Honor code: I pledge on my honor that: I have completed all steps in the below homework on my own, I have not used any unauthorized materials while completing this homework, and I have not given anyone else access to my homework.

Varun Gumma

Name and Signature

1. (1 point) Have you read and understood the honor code?

Solution: YES

Concept: Projection

2. (2 points) Consider a matrix A and a vector $\mathbf{b}$ which does not lie in the column space of A. Let $\mathbf{p}$ be the projection of $\mathbf{b}$ on to the column space of A. If $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 2 & 0 \\ 3 & 1 \end{bmatrix}$ and $\mathbf{p} = \begin{bmatrix} 7 \\ 4 \\ 2 \\ 5 \end{bmatrix}$, find $\mathbf{b}$.

Solution: Given A, $\mathbf{p}$, for projection we have $A^T(\mathbf{b} - A\hat{\mathbf{x}}) = \mathbf{0}$, where $A\hat{\mathbf{x}} = \mathbf{p}$. $A^T\mathbf{p} = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 3 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 7 & 4 & 2 & 5 \end{bmatrix}^\top = \begin{bmatrix} 26 \\ 34 \end{bmatrix}$. As $A^T\mathbf{p} = A^T\mathbf{b}$, and $\mathbf{b}$ is of the form $\begin{bmatrix} b_1 & b_2 & b_3 & b_4 \end{bmatrix}^\top$, we have $\begin{bmatrix} 1 & 0 & 2 & 3 \\ 3 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \end{bmatrix}^\top = \begin{bmatrix} 26 \\ 34 \end{bmatrix}$.

$\therefore$ we have $b_1 + 2b_3 + 3b_4 = 26$ and $3b_1 + 2b_2 + b_4 = 34$. Substituting b_1 from eq-1 into eq-2, we have $3(26 - 2b_3 - 3b_4) + 2b_2 + b_4 = 34$ or $2b_2 - 8b_4 - 6b_3 = -44$ or $b_2 - 4b_4 - 3b_3 = -22$. Here, we have 2 equations and 4 variables and as b_1 and b_2 can be expressed in terms of b_3 and b_4, we choose b_3, b_4 as free variables.

$\therefore b_1 = 26 - 2b_3 - 3b_4$ and $b_2 = 4b_4 + 3b_3 - 22$. If $b_4 = 0$, $b_3 = 1$ we have $b_1 = 26 - 2 = 24$ and $b_2 = 3 - 22 = -19$. Hence, a possible value for $\mathbf{b}$ can be $\begin{bmatrix} 24 & -19 & 1 & 0 \end{bmatrix}^\top$. In fact the general solution of $\mathbf{b}$ is $\begin{bmatrix} 26 - 2b_3 - 3b_4 & -22 + 3b_3 + 4b_4 & b_3 & b_4 \end{bmatrix}^\top = \begin{bmatrix} 26 & -22 & 0 & 0 \end{bmatrix}^\top + b_3 \begin{bmatrix} -2 & 3 & 1 & 0 \end{bmatrix}^\top + b_4 \begin{bmatrix} -3 & 4 & 0 & 1 \end{bmatrix}^\top$.

3. (2 points) Consider the following statement: Two vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ cannot have the same projection $\mathbf{p}$ on the column space of A.

   (a) Give one example where the above statement is True.

   Solution: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Here, the columns vectors of A, span the 2D xy-plane. Here, $A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Consider $\mathbf{b_1} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^\top$ and $\mathbf{b_2} = \begin{bmatrix} 3 & 4 & 5 \end{bmatrix}^\top$.

   $$\mathbf{p_1} = A(A^T A)^{-1} A^T \mathbf{b_1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

   $$\mathbf{p_2} = A(A^T A)^{-1} A^T \mathbf{b_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}.$$

   As seen above, $\mathbf{p_1} \neq \mathbf{p_2}$.

   (b) Give one example where the above statement is False.

   Solution: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Here, the columns vectors of A, span the 2D xy-plane. Here, $A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Consider $\mathbf{b_1} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^\top$ and $\mathbf{b_2} = \begin{bmatrix} 1 & 2 & 4 \end{bmatrix}^\top$.

   $$\mathbf{p_1} = A(A^T A)^{-1} A^T \mathbf{b_1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

   $$\mathbf{p_2} = A(A^T A)^{-1} A^T \mathbf{b_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

   As seen above, $\mathbf{p_1} = \mathbf{p_2}$.

   (c) Based on the above examples, state the generic condition under which the above statement will be True or False.

