Honor code: I pledge on my honor that: I have completed all steps in the below homework on my own, I have not used any unauthorized materials while completing this homework, and I have not given anyone else access to my homework.

Varun Gumma

Name and Signature

1. (1 point) Have you read and understood the honor code?

   Solution: YES

Count, Count, Count!

2. (1 point) In how many ways can 10 people be seated:

   (a) in a row such that Motu and Patlu sit next to each other (there is only one boy named Motu and only one boy named Patlu in the group)

   Solution: $9! \cdot 2 = 725760$

   (b) in a row such that there are 5 engineers and 5 doctors and no two doctors or no two engineers can sit next to each other

   Solution: $5! \cdot 5! \cdot 2 = 28800$

   (c) in a row such that there are 3 engineers, 3 doctors and 4 lawyers and all people of the same profession should sit in consecutive positions.

   Solution: $3! \cdot (3! \cdot 3! \cdot 4!) = 5184$

   (d) in a row such that there are 5 married couples and each couple must sit together.

   Solution: $5! \cdot 2^5 = 3840$

3. ($\frac{1}{2}$ point) How many unique 9 letter words can you form using the letters of the word MANMOHANA (the words can be gibberish)?

> **Solution:** If a character repeats k times we divide the result by $k!$ as the total permutations have a set of $k!$ permutations with identical characters interchanging which actually contribute to only one distinct permuatation. As frequencies of characters are M - 2, A - 3, N - 2, O - 1, H - 1, the number of distinct permutations are $\frac{9!}{2!\cdot3!\cdot2!} = 15120$.

4. (½ point) Suppose you have a class of 7 students (A,B,C,D,E,F,G) who need to be arranged in a line with the following restrictions:

   1. A has to be in one of the first 3 slots
   2. B and A are very good friends and insist on being next to each other
   3. B doesn't want to stand immediately behind C

   In how many different ways can you arrange them?

> **Solution:**
>
> 1. *A is in first place*: B can be places 2 only. Since C can appear only after B (from place 3) it doesn't violate the third condition, we have 5! combinations (rest 5 characters can be arranged in the 5 places in 5! ways), i.e. 120 cases.
> 2. *A is in second place*: B can be places 1 or 3. In first case, B is first element and in the second case B is preceded by A. Hence, both don't violate third condition and we have $5! \times 2 = 240$ cases.
> 3. *A is in third place*: B can be in places 2 or 4. If it is in the second place, C cannot be in the first place as it violates third condition. $\therefore$ we have only $4 \times 4! = 96$ cases. If B is in fourth place, it is preceded by A and doesn't violate third condition wherever C is placed. $\therefore$ $5! = 120$ cases.
>
> $\therefore$ total combinations are $120 + 240 + 96 + 120 = 576$.

The boring questions are done. I hope you find the rest of the assignment to be interesting!

The birthday problem

5. (3 points) The days of the year can be numbered 1 to 365 (ignore leap days). Consider a group of n people, of which you are not a member. Any of the 365 days is equally likely to be the birthday of any member of this group. An element of the sample space Ω will be a sequence of n birthdays (one for each person).

(a) How many elements are there in the sample space?

> **Solution:** 365^n as each person from the n people have 365 equally likely choices for their birthday and the birthdays are independent of one another.

(b) Let A be the event that at least one member of the group has the same birthday as you. What is the probability of this event A?

> **Solution:** $P(\text{atleast one member has same birthday as me}) = 1 - P(\text{none have the same birthday as me}) = 1 - \frac{364^n}{365^n} = 1 - (\frac{364}{365})^n$.

(c) What is the minimum value of n such that $P(A) \geq 0.5$?

> **Solution:** $\therefore 1 - (\frac{364}{365})^n \geq 0.5$ or $(\frac{364}{365})^n \leq 0.5$. Taking a log on both sides $n \log(\frac{364}{365}) \leq \log(0.5)$ or $n(\log(365) - \log(364)) \geq \log(2)$. $\therefore n \geq \frac{\log(2)}{\log(365)-\log(364)} = 252.651$. $\therefore n = 253$ is the minimum value that satisfies the equation.

(d) Let B be the event that at least two members of the group share the same birthday. What is the probability of this event B?

> **Solution:** $P(\text{atleast 2 people have the same birthday}) = 1 - P(\text{all have unique birthday})$. When everyone has unique birthday, first person has 365 choices, second person has 364 choices and so on till the n^{th} person who will have $365 - n + 1$ choices. As all these are independent, total number of combinations are $365 \times 364 \times \cdots \times (365 - n + 1)$. $\therefore P(\text{all have unique birthday}) = \frac{\prod_{i=1}^{n}(365-i+1)}{365^n} = \frac{n! \cdot \binom{365}{n}}{365^n}$. $P(\text{atleast two people have same birthday}) = 1 - \frac{n! \cdot \binom{365}{n}}{365^n}$.

