Honor code: I pledge on my honor that: I have completed all steps in the below homework on my own, I have not used any unauthorized materials while completing this homework, and I have not given anyone else access to my homework.

Varun Gumma

Name and Signature

1. (1 point) Have you read and understood the honor code?

   Solution: YES

Concept: System of linear equations

2. (2 points) This question has two parts as mentioned below:

   (a) Find a 2 x 3 system Ax = b whose complete solution is

$$x = \begin{bmatrix}1\\1\\1\end{bmatrix} + w\begin{bmatrix}1\\2\\1\end{bmatrix}$$

   Solution: As this system has infinite values for $\mathbf{x}$, we can conclude that this is possible when $\begin{bmatrix}1 & 2 & 1\end{bmatrix}^\top \in \mathcal{N}(A)$ and $A\begin{bmatrix}1 & 1 & 1\end{bmatrix}^\top = \mathbf{b}$ (particular solution). $\therefore \mathbf{a_1} + 2\mathbf{a_2} + \mathbf{a_3} = 0$ and we can have $A = \begin{bmatrix}1 & 0 & -1\\0 & 1 & -2\end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix}1 & 0 & -1\\0 & 1 & -2\end{bmatrix}\mathbf{x} = \begin{bmatrix}0\\-1\end{bmatrix}$.

   (b) Now find a 3 x 3 system which has these solutions exactly when $b_1 + b_2 + b_3 = 0$. (Note: $b = [b_1\ b_2\ b_3]^T$.)

   Solution: Again, we can conclude that $\begin{bmatrix}1 & 2 & 1\end{bmatrix}^\top \in \mathcal{N}(A)$ and $A\begin{bmatrix}1 & 1 & 1\end{bmatrix}^\top = \mathbf{b}$ (particular solution). $\therefore \mathbf{a_1} + 2\mathbf{a_2} + \mathbf{a_3} = 0$, which imples $\mathbf{a_3} = -\mathbf{a_1} - 2\mathbf{a_2}$. $A\begin{bmatrix}1 & 1 & 1\end{bmatrix}^\top = \mathbf{a_1} + \mathbf{a_2} + \mathbf{a_3} = \mathbf{a_1} + \mathbf{a_2} - \mathbf{a_1} - 2\mathbf{a_2} = -\mathbf{a_2} = \mathbf{b}$. $\therefore b_1 + b_2 + b_3 = 0 = a_{21} + a_{22} + a_{23}$.

We can have $\mathbf{a_2}$ as $\begin{bmatrix}-1 & 0 & 1\end{bmatrix}^\top$, $\mathbf{b} = \begin{bmatrix}1 & 0 & -1\end{bmatrix}^\top$, $\mathbf{a_1} = \begin{bmatrix}0 & 1 & 1\end{bmatrix}^\top$ and $\mathbf{a_3} = \begin{bmatrix}2 & -1 & -3\end{bmatrix}^\top$. $\therefore$ the system can be $\begin{bmatrix}0 & -1 & 2\\ 1 & 0 & -1\\ 1 & 1 & -3\end{bmatrix}\mathbf{x} = \begin{bmatrix}1\\ 0\\ -1\end{bmatrix}$.

3. (2 points) Consider the matrices A and B below

$$\text{(i) A} = \begin{bmatrix}1 & 6 & -3 & 4\\ 0 & 2 & 4 & 0\\ 2 & 12 & -6 & 8\end{bmatrix} \quad \text{(ii) B} = \begin{bmatrix}3 & 1 & 2\\ 12 & 4 & 8\\ 6 & 2 & 4\end{bmatrix}$$

(a) Write down the row reduced echelon form of matrices A and B (also mention the steps involved).

