

DS 102 Discussion 2

Monday, February 3, 2020

1. **Distribution of p -values.** Understanding p -values and their distribution can help develop intuition about how to control FDR. We want to test two hypotheses. Under H_0 the data is generated by the null distribution $\mathcal{N}(0, 1)$, and under the alternative H_1 , the data is generated by a distribution $\mathcal{N}(\mu, 1)$ where $\mu > 0$.

Suppose you have a test $T(X)$, where X is your data. Recall that the p -value P of the test is:

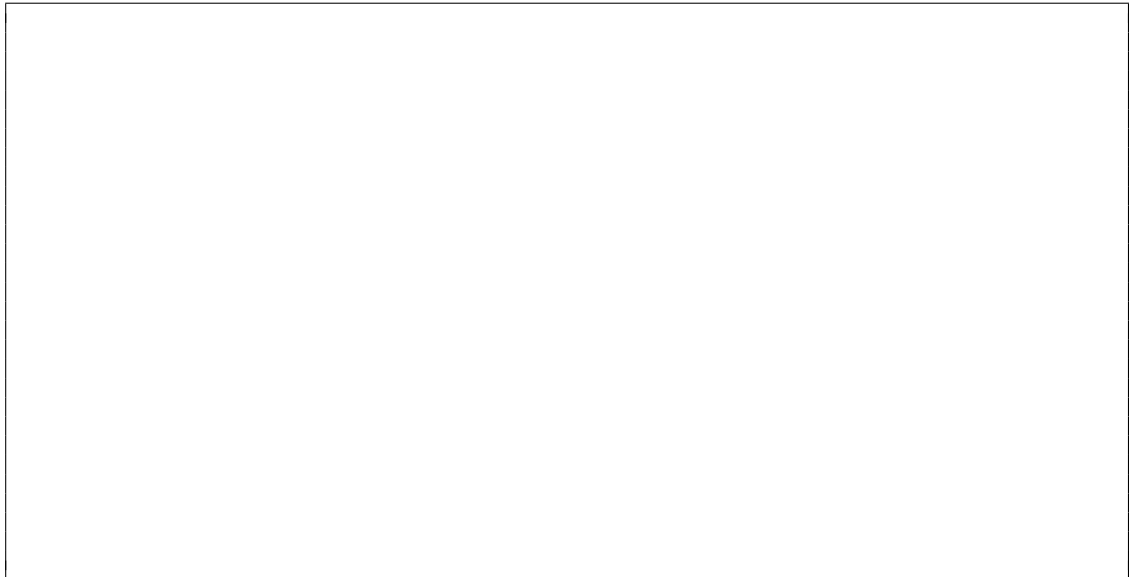
$$P(X) = \mathbb{P}(T > t | T(X) = t)$$

where \mathbb{P} denotes the probability with respect to data drawn from the null distribution. This is the probability that, under the null distribution, you see a result at least as extreme as from your data. Remember that a p -value is a random variable, since it is a function of your data.

- (a) Let $F(t)$ be the cumulative density function (CDF) of the test statistic $T(X)$ under the null distribution. Recall that F is a monotonically increasing function (if that's not clear, draw a picture to see why). Suppose that F is invertible, meaning that there exists a function F^{-1} such that:

$$F^{-1}(F(t)) = t.$$

What is the CDF of the p -value? What can we conclude about p -values under the null?



- (b) What part of the proof above fails if data do not come from the null distribution?

- (c) Suppose you have two independent p -values p_1 and p_2 . If on both hypotheses you choose a $0 < \alpha < 1$ and use the naive decision rule:

$$\delta(p; \alpha) = \begin{cases} \text{reject null} & p \leq \alpha \\ \text{accept null} & p > \alpha \end{cases}$$

what is the probability of making at least one false discovery? This probability is also known as the family-wise error rate (FWER).

- (d) Does this decision rule keep the probability of a false discovery below α ?

