Stochastic Low-rank Tensor Bandits for Multi-dimensional Online Decision Making

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Abstract

Multi-dimensional online decision making plays a crucial role in many real applications such as online recommendation and digital marketing. In these problems, a decision at each time is a combination of choices from different types of entities. To solve it, we introduce stochastic low-rank tensor bandits, a class of bandits whose mean rewards can be represented as a low-rank tensor. We consider two settings, tensor bandits without context and tensor bandits with context. In the first setting, the platform aims to find the optimal decision with the highest expected reward, a.k.a, the largest entry of true reward tensor. In the second setting, some modes of the tensor are contexts and the rest modes are decisions, and the goal is to find the optimal decision given the contextual information. We propose two learning algorithms tensor elimination and tensor epoch-greedy for tensor bandits without context, and derive finite-time regret bounds for them. Comparing with existing competitive methods, tensor elimination has the best overall regret bound and tensor epoch-greedy has a sharper dependency on dimensions of the reward tensor. Furthermore, we develop a practically effective Bayesian algorithm called tensor ensemble sampling for tensor bandits with context. Numerical experiments back up our theoretical findings and show that our algorithms outperform various state-of-the-art approaches that ignore the tensor low-rank structure. In an online advertising application with contextual information, our tensor ensemble sampling reduces the cumulative regret by 75% compared to the benchmark method.

Key Words: Bandit algorithms; Finite-time regret bounds; Online decision making; Tensor completion.

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1 Introduction

The tensor, which is also called multidimensional array, is well recognized as a powerful tool to represent complex and unstructured data. Tensor data are prevalent in a wide range of applications such as recommender systems, computer vision, bioinformatics, operations research, and etc (Frolov and Oseledets, 2017; Bi et al., 2018; Song et al., 2019; Bi et al., 2021, 2022). The growing availability of tensor data provides a unique opportunity for decisionmakers to efficiently develop multi-dimensional decisions for individuals. In this paper, we introduce tensor bandits problem where a decision, also called an arm, is a combination of choices from different entity types, and the expected rewards formulate a tensor. The problem is motivated by numerous applications in which the agent (the platform) must recommend multiple different entity types as one arm. For example, in an advertising campaign a marketer wants to promote a new product with various promotion offers. The goal is to choose an optimal triple user segment × offer × channel for this new product to boost the effectiveness of the advertising campaign. At each time, after making an action, i.e., pulling the arm (user i, offer j, channel k), the leaner receives a reward, e.g., clicking status or revenue, indicating the user segment i's feedback on promotion offer j on marketing channel k. The rewards of all these three-dimensional arms formulate an order-three tensor, see Figure 1 for an illustration. Similarly, a clothing website may want to recommend the triple $top \times bottom \times shoes$ to a user that fits the best together. Each arm is the triple of three entities. In these applications, the agent needs to pull an arm by considering multiple entities together and learn to decide which arm provides the highest reward.

Traditional tensor methods focus on static systems where agents do not interact with the environment, and typically suffer the cold-start issue in the absence of information from new customers, new products or new contexts (Song et al., 2019). However, in many real applications, agents receive feedback from the environment interactively and new subjects enter the system sequentially. See Figure 1 for an illustration of such interactive sequential decision making. In each round, the agent recommends a promotion offer to a chosen user

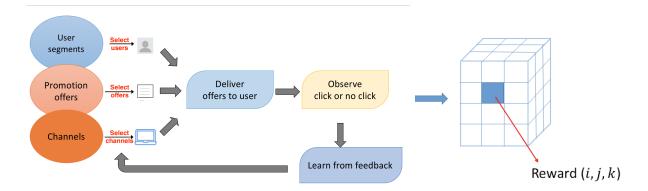


Figure 1: An example of interactive multi-dimensional online decision making. The rewards from all sequential multi-dimensional decisions formulate a tensor.

segment in a channel, and then the agent receives a feedback from this user segment. Based on this instant feedback, the agent needs to update the model to improve the user targeting accuracy in the future.

Bandit problems are basic instances of interactive sequential decision making and now play an important role in vast applications such as revenue management, online advertising, and recommender system (Li et al., 2010; Bubeck and Cesa-Bianchi, 2012; Lattimore and Szepesvári, 2020). In bandit problems, at each time step the agent chooses an arm/action from a list of choices based on the action-reward pairs observed so far, and receives a random reward that is conditionally independently drawn from the unknown reward distribution given the chosen action. The objective is to learn the optimal arm that maximizes the sum of the expected rewards. The heart of bandit problems is to address the fundamental trade-off between exploration and exploitation in sequential experiments. At each time step, after receiving the feedback from users, the agent faces a decision dilemma. The agent can either exploit the current estimates to optimize decisions or explore new arms to improve the estimates and achieve higher payoffs in the future. Our considered tensor bandits problem can be viewed as a higher-order extension of the standard bandit problem, which generalizes a scalar arm to a multi-dimensional arm and correspondingly generalizes a vector reward to a tensor reward case.

In this article, we introduce stochastic low-rank tensor bandits for multi-dimensional

online decision-making problems. These are a class of bandits whose mean rewards can be represented as a low-rank tensor and arms are selected from different entity types. This low-rank assumption is common in tensor literature, greatly reduces the model complexity and is also well motivated from various practical problems in online recommendation and digital marketing (Kolda and Bader, 2009; Allen, 2012; Jain and Oh, 2014; Xia et al., 2021; Bi et al., 2021). To balance the exploration-exploitation trade-off, we propose two algorithms for tensor bandits, tensor epoch-greedy and tensor elimination. The tensor epoch-greedy proceeds in epochs, with each epoch consisting of an exploration phase and an exploitation phase. In the exploration phase, arms are randomly selected and in the exploitation phase, arms that expect the highest reward are pulled. The number of steps in each exploitation phase increases with number of epochs, guided by the fact that, as the number of epochs increases, the estimation accuracy of the true reward improves and more exploitation steps are desirable. For tensor elimination, we incorporate the lowrank structure of reward tensor to transform the tensor bandit into linear bandit problem with low-dimension and then employ the upper confidence band (UCB) (Lai and Robbins, 1985) to enable the uncertainty quantification. The UCB has been very successful in bandit problems, leading to an extensive literature on UCB algorithms for standard multi-armed bandits (Lattimore and Szepesvári, 2020). However, employing the successful UCB strategy in low-rank tensor bandits encounters a critical challenge, as the tensor decomposition is a non-convex problem. When the data is not uniformly randomly collected but adaptively collected, the concentration results for the low-rank tensor components remain elusive thus far. Our tensor elimination approach considers a tensor spectral-based rotation strategy that preserves the tensor low-rank information and meanwhile enables uncertainty quantification.

In addition to these methodological contributions, in theory we further derive the finitetime regret bounds of our proposed algorithms and show the improvement over existing methods. Low-rank tensor structure has imposed fundamental challenges, as the proof strategies for existing bandit algorithms are not directly applicable to our tensor bandits

Algorithm	Regret bound
tensor epoch-greedy	$\widetilde{\mathcal{O}}(p^{d/2} + p^{(d+1)/3}n^{2/3})$
tensor elimination	$\widetilde{\mathcal{O}}(p^{d/2} + p^{(d-1)/2}n^{1/2})$
vectorized UCB	$\widetilde{\mathcal{O}}(p^d + p^{d/2}n^{1/2})$
matricized ESTR	$\widetilde{\mathcal{O}}(p^{d-1} + p^{3(d-1)/2}n^{1/2})$

Table 1: Regret bounds of our proposed tensor epoch-greedy and tensor elimination, as well as the competitors vectorized UCB and matricized ESTR. Here n denotes the time horizon, $p = \max\{p_1, \ldots, p_d\}$ denotes the maximum tensor dimension and d denotes the order of the reward tensor. We consider $d \geq 3$, the maximum tensor rank $r = \mathcal{O}(1)$, and use $\widetilde{\mathcal{O}}$ to denote \mathcal{O} ignoring logarithmic factors.

problem. So the regret analysis of tensor bandits demands new technical tools. In theory, we show that two existing competitors: (1) vectorized UCB which vectorizes the reward tensor into a vector and then applies UCB (Auer, 2002); and (2) matricized ESTR which unfolds the reward tensor into a matrix and then applies matrix bandit ESTR (Jun et al., 2019), both lead to sub-optimal regret bounds. Table 1 illustrates the comparison of our regret bounds and the regret bounds of these two competitors. Importantly, we prove that tensor epoch-greedy has better dependency on tensor dimensions and worse dependency on time horizon compared with the other methods. Therefore, it has superiority over other methods in two scenarios: (1) when the time horizon is short, e.g., the market campaign has a small time budget; or (2) when the dimensions are high. In contrast, tensor elimination is always better than the two existing competitors due to its sharper dependency on the dimensions, and also has advantages over tensor epoch-greedy when time horizon is long since it has better dependency on time horizon.

Finally, we consider an interesting extension of tensor bandits when the contextual information is available. In the aforementioned tensor bandits setting, the goal is to find the optimal arm corresponding to the largest entry of the reward tensor. This setting is called tensor bandits without context. When some modes of the reward tensor are contextual information, we encounter contextual tensor bandits. Take the online advertising data considered in Section 6 as an example. Users use the online platform on some day of the

week, and the platform can only decide which advertisement to show to this given user at the given time. In this example, the user mode and the day-of-week mode of the reward tensor are both contextual information and both are not decided by the platform. This is the key difference to the user targeting example shown in Figure 1. Because of this, many of the aforementioned methods are no longer applicable. In this paper we further develop tensor ensemble sampling for contextual tensor bandits that utilizes Thompson sampling (Russo et al., 2018) and ensemble sampling (Lu and Van Roy, 2017). Thompson sampling is a powerful Bayesian algorithm that can be used to address a wide range of online decision problems. The algorithm, in its basic form, first initializes a prior distribution over model parameters, and then samples from its posterior distribution calculated using past observations. Finally, an action is made to maximize the reward given the sampled parameters. The posterior distribution can be derived in closed-form in a few special cases such as the Bernoulli bandit (Russo et al., 2018). With more complex models such as our low-rank tensor bandit problem, the exact calculation of the posterior distribution becomes intractable. In this case, we consider an ensemble sampling approach (Lu and Van Roy, 2017) that aims to approximate Thompson sampling while maintaining computational tractability. In an online advertising application, our tensor ensemble sampling is empirically successful and reduces the cumulative regret by 75% compared to the benchmark method.

