), (v_7, v_5), (v_7, v_6), (v_8, v_2), (v_8, v_3), (v_8, v_4), (v_8, v_5). You can check that Ore's criterion is satisfied for these pairs of vertices.

E18) Notice that
$f(v_1 v_2) + f(v_4 v_5) = 51 + 78 = 129$
$f(v_1 v_4) + f(v_2 v_5) = 5 + 36 = 41$

We can modify the given cycle to get the following cycle of smaller weight:
$\{v_1, v_4, v_3, v_2, v_5, v_1\}$.

Fig.32

UNIT 4 GRAPH COLOURINGS

Structure **Page Nos.**

4.0 INTRODUCTION

You must have seen political maps of India with different states coloured differently to distinguish between them. Have you ever wondered what the minimum number of colours required to colour the map is for any two states with a common boundary to have two different colours? This problem of finding the minimum number of colours needed to colour a given map is called the map colouring problem.

We can formulate the problem in terms of graph theory. We can construct a graph in such a way that each state of India corresponds to a vertex of India and two states are adjacent if and only if the corresponding vertices are adjacent. So, we have to colour the vertices of the graph in such a way that any pair of adjacent vertices have different colours. In the map colouring problem, we ask for the minimum number of colours needed to carry out such a colouring.

Note that the construction mentioned above leads to a special class of graphs called planar graphs. If we are interested in the map colouring problem alone, it is enough to restrict ourselves to such graphs. However, the general vertex colouring problem, which asks for the minimum number of colours needed to colour the vertices of a given graph, not necessarily planar, is interesting in itself. So, we start our unit by discussing this problem in Sec. 4.2.

Analogous to the colouring of vertices, is the colouring of edges. In Sec. 4.3, we have a brief discussion on edge colourings. This includes the definition of edge colouring, some examples of edge colouring and statements of some of the well known results in this field.

In Sec. 4.4, as a preparation for our study of the map colouring problem, we study planar graphs. In this section, we will prove some basic results about planar graphs. We will also prove a characterization of planar graphs due to Kuratowski.

In Sec. 4.5, we study the map colouring problem. We give a brief history of the four colour theorem, which says that any map can be coloured with four colours. However, the proof of this theorem is beyond the scope of this course.

4.1 OBJECTIVES

After studying this unit, you should be able to

- compute the vertex chromatic number of some simple graphs;
- compute the edge chromatic number for some simple graphs;

- in simple cases verify whether a given graph is planar or not using Kuratowski's theorem;
- explain the map colouring problem and relate it to the study of planar graphs.

4.2 VERTEX COLOURING

In this section, we start our study of colourings by considering the graph in Fig. 1. We have given a colouring of K_3 using three colours, namely, red (r), green (g) and blue (b).

Fig.1

Why have we used three colours? It is because we want the adjacent vertices to have different colours. In K_3, any two vertices are adjacent. So we need to colour each of the vertices with different colours. Keep this example in mind when you read the definition given below.

'χ' is the Greek letter 'chi'.

Definition: A **vertex colouring** of a graph G is an assignment of colours to vertices of G in such a way that no two adjacent vertices have the same colour. A graph is called **k-vertex colourable** if it has a vertex colouring of k colours. The **minimum** number of colours required to colour a graph G is called the **vertex chromatic number of G,** usually denoted as χ**(G)**.

We will say that a graph is **k-chromatic** (or **k-colourable**) if it has chromatic number k.

In Fig. 1, we were able to use the names of the colours, red, green and blue, because we needed only three colours. Suppose we need, say, 20 colours, can we still use the names to refer to the colours? We may not remember the names of so many colours, and could call them Colour 1, Colour 2, etc. This will do just as well, because the names of the colours are not important as long as you can distinguish between the different colours. In the diagrams, we will denote the colours as $\boxed{1}$, $\boxed{2}$,….

Let us now look at some examples.

Example 1: Colour the graphs in Fig. 2 with the minimum possible number of colours. Also, find the chromatic numbers of the graphs.

(a) **(b)** **(c)** **(d)**

Fig. 2 : Some examples of colouring

Solution: In Fig. 2(a), K_1 has just one vertex. Let us colour this with $\boxed{1}$. So, one colour is clearly the minimum required for colouring this graph. Hence, its chromatic number is 1.

In Fig. 2(b), K_2 has two adjacent vertices. We assign $\boxed{1}$ to the vertex v_1 and $\boxed{2}$ to the vertex v_2. Thus, we have a 2-colouring. Is there a 1-colouring? No! The two vertices are adjacent and so we need at least two colours. In other words, the chromatic number $\chi(K_2) = 2$.

In Fig. 2(c), we have three vertices and we can colour them with three different colours. But, can we also have a two-colouring? Notice that, v_1 and v_3 are not adjacent. So, we can colour them with the same colour, say, $\boxed{1}$. v_2 is adjacent to both v_1 and v_3. So, we cannot assign $\boxed{1}$ to this. Let us assign $\boxed{2}$ to v_2. So, we have a 2-colouring. As we cannot have a 1-colouring, this graph has chromatic number 2.