   Solution: The condition is True except when $\mathbf{b_1} - \mathbf{b_2} \in \mathcal{N}(A^T)$.
   For any two distinct $\mathbf{b_1}$ and $\mathbf{b_2}$, if they have the same projection $\mathbf{p}$, we have $A^T(\mathbf{b_1} - \mathbf{p}) = \mathbf{0} = A^T(\mathbf{b_2} - \mathbf{p})$. $\therefore A^T\mathbf{b_1} - A^T\mathbf{p} = A^T\mathbf{b_2} - A^T\mathbf{p}$ or $A^T(\mathbf{b_1} - \mathbf{b_2}) = \mathbf{0}$. This means $\mathbf{b_1} - \mathbf{b_2}$ should lie in the null space of A^T (or left null space of A).

4. (2 points) (a) Find the projection matrix P_1 that projects onto the line through $\mathbf{a} =$

$\begin{bmatrix}1\\2\end{bmatrix}$ and also the matrix P_2 that projects onto the line perpendicular to $\mathbf{a}$.

Solution: For P_1: Columns space is of the form $\alpha \begin{bmatrix}1 & 2\end{bmatrix}^\top$, and hence the basis is $\begin{bmatrix}1 & 2\end{bmatrix}^\top$. Projection matrix $P_1 = A(A^TA)^{-1}A^T$, where $A = \begin{bmatrix}1 & 2\end{bmatrix}^\top$. $\therefore P_1 = \begin{bmatrix}1\\2\end{bmatrix}(\begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix})^{-1}\begin{bmatrix}1 & 2\end{bmatrix} = \frac{1}{5}\begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}1 & 2\end{bmatrix} = \begin{bmatrix}0.2 & 0.4\\0.4 & 0.8\end{bmatrix}$.

For P_2: Columns space is of the form $\alpha \begin{bmatrix}-2 & 1\end{bmatrix}^\top$, and hence the basis is $\begin{bmatrix}-2 & 1\end{bmatrix}^\top$. Projection matrix $P_2 = A(A^TA)^{-1}A^T$, where $A = \begin{bmatrix}-2 & 1\end{bmatrix}^\top$. $\therefore P_2 = \begin{bmatrix}-2\\1\end{bmatrix}(\begin{bmatrix}-2 & 1\end{bmatrix}\begin{bmatrix}-2\\1\end{bmatrix})^{-1}\begin{bmatrix}-2 & 1\end{bmatrix} = \frac{1}{5}\begin{bmatrix}-2\\1\end{bmatrix}\begin{bmatrix}-2 & 1\end{bmatrix} = \begin{bmatrix}0.8 & -0.4\\-0.4 & 0.2\end{bmatrix}$.

(b) Compute $P_1 + P_2$ and P_1P_2 and explain the result.

Solution: $P_1 + P_2 = \begin{bmatrix}0.2 & 0.4\\0.4 & 0.8\end{bmatrix} + \begin{bmatrix}0.8 & -0.4\\-0.4 & 0.2\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$. This is because for any $\mathbf{v}$, $(P_1 + P_2)\mathbf{v} = P_1\mathbf{v} + P_2\mathbf{v}$, i.e. projection onto the line through $\begin{bmatrix}1 & 2\end{bmatrix}^\top$ and line through $\begin{bmatrix}-2 & 1\end{bmatrix}^\top$ (which are orthogonal). This is the same as projecting a vector on to any basis. The components of the vector along the basis (i.e. $P_1\mathbf{v}, P_2\mathbf{v}$) add up to the vector itself (parallelogram law of addition). $\therefore$ for every vector $\mathbf{v}$, $P_1\mathbf{v_1} + P_2\mathbf{v_2} = \mathbf{v} = (P_1 + P_2)\mathbf{v}$, and hence $P_1 + P_2 = I$.

$P_1P_2 = \begin{bmatrix}0.2 & 0.4\\0.4 & 0.8\end{bmatrix} \cdot \begin{bmatrix}0.8 & -0.4\\-0.4 & 0.2\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$. For any vector $\mathbf{v}$, $P_1P_2\mathbf{v} = P_1\mathbf{w}$ ($\mathbf{w} = P_2\mathbf{v}$). Here $\mathbf{w}$ is along P_2 (as it is projected onto it). $P_2\mathbf{w}$ now tries to project $\mathbf{w}$ onto the line through $\begin{bmatrix}1 & 2\end{bmatrix}^\top$ which will be zero ($\mathbf{w}$ is along a line perpendicular to it, and projection onto a perpendicular line results in 0 as initially there was no component along that direction). As this holds for every vector in that space, i.e. $P_1P_2\mathbf{v} = 0$, P_1P_2 must be $\mathbf{0}$.

Concept: Dot product of vectors

5. (1 point) For all the vectors u, v $\in$ R^n, u^{T}v $\leq ||u||_2||v||_2$.
Prove the statement if true, or give counterexample if false.