There are several lines of research that are related to but also clearly distinctive of the problem we address. The first line is tensor completion (Yuan and Zhang, 2016; Song et al., 2019; Zhang et al., 2019; Cai et al., 2021; Xia et al., 2021; Han et al., 2022). While we employ similar low-dimensional structures as tensor completion, the two problems have fundamental difference. First, a key assumption in existing tensor completion is to assume the observed entries are collected uniformly and randomly (the only exception is Zhang et al. (2019) which assumes a special cross structure of the missing mechanism). This is largely different from our interactive online decision problem where the observed entries are collected adaptively based on some bandit policy. The difference is analogous to that between linear regression

and linear bandit (Lattimore and Szepesvári, 2020). Second, the goal of existing tensor completion is to predict all missing entries while the goal of tensor bandits is to find the *largest* entry in the reward tensor so that the cumulative regret is minimized. Third, these tensor completion algorithms are developed for off-line settings where data are collected all at once. They are not applicable to our online decision problem where data enter the system sequentially. On the other hand, existing online tensor completion (Yu et al., 2015; Ahn et al., 2021) for streaming data could not handle our interactive decision problem due to their uniform and random missing mechanism and non-interaction nature.

The second line of related work is low-rank matrix bandit. There are some works considering special rank-1 matrix bandits (Katariya et al., 2017b,a; Trinh et al., 2020). To find the largest entry of a non-negative rank-1 matrix, one just needs to identify the largest values of the left-singular and right-singular vectors. However, this is no longer applicable for higher-rank matrices. For general low-rank matrix bandits, Kveton et al. (2017) handled low-rank matrix bandits but imposed strong "hott topics" assumptions on the mean reward matrix. They assumed all rows of decomposed factor matrix can be written as a convex combination of a subset of rows. Sen et al. (2017) considered low-rank matrix bandits with one dimension choosing by the nature and the other dimension choosing by the agent. They derived a logarithmic regret under a constant gap assumption. However the gap may not be specified in advance. Lu et al. (2018) utilized ensemble sampling for low-rank matrix bandits but did not provide any regret guarantee due to the theoretical challenges in handling sampling-based exploration. Jun et al. (2019) proposed a bilinear bandit that can be viewed as a contextual low-rank matrix bandit. However, their regret bound becomes sub-optimal in the context-free setting due to the use of LinUCB (Abbasi-Yadkori et al., 2011) for linear bandits with finitely many arms. In addition, our theory shows that unfolding reward tensor into matrix and then applying algorithm proposed by Jun et al. (2019) leads to a suboptimal regret bound. Lu et al. (2021) further generalized Jun et al. (2019) to a low-rank generalized linear bandit. To the best of our knowledge, there is no existing work that systematically studies tensor bandits problem. Low-rank tensor structure has imposed fundamental challenges. It is well known that many efficient tools for matrix data, such as nuclear norm minimization or singular value decomposition, cannot be simply extended to tensor framework (Richard and Montanari, 2014; Yuan and Zhang, 2016; Friedland and Lim, 2017; Zhang and Xia, 2018). Hence existing algorithms and proof strategies for linear bandits or matrix bandits are not directly applicable to our tensor bandits problem. Our proposed algorithms and their regret analysis demand new technical tools.

The rest of the paper is organized as follows. Section 2 reviews some notation and tensor algebra. Section 3 presents our model, two main algorithms and their theoretical analysis for the tensor bandits. Section 4 considers the extension to the contextual tensor bandits. Section 5 contains a series of simulation studies. Section 6 applies our algorithm to an online advertising application to illustrate its practical advantages. All proof details are left in the supplemental material.

2 Notation and Tensor Algebra

A tensor is a multidimensional array and the order of a tensor is the number of dimensions it has, also referred to as the mode. We denote vectors using lower-case bold letters (e.g., \mathbf{x}), matrices using upper-case bold letters (e.g., \mathbf{X}), and high-order tensors using upper-case bold script letters (e.g., \mathcal{X}). We denote the cardinality of a set by $|\cdot|$ and write $[k] = \{1, 2, \dots, k\}$ for an integer $k \geq 1$. For a positive scalar x, let $\lceil x \rceil = \min\{z \in \mathbb{N}^+ : z \geq x\}$. We use $\mathbf{e}_j \in \mathbb{R}^p$ to denote a basis vector that takes 1 as its j-th entry and 0 otherwise. For a vector $\mathbf{a} \in \mathbb{R}^d$ and $s_1 \leq s_2 \in [d]$, let $\mathbf{a}_{s_1:s_2}$ be the sub-vector $(\mathbf{a}_{s_1}, \mathbf{a}_{s_1+1}, \dots, \mathbf{a}_{s_2})$. For an order-d tensor $\mathcal{X} \in \mathbb{R}^{p_1 \times \dots \times p_d}$, define its mode-j fibers as the p_j -dimensional vectors $\mathcal{X}_{i_1,\dots,i_{j-1},\cdot,i_{j+1},\dots,i_d}$, and its mode-j matricization as $\mathcal{M}_j(\mathcal{X}) \in \mathbb{R}^{p_j \times (p_1 \dots p_{j-1}p_{j+1} \dots p_d)}$, where the column vectors of $\mathcal{M}_j(\mathcal{X})$ are the mode-j fibers of \mathcal{X} . For instance, for an order-3 tensor $\mathcal{X} \in \mathbb{R}^{p_1 \times p_2 \times p_3}$, its mode-1 matricization $\mathcal{M}_1(\mathcal{X}) \in \mathbb{R}^{p_1 \times (p_2p_3)}$ is defined as, for $i \in [p_1], j \in [p_2], k \in [p_3]$,

$$\left[\mathcal{M}_1(\mathcal{X})\right]_{i,(j-1)p_3+k} = \mathcal{X}_{i,j,k}.\tag{1}$$

For a tensor $\mathcal{X} \in \mathbb{R}^{p_1 \times p_2 \times \cdots \times p_d}$ and a matrix $\mathbf{Y} \in \mathbb{R}^{r_1 \times p_1}$, we define the marginal multiplication $\mathcal{X} \times_1 \mathbf{Y} \in \mathbb{R}^{r_1 \times p_2 \times \cdots \times p_d}$ as

$$\mathcal{X} \times_1 \mathbf{Y} = \left(\sum_{i_1'=1}^{p_1} \mathcal{X}_{i_1', i_2, \dots, i_d} \mathbf{Y}_{i_1, i_1'} \right)_{i_1 \in [r_1], i_2 \in [p_2] \dots, i_d \in [p_d]}. \tag{2}$$

Marginal multiplications along other modes, i.e., $\times_2, \ldots, \times_d$, can be defined similarly. For $\mathcal{X}, \mathcal{Y} \in \mathbb{R}^{p_1 \times \cdots \times p_d}$, define the tensor inner product as $\langle \mathcal{X}, \mathcal{Y} \rangle = \sum_{i_1 \in [p_1], \ldots, i_d \in [p_d]} \mathcal{X}_{i_1, \ldots, i_d} \mathcal{Y}_{i_1, \ldots, i_d}$. The tensor Frobenius norm is defined as $\|\mathcal{X}\|_F = \sqrt{\langle \mathcal{X}, \mathcal{X} \rangle}$, and the element-wise tensor max norm is defined as $\|\mathcal{X}\|_{\infty} = \max_{i_1, \ldots, i_d} |\mathcal{X}_{i_1, \ldots, i_d}|$.

Consider again an order-d tensor $\mathcal{X} \in \mathbb{R}^{p_1 \times \cdots \times p_d}$. Letting r_j be the rank of matrix $\mathcal{M}_j(\mathcal{X})$, $j \in [d]$, the tensor Tucker rank of \mathcal{X} is the d-tuple (r_1, \ldots, r_d) . Let $\mathbf{U}_1 \in \mathbb{R}^{p_1 \times r_1}, \ldots, \mathbf{U}_d \in \mathbb{R}^{p_d \times r_d}$ be the matrices whose columns are the left singular vectors of $\mathcal{M}_1(\mathcal{X}), \ldots, \mathcal{M}_d(\mathcal{X})$, respectively. Then, there exists a core tensor $\mathcal{S} \in \mathbb{R}^{r_1 \times \cdots \times r_d}$ such that

$$\mathcal{X} = \mathcal{S} \times_1 \mathbf{U}_1 \times_2 \cdots \times_d \mathbf{U}_d,$$

or equivalently, $\mathcal{X}_{i_1,\dots,i_d} = \sum_{i'_1 \in [r_1],\dots,i'_d \in [r_d]} S_{i'_1,\dots,i'_d}[\mathbf{U}_1]_{i_1,i'_1} \cdots [\mathbf{U}_d]_{i_d,i'_d}$. The above decomposition is often referred to as the tensor Tucker decomposition (Kolda and Bader, 2009).

3 Tensor Bandits

In this section, we first introduce tensor bandits, followed by two new algorithms — tensor elimination and tensor epoch—greedy. We then establish the finite-time regret bounds of these two algorithms, which reveal their different performances under different rate conditions and provide a useful guidance for their implementations in practice.

In tensor bandits problem, the agent interacts with an environment for n time steps, and at each step, the agent faces a d-dimensional decision, indexed by $[p_1] \times \cdots \times [p_d]$. A standard multi-armed bandit can be regarded as a special case of tensor bandits with d = 1. At step $t \in [n]$ and given past interactions, the agent pulls an arm I_t , which denotes a d-tuple $(i_{1,t}, \ldots, i_{d,t}) \in [p_1] \times \ldots \times [p_d]$. Correspondingly, the agent observes a reward $y_t \in \mathbb{R}$, drawn

from a probability distribution associated with the arm I_t . Specifically, denoting the true reward tensor as $\mathcal{X} \in \mathbb{R}^{p_1 \times ... \times p_d}$, the agent at time t receives a noisy reward

$$y_t = \langle \mathcal{X}, \mathcal{A}_t \rangle + \epsilon_t, \text{ with } \mathcal{A}_t = \mathbf{e}_{i_{1,t}} \circ \cdots \circ \mathbf{e}_{i_{d,t}},$$
 (3)

where "o" denotes the vector outer product, $\mathbf{e}_{i_{j,t}} \in \mathbb{R}^{p_j}$ is a basis vector, $j \in [d]$, and \mathcal{A}_t is a tensor indicating the location of the arm I_t . For example, if the agent pulls $I_t = (i_{1,t}, \dots, i_{d,t})$, then the $(i_{1,t}, \dots, i_{d,t})$ -th entry of \mathcal{A}_t is 1 while all other entries are 0. In (3), ϵ_t is a random noise term, assumed to be sub-Gaussian in Assumption 1.

The goal of our work, aligned with the central task in bandit problems, is to strike the right balance between exploration and exploitation, and to minimize the cumulative regret.

Let the arm with the maximum true reward be

$$(i_1^*, \dots, i_d^*) = \underset{i_1 \in [p_1], \dots, i_d \in [p_d]}{\operatorname{argmax}} \langle \mathcal{X}, \mathbf{e}_{i_1} \circ \dots \circ \mathbf{e}_{i_d} \rangle$$

and correspondingly, denote $\mathcal{A}^* = \mathbf{e}_{i_1^*} \circ \cdots \circ \mathbf{e}_{i_d^*}$. Our objective is to minimize the cumulative regret (Audibert et al., 2009), defined as

$$R_n = \sum_{t=1}^n \langle \mathcal{X}, \mathcal{A}^* \rangle - \sum_{t=1}^n \langle \mathcal{X}, \mathcal{A}_t \rangle. \tag{4}$$

Naturally, at each step $t \in [n]$, the agent faces an exploitation-exploration dilemma, in that the agent can either choose the arm that expects the highest reward based on historical data (exploitation), so as to reduce immediate regret, or choose some under-explored arms to gather information about their associated reward (exploration), so as to reduce future regret.

