3 Solution of Simultaneous Linear Algebraic Equations

3.1. Introduction

In this chapter, we shall study the methods to solve the linear equations containing more than two variables.

Let there be m first degree equations with n unknowns $x_1, x_2, x_3, \ldots, x_n$, as follows :

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1j}x_j + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2j}x_j + \ldots + a_{2n}x_n &= b_2 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\
a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{ij}x_j + \ldots + a_{in}x_n &= b_i \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mj}x_j + \ldots + a_{mn}x_n &= b_m
\end{aligned}$$

Now these equations can be written in matrix form as given below :

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1j} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2j} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{i1} & a_{i2} & \ldots & a_{ij} & \ldots & a_{in} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mj} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ \ldots \\ x_i \\ \ldots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ \ldots \\ b_i \\ \ldots \\ b_m \end{bmatrix}$$

These matrices are denoted by A, X and B respectively. We have to find the solutions of the system of given equations *i.e.*, the values of $x_1, x_2, x_3, \ldots, x_n$ which satisfy the given equations. The system of the given equations is said to be ***homogeneous*** if all the coefficients of variables *i.e.*, b_i $(i = 1, 2, 3, \ldots, m)$ vanish, otherwise it is called ***non-homogeneous system***. According to the present scope of syllabus we shall restrict ourselves to the case when the number of rows is equal to the number of columns *i.e.*, $m = n$.

3.2. Different Methods of Obtaining the Solutions

If the given system of equations is AX = B

where $A = [a_{ij}]_{m \times n}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_m \end{bmatrix}_{m \times 1}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_m \end{bmatrix}_{m \times 1}$

then the following methods are to be noted :

(*i*) If B = 0, det. A = 0, then there exist infinite number of solutions besides trivial solution X = 0.

(*ii*) If B = 0 and det. A ≠ 0, then the system has only unique trivial solution, X = 0.

(*iii*) If B ≠ 0 and det. A ≠ 0, then the system has a unique solution.

There are number of methods of solving the system of linear equations. However, in this chapter we shall study the following methods :

(*i*) **Gauss Elimination Method**

(*ii*) **Triangularisation Method (LU Decomposition Method)**

(*iii*) **Gauss Seidel Method**

3.3. Gauss Elimination Method

Let the system of linear equations be

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1 \quad \text{...(1)}$$

$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = b_2 \quad \text{...(2)}$$

$$a_{31} x_1 + a_{32} x_2 + a_{33} x_3 = b_3 \quad \text{...(3)}$$

To solve these equations, we first eliminate x_1 from equations (1) and (2) and then from (1) and (3). From these two obtained equations, eliminate x_2 obtaining the value of x_3. After that by back substitutions, x_2 and x_1 can be found out.

For the elimination of x_1 from (1) and (2) multiply (1) by r_1, where

$$r_1 = \frac{a_{21}}{a_{11}} = \frac{\text{coefficient of } x_1 \text{ in equation (2)}}{\text{coefficient of } x_1 \text{ in equation (1)}}$$

and subtract the equation so obtained from equation (2)

i.e., $$\left(a_{21} - \frac{a_{21}}{a_{11}} a_{11}\right) x_1 + \left(a_{22} - \frac{a_{21}}{a_{11}} a_{12}\right) x_2 + \left(a_{23} - \frac{a_{21}}{a_{11}} a_{13}\right) x_3 = b_2 - \frac{a_{21}}{a_{11}} b_1$$

i.e., $$(a_{22} - r_1 a_{12}) x_2 + (a_{23} - r_1 a_{13}) x_3 = b_2 - r_1 b_1 \quad \text{...(4)}$$