(e) What is the minimum value of n such that $P(B) \geq 0.5$?

> **Solution:** $1 - \frac{n! \cdot \binom{365}{n}}{365^n} \geq 0.5$ or $\frac{n! \cdot \binom{365}{n}}{365^n} \leq 0.5$. $n = 23$ gives us $\frac{23! \cdot \binom{365}{23}}{365^{23}} = 0.493$ which is just less than 0.5. Hence, a minimum of 23 people are sufficient.

(f) **[Ungraded question]** Why is there a big gap between the answers to part (c) and part (e)? (although at "first glance" they look very similar problems)

A biased coin

6. (1 point) Your friend Chaman has a coin which is biased (i.e., $P(H) \neq P(T)$). He proposes that he will toss the coin twice and asks you to bet on one of these events: A: both the tosses will result in the same outcome or B: both the tosses will result in a different outcome. Which event will you bet on to maximize your chance of winning the bet. (I am looking for a precise mathematical answer. No marks for answers which do not have an explanation).

Solution: Let the coin be biased with probability of head being p and probability of tail being $1-p$ ($p \neq 0.5$). P(both tosses have same outcome) $= P(\text{HH} \cup \text{TT}) = P(\text{HH}) + P(\text{TT}) = p^2 + (1-p)^2$ and P(both tosses have different outcome) $= P(\text{HT} \cup \text{TH}) = P(\text{HT}) + P(\text{TH}) = 2p(1-p)$. Let's compare $p^2 + (1-p)^2$ and $2p(1-p)$. We know from AM-GM inequality that for any two non-negative integers a and b, $a + b \geq 2\sqrt{ab}$. If $a = p^2$ and $b = (1-p)^2$ (both are non-negative as they are square terms), $p^2 + (1-p)^2 \geq 2\sqrt{p^2(1-p)^2} = 2p(1-p)$. $\therefore p^2 + (1-p)^2 \geq 2p(1-p)$ or $P(\text{HH} \cup \text{TT}) \geq P(\text{HT} \cup \text{TH})$. Hence, if we choose event A, i.e. both tosses will result in same outcome, our chance of winning is higher.

Alice in Wonderland

7. (1 point) A bag contains one ball which could either be green on red. You take another red ball and put it in this pouch. You now close your eyes and pull out a ball from the pouch. It turns out to be red. What is the probability that the original ball in the pouch was red?

Solution: By Bayes Theorem,
$$P(Pouch = R|Out = R) = \frac{P(Out{=}R|Pouch{=}R) \cdot P(Pouch{=}R)}{P(Out{=}R|Pouch{=}R) \cdot P(Pouch{=}R) + P(Out{=}R|Pouch{=}G) \cdot P(Pouch{=}G)}$$

Here, $P(Pouch = G) = P(Pouch = R) = \frac{1}{2}$ as the pouch can equally likely have a green ball or red ball. $P(Out = R|Pouch = R) = 1$ as the bag in this case will contain two red balls (one of them additionally added) and pulling out a red ball is a sure outcome. $P(Out = R|Pouch = G) = \frac{1}{2}$ as the bag in this case will contain one green and one red (additionally added) and pulling out the red ball has a probability of $\frac{1}{2}$. Substituting the same in the expression, we have $P(Pouch = R|Out = R) = \frac{1 \cdot \frac{1}{2}}{(1+\frac{1}{2}) \cdot \frac{1}{2}} = \frac{(\frac{1}{2})}{(\frac{3}{4})} = \frac{2}{3}$.

Rock, paper and scissors

8. (2 points) Your friend Chaman has 3 strange dice: red, yellow and green. Unlike a standard die whose 6 faces are the numbers 1,2,3,4,5,6 these 3 dice have the following faces: red: 3,3,3,3,3,6, yellow: 5,5,5,2,2,2 and green: 4,4,4,4,4,1. Chaman suggests the following game: (i) You pick any one die (ii) Chaman then "carefully" picks one of the remaining two dice. Each of you will then roll your own die a 100 times. If on a given roll, the score of your die is higher than the score of Chaman's die then you get 1 INR else Chaman gets 1 INR. You play this game for many days and realise that you lose more often than Chaman.