At first glance, the tensor bandit problem posed in (3)-(4) can be re-formulated, via vectorization, as a standard multi-armed bandit problem of dimension $p_1 \times ... \times p_d$. However, applying the existing algorithms for standard multi-armed bandits to vectorized tensor bandits may be inappropriate due to several reasons. First, the majority of existing solutions for multi-armed bandits require a proper initialization phase where each arm is pulled at least once, in order to give a well-defined solution (Auer et al., 2002). For tensor bandits, such an initialization step can be computationally expensive or even infeasible, especially

when $p_1 \times ... \times p_d$ is large. Second, the vectorization approach may result in a severe loss of information, as the intrinsic structures (e.g., low-rank) of tensors are largely ignored after vectorization. Indeed, as commonly considered in recommendation systems and other applications (Kolda and Bader, 2009; Allen, 2012; Jain and Oh, 2014; Bi et al., 2018; Song et al., 2019; Xia et al., 2021; Bi et al., 2021), tensor objects usually have a low-rank structure and can be represented in a lower-dimensional space.

In this work, we propose to retain the tensor form of \mathcal{X} and assume that it admits the following low-rank decomposition,

$$\mathcal{X} = \mathcal{S} \times_1 \mathbf{U}_1 \times_2 \cdots \times_d \mathbf{U}_d, \tag{5}$$

where $S \in \mathbb{R}^{r_1 \times \cdots \times r_d}$ is a core tensor, and $\mathbf{U}_1 \in \mathbb{R}^{p_1 \times r_1}, \ldots, \mathbf{U}_d \in \mathbb{R}^{p_d \times r_d}$ are matrices with orthonormal columns; see more details on this decomposition in Section 2. The low-rank assumption in (5) exploits the structures in tensors and efficiently reduces the number of free parameters in \mathcal{X} . Furthermore, it allows us to consider an efficient initialization phase with $\mathcal{O}(r^{(d-2)/2}p^{d/2})$ steps, assuming $r_1 = \ldots = r_d = r$ and $p_1 = \ldots = p_d = p$ for simplicity, which is much reduced comparing to the p^d steps required in the simple vectorization strategy. As demonstrate in Table 1, comparing to the vectorized solutions that ignore the low-rank structure, our proposed low-rank tensor bandit algorithms have much improved finite-time regret bounds.

Before discussing the main algorithms, we first describe our initialization procedure. Thanks to the tensor low-rank structure, our initialization phase need not to pull every arm at least once, which is required in the majority of multi-armed bandit algorithms. Define an initial set of s_1 steps

$$\mathcal{E}_1 = \{ t \mid t \in [s_1] \}, \tag{6}$$

where s_1 is an integer to be specified later in Assumption 3. In the initialization phase, arms are pulled with a uniform probability, equivalent to assuming $\mathbb{P}(i_{jt} = k) = 1/p_j$, $k \in [p_j]$, in (3). If some prior knowledge about the true reward tensor is available, a non-uniform sampling (e.g., Klopp, 2014) can also be considered in the initialization phase.

3.1 Tensor Elimination

The upper confidence band (UCB) strategies (Lai and Robbins, 1985) have been very successful in bandit problems, leading to an extensive literature on UCB algorithms for standard multi-armed bandits (Lattimore and Szepesvári, 2020). These UCB algorithms balance between exploration and exploitation based on a confidence bound that the algorithm assigns to each arm. Specifically, in each round of steps, the UCB algorithm constructs an upper confidence bound for the reward associated with each arm, and the arms with the highest upper bounds are pulled, as they may be associated with high rewards and/or large uncertainties (i.e., under-explored). Many work have analyzed the regret bounds of UCB algorithms and investigated their optimality (Auer et al., 2002; Garivier and Cappé, 2011).

Employing the successful UCB strategy in low-rank tensor bandits encounters a critical challenge, as the tensor decomposition in (5) is a non-convex problem, the data is adaptively collected and the concentration results for $\hat{S}, \hat{\mathbf{U}}_1, \dots, \hat{\mathbf{U}}_d$, to our knowledge, remain elusive thus far. Without such concentration results, constructing the confidence bounds becomes a very difficult problem. One straightforward strategy is to first vectorize the tensor bandits and then treat the problem as a standard multi-armed bandit problem. However, as discussed before, this strategy incurs a severe loss of structural information and is demanding, in terms of sample complexity, in its initialization phase. In our proposed approach, we consider a tensor spectral-based rotation strategy that preserves the low-rank information and at the same time, enables uncertainty quantification. We also consider an elimination step that eliminate less promising arms based on the calculated confidence bounds, which further improves the finite-time regret bound (see Theorem 1). Taken together, the proposed tensor elimination algorithm avoids directly characterizing the uncertainty of tensor decomposition estimators, effectively utilizes the low-rank information and achieves a desirable sub-linear finite-time regret bound. Next, we discuss the tensor elimination algorithm in details.

The tensor elimination shown in Algorithm 1 starts with an initialization phase of length s_1 and then proceeds to an exploration phase of length n_1 , where arms in both phases

Algorithm 1 Tensor elimination

- 1: **Input:** number of total steps n, number of exploration steps n_1 , regularization parameters λ_1, λ_2 , length of confidence intervals ξ , ranks r_1, \dots, r_d .
- 2: # initialization and exploration phases
- 3: Initialize: $\mathcal{D} = \emptyset$.
- 4: **for** $t = 1, \dots, s_1 + n_1$ **do**
- 5: Randomly pull an arm A_t and receive its associated reward y_t . Let $\mathcal{D} = \mathcal{D} \cup \{(y_t, A_t)\}$.
- 6: end for
- 7: Calculate $\widehat{\mathbf{U}}_1, \ldots, \widehat{\mathbf{U}}_d$ using \mathcal{D} , and then find $\widehat{\mathbf{U}}_{1\perp}, \ldots, \widehat{\mathbf{U}}_{d\perp}$
- 8: # reduction phase
- 9: Construct an action set \mathbb{A}_1 as in (9) and denote $q = \prod_{j=1}^d p_j \prod_{j=1}^d (p_j r_j)$.
- 10: **for** k = 1 to $\log_2(n)$ **do**
- 11: Set $V_{t_k} = \operatorname{diag}(\underbrace{\lambda_1, \dots, \lambda_1}_{2}, \lambda_2, \dots, \lambda_2)$ and $\mathcal{D} = \emptyset$.
- 12: **for** $t = t_k$ to $\min(t_{k+1} 1, n n_1 s_1)$ **do**
- 13: Pull the arm $A_t = \operatorname{argmax}_{a \in \mathbb{A}_k} \|\mathbf{a}\|_{V_t^{-1}}$.
- 14: Receive its associated reward y_t and update $V_{t+1} = V_t + A_t A_t^{\top}$. Let $\mathcal{D} = \mathcal{D} \cup \{(y_t, A_t)\}$.
- 15: end for
- 16: Eliminate arms based on confidence intervals:

$$\mathbb{A}_{k+1} = \left\{ \mathbf{a} \in \mathbb{A}_k : \langle \widehat{\boldsymbol{\beta}}_k, \mathbf{a} \rangle + \|\mathbf{a}\|_{V_t^{-1}} \xi \ge \max_{\mathbf{a} \in \mathbb{A}_k} \left[\langle \widehat{\boldsymbol{\beta}}_k, \mathbf{a} \rangle - \|\mathbf{a}\|_{V_t^{-1}} \xi \right] \right\}, \text{ where}$$
 (7)

$$\widehat{\boldsymbol{\beta}}_{k} = \underset{\boldsymbol{\beta}}{\operatorname{argmin}} \left\{ \frac{1}{2} \sum_{(y_{t}, A_{t}) \in \mathcal{D}} \left(y_{t} - \langle A_{t}, \boldsymbol{\beta} \rangle \right)^{2} + \frac{1}{2} \lambda_{1} \|\boldsymbol{\beta}_{1:q}\|_{2} + \frac{1}{2} \lambda_{2} \|\boldsymbol{\beta}_{(q+1):\Pi_{j=1}^{d} p_{j}}\|_{2} \right\}.$$
(8)

17: end for

are selected randomly. In this algorithm, the initialization phase and exploration phase are same. We choose to separate them so that the format is consistency with the tensor epoch-greedy algorithm introduced in next subsection. Here, s_1 is set to be the minimal sample size for tensor completion and n_1 is chosen to minimize cumulative regret, both of which will be specified later in Section 3.2. Based on the random samples collected from the initialization and exploration phases, we calculate estimates $\hat{\mathbf{U}}_1, \ldots, \hat{\mathbf{U}}_d$ of the matrices $\mathbf{U}_1, \ldots, \mathbf{U}_d$ in (5) using a low-rank tensor completion method (see Appendix S.3.2). Next, we consider a rotation technique that preserves the tensor low-rank structure, and enables vectorization and uncertainty quantification (see Lemma 1). Specifically, given $\hat{\mathbf{U}}_j$, $j \in [d]$, define $\hat{\mathbf{U}}_{j\perp}$ whose columns are the orthogonal basis of the subspace complement to the column

subspace of $\widehat{\mathbf{U}}_j$. Consider a rotation to the true reward tensor \mathcal{X} calculated as

$$\mathcal{Y} = \mathcal{X} \times_1 [\widehat{\mathbf{U}}_1; \widehat{\mathbf{U}}_{1\perp}] \times_2 \cdots \times_d [\widehat{\mathbf{U}}_d; \widehat{\mathbf{U}}_{d\perp}] \in \mathbb{R}^{p_1 \times \cdots \times p_d},$$

where $\times_1, \ldots, \times_d$ are as defined in (2) and $[\widehat{\mathbf{U}}_j; \widehat{\mathbf{U}}_{j\perp}]$ is the concatenation (by columns) of $\widehat{\mathbf{U}}_j$ and $\widehat{\mathbf{U}}_{j\perp}$. Correspondingly, the reward defined in (3) can be re-written (see proof in Appendix S.3.1) as

$$y_t = \left\langle \mathcal{Y}, [\widehat{\mathbf{U}}_1; \widehat{\mathbf{U}}_{1\perp}]^{\top} \mathbf{e}_{i_{1,t}} \circ \cdots \circ [\widehat{\mathbf{U}}_d; \widehat{\mathbf{U}}_{d\perp}]^{\top} \mathbf{e}_{i_{d,t}} \right\rangle + \epsilon_t.$$

It is seen that replacing the reward tensor \mathcal{X} with \mathcal{Y} and the arm $\mathbf{e}_{i_1} \circ \cdots \circ \mathbf{e}_{i_d}$ with $[\widehat{\mathbf{U}}_1; \widehat{\mathbf{U}}_{1\perp}]^{\top} \mathbf{e}_{i_{1,t}} \circ \cdots \circ [\widehat{\mathbf{U}}_d; \widehat{\mathbf{U}}_{d\perp}]^{\top} \mathbf{e}_{i_{d,t}}$ does not change the tensor bandit problem. Define $\boldsymbol{\beta} = \text{vec}(\mathcal{Y}) \in \mathbb{R}^{\prod_{j=1}^{d} p_j}$, which vectorizes the reward tensor \mathcal{Y} such that the first $\prod_{j=1}^{d} r_j$ entries of $\text{vec}(\mathcal{Y})$ are $\mathcal{Y}_{i_1,\dots,i_d}$ for $i_j \in \{1,\dots,r_j\}$, $j \in [d]$, and denote the corresponding vectorized arm set as

$$\mathbb{A} := \left\{ \operatorname{vec} \left([\widehat{\mathbf{U}}_1; \widehat{\mathbf{U}}_{1\perp}]^{\top} \mathbf{e}_{i_1} \circ \cdots \circ [\widehat{\mathbf{U}}_d; \widehat{\mathbf{U}}_{d\perp}]^{\top} \mathbf{e}_{i_d} \right), i_1 \in [p_1], \dots, i_d \in [p_d] \right\}.$$
(9)

Correspondingly, the tensor bandits in (3) with the true reward tensor \mathcal{X} and arm set $\{\mathbf{e}_{i_1} \circ \cdots \circ \mathbf{e}_{i_d}, i_1 \in [p_1], \cdots, i_d \in [p_d]\}$ can be re-formulated as a multi-armed bandits with the reward vector $\boldsymbol{\beta}$ and arm set \mathbb{A} .