Similarly, for eliminating x_1 from equations (1) and (3), we multiply equation (1) by r_2

where $$r_2 = \frac{a_{31}}{a_{11}} = \frac{\text{coefficient of } x_1 \text{ in (3)}}{\text{coefficient of } x_1 \text{ in (1)}}$$

and subtract the equation so obtained from (3) to get

$$\left(a_{31} - \frac{a_{31}}{a_{11}} a_{11}\right) x_1 + \left(a_{32} - \frac{a_{31}}{a_{11}} a_{12}\right) x_2 + \left(a_{33} - \frac{a_{31}}{a_{11}} a_{13}\right) x_3 = b_3 - \frac{a_{31}}{a_{11}} b_1$$

i.e., $$(a_{32} - r_2 a_{12}) x_2 + (a_{33} - r_2 a_{13}) x_3 = b_3 - r_2 b_1 \quad \text{...(5)}$$

where $$a_{22} - r_1 a_{12} = p_{22}, \quad b_2 - r_1 b_1 = c_2, \quad a_{23} - r_1 a_3 = p_{23}$$

$$a_{32} - r_2 a_{12} = p_{32}, \quad a_{33} - r_2 a_{13} = p_{33}, \quad b_3 - r_2 b_1 = c_3$$

Now equations (4) and (5) can be written as

$$p_{22} x_2 + p_{23} x_3 = c_2 \quad \text{...(6)}$$

$$p_{32} x_2 + p_{33} x_3 = c_3 \quad \text{...(7)}$$

Now to eliminate x_2 from these two equations, multiply equation (6) by r_3, where

$$r_3 = \frac{p_{32}}{p_{22}} = \frac{\text{coefficient of } x_2 \text{ in (7)}}{\text{coefficient of } x_2 \text{ in (6)}}$$

and subtract the equation so obtained from (7), to get

$$\left(p_{32} - \frac{p_{32}}{p_{22}} p_{22}\right) x_2 + \left(p_{33} - \frac{p_{32}}{p_{22}} p_{23}\right) x_3 = c_3 - \frac{p_{32}}{p_{22}} c_2$$

or $$(p_{33} - r_3 p_{23}) x_3 = c_3 - r_3 c_2$$

or $$x_3 = \frac{c_3 - r_3 c_2}{p_{33} - r_3 p_{23}}$$

Putting the value of x_3 in (6) or (7), x_2 can be obtained.

Substituting these values of x_2 and x_3 in (1) or (2) or (3), x_1 can also be found out.

3.3.1. Algorithm of Gauss Elimination Method (For 3 equations)

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = a_{14}$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = a_{24}$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = a_{34}$$

1. *for* $i = 1$ to 3 and $j = 1$ to 4 in steps of 1 *do* Read a_{ij} *endfor*.
2. *for* $i = 2$ to 3 in steps of 1 *do*
3. $u \leftarrow a_{i1} / a_{11}$
4. *for* $j = 1$ to 4 in steps of 1 *do*

5. $a_{ij} \leftarrow a_{ij} - ua_{1j}$ *endfor*
6. $i = 3, \quad u \leftarrow a_{i2} / a_{22}$
7. for $j = 2$ to 4 in steps of 1 do.
8. $a_{ij} \leftarrow a_{ij} - ua_{2j}$ *endfor*
9. $x_3 \leftarrow a_{34} / a_{33}$
10. for $i = (n - 1)$ to 1 in steps of $- 1$ do
11. sum $\leftarrow 0$.
12. for $j = (i + 1)$ to 4 in steps of 1 do
13. sum $\leftarrow$ sum $\leftarrow a_{ij}x_j$ endfor
14. $x_i = (a_{i4} - \text{sum}) / a_{ii}$ endfor
15. Stop

The following solved examples will illustrate the method.

