Taylor Series

Consider the equation

$$\frac{dy}{dx} = f(x, y) \quad \text{where } y = y_0 \text{ at } x = x_0$$

Find $y(x)$

x is any known value.

Taylor's Series :-

$$y(x) = y_0 + (x - x_0)(y_1)_0 + \frac{(x - x_0)^2}{2!}(y_2)_0 + \frac{(x - x_0)^3}{3!}(y_3)_0 + \frac{(x - x_0)^4}{4!}(y_4)_0 + \ldots$$

where

$$(y_1)_0 = \left(\frac{dy}{dx}\right)_{x = x_0}$$

$$(y_2)_0 = \left(\frac{d^2y}{dx^2}\right)_{x = x_0} + \ldots$$

Note! - • Agar direct value di hai x ki without giving h to $(x - x_0)$ ki terms mai generalised formula bana ke, value direct jis point pe nikalni hai ie x ki value dalte jaao.

• Agar h given hai toh aapko intervals mai value nikalni padhegi.

For eg - Pehle y_1 nikalo, phir y_2 nikalo jismai y_1 use hoga and so on...

Q:- (youtube) Find the value of $y(x)$ at point $x=0.1$ (2) and $x=0.2$ for the orderly differential equation $\frac{dy}{dx} = x^2y - 1$, $y(0) = 1$

Sol^n Given

$$\frac{dy}{dx} = x^2 y - 1$$

$$y_1 = x^2 y - 1$$

$$x_0 = 0 \quad y_0 = 1$$

$$y(x_0) = y_0 = 1$$

To find $y(0.1)$ and $y(0.2)$

By Taylor's Method

$$y(x) = y_0 + \frac{(x-x_0)}{1!}(y_1)_0 + \frac{(x-x_0)^2}{2!}(y_2)_0 + \frac{(x-x_0)^3}{3!}(y_3)_0 + \frac{(x-x_0)^4}{4!}(y_4)_0 + \dots \quad (1)$$

Since $y_1 = x^2 y - 1$

$(y_1)_0 = x_0^2 y_0 - 1$

$= 0 \cdot 1 - 1 = -1$

$\boxed{(y_1)_0 = -1}$

$\boxed{\frac{dy}{dx} = y_1}$

$\cancel{y_2 = \frac{d^2y}{dx^2}}$

$y_2 = \frac{dy_1}{dx} = x^2 y_1 + y \times 2x$

$= x^2 y_1 + 2xy$

$(y_2)_0 = x_0^2 \cancel{\times} (y_1)_0 + 2x_0 y_0$

$= 0 \cancel{\times} (-1) + 2 \times 0 \times 1$

$\boxed{(y_2)_0 = 0}$

$y_3 = \frac{d^2 y_2}{dx} = x^2 y_2 + 2xy_1 + 2xy_1 + 2y$

$(y_3)_0 = x_0^2 (y_2)_0 + 2x_0 (y_1)_0 + 2x_0 (y_1)_0 + 2y_0$

$= 0 \times 0 + 0 \times -1 + 0 + 1$

$\boxed{(y_3)_0 = 2}$

$$y_4 = \frac{dy_3}{dx}$$

$$= \frac{d}{dx}\left(x^2 y_2 + 4xy_1 + 2y\right)$$

$$= x^2 y_3 + y_2 \times 2x + 4xy_2 + 4y_1 + 2y_1$$

$$= x^2 y_3 + 2xy_2 + 4xy_2 + 6y_1$$

$$= x^2 y_3 + 6xy_2 + 6y_1$$

$$(y_4)_0 = x_0^2 (y_3)_0 + 6x_0 (y_2)_0 + 6(y_1)_0$$

$$= 0 + 0 + -1 \times 6$$

$$\boxed{(y_4)_0 = -6}$$

Putting these values into (1)

$$y(x) = 1 + (x-0)(-1) + \frac{(x-0)^2}{2!}(0) + \frac{(x-0)^3}{3!}(2) + \frac{(x-0)^4}{4!}(-6)$$

