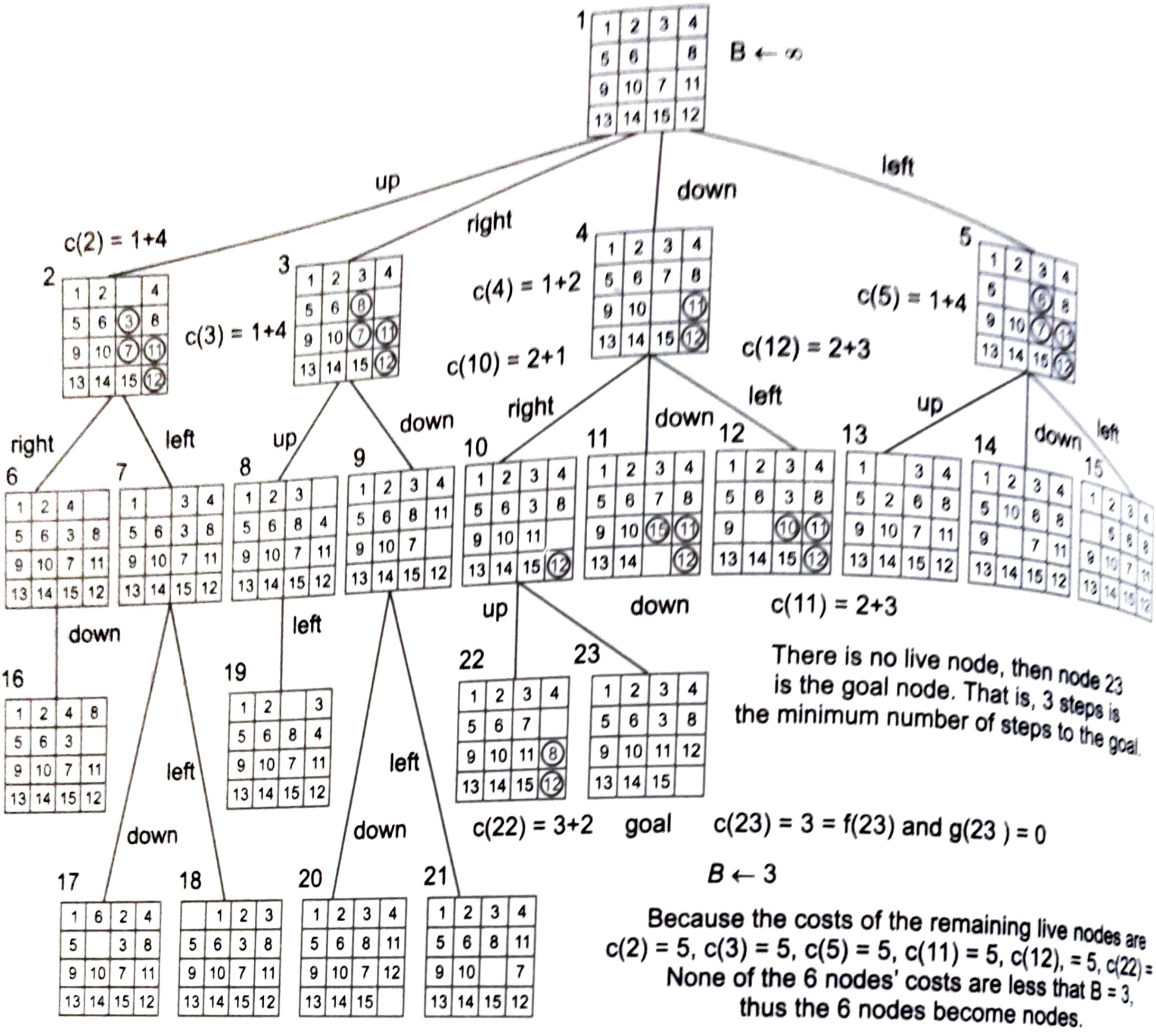


Figure 24.3

24.6 BRANCH AND BOUND FOR TSP (Traveling Salesman Problem)

TSP includes as a salesperson who has to visit a number of cities during a tour and the condition is to visit all the cities exactly once and return back to the same city where the person started.