It is easy to see that in $\operatorname{vec}(\mathcal{X} \times_1 [\mathbf{U}_1; \mathbf{U}_{1\perp}] \times_2 \cdots \times_d [\mathbf{U}_d; \mathbf{U}_{d\perp}])$, the first $\Pi_{j=1}^d r_j$ entries are nonzero and the last $\Pi_{j=1}^d (p_j - r_j)$ entries are zero. Such a sparsity pattern cannot be achieved if \mathcal{X} is vectorized directly without the rotation. From this perspective, the rotation strategy preserves the structural information in the vectorized tensor. Specifically, when estimating the reward vector $\boldsymbol{\beta}$ in (8), we apply different regularizations to the first $\Pi_{j=1}^d r_j$ entries and the remaining $\Pi_{j=1}^d (p_j - r_j)$ entries, respectively.

The algorithm then proceeds to the elimination phase, where less promising arms are identified and eliminated. This phase aims to further improve the regret bound. Given a vector \mathbf{a} , we define its \mathbf{A} -norm as $\|\mathbf{a}\|_{\mathbf{A}} = \sqrt{\mathbf{a}^{\top} \mathbf{A} \mathbf{a}}$, where \mathbf{A} is a positive definite matrix. During phase k with the arm set \mathbb{A}_k , the confidence ellipsoid of the mean reward of each

arm $\mathbf{a} \in \mathbb{A}_k$ is constructed using $\widehat{\boldsymbol{\beta}}_k$. It is show in Lemma 1 that the confidence width of the reward of arm \mathbf{a} is $\|\mathbf{a}\|_{V^{-1}}\xi$, where V is the covariance matrix and ξ is a fixed constant term that does not depend on \mathbf{a} . At each time step t, the algorithm (line 13) then pulls the arm with the largest confidence interval width. The intuition of the arm selection in this step is that arms with the highest confidence widths are likely under-explored. At the end of phase k (line 16), we implement an elimination procedure that trims less promising arms. Specifically, we first update the estimate $\widehat{\boldsymbol{\beta}}_k$ in (8) based on the pulled arms and their associated rewards during phase k. Based on the estimated reward $\widehat{\boldsymbol{\beta}}_k$, we then construct confidence interval (7) for the mean reward of each arm and eliminate the arms whose upper confidence bound is lower than the maximum of lower confidence bounds of all arms in \mathbb{A}_k .

3.2 Regret Analysis of Tensor Elimination

In this section, we carry out the regret analysis of the tensor elimination. To ease notation, we assume the tensor rank $r_1 = \ldots = r_d = r$ and the tensor dimension $p_1 = \ldots = p_d = p$. The results for general ranks and dimensions can be established similarly using a more involved notation system. We first state some assumptions.

Assumption 1 (Sub-Gaussian noise). The noise term ϵ_t is assumed to follow a 1-sub-Gaussian distribution such that, for any $\lambda \in \mathbb{R}$,

$$\mathbb{E}[\exp(\lambda \epsilon_t)] \le \exp(\lambda^2/2).$$

Assumption 2. Assume true reward tensor \mathcal{X} admits the low-rank decomposition in (5) and $\|\mathcal{X}\|_{\infty} \leq 1$.

The assumption $\|\mathcal{X}\|_{\infty} \leq 1$ assumes that the reward is bounded, and it is common in the multi-armed bandit literature (see, for example, Langford and Zhang, 2007). It implies that the immediate regret in each exploration step is $\mathcal{O}(1)$. Similar boundedness conditions on tensor entries can also be found in the tensor completion literature (see, for example, Cai et al., 2021; Xia et al., 2021).

Assumption 3. Assume the number of steps in the initialization phase s_1 is

$$s_1 = C_0 r^{(d-2)/2} p^{d/2}, (10)$$

where C_0 is a positive constant as defined in Lemma S3.

This assumption requires the minimal sample complexity for provably recovering a lowrank tensor from noisy observations when the entries are observed randomly (see Lemma S3 and Xia et al. (2021)). Such random initialization phase is standard and important in all bandit algorithms (Lattimore and Szepesvári, 2020). As discussed before, the simple vectorization strategy would require $s_1 = \mathcal{O}(p^d)$, which is significantly larger.

The next lemma provides the confidence interval for the reward of a fixed arm a.

Lemma 1. For any fixed vector $\mathbf{a} \in \mathbb{R}^{p^d}$ and $\delta > 0$, we have that, if

$$\xi = 2\sqrt{14\log(2/\delta)} + \sqrt{\lambda_1} \|\boldsymbol{\beta}_{1:q}\|_2 + \sqrt{\lambda_2} \|\boldsymbol{\beta}_{(q+1):p^d}\|_2, \tag{11}$$

with $\beta = vec(\mathcal{Y})$, $\lambda_1 > 0$ and $\lambda_2 = n/(q \log(1 + n/\lambda_1))$, then at the beginning of phase k

$$\mathbb{P}(|\mathbf{a}^{\top}(\widehat{\boldsymbol{\beta}}_k - \boldsymbol{\beta})| \le \xi ||\mathbf{a}||_{V^{-1}}) \ge 1 - \delta,$$

where $V_t = \sum_{s=1}^t A_s A_s^{\top} + \operatorname{diag}(\underbrace{\lambda_1, \dots, \lambda_1}_{s}, \lambda_2, \dots, \lambda_2).$

Next, we show the finite-time regret bound for tensor elimination. Recall $q = \prod_{j=1}^d p_j - \prod_{j=1}^d (p_j - r_j)$.

Theorem 1. Suppose Assumptions 1-3 hold. Let $t_k = 2^{k-1}$, $0 < \lambda_1 \le 1/p^d$, $\lambda_2 = n/(q \log(1 + n/\lambda_1))$, and

$$n_1 = \left\lceil n^{\frac{2}{d+2}} \frac{r^d}{\prod_{j=1}^d \sigma_j} p^{\frac{d^2+d}{2}} \log^{d/2}(p) \right\rceil, \tag{12}$$

where σ_j is the smallest non-zero singular value of $\mathcal{M}_j(\mathcal{X})$, $j \in [d]$. The cumulative regret of Algorithm 1 satisfies

$$R_n \le C \left(r^{\frac{d}{2}} p^{\frac{d}{2}} + \left(\frac{r^d}{\prod_{i=1}^d \sigma_i} \log^{d/2}(p) \right)^{\frac{2}{d+2}} p^{\frac{d^2+d}{d+2}} n^{\frac{2}{d+2}} + \sqrt{(d \log(p) + \log(n))^2 p^{d-1} n} \right),$$

with probability at least $1 - dp^{-10} - 1/n$, where C > 0 is some constant.

The detailed proof of Theorem 1 is deferred to Appendix S.1.1. Ignoring any logarithmic and constant factor, the above regret bound can be simplified to

$$R_n = \widetilde{\mathcal{O}}(r^{\frac{d-2}{2}}p^{\frac{d}{2}} + r^{\frac{2d}{d+2}}p^{\frac{d^2+d}{d+2}}n^{\frac{2}{d+2}} + p^{\frac{d-1}{2}}n^{\frac{1}{2}}).$$
(13)

The upper bound on the cumulative regret is the sum of three terms, with the first two terms characterizing regret from the s_1 initialization steps and n_1 exploration steps, respectively, and the third term quantifying the regret in the $n-s_1-n_1$ elimination steps. As the regret from the exploration phase increases with n_1 and the regret from the elimination phase decreases with n_1 , the value for n_1 in (12) is chosen to minimize the sum of these two regrets. Note that after the rotation, the order of $\|\boldsymbol{\beta}_{1:q}\|_2$ is of $\widetilde{\mathcal{O}}(p^{d/2})$ which guides the choice of λ_1 . One component of the upper bound of the cumulative regret is $\log\left(\frac{\det(V_k)}{\det(\Lambda)}\right)$ with $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_1, \lambda_2, \ldots, \lambda_2)$ and λ_2 is chosen to minimize the upper bound of the log term so as to minimize the upper bound of cumulative regret.

Remark 1. It is worth to compare the regret bound in Eq. (13) with other strategies. As summaized in Table 1, when d=3 and $r=\mathcal{O}(1)$, vectorized UCB suffers $\widetilde{\mathcal{O}}(p^3+p^{3/2}n^{1/2})$. If we unfold the tensor into a matrix and implement ESTR (Jun et al., 2019), it suffers $\widetilde{\mathcal{O}}(p^2+p^3n^{1/2})$. Both of these competitive methods obtain significantly sub-optimal regret bounds. By utilizing the low-rank tensor information, our bound greatly improves the dependency on the dimension p. Moreover, our advantage is even larger when the tensor order d is larger.

Remark 2. One may wonder whether we can extend the matrix bandit ESTR (Jun et al., 2019) to the tensor case. In this case, standard LinUCB (Abbasi-Yadkori et al., 2011) algorithm could be queried to handle the reshaped linear bandits as did in the matrix bandits (Jun et al., 2019). However, it is known that the algorithm of LinUCB is suboptimal for linear bandits with finitely many arms and the sub-optimality will be amplified as the order of tensor grows. Hence, using LinUCB in the reduction phase results in $\mathcal{O}(p^2n^{1/2})$ for the leading term that is even worse than vectorized UCB.

