Complex Numbers

I. Questions carrying 2 to 4 marks

1) Find the modulus and argument of $-\frac{1}{2}+i\frac{\sqrt{3}}{2}$
2) Find the complex number z if $\arg(z+1)=\frac{\pi}{6}$ and $\arg(z-1)=\frac{2\pi}{3}$
3) Find real part, imaginary part, modulus and argument of $(4+2i)(-3+\sqrt{2})$
4) Evaluate $(1+i)^{100}+(1-i)^{100}$
5) Express in the form $a+ib$, $\frac{(1+i)^6(1-i\sqrt{3})^4}{(1-i)^8(1+i\sqrt{3})^5}$
6) Show that $\frac{(1+i)^6(\sqrt{3}-i)^4}{(1-i)^8(\sqrt{3}+i)^5}=\frac{1}{4}$

II. Questions carrying 5 to 8 marks

1) If $\cos\alpha+\cos\beta+\cos\gamma=0$ *and* $\sin\alpha+\sin\beta+\sin\gamma=0$ then prove that
   a. $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=\cos^2\alpha+\cos^2\beta+\cos^2\gamma=\frac{3}{2}$
   b. $\cos(\alpha+\beta)+\cos(\beta+\gamma)+\cos(\gamma+\alpha)=0$
   c. $\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)=0$
2) Show that $\frac{\sin 5\theta}{\sin\theta}=16\cos^4\theta-12\cos^2\theta+1$
3) Show that $\tan 7\theta=\frac{7\tan\theta-35\tan^3\theta+21\tan^5\theta-\tan^7\theta}{1-21\tan^2\theta+35\tan^4\theta-7\tan^6\theta}$
4) If $\sin 6\theta=a\cos^5\theta\sin\theta+b\cos^3\theta\sin^3\theta+c\cos\theta\sin^5\theta$ then find a,b,c.
5) Show that $\sin^5\theta=\frac{1}{16}(\sin 5\theta-5\sin 3\theta+10\sin\theta)$
6) If $\sin^4\theta\cos^3\theta=a_1\cos\theta+a_3\cos 3\theta+a_5\cos 5\theta+a_7\cos 7\theta$ then prove that $a_1+9a_3+25a_5+49a_7=0$
7) Solve $x^6-i=0$
8) Solve completely the equation $x^{10}+11x^5+10=0$
9) If ω is a cube root of unity then prove that $(1-\omega)^6=-27$
10) Show that the roots of $(x+1)^7=(x-1)^7$ are given by $\pm i\cot\frac{r\pi}{7}$, $r=1,2,3$.
11) If $\alpha,\alpha^2,\alpha^3,\alpha^4$ are roots of $x^5-1=0$ then find them and show that $(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4)=5$
12) Find continued product of all the values of $\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{\frac{3}{4}}$
13) If $1+2i$ is a root of the equation $x^4-3x^3+8x^2-7x+5=0$ then find all other roots.
14) Solve $x^4+x^3+x^2+x+1=0$

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Hyperbolic Function

I. Questions carrying 2 to 4 marks

1) If $\tan x = \frac{1}{2}$, find the value of x and $\sinh 2x$ $\tanh x = \frac{1}{2}$
2) Solve the equation $7\cosh x + 8\sinh x = 1$ for real values of x
3) Prove that $\cosh^2 x = \frac{1}{1-\frac{1}{1-\frac{1}{1-\cosh^2 x}}}$
4) Prove that
   a. $\cosh^{-1}\sqrt{1+x^2} = \sinh^{-1} x$
   b. $\cosh^{-1}\sqrt{1+x^2} = \tanh^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$
   c. $\tanh^{-1} x = \sinh^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$
5) Separate into real and imaginary parts $\cos^{-1}\left(\frac{3i}{4}\right)$
6) Separate into real and imaginary parts $\sin^{-1}\left(e^{i\theta}\right)$

