FE AM - I IAT 2 SOLUTION

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Q.1 a) $u = \sin^{-1}\left(\frac{x}{y}\right)$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}} \cdot \frac{1}{y} = \frac{1}{\sqrt{y^2-x^2}}$$ —1m

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\left(\frac{-x}{y^2}\right) = \left(\frac{-x}{y}\right)\frac{1}{\sqrt{y^2-x^2}}$$ —2m

b) $u = r^2\cos 2\theta \quad v = r^2 \sin 2\theta$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} u_r & u_\theta \\ v_r & v_\theta \end{vmatrix}$$ —1m

$$= \begin{vmatrix} 2r\cos 2\theta & -2r^2\sin 2\theta \\ 2r\sin 2\theta & 2r^2\cos 2\theta \end{vmatrix}.$$

$$= 4r^3\cos^2 2\theta + 4r^3\sin^2 2\theta$$

$$= 4r^3$$ —2m

c) P.T : $\sin \log_e (i^{-i}) = 1$

Consider $(i^{-i}) = (e^{i\pi/2})^{-i} = e^{\pi/2}$ —1m

$$\therefore \sin\log_e (i^{-i}) = \sin\log_e (e^{\pi/2})$$

$$= \sin \cancel{\log}\left(\frac{\pi}{2}\right)$$

$$= 1$$ —2m

d) $7\cosh x + 8\sinh x = 1$

$7\left(\frac{e^x+e^{-x}}{2}\right) + 8\left(\frac{e^x-e^{-x}}{2}\right) = 1$ — 1m

$15e^x - e^{-x} = 2$

$\therefore 15e^{2x} - 1 = 2e^x$

$\therefore 15e^{2x} - 2e^x - 1 = 0$

$15e^{2x} - 5e^x + 3e^x - 1 = 0$

$5e^x(3e^x-1) + (3e^x-1) = 0$

$5e^x+1 = 0$ or $3e^x-1 = 0$

$e^x = \frac{-1}{5}$ or $e^x = \frac{1}{3}$

$e^x \neq \frac{-1}{5}$ as x is real

$\therefore e^x = \frac{1}{3}$ $\therefore x = \log\left(\frac{1}{3}\right)$ — 2m

e) Let $\cosh^{-1}\sqrt{1+x^2} = y$

$\therefore \cosh y = \sqrt{1+x^2}$ — 1m

$\cosh^2 y = 1+x^2$

$\therefore 1+\sinh^2 y = 1+x^2$

$\therefore x = \sinh y$

$\therefore y = \sinh^{-1} x$

$\therefore \cosh^{-1}\sqrt{1+x^2} = \sinh^{-1} x$. — 2m

f) By Maclaurin Series,

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots$ — 1m

— (1)

$f(x) = e^x$ $f(0) = e^0 = 1$

$f'(x) = e^x$ $f'(0) = 1$

$f''(x) = e^x$ $f''(0) = 1$

$\therefore e^x = \cancel{\beta}\, 1 + x + \frac{x^2}{2!} + \cdots$ — 2m

Q.2) a) $\cos(\theta + i\phi) = r(\cos\alpha + i\sin\alpha)$.

$\cos\theta \cdot \cos(i\phi) - \sin\theta \sin(i\phi) = r\cos\alpha + i r\sin\alpha$

$\cos\theta \cdot \cosh\phi - i\sin\theta \sinh\phi = r\cos\alpha + i r\sin\alpha$ —1m

$r\cos\alpha = \cos\theta \cosh\phi$

$r\sin\alpha = -\sin\theta \sinh\phi$ —2m

$\therefore \tanh\phi = -\dfrac{\sin\alpha \cos\theta}{\cos\alpha \sin\theta}$

$\dfrac{e^{\phi} - e^{-\phi}}{e^{\phi} + e^{-\phi}} = -\dfrac{\sin\alpha \cdot \cos\theta}{\cos\alpha \sin\theta}$ —3m

By Componendo & dividendo.

$\dfrac{\cancel{2}e^{\phi}}{-\cancel{2}e^{-\phi}} = \dfrac{-\sin\alpha\cos\theta + \cos\alpha\sin\theta}{-\sin\alpha\cos\theta - \cos\alpha\sin\theta}$

$\dfrac{e^{\phi}}{e^{-\phi}} = \dfrac{\cos\alpha\sin\theta - \sin\alpha\cos\theta}{\sin\alpha\cos\theta + \cos\alpha\sin\theta}$

$e^{2\phi} = \dfrac{\sin(\theta - \alpha)}{\sin(\theta + \alpha)}$ —4m

$2\phi = \log\left[\dfrac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}\right]$

$\phi = \dfrac{1}{2}\log\left[\dfrac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}\right]$ —5m

b) Let $a-b=u$, $a+b=v$.