$$y(x) = 1 - x + \frac{2x^3}{6} - \frac{6x^4}{24} + \ldots$$

$$\boxed{y(x) = 1 - x + \frac{x^3}{3} - \frac{x^4}{4} + \ldots}$$

(Solution of given differ

$$y(0.1) = 1 - 0.1 + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4}$$

$$= 1 - 0.1 + \frac{0.001}{3} - \frac{0.0001}{4}$$

$$= 1 - 0.1 + 0.00033 - 0.000025$$

$$= 0.900305$$

$$\boxed{y(0.1) = 0.9003} \text{ (approx)}$$

$$y(0.2) = 1 - 0.2 + \frac{(0.2)^3}{3} - \frac{(0.2)^4}{4}$$

$$= 1 - 0.2 + \frac{0.08}{3} - \frac{0.0016}{4}$$

$$= 1 - 0.2 + 0.00267 - 0.0004$$

$$\boxed{y(0.2) = 0.80227} \text{ (approx)}$$

eg-1 Pg 4.25 Solve $\frac{dy}{dx} = x+y$ using Taylor's series method. Start from $x=1, y=0$ and carry to $x=1.2$ with $h=0.1$. Compare the numerical solution obtained with the exact solution.

Sol^n: The given equation is

$$\frac{dy}{dx} = x+y$$

$$x_0 = 1 \qquad y_0 = 0$$

To find $y(1.1)$ and $y(1.2)$

Since $y_1 = x+y$

$(y_1)_0 = x_0 + y_0$

$= 1+0$

$\boxed{(y_1)_0 = 1}$

$y_2 = 1 + y_1$

$(y_2)_0 = 1+1$

$\boxed{(y_2)_0 = 2}$

$y_3 = y_2$

$\boxed{(y_3)_0 = 2}$

$y_4 = y_3$

$(y_4)_0 = (y_3)_0$

$\boxed{(y_4)_0 = 2}$

$y_5 = y_4$

$(y_5)_0 = (y_4)_0$

$\boxed{(y_5)_0 = 2}$

By Taylor Series

$$y(x) = y_0 + h(y_1)_0 + \frac{h^2}{2!}(y_2)_0 + \frac{h^3}{3!}(y_3)_0 + \frac{h^4}{4!}(y_4)_0 + \frac{h^5}{5!}(y_5)_0 + \cdots$$

$$= 0 + 0.1(1) + \frac{(0.1)^2}{2} \times 2 + \frac{(0.1)^3}{6} \times (2) + \frac{(0.1)^4}{24} \times 2 + \frac{(0.1)^5}{120} \times 2$$

$$= 0 + 0.1 + 0.01 + \frac{0.001}{3} + \frac{0.0001}{12} + \frac{0.00001}{60}$$

$$= 0 + 0.1 + 0.01 + 0.00033 + 0.0000008 + \cdots$$

$$= 0 + 0.1 + 0.01 + 0.00033 + 0.00000 + \cdots$$

$\boxed{y(1.1) = 0.11033}$ (approx)

Or

$\boxed{y(1.1) = 0.110}$ (approx)

Now $x_0 = 1$ $h = 0.1$ (5)

$x_1 = x_0 + h$
$= 1 + 0.1$
$\boxed{x_1 = 1.1}$

$\boxed{y_1 = 0.11}$

$y_1 = x + y$
$(y_1)_1 = x_1 + y_1$
$= 1.1 + 0.11$
$\boxed{(y_1)_1 = 1.21}$

$y_2 = 1 + y_1$
$(y_2)_1 = 1 + (y_1)_1$
$= 1 + 1.21 = 2.21$
$\boxed{(y_2)_1 = 2.21}$

