Honor code: I pledge on my honor that: I have completed all steps in the below homework on my own, I have not used any unauthorized materials while completing this homework, and I have not given anyone else access to my homework.

Varun Gumma

Name and Signature

1. (1 point) Have you read and understood the honor code?

Solution: YES

Concept: Linear Combinations

2. (2 points) Consider the vectors $[x, y]$, $[a, b]$ and $[c, d]$.

   (a) Express $[x, y]$ as a linear combination of $[a, b]$ and $[c, d]$.

Solution: Let $\begin{bmatrix} x \\ y \end{bmatrix} = \alpha \begin{bmatrix} a \\ b \end{bmatrix} + \beta \begin{bmatrix} c \\ d \end{bmatrix}$. In simultaneous equation form,

$\alpha a + \beta c = x,\ \alpha b + \beta d = y$
$\implies \alpha ba + \beta bc = bx,\ \alpha ab + \beta ad = ay$ − subtracting eq-1 from eq-2
$\therefore \beta(ad - bc) = (ya - xb)$ or $\beta = \frac{ya-xb}{ad-bc}$
$\implies \alpha ad + \beta cd = xd,\ \alpha bc + \beta dc = yc$ − subtracting eq-2 from eq-1
$\therefore \alpha(ad - bc) = (xd - yc)$ or $\alpha = \frac{xd-yc}{ad-bc}$
$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \frac{xd-yc}{ad-bc} \begin{bmatrix} a \\ b \end{bmatrix} + \frac{ya-xb}{ad-bc} \begin{bmatrix} c \\ d \end{bmatrix}$

   (b) Based on the expression that you have derived above, write down the condition under which $[x, y]$ cannot be expressed as a linear combination of $[a, b]$ and $[c, d]$. (Must: the condition should talk about some relation between the scalars a, b, c, d, x and y)

Solution: The linear combination can be expressed in simultaneous equation form as $\alpha a + \beta c = x$ and $\alpha b + \beta d = y$. These equations will not have a solution (i.e. $[x, y]$ cannot be a linear combination of $[a, b]$ and $[c, d]$) if the lines represented by them are parallel or $\frac{a}{b} = \frac{c}{d} \neq \frac{x}{y}$.

Concept: Elementary matrices

3. (1 point) Compute L and U for the matrix

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

Find four conditions on a, b, c, d to get A = LU with 4 pivots

Solution: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix} = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$ $(R_2 \leftarrow R_2 - R_1; E_{21})$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ a & b & c & c \\ a & b & c & d \end{bmatrix} = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & 0 & c-b & d-b \end{bmatrix} \quad (R_3 \leftarrow R_3 - R_1; E_{31})$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & 0 & c-b & d-b \end{bmatrix} = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{bmatrix} \quad (R_4 \leftarrow R_4 - R_1; E_41)$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{bmatrix} = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & b-a & c-a & d-a \end{bmatrix} \quad (R_3 \leftarrow R_3 - R_2; E_{32})$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & b-a & c-a & d-a \end{bmatrix} = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & c-b & d-b \end{bmatrix} \quad (R_4 \leftarrow R_4 - R_2; E_{42})$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & c-b & d-b \end{bmatrix} = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix} \quad (R_4 \leftarrow R_4 - R_3; E_{43})$$

$$\therefore L = E_{21}^{-1} E_{31}^{-1} E_{41}^{-1} E_{32}^{-1} E_{42}^{-1} E_{43}^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}, U = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

$\therefore$ the four conditions to have all pivots are $a \neq 0$, $a \neq b$, $b \neq c$, $c \neq d$

4. (1 point) Let $E_1, E_2, E_3, \ldots, E_n$ be n lower triangular elementary matrices. Let $(i_1, j_1), (i_2, j_2), \ldots (i_n, j_n)$ be the position of the non-zero off-diagonal element in each of these elementary matrices. Further, $if\ k \neq m\ then\ (i_k, j_k) \neq (i_m, j_m)$ (*i.e.*, no two elementary matrices in the sequence have a non-zero off-diagonal element in the same position). Prove that the product of these n elementary matrices will have all diagonal entries as 1. (Proving this will help you understand why the diagonal elements of L are always equal to 1.)