Basic steps

Let $G=(V, E)$ be a direct graph defining an instance of the TSP.

1. This graph is first represented by a *cost matrix* where
   c_{ij} = the cost of edge if there is a path between vertex i and vertex j
   $c_{ij} = \infty$ if there is no path
2. Convert cost matrix into *reduced matrix i.e.*, every row and column should contain at least one zero entry.
3. Cost of the reduced matrix is the sum of elements that are subtracted from rows and columns of cost matrix to make it reduced.
4. Make the *state space tree* for reduced matrix.

5. To find the next E-node, find the *least cost valued node* by calculating the reduced cost matrix with every node.
6. If $\langle i, j \rangle$ edge is to be included, then there are three conditions to accomplish this task :
   I. Change all entries in row i and column j of A to ∞.
   II. Set $A[j, 1] = \infty$
   III. Reduce all rows and columns in resulting matrix except for rows and columns containing ∞.
7. Calculate the cost of the matrix where
   $\text{cost} = L + \text{cost}(i, j) + r$
   where L = cost of original reduced cost matrix and r = new reduced cost matrix.
8. Repeat the above steps for all the nodes until all the nodes are generated and we get a path.

Example 1 Apply Branch and Bound technique to solve travelling salesman problem for the graph whose cost matrix given below :

$$\begin{bmatrix} \infty & 7 & 3 & 12 & 8 \\ 3 & \infty & 6 & 14 & 9 \\ 5 & 8 & \infty & 6 & 18 \\ 9 & 3 & 5 & \infty & 11 \\ 18 & 14 & 9 & 8 & \infty \end{bmatrix}$$

Solution. For finding reduced cost matrix, first we try to reduce its rows. Subtracting minimum cost from corresponding row.

(*i*) Subtract 3 from first row
(*ii*) Subtract 3 from second row
(*iii*) Subtract 5 from third row
(*iv*) Subtract 3 from fourth row
(*v*) Subtract 8 from fifth row

Resulting matrix is

$$\begin{bmatrix} \infty & 4 & 0 & 9 & 5 \\ 0 & \infty & 3 & 11 & 6 \\ 0 & 3 & \infty & 1 & 13 \\ 6 & 0 & 2 & \infty & 8 \\ 10 & 6 & 1 & 0 & \infty \end{bmatrix}$$

Now, check the resulting matrix that whenever it is reduced or not *i.e.*, all of its rows and columns have at least one zero value or not.

Here, we see that each row in resulting matrix has a zero entry. Also all the columns have a zero entry except last column. So subtracting 5 from last column.

The resulting matrix is

$$\begin{bmatrix} \infty & 4 & 0 & 9 & 0 \\ 0 & \infty & 3 & 11 & 1 \\ 0 & 3 & \infty & 1 & 8 \\ 6 & 0 & 2 & \infty & 3 \\ 10 & 6 & 1 & 0 & \infty \end{bmatrix}$$

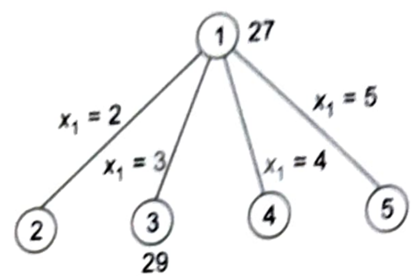
Total cost of the above matrix is the sum of total cost reduced from the original matrix *i.e.*,

$$(3+3+5+3+8+5)=27$$

Now, the above matrix has one zero value in each row and column. So this is the required reduced cost matrix with cost 27. It indicates that minimum cost path has at least cost 27.

For this, we can show a tree representation of state space generated by least cost branch and bound (LCBB). In this tree minimum cost of tour is 27 and set maximum cost bound *i.e.*, upper $=\infty$.

Starting with the root node, nodes 2, 3, 4, 5 are generated.