II. Questions carrying 5 to 8 marks

1) If $u = \log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$, prove that
   a. $\cosh u = \sec\theta$
   b. $\sinh u = \tan\theta$
   c. $\tanh u = \sin\theta$
   d. $\tanh\frac{u}{2} = \tan\frac{\theta}{2}$
   e. $\cosh u \cos\theta = 1$
2) If $\sin(\theta + i\phi) = \cos\alpha + i\sin\alpha$ then prove that $\cos^4\theta = \sin^2\alpha = \sinh^4\phi$
3) If $\cos(\theta + i\phi) = r(\cos\alpha + i\sin\alpha)$ then prove that $\phi = \frac{1}{2}\log\left[\frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha}\right]$
4) If $x + iy = c\cot(u + iv)$ then show that $\frac{x}{\sin 2u} = -\frac{y}{\sinh 2v} = \frac{c}{\cosh 2v - \cos 2u}$
5) If $u + iv = cosec(\frac{\pi}{4} + ix)$ then prove that $(u^2+v^2)^2 = 2(u^2 - v^2)$
6) Separate $\tan^{-1} e^{i\theta}$ into real and imaginary parts.
7) If $\cosh x = \sec\theta$ then prove that
   a. $x = \log(sec\,\theta + tan\,\theta)$
   b. $\theta = \frac{\pi}{2} - 2\tan^{-1}(e^{-x})$
   c. $\tanh\frac{x}{2} = \tan\frac{\theta}{2}$

Logarithms of complex numbers

I. Questions carrying 2 to 4 marks

1) Prove that $\sin\log_e\left(i^{-i}\right) = 1$
2) Find the general value of $Log(1+i) + Log(1-i)$
3) Considering principal value only prove that $\log_2(-3) = \frac{\log 3 + i\pi}{\log 2}$
4) Separate into real and imaginary parts
   a. i^i
   b. $\sqrt{i}^{\sqrt{i}}$
   c. $(1+i)^{1-i}$

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II. Questions carrying 5 to 8 marks

1) Show that for real values of a and b, $e^{2ai\cot^{-1}b}\left[\frac{bi-1}{bi+1}\right] = 1$
2) Considering only the principal values, if $(1 + i\tan\alpha)^{1+i\tan\beta}$ is real, prove that its value is $(\sec\alpha)^{\sec^2\beta}$
3) If $\tan[\log(x + iy)] = a + ib$, prove that $\tan[\log(x^2 + y^2)] = \frac{2a}{1-a^2-b^2}$ where $a^2 + b^2 \neq 1$
4) Prove that $\log(e^{i\alpha} + e^{i\beta}) = \log\left[2\cos\left(\frac{\alpha-\beta}{2}\right)e^{\frac{i(\alpha+\beta)}{2}}\right]$
5) Prove that $Log\left[\frac{(a-b)+i(a+b)}{(a+b)+i(a-b)}\right] = i\left(2n\pi + \tan^{-1}\frac{2ab}{a^2-b^2}\right)$
6) Prove that $\log\left(\frac{sin(x+iy)}{sin(x-iy)}\right) = 2i\tan^{-1}(\cot x\tanh y)$
7) Show that $\tan\left[i\log\left(\frac{a-ib}{a+ib}\right)\right] = \frac{2ab}{a^2-b^2}$

Successive Differentiation

I. Questions carrying 2 to 4 marks

1) Find the n^{th} order derivative of the following function
   a. $y = (ax + b)^m$
   b. $y = \frac{1}{(ax+b)^m}$
   c. $y = e^{ax}\sin(bx + c)$
2) Find the n^{th} order derivative of the following function (using partial fraction)
   a. $y = \frac{2}{(x-1)(x-2)(x-3)}$
   b. $y = \frac{4x}{(x-1)^2\,(x+1))}$

