Autonomous exploration for navigating in non-stationary CMPs

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Abstract

We consider a setting in which the objective is to learn to navigate in a controlled Markov process (CMP) where transition probabilities may abruptly change. For this setting, we propose a performance measure called *exploration steps* which counts the time steps at which the learner lacks *sufficient* knowledge to navigate its environment *efficiently*. We devise a learning meta-algorithm, MNM, and prove an upper bound on the exploration steps in terms of the number of changes.

1 Introduction

The ability to quickly learn to reliably control one's environment is core to the functionality of intelligent agents. Throughout the last decades, much work has been devoted to the design and testing of various algorithms targeted at this task, under various names such as learning using intrinsic motivation, intrinsic reward, curiosity-driven learning, etc. A necessarily incomplete sample of prior works in the area includes that of Schmidhuber (1991); Singh et al. (2004); Oudeyer and Kaplan (2007); Oudeyer et al. (2007); Baranes and Oudeyer (2009); Schmidhuber (2010); Singh et al. (2010); Lopes et al. (2012); Gottlieb et al. (2013); Stadie et al. (2015); Houthooft et al. (2016); Achiam and Sastry (2017); Ostrovski et al. (2017); Pathak et al. (2017); Haber et al. (2018); Burda et al. (2019); Azar et al. (2019); Hazan et al. (2019). Conceptually, the problem can be thought of as learning to reliably navigate an unknown environment. In this article we focus on this problem, and in particular, on learning to navigate in the face of a changing, or nonstationary environment. Following Lim and Auer (2012), we consider the case when an agent interacts

with a controlled Markov process (CMP) equipped with finitely many actions and at most countably many states, the state is observable after every transition and a reset action is available which brings the agent back to some initial state. The problem then is to minimize the number of steps where the agent lacks the ability to reliably navigate to safely reachable states. Since the number of states is unbounded, the agent is given as input a 'radius' L such that it needs to consider all states that are reachable within L steps (precise definitions will be given in the next section). Lim and Auer (2012) gave an algorithm that with high probability finishes the discovery task in time that is proportional to the product of L^3 and the number of states to be discovered. Unlike this previous work, we consider the case *when the transition probabilities can (abruptly) change*. This setting is important as agents with a long "lifespan" may expect their environment to change: "moving parts" can suddenly break down as commonly experienced in robotics or more generally in automation (Kober et al., 2013), or the environment may change abruptly due to the appearance or disappearance of other agents, or objects, such as in rescue robots in urban search and rescue missions in unknown environments (Niroui et al., 2019). The time when the changes happen or the nature of the changes are unknown. In this new setting, we consider the problem of minimizing the number of exploration steps: A time step is considered an exploration step if at that time step the agent lacks sufficient knowledge to navigate its current environment efficiently. The challenge is of course that the agent may not be aware of when it does not have this sufficient knowledge. For this problem we give a meta-algorithm MNM which can utilize any base algorithm designed for the stationary version of the problem and which keeps the number of exploration steps below $O(F^2)$ when the number of environment changes is F.

Changing environments have been studied in the context of reinforcement learning (see e.g., Even-dar et al. (2005); Abbasi et al. (2013); Ortner et al. (2019)). However, our problem setting fundamentally differs from these works as the external rewards are absent and as such our performance metric is incomparable.

2 Problem Setting

We consider a discrete-time controlled Markov process – a Markov decision process where rewards are absent. We assume a countable, possibly infinite state space $\mathcal{S}$ and a finite action space $\mathcal{A}$ with $A = |\mathcal{A}|$ actions. Upon executing an action $a \in \mathcal{A}$ in state $s \in \mathcal{S}$ at time t, the environment transitions into the next state $s' \in \mathcal{S}$ selected randomly according to the unknown transition probabilities $P_t(s'|s,a)$. In order to define the performance measure for our problem, we make use of some of the preliminary definitions and an assumption from Lim and Auer (2012) (Definitions 1–3 and Assumption 1 below), which assume $P_t = P$. We assume that the reader is familiar with terminology of Markov decision processes which we borrow from.

The learning agent is expected to solve the autonomous exploration problem in which the goal is to find a policy for each *reachable state* from a starting state s_0, which we will fix for the rest of the article, and hence will be omitted from any notation.

Definition 1 (Navigation time). *For any (possibly non-stationary) policy π, let $\tau(s|\pi)$ be the expected number of steps before reaching s for the first time when executing policy π starting from s_0.*

The learner will be given a number $L > 0$ and we may naively demand that it finds all states reachable in at most L steps:

Definition 2 (S_L). *We let S_L denote $S_L := \{s \in \mathcal{S} : \min_\pi \tau(s|\pi) \leq L\}$.*

Since the state space might be infinite, a learner could wander off in some direction or get stuck without being able to return to the starting state. To exclude this possibility, we make the following assumption.

Assumption 1. *In every state, there is a designated* RESET *action available, that will transition back to the starting state s_0 with probability 1.*

We define a *policy π on $\mathcal{S}' \subset \mathcal{S}$* to be a policy with $\pi(s) = $ RESET for any $s \notin \mathcal{S}'$. As it turns out, in general it is too much to ask for learners to discover all the states in S_L. Rather, following Lim and Auer (2012) we require learners to discover only the so-called incrementally discoverable states, $\mathcal{S}_L^{\rightarrow}$.