$y_3 = 0 + y_2$
$(y_3)_1 = (y_2)_1$
$\boxed{(y_3)_1 = 2.21}$

$y_4 = y_3$
$(y_4)_1 = (y_3)_1$
$\boxed{(y_4)_1 = 2.21}$

$y_5 = y_4$
$(y_5)_1 = (y_4)_1$
$\boxed{(y_5)_1 = 2.21}$

Again Applying Taylor's Series, we have

$$y_2 = y_1 + h(y_1)_1 + \frac{h^2}{2!}(y_2)_1 + \frac{h^3}{3!}(y_3)_1 + \frac{h^4}{4!}(y_4)_1 + \frac{h^5}{5!}(y_5)_1 + \cdots$$

$$y_2 = 0.11 + (0.1)(1.11) + \frac{(0.1)^2}{2} \times 2.121 + \frac{(0.1)^3}{6} \times 2.121 + \frac{(0.1)^4}{24} \times 2.121 + \frac{(0.1)^5}{120} \times 2.121 + \cdots$$

$$y_2 = 0.11 + 0.111 + \frac{0.01}{2} \times 2.121 + \frac{0.001}{6} \times 2.121 + \frac{0.0001}{24} \times 2.121 + \cdots$$

$$= 0.11 + 0.111 + \frac{0.02121}{2} + \frac{0.002121}{6} + \frac{0.0002121}{24} + \cdots$$

$$= 0.11 + 0.111 + 0.01105 + 0.00037 + 0.00000$$

$$= 0.2424792$$

$$= 0.242 \text{ (approx)}$$

$$\boxed{y(1.2) = 0.242 \text{ (approx)}}$$

To find exact value of y at $x = 1.2$

The given equation is

$$\frac{dy}{dx} = x + y$$

$$\boxed{\frac{dy}{dx} - y = x}$$

which is linear differential equation of form

$\frac{dy}{dx} \pm Py = Q$ where

$P = -1 \quad Q = x$

$I.f = e^{\int P dx}$

$= e^{\int -dx}$

$\boxed{I.f = e^{-x}}$

Thus, the solution is

$y(I.f) = \int Q(I.f)\,dx + c$

$ye^{-x} = \int x e^{-x}\,dx + c$

$ye^{-x} = -xe^{-x} + \int e^{-x}\,dx + c$

$\boxed{ye^{-x} = -xe^{-x} - e^{-x} + c}$

or

$\frac{y}{e^{x}} = -xe^{-x} - e^{-x} + c$

$y = -xe^{-x}.e^{x} - e^{-x}.e^{x} + ce^{x}$

$= -xe^{0} - e^{0} + ce^{x}$

$\boxed{y = -x - 1 + ce^{x}}$

Initially at $x = 1$ $y = 0$

Putting these value in

$y = -x - 1 + ce^{x}$

$0 = -1 - 1 + ce^{1}$

$2 = ce$

$ce = 2$

$$\boxed{c = \frac{2}{e}}$$

Putting this value of c in

$$y = -x - 1 + ce^{x}$$

$$y = -x - 1 + \frac{2}{e} e^{x}$$

Putting $x = 1.2$ in

$$y(1.2) = -1.2 - 1 + \frac{2}{e} \cdot e^{1.2}$$

$$y(1.2) = -1.2 - 1 + 2e^{0.2}$$

$$= -2.2 + 2e^{0.2}$$

$$= -2.2 + 2 \times 1.22140$$

$$= -2.2 + 2.4428$$

$$\boxed{y(1.2) = 0.2428}$$

Thus, we have $y(1.2) = 0.2428$ by exact solution and $y(1.2) = 0.24247$ by Taylor's series method

Pg 4.26 Using Taylor's Method, obtain the approximate value of y at $x=0.2$ for the differential equation $\frac{dy}{dx} = 2y+3e^x$, $y(0)=0$ and compare the Numerical solution obtained with exact solution. (7)

Soln Given $\frac{dy}{dx} = 2y+3e^x$

$x_0 = 0 \quad y_0 = 0$

To find $y(0.2) = ?$

So $\frac{dy}{dx} = 2y+3e^x$

$y_1 = 2y+3e^x$

By Taylor's Method

$$y(x) = y_0 + (x-x_0)(y_1)_0 + \frac{(x-x_0)^2}{2!}(y_2)_0 + \frac{(x-x_0)^3}{3!}(y_3)_0 + \frac{(x-x_0)^4}{4!}(y_4)_0 + \cdots \quad (1)$$