SOLVED EXAMPLES

Example 1. *Solve the following equations by the Gauss elimination method.*

$$2x + y + 4z = 12$$
$$4x + 11y - z = 33$$
$$8x - 3y + 2z = 20.$$

Solution. The given system of equations is

$$2x + y + 4z = 12 \quad \text{...(1)}$$
$$4x + 11y - z = 33 \quad \text{...(2)}$$
$$8x - 3y + 2z = 20 \quad \text{...(3)}$$

First eliminate x from (1) and (2) :

Here $$r_1 = \frac{\text{coefficient of } x \text{ in eq. (2)}}{\text{coefficient of } x \text{ in eq. (1)}} = \frac{4}{2} = 2.$$

Multiplying eq. (1) by 2 and subtracting from eq. (2), we have

$$(4x + 11y - z) - 2(2x + y + 4z) = 33 - 2(12)$$

or $$9y - 9z = 9$$

or $$y - z = 1 \quad \text{...(4)}$$

Now eliminate x from (1) and (3) :

$$r_2 = \frac{\text{coefficient of } x \text{ in eq. (3)}}{\text{coefficient of } x \text{ in eq. (1)}} = \frac{8}{2} = 4.$$

Multiplying eq. (1) by 4 and subtracting from eq. (3), we have

$$(8x - 3y + 2z) - 4(2x + y + 4z) = 20 - 4(12)$$

or $$-7y - 14z = -28$$

or $$y + 2z = 4 \qquad \ldots(5)$$

Now eliminate y from (4) and (5):

$$r_3 = \frac{\text{coefficient of } y \text{ in eq. (5)}}{\text{coefficient of } y \text{ in eq. (4)}} = \frac{1}{1} = 1.$$

Subtracting eq. (4) from eq. (5), we get

$$(y + 2z) - (y - z) = 4 - 1$$

or $$3z = 3$$

$\Rightarrow$ $$z = 1$$

Putting the value of z in (5), we get $y = 2$

and putting the values of y and z in (1), we get $x = 3$

Hence the solution is $x = 3, y = 2, z = 1$.

Aliter. This process can be carried out in the following tabular form called *Gauss Scheme* :

Table

Row No.	*Operation*	x	y	z	c	*S (sum)*	*Elimination of*
(1)	–	2	1	4	12	19	
(2)	$r_1 = \frac{4}{2} = 2$	4	11	–1	33	47	x from eq. (1) and (2)
(3)	$-2 \times (1)$	–4	–2	–8	–24	–38	
(4)	(2) + (3)	0	9	–9	9	9	
(5)	(4) ÷ 9	0	1	–1	1	1	
(6)	–	2	1	4	12	19	
(7)	$r_2 = \frac{8}{2} = 4$	8	–3	2	20	27	
(8)	$-4 \times (6)$	–8	–4	–16	–48	–76	x from eq. (1) and (3)
(9)	(7) + (8)	0	–7	–14	–28	–49	
(10)	9 ÷ (–7)	0	1	2	4	7	
(11)			1	–1	1	1	
(12)	$r_3 = \frac{1}{1} = 1$		1	2	4	7	
(13)	1 × (11)		1	1	–1	–1	y from eq. (5) and (10)
(14)	(12) + (13)		0	3	3	6	
(15)	(14) ÷ 3			1	1	2	

From row no. 15, $z = 1$

From row no. 5, $y - z = 1$

$\therefore$ $y = 1 + z = 1 + 1 = 2$

From row no. 1, $2x + y + 4z = 12$

or $2x + 2 + 4 = 12$

or $2x = 6 \Rightarrow x = 3.$

Hence the required solution is $x = 3, y = 2, z = 1$.