Solution: All elementary matrices have their diagonal elements ($i = j$) as 1 and some off-diagonal term as non-zero. We can represent $(((E_1E_2)E_3)...E_n)$ as L. Let's analyse multiplying two elementary matrices, $E = E_1E_2$. $E_{ii} = \sum_{k=1}^{n} E_{1ik}E_{2ki} = E_{1i1}E_{21i} + E_{1i2}E_{22i} + \ldots E_{1ii}E_{2ii} \cdots + E_{1in}E_{2ni} = 0 + 0 + \ldots 1 \cdots + 0 = 1$. (this is because $E_{1ii} = E_{2ii} = 1$ and rest products are 0 as either of E_{1ik} or E_{2ki} are 0 according the statement that no two matrices have non-zero off-diagonal terms in the same position) $\therefore$ all diagonal elements of the product are 1. Following the shown associativity for L, each term will have a diagonal of all 1s and so will the final product L. Hence, proved.

Concept: Inverse

5. (½ point) Show that the matrix $B^\top AB$ is symmetric if A is symmetric.

Solution: Assuming A is symmetric, we have $A^T = A$. Multiplying L.H.S with B^T and R.H.S with B, we get $B^TA^TB = B^TAB$. By using the property of transpose that $(PQR)^T = R^TQ^TP^T$, the L.H.S becomes $(B^TAB)^T = B^TAB$. This implies B^TAB is symmetric if we start with the condition A is symmetric.

6. (2 points) Prove that a $n \times n$ matrix A is invertible if and only if Gaussian Elimination of A produces n non-zero pivots.

Solution:

Proof (the if part): Given A upon Gaussian Elimination produces all n-pivots, which means all column vectors of A are independent (A has a full rank). $\therefore$ we can find unique $\mathbf{x_1}, \mathbf{x_2}...\mathbf{x_n}$, such that $A\mathbf{x_1}, A\mathbf{x_2}, ...A\mathbf{x_n}$ equal $\mathbf{e_1}, \mathbf{e_2}...\mathbf{e_n}$ (standard basis vectors) respectively. Stacking $\mathbf{x_1}, \mathbf{x_2}...\mathbf{x_n}$ into a single unique matrix X, we get $AX = I$, which implies A is invertible and $A^{-1} = X$ (in our case).

Proof (the only if part): Given A is invertible, i.e. there exists some matrix B of appropriate dimensions such that $AB = I$. Let's assume A does not have all n pivots, $\therefore$ atleast one row of A upon Gaussian Elimination has all 0s. Since, an inverse must exist for this upper triangular form of A (let's call it A'), there must be some B' such that $A'B' = I$. But we can observe that B' cannot exist as we require to multiply columns of B' with a row of $0s$ of A' to get the row of an identity matrix (i.e. $k * 0 = 1$, which is a contradiction). Hence, the assumption is false, A must have all n-pivots.

7. (1 ½ points) If A and B are $n \times n$ and $n \times m$ matrices respectively and a and b are $n \times 1$ and $m \times 1$ vectors respectively, then what is the cost of:

   (a) Computing AB

   Solution: To compute a single element of AB requires n operations and hence to compute all nm terms requires a total cost of n^2m.

   (b) Computing $B^T a$

   Solution: Computing a single element of $B^T a$ requires n operations and hence to compute all m terms of the product requires a total cost of nm.