Definition 3 ($\mathcal{S}_L^{\rightarrow}$). *Let $\prec$ be some partial order on $\mathcal{S}$. The set $\mathcal{S}_L^{\prec}$ of states reachable in L steps with respect to $\prec$, is defined inductively as follows:*

- *$s_0 \in \mathcal{S}_L^{\prec}$,*

- *if there is a policy π on $\{s' \in \mathcal{S}_L^{\prec} : s' \prec s\}$ with $\tau(s|\pi) \leq L$, then $s \in \mathcal{S}_L^{\prec}$.*

The set $\mathcal{S}_L^{\rightarrow}$ of states reachable in L steps in respect to some partial order *is given by $\mathcal{S}_L^{\rightarrow} := \bigcup_{\prec} \mathcal{S}_L^{\prec}$, where the union is over all possible partial orders.*

Back to the nonstationary case, we define the number of changes in the environment as

$$F := \#\{1 \leq t | \exists s', s, a : P_{t-1}(s'|s,a) \neq P_t(s'|s,a)\}.$$

For notational convenience, we assume that $P_0(s'|s,a) \neq P_1(s'|s,a)$ for some (s,a), thereby always counting the first change at $t = 1$. Therefore,

$$\#\text{changes} = \# \text{ different CMP settings} = F. \quad (1)$$

Next we define the performance measure we propose for the considered problem setting.

Definition 4 (Exploration steps). *The (L, ϵ)-exploration steps are the complement of the set $\mathcal{T}$, where $\mathcal{T}$ contains the time steps t at which the learner*

- *has identified a set $\mathcal{K} \supseteq \mathcal{S}_L^{\rightarrow}$ for the CMP with transition probabilities P_t, and*

- *has a policy π_s for every state $s \in \mathcal{K}$ with $\tau(s|\pi_s) \leq (1+\epsilon)L$ for the transition probabilities P_t.*

The set of exploration steps contains the time steps for which the learner doesn't have *sufficient* knowledge about the current CMP structure in order to navigate to reachable states from s_0 efficiently. The learner's aim is to be able to efficiently navigate the current CMP structure at most of the time steps, or equivalently to minimize the number of exploration steps.

Introduction to UcbExplore(Lim and Auer, 2012): Before we illustrate our meta-algorithm using UCBEXPLORE as a subroutine, let us take a look at a few relevant details. UCBEXPLORE alternates between two phases: *state discovery* and *policy evaluation*. In a state discovery phase, new candidate states are discovered as potential members of the set of reachable states. In a policy evaluation phase, the optimistic policy π_s for reaching one of the candidate states s is evaluated to verify if π_s is *acceptable*[1]. A policy evaluation phase for any π_s lasts for a certain number of episodes. Each episode begins at s_0 and ends either when π_s successfully reaches s or $\lceil (1 + \frac{1}{\epsilon}) L \rceil$ steps have been executed. If s is not reached in a suitably high number of episodes, policy evaluation for π_s is said to have *failed*. A successful policy evaluation means a new reachable state and an acceptable policy have been discovered. A failed policy evaluation leads to selecting another candidate state-optimistic policy

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[1]By an acceptable policy, we mean any policy π_s such that $\tau(s|\pi_s) \leq (1+\epsilon)L$.

pair for evaluating while a successful policy evaluation leads to a state discovery phase which in turn adds more candidate states for the subsequent policy evaluation phases. We restate the main result of Lim and Auer (2012) below for reference.

Theorem 1. *[Lim and Auer (2012, Theorem 8)] When algorithm* UCBEXPLORE *is run on a stationary CMP problem (i.e $\forall t, P_t(\cdot|s,a) = P(\cdot|s,a)$) with inputs s_0, $\mathcal{A}$, $L \geq 1$, $\varepsilon > 0$, and δ, then with probability $1 - \delta$*

- *it discovers a set of states $\mathcal{K} \supseteq \mathcal{S}_L^{\rightarrow}$;*

- *for each $s \in \mathcal{K}$, it outputs a policy π_s with $\tau(s|\pi_s) \leq (1 + \varepsilon)L$, and*

- *it terminates after $O\left(\frac{SAL^3}{\varepsilon^3}\left(\log \frac{SAL}{\varepsilon\delta}\right)^3\right)$ exploration steps, where $S = |\mathcal{K}| \leq |\mathcal{S}_{(1+\varepsilon)L}^{\rightarrow}|$.*

3 Meta-algorithm for autonomous exploration in non-stationary CMPs

Our meta-algorithm (<u>M</u>eta-algorithm for autonomous exploration in <u>n</u>on-stationary C<u>M</u>Ps or MNM) can use any algorithm designed for autonomous exploration in a stationary CMP as a *subroutine*. In Figure 1, for the sake of specificity, we describe the algorithm using UCBEXPLORE (Lim and Auer, 2012) as a subroutine.

The algorithm proceeds in rounds and each round consists of two phases: a building phase and a checking phase. In a building phase, we build a hypothesis which consists of a set of states and an acceptable policy for each of them. In a checking phase, we check if the hypothesis we built in this round is still valid. In any building phase, the algorithm initiates several copies of the subroutine at different time steps (see 2(a) in Figure 1) and switches back and forth between them (see 2(b) in Figure 1). Once it switches to a copy of the subroutine, that subroutine is said to be to active and it remains so until the next switch. To simulate this approach, our algorithm proceeds in *streams*. A stream is a single run of the subroutine acting only according to the previous time-steps for which the said stream is active. At any time step, only a single stream is active. Once a stream is active it stays so for a quantum of time steps, the length of which is determined dynamically (see 2(c) in Figure 1). When a hypothesis is formed in the building phase of a round r, it is stored in $\mathcal{K}_r$ and $\mathcal{P}_r$ (see 2(d) in Figure 1) and the algorithm moves on to the checking phase.