II. Questions carrying 5 to 8 marks

1) If $y = \sin rx + \cos rx$, prove that $y_n = r^n[1 + (-1)^n\sin 2rx]^{\frac{1}{2}}$
2) If $= e^x\cos 2x\cos x$, find y_n.
3) If $y = \sin x\sin 2x\sin 3x$ find y_n.
4) If $y = \frac{x}{x^2+a^2}$, prove that $y_n = (-1)^n\, n!\, a^{-(n+1)}\sin^{n+1}\theta\cos(n + 1)\theta$ where $\theta = \tan^{-1}\left(\frac{a}{x}\right)$
5) Expand $y = \cos^9 x$ in cosine of multiples of x and then find y_n
6) Find the n^{th} order derivative of the following function (using Leibnitz's theorem)
   a. If $y = x^n\log x$, prove that $y_{n+1} = \frac{n!}{x}$
   b. If $y = \sin(msin^{-1}x)$, Prove that
      $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} + (m^2 - n^2)y_n = 0$
   c. If $y = e^{(m\,sin^{-1}x)}$, Prove that
      $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - (m^2 + n^2)y_n = 0$
   d. If $\log y = \tan^{-1}x$, Prove that
      $(1 + x^2)y_{n+2} + [2(n + 1)x - 1]y_{n+1} + n(n + 1)y_n = 0$
   e. If $y = \sin[\log(x^2 + 2x + 1)]$, Prove that
      $(x + 1)^2y_{n+2} + (2n + 1)(x + 1)y_{n+1} + (n^2 + 4)y_n = 0$

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7) If $y = (\sin^{-1} x)^2$prove that$y_n(o) = 0$, if n is odd
$y_n(0) = 2\,.\,2^2.\,4^2\,.\,6^2 \ldots\ldots\ldots\ldots.(n-2)^2$, if $n \neq 2$ and n is even.

8) If $y = (x-1)^n$ Prove that $y + \frac{y_1}{1!} + \frac{y_2}{2!} + \frac{y_3}{3!} + - - - - - + \frac{y_n}{n!} = x^n$

Partial Differentiation

I.Questions carrying 2 to 4 marks

1) Find all first order partial derivatives of the following
   a. $u = x^2y^3 + x^3y^2$
   b. $u = \log x \sin y$
   c. $u = 2^x \sin y \cos z$
2) Find $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}$ for the following functions
   a. $u = x^2y^3 + xy^2$
   b. $e^x \log y + \sin y \log x$
3) If $u = \sin^{-1}\left(\frac{x}{y}\right)$ then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$
4) If $u = \log(\tan x + \tan y)$ then prove that $\sin 2x\frac{\partial u}{\partial x} + \sin 2y\frac{\partial u}{\partial y} = 2$
5) If $u = \frac{e^{x+y}}{e^x + e^y}$ then prove that $u_x + u_y = u$
6) If $z = \sin^{-1}(x-y), x = 3t, y = 4t^3$, prove that $\frac{dz}{dt} = \frac{3(1-4t^2)}{\sqrt{1-t^2}}$
7) Using partial derivatives find $\frac{dy}{dx}$ for the function $xy + y^2 - 3x - 5 = 0$
8) If $z^3 - xz - y = 0$, find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

II. Questions carrying 5 to 8 marks

1) If $\theta = t^n e^{-\frac{r^2}{4t}}$ then find value of n which will make $\frac{\partial \theta}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right)$
2) If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, prove that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$
3) If $u = f(r)$ and $r^2 = x^2 + y^2 + z^2$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f''(r) + \frac{2}{r}f'(r)$
4) If $z = x\log(x+r) - r$,where $r^2 = x^2 + y^2$, prove that
   $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{x+r}, \frac{\partial^3 z}{\partial x^3} = -\frac{x}{r^3}$
5) If $x = e^u \tan v$, $y = e^u \sec v$, find $\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right)$
6) If $z = f(x,y), x = e^u + e^{-v}, y = e^{-u} - e^v$, prove that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$
7) If $u = f(x-y, y-z, z-x)$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$
8) If $u = f(e^{x-y}, e^{y-z}, e^{z-x})$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$
9) If $x = \sqrt{vw}, y = \sqrt{wu},\ z = \sqrt{uv}$, prove that
   $x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y} + z\frac{\partial \phi}{\partial z} = u\frac{\partial \phi}{\partial u} + v\frac{\partial \phi}{\partial v} + w\frac{\partial \phi}{\partial w}$