Solution: Here, we take $||\mathbf{u}||_2 = \sqrt{u_1^2 + u_2^2 + u_3^2 \dots u_n^2}$, $||\mathbf{v}||_2 = \sqrt{v_1^2 + v_2^2 + v_3^2 \dots v_n^2}$. Let's consider $||\mathbf{u} - \mathbf{v}||_2^2 = (\mathbf{u} - \mathbf{v})^T(\mathbf{u} - \mathbf{v}) = ||\mathbf{u}||_2^2 - \mathbf{u}^T\mathbf{v} - \mathbf{v}^T\mathbf{u} + ||\mathbf{v}||_2^2$. Since, $\mathbf{u}^T\mathbf{v}$ is a scalar $(u_1v_1 + u_2v_2 \cdots + u_nv_n)$ which is the same as $\mathbf{v}^T\mathbf{u}$, the above expression reduces to $||\mathbf{u}||_2^2 - 2\mathbf{u}^T\mathbf{v} + ||\mathbf{v}||_2^2$. As this expression is the square of some value, it is always non-negative or $||\mathbf{u}||_2^2 - 2\mathbf{u}^T\mathbf{v} + ||\mathbf{v}||_2^2 \geq 0$. Dividing throughout by $\frac{1}{||\mathbf{v}||_2^2}$ we get, $\frac{||\mathbf{u}||_2^2}{||\mathbf{v}||_2^2} + 2\frac{\mathbf{u}^T\mathbf{v}}{||\mathbf{v}||\cdot||\mathbf{v}||} + 1 \geq 0$. With $\frac{1}{||\mathbf{v}||_2} = \alpha$, the equation reduces to $\alpha^2||\mathbf{u}||_2^2 + 2\alpha\frac{\mathbf{u}^T\mathbf{v}}{||\mathbf{v}||_2} + 1 \geq 0$. This is a quadratic which is always non-negative for any value of α hence, it has no real roots or equal roots, i.e. its discriminant is non-negative. $\therefore 4\frac{(\mathbf{u}^T\mathbf{v})^2}{||\mathbf{v}||_2^2} - 4 \cdot 1 \cdot ||\mathbf{u}||_2^2 \leq 0$ or $\frac{(\mathbf{u}^T\mathbf{v})^2}{||\mathbf{v}||_2^2} \leq ||\mathbf{u}||_2^2$ or $(\mathbf{u}^T\mathbf{v})^2 \leq ||\mathbf{u}||_2^2||\mathbf{v}||_2^2$. Taking a square root in both sides, we get the final result $\mathbf{u}^T\mathbf{v} \leq ||\mathbf{u}||_2||\mathbf{v}||_2$.

Concept: Vector norms

6. (1 point) The L_p-norm of a vector $\mathbf{x} = [x_1, x_2, \ldots, x_n] \in \mathbb{R}^n$ is defined as:

$$||\mathbf{x}||_p = (|x_1|^p + |x_2|^p + |x_3|^p + \cdots + |x_n|^p)^{\frac{1}{p}}$$

(a) Prove that $||\mathbf{x}||_\infty = max_{1\leq i\leq n}|x_i|$

Solution: As we cannot directly compute $||\mathbf{x}||_\infty$, we compute it as $\lim_{p\to\infty} ||\mathbf{x}||_p$. Without loss of Generality, let's assume the maximum of $|x_1| \ldots |x_n|$ is $|x_k|$.
$\lim_{p\to\infty} ||\mathbf{x}||_p = |x_k| \cdot (\frac{|x_1|^p}{|x_k|^p} + \frac{|x_2|^p}{|x_k|^p} + \cdots + 1 + \ldots \frac{|x_n|^p}{|x_k|^p})^{\frac{1}{p}}$, which equals,
$\lim_{p\to\infty} ||\mathbf{x}||_p = |x_k| \cdot (|\frac{x_1}{x_k}|^p + |\frac{x_2}{x_k}|^p + \cdots + 1 + \ldots |\frac{x_n}{x_k}|^p)^{\frac{1}{p}}$
As x_k is the maximum, $|\frac{x_i}{x_k}|$ $(i \neq k) < 1$ and for very large values of p, $|\frac{x_i}{x_k}|^p \approx 0$ and $\frac{1}{p} \approx 0$. $\therefore \lim_{p\to\infty} ||\mathbf{x}|| = |x_k| \cdot (1)^0 = |x_k| = max_{1\leq i\leq n}|x_i|$. Hence, proved.

(b) True or False (explain with reason): $||\mathbf{x}||_0$ is a norm.

