Ch-4 Numerical Solution of Ordinary Differential Equations ①

Euler's Method – For Approximately solving Ordinary Differential Equation. With this method, we solve ordinary differential equation and find out the value in a particular range and that value will be the approximate value

For Example:

Consider the differential equation / Initial Value Problem (IVP) –

$$\frac{dy}{dx} = f(x,y) \qquad y(x_0) = y_0 \text{ – given}$$

For particular value of $x$, we need to find value of $y$

$$y(x_n) = y_n$$

ie for $x_n$ value, we need to find $y_n$.

Euler's Method

$$y_n = y(x_n) = y_{n-1} + hf(x_{n-1}, y_{n-1}) \quad \text{--- (1)}$$

where

$$h = x_n - x_{n-1}$$

ie $\boxed{x_n = x_0 + nh}$

$\boxed{y_{n+1} = y_n + hf(x_n, y_n)}$

If $n = 1$

$$y_1 = y(x_1) = y_0 + hf(x_0, y_0)$$

If $n = 2$

$$y_2 = y(x_2) = y_1 + hf(x_1, y_1)$$

Similarly, we can calculate $y_3, y_4 \ldots$

For calculating $y_2$, you need $y_1$,
For calculating $y_3$, you need $y_2$

ie $y_3 = y(x_3) = y_2 + hf(x_2, y_2)$

For Example:

youtube

Find $y(2.2)$ using Euler's method from the equation

$$\frac{dy}{dx} = -xy^2 \quad \text{with } y(2) = 1$$

Sol$^n$ Given

$$f(x, y) = -xy^2$$

$$\boxed{x_0 = 2} \quad \boxed{y_0 = 1}$$

Find $y(2.2)$

Distance b/w 2 ad 2.2 is 0.2.

Now, divide this difference is 0.2 into small parts in order to find h. Jitna ch small parts mai divide karoge, utna approximate answer hai

If we direct $h = 0.2$, ek hi baar mai answer aa jaayega ad that is not very much accurate and approximate answer.

That is upto you, what value of h you want to choose. So take small value of h

Jitna chota interval hai, utna approximate answer hai

$\Rightarrow$ Let $n = 4$

$$h = \frac{2.2-2}{4} = \frac{0.2}{4} = 0.05$$

$x_1 = x_0 + h$
$= 2 + 0.05 = 2.05$

$x_2 = x_0 + 2h$
$= 2 + 2 \times 0.05 = 2 + 0.1 = 2.1$

$x_3 = x_0 + 3h$
$= 2 + 3 \times 0.05 = 2 + 0.15 = 2.15$

$x_4 = x_0 + 4h$
$= 2 + 4 \times 0.05 = 2 + 0.2 = 2.2$

On $x_4$, we have to calculate value of $y$.

By Euler's Formula

$$y_n = y_{n-1} + h f(x_{n-1}, y_{n-1}) \quad (1)$$

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When $n=1$

$$y_1 = y(x_1) = y(2.05) = y_0 + hf(x_0, y_0)$$
$$= y_0 + h[-x_0 y_0^2]$$
$$= 1 + 0.05(-2 \times 1^2)$$
$$= 1 - 0.1$$
$$\boxed{y_1 = 0.9}$$

When $n=2$ $\quad x_2 = 2.1$

$$y_2 = y(x_2) = y(2.1) = y_1 + hf(x_1, y_1)$$
$$= 0.9 + 0.05[-x_1 y_1^2]$$
$$= 0.9 + 0.05[-2.05 \times (0.9)^2]$$
$$= 0.9 - 0.05[2.05 \times 0.81]$$
$$= 0.9 - 0.05 \times 1.6605$$
$$= 0.9 - 0.08303$$
$$\boxed{y_2 = 0.81697}$$