In the checking phase, recent history is examined, by employing a sliding window, to detect various kinds of changes in the hypothesis. When the hypothesis is found to be valid no more on account of a change,

the algorithm terminates the checking phase and proceeds to the next round. In the checking phase, our algorithm employs the subroutine as a black-box using the upper bound on the exploration steps required by the subroutine for a stationary CMP problem. Using Theorem 1, the upper bound is $O(\frac{SAL^3}{\varepsilon^3}\left(\log \frac{SAL}{\varepsilon\delta}\right)^3)$ for UCBEXPLORE. We use this bound to compute W_r for each round r with suitable constants C_1 and C_2.

At any time step t, our algorithm's knowledge of the current CMP structure is represented by $\mathcal{K}_{r-1}$ and $\mathcal{P}_{r-1}$ where r the current round at t. When the current round $r = 1$, the algorithm is yet to learn the present CMP structure.

MNM can use any algorithm designed for autonomous exploration in a stationary CMP as a subroutine if it is provided with two values:

- the length of the quantum i.e. the number of contiguous time steps for which a copy of the subroutine (i.e., a stream) must be active, and

- a high-probability upper bound on the number of exploration steps required by the subroutine for a stationary CMP problem.

These two values are used in Step 2(c) and the computation of W_r at the beginning of a checking phase respectively (see Figure 1). Using another algorithm as a subroutine instead of UCBEXPLORE would only cause these two changes with the rest of MNM remaining the same.

Our main result, stated in Theorem 2, upper-bounds the number of exploration steps required by MNM using UCBEXPLORE as a subroutine. The corresponding result while using other subroutines could simply be obtained by replacing the upper bound of exploration steps required by UCBEXPLORE for a stationary CMP with the analogous bound of the subroutine being used.

Theorem 2. *With probability $1 - \delta$, the total number of exploration steps for* MNM *using* UCBEXPLORE *as a subroutine and with inputs s_0, $L \geq 1$, ϵ, δ, $C_1 = 216 \cdot (15)^2 + 61$ and $C_2 = 225$ is upper bounded by*

$$\left(\sum_{f=1}^{F} \frac{C_1 S_f AL^3}{\epsilon^3}\left(\log \frac{4\pi^2 C_2 F^2 S_f AL}{3\epsilon\delta}\right)^3\right)^2$$
$$+ F \max_{f \in \{1,\ldots,F\}}\left[\frac{2C_1 S_f AL^3}{\epsilon^3}\left(\log \frac{4\pi^2 C_2 F^2 S_f AL}{3\epsilon\delta}\right)^6\right],$$

where $S_f = |S_{(1+\epsilon)L}^{\rightarrow}(f)|$ is the number of incrementally discoverable states reachable in $(1+\epsilon)L$ time steps in the fth CMP setting, and #changes $= F$.

Input: A confidence parameter δ, an error threshold $\varepsilon > 0$, $L \geq 1$, $\mathcal{A}$, s_0, constants $C_1 > 0$ and $C_2 > 0$.

For round $r = 1, 2, \ldots$

Building phase:

1: Initialize STM = {}. The set STM is used to store the set of initiated streams in round r so far.

2: **Stream handling:** Let q_r indicate the current quantum of time steps within the building phase of round r. The length of q_r is determined dynamically (explained below in step (c)) but is at most $\left\lceil \left(1 + \frac{1}{\epsilon}\right) L \right\rceil$. Let $\delta'_r = \frac{3\delta}{4\pi^2 r^2}$.

For $q_r = 1, 2, \ldots$

(a) **Initiation rule:** For any integer p ≥ 1, if $q_r = (\text{p}-1)^2+1$, then add p to STM. Initiate a new copy of UcbExplore$(\delta'_r, \epsilon, L, \mathcal{A}, s_0)$ and associate it with stream p. This copy of Ucb-Explore acts only according to the samples taken from the time steps at which p is active.

(b) **Allocation rules:**

(i) If $q_r = 1$, activate the only initiated stream in STM so far i.e. $\text{p}_{q_r} = 1$.

(ii) Otherwise if all the initiated streams in STM have been active for equal number of quantums previously, then $\text{p}_{q_r} = $ least recently active stream in STM.

(iii) Otherwise $\text{p}_{q_r} = $ the stream in STM which has been active for the least number of quantums previously.

(c) If the copy of UcbExplore associated with the stream p_{q_r} is in a state discovery phase, run it for $\left\lceil \left(1 + \frac{1}{\epsilon}\right) L \right\rceil$ time steps. Otherwise the copy of UcbExplore associated with the stream p_{q_r} is in a policy evaluation phase, and then run it for an episode (which is always $\leq \left\lceil \left(1 + \frac{1}{\epsilon}\right) L \right\rceil$ time steps) of policy evaluation of UcbExplore i.e.,

$$|q_r| = \begin{cases} \left\lceil \left(1 + \frac{1}{\epsilon}\right) L \right\rceil, \text{if } \text{p}_{q_r} \text{ in state discovery} \\ |\text{episode of policy evaluation of } \text{p}_{q_r}| \end{cases}$$

(d) **Check for the end of building phase:** If during q_r, the copy of UcbExplore associated with the active stream terminates and provides a set of reachable states and acceptable policies for them, record them in $\mathcal{K}_r$ and

$\mathcal{P}_r = \{\pi_s, \forall s \in \mathcal{K}_r\}$ respectively. Terminate all the other initiated streams in STM and proceed to the checking phase. Otherwise proceed to next q_r.