Solution: False, as it fails the condition $||k\mathbf{x}||_0 = |k| \cdot ||\mathbf{x}||_0, \forall k \neq 0$.
$||k\mathbf{x}|| = ((kx_1)^p + (kx_2)^p + \cdots + (kx_n)^p)^{\frac{1}{p}} = (k^px_1^p + k^px_2^p + \cdots + k^px_n^p)^{\frac{1}{p}}$. When $p = 0$, we have $k^p = 1$, hence, $||k\mathbf{x}||_0 = (x_1^p + x_2^p + \cdots + x_n^p)^{\frac{1}{p}} = ||\mathbf{x}||_0$. As seen, the statement $||k\mathbf{x}||_0 = |k| \cdot ||\mathbf{x}||_0$ holds only when $k = 1$ but not $\forall k \neq 0$.

Concept: Orthogonal/Orthornormal vectors and matrices

7. (1 point) Consider the following questions:

(a) Construct a 2×2 Orthonormal matrix, such that none of its entries are real.

Solution: Let that matrix be of the form $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (all entries are complex). Since M must be orthonormal $MM^T = I$ or $M^T = M^{-1}$. $\therefore \begin{bmatrix} a & c \\ b & d \end{bmatrix} =$

$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. $\therefore a = d$ and $c = -b$. Since, determinant of $M = 1$, $ad - bc = a^2 + c^2 = 1$. If $c = (1 - i)$, $a = \sqrt{1 + 2i}$. Hence, $M = \begin{bmatrix} \sqrt{1+2i} & 1-i \\ -1+i & \sqrt{1+2i} \end{bmatrix}$

(b) Construct a 4×4 matrix such that all of its columns are orthogonal and all its entries are +1, -1, +2 or -2.

Solution: A possible combination of the column vectors can be $\begin{bmatrix} -1 & 1 & -1 & 1 \end{bmatrix}^\top$, $\begin{bmatrix} -1 & -1 & -1 & 1 \end{bmatrix}^\top$, $\begin{bmatrix} 2 & 2 & -2 & 2 \end{bmatrix}^\top$, $\begin{bmatrix} -2 & 2 & 2 & -2 \end{bmatrix}^\top$ and hence the matrix $\begin{bmatrix} -1 & -1 & 2 & -2 \\ 1 & -1 & 2 & 2 \\ -1 & -1 & -2 & 2 \\ 1 & 1 & -2 & -2 \end{bmatrix}$ satisfies the required condition.

8. (1 point) Consider the vectors $\mathbf{a} = \begin{bmatrix} 4 \\ 6 \\ 2 \\ 5 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

(a) What multiple of $\mathbf{a}$ is closest to $\mathbf{b}$?

Solution: Let that vector $\mathbf{v} = \alpha \begin{bmatrix} 4 & 6 & 2 & 5 \end{bmatrix}^\top$. If this vector has to be closest to $\begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^\top$, then the length of the difference vector must be minimum, i.e. $min\ ||\alpha \begin{bmatrix} 4 & 6 & 2 & 5 \end{bmatrix}^\top - \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^\top||$.

$\therefore min\ \sqrt{(4\alpha - 1)^2 + (6\alpha)^2 + (2\alpha)^2 + (5\alpha - 1)^2}$
$\therefore min\ \sqrt{81\alpha^2 - 18\alpha + 2}$
$\therefore min\ 81\alpha^2 - 18\alpha + 2$
$\therefore$ taking a derivative and setting it to 0, we have $162\alpha - 18 = 0$ or $\alpha = \frac{1}{9}$
$\therefore \mathbf{v} = \begin{bmatrix} \frac{4}{9} & \frac{6}{9} & \frac{2}{9} & \frac{5}{9} \end{bmatrix}^\top$

(b) Find orthonormal vectors $\mathbf{q_1}$ and $\mathbf{q_2}$ that lie in the plane formed by $\mathbf{a}$ and $\mathbf{b}$?

Solution: We can use Gram-Schmidt to reduce $\mathbf{a}$, $\mathbf{b}$ to an orthonormal pair. Since the formed vectors are derived from $\mathbf{a}$, $\mathbf{b}$ by projection and subtraction, the pair will lie in the plane formed by $\mathbf{a}$, $\mathbf{b}$. $\therefore \hat{\mathbf{q_1}} = \frac{\mathbf{a}}{||\mathbf{a}||} = \begin{bmatrix} \frac{4}{9} & \frac{6}{9} & \frac{2}{9} & \frac{5}{9} \end{bmatrix}^\top$. $\mathbf{b_2} = \mathbf{b} - \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}} \mathbf{a} = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^\top - \frac{9}{81} \begin{bmatrix} 4 & 6 & 2 & 5 \end{bmatrix}^\top = \begin{bmatrix} \frac{5}{9} & -\frac{6}{9} & -\frac{2}{9} & \frac{4}{9} \end{bmatrix}^\top$. $\therefore \hat{\mathbf{q_2}} = \frac{\mathbf{b_2}}{||\mathbf{b_2}||} = \begin{bmatrix} \frac{5}{9} & -\frac{6}{9} & -\frac{2}{9} & \frac{4}{9} \end{bmatrix}^\top$

9. (1 point) True or False : If A is Unitary matrix then A^2 must be an Unitary matrix. Prove the statement if True, or give counterexample if false.