   (a) Why are you losing more often? or What is Chaman's "carefully" planned strategy? (the key thing to note is that he lets you choose first)

Solution: We solve this by cases to see which die Chaman chooses when we choose a particular die. As Chaman wants to win always, we chooses the die which will give him higher expected earning than us. Assuming Chaman has a probability p of winning (then we have a probability of $1-p$ of winning), $\mathbb{E}$(Chaman's earning with 100 rolls) $= 100 \cdot (p \cdot 1 + (1-p) \cdot 0) = 100p$ and $\mathbb{E}$(our earning with 100 rolls) $= 100 \cdot (p \cdot 0 + (1-p) \cdot 1) = 100(1-p)$. $\therefore$ as Chaman wants $100p > 100(1-p)$ or $p > 1-p$ or $p > 0.5$, he chooses the optimal die which would have a better probability of landing a higher number than our die or with which his probability of winning is greater than $\frac{1}{2}$.

For red die, $P(3) = \frac{5}{6}$, $P(6) = \frac{1}{6}$.
For yellow die, $P(5) = \frac{1}{2}$, $P(2) = \frac{1}{2}$.
For green die, $P(4) = \frac{5}{6}$, $P(1) = \frac{1}{6}$.

Let $P(X, Y)$ denote the probability that we roll X and Chaman rolls Y on our respective dice. But as our rolls are independent $P(X, Y) = P(X) \cdot P(Y)$

1. *We choose red*:
   (a) *Chaman chooses yellow*: P(Chaman gets a higher digit) $= P(3, 5) = P(3) \cdot P(5) = \frac{5}{12}$. $\therefore P$(we get a higher digit) $= 1 - \frac{5}{12} = \frac{7}{12}$.
   (b) *Chaman chooses green*: P(Chaman gets a higher digit) $= P(3, 4) = P(3) \cdot P(4) = \frac{25}{36}$. $\therefore P$(we get a higher digit) $= 1 - \frac{25}{36} = \frac{11}{36}$.

   Hence, if Chaman chooses green die, his chance of winning is more than us and he chooses the same. $\therefore$ his expected earning with 100 rolls is, $100 \cdot \frac{25}{36} = 69.45$ INR and our expected earning is $100 \cdot \frac{11}{36} = 30.56$ INR.

2. *We choose yellow*:
   (a) *Chaman chooses red*: P(Chaman gets a higher digit) $= P(5, 6) + P(2, 3) + P(2, 6) = P(5) \cdot P(6) + P(2) \cdot P(3) + P(2) \cdot P(6) = \frac{1}{12} + \frac{5}{12} + \frac{1}{12} = \frac{7}{12}$. $\therefore P$(we get a higher digit) $= 1 - \frac{7}{12} = \frac{5}{12}$.
   (b) *Chaman chooses green*: P(Chaman gets a higher digit) $= P(2, 4) = P(2) \cdot P(4) = \frac{5}{12}$. $\therefore P$(we get a higher digit) $= 1 - \frac{5}{12} = \frac{7}{12}$.

   Hence, if Chaman chooses red die, his chance of winning is more than us and he chooses the same. $\therefore$ his expected earning with 100 rolls is, $100 \cdot \frac{7}{12} = 58.34$ INR and our expected earning is $100 \cdot \frac{5}{12} = 41.67$ INR.

3. *We choose green*:
   (a) *Chaman chooses yellow*: P(Chaman gets a higher digit) $= P(4, 5) + P(1, 5) + P(1, 2) = P(4) \cdot P(5) + P(1) \cdot P(5) + P(1) \cdot P(2) = \frac{5}{12} + \frac{1}{12} + \frac{1}{12} = \frac{7}{12}$. $\therefore P$(we get a higher digit) $= 1 - \frac{7}{12} = \frac{5}{12}$.

(b) *Chaman chooses red*: P(Chaman gets a higher digit) $= P(4,6) + P(1,3) + P(1,6) = P(4) \cdot P(6) + P(1) \cdot P(3) + P(1) \cdot P(6) = \frac{5}{36} + \frac{5}{36} + \frac{1}{36} = \frac{11}{36}$. $\therefore$ P(we get a higher digit) $= 1 - \frac{11}{36} = \frac{25}{36}$.

Hence, if Chaman chooses yellow die, his chance of winning is more than us and he chooses the same. $\therefore$ his expected earning with 100 rolls is, $100 \cdot \frac{7}{12} = 58.34$ INR and our expected earning is $100 \cdot \frac{5}{12} = 41.67$ INR.

The strategy of choosing the optimal die is as follows:

1. If we choose red die, Chaman chooses green die.
2. If we choose yellow die, Chaman chooses red die.
3. If we choose green die, Chaman chooses yellow die.