Checking phase:

Compute $W_r = \frac{C_1 |\mathcal{K}_r| AL^3}{\epsilon^3} \left(\log \frac{C_2 |\mathcal{K}_r| AL}{\epsilon \delta'_r} \right)^3$ and $\alpha_r = \sqrt{\frac{\log(1/\delta'_r)}{2(\log(|\mathcal{K}_r| AL/\epsilon \delta'_r))^3}}$. Let a single check-run consist of the following two parts in the given order: a new copy of UcbExplore$(\delta'_r, \epsilon, L, \mathcal{A}, s_0)$ running for up to W_r time steps and a policy evaluation phase of UcbExplore for each of the policies in $\mathcal{P}_r$. If the first part of any check-run doesn't terminate within W_r time steps, then terminate it manually and proceed to the second part of the check-run. Set $n_r = \left(\log \frac{|\mathcal{K}_r| AL}{\epsilon \delta'_r} \right)^3$. Execute n_r check-runs. Then:

3: Let b be the number of times UcbExplore has failed to terminate within W_r time steps in the first part during the last n_r check-runs. If

$$\frac{b}{n_r} > \alpha_r + \delta'_r, \qquad (2)$$

then stop the checking phase, set $r \leftarrow r + 1$ and start a new round, otherwise proceed to next step.

4: For every state s in $\mathcal{K}_r$, let b'_s be the number of times policy evaluation fails for $\pi_s \in \mathcal{P}_r$ in the second part of the last n_r check-runs. If

$$\frac{b'_s}{n_r} > \alpha_r + \delta'_r, \qquad (3)$$

delete s and π_s from $\mathcal{K}_r$ and $\mathcal{P}_r$ respectively. Proceed to next step.

5: Let s be any state which was absent in $\mathcal{K}_r$, but has appeared in the output of at least one of the first part of the last n_r check-runs. For every such state s, let v_s be the number of times s was present in the output of the first part of the last n_r check-runs. If

$$\delta'_r - \left(1 - \frac{v_s}{n_r}\right) > \alpha_r, \qquad (4)$$

add s and the last found policy for s to $\mathcal{K}_r$ and $\mathcal{P}_r$ respectively. Proceed to next step.

6: Execute a check-run one more time. Go back to step 3 of checking phase.

Figure 1: <u>M</u>eta-algorithm for autonomous exploration in <u>n</u>on-stationary C<u>M</u>Ps or MNM

Note that a change in this context affects the set of reachable states in $(1 + \epsilon)L$ steps from s_0 and/or the acceptable policies for reaching them. The reason, as noted by Lim and Auer (2012), is that the learner cannot distinguish between the states reachable in L steps and those reachable in $(1 + \epsilon)L$ steps (given a reasonable amount of exploration).

Motivating factors for the construction of our algorithm

• Before an algorithm forms a hypothesis i.e., it determines a set of reachable states and acceptable policies, it might not be possible to detect a change. Consider an algorithm still in the process of building a hypothesis. During this process, the algorithm must proceed and inspect states in some order. Suppose that it has found acceptable polices for some reachable states. When it finds a new reachable state, there are two plausible scenarios: *a)* this state was not reachable when the algorithm was in the process of inspecting other states earlier i.e., there was a change, or *b)* this state was reachable when the algorithm was in the process of inspecting other states earlier i.e., there was no change. It is not possible to distinguish between these two scenarios.

• Since it might not be possible to detect a change during the hypothesis building phase and a change can occur at any time, the algorithm needs to start several processes during the hypothesis building phase. Each process aims to form a hypothesis for a particular CMP setting and to be able to do that, it needs to act only on the time steps for which that CMP setting is in effect. On one hand, since a change can occur at any time step, the algorithm needs to start these processes at several time steps along the way. On the other hand, if too many processes are started, each process will not get enough time to form its hypothesis. Therefore sufficient time should be allocated to each process. Both of these diverging requirements can be balanced, if both the number of processes and the time allocated to each process grow asymptotically as the square-root of time, as done in MNM.

• **Using a sliding window in the building phase is not possible:** Using a sliding window in the checking phase is possible as each check-run is verifying the same hypothesis (the one found in the preceding building phase) and hence findings from successive check-runs can be shared. In the building phase however, each stream with its own copy of the subroutine might be attempting to build different hypotheses and hence findings from two different streams cannot be shared.

4 Analysis and Proof of Theorem 2

First, we bound the number of exploration steps in a single building phase. Then we prove that the number of rounds is upper-bounded by #changes F. Combining these two, we prove an upper bound on the number of exploration steps for all the building phases. Next, we prove an upper bound on the number of exploration steps in a checking phase caused by a single change. Summing this over all the changes in all the rounds gives us an upper bound on the number of exploration steps in all the checking phases. Finally, we add the respective upper bounds for all the building phases and all the checking phases to arrive at the bound given by Theorem 2.

4.1 Bounding the exploration steps in a single building phase

First, we state a couple of preliminary lemmas about stream handling to be used later.

Lemma 1. *At the end of any quantum q in a building phase, the number of initiated streams is equal to $\lceil \sqrt{q} \rceil$.*

The proof for Lemma 1 is given in Appendix I. Here, we provide a brief overview. We use the fact that the #initiated streams is equal to the highest stream number initiated so far and the stream initiation rule (2(a) in Figure 1) to arrive at this claim.