In Fig. 2(d), we have K_5. In this any two vertices are adjacent, so we need as many colours as there are vertices, that is, we need five colours. So K_5 has chromatic number 5.

$\chi(K_n) = n \forall n \geq 1$ because any pair of vertices are adjacent in K_n.

***

Remark: In the example above, we saw that the chromatic number of K_1 is 1. More generally, **a graph consists of isolated vertices, if and only if its chromatic number is 1.**

Example 2: Find the chromatic number of a bipartite graph with a non-empty edge set.

Solution: From Unit 2, you may recall that a graph G is bipartite if the vertex set of G can be partitioned into two **non-empty** disjoint subsets A and B
Such that any two vertices in a given set are non-adjacent. We get a 2-colouring of G by assigning $\boxed{1}$ to all the vertices in A and $\boxed{2}$ to all the vertices in B. (This is illustrated in a particular case in Fig. 3.) Further, note that, since A and B are non-empty and since the edge set of G is non-empty, at least one vertex in A is adjacent to a vertex in B and these two vertices must have different colours. So, we cannot manage with less than two colours. So, $\chi(G) = 2$ if G is a bipartite graph with a non-empty edge set.

Fig.3

***

Remark: We saw, in Example 2, that the chromatic number of a bipartite graph with non-empty edge set is 2. The converse is also true. Given a graph G and a 2-colouring of G, we can partition the edge set of G into two non-empty sets A and B defined as follows:

$A = \{ v \in V(G) \mid v \text{ is assigned the colour } \boxed{1} \}$

$B = \{ v \in V(G) \mid v \text{ is assigned the colour } \boxed{2} \}$

By the definition of colouring, no two vertices in A are adjacent, and similarly for B. Since A and B are disjoint, G is bipartite, by definition.

Here are some exercises to test your understanding of the examples above.

E1) What is the chromatic number of
- i) a tree with at least two vertices?
- ii) an even cycle C_{2n}, $n \geq 2$?
- iii) an odd cycle C_{2n+1}, $n \geq 1$?

Now, if a graph is k-colourable, are all its subgraphs k-colourable? Let us see. Let G be a k-colourable graph and H be its subgraph. We assign to each vertex of H the same colour that we assigned to it, considered as a vertex of G. If two vertices are non-adjacent in G, they are non-adjacent in H, and therefore this gives a colouring of H. In other words, $\chi(H) \leq k = \chi(G)$ for every subgraph H of G. We can also recast this statement in the following form. **If a graph G has a subgraph H with chromatic number k, the chromatic number of G must be at least k.** This fact helps us in finding the chromatic number of a graph sometimes. We illustrate this in the next example.

Example 3: Find the chromatic number of the Grötzsch graph (see Fig 4(a).)

Without colouring **With 4-colouring**
(a) **(b)**

Fig. 4 : The Grötzsch graph

Solution: Fig.4(b) gives a 4-colouring of this graph. Can this graph have a 3-colouring? Let us see. Since the outer 5-cycle is an odd cycle, it needs three colours. So, we need at least three colours. Let us suppose the colours of $x_1, \ldots, x_5$ are as shown in Fig.4(b). Since y_1 is adjacent to x_2 and x_5, we have to give it a colour different from $\boxed{2}$ and $\boxed{3}$. So, we assign $\boxed{1}$ to it. Similarly, the colours of y_4 and y_5 must be $\boxed{2}$ and $\boxed{3}$, respectively. Since the vertex z is adjacent to vertices to which the colours $\boxed{1}$, $\boxed{2}$ and $\boxed{3}$ have been allotted, we have to use a fourth colour for this vertex. So, this graph is not 3-colourable. Therefore, this has chromatic number 4.

***

In the examples and exercises above, we saw that if a graph G has a subgraph H with chromatic number $\chi(H) = n$, then $\chi(G) \geq n$. In particular, if a graph G has a subgraph H which is isomorphic to K_n (such a subgraph H is known as a **clique of size n**), the chromatic number of G is at least n.

However, **the converse is not true**, i.e., if a graph has chromatic number $\geq n$, it need not have a clique of size n. The Petersen graph provides a counter-example for this. Its chromatic number is 3 (see E2). Convince yourself (you need not prove it) that it does not contain a clique of size 3, i.e., a subgraph isomorphic to K_3.

More generally, in 1955, Mycielski proved that, for any integer k, there exists a k-chromatic graph without triangles. The proof of this result is beyond the scope of this course. However, it is not difficult to prove the much weaker result that if the chromatic number of a connected graph is greater than 2, it contains an odd cycle. We leave this as an exercise for you (see E3), along with some more exercises, to test your understanding of the material we have covered so far.

E2) Find a 3-colouring of the graph in Fig.5, and its chromatic number.

Fig.5

E3) Show that the chromatic number of the Petersen graph, given in Fig. 6, is 3. Also check that it does not contain K_3.

Fig.6 : The Petersen graph

E4) Show that if $\chi(G) \geq 3$ for a graph G, it contains an odd cycle

E5) Find the chromatic number of the following graph.

Fig. 7

E6) Construct a graph with chromatic number 5.