Note. *The column 7 (S or sum) shows the sum of the coefficients and the constants. This column is meant for checking the computation work.*

Example 2. *Solve the following equations by Gauss-Elimination method :*

$$4x_1 + x_2 + 3x_3 = 11$$
$$3x_1 + 4x_2 + 2x_3 = 11$$
$$2x_1 + 3x_2 + x_3 = 7$$

Solution. The given system of equations is

$$2x_1 + 3x_2 + x_3 = 7 \quad \text{...(1)}$$
$$3x_1 + 4x_2 + 2x_3 = 11 \quad \text{...(2)}$$
$$4x_1 + x_2 + 3x_3 = 11 \quad \text{...(3)}$$

First eliminate x_1 from (1) and (2) :

Here $r_1 = \dfrac{\text{coefficient of } x_1 \text{ in eq.(2)}}{\text{coefficient of } x_1 \text{ in eq. (1)}} = \dfrac{3}{2}$

Multiplying (1) by $\dfrac{3}{2}$ and subtracting from (2), we have

$$\left(3 - 2\cdot\frac{3}{2}\right)x_1 + \left(4 - 3\cdot\frac{3}{2}\right)x_2 + \left(2 - \frac{3}{2}\cdot 1\right)x_3 = 11 - \frac{3}{2}\cdot 7$$

or $$-\frac{1}{2}x_2 + \frac{1}{2}x_3 = \frac{1}{2}$$

or $$x_2 - x_3 = -1 \quad \text{...(4)}$$

Now eliminate x_1 from (1) and (3) :

$$r_2 = \frac{\text{coefficient of } x \text{ in eq. (3)}}{\text{coefficient of } x \text{ in eq. (1)}} = \frac{4}{2} = 2$$

Multiplying (1) by 2 and subtracting from (3), we have

$$(4 - 2.2)\,x_1 + (1 - 3.2)x_2 + (3 - 1.2)\,x_3 = 11 - 2.7$$

or
$$-5x_2 + x_3 = -3$$

$\Rightarrow$
$$5x_2 - x_3 = 3 \quad \text{...(5)}$$

Now eliminate x_3 from (4) and (5) :

$$r_3 = \frac{\text{coefficient of } x_3 \text{ in (5)}}{\text{coefficient of } x_3 \text{ in (4)}} = \frac{-1}{-1} = 1$$

Multiplying (4) by 1 and subtracting from (5), we have

$$4x_2 = 4 \quad \Rightarrow \quad x_2 = 1$$

Substituting the value of x_2 in (4), we get

$$1 - x_3 = -1 \quad \Rightarrow \quad x_3 = 2$$

Again substituting the values of x_2 and x_3 in (1), we get

$$2x_1 + 3.1 + 2 = 7$$

or
$$2x_1 = 2 \quad \Rightarrow \quad x_1 = 1$$

Hence, the required solution is $x_1 = 1,\ x_2 = 1$ and $x_3 = 2$.

3.4. Pivoting

In Gauss Elimination method, the element a_{kk} in every equation is called the pivot element *i.e.*, the elements $a_{11}, a_{22}, a_{33}, \ldots\ldots, a_{nn}$ are called **pivot elements**.

The equation, the pivot element of which is eliminated from all other equations, is called *pivot equation*.

It should be noted that the pivot element of any pivot equation is non-zoro. In elimination process, if any one of the pivot element $a_{11}, a_{22}, a_{33}, \ldots\ldots, a_{nn}$ becomes zero or very small compared to other elements in that row, then we attempt to rearrange or reorder the remaining rows to obtain a non-zoro pivot. This process is called pivoting. Thus, for solving a system of linear equations by the concept of pivoting we have to rearrange the equations to improve the accuracy in the result, even if the pivot element in any row is zero.

There are mainly two types of pivoting :

(*i*) *Partial pivoting*

(*ii*) *Complete pivoting*

Rearranging of the equations can be made either by partial pivoting or by complete pivoting.

3.4.1. Partial Pivoting. In first stage of elimination, we select the largest element in magnitude in the first column of the equations and this element is brought as the first pivot by interchanging the first equation with the equation having the largest element in magnitude. In second elimination stage, the second column is searched for the largest element in magnitude among the $n - 1$ elements leaving the first element and this element is brought as the second pivot by an interchange of second equation with the equation having the largest element in magnitude. This process is continued till the $(n-1)$ unknowns are eliminated and the system of equations results in a triangular form.