Lemma 2. *At the end of a quantum $q = b^2$ for some integer $b \geq 1$,*

1. *b streams have been initiated, and*

2. *each initiated stream has been active for exactly b quantums.*

The proof for Lemma 2 is given in Appendix II. We only provide a proof sketch here. Claim 1 is a direct result of Lemma 1. Claim 2 can be proved by induction on b and considering the initiation rule and allocation rules (ii) and (iii) (see 2(a), 2(b)(ii) and 2(b)(iii) in Figure 1 respectively).

Lemma 3. *In a round r, with probability at least $1 - \delta_r'$,*

1. *the length of the building phase is at most*

$$\left(\sum_{m \in \mathcal{M}_r^b} \frac{C_1 S_{(m)} A L^3}{\epsilon^3} \left(\log \frac{C_2 S_{(m)} A L}{\epsilon \delta_r'} \right)^3 \right)^2,$$

2. *the building phase discovers a set of states $\mathcal{K}_r \supseteq \mathcal{S}_L^{\rightarrow}(\bar{m}_r)$ for some CMP setting $\bar{m}_r \in \mathcal{M}_r^b$ (the set of underlying CMP settings during the building phase of round r),*

3. and for each $s \in \mathcal{K}_r$, $\mathcal{P}_r$ contains a policy π_s with $\tau(s|\pi_s) \leq (1+\varepsilon)L$ for the CMP setting $\bar{m}_r \in \mathcal{M}_r^b$,

where $S_{(m)} = |S^{\rightarrow}_{(1+\epsilon)L}(m)|$ is the number of incrementally discoverable states reachable in $(1+\epsilon)L$ time steps in the CMP m, $C_1 = 216 \cdot (15)^2 + 61$ and $C_2 = 225$.

Proof. Consider the CMP setting $m_{r,1}$ at the start of the building phase in round r. Assume that UCBEXPLORE requires at most $x_{r,1}$ quantums as exploration steps for $m_{r,1}$ (without any change) with high probability. Theorem 1 shows that the exploration steps for the CMP setting $m_{r,1}$ required by UCBEXPLORE are at most $\frac{C_1 S_{(m_{r,1})} A L^3}{\epsilon^3} \left(\log \frac{C_2 S_{(m_{r,1})} A L}{\epsilon \delta_r'} \right)^3$ with high probability. There are two possible cases:

Case 1: The problem doesn't change for the duration of $x_{r,1}^2$ quantums.
Our meta-algorithm initiates stream 1 at $q = 1$ and this stream will have been active for $x_{r,1}$ quantums at the end of quantum $x_{r,1}^2$ (using Lemma 2). Since the problem doesn't change for this entire duration, the copy of UCBEXPLORE for stream 1 has samples only of $m_{r,1}$. Thus, stream 1 terminates at the end of $x_{r,1}^2$ quantums of our meta-algorithm with probability $1 - \delta_r'$ and the building phase of round r ends. The three claims of the lemma follow from the respective claims[2] of Theorem 1 with $\mathcal{M}_r^b = \{m_{r,1}\}$.

Case 2: The problem changes at any point before the end of $x_{r,1}^2$ quantums.
Let $m_{r,1}, m_{r,2}, \dots$ be the successive CMP settings. Let $x_{r,i}$ be the required number of quantums needed by UCBEXPLORE for each $m_{r,i}$. Let $m_{r,k}$ be the first problem setting which doesn't change from quantum $(x_{r,1} + \cdots + x_{r,k-1})^2 + 1$ to quantum $(x_{r,1} + \cdots + x_{r,k})^2$. The stream $x_{r,1} + \cdots + x_{r,k-1} + 1$ starting at $(x_{r,1} + \cdots + x_{r,k-1})^2 + 1$ will have been active for $x_{r,k}$ quantums at the end of quantum $(x_{r,1} + \cdots + x_{r,k})^2$ (using Lemma 2). That stream will therefore terminate and output the set of reachable states and acceptable policies for $m_{r,k}$ at the end of quantum $(x_{r,1} + \cdots + x_{r,k})^2$ with probability $1 - \delta_r'$ and the building phase will terminate. The three claims of the lemma follow from the respective claims of Theorem 1 with $\mathcal{M}_r^b = \{m_{r,1}, \dots, m_{r,k}\}$.

□

4.2 Bounding the number of rounds

Lemma 4. *With probability at least $1 - \delta/4$, the total number of rounds $R \leq F$.*

[2]The referred theorem only mentions the upper bound in terms of big-O notation. However, the constants C_1 and C_2 can be computed from its proof in Lim and Auer (2012, Section 4.6).

Proof. There is always at least one change in the building phase of the first round as the first change is counted at $t = 1$ by default (see Section 2). For $1 < r < R$, we consider the following two mutually exclusive cases.

Case 1: There exists no round $1 < r < R$ which has no change in its building phase.
In this case, every round contains at least one change and the total number of rounds is immediately upper-bounded by $F - 1$.

Case 2: There exists at least one round $1 < r < R$ which has no change in its building phase.
Let r be a round such that there is no change during its building phase. For all such rounds r which contain no change in the building phase, with probability at least $1 - \delta/4$, we prove that the checking phase of r contains at least one change.

Recall from Lemma 3 that the sole CMP setting during the building phase of round r is denoted as $\bar{m}_r$ and the building phase discovers the reachable states for $\bar{m}_r$ with probability at least $1 - \delta_r'$. Theorem 1 shows that for the CMP setting $\bar{m}_r$, if UCBEXPLORE with run for $W_r = \frac{C_1 |\mathcal{K}_r| A L^3}{\epsilon^3} \left(\log \frac{C_2 |\mathcal{K}_r| A L}{\epsilon \delta_r'} \right)^3$ steps, then the failure probability (i.e., the probability with which UCBEXPLORE doesn't terminate at the end of at most W_r time steps) is at most δ_r' where $C_1 = 216 \cdot (15)^2 + 61$ and $C_2 = 225$.