One of the key challenges in our theoretical analysis is to quantify the cumulative regret in the elimination phase. Existing techniques are not applicable as we utilize a different eliminator with a modified regularization strategy. Furthermore, to bound the cumulative regret in the elimination phase, we need to bound the norm $\|\beta_{(q+1):p^d}\|_2$ which is the last $p^d - q$ entries of $vec(\mathcal{Y})$. Recall that the reward tensor \mathcal{Y} is a rotation of true reward tensor \mathcal{X} . We need to derive the upper bound of the norm of rotated reward vector by exploiting the knowledge of estimation error of \mathcal{X} . We use the elliptical potential lemma to bound the cumulative regret in elimination phase. Furthermore, all parameters such as the penalization parameter λ_2 and exploration phase length n_1 are carefully selected to obtain the best bound.

3.3 Tensor Epoch-greedy and Regret Analysis

Next, we propose an epoch-greedy type algorithm for low-rank tensor bandits, and compare its performance with tensor elimination. The epoch-greedy algorithm (Langford and Zhang, 2007) proceeds in epochs, with each epoch consisting of an exploration phase and an exploitation phase. One advantage of this epoch-greedy algorithm is that we do not need to know the total time horizon n in advance. In the exploration phase, arms are randomly selected and in the exploitation phase, arms that expect the highest reward are pulled. The number of steps in each exploitation phase increases with number of epochs, guided by the fact that, as the number of epochs increases, the estimation accuracy of the true reward improves and more exploitation steps are desirable. The epoch-greedy algorithm is straightforward to implement, and we find that compared to tensor elimination, tensor epoch-greedy algorithm has a better dependence on dimension p and a worse dependence on time horizon n.

The detailed steps of tensor epoch-greedy are given in Algorithm 2. In the initialization phase, i.e., $t \in \mathcal{E}_1$, arms are randomly pulled to collect samples for tensor completion. Recall the initialization phase has s_1 steps. Let the index set of steps in the exploration phases be

$$\mathcal{E}_2 = \left\{ s_1 + l + 1 + \sum_{k=0}^{l} s_{2k} \mid l = 0, 1, \dots \right\}, \tag{14}$$

Algorithm 2 Tensor epoch-greedy

```
1: Input: initial set \mathcal{E}_1, exploration set \mathcal{E}_2.
 2: Initialize \mathcal{D} = \emptyset.
 3: for t = 1, 2, ..., n do
            # initialization and exploration phases
           if t \in \mathcal{E}_1 \cup \mathcal{E}_2 then
                Randomly pull an arm \mathcal{A}_t and receive its associated reward y_t = \langle \mathcal{X}, \mathcal{A}_t \rangle + \epsilon_t.
 6:
                 Let \mathcal{D} = \mathcal{D} \cup \{(y_t, \mathcal{A}_t)\}.
 7:
 8:
            end if
 9:
            # exploitation phase
            if t \notin \mathcal{E}_1 \cup \mathcal{E}_2 then
10:
                Based on \mathcal{D}, calculate a low-rank tensor estimate \widehat{\mathcal{X}}_t.
11:
                Pull the arm (i_{1,t}, \ldots, i_{d,t}) = \operatorname{argmax}_{i_1, \ldots, i_d} \langle \widehat{\mathcal{X}}_t, \mathbf{e}_{i_1} \circ \ldots \circ \mathbf{e}_{i_d} \rangle.
Receive the associated reward y_t = \langle \mathcal{X}, \mathbf{e}_{i_{1,t}} \circ \ldots \circ \mathbf{e}_{i_{d,t}} \rangle + \epsilon_t.
12:
13:
14:
15: end for
```

where s_{2k} denotes the number of exploitation steps in the kth epoch and it increases with k. In the exploration phase, i.e., $t \in \mathcal{E}_2$, an arm \mathcal{A}_t is pulled (or sampled) randomly. These random samples collected in the exploration phases are important for unbiased estimation, as they do not dependent on historical data, and their accumulation can improve estimation accuracy of the reward tensor. Meanwhile, as the exploration phase does not focus on the best arm, each step $t \in \mathcal{E}_2$ is expected to result in a large immediate regret, though it can potentially reduce regret from future exploitation steps. In the exploitation phase, i.e., $t \notin \mathcal{E}_1 \cup \mathcal{E}_2$, we construct a low-rank estimate $\widehat{\mathcal{X}}_t$ of the reward tensor using the random samples collected thus far in \mathcal{D} . Then, the arm $(i_{1,t}, \ldots, i_{d,t})$ with the highest estimated reward in $\widehat{\mathcal{X}}_t$ is selected, i.e.,

$$(i_{1,t},\ldots,i_{d,t}) = \underset{i_1,\ldots,i_d}{\operatorname{argmax}} \langle \widehat{\mathcal{X}}_t, \mathbf{e}_{i_1} \circ \ldots \circ \mathbf{e}_{i_d} \rangle.$$

Samples in the exploitation phase will not be used to estimate the reward tensor as they are biased and thus exploitation steps cannot improve estimation accuracy of the reward tensor.

We next derive the regret bound of proposed tensor epoch-greedy.

Theorem 2. Suppose Assumptions 1-3 hold. Let

$$s_{2k} = \left[C_2 p^{-\frac{d+1}{2}} r^{-\frac{1}{2}} (\log p)^{-\frac{1}{2}} (k + s_1)^{\frac{1}{2}} \right], \tag{15}$$

for some small constant $C_2 > 0$. When $n \ge C_0 r^{\frac{d-2}{2}} p^{\frac{d}{2}}$, the cumulative regret of Algorithm 2 satisfy, with probability at least $1 - p^{-10}$,

$$R_n \le C_0 r^{\frac{d-2}{2}} p^{\frac{d}{2}} + 8n^{\frac{2}{3}} p^{\frac{d+1}{3}} (r \log p)^{\frac{1}{3}}.$$

$$(16)$$

The regret bound has two terms with the first term characterizing the regret accumulated during the initialization phase and the second term characterizing the regret accumulated over the exploration and exploitation phases. The first term depends on the tensor rank r and dimension p, but not n. It clearly highlights the benefit of exploiting a tensor low-rank structure since unfolding the tensor into a vector or a matrix requires much longer initialization phase. The second term in the regret bound is related to time horizon n and it increases with n at a rate of $n^{\frac{2}{3}}$.

It is worth to compare the leading term of regret bounds for high-order tensor bandits of tensor elimination in Eq. (13) and tensor epoch-greedy in Eq. (16). As summaized in Table 1, when $d \geq 3$ and $r = \mathcal{O}(1)$, tensor elimination suffers $\widetilde{\mathcal{O}}(p^{(d-1)/2}\sqrt{n})$ regret while tensor epoch-greedy suffers $\widetilde{\mathcal{O}}(p^{(d+1)/3}n^{2/3})$ regret. Although the latter one has a sub-optimal dependency on the horizon due to the ε -greedy paradigm, it enjoys a better regret than the prior one in the high-dimensional regime $(n \leq p^{d-5})$.

In the theoretical analysis, a key step is to determine the switch time between the two phases, i.e., s_{2k} . We set the length of exploitation phase to be the inverse of tensor estimation error. Intuitively, when the tensor estimation error is large, more exploration can increase the sample size and improve the estimation. When the tensor estimation error is small, there is no need to perform more randomly exploration. Instead, we exploit more to reduce instant regrets. After obtaining the regret in epoch, we need to derive the upper bound of number of epochs. Similar to the optimal tuning procedure in explore-then-commit regret analysis, we tune the parameter to determine the final bound of total number of exploration steps.

4 Contextual Tensor Bandits

In this section, we consider an extension of tensor bandits to contextual tensor bandits where some modes of the reward tensor are contextual information. Take the online advertising data considered in Section 6 as an example. Users use the online platform on some day of the week, and the platform can only decide which advertisement to show to this given user at the given time. In this example, the user mode and the day-of-week mode of the reward tensor are both contextual information and both are not decided by the platform.

The above example can be formalized as contextual tensor bandits. Specifically, at time t, the agent observes a d_0 -dimensional context $(i_{1,t}, \dots, i_{d_0,t}) \in [p_1] \times \dots \times [p_{d_0}]$ and given the observed context, pulls an $(d-d_0)$ -dimensional arm $(i_{d_0+1,t}, \dots, i_{d,t}) \in [p_{d_0+1}] \times \dots \times [p_d]$. Let $I_t = (i_{1,t}, \dots, i_{d,t})$ collect the $context \times arm$ information at time step t. Correspondingly, the agent observes a noisy reward y_t drawn from a probability distribution associated with I_t . The objective is to maximize the cumulative reward over the time horizon. This contextual tensor bandit problem is different from the tensor bandit problems considered in Section 3, as the agent does not have the ability to choose the context. Therefore, the tensor elimination algorithm can not be applied to contextual tensor bandits. To tackle this problem, we introduce a heuristic solution to contextual tensor bandits that utilizes Thompson sampling (Russo et al., 2018) and ensemble sampling (Lu and Van Roy, 2017).

Thompson sampling is a powerful Bayesian algorithm that can be used to address a wide range of online decision problems. The algorithm, in its basic form, first initializes a prior distribution over model parameters, and then samples from its posterior distribution calculated using past observations. Finally, an action is made to maximize the reward given the sampled parameters. The posterior distribution can be derived in closed-form in a few special cases such as the Bernoulli bandit (Russo et al., 2018). With more complex models such as our low-rank tensor bandit problem, the exact calculation of the posterior distribution may become intractable. In this case, we consider an ensemble sampling approach that aims to approximate Thompson sampling while maintaining computational tractability. Specifically,

ensemble sampling aims to maintain, incrementally update, and sample from a finite ensemble of models; and this ensemble of models approximates the posterior distribution (Lu and Van Roy, 2017).