Recall that, we have shown that any 2-colourable graph is bipartite. How was this done? We had put all the vertices having the same colour in a single set. There were two colours and so we got two subsets. They were disjoint because no vertex can be assigned two colours.

The colour classes can be defined for any colouring of a graph G, not just for a χ (G)-colouring.

We are going to extend these ideas to n-colourable graphs. We do this through the concept of colour classes, which we now define.

Definition: For a k-colouring of a graph G, consider the set
$C_i = \{ x \in V(G) \mid x \text{ is assigned the colour } \boxed{i} \}$, for $1 \le i \le k$.
Clearly, $C_i \cap C_j = \phi$, for every $i \neq j$, and $V(G) = C_1 \cup \ldots\ldots \cup C_k$.
In particular, if $\mathcal{X}(G) = n$, each of the n colours is assigned to at least one vertex. (Why?) So none of these subsets is empty. Therefore, we get a partition of the vertex set V(G) into n mutually disjoint non-empty subsets. The subsets $C_1, \ldots, C_n$ are called **the colour classes** of G given by the n-colouring.

For example, the colour classes of a 2-colourable graph give a bipartition of the vertex set of the graph, making it bipartite.

Let us now look at some examples of colour classes.

Example 4: Find the colour classes in the two different colourings of the graph given in Fig.8.

Fig. 8

Solution : The colour classes given by the colouring in Fig. 9(a) are $C_1 = \{x_1\}$, $C_2 = \{x_4, x_6, x_8, x_{10}, x_{15}, x_{16}\}$, $C_3 = \{x_3, x_{12}, x_{13}, x_{14}\}$ and $C_4 = \{x_2, x_5, x_7, x_9, x_{11}\}$. Check that $C_1 = \{x_7, x_9, x_{11}, x_{15}\}$, $C_2 = \{x_1, x_5, x_8, x_{12}, x_{14}\}$, $C_3 = \{x_4, x_{10}, x_{16}\}$, $C_4 = \{x_2, x_6\}$, and $C_5 = \{x_3, x_{13}\}$ are the colour classes corresponding to the colouring in Fig. 9(b).

The colour classes can be defined for any colouring of a graph G, not just for a $\mathcal{X}$(G)-colouring.

(a)

(b)
Fig.9

***

Try some exercises to test your understanding of the example above.

E7) Check whether 'xRy iff x and y lie in the same colour class' defines an equivalence relation on the vertices of a graph G.

E8) Colour the following graph in two different ways, and give the colour classes in each case.

Fig. 10

We have seen that any colouring of a graph gives rise to colour classes.
You know that, if x, y are two vertices in a colour class C_i, then $xy \notin E(G)$. So, **each colour class consists of mutually non-adjacent vertices.** We now give a name to those subsets of the vertex set of a graph with this property.

Definition: A subset S of the vertex set V(G) of a graph G, is said to be an **independent set** if any two vertices in S are non-adjacent. An independent set is called **maximal** if it is not contained in any other independent set. The number of vertices in a largest independent set of G, is called the **independence number** of the graph G, and is denoted by **α(G)**.

So, for example, each colour class is an independent set, and, in fact, a maximal independent set. However, all independent sets are not dependent on a particular colouring.

Example 5 : Find three different maximal independent sets in the graph given in Fig. 11.

Fig. 11

Solution : The graph has the following maximal independent sets :
$\{v_8, v_5\}$, $\{v_5, v_1, v_2, v_3, v_7\}$, $\{v_1, v_2, v_3, v_4, v_6, v_7\}$
Let us check this.

Firstly, $\{v_8, v_5\}$ is a maximal independent set because all the other vertices are adjacent to one of these two vertices. So, if any more vertices are added, the resulting set will no longer be an independent set.

In the same way, you can check that the other two sets are also maximal independent sets.

***

Now test your understanding of independent sets by trying the following exercises.

E9) Find an independent set of cardinality 4 in the graph given below:

Fig. 12

E10) Find α(G) for the graphs given in Fig.8 and Fig. 10.

If we can colour the vertices of a graph, can we colour the edges of a graph? Is it interesting, or useful, to do so? In the next chapter, we will answer these questions.

4.3 EDGE COLOURING

In this section, we consider the problem of colouring the edges of a graph in such a way that no two edges with a common vertex receive the same colour. We will not prove any of the important results in this subject although we will state some of them. The purpose of this section is to give a brief introduction to edge colouring. We begin by defining edge colouring.

Definition: A **k-edge colouring** of a graph G is an assignment of k colours to the edges of G in such a way that no two edges incident with the same vertex have the same colour. A graph is **k-edge colourable** if it has a k-edge colouring. The **minimum** number of colours required to colour the edges of a graph is called the **edge chromatic number of G,** usually denoted by $\chi'(G)$.

Let us now look at some examples of edge colouring. The easiest case is the edge colouring of those graphs which have edge chromatic number 1.

Example 6 : Find all the graphs that have edge chromatic number 1.

Solution: Suppose a graph G has edge chromatic number 1. Since the edge chromatic number is one, the graph is 1-edge colourable, and no two edges share an end vertex, that is, the graph must be the union of some isolated vertices and some components consisting of two vertices and an edge (see Fig.13).
Conversely, a graph which is the union of isolated vertices and components having two vertices each has edge chromatic number 1.