These selections ensure than Chaman always has a higher earning than us over many days.

(b) You realise what is happening and decide to turn the tables on Chaman. You buy 3 dice which are identical to Chaman's red, yellow and green dice. You now propose that instead of rolling a single die each of you will roll two dice of the same color. The rest of the rules remain the same (i) You pick any one color (ii) Chaman then uses his original strategy to carefully pick a different color (he is overconfident and simply uses the same strategy that he used when you were rolling only one die) (iii) If on a given roll, the sum of your two dice is greater than the sum of Chaman's two dice then you get 1 INR else Chaman gets 1 INR. To his horror Chaman realises that now he is loosing more often. Explain why?

Solution:

1. If 2 red die are rolled, the sum of the outcomes can be 6, 9, 12 and $P(6) = P(3) \cdot P(3) = \frac{25}{36}$, $P(9) = 2 \cdot P(3) \cdot P(6) = \frac{10}{36}$, $P(12) = P(6) \cdot P(6) = \frac{1}{36}$.
2. If 2 yellow die are rolled, the sum of the outcomes can be 10, 7, 4 and $P(10) = P(5) \cdot P(5) = \frac{1}{4}$, $P(7) = 2 \cdot P(5) \cdot P(2) = \frac{2}{4}$, $P(4) = P(2) \cdot P(2) = \frac{1}{4}$.
3. If 2 green die are rolled, the sum of the outcomes can be 8, 5, 2 and $P(8) = P(4) \cdot P(4) = \frac{25}{36}$, $P(5) = 2 \cdot P(3) \cdot P(6) = \frac{10}{36}$, $P(2) = P(1) \cdot P(1) = \frac{1}{36}$.

Let $P(X,Y)$ denote the probability that the sum of digits on our die is X and sum of digits of Chaman's die is Y. Since, our rolls are independent, $P(X,Y) = P(X) \cdot P(Y)$.

1. *We choose red*: For this Chaman chooses a green die. P(Chaman gets a larger sum) $= P(6,8) = P(6) \cdot P(8) = \frac{625}{1296}$. $\therefore$ P(we get a larger sum)

$= 1 - \frac{625}{1296} = \frac{671}{1296}$. Hence, we have a higher chance of winning. Here, our expected earning is $100 \cdot \frac{671}{1296} = 51.774$ INR and Chaman's expected earning is $100 \cdot \frac{625}{1296} = 48.226$ INR.

2. *We choose yellow*: For this Chaman chooses a red die. P(we get a larger sum) $= P(10,6) + P(10,9) + P(7,6) = P(10) \cdot P(6) + P(10) \cdot P(9) + P(7) \cdot P(7) \cdot P(6) = \frac{25}{144} + \frac{10}{144} + \frac{50}{144} = \frac{85}{144}$. $\therefore$ P(Chaman gets a larger sum) $= 1 - \frac{85}{144} = \frac{59}{144}$. Hence, we have a higher chance of winning. Here, our expected earning is $100 \cdot \frac{85}{144} = 51.027$ INR and Chaman's expected earning is $100 \cdot \frac{69}{1296} = 40.973$ INR.

3. *We choose green*: For this Chaman chooses yellow die. P(we get a larger sum) $= P(8,7) + P(8,4) + P(5,4) = P(8) \cdot P(7) + P(8) \cdot P(4) + P(5) \cdot P(4) = \frac{50}{144} + \frac{25}{144} + \frac{10}{144} = \frac{85}{144}$. $\therefore$ P(Chaman gets a larger sum) $= 1 - \frac{85}{144} = \frac{59}{144}$. Hence, we have a higher probability of winning. Here, our expected earning is $100 \cdot \frac{85}{144} = 51.027$ INR and Chaman's expected earning is $100 \cdot \frac{59}{1296} = 40.973$ INR.

Even if we randomly choose a die at first, we always have a higher expected earning. $\therefore$ on a long stretch of rolls, we can earn more than Chaman.

Sitting under an apple tree

9. (1 point) Which of the following has a greater chance of success?

A. Six fair dice are tossed independently and at least one "6" appears.

B. Twelve fair dice are tossed independently and at least two "6"s appear.

C. Eighteen fair dice are tossed independently and at least three "6"s appear.

Explain your answer.

Solution: Each of the following follows a binomial distribution.

A. P(atleast one 6 appears) $= 1 - P$(no 6 appears). If no 6 appears means all six dice have numbers 1 to 5 and these have a probability of $\frac{5}{6}$. Since, each roll is independent of the other, the probabilities can be multiplied. P(no 6 appears) $= (\frac{5}{6})^6$. P(atleast one 6) $= 1 - (\frac{5}{6})^6 = 0.6651$.