3.4.2. Complete Pivoting. In this, we search the coefficient matrix for the largest element in magnitude and bring it as the first pivot. This requires not only an interchange of equation but also an interchange of the position of the variables. A complete pivoting gives the better numerical stability but it is not generally used as it is difficult and requires a lots of overheads.

3.5. Ill-Conditioned Equations

A linear system is said to be *ill-conditioned* if small changes in the coefficients of the equations result in large changes in the values of the unknowns. On the contrary, a system is well-conditioned if small changes in the coefficients of the system also produce small changes in the solution. We come across many ill-conditioned systems in practical applications. Ill-conditioning of a system is usually expected when the determinant of the coefficient matrix is small. The coefficient matrix of an ill-conditioned system is called an ill-conditioned matrix.

While solving simultaneous equations, we also come across two forms of instabilities: *Inherent* and *Induced*. Inherent instability of a system is a property of the given problem and occurs due to the problem being ill-conditioned. It can be avoided by reformulation of the problem suitably. Induced instability occurs due to the incorrect choice of method.

3.6. Refinement of Solution

Consider the system of equations

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \quad ...(1)$$

Let x', y', z' be an approximate solution. Substituting these value on the left-hand sides, we get new values of d_1, d_2, d_3 as d_1', d_2', d_3' so that the new system of equations is

$$\left.\begin{aligned} a_1x' + b_1y' + c_1z' &= d_1' \\ a_2x' + b_2y' + c_2z' &= d_2' \\ a_3x' + b_3y' + c_3z' &= d_3' \end{aligned}\right\} \quad ...(2)$$

Subtracting each equation in (2) from the corresponding equation in (1), we obtain

$$\left.\begin{aligned} a_1X_e + b_1Y_e + c_1Z_e &= l_1 \\ a_2X_e + b_2Y_e + c_2Z_e &= l_2 \\ a_3X_e + b_3Y_e + c_3Z_e &= l_3 \end{aligned}\right\} \quad ...(3)$$

where $X_e = x - x'$, $Y_e = y - y'$, $Z_e = z - z'$ and $l_i = d_i - d_i'$

Now, solve the system (3) for x, y, z by taking $x = x' + X_e$, $y = y' + Y_e$ and $z = z' + Z_e$ which will be better approximations for x, y, z. We can repeat the procedure for improving the accuracy. This method is also called *Iterative method*.

Example 3. *Establish whether the system*

$1{\cdot}01x + 2y = 2{\cdot}01; \quad x + 2y = 2$ *is well-condition or not.*

Solution. The solution of given system is $x = 1$ and $y = 0{\cdot}5$

Now, consider the system $\quad x + 2{\cdot}01y = 2{\cdot}04$

$$x + 2y = 2$$

which has the solution $x = -6$ and $y = 4$

Hence the system is ill-conditioned.

Example 4. *An approximate solution of the system*

$2x + 2y - z = 6; \quad x + y + 2z = 8; \quad -x + 3y + 2z = 4$ *is given by* $x = 2{\cdot}8,\ y = 1,\ z = 1{\cdot}8.$

Using the above iterative method, improve this solution.

Solution. Substituting the approximate values

$x' = 2{\cdot}8,\ y' = 1,\ z' = 1{\cdot}8$ in the given equations, we get

$$\left.\begin{aligned} 2(2{\cdot}8) + 2(1) - 1{\cdot}8 &= 5{\cdot}8 \\ 2.8 + 1 + 2(1{\cdot}8) &= 7{\cdot}4 \\ -2{\cdot}8 + 3(1) + 2(1{\cdot}8) &= 3{\cdot}8 \end{aligned}\right\} \quad \text{...(1)}$$

Subtracting each equation in (1) from the corresponding given equations, we obtain

$$\left.\begin{aligned} 2X_e + 2Y_e - Z_e &= 0{\cdot}2 \\ X_e + Y_e + 2Z_e &= 0{\cdot}6 \\ -X_e + 3Y_e + 2Z_e &= 0{\cdot}2 \end{aligned}\right\} \quad \text{...(2)}$$

where $X_e = x - 2{\cdot}8, \quad Y_e = y - 1, \quad Z_e = 1{\cdot}8$

Solving equations in (2), we get $X_e = 0{\cdot}2,\ Y_e = 0,\ Z_e = 0{\cdot}2$. This gives the better solution $x = 3,\ y = 1,\ z = 2$, which is the exact solution.