Fig.13

***

Example 7 : Colour the edges of the graphs K_3, K_4, K_5.
Solution : The colouring of K_3, K_4, K_5 is given in Fig. 14. Here no two adjacent edges have received the same colour. In all the cases, we have used the least possible colours.

Fig.14

***

Example 8 : Give an edge colouring of the Petersen graph.

Solution: Fig. 15 gives a 4-edge colouring of the Petersen graph.

Fig. 15

Again no two adjacent edges have received the same colour. You can quickly check that three colours will not be enough. So $\chi'(G) = 4$.

***

Example 9: Give edge colourings of all the trees on 5 vertices.

Solution : In Fig.16, we have presented all types of possibilities, with colourings.

Fig. 16

Again we have used the least possible number of colours.

***

Example 10: Find the edge-chromatic number of C_n.

Solution: As in the case of vertex colouring, if n is even, the edge chromatic number is 2. We can colour the edges alternately with the two colours. If n is odd, the edge chromatic number is 3. We have illustrated this in the case of C_4 and C_5 in Fig.17 .

Fig. 17

***

If G is a graph and $v \in V(G)$ such that $d_G(v) = \Delta(G)$, then all the edges incident on v must receive different colours. Hence, any edge colouring of G will need at least $\Delta(G)$ colours, that is, $\Delta(G) \leq \chi'(G)$.

Regarding an upper bound for $\chi'(G)$, in 1964 Vizing proved the following result.

Theorem 1: For any graph G, $\chi'(G) \leq \Delta(G) + 1$.

So, for any graph G, $\mathbf{\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1}$.

Therefore, there are only two possibilities for the edge chromatic number of a graph G, either $\Delta(G)$ or $\Delta(G) + 1$. We now present some of the results known in this direction, without proof.

1) The **edge chromatic number of K_n** is n, if n is odd ($\neq 1$) and $n - 1$ if n is even. Recall that, K_n is $(n-1)$-regular. So $\Delta(K_n) = n - 1$.

2) For a **bipartite graph** G, $\chi'(G) = \Delta(G)$ (proved by König in 1916).

Here are some related exercises now.

E11) What is the edge chromatic number of $K_{m,n}$?

E12) Consider the tree T given in Fig.18. Give an explicit $\Delta(T)$-edge colouring of T.

Fig.18

In the introduction, we mentioned that the map colouring problem can be reduced to finding the minimum number of colours needed to colour a special class of graphs called planar graphs. In the next section, we define planar graphs and prove some basic results that will be useful in the study of the map colouring problem.

4.4 PLANAR GRAPHS

In transistor radios and television sets, you must have seen printed circuit boards. These boards have slots for various components and these slots are connected to each other. The connection between these slots must be made in such a way that no two connections cross each other. Given an electronic circuit, is it always possible to design a printed circuit board corresponding to it?

This can be formulated as a problem in graph theory. We replace the electronic components by vertices and the connections between them by edges. If the resulting graph can be drawn in such a way that no two of the edges cross each other except at the vertices, then we can design a printed circuit board for the given circuit. Let's start by seeing what such graphs are called.

Definition: A graph G is called **planar** if it can be drawn on a plane in such a way that no two edges cross each other at any point except possibly at a common end vertex. Such a drawing is called a **plane drawing.**

To see some examples of planar graphs, consider Fig.19. In this figure, we have given the five regular solids called platonic solids. In the second row, we have given the corresponding planar graphs. In each of these graphs, the vertices correspond to the vertices of the associated solid and the edges correspond to the edges of the solid.

Fig. 18
Note that x and y are just points in the plane, they are not vertices.

tetrahedron **cube** **octahedron** **dodecahedron** **icosahedron**

tetrahedron **cube** **octahedron** **dodecahedron** **icosahedron**

Fig. 19 : Regular solids and the corresponding planar graphs

Next, we introduce the concept of a region. Look at the tetrahedron in Fig. 19 (a). It has four faces. The planar graph corresponding to it is given in Fig. 19(A). It divides the plane into four faces, or regions, which we have numbered from 1 to 4 in the figure. Similarly, the graph of the cube, given in Fig. 19(B) divides the plane into six regions.

In all the cases above, it is very clear what the different regions are. But, look at the graph in Fig. 20. Into how many regions does it divide the plane? Two or three? Do the points x and y lie in the same region or in different regions? To avoid such confusion we need to define the concept of a region carefully. Here is the definition.

Fig.20 : x and y are just points in the plane, they are not vertices.

Definition: Given a plane drawing of a planar graph G, by a **region** (or **face)** of G, we mean a maximal portion of the plane for which any two points a, b in it can be joined by a curve which lies completely in that portion of the plane.

If R is a region of a planar graph G, by the **boundary of R** we mean all those points x in the plane corresponding to the vertices and edges of G having the property that x can be joined to any point in that region by a simple curve all of whose points, except x, are in that region.

There is always one unbounded region of G, and it is called the **exterior region** of G. Any other region is called an **interior region**.