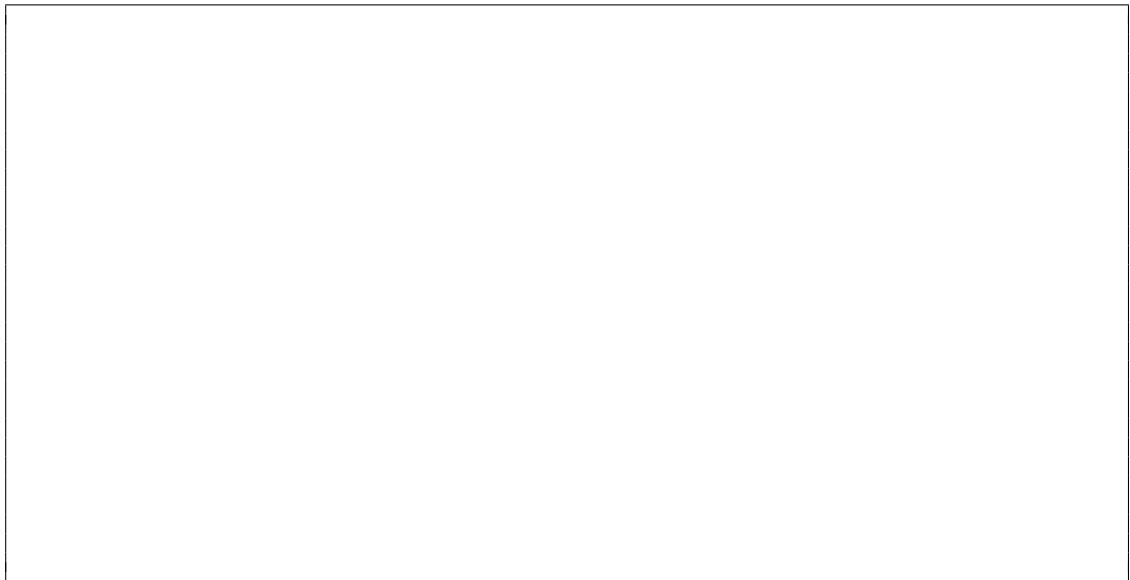
2. False discoveries in genome-wide association studies (GWAS).

In “classical” genetics, researchers identify one candidate mutation they believe is associated with a trait (*e.g.* a disease), then conduct an experiment to test that hypothesis. In contrast, the recent explosion of genomic data has led to the rise of genome-wide association studies (GWAS), where researchers search for associations between a trait and mutations across the entire genome. For example, in one of the first GWAS (Klein et al. 2005), 103,611 mutations across the genome were tested for association with macular degeneration. Methods for controlling FWER or false-discovery rate (FDR) are critical for ensuring the findings of GWAS are meaningful and actionable: today, the mutation identified in that first GWAS is showing promising results as a therapeutic target.

- (a) The Bonferroni correction, which uses the decision rule

$$\delta\left(p; \frac{\alpha}{n}\right)$$

controls the FWER described in the previous problem. Suppose you have n independent p -values: p_1, \dots, p_n . Show that the Bonferroni correction controls the probability of at least one false discovery. *Hint: Let E_i be the event that $p_i < \frac{\alpha}{n}$.*



- (b) Klein and his colleagues tested 103,611 hypotheses in their GWAS. Using the Bonferroni correction, what threshold should be used for the decision rule such that the FWER is less than 0.05?