The only condition to trigger the next round is given by Eq. (2). Therefore, when round r ends,

$$\frac{b}{n_r} > \alpha_r + \delta_r',$$

where b is the number of times UCBEXPLORE has failed to stop and return a set of reachable states within W_r time steps during the first part of the last n_r check-runs. If the CMP setting was indeed $\bar{m}_r$ (i.e. there was no change) in the last n_r check-runs, then by Hoeffding's inequality,

$$\mathbb{P}\left\{ \frac{b}{n_r} > \alpha_r + \delta_r' \right\} \leq \exp(-2\alpha_r^2 n_r) = \delta_r'.$$

Therefore when the round r stops, there has been a change in its checking phase with probability at least $(1 - (\delta_r' + \delta_r'))$. Below we use that $\sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6}$. With a union bound and using $\sum_{r=1}^{R-1} 2\delta_r' < \frac{3\delta}{2\pi^2} \sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\delta}{4}$, we can claim that for all rounds $r < R$ which do not contain a change in the building phase, there is at least one change in each of their respective checking phases with probability at least $1 - \frac{\delta}{4}$.

Considering both the cases, we get that the total number of rounds is upper-bounded by the total number of changes F with probability at least $1 - \frac{\delta}{4}$. □

4.3 Bounding the exploration steps in all the building phases

Lemma 5. *With probability at least* $1 - \frac{\delta}{2}$, *the total number of exploration steps in all the building phases is at most*

$$\left(\sum_{f=1}^{F} \frac{C_1 S_f A L^3}{\epsilon^3} \left(\log \frac{4\pi^2 C_2 F^2 S_f A L}{3\epsilon\delta} \right)^3 \right)^2,$$

where $S_f = |S_{(1+\epsilon)L}^{\rightarrow}(m)|$ *is the number of incrementally discoverable states reachable in* $(1+\epsilon)L$ *time steps in the* f^{th} *CMP setting and* #*changes* $= F$.

Proof. We count all the steps in each building phase as exploration steps. Lemma 3 provides an upper bound on the number of exploration steps in the building phase of a single round r with the error probability limited to δ'_r. Therefore, the total number of exploration steps in all the building phases is at most

$$\sum_{r=1}^{R} \left(\sum_{m \in \mathcal{M}_r^b} \frac{C_1 S_{(m)} A L^3}{\epsilon^3} \left(\log \frac{C_2 S_{(m)} A L}{\epsilon\delta'_r} \right)^3 \right)^2$$

$$= \sum_{r=1}^{R} \left(\sum_{m \in \mathcal{M}_r^b} \frac{C_1 S_{(m)} A L^3}{\epsilon^3} \left(\log \frac{4\pi^2 C_2 r^2 S_{(m)} A L}{3\epsilon\delta} \right)^3 \right)^2$$

$$\leq \left(\sum_{f=1}^{F} \frac{C_1 S_f A L^3}{\epsilon^3} \left(\log \frac{4\pi^2 C_2 F^2 S_f A L}{\epsilon\delta} \right)^3 \right)^2$$

with error probability limited to $\left(\frac{\delta}{4} + \sum_{r=1}^{R} \delta'_r \right) < \frac{\delta}{2}$. In the last inequality, we use that $r \leq R \leq F$ with probability $1 - \frac{\delta}{4}$ (Lemma 4) and the number of different CMP settings in all the rounds is F (Eq. (1)). □

4.4 Analyzing the checking phase

We first bound the number of exploration steps in a checking phase caused due to a single change.

Lemma 6. *With probability* $1 - \delta'_r$, *the total number of exploration steps in the checking phase of a round r due to a single change is at most*

$$\frac{2 C_1 S_{(\bar{m}_r)} A L^3}{\epsilon^3} \left(\log \frac{C_2 S_{(\bar{m}_r)} A L}{\epsilon\delta'_r} \right)^6$$

where $S_{(\bar{m}_r)} = |S_{(1+\epsilon)L}^{\rightarrow}(\bar{m}_r)|$ *is the number of incrementally discoverable states reachable in* $(1+\epsilon)L$ *time steps in the CMP setting* $\bar{m}_r$.

Proof. Recall from Lemma 3 that the CMP setting for which the building phase in round r has found the reachable states and acceptable policies is denoted

as $\bar{m}_r$. Below we use that the number of time steps in a single check-run of round r is upper-bounded by $2 \times \frac{C_1 S_{(\bar{m}_r)} A L^3}{\epsilon^3} \left(\log \frac{C_2 S_{(\bar{m}_r)} A L}{\epsilon\delta'_r} \right)^3$ as $|\mathcal{K}_r| \leq S_{(\bar{m}_r)}$ from Theorem 1. Till the CMP setting is $\bar{m}_r$ in the checking phase, the algorithm does not incur any exploration steps. For a change to $m' \neq \bar{m}_r$, the following mutually exclusive and exhaustive cases are possible:

Case 1: m' doesn't last for $\left(\log \frac{|\mathcal{K}_r| A L}{\epsilon\delta'_r} \right)^3$ check-runs. Then all the time steps for which m' is active are considered as exploration steps and they are are upper bounded by

$$\frac{2 C_1 S_{(\bar{m}_r)} A L^3}{\epsilon^3} \left(\log \frac{C_2 S_{(\bar{m}_r)} A L}{\epsilon\delta'_r} \right)^3 \times \left(\log \frac{S_{(\bar{m}_r)} A L}{\epsilon\delta'_r} \right)^3.$$

Case 2: m' lasts for at least $\left(\log \frac{|\mathcal{K}_r| A L}{\epsilon\delta'_r} \right)^3$ check-runs. There are three possible subcases.