   (c) Computing A^{-1}

   Solution: Solving A^{-1} by Gauss-Jordan requires solving $[A|I_n]$ to $[I_n|A^{-1}]$. Every row is used to adjust the remaining rows and get 0s in the required positions, $\therefore n - 1$ row operations and each row operation has $2n$ additions/subtractions. $\therefore$ to work with all n rows requires $n(n-1)(2n)$. Further, we might need to divide a row throughout by the pivot to make the pivot 1, which will contribute to extra $n(2n)$ in the total. $\therefore$ total cost is $2n^2(n-1) + 2n^2 = 2n^3$ (worst case) $= O(n^3)$.

Concept: LU factorisation

8. (1 ½ points) (a) Under what conditions is the would A have a full set of pivots ?

$$A=\begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 0 & -1 & 1\end{bmatrix}\begin{bmatrix}d_1 & & \\ & d_2 & \\ & & d_3\end{bmatrix}\begin{bmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}$$

Solution: Here matrix $U = \begin{bmatrix}d_1 & -d_1 & 0\\ 0 & d_2 & -d_2\\ 0 & 0 & d_3\end{bmatrix}$.
If all pivots for this matrix should exist, $d_1 \neq 0$, $d_2 \neq 0$ and $d_3 \neq 0$.

(b) Solve as two triangular systems, without multiplying LU to find A:

$$LUx=\begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 4 & 4\\ 0 & 1 & 2\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}2\\ 0\\ 2\end{bmatrix}$$

Solution: Assuming $\begin{bmatrix}2 & 4 & 4\\ 0 & 1 & 2\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \mathbf{y}$, we have $\begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}\mathbf{y} = \begin{bmatrix}2\\ 0\\ 2\end{bmatrix}$.
$\therefore y_1 = 2,\ y_1 + y_2 = 0 \implies y_2 = -y_1 = -2,\ y_1 + y_3 = 2 \implies y_3 = 2 - y_1 = 0$

$\therefore \begin{bmatrix}2 & 4 & 4\\ 0 & 1 & 2\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \mathbf{y} = \begin{bmatrix}2\\ -2\\ 0\end{bmatrix}$. Solving for $\mathbf{x}$, we get,
$x_3 = 0,\ x_2 + 2x_3 = -2 \implies x_2 = -2,\ 2x_1 + 4x_2 + 4x_3 = 2 \implies x_1 = 5.$

$\therefore \mathbf{x} = \begin{bmatrix}5\\ -2\\ 0\end{bmatrix}$

9. (2 points) Consider the following system of linear equations. Find the LU factorisation of the matrix A corresponding to this system of linear equations. Show all the steps involved. (this is where you will see what happens when you have to do more than 1 permutations).

$$\begin{aligned} x + y &= -3\\ w - x - y &= +2\\ 3w - 3x - 3y - z &= -19\\ -5x - 3y - 3z &= -2 \end{aligned}$$

Solution: The above system can be written as $\begin{bmatrix}0 & 1 & 1 & 0\\ 1 & -1 & -1 & 0\\ 3 & -3 & -3 & -1\\ 0 & -5 & -3 & -3\end{bmatrix}\begin{bmatrix}w\\ x\\ y\\ z\end{bmatrix} = \begin{bmatrix}-3\\ 2\\ -19\\ -2\end{bmatrix}$

$$\begin{bmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 1 & 1 & 0\\ 1 & -1 & -1 & 0\\ 3 & -3 & -3 & -1\\ 0 & -5 & -3 & -3\end{bmatrix} = \begin{bmatrix}1 & -1 & -1 & 0\\ 0 & 1 & 1 & 0\\ 3 & -3 & -3 & -1\\ 0 & -5 & -3 & -3\end{bmatrix} \quad (R_1 \leftrightarrow R_2;\ P_{12})$$

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ -3 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & -1 & -1 & 0\\ 0 & 1 & 1 & 0\\ 3 & -3 & -3 & -1\\ 0 & -5 & -3 & -3\end{bmatrix} = \begin{bmatrix}1 & -1 & -1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & -1\\ 0 & -5 & -3 & -3\end{bmatrix} \quad (R_3 \leftarrow R_3 - 3R_1;\ E_{31})$$