The reduced matrix for these nodes can be obtained in following manner :

Reduced cost matrix for node 2 *i.e.*, path 1, 2 will be obtained by :

(*i*) Setting all the entries in row 1 to ∞.

(*ii*) Setting all the entries in column 2 to ∞.

(*iii*) Setting entry (2, 1) to ∞.

(*iv*) Reducing the resulting matrix

(*v*) Find the cost matrix.

Now
$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 3 & 11 & 1 \\ 0 & \infty & \infty & 1 & 8 \\ 6 & \infty & 2 & \infty & 3 \\ 10 & \infty & 1 & 0 & \infty \end{bmatrix}$$

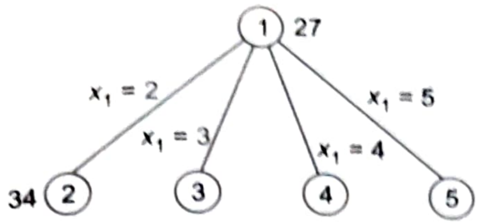
This matrix is not reduced, so reducing it by subtracting 1 from second row and 2 from fourth row, we get

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 2 & 10 & 0 \\ 0 & \infty & \infty & 1 & 8 \\ 4 & \infty & 0 & \infty & 1 \\ 10 & \infty & 1 & 0 & \infty \end{bmatrix}$$

Thus, total cost is

$$C(2)=C(1)+A(1,2)+r$$
$$=27+4+(1+2)=27+4+3=34$$

So edge (1, 2) having cost 34, *i.e.*,



Similarly, we try to find path (1, 3) *i.e.*, for $x_1 = 3$

(*i*) Make all the entries in row 1 and column 3 to ∞ and set (3, 1) to ∞.

(*ii*) Reduce the matrix.

(*iii*) Find the cost of matrix.

i.e.,

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 0 & \infty & \infty & 11 & 1 \\ \infty & 3 & \infty & 1 & 8 \\ 6 & 0 & \infty & \infty & 3 \\ 10 & 6 & \infty & 0 & \infty \end{bmatrix}$$

Subtract 1 from third row

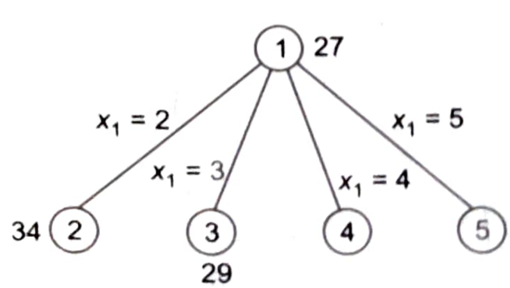
$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 0 & \infty & \infty & 11 & 1 \\ \infty & 2 & \infty & 0 & 7 \\ 6 & 0 & \infty & \infty & 3 \\ 10 & 6 & \infty & 0 & \infty \end{bmatrix}$$

Now subtract 1 from column 5

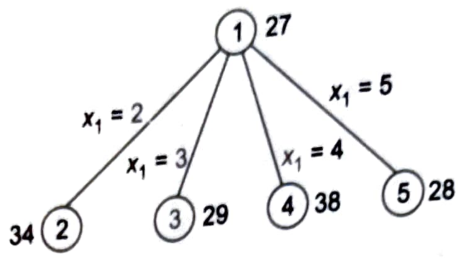
$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 0 & \infty & \infty & 11 & 0 \\ \infty & 2 & \infty & 0 & 6 \\ 6 & 0 & \infty & \infty & 2 \\ 10 & 6 & \infty & 0 & \infty \end{bmatrix}$$

Total cost of this matrix

$$\begin{aligned} C(3) &= C(1) + A(1,3) + r \\ &= 27 + 0 + (1+1) = 27 + 0 + 2 = 29 \end{aligned}$$