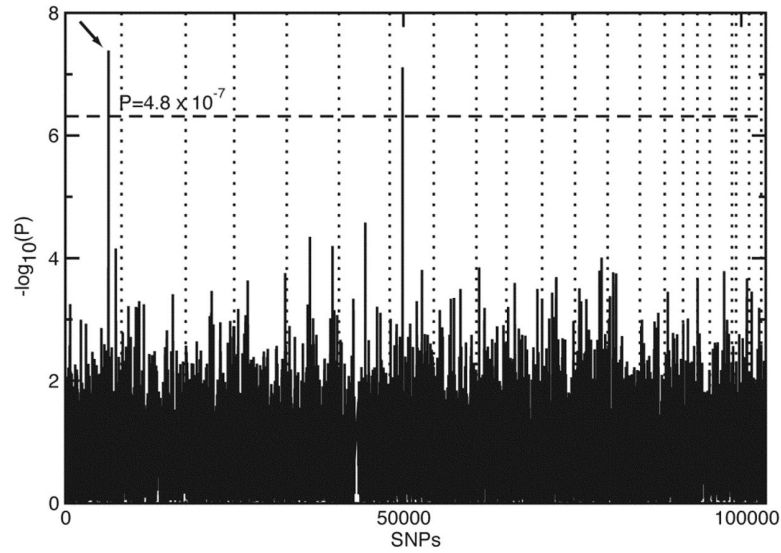
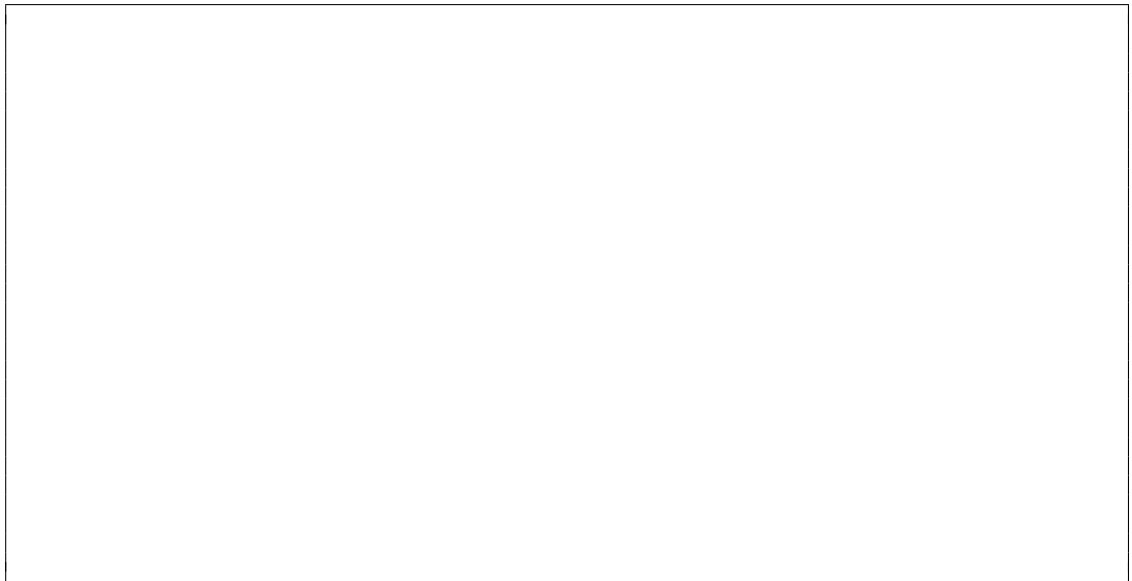


Figure 1: Figure 1(a) from Klein et al. (2005). From their caption: *P values of genome-wide association scan for genes that affect risk of developing macular degeneration. . . . The dotted horizontal line shows the cutoff for $P = 0.05$ after Bonferroni correction. The vertical dotted lines show chromosomal boundaries. The arrow indicates the peak for SNP rs380390, the most significant association, which was studied further.*



References

R. J. Klein, C. Zeiss, E. Y. Chew, J.-Y. Tsai, R. S. Sackler, C. Haynes, A. K. Henning, J. P. SanGiovanni, S. M. Mane, S. T. Mayne, M. B. Bracken, F. L. Ferris, J. Ott, C. Barnstable, J. Hoh. 2005. Complement Factor H Polymorphism in Age-Related Macular Degeneration.

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