Honor code: I pledge on my honor that: I have completed all steps in the below homework on my own, I have not used any unauthorized materials while completing this homework, and I have not given anyone else access to my homework.

Varun Gumma

Name and Signature

1. (1 point) Have you read and understood the honor code?

Solution: YES

Concept: Linear Transformation

2. (1 point) Consider a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$. Suppose $T(\begin{bmatrix}2 & 3 & -6\end{bmatrix}^\top) = \begin{bmatrix}2 & 5 & 1\end{bmatrix}^\top$ and $T(\begin{bmatrix}4 & -2 & 8\end{bmatrix}^\top) = \begin{bmatrix}7 & -2 & -2\end{bmatrix}^\top$. Find $T(\begin{bmatrix}-2 & 13 & -34\end{bmatrix}^\top)$

Solution: Assuming $\begin{bmatrix}2 & 3 & -6\end{bmatrix}^\top = \mathbf{a}$ and $\begin{bmatrix}4 & -2 & 8\end{bmatrix}^\top = \mathbf{b}$, vector $\begin{bmatrix}-2 & 13 & -34\end{bmatrix}^\top$ can be written as $3\mathbf{a} - 2\mathbf{b}$. Therefore, $T(3\mathbf{a} - 2\mathbf{b}) = 3T(\mathbf{a}) - 2T(\mathbf{b}) = \begin{bmatrix}-8 & 19 & 7\end{bmatrix}^\top$

3. (1 point) Prove that if $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ and $T(a\mathbf{x}) = aT(\mathbf{x})$ then $T(b\mathbf{x} + c\mathbf{y}) = bT(\mathbf{x}) + cT(\mathbf{y})$.

Solution: Taking $b\mathbf{x}$ as $\mathbf{u}$ and $c\mathbf{y}$ as $\mathbf{v}$, we can write the operation as $T(\mathbf{u}+\mathbf{v})$. Now, according to the rules $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) = T(b\mathbf{x}) + T(c\mathbf{y}) = bT(\mathbf{x}) + cT(\mathbf{y})$.

4. (2 points) Let T be a transformation defined from $\mathbb{R}^n \to \mathbb{R}^n$ given by
$T(\mathbf{X}) = \mathbf{X} + 1$ where $\mathbf{X} \in \mathbf{R}^n$
Formally argue if T is a linear transformation or not. If Yes, then give the matrix representing that transformation.

Solution: Assume this represents a valid linear transformation. $T(a\mathbf{X}) = a\mathbf{X} + \mathbf{1}$ (according to given statement, here $\mathbf{1}$ represents an n-dim vector of all ones). But as it is a linear transformation, $T(a\mathbf{X}) = aT(\mathbf{X}) = a(\mathbf{X} + \mathbf{1}) = a\mathbf{X} + a\mathbf{1} \neq a\mathbf{X} + \mathbf{1}$ $\forall a \in \mathbb{R}$. This is a contradiction, hence, the assumption that T represents a valid transformation is wrong. T is not a linear transformation.

5. (2 points) Suppose $A \in \mathbf{R}^{3\times3}$ and $\mathbf{x}, \mathbf{y} \in \mathbf{R}^3 (\mathbf{x} \neq \mathbf{0}, \mathbf{y} \neq \mathbf{0})$. Further, suppose $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{y} = \begin{bmatrix}0 & 0 & 0\end{bmatrix}^\top$. If $\begin{bmatrix}1 & 1 & 1\end{bmatrix}^\top$ is one solution for $A\mathbf{x} = b$, write down at least one more solution (you are welcome to write down all the infinite solutions if you want :-))

Solution: Since $\mathbf{Ay} = \begin{bmatrix}0 & 0 & 0\end{bmatrix}^\top$, $\mathbf{A}(\begin{bmatrix}1 & 1 & 1\end{bmatrix}^\top + \mathbf{y}) = \mathbf{A}\begin{bmatrix}1 & 1 & 1\end{bmatrix}^\top + \mathbf{Ay} = \mathbf{b} + \mathbf{0} = \mathbf{b}$. Hence, another solution can be $\begin{bmatrix}1 & 1 & 1\end{bmatrix}^\top + \mathbf{y}$. In fact, a general solution can be $\begin{bmatrix}1 & 1 & 1\end{bmatrix}^\top + \lambda\mathbf{y}$, $\lambda \in \mathbf{R}$.