Let us go back to Fig. 20 again. Armed with this definition, we can answer the question we raised. As you can see in Fig. 21, the points x and y can be joined by a curve that does not cross any of the edges. So, there are only two regions, the region inside the triangle and the region outside it. Both the points lie in the exterior region of the triangle.

Let us now look at an example to understand these concepts better.

Example 11: Find the number of regions in the graphs given in Fig.22.

(a) **(b)**

Fig. 22

Solution: The graph in Fig. 22 (a) has 8 regions, 7 interior and 1 exterior region. In the graph in Fig. 22(b), there are 3 regions.

***

Now, try the following exercises.

E13) Find the number of regions in a tree, a cycle and K_4.

E14) Is a subgraph of a planar graph planar ? Why ?

Now, given a planar (p,q)-graph, is there any relationship between the number of regions, r, and p,q ? Let us calculate the quantity $p - q + r$ for all the planar graphs in Fig. 13 and for the graph in Fig. 16(b), to see if we can find an answer to this.

Graph	**P**	**q**	**r**	**p − q + r**
Fig. 16(b)	6	7	3	2
Tetrahedron	4	6	4	2
Cube	8	12	6	2
Octahedron	6	12	8	2
Dodecahedron	20	30	12	2

Icosahedron	12	24	18	2

As you can see, $p - q + r$ is 2 for all these planar graphs. In fact, the following theorem, proved by Euler in 1736, shows that this is true for all such graphs.

Theorem 2 (Euler's formula) : If G is a connected planar (p,q)-graph, then the number r of the regions of G is given by $r = q - p + 2$.

Proof: We apply induction on q, the number of edges of G, to show that $p - q + r = 2$.
If q = 0, then G just consists of 1 isolated vertex since it is connected. Hence, r = 1 and the formula holds.

Now, assume that the formula holds for any plane drawing of a (p,t)-graph for every $t \le (q - 1)$, and suppose G is a (p, q)-graph.

If G is a tree, then $p = q + 1$ and $r = 1$, so that the formula holds.
If G is not a tree, then it contains a cycle. Let e be an edge of G that lies on a cycle of G, and consider the subgraph $G - e$ of G. When we remove this edge e on the cycle, we are joining two regions to make one region out of them (e.g., if we remove e in Fig.23(a), the earlier regions 1 and 2 join to become one region in Fig.23(b).) So, $G - e$ has p vertices, $(q - 1)$ edges and $(r - 1)$ regions.

(a) **(b)**

Fig. 23

Now, by the induction assumption, Euler's formula holds for $G - e$. So, the number of regions = number of edges – number of vertices + 2.
i.e., $r - 1 = (q - 1) - p + 2$.
$\Rightarrow r = q - p + 2$.

Hence, by induction, the result is true for any connected planar graph.

Remark : Since $r = p - q + 2$, where p and q are fixed once we fix a graph, the number of regions in a plane drawing of a planar graph is independent of the plane drawing.

The result we have just proved does not immediately help us to tell whether a graph is planar or not. However, we derive a necessary condition for planarity from it that is very useful. It tells us intuitively, that a planar graph can't have 'too many' edges. Recall that a graph on p vertices can have up to $\frac{p(p-1)}{2}$ edges. In the case of planar graphs, there is a much better bound. We give this bound in the next theorem.

Theorem 3: If G is a planar (p, q)-graph, with $p \ge 3$, then $q \le 3p - 6$. Further, if G is also bipartite, we have $q \le 2p - 4$.

Proof : For p = 3, the result is clearly true. So, we assume $p \ge 4$. Let G have r regions. For each region R of G, the number of edges lying on its boundary is at least 3. So, if S is the sum of the number of edges of each region, then $S \ge 3r$.
Also, every edge of G is counted once or twice while obtaining S. So $S \le 2q$. Thus, $3r \le 2q$. Using Euler's formula, we obtain $3(q - p + 2) \le 2q$, which gives us $q \le 3p - 6$.

Now, if G is also bipartite, it will not contain any 3-cycle. Therefore, a region will be bounded by at least 4 edges, so that $S \geq 4r$. Then, as argued above, we get $q \leq 2p - 4$.

Let us see how this result helps us to check whether a graph is planar or not.

Example 12: Show that K_5 is not planar.

Solution: Suppose K_5 is planar. Then the number of edges and vertices in K_5 satisfy the relation $q \leq 3p - 6$ given in Theorem 3. K_5 has 5 vertices and 10 edges, so $10 \leq 3 \times 5 - 6$, i.e. $10 \leq 9$, a contradiction. Thus, K_5 is non-planar.

Try the next exercise to check your understanding of Theorem 3.

E15) Show that $K_{3,3}$ is non-planar. Hence, show that given 3 houses, each having 3 outlets for electricity, gas and water, respectively, it is not possible to connect each of these utilities to each of the houses without the lines or mains crossing.

We have already seen that K_5 and $K_{3,3}$ are not planar. To prove this we used either of two necessary conditions. However, these conditions are not sufficient. For example, in the Grötzsch graph (see Fig.4), $p = 11$, $q = 20$ and $20 \leq 33 - 6 = 27$. So, the condition in Theorem 3 is satisfied. But, as we shall show later, the Grötzsch graph is not planar.