Solution: $\begin{bmatrix}1 & 6 & -3 & 4\\ 0 & 2 & 4 & 0\\ 2 & 12 & -6 & 8\end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - 2R_1} \begin{bmatrix}1 & 6 & -3 & 4\\ 0 & 2 & 4 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \xrightarrow{R_2 \leftarrow R_2/2}$
$\begin{bmatrix}1 & 6 & -3 & 4\\ 0 & 1 & 2 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 - 6R_2} \begin{bmatrix}1 & 0 & -15 & 4\\ 0 & 1 & 2 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$

$\begin{bmatrix}3 & 1 & 2\\ 12 & 4 & 8\\ 6 & 2 & 4\end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 4R_1} \begin{bmatrix}3 & 1 & 2\\ 0 & 0 & 0\\ 6 & 2 & 4\end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - 2R_1} \begin{bmatrix}3 & 1 & 2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \xrightarrow{R_1 \leftarrow R_1/3} \begin{bmatrix}1 & \frac{1}{3} & \frac{2}{3}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$

(b) Find all solutions to $A\mathbf{x} = 0$ and $B\mathbf{x} = 0$.

Solution: For $A\mathbf{x} = 0$, we have two free variables (let them be x_3 and x_4). $\therefore$ from the above RREF we have $x_2 = -2x_3$ and $x_1 = 15x_3 - 4x_4$. Null space solution in parametric form can be written as $\begin{bmatrix}15x_3 - 4x_4\\ -2x_3\\ x_3\\ x_4\end{bmatrix} = x_3\begin{bmatrix}15\\ -2\\ 1\\ 0\end{bmatrix} + x_4\begin{bmatrix}-4\\ 0\\ 0\\ 1\end{bmatrix}$

For $B\mathbf{x} = 0$, we have two free variables (let them be x_2 and x_3). $\therefore$ from the above RREF we have $x_1 = -\frac{1}{3}x_2 - \frac{2}{3}x_3$. Null space solution in parametric form can be written as $\begin{bmatrix}-\frac{1}{3}x_2 - \frac{2}{3}x_3\\ x_2\\ x_3\end{bmatrix} = x_2\begin{bmatrix}-\frac{1}{3}\\ 1\\ 0\end{bmatrix} + x_3\begin{bmatrix}-\frac{2}{3}\\ 0\\ 1\end{bmatrix}$

(c) Write down the basis for the four fundamental subspaces of A.

Solution: Basis of $\mathcal{C}(A)$ are pivot columns of A: $\begin{bmatrix}1 & 0 & 2\end{bmatrix}^\top$, $\begin{bmatrix}6 & 2 & 12\end{bmatrix}^\top$
Basis of $\mathcal{N}(A)$ are solutions of $A\mathbf{x} = 0$: $\begin{bmatrix}15 & -2 & 1 & 0\end{bmatrix}^\top$, $\begin{bmatrix}-4 & 0 & 0 & 1\end{bmatrix}^\top$
Basis of $\mathcal{R}(A)$ are pivot rows of R: $\begin{bmatrix}1 & 0 & -15 & 4\end{bmatrix}^\top$, $\begin{bmatrix}0 & 1 & 2 & 0\end{bmatrix}^\top$
Basis of $\mathcal{N}(A^T)$ are solutions of $A^T\mathbf{x} = 0$: $\begin{bmatrix}-2 & 0 & 1\end{bmatrix}^\top$

(d) Write down the basis for the four fundamental subspaces of B.

Solution: Basis of $\mathcal{C}(B)$ are pivot columns of B: $\begin{bmatrix}3 & 12 & 6\end{bmatrix}^\top$
Basis of $\mathcal{N}(B)$ are solutions of $B\mathbf{x} = 0$: $\begin{bmatrix}\frac{-1}{3} & 1 & 0\end{bmatrix}^\top$, $\begin{bmatrix}\frac{-2}{3} & 0 & 1\end{bmatrix}^\top$
Basis of $\mathcal{R}(B)$ are pivot rows of R: $\begin{bmatrix}1 & \frac{1}{3} & \frac{2}{3}\end{bmatrix}^\top$
Basis of $\mathcal{N}(B^T)$ are solutions of $B^T\mathbf{x} = 0$: $\begin{bmatrix}-4 & 1 & 0\end{bmatrix}^\top$, $\begin{bmatrix}-2 & 0 & 1\end{bmatrix}^\top$

Concept: Rank

4. (1 ½ points) Consider the matrices A and B as given below:

$$A = \begin{bmatrix}3 & 2 & 1\\ -6 & -4 & -2\\ 3 & 2 & x\end{bmatrix} \text{ and B} = \begin{bmatrix}7 & 2 & 7\\ y & 2 & y\end{bmatrix}$$

Give the values for entries x and y such that the ranks of the matrices A and B are

(a) 1

Solution: After Gaussian Elimination, A and B can be written as $\begin{bmatrix}3 & 2 & 1\\ 0 & 0 & 0\\ 0 & 0 & x-1\end{bmatrix}$ and $\begin{bmatrix}7 & 2 & 7\\ y-7 & 0 & y-7\end{bmatrix}$ respectively. To have rank one, we need only one row with non-zero pivot, hence, $x = 1$, $y = 7$.