(a) W_r time steps are *insufficient* for m'. By insufficient we mean that

$$\frac{C_1 S_{(\bar{m}_r)} A L^3}{\epsilon^3} \left(\log \frac{C_2 S_{(\bar{m}_r)} A L}{\epsilon\delta'_r} \right)^3$$
$$< \frac{C_1 S_{(m')} A L^3}{\epsilon^3} \left(\log \frac{C_2 S_{(m')} A L}{\epsilon\delta'_r} \right)^3$$

i.e. $S_{(\bar{m}_r)} < S_{(m')}$.

Eq.(2) verifies if change to a m' such that $S_{(\bar{m}_r)} < S_{(m')}$ has occurred. Our algorithm keeps a count of the empirical failures in the last n_r check-runs where a failure means that the first part of a check-run has failed to terminate within W_r time steps (and thus had to be manually terminated at W_r). From Theorem 1, we know that that if no change has occurred then the true failure probability is δ'_r. By Hoeffding's inequality,

$$\mathbb{P}\left\{ \frac{b}{n_r} > \alpha_r + \delta'_r, \right\} \leq \exp(-2\alpha_r^2 n_r) = \delta'_r.$$

Therefore, with probability $1 - \delta'_r$, we detect a change to m' such that $S_{(\bar{m}_r)} < S_{(m')}$ and the number of exploration steps added are at most

$$2 C_1 \frac{S_{(\bar{m}_r)} A L^3}{\epsilon^3} \left(\log \frac{C_2 S_{(\bar{m}_r)} A L}{\epsilon\delta'_r} \right)^3 \times \left(\log \frac{S_{(\bar{m}_r)} A L}{\epsilon\delta'_r} \right)^3.$$

(b) $\{W_r$ time steps are *sufficient* for $m'\}$ and $\{$a previously reachable state becomes unreachable in m' or the previously acceptable policy $\pi_s \in \mathcal{P}_r$ to a reachable state is not acceptable in $m'\}$.
Eq.(3) checks for such scenarios. As it keeps verifying if the policy evaluation of $\{\pi \in \mathcal{P}_r\}$ succeeds in the last n_r check-runs, it checks for both - i) if a previously reachable state is still reachable and ii) if the previously acceptable policy is still acceptable. Proceeding in a similar manner to the

previous subcase, we can show that, with probability $1 - \delta'_r$, the number of exploration steps added is at most

$$2C_1 \frac{S_{(\bar{m}_r)}AL^3}{\epsilon^3} \left(\log \frac{C_2 S_{(\bar{m}_r)}AL}{\epsilon \delta'_r}\right)^3 \times \left(\log \frac{S_{(\bar{m}_r)}AL}{\epsilon \delta'_r}\right)^3.$$

(c) W_r time steps are *sufficient* for m' and a previously unreachable state becomes reachable in m'. Let's assume that a previously unreachable state s is reachable in m'. Either $s \in \mathcal{K}_r$ or $s \notin \mathcal{K}_r$. In the former case, policy evaluation (i.e. 2nd part of a check-run) continues to check if $\pi_s \in \mathcal{P}_r$ is still acceptable. If $\pi_s \in \mathcal{P}_r$ is found to be acceptable no more, then the check given by Eq. (3) will be triggered, the change will be detected and the number of exploration steps added are given by the previous subcase. If $\pi_s \in \mathcal{P}_r$ is still acceptable, it leads to no additional exploration steps (see Definition 4). Eq. (4) checks for scenarios where $s \notin \mathcal{K}_r$. Theorem 1 guarantees that if a state is in $S_L^{\rightarrow}(m')$, the probability that it fails to appear in the output of $\textsc{UcbExplore}(\delta'_r, \epsilon, L, \mathcal{A}, s_0)$ is at most δ'_r. For every state $s \notin \mathcal{K}_r$, but which has appeared in the output of the first part in one of the last n_r check-runs, we can compute the empirical failures as $n_r - v_s$. Then, by Hoeffding's inequality

$$\mathbb{P}\left\{\delta'_r - \left(1 - \frac{v_s}{n_r}\right) > \alpha_r\right\} \leq \exp(-2\alpha_r^2 n_r) = \delta'_r.$$

Therefore, with probability at least $1 - \delta'_r$, we detect such a change and the number of exploration steps added is at most

$$2C_1 \frac{S_{(\bar{m}_r)}AL^3}{\epsilon^3} \left(\log \frac{C_2 S_{(\bar{m}_r)}AL}{\epsilon \delta'_r}\right)^3 \times \left(\log \frac{S_{(\bar{m}_r)}AL}{\epsilon \delta'_r}\right)^3.$$

Considering all the cases, the number of exploration steps added is at most $2C_1 \frac{S_{(\bar{m}_r)}AL^3}{\epsilon^3} \left(\log \frac{C_2 S_{(\bar{m}_r)}AL}{\epsilon \delta'_r}\right)^6$ with probability at least $1 - \delta'_r$. $\qquad\square$

Now we can bound the number of exploration steps for all the checking phases.