When $n=3$

$$y_3 = y(x_3) = y(2.15) = y_2 + hf(x_2, y_2)$$
$$= 0.81697 + 0.05(-x_2 y_2^2)$$
$$= 0.81697 + 0.05(-2.1 \times (0.81697)^2)$$
$$= 0.81697 - 0.05(2.1 \times 0.66744)$$
$$= 0.81697 - 0.05(1.40162)$$
$$= 0.81697 - 0.07008$$

$$\boxed{y_3 = 0.74689}$$

When $n = 4$

$$y_4 = y(x_4) = y(2.2) = y_3 + hf(x_3, y_3)$$

$$= y_3 + h[-x_3 y_3^2]$$

$$= 0.74689 + 0.05[-2.15 \times (0.74689)^2]$$

$$= 0.74689 - 0.05(2.15 \times 0.55784)$$

$$= 0.74689 - 0.05 \times 1.19936$$

$$= 0.74689 - 0.05997$$

$$\boxed{y_4 = 0.68692}$$ Hence at $x = 2.2$ the approximate value of $y$ is 0.68692

Note:- Interval (h) ko kitne part mai divide karna hai, ye aapke haath mai hai. Agar ques. mai nhi mention, then it is upto you. Try to divide in zaada se zaada intervals, utna accurate answer aayega.

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eg-1 Pg 43



Given $\frac{dy}{dx} = \frac{y-x}{y+x}$ with initial conditions $y=1$ at $x=0$. Find $y$ for $x=0.1$ by Euler's Method.

Sol^n

$$\frac{dy}{dx} = \frac{y-x}{y+x} \qquad \text{find } y(0.1)$$

$$f(x,y) = \frac{y-x}{y+x}$$

$$x_0 = 0 \qquad y_0 = 1$$

$$h = \frac{x - x_0}{n}$$

Let $n=5$

$$h = \frac{0.1-0}{5} = \frac{0.1}{5} = 0.02$$

$$\boxed{h = 0.02}$$

$$x_1 = x_0 + h$$
$$= 0 + 0.02 = 0.02$$

$$x_2 = x_0 + 2h$$
$$= 0 + 2(0.02) = 0.04$$

$$x_3 = x_0 + 3h$$
$$= 0 + 3(0.02) = 0.06$$

$$x_4 = x_0 + 4h$$
$$= 0 + 4(0.02)$$
$$x_4 = 0.08$$

$$x_5 = x_0 + 5h$$
$$= 0 + 5(0.02)$$
$$\boxed{x_5 = 0.1}$$

On $x_5$, we have to calculate value of $y$.