So, the question is whether there is a necessary and sufficient condition for a graph to be planar. In 1930, K. Kuratowski, a Polish mathematician, proved a necessary and sufficient condition for a graph to be planar. We will state this theorem and illustrate its application through an example. To understand the statement, let us first consider Fig. 24 below.

Fig. 24 : Subdivision of a graph

In this figure, we have started with K_4 and inserted vertices of degree 2 in some of the existing edges. For example, in Fig. 24(b), we have inserted a vertex a on the edge uv. In effect, this replaces the edge uv with two new edges va and au. We have made similar changes in the graphs in Fig. 24(b), Fig. 24(c), Fig. 24(d) and Fig. 24(e). In this way we have got subdivisions of the graph in Fig. 24(a), as you shall now see.

Definition: A graph G′ is a **subdivision** of a graph G if it can be obtained by adding one or more vertices of degree 2 on the existing edges of G. In other words, we 'subdivide' some of the existing edges.

Note : If a graph is planar, all its subdivisions are planar. If a graph G is non-planar, any subdivision of G is also non-planar. So, if a graph contains a non-planar subgraph or a subgraph which is a subdivision of a non-planar graph, it is non-planar. For example, the graph in Fig. 25 is non-planar since it contains as a subgraph a subdivision of K_5 (shown by dotted lines), which is a non-planar graph.

Fig.25

In proving the non-planarity of the graph in Fig. 25, is it just a coincidence that we found that it had a subdivision of K_5 as a subgraph? Here's an exercise about this.

E16) Check whether the graph given in Fig.26

i) is planar;

ii) has a subdivision of K_5.

Fig.26

From E16 you see that non-planar graphs do not necessarily contain a subdivision of K_5. However, Kuratowski's theorem (stated below) says that a non-planar graph has to contain a subgraph which is a subdivision of K_5 or $K_{3,3}$. So, we need to restrict our search for non-planar subgraphs (or their subdivisions) to only these two graphs.

Theorem 4 (Kuratowski) : A graph G is non-planar if and only if it contains a subdivision of K_5 or $K_{3,3}$ as a subgraph.

We will not present the proof of this statement here. But, we shall look at an example to see how this theorem can be used to prove non-planarity.

Example 13: Show that the Grötzsch graph (see Fig. 4) is non-planar.

Solution: From Kuratowski's theorem we know that we have to look for a subgraph which is a subdivision of K_5 or K_4. But, in this case, which of these two should we look for? Note that a subdivision of a graph does not affect the degree of any of the vertices of a graph; it only introduces new vertices of degree 2.

So, if our graph contains a subdivision of K_5, it will contain at least 5 vertices of degree 4. If it contains a subdivision of $K_{3,3}$ it will have at least six vertices of degree 3.

Let us first check if our graph contains a subdivision of $K_{3,3}$. Since it contains only five vertices of degree 3, namely, y_1,y_2, y_3, y_4 and y_5, it cannot contain a subdivision of $K_{3,3}$.

So, let us check if it contains a subdivision of K_5. K_5 contains 5 vertices of degree 4. In the Grötzsch graph also there are 5 vertices of degree 4, namely x_1, x_2, x_3, x_4 and x_5. Let us remove the middle vertex, labelled as z. We get the graph given in Fig. 27(a).

Fig. 27 : Non-planarity of the Grötzsch graph.

As you can see, it can be obtained from K_5 in Fig. 27(b) by adding degree two vertices to x_1x_3, x_1x_4, x_2x_4, x_2x_5, x_3x_5 and x_1x_5. So, it is non-planar.

***

Now, an exercise for you to try!

E17) Show that the Petersen graph (in Fig.5) is non-planar.
(**Hint** : Consider the graph obtained by removing the two horizontal edges.)

In the next section we will discuss the map colouring problem. We will show that this can be reduced to a colouring of planar maps.

4.5 MAP COLOURING PROBLEM

The four colour problem asks whether any map of a part of the world can be coloured with 4 colours. We begin this section with a brief discussion of the history of the four colour problem. We then show how to construct a planar graph corresponding to a given map in such a way that colouring the graph is equivalent to colouring the map. So, if we can prove that any planar map can be coloured with four colours, we would have proved that any map can be coloured with four colours. In 1976, the American mathematicians, Kenneth Appel and Wolfgang Haken, proved that four colours are enough to colour planar graphs. They used nearly 1200 hours of computer time on some of the fastest computers available at that time to prove this by doing a case-by-case analysis. This gives an idea about the complexity of the proof, which we will not be giving in this course.

Now, for some background about this problem. In 1852, Francis Guthrie communicated the four colour problem to De Morgan through his brother Fredrick Guthrie, who was a student at the University College, London at that time. It appeared in print for the first time when Cayley published a paper on this problem in the Royal Geographical Society in 1879. In this paper, he outlines where the difficulties lie in this problem. In the same year, A.B. Kempe published a proof of the theorem in the American Journal of Mathematics. However, in 1890, P.J. Heawood pointed out a mistake in Kempe's proof. He also showed that the proof can be modified to show that five colours are enough to colour any map. Since then many mathematicians, G.D. Birkhoff, Veblen, Ore, Franklin among others, contributed to the solution of the problem. Appel and Haken finally solved the problem in 1976.