Lemma 7. *With probability at least $1 - \frac{\delta}{2}$, the total number of exploration steps in all the checking phases is upper-bounded by*

$$F \cdot \max_{f \in \{1, \ldots, F\}} \left[\frac{2C_1 S_f AL^3}{\epsilon^3} \left(\log \frac{4\pi^2 C_2 F^2 S_f AL}{3\epsilon \delta}\right)^6\right]$$

where $S_f = |S_{(1+\epsilon)L}^{\rightarrow}(f)|$ is the number of incrementally discoverable states reachable in $(1+\epsilon)L$ time steps in the fth CMP setting and # changes = F.

Proof. Lemma 6 provides an upper bound on the number of exploration steps in the checking phase of a round r due to a single change with error probability limited to δ'_r. Due to the construction of our algorithm, only the changes in round r can lead to exploration steps in the checking phase of round r. Let F_r be the number of changes in round r. Then, the total number of exploration steps are at most

$$\sum_{r=1}^{R} \frac{2F_r C_1 S_{(\bar{m}_r)}AL^3}{\epsilon^3} \left(\log \frac{C_2 S_{(\bar{m}_r)}AL}{\epsilon \delta'_r}\right)^6$$

$$\leq \max_f \left[\frac{2C_1 S_f AL^3}{\epsilon^3} \left(\log \frac{4\pi^2 C_2 F^2 S_f AL}{3\epsilon \delta}\right)^6\right] \cdot \sum_{r=1}^{R} F_r$$

$$\leq F \cdot \max_f \left[\frac{2C_1 S_f AL^3}{\epsilon^3} \left(\log \frac{4\pi^2 C_2 F^2 S_f AL}{3\epsilon \delta}\right)^6\right]$$

with error probability limited to $\left(\frac{\delta}{4} + \sum_{r=1}^{F} \delta'_r\right) < \frac{\delta}{2}$. In the first inequality, we use that $r \leq R \leq F$ with probability $1 - \frac{\delta}{4}$ (Lemma 4) and $S_{(\bar{m}_r)} \leq \max_f S_f$. $\qquad\square$

4.5 Proof of Theorem 2

Proof. The total number of exploration steps in all the rounds is simply the sum of the exploration steps in all the building phases and all the checking phases given by Lemma 5 and Lemma 7 respectively. Therefore the number of total exploration steps for all the rounds is at most

$$\left(\sum_{f=1}^{F} \frac{C_1 S_f AL^3}{\epsilon^3} \left(\log \frac{4\pi^2 C_2 F^2 S_f AL}{3\epsilon \delta}\right)^3\right)^2$$

$$+ F \cdot \max_{f \in \{1, \ldots, F\}} \left[\frac{2C_1 S_f AL^3}{\epsilon^3} \left(\log \frac{4\pi^2 C_2 F^2 S_f AL}{3\epsilon \delta}\right)^6\right]$$

with probability at least $\left(1 - \left(\frac{\delta}{2} + \frac{\delta}{2}\right)\right)$ using a union bound. $\qquad\square$

5 Concluding remarks

We considered the problem of learning to explore autonomously in a non-stationary environment and proposed a pertinent performance measure. We gave a natural algorithm for the considered problem and proved an upper bound on the performance measure that scales with the square of the number of changes.

Proving a lower bound for this problem setting remains for future work. The solution strategy of first having a building phase (with multiple processes trying to build a hypothesis) and then a checking phase (where it is verified if the last built hypothesis is still true) could be used for other non-stationary learning problems. In particular, this strategy could be useful for the learning problems where each hypothesis building-process needs to act independently and cannot share findings.

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I Proof of Lemma 1

Proof. The number of initiated streams is equal to the highest stream number initiated so far. Let that be $\hat{p}$. Since $\hat{p}$ is initiated on or before q, $(\hat{p}-1)^2+1 \le q$ (see 2(a) in Figure 1) which is equivalent to,

$$\hat{p} < \sqrt{q} + 1. \tag{5}$$

Since $\hat{p}+1$ has not been initiated yet, $q \le (\hat{p}+1-1)^2$ which translates to,

$$\hat{p} \ge \sqrt{q}. \tag{6}$$

Recall that both $\hat{p}$ and q are integers ≥ 1. If q is a perfect square, the only integer satisfying both Eq. (5) and (6) is $\sqrt{q} = \lceil \sqrt{q} \rceil$. If q is not a perfect square, then Eq. (6) reduces to $\hat{p} > \sqrt{q}$. And the only integer satisfying $\sqrt{q} < p < \sqrt{q} + 1$ is $\lceil \sqrt{q} \rceil$. ☐

II Proof of Lemma 2

Proof. Claim 1 is a direct result of Lemma 1. We prove claim 2 by induction on b. Base case: $b = 1$. At the end of $q = b^2 = 1$, only 1 stream has been initiated and it has been active for 1 quantum.

Inductive case: Let's assume that the claim is true for $b = \hat{b}$ i.e at the end of quantum $q = \hat{b}^2$, exactly $\hat{b}$ streams have been initiated and each of them has been active for $\hat{b}$ quantums. At the next quantum i.e $\hat{b}^2+1$, stream $\hat{b}+1$ will be initiated by the initiation rule and it will be active for the next b quantums due to the allocation rule (ii). At this point, we are at the end of quantum $(\hat{b}+1) \cdot \hat{b}$ and all the $\hat{b}+1$ initiated streams have each been active for $\hat{b}$ quantums. Next, by virtue of the allocation rule (iii), each of the $(b+1)$ streams will be allocated 1 quantum each till we are the end of quantum $((\hat{b}+1) \cdot \hat{b}) + (\hat{b}+1) = (\hat{b}+1)^2$. ☐