When $n = 1$

$$y_1 = y(x_1) = y(0.02) = y_0 + h f(x_0, y_0)$$

$$= 1 + 0.02\left(\frac{y_0 - x_0}{y_0 + x_0}\right)$$

$$= 1 + 0.02\left(\frac{1-0}{1+0}\right)$$

$$= 1 + 0.02$$

$$\boxed{y_1 = 1.02}$$

When $n = 2$

$$y_2 = y(x_2) = y(0.04) = y_1 + h f(x_1, y_1)$$

$$= 1.02 + 0.02\left(\frac{y_1 - x_1}{y_1 + x_1}\right)$$

$$= 1.02 + 0.02\left(\frac{1.02 - 0.02}{1.02 + 0.02}\right)$$

$$= 1.02 + 0.02\left(\frac{1}{1.04}\right)$$

$$= 1.02 + 0.0192$$

$$\boxed{y_2 = 1.0392}$$

When $n = 3$

$$y_3 = y(x_3) = y(0.06) = y_2 + h f(x_2, y_2)$$

$$= 1.0392 + 0.02\left[\frac{y_2 - x_2}{y_2 + x_2}\right]$$

$$= 1.0392 + 0.02\left[\frac{1.0392 - 0.04}{1.0392 + 0.04}\right]$$

$$= 1.0392 + 0.02\left[\frac{0.9992}{1.0792}\right]$$

$$= 1.0392 + 0.02(0.9259)$$

$$= 1.0392 + 0.0185$$

$$\boxed{y_3 = 1.0577}$$

When $n = 4$

$$y_4 = y(x_4) = y(0.08) = y_3 + h f(x_3, y_3)$$

$$= y_3 + h\left[\frac{y_3 - x_3}{y_3 + x_3}\right]$$

$$= 1.0577 + 0.02\left[\frac{1.0577 - 0.06}{1.0577 + 0.06}\right]$$

$$= 1.0577 + 0.02\left[\frac{0.9977}{1.1177}\right]$$

$$= 1.0577 + 0.02(0.8926)$$

$$= 1.0577 + 0.0179$$

$$\boxed{y_4 = 1.0756}$$

When $n = 5$

$$y_5 = y(x_5) = y(0.1) = y_4 + h f(x_4, y_4)$$

$$= y_4 + h\left[\frac{y_4 - x_4}{y_4 + x_4}\right]$$

$$= 1.0756 + 0.02\left[\frac{1.0756 - 0.08}{1.0756 + 0.08}\right]$$

$$= 1.0756 + 0.02\left[\frac{0.9956}{1.1556}\right]$$

$$= 1.0756 + 0.02 \times 0.8615$$

$$y_5 = 1.0756 + 0.0172$$

$$\boxed{y_5 = 1.0928}$$

Hence at $x = 0.1$,

the required approximate value of $y$ is $1.0928$.

Eg-2 Pg 45 (6)

Apply Euler's method to solve $\frac{dy}{dx} = x+y$, $y(0)=0$, choosing the step length $=0.2$. Find $y(1.4)$

Sol^n We have

$$\frac{dy}{dx} = x+y$$

ie $f(x,y) = x+y$

given $\boxed{h=0.2}$ $x_0=0$ $y_0=0$

Find $y(1.4)$

Here $n=7$ as $\boxed{h=0.2 \text{ given}}$

Here,

$x_1 = x_0+h$
$= 0+0.2$
$\boxed{x_1 = 0.2}$

$x_2 = x_0+2h$
$= 0+2(0.2)$
$\boxed{x_2 = 0.4}$

$x_3 = x_0+3h$
$= 0+3(0.2)$
$\boxed{x_3 = 0.6}$

$x_4 = x_0+4h$
$= 0+4(0.2)$
$\boxed{x_4 = 0.8}$

$x_5 = x_0+5h$
$= 0+5(0.2)$
$\boxed{x_5 = 1.0}$

$x_6 = x_0+6h$
$= 0+6(0.2)$
$\boxed{x_6 = 1.2}$

$x_7 = x_0+7h$
$= 0+7(0.2)$
$\boxed{x_7 = 1.4}$

$= 0.128 + 0.2(0.728)$ (7)

$= 0.128 + 0.1456$

$\boxed{y_4 = 0.2736}$

When $n = 5$

$y_5 = y(x_5) = y(1.0) = y_4 + h f(x_4, y_4)$

$= 0.2736 + 0.2[x_4 + y_4]$

$= 0.2736 + 0.2[0.8 + 0.2736]$

$= 0.2736 + 0.2[1.0736]$

$= 0.2736 + 0.21472$

$= 0.48832$

$\boxed{y_5 = 0.4883}$

When $n = 6$

$y_6 = y(x_6) = y(1.2) = y_5 + h f(x_5, y_5)$

$= 0.4883 + 0.2[x_5 + y_5]$

$= 0.4883 + 0.2[1.0 + 0.4883]$

$0.4883 + 0.2 \times 1.4883$

$0.4883 + 0.29766$

$= 0.4883 + 0.29766$

$= 0.78596$

$\boxed{y_6 = 0.786}$

When $n=7$

$$y_7 = y(x_7) = y(1.4) = y_6 + h f(x_6, y_6)$$
$$= 0.786 + 0.2[x_6 + y_6]$$
$$= 0.786 + 0.2[1.2 + 0.786]$$
$$= 0.786 + 0.2[1.986]$$
$$= 0.786 + 0.3972$$
$$\boxed{y_7 = 1.1832}$$

$\therefore$ At $x = 1.4$, the approximate value of $y$ is $1.1832$

eg-3 Pg 4·6 

Using Euler's method, find an approximate value of $y$ corresponding to $x=1$, given that $\frac{dy}{dx} = x+y$ with initial condition $y=1$ $x=0$

