L.U Decomposition Method / Triangularization Method

①

Working Rule :-

Consider the system of equation

$$AX = B \quad \text{—— (1)}$$

where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Let $A = LU$ —— (2)

where

L = Lower Triangular Matrix

U = Upper Triangular Matrix

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

and $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

By (1) & (2), we get

$$LUX = B \quad \text{--- (3)}$$

Put $UX = Y$ where --- (4)

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Then (3) becomes

$$LY = B \quad \text{--- (5)}$$

Solving (5) for Y

Put the value of Y into (4) and solving it for X

3- Solve

$$3x + y + z = 4$$
$$x + 2y + 2z = 3$$
$$2x + y + 3z = 4$$

Solⁿ The given system of equations

$$AX = B \quad \text{--- (1)}$$

Where

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$$

Let $A = LU$ --- (2)

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & u_{12}l_{31}+l_{32}u_{22} & u_{13}l_{31}+u_{23}l_{32}+u_{33} \end{bmatrix}$$

On solving / Equating both sides

Ut=1 $\boxed{u_{11} = 3}$ $\boxed{u_{12} = 1}$ $\boxed{u_{13} = 1}$

$l_{21}u_{11} = 1$

$l_{21}(3) = 1$

$\boxed{l_{21} = \frac{1}{3}}$

$l_{21}u_{12} + u_{22} = 2$

$\frac{1}{3} \times 1 + u_{22} = 2$

$u_{22} = 2 - \frac{1}{3}$

$\boxed{u_{22} = \frac{5}{3}}$

$l_{21}u_{13} + u_{23} = 2$

$\frac{1}{3} \times 1 + u_{23} = 2$

$u_{23} = 2 - \frac{1}{3}$

$\boxed{u_{23} = \frac{5}{3}}$

$l_{31}u_{11} = 2$

$l_{31} \times 3 = 2$

$\boxed{l_{31} = \frac{2}{3}}$

$u_{12}l_{31} + l_{32}u_{22} = 1$

$1 \times \frac{2}{3} + l_{32} \times \frac{5}{3} = 1$

$\frac{5}{3} l_{32} = 1 - \frac{2}{3}$

$\Rightarrow \frac{5}{3} l_{32} = \frac{1}{3}$

$\boxed{l_{32} = \frac{1}{5}}$

$u_{13} l_{31} + u_{23} l_{32} + u_{33} = 3$ (3)

$1 \times \frac{2}{3} + \frac{\cancel{8}}{3} \times \frac{1}{\cancel{8}} + u_{33} = 3$

$\frac{2}{3} + \frac{1}{3} + u_{33} = 3$

$u_{33} = 3 - \frac{2}{3} - \frac{1}{3}$

$\boxed{u_{33} = 2}$

$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 2/3 & 1/5 & 1 \end{bmatrix}$

$U = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 5/3 & 5/3 \\ 0 & 0 & 2 \end{bmatrix}$

By Equation (1) & (2)

$LUX = B$ ——— (3)

Let $UX = Y$ ——— (4)

where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Then (3) becomes

$LY = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 2/3 & 1/5 & 1 \end{bmatrix}_{3\times 3} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}_{3\times 1} = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ 1/3\, y_1 + y_2 \\ 2/3\, y_1 + 1/5\, y_2 + y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$$

On Equatng

$\boxed{y_1 = 4}$

$\frac{1}{3} \times y_1 + y_2 = 3$

$\frac{4}{3} + y_2 = 3$

$y_2 = 3 - \frac{4}{3}$

$\boxed{y_2 = \frac{5}{3}}$

$\frac{2}{3} y_1 + \frac{1}{5} y_2 + y_3 = 4$

$\frac{2}{3} \times 4 + \frac{1}{\cancel{5}} \times \frac{\cancel{5}}{3} + y_3 = 4$