Consider the true reward tensor $\mathcal{X} \in \mathbb{R}^{p_1 \times ... \times p_d}$ that admits the decomposition in (5), where the first d_0 dimensions of \mathcal{X} correspond to the context and the last $d - d_0$ dimensions correspond to the decision (or arm). At time t and given the arm $\mathcal{A}_t = \mathbf{e}_{i_{1,t}} \circ \cdots \circ \mathbf{e}_{i_{d,t}}$, the reward y_t is assumed to follow $y_t = \langle \mathcal{X}, \mathcal{A}_t \rangle + \epsilon_t$. To ease the calculation of the posterior distribution, in contextual tensor bandits we consider $\epsilon_t \sim N(0, \sigma^2)$. For the prior distribution over model parameters, we assume the rows of \mathbf{U}_k are drawn independently from

$$[\mathbf{U}_k]_{i,\cdot} \sim N(\boldsymbol{\mu}_{k,i}, \sigma_k^2 \mathbf{I}), \quad i \in [p_k], \ k \in [d].$$

Let $\mathcal{H}_{t-1} = \{(\mathcal{A}_s, y_s)\}_{s=1}^{t-1}$ denote the history of action×reward up to time t. Given the prior distribution, the posterior density function can be calculated as

$$f(\mathcal{X}|y_1,\cdots,y_{t-1}) \propto f(y_1,\cdots,y_{t-1}|\mathcal{X}) \prod_{k,i} f([\mathbf{U}_k]_{i,\cdot}).$$

We maximize $f(\mathcal{X}|y_1,\dots,y_{t-1})$ to obtain the maximum the posteriori (MAP) estimate as

$$(\widehat{\mathcal{S}}^{(t)}, \widehat{\mathbf{U}}_1^{(t)}, \cdots, \widehat{\mathbf{U}}_d^{(t)}) = \underset{\mathcal{S}, \mathbf{U}_1, \cdots, \mathbf{U}_d}{\operatorname{argmin}} \left(\frac{1}{\sigma^2} \sum_{s=1}^{t-1} (y_s - \langle \mathcal{X}, \mathcal{A}_s \rangle)^2 + \sum_{k=1}^d \frac{1}{\sigma_k^2} \sum_{i=1}^{p_k} \left\| [\mathbf{U}_k]_{i,\cdot} - \boldsymbol{\mu}_{k,i} \right\|_2^2 \right). (17)$$

The objective function in (17) can be equivalently written as

$$\frac{1}{\sigma^2} \sum_{s=1}^{t-1} (y_s - \mathcal{S} \times_1 [\mathbf{U}_1]_{i_{1s},\cdot} \times \cdots \times_d [\mathbf{U}_d]_{i_{ds},\cdot})^2 + \sum_{k=1}^d \frac{1}{\sigma_k^2} \sum_{i=1}^{p_k} \left\| [\mathbf{U}_k]_{i,\cdot} - \boldsymbol{\mu}_{ki} \right\|_2^2,$$

which is a non-convex optimization problem. In our proposed algorithm, we alternatively optimize \mathbf{U}_k , $k \in [d]$ and \mathcal{S} . Given all \mathbf{U}_l such that $l \neq k$ and \mathcal{S} , we estimate the *i*-th row of \mathbf{U}_k as

$$[\mathbf{U}_{k}]_{i,\cdot}^{(t)} = \left[\frac{1}{\sigma^{2}} \sum_{s=1}^{t-1} \mathbf{1}_{(i_{ks}=i)} \mathbf{v}^{(t-1)} (\mathbf{v}^{(t-1)})^{\top} + \frac{1}{\sigma_{1}^{2}} \mathbf{I}\right]^{-1} \left\{\frac{1}{\sigma^{2}} \sum_{s=1}^{t-1} \mathbf{1}_{(i_{1,s}=i)} y_{s} \mathbf{v}^{(t-1)} + \frac{1}{\sigma^{2}} \boldsymbol{\mu}_{k,i}\right\},$$
where $\mathbf{v}^{(t-1)} = \left\{\mathcal{S}^{(t-1)} \times_{1} [\mathbf{U}_{1}]_{i_{1s},\cdot}^{(t-1)} \times \cdots \times_{k-1} [\mathbf{U}_{k-1}]_{i_{k-1,s},\cdot}^{(t-1)} \times_{k+1} [\mathbf{U}_{k+1}]_{i_{k+1,s},\cdot}^{(t-1)} \times \cdots \times_{d} [\mathbf{U}_{d}]_{i_{ds},\cdot}^{(t-1)}\right\}.$

After updating all rows of U_k for $k \in [d]$, we then estimate S by solving (17).

Algorithm 3 Tensor ensemble sampling

- 1: Input: rank $r_1, \ldots, r_d, \sigma^2, \{\mu_{ki}\}_{i \in [p_k], k \in [d]}, \{\sigma_k^2\}_{k \in [d]}, \text{ number of models } M, \text{ variance of } M$ perturbed noise $\tilde{\sigma}^2$.
- 2: # initialize M models from prior distributions
- 3: **Initialize** sample $[\widehat{\mathbf{U}}_{km}]_{i,\cdot}^{(0)} \sim N(\boldsymbol{\mu}_{ki}, \sigma_k^2 \mathbf{I})$ for $m \in [M], i \in [p_k], k \in [d]$. Normalize each column of matrix $\hat{\mathbf{U}}_{km}^{(0)}$. Initialize the core tensor $\mathcal{S}_m^{(0)} = \mathbf{1} \circ \cdots \circ \mathbf{1} \in \mathbb{R}^{r_1 \times_1 \cdots \times_d r_d}$.
- 4: **for** $t = 0, 1, 2 \cdots$ **do**
- 5: # exploitation phase
- Sample $\widetilde{m} \sim \text{Unif}\{1, \cdots, M\}$
- 7:
- Observe context $\boldsymbol{x}_t = (i_{1t}, \cdots, i_{d_0t})$ Update $(\widehat{\mathcal{S}}_{\widetilde{m}}^{(t)}, \widehat{\mathbf{U}}_{1\widetilde{m}}^{(t)}, \cdots, \widehat{\mathbf{U}}_{d\widetilde{m}}^{(t)})$ by solving (18). 8:
- Choose $\mathbf{a}_t = (i_{d_0+1,t}, \cdots, i_{dt}) = \operatorname{argmax}_{\mathbf{a} = (i_{d_0+1}, \cdots, i_d) \in [p_{d_0+1}] \times \cdots \times [p_d]} \widehat{R}_{\widetilde{m}}(\boldsymbol{x}_t, \mathbf{a}), \text{ where}$ 9:

$$\widehat{R}_{\widetilde{m}}(\boldsymbol{x}_{t}, \mathbf{a}) = \widehat{\mathcal{S}}_{\widetilde{m}}^{(t)} \times_{1} [\widehat{\mathbf{U}}_{1\widetilde{m}}]_{i_{1t}, \cdot}^{(t)} \times \cdots \times_{d} [\widehat{\mathbf{U}}_{d\widetilde{m}}^{(t)}]_{i_{dt}, \cdot}$$

- Receive reward y_t . 10:
- 11: # perturbation phase
- Sample perturbation noise $\omega_{tm} \sim N(0, \tilde{\sigma}^2)$ for $m \in [M]$.
- Obtain perturbed rewards $\widetilde{y}_{tm} = y_t + \omega_{tm}$ for $m \in [M]$. 13:
- 14: end for

Tensor ensemble sampling in Algorithm 3 consists of initialization, exploitation and perturbation phases. In the initialization phase, we sample M models from the prior distributions. The mean μ_{ki} and variance σ_k^2 in the prior distributions could be determined from prior knowledge or specified so that the range of models spans plausible outcomes. Then, at each time step t, a model \widetilde{m} is uniformly sampled from the ensemble of M models. After observing a context $x_t = (i_{1t}, \dots, i_{d_0t})$, the agent exploits the history data of model \widetilde{m} to estimate the low-rank component of the reward tensor via

$$\min_{\mathcal{S}, \mathbf{U}_{1}, \cdots, \mathbf{U}_{d}} \frac{1}{\sigma^{2}} \sum_{s=1}^{t-1} \left\{ \widetilde{y}_{sm} - \langle \mathcal{X}, \mathcal{A}_{s} \rangle \right\}^{2} + \sum_{k=1}^{d} \frac{1}{\sigma_{k}^{2}} \sum_{i=1}^{p_{k}} \left\| [\mathbf{U}_{k}]_{i, \cdot} - [\widehat{\mathbf{U}}_{k, m}]_{i, \cdot}^{(0)} \right\|_{2}^{2}.$$
(18)

Compared to (17), the objective in (18) uses perturbed rewards and perturbed priors, which helps to diversify the models and capture model uncertainty. The goal is for the ensemble to approximate the posterior distribution and the variance among models to diminish as the posterior concentrates. Based on the sampled model \widetilde{m} , we pull the optimal arm \mathbf{a}_t given the observed context x_t . At the end of each time step, we perturb observed rewards for all M models to diversify the ensemble. Our tensor ensemble sampling can be viewed as an extension of ensemble sampling (Lu and Van Roy, 2017) for contextual bandits problem.

It is worth to mentioning that although tensor ensemble sampling is motivated from contextual tensor bandits problem, it can also be used to solve tensor bandits without context. In this case, the context dimension $d_0 = 0$ and arm \mathcal{A}_t consists of all decisions to be made. As we will show in the experiments, the tensor ensemble sampling is an empirically very successful algorithm. We also would like to mention that the theoretical investigation of the tensor ensemble sampling is very challenging due to the nascent ensemble sampling framework (Lu and Van Roy, 2017) and the involved non-convex optimization. We leave its regret analysis for future work.

5 Simulations

We carry out some preliminary experiments to compare the numerical performance of tensor epoch-greedy, tensor elimination and tensor ensemble sampling with two competitive methods: vectorized UCB which unfolds the tensor into a long vector and then implements standard UCB (Auer, 2002) for multi-armed bandits, and matricized ESTR (Jun et al., 2019) which unfolds the tensor into a matrix along an arbitrary mode and implements ESTR for low-rank matrix bandits.

We first describe the way to generate an order-three true reward tensor (d=3) according to Tucker decomposition in (5). The tensor dimensions are set to be same, i.e., $p_1 = p_2 = p_3 = p$. The triplet of tensor Tucker rank is fixed to be $r_1 = r_2 = r_3 = r = 2$. Denote $\widetilde{U}_j \in \mathbb{R}^{p_j \times r_j}$ as i.i.d standard Gaussian matrices. Then we apply QR decomposition on \widetilde{U}_j , and assign the Q part as the singular vectors U_j . The core tensor $S \in \mathbb{R}^{r \times r \times r}$ is constructed as a diagonal tensor with $S_{iii} = wp^{1.5}$, for $1 \le i \le r$. Here, $wp^{1.5}$ indicates the signal strength (Zhang and Xia, 2018). The random noise ϵ_t is generated i.i.d from a standard Gaussian distribution.

All algorithms involve some hyper-parameters, such as initial exploration length, confidence interval width, the round of pure exploration and etc. We next discuss the choice of hyper-

parameters for tensor elimination, tensor epoch-greedy, tensor ensemble sampling, and matricized ESTR respectively. For a fair comparison, we do a grid search of some hyper-parameters and report the ones with lowest cumulative regret for each algorithm. Both matricized ESTR and tensor elimination are optimism-based algorithms that utilize confidence intervals. In practice, the theoretically suggested confidence interval, i.e. Theorem 1, may be conservative. So in the experiments, we set a base ξ according to its theoretical result and find the best multiplier c by grid search from $\{0.01, 0.05, 0.1, 0.5\}$. For both algorithms, the initial exploration length is set to be c_0n_1 where n_1 follows the theoretical value. We tune the unknown constant by varying $c_0 \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$. Besides, we set the ridge regularization parameter $\lambda_1 = 0.1$ for both algorithms. For tensor epoch-greedy, we set $C_0 = 1$ for the initialization phase length s_1 defined in (6) and tune $C_2 \in \{1, 5, 10, 20\}$ for the exploitation parameter s_{2k} defined in (15). Here we do not tune C_0 , the initial exploration length, for tensor epoch-greedy since we find that it is not sensitive to C_0 and a relatively small number is enough to ensure model performance. For tensor ensemble sampling, we set ensemble size M as a relatively large size M=100 and tune variance of perturbation noise as $\tilde{\sigma}^2 \in \{0, 0.01, 0.1, 1, 2, 5, 10\}.$

In Figure 2, we report the cumulative regrets of all five algorithms for four settings with $w \in \{0.5, 0.8\}$ and $p \in \{15, 20\}$. All the results are based on 30 replications. Figure 2 shows that tensor ensemble sampling outperforms all other methods in different settings. Tensor elimination behaves worse than tensor ensemble sampling but better than other methods for a long time horizon. It aligns with our theoretical findings that tensor elimination has a better overall regret bound for long time horizon, while tensor epoch-greedy is more competitive for small time horizon. When the tensor dimension p increases, the advantage of tensor epoch-greedy in early stage is more apparent. This result further backs up our theoretical finding. In our theory, the regret bound of tensor epoch-greedy has a lower dependency on dimension compared with other methods.