Concept: Matrix multiplication

6. (1 point) Statement: If A and B are matrices such that A is not a Null or Identity matrix and B is not a Null matrix,
if $AB = A^2$ then A = B.
options:
a) always true,
b) always false,
c) sometimes can be true , sometimes can be false also
Explain your answer based on the option you have chosen.

Solution: C - sometimes can be true, sometimes can be false also. The statement holds when A is invertible. $A^{-1}AB = A^{-1}A^2$, which means $A = B$. An example where the statement does not hold is when $A = \begin{bmatrix}1 & 2 \\ 1 & 2\end{bmatrix}$. Here $B = \begin{bmatrix}-1 & -6 \\ 2 & 6\end{bmatrix}$ can result in $AB = A^2 = \begin{bmatrix}3 & 6 \\ 3 & 6\end{bmatrix}$ and clearly $A \neq B$.

7.

$$A = \begin{bmatrix} 1 & 2 & 5 & 6 \\ -1 & 3 & -2 & 1 \\ 3 & 0 & 0 & 1 \\ 1 & 5 & 4 & -14 \end{bmatrix}$$

For each of the equations below, find $\mathbf{x}$

(a) (½ point) $A\mathbf{x} = \begin{bmatrix}1 & 10 & 4 & -11\end{bmatrix}^\top$

Solution: $\mathbf{x} = \begin{bmatrix}1 & 2 & -2 & 1\end{bmatrix}^\top$

(b) (½ point) $A\mathbf{x} = \begin{bmatrix}16 & 4 & -30 & 81\end{bmatrix}^\top$

Solution: $\mathbf{x} = \begin{bmatrix} -9 & 4 & 7 & -3 \end{bmatrix}^\top$

8. (1 point) Give two matrices A and B (of appropriate dimensions) such that $A \neq B$ and,

   (a) (½ point) $AB = BA$

   Solution: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Here $AB = BA = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

   (b) (½ point) $AB \neq BA$

   Solution: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Here $AB = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ and $BA = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$

9. If A, B & C are matrices (assume appropriate dimensions) prove that,

   (a) (½ point) $(A + B)^T = A^T + B^T$

   Solution: Let $A + B = C$ and $C^T = D$, $C_{ij} = A_{ij} + B_{ij}$. Upon transpose, C_{ji} becomes the ij^{th} element of D. $\therefore D_{ij} = C_{ji} = A_{ji} + B_{ji}. = A^T_{ij} + B^T_{ji}$. $\therefore C^T = (A + B)^T = A^T + B^T$.

   (b) (½ point) $(AB)^T = B^T A^T$

   Solution: Let $AB = C$, $C_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$. $(AB)^T = C^T$, $\therefore C_{ij}$ becomes ji^{th} element upon transpose which is still numerically equal to $\sum_{k=1}^{n} A_{ik}B_{kj}$. Adjusting the terms to get ji, the formula can be re-written as $\sum_{k=1}^{m} B^T_{jk}A^T_{ki}$ $\therefore C^T = (AB)^T = B^T A^T$.

10. (1 point) Let A be any matrix. In the lecture we saw that $A^\top A$ is a square symmetric matrix. Is $AA^\top$ also a square symmetric matrix? (Hint: The answer is either "Yes, except when ..." or "No, except when ...".)

    Solution: Yes, it is symmetric as $(AA^T)^T = (A^T)^T A^T = AA^T$ [$\because (A^T)^T = A$].

Concept: Inverse

11. (1 point) If $(A+B)^2 = A^2+2AB+B^2$. Show that $AB = BA$. (assume AB, BA exists).

Solution: $(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2 = A^2 + 2AB + B^2$. Cancelling square terms, $AB + BA = AB + AB$ or $AB = BA$.