> **Solution:** A matrix A is unitary, if $AA^* = A^*A = I$, where A^* is the conjugate transpose of A ($a^*_{ij} = \bar{a}_{ji}$) or $A^* = \bar{A}^T$.
>
> $\therefore$ to check A^2 is unitary or not, we perform $A^2(A^2)^*$. But $(A^2)^* = \overline{(A^2)^T} = \overline{(AA)^T} = \overline{A^T A^T} = \bar{A^T}\bar{A^T}$ (conjugate of products is product of conjugates) $= A^*A^*$.
>
> $\therefore A^2(A^2)^* = A^2A^*A^* = AAA^*A^* = AIA^* = AA^* = I$. Hence, A^2 is also Unitary.

10. (1 point) If Q is an orthogonal matrix, show that for any two vectors x and y of the proper dimension :

$$||Qx - Qy|| = ||x - y||$$

> **Solution:** As Q is orthogonal, $QQ^T = Q^TQ = I$.
>
> $||Qx - Qy||^2 = (Qx - Qy)^T(Qx - Qy) = ((Qx)^T - (Qy)^T)(Qx - Qy) = (Qx)^TQx - (Qy)^TQx - (Qx)^TQy + (Qy)^TQy = x^TQ^TQx - y^TQ^TQx - y^TQ^TQy + y^TQ^TQy = x^Tx - x^Ty - y^Tx + y^T = (x^T - y^T)(x - y) = (x - y)^T(x - y) = ||x - y||^2$.
>
> $\therefore ||Qx-Qy||^2 = ||x-y||^2$ and taking a square root in both sides we have $||Qx-Qy|| = ||x - y||$. Hence, proved.

Concept: Determinants

11. (2 points) Let A be a n $\times$ n matrix such that $A[i][j] = \begin{cases} 1 & i - j = 1 \text{ OR } i = j \\ -1 & j - i = 1 \\ 0 & otherwise \end{cases}$

Prove $|A_n| = |A_{n-1}| + |A_{n-2}|$.

> **Solution:** The matrix formed by the given conditions is tridiagonal and has 1s on the main and lower diagonal, and -1s on the upper diagonal and rest all elements are 0.
>
> $|A_n| = a_{11}C^n_{11} + a_{12}C^n_{12} + \cdots + a_{1n}C^n_{1n}$. But according to give conditions, $a_{13}, a_{14}, \ldots a_{1n}$ all are zero, $a_{11} = 1$ and $a_{12} = -1$. $\therefore |A_n| = C^n_{11} - C^n_{12}$.
>
> To compute C^n_{11}, we remove the first column and row of A_n and hence the minor formed will still have the main diagonal and lower diagonal as 1s and upper diagonals as -1s with rest all elements as 0. Hence $C^n_{11} = 1 \cdot |A_{n-1}|$.

To compute C_{12}^n, we remove the first row and second column of A_n. This does alter the lower diagonal, as the formed minor misses a 1 the $(2,1)$ location and instead has a 0 in its place (which was initially the 0 in the (3, 1) location of A_n). The main and upper diagonals are unaffected. Now expanding this minor along the first column we have, $C_{12}^n = -1 \cdot |A_{12}| = -1 \cdot C_{11}^{n-1}$. (rest terms in that columns are zero as they are off-diagonal terms in A_n).

To compute C_{11}^{n-1}, we remove the first row and first column of minor matrix A_{12}. This is same as removing first two rows and columns of A_n (initially column-2 and row-1 were removed, now column-1 and row-2 are removed from A_n). This again follows the conditions for main, upper and lower diagonals as given in the problem and its determinant can be given as $|A_{n-2}|$.

$\therefore C_{11}^n = 1 \cdot |A_{n-2}|$.
$\therefore |A_n| = |A_{n-1}| - (-1 \cdot 1 \cdot |A_{n-2}|) = |A_{n-1}| + |A_{n-2}|$.

12. (1 point) What is the least number of zeros in a $n \times n$ matrix that will guarantee $det(A) = 0$. Construct such matrix for n = 4.
On the other hand, what is the maximum numbers of zeros in a $n \times n$ matrix that will guarantee $det(A) \neq 0$. Construct such matrix for n = 4.

