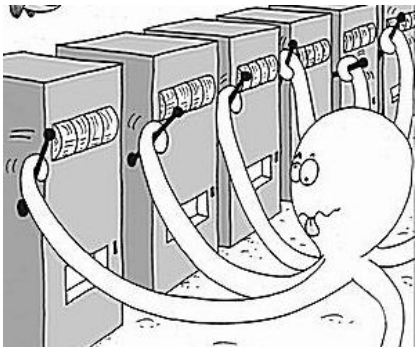
DS 102 Discussion 7

Monday, March 30, 2020

Multi-Armed Bandits: UCB and Hoeffding's inequality

In the last lecture, we began talking about multi-armed bandits. In a multi-armed bandit problem, we consider having K options to choose from. We refer to these options as "arms". Associated with each arm is a probability distribution over rewards. Our goal is to figure out, and draw rewards from, the arm with the highest expected reward.

Initially, these reward distributions are unknown. However, when we choose an arm, referred to as "pulling an arm", we receive a reward sampled from the corresponding reward distribution. We want to develop an algorithm for which arm to pull at each time step, so that we can efficiently figure out which arm gives the highest expected reward.



One algorithm for deciding which arm to pull is the Upper Confidence Bound (UCB) algorithm presented in lecture. If we assume that the reward of each arm is bounded (e.g. the slot machine returns between \$0 and \$100), then we proved in lecture that the UCB algorithm has bounded regret over time.

One thing that we didn't do in lecture is actually derive where the UCB algorithm came from. In this discussion, we will derive the UCB algorithm by leveraging the Hoeffding bound. As we did in lecture, we will assume that the reward of each arm is bounded.

1. Suppose there are K arms, $\mathcal{A} = \{1, 2, ..., K\}$. Each arm $a \in \mathcal{A}$ has its own reward distribution $X_a \sim \mathbb{P}_a$ with mean $\mu_a = \mathbb{E}[X_a]$. We do not know μ_a, but we would like to efficiently find the arm with the maximum μ_a by creating an algorithm that balances *exploration* of the arms with *exploitation*. The efficiency of the algorithm is measured by a quantity known as *regret*, which measures how well the algorithm performs in expectation against an "oracle" that knows the arm with highest mean and always pulls it.

   We will now derive the upper confidence bound that yields the UCB algorithm we saw in the last lecture.

   Define the number of times arm a has been pulled, up to and including time t, as $T_a(t)$.

For any arm a, to find an *upper confidence bound* for μ_a given $T_a(t)$ samples $X_a^{(1)}, ..., X_a^{(T_a(t))}$, we want to find a value $C_a(T_a(t), \delta)$ such that

$$P(\mu_a < \hat{\mu}_{a,T_a(t)} + C_a(T_a(t), \delta)) > 1 - \delta \tag{1}$$

where $\hat{\mu}_{a,T_a(t)}$ is the sample mean of the rewards from arm a:

$$\hat{\mu}_{a,T_a(t)} = \frac{1}{T_a(t)} \sum_{i=1}^{T_a(t)} X_a^{(i)}.$$

In words, Equation (1) says that with probability at least $1 - \delta$, the true mean μ_a is no bigger than the sample mean $\hat{\mu}_{a,T_a(t)}$ plus the term $C_a(T_a(t), \delta)$.

(a) Suppose that you know that the reward of any arm is between 0 and 1. That is:

$$X_a \in [0, 1]$$

Construct an upper confidence bound for the mean of arm a, after observing t samples from arm a.

Solution: Hoeffding's inequality bounds the difference between $\hat{\mu}_{a,T_a(t)}$ and μ_a. Hoeffding's inequality works in two directions:

$$P(\hat{\mu}_{a,T_a(t)} - \mu_a \geq \epsilon) \leq e^{-2T_a(t)\epsilon^2} \tag{2}$$

$$P(\hat{\mu}_{a,T_a(t)} - \mu_a \leq -\epsilon) \leq e^{-2T_a(t)\epsilon^2} \tag{3}$$

The goal is to find the unknown quantity $C_a(T_a(t), \delta)$ from Equation (1) in terms of δ and $T_a(t)$. We now rearrange Equation (1) so that we can apply Hoeffding's inequality.

$$P(\mu_a < \hat{\mu}_{a,T_a(t)} + C_a(T_a(t), \delta)) > 1 - \delta$$
$$P(\hat{\mu}_{a,T_a(t)} - \mu_a > -C_a(T_a(t), \delta)) > 1 - \delta$$

Subtracting both sides from 1,

$$1 - P(\hat{\mu}_{a,T_a(t)} - \mu_a > -C_a(T_a(t), \delta)) \leq \delta$$
$$\implies P(\hat{\mu}_{a,T_a(t)} - \mu_a \leq -C_a(T_a(t), \delta)) \leq \delta$$

This looks a lot more like the form of Hoeffding's inequality in Equation (**??**). Remember that the goal here is to compute the unknown quantity $C_a(t, \delta)$ in terms of δ and t. If we replace ϵ in Hoeffding's inequality (Equation (**??**)) with $C_a(T_a(t), \delta)$, we have

$$P(\hat{\mu}_{a,T_a(t)} - \mu_a \leq -C_a(T_a(t), \delta)) \leq e^{-2T_a(t)C_a(T_a(t),\delta)^2} \tag{4}$$

To compute the value of $C_a(T_a(t), \delta)$ that achieves the desired probability bound of δ from Equation (1), we can set the right hand side of Equation (**??**) to δ:

$$e^{-2T_a(t)C_a(T_a(t),\delta)^2} = \delta$$
$$\implies C_a(T_a(t), \delta) = \sqrt{\frac{-\log \delta}{2T_a(t)}}$$

Plugging this value of $C_a(T_a(t), \delta)$ back into Equation (1), we have

$$P\left(\mu_a < \hat{\mu}_{a,T_a(t)} + \sqrt{\frac{-\log \delta}{2T_a(t)}}\right) > 1 - \delta.$$

(b) Suppose we set $\delta = \frac{1}{t^3}$. This controls the probability that the true mean μ_a is greater than our upper confidence bound $C_a(T_a(t), \delta)$ on the estimated mean $\hat{\mu}_{a,T_a(t)}$. What rule does the UCB algorithm use to choose an arm A_t at each iteration t?

Solution: The UCB algorithm optimistically chooses the arm with the highest upper confidence bound. At time step t, we have access to upper confidence bounds from all time steps up to $t-1$. Therefore, at time step t, the UCB algorithm chooses arm A_t as follows:

$$A_t = \underset{a \in \mathcal{A}}{\text{argmax}} \quad \hat{\mu}_{a,T_a(t-1)} + C_a(T_a(t-1), \delta)$$

Plugging in the upper confidence bound from Part (a), we have

$$A_t = \underset{a \in \mathcal{A}}{\text{argmax}} \quad \hat{\mu}_{a,T_a(t-1)} + \sqrt{\frac{-\log \delta}{2T_a(t-1)}}$$

Substituting in our choice of small $\delta = \frac{1}{t^3}$,

$$A_t = \underset{a \in \mathcal{A}}{\text{argmax}} \quad \hat{\mu}_{a,T_a(t-1)} + \sqrt{\frac{-\log \frac{1}{t^3}}{2T_a(t-1)}}$$
$$\implies A_t = \underset{a \in \mathcal{A}}{\text{argmax}} \quad \hat{\mu}_{a,T_a(t-1)} + \sqrt{\frac{3\log t}{2T_a(t-1)}}$$

This final result matches the UCB algorithm presented in lecture.