(b) 2

Solution: To have rank two, $x \neq 1$ and $y \neq 7$, i.e. two rows with non-zero pivots. $\therefore x = 0$, $y = 0$ can satisfy.

(c) 3

Solution: No values for x and y can give us rank three as A already has one dependent row and B has only two rows which means they can span a plane (2D) in the best case (when those have non-zero pivots) but not 3D.

Concept: Nullspace and column space

5. (½ point) State True or False and explain you answer: The nullspace of R is the same as the nullspace of U (where R is the row reduced echelon form of A and U is the matrix in LU decomposition of A).

> **Solution:** True, as we can obtain R from U by elementary row operations. $\therefore R = E_1E_2\dots E_kU$, where $E_1\dots E_k$ represents elementary matrices. To obtain null space of R, we solve $R\mathbf{y} = E_1E_2\dots E_kU\mathbf{y} = \mathbf{0}$ ($\mathbf{y} \in \mathcal{N}(R)$). If we represent $E_1E_2\dots E_k$ as E and $U\mathbf{y}$ as $\mathbf{x}$, we have to solve $E\mathbf{x} = \mathbf{0}$. As we know, E is obtained by as product of elementary matrices and elementary matrices are invertible, E is invertible and full rank, or all columns of E are independent. $\therefore E\mathbf{x} = \mathbf{0}$ is possible only when $\mathbf{x} = 0$ or $E\mathbf{y} = \mathbf{0}$. This implies that $\mathbf{y} \in \mathcal{N}(U)$. Since every vector in null space of R also belongs to null space of U, $\mathcal{N}(U) = \mathcal{N}(R)$.

6. (1 point) Construct a matrix whose column space contains $[2, 5, 3]^\top$ and $[0, 3, 1]^\top$ and whose null space contains $[1, 3, 2]^\top$

> **Solution:** Since $\mathcal{C}(A)$ contains, $\begin{bmatrix} 2 & 5 & 3 \end{bmatrix}^\top$ and $\begin{bmatrix} 0 & 3 & 1 \end{bmatrix}^\top$ (which are independent), we can write the matrix A as $\begin{bmatrix} 2 & 0 & a_{13} \\ 5 & 3 & a_{23} \\ 3 & 1 & a_{33} \end{bmatrix}$. Also, as $\begin{bmatrix} 1 & 3 & 2 \end{bmatrix}^\top \in \mathcal{N}(A)$. $\therefore \mathbf{a_3}$ can be written as $-\frac{1}{2}\mathbf{a_1} - \frac{3}{2}\mathbf{a_2}$. $\therefore \mathbf{a_3} = \begin{bmatrix} -1 & -7 & -3 \end{bmatrix}^\top$ and $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 3 & -7 \\ 3 & 1 & -3 \end{bmatrix}$

7. (2 points) Consider the matrix $A = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ 1 & 9 \end{bmatrix}$. The column space of this matrix is a 2 dimensional plane. What is the equation of this plane? (You need to write down the steps you took to arrive at the equation)

> **Solution:** Assuming the plane equation to be of the form $ax + by + cz = 0$. As this plane is formed by the column space of the given matrix, it will contain both the vectors (points), i.e. $\begin{bmatrix} 3 & 2 & 1 \end{bmatrix}^\top$, $\begin{bmatrix} 0 & 1 & 9 \end{bmatrix}^\top$ and all their linear combinations. $\therefore$ substituting them in the assumed plane equation we get, $3a + 2b + c = 0$ and $b + 9c = 0$. Now we have one free variable that is c. Assuming $c = 3$, we have $b = -27$ and $a = \frac{-3+54}{3} = 17$. $\therefore$ the plane equation can be $17x - 27y + 3z = 0$.