$y_1 = 2y+3e^x$

$(y_1)_0 = 2y_0+3e^0$

$= 0+3$

$\boxed{(y_1)_0 = 3}$

$\boxed{e^0 = 1}$

$y_2 = 2y_1 + 3e^x$

$(y_2)_0 = 2(y_1)_0 + 3e^{x_0}$

$= 2\times 3 + 3 = 6 + 3 = 9$ $\boxed{(y_2)_0 = 9}$

$y_3 = 2y_2 + 3e^x$

$(y_3)_0 = 2(y_2)_0 + 3e^{x_0}$

$= 2\times 9 + 3\times 1 = 18 + 3 = 21$

$\boxed{(y_3)_0 = 21}$

$y_4 = 2y_3 + 3e^x$

$(y_4)_0 = 2(y_3)_0 + 3e^{x_0}$

$= 2\times 21 + 3\times 1$

$= 42 + 3 = 45$

$\boxed{(y_4)_0 = 45}$

By Taylor's series, we have

$$y(x) = y_0 + (x - x_0)(y_1)_0 + \frac{(x-x_0)^2}{2!}(y_2)_0 + \frac{(x-x_0)^3}{3!}(y_3)_0 + \frac{(x-x_0)^4}{4!}(y_4)_0 + \cdots$$

$$y(x) = 0 + (x-0)3 + \frac{(x-0)^2}{2}\times 9 + \frac{(x-0)^3}{6}\times 21 + \frac{(x-0)^4}{24}\times 45 + \cdots$$

$$y(x) = 0 + 3x + \frac{9x^2}{2} + \frac{21x^3}{6} + \frac{45}{24}x^4 + \cdots \quad (8)$$

$$= 0 + 3x + \frac{9x^2}{2} + \frac{7}{2}x^3 + \frac{15}{8}x^4 + \cdots$$

$$\boxed{y(x) = 0 + 3x + \frac{9}{2}x^2 + \frac{7}{2}x^3 + \frac{15}{8}x^4 + \cdots} \quad (2)$$

This is the approximate solution of Ordinary Differential Equation

Putting $x = 0.2$ in (2)

$$y(0.2) = 3(0.2) + \frac{9}{2}(0.2)^2 + \frac{7}{2}(0.2)^3 + \frac{15}{8}(0.2)^4 + \cdots$$

$$= 0.6 + \frac{9}{2} \times 0.04 + \frac{7}{2} \times 0.008 + \frac{15}{8} \times 0.0016 + \cdots$$

$$= 0.6 + \frac{0.36}{2} + \frac{0.056}{2} + \frac{0.024}{8} + \cdots$$

$$= 0.6 + 0.18 + 0.028 + 0.003 + \cdots$$

$$\boxed{y(0.2) = 0.811}$$

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To find exact value of y at $x = 0.2$

$$\frac{dy}{dx} = 2y + 3e^x$$

$$\frac{dy}{dx} - 2y = 3e^x$$

which is linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

where $P = -2$ $Q = 3e^x$

$$\text{I.F (Integrating Factor)} = e^{\int P dx}$$

$$= e^{\int -2 dx}$$

$$= e^{-2\int dx}$$

$$\boxed{IF = e^{-2x}}$$

Thus the solution is

$$y(\text{I.F}) = \int Q(\text{I.F})dx + C$$

$$ye^{-2x} = \int 3e^x \cdot e^{-2x} dx + C$$

$$= 3\int e^{-x} dx + C$$

$$\boxed{ye^{-2x} = -3e^{-x} + C}$$

Scanned with CamSc

$$\frac{y}{e^{2x}} = -3e^{-x} + c$$

$$y = e^{2x}(-3e^{-x} + c)$$

$$\boxed{y = -3e^{x} + ce^{2x}} \quad \text{--- (3)}$$

Putting $x = 0$ $y = 0$ in (3)

$$0 = -3 + c$$

$$\boxed{c = 3}$$

Putting this value in (3)

$$y = -3e^{x} + 3e^{2x}$$

Hence the exact solution of equation (1) is,

$$y = 3e^{2x} - 3e^{x}$$

$$y = 3(e^{2x} - e^{x}) \quad \text{--- (4)}$$

Put $x = 0.2$ in (4)

$$y = 3(e^{2(0.2)} - e^{0.2})$$

$$y = 3(e^{0.4} - e^{0.2})$$

$$y = 3(1.4918 - 1.2214)$$

$$y = 3(0.2704)$$

$$\boxed{y = 0.8112}$$

Hence, we see that Numerical solution approximates to exact value upto 3 decimal places