$\mathrm{Log}\left[\dfrac{u+iv}{v+iu}\right] = \log\left[\dfrac{u+iv}{v+iu}\right] + i2n\pi$ —1m

$= \log(u+iv) - \log(v+iu) + i2n\pi$ —2m

$= \log(\sqrt{u^2+v^2}) + i\tan^{-1}\left(\dfrac{v}{u}\right) - \log(\sqrt{v^2+u^2}) - i\tan^{-1}\left(\dfrac{u}{v}\right) + i2n\pi$ —3m

$$= i\tan^{-1}\left(\frac{v}{u}\right) - i\tan^{-1}\left(\frac{u}{v}\right) + i2n\pi$$

$$= i\tan^{-1}\left[\frac{v/u - u/v}{1+\left(\frac{v}{u}\right)\left(\frac{u}{v}\right)}\right] + i2n\pi$$ — 4m

$$= i\tan^{-1}\left[\frac{v^2-u^2}{2uv}\right] + i2n\pi$$

$$= i\tan^{-1}\left[\frac{(a+b)^2-(a-b)^2}{2(a+b)(a-b)}\right] + i2n\pi$$

$$= i\tan^{-1}\left[\frac{4ab}{2(a^2-b^2)}\right] + i2n\pi$$

$$= i\left[\tan^{-1}\left[\frac{2ab}{a^2-b^2}\right] + 2n\pi\right]$$ — 5m

Q3.

a. Let $y = \sin^{-1} x$ —— ①

$\therefore \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2}$ —1m

$= 1 - \left(\frac{-1}{2}\right)x^2 + \dfrac{\left(\frac{-1}{2}\right)\left(\frac{-1}{2}-1\right)}{2!}x^4 - \dfrac{\left(\frac{-1}{2}\right)\left(\frac{-1}{2}-1\right)\left(\frac{-1}{2}-2\right)}{3!}x^6$ —2m

$+ \cdots$

$\therefore \dfrac{dy}{dx} = 1 + \frac{1}{2}x^2 + \frac{1}{2}\cdot\frac{3}{4}x^4 + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}x^6 + \cdots$ —3m

Integrating,

$\therefore y = A_0 + x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\frac{x^5}{5} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\frac{x^7}{7} + \cdots$ —4m

By (1), when $x = 0 \Rightarrow y = 0$

$\therefore A_0 = 0$

$\therefore y = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\frac{x^7}{7} + \cdots$ —5m

b. $f(x,y) = x^3 + xy^2 - 12x^2 - 2y^2 + 21x + 10$

i) $\therefore \dfrac{\partial f}{\partial x} = 3x^2 + y^2 - 24x + 21$

$\dfrac{\partial f}{\partial y} = 2xy - 4y$ —1m

$r = \dfrac{\partial^2 f}{\partial x^2} = 6x - 24$

$t = \dfrac{\partial^2 f}{\partial y^2} = 2x - 4$

$s = \dfrac{\partial^2 f}{\partial x \partial y} = 2y - 0$ 2m

ii) Consider $\dfrac{\partial f}{\partial x} = 0$ & $\dfrac{\partial f}{\partial y} = 0$

$3x^2 + y^2 - 24x + 21 = 0$, $2xy - 4y = 0$

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$2xy - 4y = 0$

$\therefore 2y(x-2) = 0$

$\therefore y = 0$ or $x = 2$

when, $y = 0$, $3x^2 - 24x + 21 = 0$

$\therefore x^2 - 8x + 7 = 0$

$\therefore x = 7, 1$

$\therefore$ points are $(7,0)$, $(1,0)$.

when $x = 2$, $12 + y^2 - 48 + 21 = 0$

$\therefore y^2 - 15 = 0$

$\therefore y^2 = 15$

$\therefore y = \pm\sqrt{15}$

$\therefore$ points are $(2, \sqrt{15})$, $(2, -\sqrt{15})$ —3m

point (x, y)	r	t	s	$rt - s^2$	r	maxima/minima	value
$(7,0)$	18	10	0	$180 > 0$	$18 > 0$	minima	-88
$(1,0)$	-18	-2	0	$36 > 0$	$-18 < 0$	maxima	20
$(2, \sqrt{15})$	-12	0	$2\sqrt{15}$	$-60 < 0$	-12	neither maxima nor minima	—
$(2, -\sqrt{15})$	-12	0	$-2\sqrt{15}$	$-60 < 0$	-12	"	—

5m