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & -1 & -1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & -1\\ 0 & -5 & -3 & -3\end{bmatrix} = \begin{bmatrix}1 & -1 & -1 & 0\\ 0 & 1 & 1 & 0\\ 0 & -5 & -3 & -3\\ 0 & 0 & 0 & -1\end{bmatrix} \quad (R_3 \leftrightarrow R_4;\ P_{34})$$

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 5 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & -1 & -1 & 0\\ 0 & 1 & 1 & 0\\ 0 & -5 & -3 & -3\\ 0 & 0 & 0 & -1\end{bmatrix} = \begin{bmatrix}1 & -1 & -1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 2 & -3\\ 0 & 0 & 0 & -1\end{bmatrix} \quad (R_3 \leftarrow R_3 + 5R_2;\ E_{32})$$

$\therefore U = E_{32}P_{34}E_{31}P_{12}A$
$\therefore U = (E_{32}P_{34}E_{31}P_{34}^{-1})P_{34}P_{12}A.$
$\therefore (P_{34}P_{12})A = (P_{34}E_{31}^{-1}P_{34}^{-1}E_{32}^{-1})U = (P_{34}E_{31}^{-1}P_{34}E_{32}^{-1})U.$ (since, $P^{-1} = P$)

With $L = P_{34}E_{31}^{-1}P_{34}E_{32}^{-1}$ and $P = P_{34}P_{12}$, we have $PA = LU$

$$\therefore L = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 3 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & -5 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & -5 & 1 & 0\\ 3 & 0 & 0 & 1\end{bmatrix}$$

$$\therefore P = \begin{bmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix}, L = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & -5 & 1 & 0\\ 3 & 0 & 0 & 1\end{bmatrix}, U = \begin{bmatrix}1 & -1 & -1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 2 & -3\\ 0 & 0 & 0 & -1\end{bmatrix}.$$

Using these to solve the system, we get $\begin{bmatrix}w & x & y & z\end{bmatrix}^\top = \begin{bmatrix}-1 & -41 & 38 & 31\end{bmatrix}^\top$

10. (1 point) For a square matrix A, prove that LDU factorisation is unique.

Solution: Lets assume LDU factorization is not unique (here L and U matrices have diagonal entries as 1). $\therefore$ for a matrix A we can have more than one way to decompose it, i.e. $A = L_1D_1U_1 = L_2D_2U_2$. $\therefore$ rearranging the terms we get $L_2^{-1}L_1 = D_2U_2U_1^{-1}D_1^{-1}$. Now we can observe that the L.H.S is a lower triangular matrix (with $1s$ in the diagonal) and R.H.S is an upper triangular matrix and they are both equal which implies that matrix on L.H.S and R.H.S must be identity. $\therefore L_2^{-1}L_1 = I$ or $L_2 = L_1$. Given this, we have $D_1U_1 = D_2U_2$ or $D_2^{-1}D_1 = U_2U_1^{-1}$. $\therefore$ the R.H.S must be a diagonal matrix with diagonal entries at 1 which is in fact identity, which implies $U_2 = U_1$. $\therefore L_1 = L_2, D_1 = D_2, U_1 = U_2$, and hence LDU decomposition must be unique as another set of matrices cannot be found.

11. (1 ½ points) Consider the matrix A which factorises as:

$$\begin{bmatrix}1 & 0 & 0\\ 7 & 1 & 0\\ 0 & 5 & 1\end{bmatrix}\begin{bmatrix}1 & 7 & 0\\ 0 & 1 & 5\\ 0 & 0 & 1\end{bmatrix}$$

Without computing A or A^{-1} argue that

(a) A is invertible (I am looking for an argument which relies on a fact about elementary matrices)