B. P(atleast two 6 appear) $= 1-$ (P(no 6 appears) + P(only one 6 appears)) $= 1 - P$(no 6 appears) $-P$(only one 6 appears). Since, each roll is independent of the other, the probabilities can be multiplied. P(no 6 appears) $= (\frac{5}{6})^{12}$ and P(only one 6 appears) $= \binom{18}{1} \cdot (\frac{5}{6})^{12} \cdot (\frac{1}{6})$. $\therefore$ P(atleast two sixes) $= 1 - (\frac{5}{6})^{12} - 12 \cdot (\frac{5}{6})^{11} \cdot (\frac{1}{6}) = 0.6186$.

C. P(atleast three 6 appear) $= 1-(P$(no 6 appears) $+$ P(only one 6 appears) $+$ P(only two 6 appear)) $= 1 - P$(no 6 appears) $-P$(only one 6 appears) $-P$(only two 6 appear). Since, each roll is independent of the other, the probabilities can be multiplied. P(no 6 appears) $= (\frac{5}{6})^{18}$, P(only one 6 appears) $= \binom{18}{1}(\frac{5}{6})^{17}(\frac{1}{6})$ and P(only two 6 appear) $= \binom{18}{2}(\frac{5}{6})^{16}(\frac{1}{6})^2$. $\therefore$ P(atleast two 6 appear) $= 1 - (\frac{5}{6})^{18} - 18 \cdot (\frac{5}{6})^{17} \cdot (\frac{1}{6}) - 153 \cdot (\frac{5}{6})^{16} \cdot (\frac{1}{6})^2 = 0.5973$.

From the above calculations, we have, $P_A > P_B > P_C$. $\therefore$ event-A is, i.e. six fair dice are tossed independently and at least one 6 appears has a greater chance of success.

With love from Poland

10. (1 point) A chain smoker carries two matchboxes - one in his left pocket and another in his right pocket. Every time he wants to light a cigarette he randomly selects a matchbox from one of the two pockets and then uses a matchbox from that box to light his cigarette. Suppose he takes out a matchbox and sees for the first time that it is empty, what is the probability that the matchbox in the other pocket has exactly one matchstick left?

Solution: P(left matchbox) $= P$(right matchbox) $= \frac{1}{2}$. Let's assume each matchbox had n matchstickes. W.L.O.G, let the matchbox which was observed to be empty for the first time be the left one. $\therefore$ as n matchstickes from the left and $n-1$ matchstickes from the right have been exhausted and the left box was chosen one extra time, the total number of trials are $n + n - 1 + 1 = 2n$.

Let us denote this trial by a string S where the i^{th} character of the string is **L** if the left matchbox was chosen in the i^{th} trail or **R** if the right matchbox was chosen in the i^{th} trail. As at the end, the left matchbox was chosen, hence the last character of S (S has a length $2n$ as it has one character for each trail and we have $2n$ trials) is **L**. $\therefore$ S is a random permutation of n **L**s and $n-1$ **R**s. As probability of **L** and **R** is 0.5, this arrangement has a probability of $\frac{(2n-1)!}{n!(n-1)!} \cdot P(left)^n \cdot P(right)^{n-1} \cdot P(left) = \binom{2n-1}{n} \cdot (\frac{1}{2})^n \cdot (\frac{1}{2})^{n-1} \cdot \frac{1}{2} = \binom{2n-1}{n} \cdot (\frac{1}{2})^{2n}$ (as picking matchboxes are independent of each other, their probabilities can be multiplied). If the last chosen matchbox was the right one, the same argument holds and the last character of S will be **R** instead of **L**. Hence, it has the same probability. $\therefore$ the total probability for this problem $\binom{2n-1}{n} \cdot (\frac{1}{2})^{2n} + \binom{2n-1}{n} \cdot (\frac{1}{2})^{2n} = \binom{2n-1}{n} \cdot (\frac{1}{2})^{2n} \cdot 2 = \binom{2n-1}{n} \cdot (\frac{1}{2})^{2n-1}$.

A paradox

11. (1 point) Suppose there are 3 boxes:

1. a box containing two gold coins,
2. a box containing two silver coins,
3. a box containing one gold coin and one silver coin.

You select one box at random and draw a coin from it. The coin turns out to be a gold coin. You remove this coin and draw another coin from the same box. What is the probability that the second coin is also a gold coin?