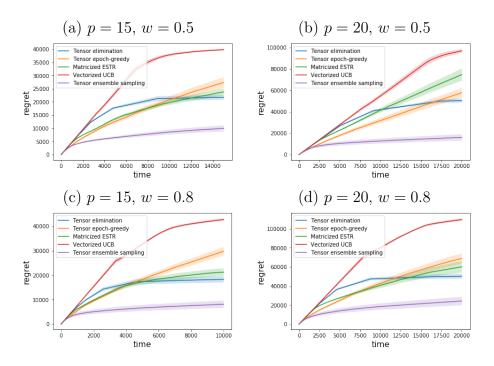


Figure 2: Cumulative regrets with varying dimension p and signal strength w. The shaded areas represent the confidence bands.

6 Application to Online Advertising

Our data set comes from a major internet company and contains the impressions for advertisements displayed on the company's webpages over four weeks in May to June, 2016. The impression is the number of times the advertisement has been displayed. It is a crucial measure to evaluate the effectiveness of an advertisement campaign, and plays an important role in digital advertising pricing. Studying online advertisement recommendation not only brings opportunities for advertisers to increase their ad exposures but also allows them to efficiently study individual-level behavior.

The impressions of 20 advertisements were recorded for 20 most active users. In order to understand the user behavior over different days of a week, the data were aggregated by days of a week. Thus, the data forms an order-three tensor of dimension $20 \times 7 \times 20$ where each entry in the tensor corresponds to the impression for the given combination of user, day of week and advertisement. The goal of this real application is to recommend advertisement to a selected user on a specific day to achieve maximum reward (impression). The user mode

and the day-of-week mode are both contextual information and the agent recommends the corresponding optimal advertisement. Tensor elimination and matricized ESTR can only handle the setting where the agent chooses arms without contextual information. Tensor epoch-greedy is for context-free tensor bandits in our theory but it can also be extended to tensor bandit with contextual information. Therefore, we compare the performance of tensor epoch-greedy, tensor ensemble sampling and vectorized UCB in this contextual tensor bandits problem. The cumulative regrets of all these algorithms are shown in Figure 3.

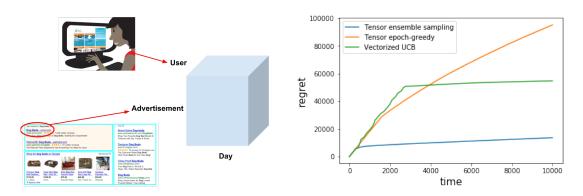


Figure 3: The left plot illustrates the reward tensor formulation in our online advertising data. The right plot shows cumulative regrets of tensor epoch-greedy, tensor ensemble sampling and vectorized UCB on this data.

From the right plot of Figure 3, we can observe that tensor ensemble sampling achieves the lowest regret for a long time horizon. Comparing tensor epoch-greedy and vectorized UCB, the former is better for a short time horizon. At the last time horizon, tensor ensemble sampling is 75% lower than that of vectorized UCB and is 85.6% lower than that of tensor epoch-greedy. The t-test of difference between the mean of final regret for tensor epoch-greedy and tensor ensemble sampling indicates that the two means are significantly different (t-statistic is 1191.37 and p-value is 0). The t-test between tensor ensemble sampling and vectorized UCB also shows significantly improvement is achieved by tensor ensemble sampling (t-statistic is 1770.33 and p-value is 0). The success of tensor ensemble sampling helps advertisers to better optimize the allocation of ad resources for different users on different days. By tracking users' behavior on ad exposures and conversions over time,

advertises can make personalized recommendation based on individual-level data. Besides, our models are maintained and updated based on users' feedback. Such interactive models can be applied to other dynamic and online learning real problems.

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Supplementary Materials

"Stochastic Low-rank Tensor Bandits for Multi-dimensional Online Decision Making"

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In the appendix, we provide detailed proofs of Theorems 1-2 in Section S.1, proof of the main lemma in Section S.2, the equivalent formulation of tensor bandits in Section S.3.1, and the algorithm for low-rank tensor completion in Section S.3.2.

S.1 Proofs of Main Theorems

S.1.1 Proof of Theorem 1

From Lemma S3 and the assumption $\|\mathcal{X}\|_{\infty} \leq 1$, we know that with probability at least $1 - p^{-10}$,

$$\|\widehat{\mathcal{X}}_{n_1} - \mathcal{X}\|_F \le C_1 \sqrt{\frac{p^{d+1}r\log(p)}{n_1}}.$$

By definitions, U_i , $\widehat{\mathbf{U}}_i$ are left singular vectors of $\mathcal{M}_i(\mathcal{X})$ and $\mathcal{M}_i(\widehat{\mathcal{X}}_{n_1})$, respectively. Here, the matricization operator $\mathcal{M}(\cdot)$ is defined in (1). Then we can verify

$$U_i U_i^{\mathsf{T}} \mathcal{M}_i(\mathcal{X}) = U_i U_i^{\mathsf{T}} U_i \Sigma V_i^{\mathsf{T}} = U_i^{\mathsf{T}} \Sigma V_i^{\mathsf{T}} = \mathcal{M}_i(\mathcal{X}).$$

Let $\widehat{\mathbf{U}}_{i\perp} \in \mathbb{R}^{p \times (p-r)}$ be the orthogonal complement of $\widehat{\mathbf{U}}_i$ for $i \in [d]$. For an orthogonal matrix U and an arbitary matrix X, Y, we have $\|UX\|_F \leq \|U\|_2 \|X\|_F = \|X\|_F$ and $\|XY\|_F \geq \|X\|\sigma_{\min}(Y)$. Suppose σ_i is the r-th singular value of $\mathcal{M}_i(\mathcal{X})$. Using the above fact, we have

$$\|\mathcal{M}_{i}(\widehat{\mathcal{X}}_{n_{1}}) - \mathcal{M}_{i}(\mathcal{X})\|_{F}$$

$$\geq \|\widehat{\mathbf{U}}_{i\perp}^{\top}(\mathcal{M}_{i}(\widehat{\mathcal{X}}_{n_{1}}) - U_{i}U_{i}^{\top}\mathcal{M}_{i}(\mathcal{X}))\|_{F}$$

$$= \|\widehat{\mathbf{U}}_{i\perp}^{\top}U_{i}U_{i}^{\top}\mathcal{M}_{i}(\mathcal{X})\|_{F}$$

$$\geq \|\widehat{\mathbf{U}}_{i\perp}^{\top}U_{i}\|_{F}\sigma_{r}(U_{i}^{\top}\mathcal{M}_{i}(\mathcal{X})) = \|\widehat{\mathbf{U}}_{i\perp}^{\top}U_{i}\|_{F}\sigma_{i}.$$

Therefore we have,

$$\|\widehat{\mathbf{U}}_{i\perp}^{\top} U_i\|_F \le \frac{\|\mathcal{M}_i(\mathcal{X}) - \mathcal{M}_i(\widehat{\mathcal{X}}_{n_1})\|_F}{\sigma_i} = \frac{\|\mathcal{X} - \widehat{\mathcal{X}}_{n_1}\|_F}{\sigma_i} \le \frac{C_1}{\sigma_i} \sqrt{\frac{p^{d+1} r \log(p)}{n_1}}, \quad (S1)$$

with probability at least $1 - p^{-\alpha}$. As discussed in Section 3.1, we reformulate original tensor bandits into a stochastic linear bandits with finitely many arms. Recall that $\beta = \text{vec}(\mathcal{Y})$ with

$$\mathcal{Y} = \mathcal{X} \times_1 [\widehat{\mathbf{U}}_1; \widehat{\mathbf{U}}_{1\perp}] \cdots \times_d [\widehat{\mathbf{U}}_d; \widehat{\mathbf{U}}_{d\perp}] \in \mathbb{R}^{p_1 \times \cdots \times p_d}$$

and the corresponding action set

$$\mathbb{A} := \left\{ \operatorname{vec} \left([\widehat{\mathbf{U}}_1; \widehat{\mathbf{U}}_{1\perp}]^\top \boldsymbol{e}_{\mathbf{i}_1} \circ \cdots \circ [\widehat{\mathbf{U}}_d; \widehat{\mathbf{U}}_{d\perp}]^\top \boldsymbol{e}_{\mathbf{i}_d} \right), \mathbf{i}_1 \in [p_1], \dots, \mathbf{i}_d \in [p_d] \right\}.$$

From Eq. (S1), we have

$$\|\boldsymbol{\beta}_{(q+1):p^{d}}\|_{2} \leq \prod_{i=1}^{d} \|\widehat{\mathbf{U}}_{i\perp}^{\top} U_{i}\|_{F} \|\mathcal{S}\|_{F}$$

$$\leq \frac{\|\widehat{X}_{n_{1}} - X\|_{F}^{d}}{\Pi_{i=1}^{d} \sigma_{i}} \|\mathcal{S}\|_{F}$$

$$\leq \frac{r^{d/2}}{\Pi_{i=1}^{d} \sigma_{i}} \frac{C_{1}^{d} r^{d/2} p^{\frac{d^{2}+d}{2}} \log^{d/2}(p)}{n_{1}^{d/2}}, \tag{S2}$$

with probability at least $1 - dp^{-\alpha}$. Thus it is equivalent to consider the following linear bandit problem:

$$y_t = \langle A_t, \boldsymbol{\beta} \rangle + \epsilon_t,$$

where $\|\boldsymbol{\beta}_{(q+1):p^d}\|_2$ satisfies Eq. (S2) and A_t is pulled from action set \mathbb{A} . To better utilize the information coming from low-rank tensor completion, we present the following regret bound for the elimination-based algorithm for stochastic linear bandits with finitely-many arms. The detailed proof is deferred to Section S.2.

Lemma S2. Consider the the elimination-based algorithm in Algorithm 1 with $\lambda_2 = n/(k \log(1 + n/\lambda_1))$ and $\lambda_1 > 0$. With the choice of $\xi = 2\sqrt{14\log(2/\delta)} + \sqrt{\lambda_1}||\boldsymbol{\beta}_{1:q}||_2 + \sqrt{\lambda_2}||\boldsymbol{\beta}_{(q+1):p^d}||_2$, the upper bound of cumulative regret of n rounds satisfies

$$R_n \leq 8 \left(2\sqrt{14 \log(2 \log(n) p^d / \delta)} + \sqrt{\lambda_1} \|\beta_{1:q}\|_2 \right) \sqrt{2qn \log(1 + \frac{n}{\lambda_1})} + 8\sqrt{2}n \|\beta_{(q+1):p^d}\|_2.$$
with probability at least $1 - \delta$, where $q = p^d - (p - r)^d$.