8. (1 point) True or false? (If true give logical, valid reasoning or give a counterexample if false)

a. If the row space equals the column space then $A^T = A$

Solution: False, as we can have $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix}$. Here, both row space and column space are $\mathbf{R}^3$ (whole 3D space), but $A \neq A^T$.

b. If $A^T = -A$ then the row space of A equals the column space.

Solution: True, as the row space of A can be written as $A^T\mathbf{x}$. But since $A^T = -A$, the row space becomes $-A\mathbf{x}$. Now replacing $-\mathbf{x}$ by $\mathbf{y}$, row space is $A\mathbf{y}$ which is linear combination of columns of A, i.e. the column space of A. $\therefore$ if $A^T = -A$, row space equals column space. Infact, if $A^T = kA$, this statement holds.

9. (1 point) What are the dimensions of the four subspaces for **A, B,** and **C,** if I is the 3×3 identity matrix and 0 is the 3×2 zero matrix?

A=$\begin{bmatrix} I & 0 \end{bmatrix}$ and B $= \begin{bmatrix} I & I \\ 0^\top & 0^\top \end{bmatrix}$ and C=$\begin{bmatrix} 0 \end{bmatrix}$

Solution: Given $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

For $A_{3\times5}$:
dim($\mathcal{C}(A)$): 3 (as it has 3 independent columns)
dim($\mathcal{N}(A)$): $5 - 3 = 2$ (rank-nullity theorem)
dim($\mathcal{R}(A)$): 3 (dim of row space = dim of column space)
dim($\mathcal{N}(A^T)$): $3 - 3 = 0$ (rank-nullity theorem)

For $B_{5\times6}$:
dim($\mathcal{C}(B)$): 3 (as it has 3 independent columns)
dim($\mathcal{N}(B)$): $6 - 3 = 3$ (rank-nullity theorem)
dim($\mathcal{R}(B)$): 3 (dim of row space = dim of column space)
dim($\mathcal{N}(B^T)$): $5 - 3 = 2$ (rank-nullity theorem)

For $C_{3\times 2}$:
$\dim(\mathcal{C}(C))$: 0 (no non-zero pivots)
$\dim(\mathcal{N}(C))$: $2 - 0 = 2$ (rank-nullity theorem)
$\dim(\mathcal{R}(C))$: 0 (dim of row space = dim of column space)
$\dim(\mathcal{N}(C^T))$: $3 - 0 = 3$ (rank-nullity theorem)

10. (2 points) Solve the following questions.

(a) If A is an m×n matrix, find $\dim(\mathcal{R}(A)) + \dim(\mathcal{C}(A)) + \dim(\mathcal{N}(A)) + \dim(\mathcal{N}(A^\top))$. (in terms of n & m)

Solution: Let the $\dim(\mathcal{C}(A)$ be r. Then,
$\dim(\mathcal{N}(A)) = n - r$ (Rank-Nullity Theorem)
$\dim(\mathcal{R}(A)) = r$ (row rank = column rank)
$\dim(\mathcal{N}(A^T)) = m - r$ (Rank-Nullity Theorem).
$\therefore \dim(\mathcal{R}(A)) + \dim(\mathcal{C}(A)) + \dim(\mathcal{N}(A)) + \dim(\mathcal{N}(A^\top)) = r+r+n-r+m-r = n+m$.

(b) Let A and B be two $n \times n$ matrices such that $AB = 0$. Show that the row space of A is contained in the left null space of B.

Solution: As $AB = 0$, $(AB)^T = B^T A^T = \mathbf{0}^T_{n\times n} = \mathbf{0}_{n\times n}$. If columns of A^T are $\mathbf{a_1}, \mathbf{a_2} \dots \mathbf{a_n}$, then $B^T\mathbf{a_1} = \mathbf{0}, B^T\mathbf{a_2} = \mathbf{0}, \dots B^T\mathbf{a_n} = \mathbf{0}$, which imples that $\mathbf{a_1}, \mathbf{a_2} \dots \mathbf{a_n} \in \mathcal{N}(B^T)$. Since all the columns of A^T belong to null space of B^T, any linear combination of those columns of A^T, i.e. the column space of A^T also belongs to the null space of B^T, i.e. $\mathcal{C}(A^T) \in \mathcal{N}(B^T)$. But as we know $\mathcal{C}(A^T) = \mathcal{R}(A)$, we can conclude that $\mathcal{R}(A) \in \mathcal{N}(A^T)$, i.e. row space of A is contained in left null space of B.