Solution: If an $n \times n$ matrix has a complete row/column of zeros, i.e. n zeros, then it is a guarantee that its determinant is 0. An example of such a 4×4 matrix is $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 7 & 6 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

For maximum number of zeros, we can have all non-diagonal terms to be 0. i.e. $n^2 - n$ zeros. Since the determinant of a diagonal matrix is the product of its diagonal entries, all diagonal entries in this case must be non-zero. An example of such 4×4 matrix is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ and its det is 24.

13. (1 point) This question is about properties 9 and 10 of determinants.
    (a) Prove that $det(AB) = det(A)det(B)$

Solution: Before we solve the problem, some properties:

1. *E is a matrix that swaps two rows*: Let V be an $n \times n$ matrix and $W = EV$. $det(W) = -det(V)$ (as W has two rows swapped from V and swapping rows inverts the sign of the determinant). $\therefore\ det(W) = -det(V) = det(EV)$. As E is a permutation matrix (with two rows swapped from identity), $det(E) = -det(I) = -1$. $\therefore det(EV) = det(W) = -1 \cdot det(V) = det(E)det(V)$.

2. *E is a matrix that multiplies a row by a scalar k*: Let V be an $n \times n$ matrix and $W = EV$. As W has one row that is k times the corresponding row in V, $det(W) = k \cdot det(V)$ (pull out k from that row). But since E is a matrix that has one row that is k times identity (i.e. k in some ii^{th} position, with rest all diagonals as 1), $det(E) = k \cdot 1 \cdot 1 \ldots 1 = k$ (for a diagonal matrix the det is product of diagonal terms). $\therefore det(EV) = det(W) = k \cdot det(V) = det(E)det(V)$.

3. *E is a matrix that adds k times one row to another*: Let V be an $n \times n$ matrix and $W = EV$. Since W is obtained by row-ops on V, $det(W) = det(V)$. E is an elementary matrix which is similar to identity but has a k at the ij^{th} position (assuming E swaps row-i and row-j). $\therefore\ E$ is a upper/lower triangular matrix and its determinant is the product of its diagonal entries which which are all 1s, $\therefore det(E) = 1$. Hence, $det(EV) = det(W) = 1 \cdot det(V) = det(E)det(V)$.

Having proved these cases, we have: (here, W.L.O.G we work with A)

1. *A is singular*: If A is singular, $det(A) = 0$, i.e. A has dependent columns and even AB has dependent columns, $\therefore det(AB) = 0$ and $det(AB) = 0 = 0 \cdot det(B) = det(A)det(B)$.

2. *A is invertible*: A can be reduced to RREF with all pivots, i.e. $A = E_1E_2E_3 \ldots E_mI$, where $E_1, E_2, \ldots E_m$ are elementary/permutation matrices. $\therefore\ det(AB) = E_1E_2E_3E_m \ldots B$. If we take $E_2E_3 \ldots E_mB = B'$, we have $det(AB) = det(E_1B') = det(E_1)det(B')$, now if we write $B' = E_2B''$, $det(AB) = det(E_1)det(E_2)det(B'')$. Similarly, pulling out all matrices we have $det(AB) = det(E_1)det(E_2)det(E_3) \ldots det(E_m)det(B)$. Now clubbing the terms, $det(E_1)det(E_2) = det(E_1E_2)$. $det(E_1E_2)det(E_3) = det(E_1E_2E_3)$ (since, E_3 is elementary/permutation matrix, it can be clubbed according to the rules proved earlier). Similarly, we have $det(E_1)det(E_2) \ldots det(E_m) = det(E_1E_2 \ldots E_m) = det(A)$. $\therefore det(AB) = det(E_1)det(E_2) \ldots det(E_m)det(B) = det(E_1E_2 \ldots E_m)det(B) = det(A)det(B)$, QED.

(b) Prove that $det(A^\top) = det(A)$

Solution:
Before we solve the problem, some properties:

1. *E is a matrix that swaps two rows*: Let V be an $n \times n$ matrix and $W = EV$. $det(W) = -det(V)$ (as W has two rows swapped from V and swapping rows inverts the sign of the determinant). $\therefore det(W) = -det(V) = det(EV)$. As E is a permutation matrix (with two rows swapped from identity), $det(E) = -det(I) = -1$. $\therefore det(EV) = det(W) = -1 \cdot det(V) = det(E)det(V)$. Further, as $E = E^T, det(E) = det(E^T)$.