Sol$^n$ $\frac{dy}{dx} = x+y$

$f(x,y) = x+y$

$n=5$ $x_0=0$ $y_0=1$ Find $y(1)$

$h = \frac{1-0}{5} = 0.2$ $\boxed{h=0.2}$

$x_1 = x_0 + h$
$= 0+0.2 = 0.2$
$\boxed{x_1=0.2}$

$x_2 = x_0 + 2h$
$= 0+2(0.2) = 0.4$
$\boxed{x_2=0.4}$

$x_5 = x_0 + 5h$
$= 0+5(0.2)$
$\boxed{x_5 = 1.0}$

$x_3 = x_0 + 3h$
$= 0+3(0.2)$
$\boxed{x_3 = 0.6}$

$x_4 = x_0 + 4h$
$= 0+4(0.2)$
$\boxed{x_4 = 0.8}$

For $n=1$

$$y_1 = y(x_1) = y(0.2) = y_0 + h f(x_0, y_0)$$
$$= 1 + 0.2[x_0 + y_0]$$
$$= 1 + 0.2[0+1]$$
$$= 1 + 0.2 = 1.2$$

$$\boxed{y_1 = 1.2}$$

For $n=2$

$$y_2 = y(x_2) = y(0.4) = y_1 + h f(x_1, y_1)$$
$$= 1.2 + 0.2[x_1 + y_1]$$
$$= 1.2 + 0.2[0.2 + 1.2]$$
$$= 1.2 + 0.2[1.4]$$
$$= 1.2 + 0.28$$

$$\boxed{y_2 = 1.48}$$

For $n=3$

$$y_3 = y(x_3) = y(0.6) = y_2 + h f(x_2, y_2)$$
$$= 1.48 + 0.2[x_2 + y_2]$$
$$= 1.48 + 0.2[0.4 + 1.48]$$
$$= 1.48 + 0.2[1.88]$$
$$= 1.48 + 0.376$$

$$\boxed{y_3 = 1.856}$$

for $n = 4$

$$y_4 = y(x_4) = y(0.8) = y_3 + hf(x_3, y_3)$$
$$= 1.856 + 0.2[x_3 + y_3]$$
$$= 1.856 + 0.2[0.6 + 1.856]$$
$$= 1.856 + 0.2[2.456]$$
$$= 1.856 + 0.4912$$
$$\boxed{y_4 = 2.3472}$$

for $n = 5$

$$y_5 = y(x_5) = y(1.0) = y_4 + hf(x_4, y_4)$$
$$= 2.3472 + 0.2[x_4 + y_4]$$
$$= 2.3472 + 0.2[0.8 + 2.3472]$$
$$= 2.3472 + 0.2[3.1472]$$
$$= 2.3472 + 0.62944$$
$$= 2.97664$$
$$\boxed{y_5 = 2.977}$$

When $n = 1$

$$y_1 = y(x_1) = y(0.2) = y_0 + h f(x_0, y_0)$$
$$= 0 + 0.2 [x_0 + y_0]$$
$$= 0.2 (0 + 0)$$
$$\boxed{y_1 = 0}$$

When $n = 2$

$$y_2 = y(x_2) = y(0.4) = y_1 + h f(x_1, y_1)$$
$$= 0 + 0.2 [x_1 + y_1]$$
$$= 0.2 [0.2]$$
$$\boxed{y_2 = 0.04}$$

When $n = 3$

$$y_3 = y(x_3) = y(0.6) = y_2 + h f(x_2, y_2)$$
$$= 0.04 + 0.2 [x_2 + y_2]$$
$$= 0.04 + 0.2 [0.4 + 0.04]$$
$$= 0.04 + 0.2 [0.44]$$
$$= 0.04 + 0.088$$
$$\boxed{y_3 = 0.128}$$

When $n = 4$

$$y_4 = y(x_4) = y(0.8) = y_3 + h f(x_3, y_3)$$
$$= 0.128 + 0.2 (x_3 + y_3)$$
$$= 0.128 + 0.2 (0.6 + 0.128)$$