11. (1 point) True or false? If A is a $n \times n$ square matrix then $\mathcal{N}(A) = \mathcal{N}(AA^T)$ (If true give logical, valid reasoning or give a counterexample if false)

Solution: False, as we can find the following counter example.
Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ which upon Guassian elimination yields $U_A = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$, which implies the null space is of the form $\alpha \begin{bmatrix} -3 & 1 \end{bmatrix}^\top$. $AA^T = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}$ which upon Gaussian Elimination yields $U_{AA^T} = \begin{bmatrix} 10 & 20 \\ 0 & 0 \end{bmatrix}$, which implies the null space is of the form $\alpha \begin{bmatrix} -2 & 1 \end{bmatrix}^\top$. Hence, we can observe that $\mathcal{N}(A) \neq \mathcal{N}(AA^T)$.

12. (2 points) Without explicitly computing the product of given two matrices, find bases for each of its four sub-spaces.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

And also explain the four sub-spaces along with the method you followed to compute them.

Solution: Given $A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $M = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, let's denote the product by C, i.e. $C = MA$. Also, we have RREF of A as $\begin{bmatrix} 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and RREF of M as $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

As M performs row operations on A to get C, i.e. $R_2 \leftarrow R_1 + R_2$ and $R_3 \leftarrow R_2 + R_3$, the rows of C and A are linear combinations of each other. $\therefore$ the space spanned by the rows of A, i.e. $\mathcal{R}(A)$ is the same as the space spanned by rows of C, i.e. $\mathcal{R}(C)$. $\therefore$ basis for $\mathcal{R}(C)$ is same as basis for $\mathcal{R}(A)$ and is $\begin{bmatrix} 0 & 1 & 2 & 0 & -2 \end{bmatrix}^\top$ and $\begin{bmatrix} 0 & 0 & 0 & 1 & 2 \end{bmatrix}^\top$ (pivot rows in RREF).

To compute null space of C, we solve $C\mathbf{y} = MA\mathbf{y} = \mathbf{0}$ ($\mathbf{y} \in \mathcal{N}(C)$). If we denote $A\mathbf{y}$ as $\mathbf{x}$, we need to now solve $M\mathbf{x} = \mathbf{0}$. As M is full rank, $\mathbf{x}$ must be $\mathbf{0}$. $\therefore A\mathbf{y} = \mathbf{0}$, i.e. $\mathbf{y} \in \mathcal{N}(A)$. This implies that $N(C) = N(A)$. Here for A, x_1, x_3, x_5 are free variables. From the RREF of A, we have $x_4 = -2x_5$ and $x_2 = -2x_3 + 2x_5$. $\therefore$ parametric form of $\mathcal{N}(A)$ is $\begin{bmatrix} x_1 & -2x_3 + 2x_5 & x_3 & -2x_5 & x_5 \end{bmatrix}^\top = x_1 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}^\top + x_3 \begin{bmatrix} 0 & -2 & 1 & 0 & 0 \end{bmatrix}^\top + x_5 \begin{bmatrix} 0 & 2 & 0 & -2 & 1 \end{bmatrix}^\top$. $\therefore$ we can conclude that basis for $N(C)$ which is also basis for $\mathcal{N}(A)$ is $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}^\top$, $\begin{bmatrix} 0 & -2 & 1 & 0 & 0 \end{bmatrix}^\top$ and $\begin{bmatrix} 0 & 2 & 0 & -2 & 1 \end{bmatrix}^\top$.