2. *E is a matrix that multiplies a row by a scalar k*: Let V be an $n \times n$ matrix and $W = EV$. As W has one row that is k times the corresponding row in V, $det(W) = k \cdot det(V)$ (pull out k from that row). But since E is a matrix that has one row that is k times identity (i.e. k in some ii^{th} position, with rest all diagonals as 1), $det(E) = k \cdot 1 \cdot 1 \ldots 1 = k$ (for a diagonal matrix the det is product of diagonal terms). $\therefore det(EV) = det(W) = k \cdot det(V) = det(E)det(V)$. Further as E is a diagonal matrix, $E^T = E, det(E^T) = det(E) = k$.

3. *E is a matrix that adds k times one row to another*: Let V be an $n \times n$ matrix and $W = EV$. Since W is obtained by row-ops on V, $det(W) = det(V)$. E is an elementary matrix which is similar to identity but has a k at the ij^{th} position (assuming E swaps row-i and row-j). $\therefore E$ is a upper/lower triangular matrix and it's determinant is the product of its diagonal entries which which are all 1s, $\therefore det(E) = 1$. Hence, $det(EV) = det(W) = 1 \cdot det(V) = det(E)det(V)$. Further, as E^T is also a lower/upper triangular matrix, with 1s in the diagonal, $det(E^T) = det(E) = 1$.

Solving the above problem, we have:

1. *A is singular*: If A is singular, $det(A) = 0$, i.e. A is not full rank. $\therefore A^T$ will also not be full rank (rank(A) = rank(A^T)) and $det(A^T)$ will also be zero. $\therefore det(A) = det(A^T) = 0$.

2. *A is invertible*: If A is invertible, it has as a conversion to RREF using elementary row-ops and those can be undone as $A = E_1 E_2 \ldots E_m I$. $A^T = E_m^T E_{m-1}^T \ldots E_1^T$ and $det(A^T) = det(E_m^T E_{m-1}^T \ldots E_1^T)$. As $E_m^T \ldots E_1^T$ are also elementary/permutation matrices we have $det(E_m^T E_{m-1}^T \ldots E_1^T)$ can be split as per the above rules proved. Further as seen above, $det(E_i^T) = det(E)$.

   $\therefore det(A^T)$
   $= det(E_m^T E_{m-1}^T \ldots E_1^T)$

$$= det(E_m^T)det(E_{m-1}^T E_{m-2}^T \dots E_1^T)$$
$$= det(E_m^T)(det(E_{m-1}^T)det(E_{m-2}^T E_{m-3}^T \dots E_1^T))$$
$$= det(E_m^T)det(E_{m-1}^T)(det(E_{m-2}^T)det(E_{m-3}^T E_{m-4}^T \dots E_1^T))$$

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$$= det(E_m^T)det(E_{m-1}^T)\dots det(E_1^T)$$
$$= (det(E_m)det(E_{m-1}))det(E_{m-2})\dots det(E_1)$$
$$= (det(E_m E_{m-1})det(E_{m-2}))det(E_{m-3})\dots det(E_1)$$
$$= (det(E_m E_{m-1} E_{m-2})det(E_{m-3}))det(E_{m-4})\dots det(E_1)$$

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$$= det(E_m E_{m-1} \dots E_1)$$
$$= det(A)$$

Hence, $det(A^T) = det(A)$, QED.

14. (1 point) Let $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$.

(a) Find the area of the triangle whose vertices are $\mathbf{u}, \mathbf{v}, \mathbf{u} + \mathbf{v}$

Solution: Here, $\mathbf{u}$ and $\mathbf{v}$ are collinear (as $\mathbf{v} = 2\mathbf{u}$) and do not span a plane (2D surface). Even the vector $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 9 \\ 6 \end{bmatrix}$ lies on the same line i.e. span of $\mathbf{u}$ (as $\mathbf{u} + \mathbf{v} = 3\mathbf{u}$). As all points are collinear the area of the triangle formed is 0.

(b) Suppose you rotate these vectors along the origin such that the heads of vectors $\mathbf{u}$ and $\mathbf{v}$ trace two concentric circles, then find the area of figure trapped between circles

Solution: Radius of circle traced out by vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is $\sqrt{9+4} = \sqrt{13}$ (i.e. $||\mathbf{u}||_2$) and area of that circle is 13π. Radius of circle traced out by vector $\begin{bmatrix} 6 \\ 4 \end{bmatrix}$ is $\sqrt{36+16} = \sqrt{52}$ (i.e. $||\mathbf{v}||_2$) and area of that circle is 52π. $\therefore$ area of annular region is $52\pi - 13\pi = 39\pi$.