We now show how to construct a planar graph corresponding to a given map in such a way that colouring the vertices of the graph is equivalent to colouring the map.

Consider the map given in Fig. 28(a) below. There are 10 regions in the map, A, B, C, D, E, F, G, H, I and J, including the exterior region. In this map we add a vertex corresponding to each region of the map (see Fig.28(b)). Note that we have added a vertex corresponding to the exterior region, namely, J.

(a) **(b)**

Fig. 28

We join two vertices if the corresponding regions have an edge in common. For example, we have connected a and c because they have a common boundary (see Fig.

29 below). We have not connected the vertices a and e because they do not have a common boundary. We do not connect two vertices if the corresponding regions share only a point and not a boundary. For example, we have not connected c and g by an edge for this reason.

As you can see, we get a planar graph, and colouring this graph is equivalent to colouring the map. (We assume that the exterior region of the map is coloured with a single colour.) So, the four colour problem can be stated as follows:

Fig.29

Is it possible to colour any planar graph with four colours?

The following theorem answers this question.

Theorem 5 (Appel-Haken) : Any planar graph can be coloured with four colours.

As we mentioned in the introduction we will not be proving this theorem.

Now, the question is whether we can get a better result. Can we colour a map with three colours always? No! For example, K_4 is planar, being the graph corresponding to a tetrahedron, and it cannot be coloured with three colours. So, we cannot improve the result in Theorem 5.

Why don't you try an exercise now ?

E18) Show that $\chi(K_5) = 5$. Does this contradict Theorem 5 ? Give reasons for your answer.

E19) Check whether there exists a non-planar graph with vertex chromatic number i) 1, ii) 2, iii) 3.

On doing E18, you will have realised that the result in Theorem 5 would not hold true for non-planar graphs.

We have now reached the end of this unit. Let us briefly summarise what we have learnt so far.

4.6 SUMMARY

In this unit we discussed the following concepts along with several examples and exercises.

i) **Vertex colouring of a graph:** A vertex colouring of a graph is an assignment of colours to its vertices in such a way that no two adjacent vertices receive the same colouring.

ii) **Vertex chromatic number of graph:** The chromatic number of a graph is the minimum number of colours required to colour the graph.

iii) **A colour class of a colouring:** For each colour of a colouring, the set of all vertices that are coloured with that colour is the colour class of that colour.

iv) **Independent set:** A subset of the vertex set is independent if any two vertices in the set are non-adjacent.

v) **Edge colouring of a graph:** An edge colouring of a graph is an assignment of colours to its edges in such a way that no two edges with a common vertex are given the same colour.

vi) **Edge chromatic number of a graph:** The edge chromatic number of a graph is the minimum number of colours needed to colour the edges of graph.

vii) **Planar graph:** A graph is planar if there is a plane drawing in which no two edges cross each other, except at vertices.

viii) **Subdivision of a graph:** A graph G_2 is a subdivision of another graph G_1 if it can be obtained from G_1 by inserting vertices of degree two in the existing edges.

In the process we also studied the following matters.

1) $\chi(K_n) = n$; and $\chi(G) = 2$ **iff** G is a bipartite graph with $E(G) \neq \phi$.

2) If G has a subgraph isomorphic to K_n, then $\chi(G) \geq n$. But the converse is not true. However, if $\chi(G) \geq 3$, then G contains an odd cycle.

3) For a graph G, the colour classes corresponding to each of the $\chi(G)$ colours give maximal independent sets of V(G).

4) Vizing's bound for the edge chromatic number of a graph, namely, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

5) Euler's formula for planar graphs, which states that Number of vertices − Number of edges + Number of regions = 2, for any planar graph.

6) If G is a planar (p, q)-graph, with $p \geq 3$, then $q \leq 3p - 6$. Further, if G is also bipartite, we have $q \leq 2p - 4$.

7) Kuratowski's characterization of planar graphs, which says that a graph is planar if and only if it does not contain a subdivision of $K_{3,3}$ or K_5.

8) The four colour theorem (without proof), which says that any planar graph can be coloured with four colours.

4.7 SOLUTIONS/ANSWERS

E1) i) Trees do not contain cycles as subgraphs, and therefore, in particular, they are bipartitie. Since a tree is connected and we have assumed it has at least two vertices, it has chromatic number 2.

ii) Even cycles do not contain odd cycles as subgraphs. So, they are bipartite. Therefore, they have chromatic number 2.

iii) The chromatic number of an odd cycle is 3. Since it is not bipartite, its chromatic number is at least 3. We get a 3-colouring of C_{2n+1} as follows:Let $\{v_1, v_2,\ldots,v_{2n+1}\}$ be the vertex set of C_{2n+1}.
We assign $\boxed{1}$ to all the vertices in the set $\{v_i \in V(C_{2n+1}) \mid i \text{ odd}, 1 \le i \le 2n\}$, and $\boxed{2}$ to all the vertices in the set $\{v_i \mid i \text{ even}, 2 \le i \le 2n\}$.
Now, v_{2n+1} is adjacent to both v_1 and v_{2n}. So, we cannot assign $\boxed{1}$ or $\boxed{2}$ to this vertex. Therefore, we assign the colour $\boxed{3}$ to v_{2n+1}.