To compute column space space of C we solve $C\mathbf{y} = MA\mathbf{y} = \mathbf{b}$ ($\mathbf{b} \in \mathcal{C}(C)$). As $A\mathbf{y}$ denotes column space of A, we have column space of $C =$ M(column space of A). From the above RREF, $\mathcal{C}(A) = \alpha_1 \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\top + \alpha_2 \begin{bmatrix} 3 & 1 & 0 \end{bmatrix}^\top$. $\therefore \mathcal{C}(C) = M(\alpha_1 \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\top + \alpha_2 \begin{bmatrix} 3 & 1 & 0 \end{bmatrix}^\top) = \alpha_1 \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^\top + \alpha_2 \begin{bmatrix} 3 & 4 & 1 \end{bmatrix}^\top$. Hence basis for $\mathcal{C}(C)$ is $\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^\top$ and $\begin{bmatrix} 3 & 4 & 1 \end{bmatrix}^\top$.

The left null space of C is obtained by solving $C^T\mathbf{y} = \mathbf{0}$ ($\mathbf{y} \in \mathcal{N}(C^T)$). $C^T\mathbf{y} = A^T M^T \mathbf{y} = 0$. If we denote $M^T\mathbf{y}$ as $\mathbf{x}$, we have $A^T\mathbf{x} = \mathbf{0}$, i.e. $\mathbf{x} \in \mathcal{N}(A^T)$. From the RREF of A^T, we have one free variable x_3, and $x_1 = x_2 = 0$, $\therefore \mathcal{N}(A^T)$ is of the form $x_3 \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$. $\therefore$ we have $M^T\mathbf{y} = \begin{bmatrix} y_1 + y_2 & y_2 + y_3 & y_3 \end{bmatrix}^\top = x_3 \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$. To obtain the basis of $\mathcal{N}(C^T)$, we solve $\begin{bmatrix} y_1 + y_2 & y_2 + y_3 & y_3 \end{bmatrix}^\top = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$, $\therefore y_3 = 1, y_2 = -1, y_1 = 1$. $\therefore$ basis for $\mathcal{N}(C^T)$ is $\begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^\top$.

Concept: Free variables

13. (2 ½ points) True or False (with reason if true or example to show it is false).

(a) An matrix $m \times n$ can have zero pivots.

Solution: True, as we can have zero matrix, i.e. $\mathbf{0}_{m\times n}$. This matrix has all elements as 0 and hence no pivots.

(b) A real-symmetric matrix $m \times m$ has no free variables.

Solution: False. Consider the real symmetric matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix}$, which upon Guassian Elimination produces $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. This matrix has two pivots and one free variable, which disproves the given statement.

(c) If A & B be are two $m \times n$ matrices with non-zero pivots, then a matrix C = A + B can have zero pivots

Solution: True. Let A be a matrix with rank r ($r > 0$). Now if $B = -A$, B is also a matrix with rank r (r independent columns). Then $C = A + B = A - A = \mathbf{0}_{m\times n}$ which has no pivots.

(d) A free variable in a matrix always implies that there is either a zero-row or zero-column in the matrix.

Solution: False. Consider $A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. This matrix is in Row Reduced Echelon Form (RREF) and has one free-variable, x_4 (null space is of the form $x_4 \begin{bmatrix} -3 & -5 & -1 & 1 \end{bmatrix}^\top$), but has no zero-row/zero-column. Hence, disproved.

(e) For any matrix A, does A^T and A^{-1} have the same number of pivots.

Solution: False, as A^T will have the same number of pivots as A (as $\dim(\mathcal{R}(A)) = \dim(\mathcal{C}(A))$), but if A is rectangular or non-invertible, we cannot have its inverse and cannot find pivots for A^{-1}. Hence, for all matrices, we cannot conclude the given statement.

Concept: Reduced Echelon Form

14. (½ point) Suppose R is $m \times n$ matrix of rank r, with pivot columns first:

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

(a) Find a right-inverse B with $RB = I$ if $r = m$.

Solution: If $r = m$, then R is of the form $\begin{bmatrix} I & F \end{bmatrix}$ (no zeros below). A possible value for B can be $\begin{bmatrix} I \\ \mathbf{0} \end{bmatrix}$ where I is identity matrix of size $m \times m$ and $\mathbf{0}$ is a zero matrix of size $m \times (n-m)$. $\therefore RB = \begin{bmatrix} I & F \end{bmatrix} \begin{bmatrix} I \\ \mathbf{0} \end{bmatrix} = I \cdot I + F \cdot \mathbf{0} = I$.