12. What is the inverse of the following two matrices? (Hint: I don't want you to compute the inverse using some method. Instead think of the linear transformation that these matrices do and think how you would reverse that transformation. **You will have to explain your answer in words clearly stating the linear transformations being performed.**)

(a) (½ point)

$$A = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Solution: This matrix scales all the four standard basis vectors by $\frac{1}{2}$. $\therefore$ the inverse will need to re-scale the vectors two times to the get original standard basis. Hence, $A^{-1} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

(b) (½ point)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution: This matrix causes twice the second element to be added to the first element. The second and third element remain unchanged. (i.e. if $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\top$, $A\mathbf{x} = \begin{bmatrix} x_1 + 2x_2 & x_2 & x_3 \end{bmatrix}^\top$). Hence, the inverse of A must subtract twice second element from the first, without altering the second and third elements. $\therefore A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) (1 point)

$$A = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$

Solution: This matrix rotates the standard basis by θ anti-clockwise. Hence the inverse will be to rotate back the axis by θ in the clockwise direction ($-\theta$ in the anti-clockwise direction). $\therefore A^{-1} = \begin{bmatrix} cos(-\theta) & -sin(-\theta) \\ sin(-\theta) & cos(-\theta) \end{bmatrix} = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}$

Concept: System of linear equations

13. (1 point) Argue why the following system of linear equations will not have any solutions.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ -7 & -7 & -7 & -7 \\ 2 & 4 & 6 & 9 \\ 1 & 2 & 3 & 4 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 5 \\ -35 \\ 10 \\ 6 \end{bmatrix}$$

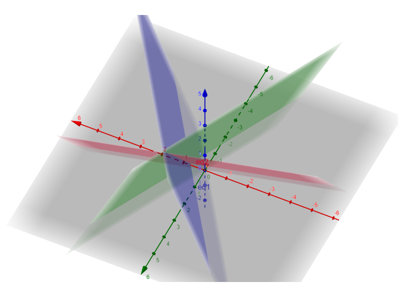
Solution: Assuming $\mathbf{x}$ to be $\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^\top$, the equations $x_1+2x_2+3x_3+4x_4 = 5$ and $x_1+2x_2+3x_3+4x_4 = 6$ represent parallel hyperplanes and hence this system cannot have a solution.

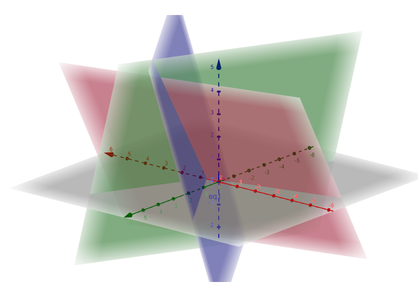
14. Consider the following 3 planes

$$\begin{aligned} 3x + 2y - z &= 2 \\ x - 4y + 3z &= 1 \\ 4x - 2y + 2z &= 3 \end{aligned}$$

(a) ($\frac{1}{2}$ point) Plot these planes in geogebra and paste the resulting figure here (you can download the figure as .png and paste it here)

Solution:





(b) ($\frac{1}{2}$ point) How many solutions does the above system of linear equations have? (based on visual inspection in geogebra)

Solution: Infinitely many solutions.

(c) (1 point) Notice that the third equation can be obtained by adding the first two equations. Based on this observation, can you explain your answer for the number of solutions in the previous part of the question. (Note that I am looking for an answer in plain English which does not include terms like "linear independence" or "dependence of columns/rows". In other words, your answer should be based only on concepts/ideas which have already been discussed in the class)

Solution: Here P_1 and P_2 are definitely known to intersect in line (infinite solutions) as they are not parallel. As $P_3 = P_1 + P_2$, the points which satisfy P_1, P_2 (points for which $P_1 = 0$ and $P_2 = 0$, i.e. the line of intersection) also satisfy P_3 as for those points $P_3 = 0 + 0 = 0$. This means, P_3 plane also passes through the points of intersection of P_1 and P_2 which yields infinite solutions for this system.

15. Consider the following system of linear equations:

$$\begin{aligned} x + 2y + 4z &= 1 \\ x + 5y - 2z &= 2 \end{aligned}$$

Add one more equation to the above system such that the resulting system of 3 linear equations has

(a) ($\frac{1}{2}$ point) 0 solutions

Solution: There will be no solutions in a case when the third plane is parallel on either of the planes. $\therefore$ P_3 can be $x + 5y - 2z = 1$.

(b) (1 point) exactly 1 solution

Solution: We have exactly one solution when the third plane is not parallel or a linear combination of the first two planes. $\therefore$ we can have P_3 as $x + y + z = 10$.

(c) ($\frac{1}{2}$ point) infinite solutions

Solution: Here, if the third plane passes through the intersection of the first two planes, then we have infinite solutions. $\therefore$ $P_3 = P_1 + \lambda P_2$. With $\lambda = 1$, we can have P_3 as $2x + 7y + 2z = 3$.