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10) If $f(x,y) = \phi(u,v)$ and $u = x^2 - y^2, v = 2xy$, prove that
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4(x^2+y^2)\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right)$$

11) If $u = \sin(x^2+y^2)$and $ax^2 + by^2 = c^2$, find $\frac{du}{dx}$

12) If $x^x y^y z^z = c$ at $x = y = z$, show that $\frac{\partial^2 z}{\partial x^2} - 2xy\frac{\partial^2 z}{\partial x\,\partial y} + \frac{\partial^2 z}{\partial y^2} = \frac{2(x^2-2)}{x(1+\log x)}$

13) If $\phi\left(\frac{z}{x^3}, \frac{y}{x}\right) = 0$,prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z$

14) If $f(xy^2, z-2x) = 0$, prove that $2x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 4x$

Homogeneous functions

I. Questions carrying 2 to 4 marks

1) If $u = \cos\left(\frac{xy+yz+zx}{x^2+y^2+z^2}\right)$ then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$
2) If $u = \frac{\sqrt{x}+\sqrt{y}}{x+y}$ then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$
3) If $u = \log(x^2+y^2) + \frac{x^2+y^2}{x+y} - 2\log(x+y)$then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{x^2+y^2}{x+y}$
4) If $u = x^2\tan^{-1}\left(\frac{y}{x}\right) + y^2\sin^{-1}\left(\frac{x}{y}\right)$ then prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2u$
5) If $u = \log\left(\frac{x^3+y^3}{x^2+y^2}\right)$, find value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

II. Questions carrying 5 to 8 marks

1) State and prove Euler's theorem for two and three variables.
2) Verify Euler's theorem for $u = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$ and also prove that $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{x^2-y^2}{x^2+y^2}$
3) If $u = x^3\sin^{-1}\left(\frac{y}{x}\right) + x^4\tan^{-1}\left(\frac{y}{x}\right)$ then find value of $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ at $x = 1, y = 1$
4) If $u = \frac{x^2y^2z^2}{x^2+y^2+z^2} + \cos^{-1}\left(\frac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\right)$ then find value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$
5) If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ then prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2\sin u\cos 3u$
6) If $u = \operatorname{cosec}^{-1}\left(\sqrt{\frac{x^{\frac{1}{2}}+y^{\frac{1}{2}}}{x^{\frac{1}{3}}+y^{\frac{1}{3}}}}\right)$ then prove that
$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}(13+\tan^2 u)$$

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Jacobian

I.Questions carrying 2 to 4 marks

1) If $u = r^2 \cos 2\theta$, $v = r^2 \sin 2\theta$ then find $\frac{\partial(u,v)}{\partial(x,y)}$
2) If $u = \frac{x+y}{1-xy}$, $v = \tan^{-1} x + \tan^{-1} y$ then find $\frac{\partial(u,v)}{\partial(x,y)}$
3) If $u = xyz,\ v = x^2 + y^2 + z^2,\ w = x + y + z$ then find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

II. Questions carrying 5 to 8 marks

1) If $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r \cos\theta$ then evaluate $\frac{\partial(r,\theta,\phi)}{\partial(x,y,z)}$ and $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$
2) If $x = uv,\ y = \frac{u}{v}$ then prove that $JJ' = 1$
3) Show that $JJ' = 1$ for $x = e^v \sec u$, $y = e^v \tan u$
4) If $u = x^2 - y^2$, $v = 2x^2 + y^2$ and $x = r\cos\theta$, $y = r \sin\theta$ then find $\frac{\partial(u,v)}{\partial(r,\theta)}$