Solution: Let G_1 denote the event that the first coin chosen in golden and G_2 denote the event the second coin drawn after removing the first coin in golden. $\therefore$ $P(G_2|G_1) = \frac{P(G_1 \cap G_2)}{P(G_1)}$. But $P(G_1) = P(G_1|Box_1) \cdot P(Box_1) + P(G_1|Box_2) \cdot P(Box_2) + P(G_1|Box_3) \cdot P(Box_3) = 1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{2}$. $G_1 \cap G_2$ is only possible when the first box is selected, only then even after discarding the first gold coin, we can get another from it. $\therefore P(G_1 \cap G_2) = P(Box_1) = \frac{1}{3}$. Substituting these in the equation, we have $P(G_2|G_1) = \frac{(\frac{1}{3})}{(\frac{1}{2})} = \frac{2}{3}$.

Once upon a time in Goa

12. (1 point) You are in one of the famous casinos in Goa[1]. You are observing the game of roulette. A roulette has 36 slots of which 18 are red and the remaining 18 are black. Each slot is equally likely. The manager places a ball on the roulette and then spins the roulette. When the roulette stops spinning, the ball lands in one of the 36 slots. If it lands in a slot which has the same color as what you bet on then you win. You do not believe in gambling but you are a student of probability[2]. You observe that the ball has landed in a black slot for the 26 consecutive rounds. Based on what you have learned in CS6015 you predict that there is a much higher chance of the ball landing in a red slot in the next round (since the probability of 27 consecutive black slots is very very low). You bet all your life's savings on red. What is the probability that you will win?

Solution: $P(R) = P(B) = \frac{18}{36} = \frac{1}{2}$. By conditional probability, $P(R|26B) = \frac{P(26B \cap R)}{P(26B)}$. Since the roulette outcomes are independent, $P(26B) = P(B \cap B \cap B \ldots B) = P(B) \cdot P(B) \ldots P(B)$ (26 times) $= \frac{1}{2} \cdot \frac{1}{2} \ldots \frac{1}{2} = (\frac{1}{2})^{26}$. $P(26B \cap R) = P(26B) \cdot P(R) = (\frac{1}{2})^{26}(\frac{1}{2})$. $\therefore P(R|26B) = \frac{P(26B \cap R)}{P(26B)} = \frac{(\frac{1}{2})^{26}(\frac{1}{2})}{(\frac{1}{2})^{26}} = \frac{1}{2} = P(R)$. This shows that, as the outcomes are independent, the previously obtained results (no matter what they are or how many occurences they have), they do not influence the current trial and the odds of winning the current trial are always 50%.

[1] I know about casinos in Goa purely out of academic interest.

[2] Ah! That's why you are in a casino! That makes perfect sense!

Oh Gambler! Thy shall be ruined!

13. (2 points) You play a game in a casino[3] where your chance of winning the game is p. Every time you win, you get 1 rupee and every time you lose the casino gets 1 rupee. You have i rupees at the start of the game and the casino has $N - i$ rupees (obviously, $N >> i$). The game ends when you go bankrupt or the casino goes bankrupt. In either case, the winner will walk away with a total of N rupees.

   (a) Find the probability p_i of winning when you start the game with i rupees.

   Solution: We can win the game with i rupees in disjoint ways:

   1. Win the first trial, get 1 rupee and then win the rest of the game with $i+1$ rupees. Since, both events are independent, the probability for this case is $p \cdot p_{i+1}$.

   2. Lose the first trail, give up 1 rupee and then win the rest of the game with $i-1$ rupees. Since, both events are independent, the probability for this case is $(1-p) \cdot p_{i-1}$

   $\therefore p_i = p \cdot p_{i+1} + p \cdot p_{i-1} = p \cdot p_{i+1} + (1-p) \cdot p_{i-1}$ or $p_{i+1} - p_i = \frac{1-p}{p} \cdot (p_i - p_{i-1})$. Here, $p_0 = 0$ as we can never win without money and $p_N = 1$ as we have already won. $\therefore \frac{p_{i+1}-p_i}{p_i - p_{i-1}} = \frac{1-p}{p}$ $(0 < i < N)$. Let's denote $\frac{1-p}{p}$ as α from now on.