EXERCISE 3.1

***Solve the following equations by Gauss Elimination method* [Q1 – 10] :**

1. $2x + 3y - z = 5$

$4x + 4y - 3z = 3$

$2x - 3y + 2z = 2$

2. $5x - y - 2z = 142{\cdot}2$

$x - 3y - z = -30$

$2x - y - 3z = 5$

3. $3x - 5y + z = 6$

$2x + 4y + z = 1$

$x + 2y + 2z = -2$

4. $x + y + z = 10$

$2x + y + 2z = 17$

$3x + 2y + z = 17$

5. $2x_1 - 3x_2 + x_3 = -0{\cdot}8$

$3x_1 + 4x_2 - 2x_3 = 14{\cdot}2$

$x_1 + 2x_2 + 3x_3 = 2{\cdot}0$

6. $2x_1 + 4x_2 + 2x_3 = 15$

$2x_1 + x_2 + 2x_3 = -5$

$4x_1 + x_2 - 2x_3 = 0$

7. $4x + 3y + 2z = 8$

$x + y + 2z = 7$

$3x + 2y + 4z = 13$

8. $10x + 3y + z = 67$

$2x + 5y + 2z = 10$

$3x - 2y + 5z = 40$

9. $2x_1 + 4x_2 + x_3 = 3$

$3x_1 + 2x_2 - 2x_3 = -2$

$x_1 - x_2 + x_3 = 6$

10. $2x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 = 10$

$\frac{1}{2}x_1 + \frac{1}{3}x_2 + \frac{1}{4}x_3 = 0$

$\frac{1}{3}x_1 + \frac{1}{4}x_2 + \frac{1}{5}x_3 = 0$

11. Establish whether the system of equations

$$10x + 8y + 9z + 6w = 33$$
$$6x + 7y + 5z + 5w = 23$$
$$8x + 10y + 7z + 7w = 32$$
$$9x + 7y + 10z + 5w = 31$$

is well-conditioned or not ?

12. An approximate solution of the equations

$$x + 4y + 7z = 5; \quad 2x + 5y + 8z = 7; \quad 3x + 6y + 9{\cdot}1z = 9{\cdot}1$$

is given by $x = 1{\cdot}8$, $y = -1{\cdot}2$, $z = 1$

Improve this solution by using the Iterative method.

ANSWERS

1. (1, 2, 3) **2.** (39·4, 16·8, 19·0) **3.** $\left(2, -\frac{1}{3}, -\frac{5}{3}\right)$

4. (2, 3, 5) **5.** (2·2, 1·4, – 1) **6.** $\left(-\frac{55}{18}, \frac{20}{3}, -\frac{25}{9}\right)$

7. (– 1, 2, 3) **8.** (7, – 2, 3) **9.** (2, – 1, 3)

10. (9, – 36, 30). **11.** ill-conditioned **12.** $x = 2$, $y = -1$, $z = 1$

3.7. Triangularization Method Or LU Decomposition Method

Consider a system of n linear equations in n unknowns. In matrix notation this system of equations can be written as AX = B where A is the coefficient matrix, B is the matrix of constants and X is the matrix of unknowns.

In triangularization method, the square matrix A (coefficient matrix) is factorized or decomposed into the product of two matrices L and U, where L is a lower triangular matrix and U is an upper triangular matrix.