Maxima and Minima

II. Questions carrying 5 to 8 marks

1) Discuss maxima and minima of $x^2 + y^2 + 8x + 6y + 6$
2) Discuss maxima and minima of $x^2 + y^2 + 6x + 12$
3) Discuss maxima and minima of $x^3 + xy^2 - 12x^2 - 2y^2 + 21x + 10$
4) Find maximum and minimum values of $x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$
5) Find the stationary values of $\sin x \sin y \sin(x + y)$
6) If a real number k is divided into three parts such that the sum of their products taken two at a time is maximum, find the numbers.
7) Find the point on the plane $2x + y - z - 5 = 0$ which is nearest to the origin.
8) Show that the rectangular solid that can be inscribed in a given sphere is a cube.

Expansions of Functions

I.Questions carrying 2 to 4 marks

1) Find the expansion of following function in power of x using Maclaurin's Series
   a. e^x
   b. $\sin x$
   c. $\log(\sec x)$
   d. $\log(1 + x)$
   e. $\sec^2 x$
2) Find the expansion of following function in power of x using standard series
   a. $e^{\sin x}$
   b. $(1 + x)^{\frac{1}{x}}$
   c. $\sin(e^x - 1)$
   d. e^{e^x}
3) Find the expansion of following function in power of x using differentiation and integration
   a. $sin^{-1}x$
   b. $\cos^{-1} x$
   c. $\tan^{-1} x$

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II. Questions carrying 5 to 8 marks

1) Find the expansion of following function in power of x using substitution
   a. $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$
   b. $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$
   c. $\cos^{-1}(\tanh(\log x))$

   $f(x) = f(a) + (x-a)f'(a) +$ $a = -3$
2) Expand e^x in power of $(x+3)$
3) Expand $\tan\left(x+\frac{\pi}{4}\right)$ upto term containing x^4 and evaluatetan 46.5^o. $f\left(\frac{\pi}{4}+x\right) = f\left(\frac{\pi}{4}\right) + x f'\left(\frac{\pi}{4}\right)$
4) Expand $f(x) = x^5 - x^4 + x^3 - x^2 + x - 1$ in a power of (x-1) and also find the $f\left(\frac{11}{10}\right)$ $f(x) = f(a) + (x-a)f'(a) + \dots$
5) Calculate the approximate value of $\sqrt{10}$ to four decimal points by using Taylor's expansion. $\sqrt{x+1} = f(1) + x f'(1)$

Indeterminate forms

I.Questions carrying 2 to 4 marks

Evaluate the following limit:

1) $\lim\limits_{x\to 1}\frac{x^x - x}{x-1-\log x}$
2) $\lim\limits_{x\to 0}\frac{\log(\sin 2x)}{\log x}$
3) $\lim\limits_{x\to 0}\frac{1}{x}(1 - x\cot x)$
4) $\lim\limits_{x\to\frac{\pi}{2}}(\cos x)^{\cos^2 x}$
5) $\lim\limits_{x\to 0}\left(\frac{1}{x}\right)^{1-\cos x}$
6) $\lim\limits_{x\to 0}(1+\sin x)^{\cot x}$
7) $\lim\limits_{x\to y}\frac{x^y - y^x}{x^x - y^y}$

II. Questions carrying 5 to 8 marks

1) Evaluate $\lim\limits_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}$
2) If $\lim\limits_{x\to 0}\frac{ae^x - b\cos x + c\,e^{-x}}{x\sin x} = 2$, find a, b, c
3) If $\lim\limits_{x\to 0}\frac{x(1+a\cos x) - b\sin x}{x^3} = 1$, find a and b
4) Evaluate $\lim\limits_{x\to 0}\frac{\tan x\tan^{-1}x - x^2}{x^6}$
5) Evaluate $\lim\limits_{x\to 0}\frac{\int_1^{1+x} e^{t^2}\,dt}{\int_2^{2+x} e^{t^2}\,dt}$