   $\frac{p_2-p_1}{p_1-p_0} = \frac{p_2-p_1}{p_1} = \alpha$ or $p_2 = p_1(1+\alpha)$.
   $\frac{p_3-p_2}{p_2-p_1} = \frac{p_3-p_1(1+\alpha)}{p_1\alpha^2} = \alpha$ or $p_3 = p_1(1+\alpha+\alpha^2)$.
   $\frac{p_4-p_3}{p_3-p_2} = \frac{p_4-p_1(1+\alpha+\alpha^2)}{p_1\alpha^3} = \alpha$ or $p_4 = p_1(1+\alpha+\alpha^2+\alpha^3)$

   $\vdots$

   Following this pattern, we have $p_N = 1 = p_1(1+\alpha+\alpha^2+\cdots+\alpha^{N-1})$ or $p_1 = \frac{1}{1+\alpha+\alpha^2+\cdots+\alpha^{N-1}}$. Here we get p_1 by two cases, $\alpha = 1$ or $\alpha \neq 1$.

   1. *if* $\alpha = 1$: $p_1 = \frac{1}{1+1+\cdots+1} = \frac{1}{N}$, $\therefore p_i = p_1(1+\alpha+\alpha^2+\cdots+\alpha^{i-1}) = \frac{1}{N} \cdot (1+1+1+\cdots+1) = \frac{1}{N} \cdot i = \frac{i}{N}$.

   2. *if* $\alpha \neq 1$: $p_1 = \frac{1}{1+\alpha+\alpha^2+\cdots+\alpha^{N-1}}$. As the denominator is a geometric progression, we have $p_1 = \frac{1-\alpha}{1-\alpha^N}$, $\therefore p_i = \frac{1-\alpha}{1-\alpha^N} \cdot (1+\alpha+\alpha^2+\cdots+\alpha^{i-1}) = \frac{1-\alpha}{1-\alpha^N} \cdot \frac{1-\alpha^i}{1-\alpha} = \frac{1-\alpha^i}{1-\alpha^N}$.

   (b) What happens if $p = \frac{1}{2}$?

[3] Again, my interest in casinos in purely academic

Solution: If $p = 0.5$, $\alpha = \frac{1-p}{p} = \frac{0.5}{0.5} = 1$. From the above proofs, we have if $\alpha = 1$, $p_1 = \frac{1}{1+1+\cdots+1} = \frac{1}{N}$, $\therefore p_i = p_1(1 + \alpha + \alpha^2 + \cdots + \alpha^{i-1}) = \frac{1}{N} \cdot (1 + 1 + 1 + \cdots + 1) = \frac{1}{N} \cdot i = \frac{i}{N}$.

(c) [**Ungraded question**] Can you reason why it does not make sense to take on a casino ($N >> i$)? Will you always go bankrupt in the long run?

Solution: If $N >> i$, the probability of winning with i rupees, with $p < 0.5$ ($\alpha > 1$), the denominator of $p_i = \frac{1-\alpha^i}{1-\alpha^N} = \frac{\alpha^i-1}{\alpha^N-1}$ takes on a much much larger value than the numerator and the probability p_i is almost zero. Even if $p = 0.5$ ($\alpha = 1$) $p_i = \frac{i}{N} \approx 0$.

The disappointed professor

14. (1 point) A particular class has had a history of low attendance. The dejected professor decides that he will not lecture unless at least k of the n students enrolled in the class are present. Each student will independently show up with probability p if the weather is good, and with probability q if the weather is bad. Given that the probability of bad weather on a given day is r, obtain an expression for the probability that the professor will teach his class on that day. [Bertsekas and Tsitsikilis, Introduction to Probability, 2nd edition.]

Solution: From total probability rule, P(teacher lectures that day) = P(atleast k students attend class) = P(atleast k students attend class | good weather)$\cdot P$(good weather) + P(atleast k students attend class | bad weather)$\cdot P$(bad weather).

Since given good weather, each student has a probability p of coming or $1-p$ of not coming to class, P(atleast k students attend class | good weather) = $\sum_{i=k}^{n} \binom{n}{i} \cdot p^i \cdot (1-p)^{n-i}$.

Since given bad weather, each student has a probability q of coming or $1-q$ of not coming to class, P(atleast k students attend class | bad weather) = $\sum_{i=k}^{n} \binom{n}{i} \cdot q^i \cdot (1-q)^{n-i}$.

Substituting these in the above formula, we have, P(atleast k students attend class) = P(teacher lectures that day) = $(1-r)\cdot\sum_{i=k}^{n} \binom{n}{i}\cdot p^i\cdot(1-p)^{n-i} + r\cdot\sum_{i=k}^{n} \binom{n}{i}\cdot q^i\cdot(1-q)^{n-i}$

The John von architecture

15. (1 point) Suppose you have a biased coin ($P(H) \neq P(T)$). How will you use it to make unbiased decision. (hint: you can toss the coin multiple times)

Solution: Let for the given unbiased coin $P(H) = p$ and $P(T) = 1 - p$ and $p \neq \frac{1}{2}$. Now we need to find a random experiment with this coin which has same probability of occurence and non-occurence (that experiment is then unbiased). For this we toss the coin twice and if the outcome is Head-Tail (HT), we take it as a success, if the outcome is Tail-Head (TH), we take it as failure else we toss again. $P(\text{HT}) = P(\text{TH})$ $= p \cdot (1 - p)$. Let A_i denote the event that the i^{th} toss-set we get HT or TH.