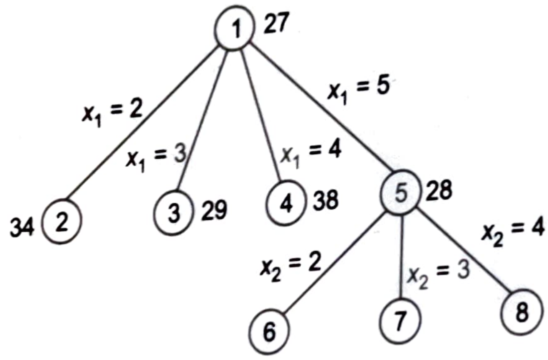


Further we try to find for path (1, 4) *i.e.*, $x_1 = 4$ and cost $= 38$ and for path (1, 5) *i.e.*, $x_1 = 5$ cost is 28.



Further, we continue with node of minimum cost *i.e.*, node 5 of cost 28.

So, the possible values of x_2 are 2, 3 and 4.



If $x_2 = 2$, we try for the path (1, 5, 2), steps to be followed are :

(*i*) Set all entries in row 5 and column 2 to ∞.

(*ii*) Set entry (2, 1) to ∞.

(*iii*) Reduce the matrix.

(*iv*) Find the cost of reduced matrix.

For path (1, 5) reduced matrix is

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 0 & \infty & 2 & 11 & \infty \\ 0 & 3 & \infty & 1 & \infty \\ 6 & 0 & 1 & \infty & \infty \\ \infty & 6 & 0 & 0 & \infty \end{bmatrix}$$

Now, we consider above matrix as a root matrix. So,

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 2 & 11 & \infty \\ 0 & \infty & \infty & 1 & \infty \\ 6 & \infty & 1 & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \end{bmatrix}$$

Now subtracting 2 from second row and 1 from 4th row

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 0 & 9 & \infty \\ 0 & \infty & \infty & 1 & \infty \\ 5 & \infty & 0 & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \end{bmatrix}$$

Again reducing 1 from column 4

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 0 & 8 & \infty \\ 0 & \infty & \infty & 0 & \infty \\ 5 & \infty & 0 & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \end{bmatrix}$$

Total cost is $C(6) = C(5) + A(5,2) + r$

$$= 28 + 6 + (2 + 1 + 1) = 28 + 6 + 4 = 38$$

For finding path (1, 5, 3) *i.e.*, $x_2 = 3$

Make row 5 and column 3 to ∞ and make entry (3, 1) to ∞. Reduce the matrix and find the cost. This is 30.

Now, we try for path (1, 5, 4).

Make row 5 and column 4 to ∞ and make entry (4, 1) to ∞. Reduce the matrix and find cost of matrix.

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 0 & \infty & 2 & \infty & \infty \\ 0 & 3 & \infty & \infty & \infty \\ \infty & 0 & 1 & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \end{bmatrix}$$

Reduce this matrix, so subtract 1 from column 3.

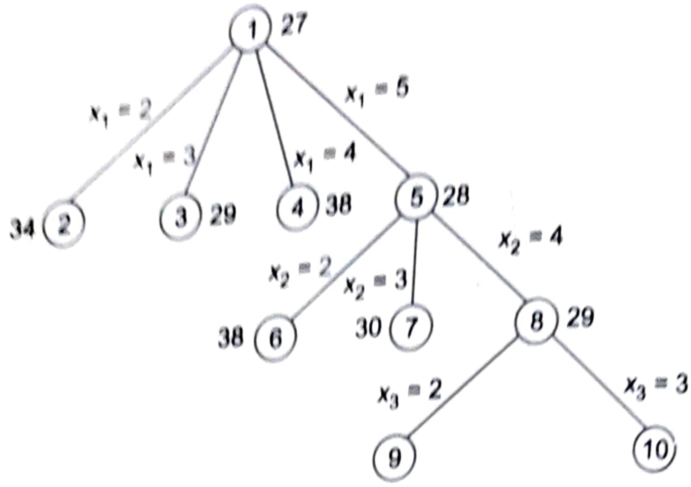
We get,

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 0 & \infty & 1 & \infty & \infty \\ 0 & 3 & \infty & \infty & \infty \\ \infty & 0 & 0 & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \end{bmatrix}$$

Total cost is

$$C(8) = C(5) + A(5,4) + r$$
$$= 28 + 0 + 1 = 29$$

We choose the path with minimum cost *i.e.*, node 8 or path (1, 5, 8)



We now find further possible path nodes *i.e.*, path (1, 5, 4, 2) and path (1, 5, 4, 3).