Solution: Here we can observe that $U = L^T$ and vice-versa. Let L be written in terms of elementary matrices as $E_1^{-1}E_2^{-1}E_3^{-1}$.
$\therefore U = (E_1^{-1}E_2^{-1}E_3^{-1})^T = (E_3^{-1})^T(E_2^{-1})^T(E_1^{-1})^T = (E_3^T)^{-1}(E_2^T)^{-1}(E_1^T)^{-1}$.
$\therefore A = LU = E_1^{-1}E_2^{-1}E_3^{-1}(E_3^T)^{-1}(E_2^T)^{-1}(E_1^T)^{-1} = (E_1^TE_2^TE_3^TE_3E_2E_1)^{-1}$.
Since, we know all elementary matrices are invertible, this inverse exists and $A^{-1} = E_1^TE_2^TE_3^TE_3E_2E_1$. Another way to verify the same is that A has all non-zero pivots which implies A in invertible.

(b) A is symmetric (convince me that $A_{ij} = A_{ji}$ without computing A)

Solution: Here we can observe that $U = L^T$ and vice-versa. $\therefore A = LU = LL^T$, and we know LL^T is symmetric as $(LL^T)^T = LL^T$. Hence, A is symmetric.

(c) A is tridiagonal (again, without computing A convince me that all elements except along the 3 diagonals will be 0.)

Solution: As all elements of L and U are non-negative, all elements of A will be non-negative. $A_{ij} = \sum_{k=1}^{n} L_{ik}U_{kj} = 0$, when alternate terms in the i^{th} row and j^{th} column of L and U respectively are 0. This occurs for A_{31} and A_{13} and both terms are 0. Rest elements of A are non-zero and positive. Hence, all elements except the 3 diagonals are 0 (i.e. A is tridiagonal).

Concept: Lines and planes

12. (1 ½ points) Consider the following system of linear equations

$$\begin{aligned} a_1x_1 + b_1y_1 + c_1z_1 &= 1 \\ a_2x_2 + b_2y_2 + c_2z_2 &= 2 \\ a_3x_3 + b_3y_3 + c_3z_3 &= 3 \end{aligned}$$

Each equation represents a plane, so find out the values for the coefficients such that the following conditions are satisfied:

1. All planes intersect at a line
2. All planes intersect at a point
3. Every pair of planes intersects at a different line.

Solution: For $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} =$

1. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$, the planes intersect in a line (here, $P_3 = P_1 + P2$)
2. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & -1 \end{bmatrix}$, these planes intersect at the point $(-1, 3, -1)$.
3. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$, these planes have no solution, all intersect in different lines.

13. (1 ½ points) Starting with a first plane $u - v - w = -1$, find the equation for
    (a) the parallel plane through the origin.

    Solution: $u - v - w = 0$

    (b) a second plane that also contains the points (-1,-1,1) and (-7,-5,-1).

    Solution: $3u - 4v - w = 0$

(c) a third plane that meets the first and second in the point (2, 1, 2).

> **Solution:** $u + v + w = 5$

Concept: Transpose

14. (2 points) Consider the transpose operation.

(a) Show that it is a linear transformation.

> **Solution:** As transpose satisfies $(A+B)^T = A^T + B^T$ i.e. $T(X+Y) = T(X) + T(Y)$ (property-1) and $(\alpha A)^T = \alpha A^T$ i.e. $T(\alpha X) = \alpha T(X)$ $\alpha \in \mathbf{R}$ (property-2), it is a linear transformation.

(b) Find the matrix corresponding to this linear transformation.

> **Solution:** If a matrix A were to be possible, it implies $AB = B^T$.
>
> For rectangular matrices (where B is of the form $n \times m$) we cannot find such an A such that it transforms an $n \times m$ matrix to a $m \times n$ matrix as corresponding dimensions of input and output matrices (B and B^T) are not same. For square matrices (where B is of the form $n \times n$), if B is invertible, $A = B^T B^{-1}$ (which is not unique for all $n \times n$ invertible matrices). We cannot find such matrix if B is non-invertible. $\therefore$ We cannot find a matrix, that always satisfies the transpose transformation, i.e. converts B to B^T for all matrices available (Square or Rectangular, individually).