$P(\text{success}) = \sum_{i=1}^{\infty} P(\text{success} \mid A_i)\ P(A_i)$. Here, $P(\text{success} \mid A_i)$ is $\frac{1}{2}$, as only one of HT or TH corresponds to success and other corresponds to failure. $\therefore P(\text{success}) = \sum_{i=1}^{\infty} \frac{1}{2} P(A_i) = \frac{1}{2} \sum_{i=1}^{\infty} A_i = \frac{1}{2}$ ($\sum_{i=1}^{\infty} A_i = 1$ as with infinite trials, we are bound to encounter TH or HT somewhere as it has non-zero probability).

Similary, $P(\text{failure}) = \sum_{i=1}^{\infty} P(\text{failure} \mid A_i) = \sum_{i=1}^{\infty} P(\text{failure}) P(A_i) = \sum_{i=1}^{\infty} \frac{1}{2} P(A_i) = \frac{1}{2} \sum_{i=1}^{\infty} A_i = \frac{1}{2}$. $\therefore$ this is an unbiased decision.

Pascal to the rescue

16. (1 point) A six-side die is rolled three times independently. What is more likely: a sum of 11 or 12?

Solution:

1. *Combinations which give a sum of 11 are*:
   (a) $(6, 4, 1) \rightarrow 3! = 6$ cases
   (b) $(6, 3, 2) \rightarrow 3! = 6$ cases
   (c) $(5, 5, 1) \rightarrow \frac{3!}{2!} = 3$ cases
   (d) $(5, 4, 2) \rightarrow 3! = 6$ cases
   (e) $(5, 3, 3) \rightarrow \frac{3!}{2!} = 3$ cases
   (f) $(4, 4, 3) \rightarrow \frac{3!}{2!} = 3$ cases

   There are a total of $6+6+3+6+3+3 = 27$ combinations and the probability is $\frac{27}{216} = 0.125$

2. *Combinations which give a sum of 12 are*:
   (a) $(6, 5, 1) \rightarrow 3! = 6$ cases
   (b) $(6, 4, 2) \rightarrow 3! = 6$ cases
   (c) $(6, 3, 3) \rightarrow \frac{3!}{2!} = 3$ cases
   (d) $(5, 5, 2) \rightarrow \frac{3!}{2!} = 3$ cases

(e) $(4,5,3) \rightarrow 3! = 6$ cases

(f) $(4,4,4) \rightarrow \frac{3!}{3!} = 1$ case

There are a total of $6+6+3+3+6+1 = 25$ combinations and the probability is $\frac{25}{216} = 0.1157$.

$\therefore$ sum of 11 has a higher probability.

Enemy at the gates

17. (1 point) There are 41 soldiers surrounded by the enemy. They would rather die than get captured. They sit around in a circle and devise the following plan. Each soldier will kill the person to his immediate left. They will continue this till only one soldier remains who would then commit suicide. For example, if there are 7 soldiers numbered 1, 2, 3, 4, 5, 6, 7 sitting in a circle then they proceed as follows: 1 kills 2, 3 kills 4, 5 kills 6, 7 kills 1, 3 kills 5, 7 kills 3, 7 commits suicide.

(a) In how many ways can 41 soldiers be arranged around a circle?

Solution: Since we have a circular permutation, we will have $\frac{n!}{n}$ arrangements. We divide by n as every circular permutation will have n identical versions of itself with each member being displaced left or right keeping the relative order the same. $\therefore$ with 41 soldiers we have $\frac{41!}{41} = 40!$.

(b) If you were one of the 41 soldiers and the soldiers were randomly arranged in the circle, what is the probability that you would survive?

Solution: Since there is only one fixed position for survival, we need to sit there and arrange the rest 40 soldiers in the remaining 40 places. $\therefore$ number of favourable cases are 40!. Probability $= \frac{\#favourable}{\#total} = \frac{40!}{41!} = \frac{1}{41}$.

(c) **[Ungraded question]** Is there a specific position in which you can sit so that you are the last surviving soldier?

Solution: The last surviving soldier will be at the 19^{th} position.