Now consider

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 0 & \infty & 1 & \infty & \infty \\ 0 & 3 & \infty & \infty & \infty \\ \infty & 0 & 0 & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \end{bmatrix} \text{ as a root matrix.}$$

For path (1, 5, 4, 2)

(*i*) Make the entries of row 4 and column 2 as ∞.

(*ii*) Make entry (2, 1) to ∞.

(*iii*) Reduce the matrix.

(*iv*) Find the cost.

Subtract 1 from row second, so we get

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 0 & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \end{bmatrix}$$

Total cost

$$C(9) = C(8) + A(4,2) + r$$
$$= 29 + 0 + 1 = 30$$

Now try for path (1, 5, 4, 3)

(*i*) Make the entries of row 4 and column 3 as ∞ and make entry (3, 1) as ∞.

(*ii*) Reduce the matrix.

(*iii*) Find the cost

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 0 & \infty & \infty & \infty & \infty \\ \infty & 3 & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \end{bmatrix}$$

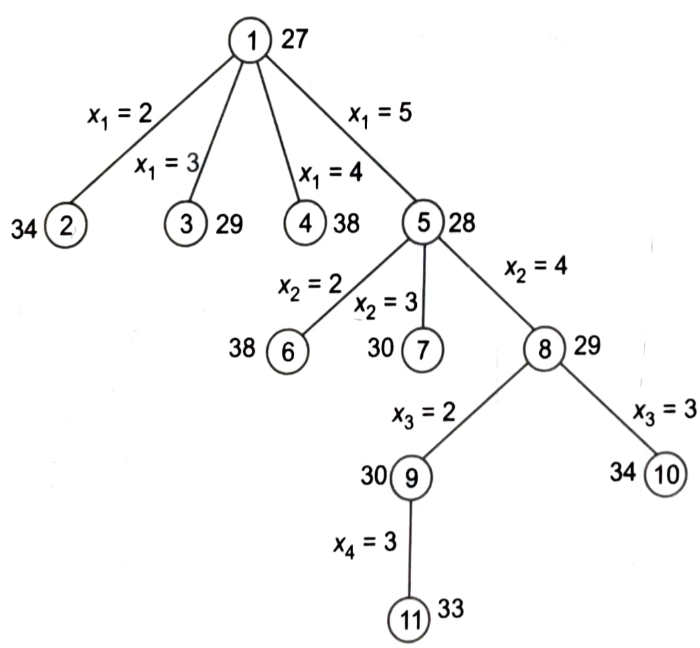
For reducing the matrix, subtract 3 from row 3, we get

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 0 & \infty & \infty & \infty & \infty \\ \infty & 0 & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \end{bmatrix}$$

Total cost $C(10) = C(8) + A(4,3) = r$

$$= 29 + 2 + 3 = 64$$

Now, we proceed with node having minimum cost *i.e.*, node 9 having cost 30.



Now, we have last choice (1, 5, 4, 2, 3) *i.e.*, $x_4 = 3$

(*i*) Make row 2 and column 3 as ∞ and make entry (3, 1) to ∞.

(*ii*) Reduce the matrix.

(*iii*) Find the cost.

For (1, 5, 4, 2, 3)

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 0 & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \end{bmatrix} \text{ as a root matrix.}$$

This matrix is already reduced. So, the total cost is

$$C(11) = C(9) + A(2,3) + r$$
$$= 30 + 3 + 0 = 33.$$

We know that in TSP initial and final positions are same so path is (1, 5, 4, 2, 3, 1) with minimum cost is 33.