15. (2 points) The determinant of the following matrix can be computed as a sum of 120 (5!) terms.

$$A = \begin{bmatrix} x & x & x & x & x \\ x & x & x & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \end{bmatrix}$$

State true or false with an appropriate explanation: All the 120 terms in the determinant will be 0.

Solution: True, all terms are 0. Let's expand the determinant along the first column. Hence, we have $det(A) = a_{11}det(A_{11}) - a_{21}det(A_{21}) + a_{31}det(A_{31}) - a_{41}det(A_{41}) + a_{51}det(A_{51})$. Each term in the $det(A)$ is of the form $a_{1i}a_{2j}a_{3k}a_{4l}a_{5m} \cdot det(P)$. So, even if one factor in this term is 0, the whole term becomes 0. Hence, we count the number of terms with atleast one zero.

1. Here, $det(A_{11}) \ldots det(A_{51})$ are determinants of dimension 4×4 and each has $4! = 24$ terms. Since $a_{31} = a_{41} = a_{51} = 0$, each term in $a_{31}det(A_{31})$, $a_{41}det(A_{41})$ or $a_{51}det(A_{51})$ contains a zero (which is either a_{31}, a_{41} or a_{51}) and so we have $3 \cdot 4! = 72$ terms which are zero.

2. Since, the first and second row/column are the same, W.L.O.G let's analyse the number of terms obtained in $a_{11}det(A_{11})$. If we A_{11} denoted by B (which is of size 3×3), $det(A_{11}) = det(B) = b_{11}det(B_{11}) - b_{21}det(B_{21}) + b_{31}det(B_{31})$, we have $b_{21} = b_{31} = b_{41} = 0$ and hence the terms in $b_{11}det(B_{11})$, $b_{21}det(B_{21})$ and $b_{31}det(B_{31})$ are all zero, which are $3 \cdot 3! = 18$ in number. If we denote B_{11} as C, $det(B_{11}) = det(C)$ and we can observe that C (which is of size 3×3) has a complete column of zeros and hence has all $3! = 6$ terms as zero. $\therefore$ the number of terms which are zero in B are $18 + 6 = 24$. Going by a similar logic for $det(A_{21})$, we get that it has another 24 terms which are zero.

$\therefore$ we have 72 zero terms (24 each from $a_{31}det(A_{31})$, $a_{41}det(A_{41})$ or $a_{51}det(A_{51})$) + 48 zero terms (24 each from $det(A_{11})$, $det(A_{21})$) = 120 zero terms.

ANOTHER APPROACH

$det(A) = \sum a_{1\alpha}a_{2\beta}a_{3\gamma}\ldots a_{n\omega} \cdot det(P)$ (here $\alpha, \beta, \gamma, \ldots \omega$ is a random permutation from $1, 2, \ldots, n$). If this term has to be 0, either one or more terms have to be 0. To count check how many terms ($a_{1\alpha}a_{2\beta}a_{3\gamma}a_{4\delta}a_{5\omega}$) have atleast one 0 in them, we use combinatorics. Each term will have atmost 3 0s as we have only 3 rows with zeros in them.

1. *Terms with 3 zeros*: Since the zeros in the matrix are present in the last 3 rows, the zeros in the terms should be in the last three positions as well (i.e. $a_{3\gamma}a_{4\delta}a_{5\omega}$ as we select one index from each row). This can happen in $3 \cdot 2 \cdot 1 = 6$ ways, i.e.

last three indices can be $((1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,2,1),(3,1,2))$ and the first two locations can be filled with remaining two indices in $2!$ ways. $\therefore$ we have $6 \cdot 2 = 12$ such combinations.

2. *Terms with 2 zeros*: Again both zeros must occur in the last three positions and have 3 index choices for the first occurrence $(3,4,5)$ and 2 index choices for the second occurrence. The remaining place should not have a zero and hence has 2 choices (4 or 5). Now the first two positions can be filled in $2!$ ways with the final remaining two indices. As the zeros can occur like this in 3 ways, i.e. $((3,4),(4,5),(3,5))$, we have $3 \cdot 3 \cdot 2 \cdot 2 \cdot 2 = 72$ combinations.

3. *Terms with 1 zero*: Again the single zero must occur in one the last three positions of the term. A zero at a position has 3 index choices $(3,4,5)$. The remaining two places in the last three positions should not have 0s and hence can be filled with indices 4 or 5, i.e. in $2!$ ways and the final remaining indices can we filled in the first two places in $2!$ ways. As that zero can occur like this in either the $3^{rd}, 4^{th}$ or 5^{th} position, we have a total of $3 \cdot 3 \cdot 2 \cdot \cdot 2 = 36$ combinations.

$\therefore$ we observe that we have $12 + 72 + 36 = 120$ terms with atleast one zero in product and hence every term in the determinant is 0.