Q- youtube Solve by Taylor series method (10)

$\frac{dy}{dx} = 3x+y^2$; $y(0) = 1$. Compute $y(0.1)$

Sol^n

$\frac{dy}{dx} = (y_1) = 3x+y^2$

Given ⇒ $y_1 = 3x+y^2$ Here $y_1 = \frac{dy}{dx}$

$x_0 = 0 \quad y_0 = 1$ Initial value daalegi for calculating y_1 and so on...

To find $y(0.1) = ?$

By Taylor's Method

$$y(x) = y_0 + (x-x_0)(y_1)_0 + \frac{(x-x_0)^2}{2!}(y_2)_0 + \frac{(x-x_0)^3}{3!}(y_3)_0 + \frac{(x-x_0)^4}{4!}(y_4)_0 + \dots \quad (1)$$

$y_1 = 3x+y^2$

$(y_1)_0 = 3x_0 + y_0^2$

$= 0+1 = 1 \quad \boxed{(y_1)_0 = 1}$

$y_2 = \frac{dy_1}{dx} = 3+2yy_1$

$(y_2)_0 = 3+2y_0(y_1)_0$

$= 3+2(1)(1)$

$\boxed{(y_2)_0 = 5}$

$$y_3 = \frac{dy_2}{dx} = 2yy_2 + 2y_1y_1$$
$$= 2yy_2 + 2(y_1)^2$$
$$(y_3)_0 = 2y_0(y_2)_0 + 2((y_1)_0)^2$$
$$= 2\times1\times5 + 2(1)^2$$
$$= 10+2$$
$$\boxed{(y_3)_0 = 12}$$

$$y_4 = 2yy_3 + 2y_2y_1 + 4y_1y_2$$
$$= 2yy_3 + 6y_1y_2$$
$$(y_4)_0 = 2y_0(y_3)_0 + 6(y_1)_0(y_2)_0$$
$$= 2\times1\times12 + 6\times1\times5$$
$$24 + 30$$
$$\boxed{(y_4)_0 = 54}$$

Note:- Atleast calculate upto y_4 when you get 4 non-zeroes

Putting the values of y_0, $(y_1)_0$, $(y_2)_0$, $(y_3)_0$, $(y_4)_0$ into (1) Taylor's Formula

$$y(x) = 1 + (x-0)\times 1 + \frac{(x-0)^2}{2!}\times 5 + \frac{(x-0)^3}{3!}\times 12 + \frac{(x-0)^4}{4!}\times 54 + \ldots$$

$$= 1 + x + \frac{5x^2}{2} + \frac{x^3}{\cancel{6}}\times \cancel{12}^{\,2} + \frac{x^4}{\cancel{24}_{\,4}}\times \cancel{54}^{\,9} + \ldots$$

$$\boxed{y(x) = 1 + x + \frac{5x^2}{2} + 2x^3 + \frac{9}{4}x^4 + \ldots}$$

This is the approximate solution of given ordinary differential equation.

Now calculate $y(0.1)$

$$y(0.1) = 1 + 0.1 + \frac{5}{2}(0.1)^2 + 2(0.1)^3 + \frac{9}{4}(0.1)^4 + \ldots$$

$$= 1 + 0.1 + \frac{5}{2}\times 0.01 + 2(0.001) + \frac{9}{4}\times 0.0001 + \ldots$$

$$= 1 + 0.1 + 0.025 + 0.002 + 0.000225 + \ldots$$

$$= 1.127225$$

$$\boxed{y(0.1) = 1.1272}$$