$\frac{8}{3} + \frac{1}{3} + y_3 = 4$

$3 + y_3 = 4$

$\boxed{y_3 = 1}$

By equation (4)

$$\begin{bmatrix} 3 & 1 & 1 \\ 0 & 5/3 & 5/3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5/3 \\ 1 \end{bmatrix}$$

$$2z = 1$$

$$\boxed{z = \frac{1}{2}}$$

$$\frac{5}{3}y + \frac{5}{3}z = \frac{5}{3}$$

$$\frac{5}{3}y + \frac{5}{3} \times \frac{1}{2} = \frac{5}{3}$$

$$\frac{5}{3}y + \frac{5}{6} = \frac{5}{3}$$

$$\frac{5}{3}y = \frac{5}{3} - \frac{5}{6}$$

$$\frac{5}{3}y = \frac{10-5}{6}$$

$$\frac{5}{3}y = \frac{5}{6}$$

$$\boxed{y = \frac{1}{2}}$$

$$3x + y + z = 4$$

$$3x + \frac{1}{2} + \frac{1}{2} = 4$$

$$3x + \frac{1}{4} = 4$$

$$3x = 4 - \frac{1}{4}$$

$$3x = \frac{15}{4}$$

$$\boxed{x = \frac{5}{4}}$$

Hence the solution is

$x = 1$ $\quad y = \frac{1}{2}$ $\quad z = \frac{1}{2}$

Example:-1

Pg 3.16

$$x+y+z=1$$
$$4x+3y-z=6$$
$$3x+5y+3z=4$$

Soln

The given system of equations

$$AX=B \quad \text{—(1)}$$

where

$$A=\begin{bmatrix}1&1&1\\4&3&-1\\3&5&3\end{bmatrix} \quad X=\begin{bmatrix}x\\y\\z\end{bmatrix} \quad B=\begin{bmatrix}1\\6\\4\end{bmatrix}$$

Let $A=LU$ —(2) where

$$L=\begin{bmatrix}1&0&0\\l_{21}&1&0\\l_{31}&l_{32}&1\end{bmatrix} \quad U=\begin{bmatrix}u_{11}&u_{12}&u_{13}\\0&u_{22}&u_{23}\\0&0&u_{33}\end{bmatrix}$$

$$A=LU$$

$$\begin{bmatrix}1&1&1\\4&3&-1\\3&5&3\end{bmatrix}=\begin{bmatrix}1&0&0\\l_{21}&1&0\\l_{31}&l_{32}&1\end{bmatrix}\begin{bmatrix}u_{11}&u_{12}&u_{13}\\0&u_{22}&u_{23}\\0&0&u_{33}\end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix} \quad (5)$$

On equating both side

$\boxed{u_{11}=1}$ $\boxed{u_{12}=1}$ $\boxed{u_{13}=1}$

$l_{21}u_{11}=4$ $l_{31}u_{11}=3$ $l_{21}u_{12}+u_{22}=3$

$\boxed{l_{21}=4}$ $\boxed{l_{31}=3}$ $4\times1+u_{22}=3$

$\boxed{u_{22}=-1}$

$l_{21}u_{13}+u_{23}=-1$

$4\times1+u_{23}=-1$

$\boxed{u_{23}=-5}$

$l_{31}u_{12}+l_{32}u_{22}=5$

$3\times1+l_{32}(-1)=5$

$-l_{32}=2$

$\boxed{l_{32}=-2}$

$l_{31}u_{13}+l_{32}u_{23}+u_{33}=3$

$3\times1+(-2)\times(-5)+u_{33}=3$

$3+10+u_{33}=3$

$\boxed{u_{33}=-10}$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix}$$

By Equation (1) & (2)

$LUX = B$ —— (3)

Let $\tilde{U}X = Y$ —— (4)

where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Then eq^n (3) become

$$LY = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ 4y_1 + y_2 \\ 3y_1 - 2y_2 + y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

$y_1 = 1$

$4y_1 + y_2 = 6$

$4 + y_2 = 6$

$y_2 = 2$

$3y_1 - 2y_2 + y_3 = 4$

$3 - 4 + y_3 = 4$

$-1 + y_3 = 4$

$y_3 = 5$

By Eqn (4)

$UX = Y$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x + y + z \\ -y - 5z \\ -10z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

On Equating

$-10z = 5$

$\boxed{z = -\frac{1}{2}}$

$-y - 5z = 2$

$-y + \frac{5}{2} = 2$

$-y = 2 - \frac{5}{2}$

$-y = -\frac{1}{2}$

$\boxed{y = \frac{1}{2}}$

$x + y + z = 1$

$x + \frac{1}{2} - \frac{1}{2} = 1$

$x = 1$

Hence the Required Solution is

$x = 1$

$y = \frac{1}{2}$

$z = -\frac{1}{2}$