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Rank of Matrix

I.Questions carrying 2 to 4 marks

1) Show that every square matrix can be uniquely expressed as the sum of a symmetric matrix and a skew-symmetric matrix.
2) Show that every square matrix can be uniquely expressed as the sum of a Hermitian matrix and a skew-Hermitian matrix.
3) Show that every square matrix A can be uniquely expressed as P + iQ. when P & Q are Hermitian matrix.
4) Show that every Hermitian matrix A can be expressed as P + iQ.when P real symmetric & Q real Skew-Symmetric matrix.
5) Show that every Skew-Hermitian matrix A can be expressed as P + iQ.when P real Skew-Symmetric & Q real Symmetric matrix.
6) Prove the following matrix orthogonal and hence find A^{-1}

   a. $A=\frac{1}{3}\begin{bmatrix}-2 & 1 & 2\\ 2 & 2 & 1\\ 1 & -2 & 2\end{bmatrix}$ b. $A=\frac{1}{\sqrt{6}}\begin{bmatrix}\sqrt{2} & 1 & \sqrt{3}\\ \sqrt{2} & -2 & 0\\ \sqrt{2} & 1 & -\sqrt{3}\end{bmatrix}$

7) If $A=\begin{bmatrix}1/3 & 2/3 & a\\ 2/3 & 1/3 & b\\ 2/3 & -2/3 & c\end{bmatrix}$ is orthogonal find a, b, c.
8) Prove the following matrix Unitary and hence find A^{-1}

   a. $A=\frac{1}{3}\begin{bmatrix}2+i & 2i\\ 2i & 2-i\end{bmatrix}$ b. $A=\frac{1}{2}\begin{bmatrix}\sqrt{2} & -i\sqrt{2} & 0\\ i\sqrt{2} & -\sqrt{2} & 0\\ 0 & 0 & 2\end{bmatrix}$

9) If N= $\begin{bmatrix}0 & 1+2i\\ -1+2i & 0\end{bmatrix}$ that show that $(I-N)(I+N)^{-1}$is an unitary matrix.

II. Questions carrying 5 to 8 marks

1) Show that matrix A= $\begin{bmatrix}\alpha-i\gamma & -\beta+i\delta\\ \beta+i\delta & \alpha+i\gamma\end{bmatrix}$ is unitary if $\alpha^2+\beta^2+\gamma^2+\delta^2=1$.
2) Reduce the following matrix to echelon form and hence find its Rank

   a. $\begin{bmatrix}0 & 2 & -6 & -2\\ 1 & 0 & 1 & 1\\ 3 & 1 & 0 & 2\\ 1 & 1 & -2 & 0\end{bmatrix}$ b. $\begin{bmatrix}1 & 1 & 1 & 6\\ 1 & 2 & 3 & 14\\ 1 & 4 & 7 & 30\end{bmatrix}$

3) Reduce the following matrix to Normal form and hence find its Rank

   a. $\begin{bmatrix}2 & 3 & 4 & 5\\ 3 & 4 & 5 & 6\\ 4 & 5 & 6 & 7\\ 9 & 10 & 11 & 12\end{bmatrix}$ b. $\begin{bmatrix}1 & -1 & 3 & 6\\ 1 & 3 & -3 & -4\\ 5 & 3 & 3 & 11\end{bmatrix}$

4) Find the value of p for which the following matrix A will have (i) Rank1,(ii)Rank2, (iii) Rank3.

   a. $A=\begin{bmatrix}3 & p & p\\ p & 3 & p\\ p & p & 3\end{bmatrix}$ b. $A=\begin{bmatrix}1 & 3p & 3p+4\\ 1 & p+4 & 4p+2\\ 1 & 2p+2 & 3p+4\end{bmatrix}$