E2) A three-colouring of the Petersen graph is given below.

Fig. 30

Further, the Petersen graph contains a 5-cycle which has chromatic number three. So, the Petersen graph has chromatic number three.

E3) Since it has chromatic number greater than 2, it cannot be bipartite. So, it must contain an odd cycle.

E4) A 3-colouring of the graph is given in Fig.31. Also, it has cycles of length 5 as subgraphs, and we have already seen that cycles of odd length have chromatic number 3. Therefore, the chromatic number of this graph is 3.

Fig. 31

E5) In the graph in Fig. 7, the graph induced by v_4, v_5, v_6, v_7 is K_4. So, it has a clique of size 4 and therefore we need at least 4 colours. We get a 4-colouring by assigning $\boxed{1}$ to v_1, $\boxed{2}$ to v_2, $\boxed{3}$ to v_3, $\boxed{2}$ to v_4 $\boxed{3}$ to v_5, $\boxed{1}$ to v_6 and $\boxed{4}$ to v_7. So, the chromatic number is 4.

E6) The figure given in Fig. 32 is 5-chromatic. It contains a clique of size 5, namely, the subgraph induced by the vertices w_1, w_2, w_3, w_4, w_5. So, we need at least 5 colours. We first give a 5-colouring to the subgraph isomorphic to

K_5 by assigning $\boxed{i}$ to w_i, $1 \le i \le 5$. Next, we assign $\boxed{2}$ to v_1, $\boxed{3}$ to v_2, $\boxed{1}$ to v_3, $\boxed{2}$ to v_4, and $\boxed{1}$ to v_5.

Fig. 32

E7) Firstly, every vertex lies in some colour class.
Next, $(x,x) \in R \, \forall x \in V(G)$.
Then, $(x,y) \in R \Leftrightarrow (y,x) \in R, x, y \in V(G)$.
Finally, you can see that $(x,y) \in R$ and $(y,z) \in R \Rightarrow (x,z) \in R, x, y, z \in V(G)$.

E8) Two different colourings are given in Fig. 33.

(a) **(b)**

Fig. 33

The colour classes for the colouring in Fig. 33(a) are $\{x_1\}$, $\{x_2\}$, $\{x_7\}$, $\{x_3, x_5\}$, $\{x_4, x_6\}$. The colour classes for the colouring in Fig. 33 (b) are $\{x_1\}$, $\{x_2\}$, $\{x_3\}$, $\{x_4, x_7\}$ and $\{x_5, x_6\}$.

E9) $\{v_1, v_2, v_4, v_6\}$

E10) For the graph in Fig.8, you can see that $\{x_{13}, x_{14}, x_3, x_5, x_7, x_9, x_{11}\}$ is an independent set and any other set has ≤ 7 elements. Thus α (G) = 7.

For the graph in Fig.10, note that for every vertex x_i, there are precisely two vertices in G not adjacent to x_i. But those two are adjacent. Hence, α (G) = 2.

E11) Since $K_{m,n}$ is a bipartite graph, by König's result,
$\chi'(K_{m,n}) = \Delta(K_{m,n}) = \min(m,n)$.

E12) The required Δ (T)- colouring is given in Fig. 34. You must remember that it is not unique.

Fig.34

Fig.35

E13) For a tree, the number of regions is 1; for a cycle it is 2; for K_4 it is 4 (see Fig.35).

E14) Yes. This is because, if G can be drawn without its edges crossing each other, this would also hold for any subgraph of G, as its edge set is a subset of the edge set of G.

E15) Since $K_{3,3}$ is bipartite, we can apply Theorem 5. Here p = 6 and q = 9. But, $2p - 4 = 10 > 9 = q$. So, $K_{3,3}$ is not planar.
The situation given is modelled by $K_{3,3}$. Since $K_{3,3}$ is non-planar, some of its edges will cross each other. Therefore, the corresponding lines or mains of the utility will cross.

E16) If you consider the vertex sets $\{x_1, x_3, x_4\}$ and $\{x_2, x_5, x_6\}$, you will see that $K_{3,3}$ is a subgraph of this graph. Thus, the graph is non-planar. You can also check that it is not a subdivision of K_5.

E17) The graph obtained by deleting the two horizontal edges is shown in Fig. 36(a). We have redrawn Fig. 36(a) in Fig. 36(b) so that you can clearly see that it is a subdivision of $K_{3,3}$.

Fig. 36

E18) Since K_5 is non-planar, there is no contradiction. Use the arguments given in Sec.4.2 to show $\chi(K_5) = 5$.

E19) $\chi(G) = 1$ iff G consists of isolated vertices. In this case, there is no question of non-planarity.
$\chi(K_{3,3}) = 2$.
$\chi(G) = 3$, where G is the Petersen graph.