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5) Find non-singular matrices P and Q such that PAQ is in normal form. Also find their rank

a. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$ c. $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

Linear Equations

I.Questions carrying 2 to 4 marks

1) Examine whether the vectors $X_1 = (3,1,1)\,, X_2 = (2,0,-1), X_3 = (4,2,1)$ are linearly independent.
2) Using the encoding matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ encode and decode the message 'MOVE'.
3) Encode and decode the message ' MY GOD ANIRUDHA ' using encoding matrix $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$
4) Encode and decode the message ' WHO IS CARL GAUSS ' using encoding matrix $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$

II. Questions carrying 5 to 8 marks

1) Are the vectors $X_1 = (1,3,4,2)\,, X_2 = (3,-5,2,6), X_3 = (2,-1,3,4)$ linearly dependent? If so, express X_1 as a liner combination of the others.
2) Show that the following vectors are linearly independent
$X_1 = (1,2,-1,0)\,, X_2 = (1,3,1,2), X_3 = (4,2,1,0), X_4 = (6,1,0,1)$
3) Test for consistency the following equations and solve them if consistent
$x - 2y + 3t = 2\ , 2x + y + z + t = -4\, , 4x - 3y + z + 7t = 8$
4) Test for consistency the following equations and solve them if consistent
$2x - y + z = 9\ , 3x - y + z = 6, 4x - y + 2z = 7\, , -x + y - z = 4$
5) For what value of λ the equations $x + y + z = 1, x + 2y + 4z = \lambda, x + 4y + 10z = \lambda^2$ have a solution and solve them completely in each case
6) For what value of λ the equations $3x - 2y + \lambda z = 1, 2x + y + z = 2, x + 2y - \lambda z = -1$ will have no unique solution? Will the equation have any solution for this value of λ ?
7) Investigate for what values of λ and μ the equations
$x + y + z = 6, x + 2y + 3z = 10, x + 2y + \lambda z = \mu$ have (i) no solution (ii) unique solution (iii) infinite number of solution.
8) Solve the equations
$x_1 + x_2 - x_3 + x_4 = 0\, , x_1 - x_2 + 2x_3 - x_4 = 0\, , 3x_1 + x_2 + x_4 = 0\,.$
9) For what value of λ ,the following system of equations possesses a non-trivial solution ? Obtain the solution for real values of λ.
$3x_1 + x_2 - \lambda x_3 = 0\ , 4x_1 - 2x_2 - 3x_3 = 0\, , 2\lambda x_1 + 4x_2 + \lambda x_3 = 0\,.$

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10) Show that the system of equation $ax + by + cz = 0\,, bx + cy + az = 0\,, cx + ay + bz = 0$ has a non-trivial solution if $a + b + c = 0 \text{ or } a = b = c$. Find the non-trivial solution when the condition is satisfied.

Solution of algebraic equation and transcendental equations (Using Numerical Methods)

Questions carrying 5 to 8 marks

1) Evaluate the root of $x \log_{10} x = 1.2$ lying between 2 and 3 by Regula-Falsi method correct upto 4 places of decimal.
2) Find the root of $x^3 + 2x - 20 = 0$ 3 by Regula-Falsi method correct upto 4 places of decimal.
3) Find root of $e^x \sin x = 1$ correct upto 4 decimal places by Newton-Raphson method.
4) Find cube root of 79 correct upto 5 decimal places by Newton-Raphson method.
5) Solve the given system of equations using Gauss Elimination method.
   $2x + y + z = 10, 3x + 2y + 3z = 18, x + 4y + 9z = 16$
6) Solve the given system of equations using Jacobi's method.
   $5x - y + z = 10, 2x + 4y = 12, x + 5y + 5z = -1$. Start with $(2,3,0)$
7) Solve the given system of equations using Gauss-Siedel method.
   $3x - 0.1y - 0.2z = 7.85, 0.1x + 7y - 0.3z = 19.3, 0.3x - 0.2y + 10z = 71.4$

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