Overall, we can decompose the pseudo regret Eq. (4) into two parts:

$$R_n = R_{1n} + R_{2n} + R_{3n},$$

where R_{1n} quantifies the regret during initialization phase, R_{2n} quantifies the regret during exploration phase and R_{3n} quantifies the regret during commit phase (linear bandits reduction). Note that $q \leq C_1 p^{d-1}$ for sufficient large C_1 . Denote

$$\delta_{p,r} = \frac{r^d}{\prod_{i=1}^d \sigma_i} p^{\frac{d^2+d}{2}} \log^{d/2}(p),$$

such that $\|\boldsymbol{\beta}_{(q+1):p^d}\|_2 \leq \delta_{p,r}/n_1^{d/2}$ from Eq. (S2). Applying the result in Lemma S2 to bound R_{3n} and properly choosing $0 < \lambda_1 \leq 1/p^d$, we have the following holds with probability at least $1 - dp^{-10} - 1/n$,

$$R_n \le C\left(\underbrace{r^{d/2}p^{d/2}}_{R_{1n}} + \underbrace{n_1}_{R_{2n}} + \underbrace{\delta_{p,r}n_2/n_1 + \sqrt{\log(\log(n_2)) + \log(n_2p^d)}}_{R_{3n}}\sqrt{p^{d-1}n_2\log(n_2p^d)}\right),$$

where $n_2 = n - n_1 - Cr^{d/2}p^{d/2}$ and C > 0 is an universal constant. Here, R_{3n} is due to the fact that we run elimination-based algorithm for the rest n_2 rounds. For simplicity, we bound all n_2 by n as usually did for the proof of explore-then-commit type algorithm.

We optimize with respect to n_1 such that

$$n_1 = (n\delta_{p,r})^{\frac{2}{d+2}}$$

It implies the following bound holds with probability at least $1 - dp^{-10} - 1/n$,

$$R_{n} \leq C \left(r^{d/2} p^{d/2} + \left(\frac{r^{d}}{\prod_{i=1}^{d} \sigma_{i}} p^{\frac{d^{2}+d}{2}} \log^{d/2}(p) \right)^{\frac{2}{d+2}} n^{\frac{2}{d+2}}$$

$$+ \sqrt{\log(\log(n)) + \log(np^{d})} \sqrt{p^{d-1} n \log(np^{d})} \right)$$

$$\leq C \left(r^{d/2} p^{d/2} + \left(\frac{r^{d}}{\prod_{i=1}^{d} \sigma_{i}} \log^{d/2}(p) \right)^{\frac{2}{d+2}} p^{\frac{d^{2}+d}{d+2}} n^{\frac{2}{d+2}} + \sqrt{(d \log(p) + \log(n))^{2} p^{d-1} n} \right).$$

This ends the proof.

S.1.2 Proof of Theorem 2

The proof uses the trick that couples epoch-greedy algorithm with explore-then-commit algorithm with an optimal tuning.

Step 1. We decompose the pseudo regret defined in (4) as:

$$R_{n} = \sum_{t=1}^{n} \langle \mathcal{A}^{*} - \mathcal{A}_{t}, \mathcal{X} \rangle$$
$$= \sum_{t=1}^{s_{1}} \langle \mathcal{A}^{*} - \mathcal{A}_{t}, \mathcal{X} \rangle + \sum_{t=s_{1}+1}^{n} \langle \mathcal{A}^{*} - \mathcal{A}_{t}, \mathcal{X} \rangle,$$

where s_1 is the number of initialization steps. After initialization phase, from the definition of exploration time index set in (14), the algorithm actually proceeds in phases and each phase contains $(1+\lceil s_{2k}\rceil)$ steps: one step random exploration plus $\lceil s_{2k}\rceil$ steps greedy actions. By algorithm, at phase k, the greedy action \mathcal{A}_t is taken to maximize $\langle \mathcal{A}_t, \widehat{\mathcal{X}}_{k+s_1} \rangle$ where $\widehat{\mathcal{X}}_{k+s_1}$ is the low-rank tensor completion estimator at phase k based on $(k+s_1)$ random samples. Therefore, we have $\langle \mathcal{A}_t - \mathcal{A}^*, \widehat{\mathcal{X}}_{k+s_1} \rangle \geq 0$ and

$$\langle \mathcal{A}^* - \mathcal{A}_t, \mathcal{X} \rangle \leq \langle \mathcal{A}^* - \mathcal{A}_t, \mathcal{X} - \widehat{\mathcal{X}}_{k+s_1} \rangle.$$

By Lemma S3 and the choice of s_{2k} in (15), it is sufficient to guarantee

$$\|\widehat{\mathcal{X}}_{k+s_1} - \mathcal{X}\|_F \le 1/s_{2k},$$

holds with probability at least $1-p^{-\alpha}$ from Lemma S3 for any $\alpha > 1$. By the Cauchy-Schwarz inequality, we have

$$\langle \mathcal{A}^* - \mathcal{A}_t, \mathcal{X} \rangle \le \|\mathcal{A}^* - \mathcal{A}_t\|_F \|\widehat{\mathcal{X}}_{k+s_1} - \mathcal{X}\|_F \le 2/s_{2k},$$

where for the second inequality we use the fact that both tensors \mathcal{A}^* and \mathcal{A}_t have only one entry equal to 1 and others are 0. Denote $n_2 = n - s_1$ and $K^* = \min\{K : \sum_{k=1}^K (1 + \lceil s_{2k} \rceil) \ge n_2\}$. Since we assume $\|\mathcal{X}\|_{\infty} \le 1$, the maximum gap Δ_{\max} is bounded by 2. Then we have

$$R_n \le s_1 \Delta_{\max} + \sum_{k=1}^{K^*} \left(1 \cdot \Delta_{\max} + \lceil s_{2k} \rceil 2 / s_{2k} \right)$$

$$\le (s_1 + K^*) \Delta_{\max} + 2K^* \le 2s_1 + 4K^*,$$
(S3)

with probability at least $1 - K^*p^{-\alpha}$.

Step 2. We will derive an upper bound for K^* . Let $n_2^* = \operatorname{argmin}_{u \in [0, n_2]} [u + (n_2 - u)/s_{2u}]$. Consider the following two cases.

1. If $n_2^* \ge K^*$, it is obvious that

$$K^* \le n_2^* + (n_2 - n_2^*)/s_{2n_2^*}$$

2. If $n_2^* \leq K^* - 1$, it holds that

$$\sum_{k=1}^{K^*-1} s_{2k} \ge \sum_{k=n_2^*}^{K^*-1} s_{2k} \ge (K^* - n_2^*) s_{2n_2^*},$$

where the second inequality is from the fact that s_{2k} is monotone increasing. By the definition of K^* , it holds that

$$n_2 - 1 \ge \sum_{k=1}^{K^* - 1} (1 + \lceil s_{2k} \rceil) \ge \sum_{k=1}^{K^* - 1} (1 + s_{2k}) \ge K^* - 1 + (K^* - n_2^*) s_{2n_2^*},$$

which implies

$$K^* \le n_2^* + (n_2 - n_2^*)/s_{2n_2^*}$$

Overall, K^* is upper bounded by $n_2^* + (n_2 - n_2^*)/s_{2n_2^*}$.

Step 3. From (S3), the cumulative regret can be bounded by

$$R_n \le 2s_1 + 4 \min_{u \in [0, n_2]} \left(u + (n_2 - u)/s_{2u} \right).$$

The second term above is essentially the regret for explore-then-comment type algorithm with the optimal tuning for the length of exploration. Plugging the definition of s_{2u} in (15) and letting $u = n/s_{2u}$, we have

$$K^*/2 \le n_2^* \le n^{2/3} p^{\frac{d+1}{3}} (r \log p)^{\frac{1}{3}}.$$

Thus, we choose $\alpha = \log(2n^{2/3}p^{\frac{d+1}{3}}(r\log p)^{\frac{1}{3}}p)$ such that $K^*p^{-\alpha} \leq 1/p$. Plugging in $s_1 = C_0r^{d/2}p^{d/2}$, we have

$$R_n \le C_0 r^{d/2} p^{d/2} + 8 \left(n^{2/3} p^{\frac{d+1}{3}} (r \log p)^{\frac{1}{3}} \right),$$

with probability at least 1 - 1/p. This ends the proof.

S.2 Proof of Lemma S2

Before we prove it, we introduce some notations first. For a vector x and matrix V, we define $||x||_V = \sqrt{x^\top V x}$ as the weighted ℓ_2 -norm and $\det(V)$ as its determinant. Let $K = \lfloor \log_2(n) \rfloor$ and $t_k = 2^{k-1}$. Denote $x^* = \operatorname{argmax}_{a \in \mathbb{A}} \langle a, \beta \rangle$.

We have the following regret decomposition by phases:

$$R_{n} = \sum_{t=1}^{n} \langle x^{*} - A_{t}, \boldsymbol{\beta} \rangle = \sum_{k=0}^{K} \sum_{t=t_{k}}^{t_{k+1}-1} \langle x^{*} - A_{t}, \boldsymbol{\beta} \rangle$$
$$= \sum_{k=0}^{K} \sum_{t=t_{k}}^{t_{k+1}-1} \left(\langle x^{*} - A_{t}, \widehat{\boldsymbol{\beta}}_{k} \rangle - \langle x^{*} - A_{t}, \widehat{\boldsymbol{\beta}}_{k} - \boldsymbol{\beta} \rangle \right),$$

where $\widehat{\beta}_k$ is the ridge estimator only based on the sample collected in the current phase, defined in Eq. (8). According to Lemma 7 in (Valko et al., 2014), for any fixed $x \in \mathbb{R}^p$ and any $\delta > 0$, we have, at phase k,

$$\mathbb{P}\left(|x^{\top}(\widehat{\boldsymbol{\beta}}_k - \boldsymbol{\beta})| \le ||x||_{V_k^{-1}} \xi\right) \ge 1 - \delta,\tag{S4}$$

where $\xi = 2\sqrt{14\log(2/\delta)} + \sqrt{\lambda_1}\|\boldsymbol{\beta}_{1:q}\|_2 + \sqrt{\lambda_2}\|\boldsymbol{\beta}_{(q+1):p^d}\|_2$. Applying Eq. (S4) for x^* and A_t , we have with probability at least $1 - Kp^d\delta$,

$$R_n \le \sum_{k=0}^K \sum_{t=t_k}^{t_{k+1}-1} \langle x^* - A_t, \widehat{\beta}_k \rangle + \sum_{k=0}^K (t_{k+1} - t_k) \Big(\|x^*\|_{V_k^{-1}} + \|A_t\|_{V_k^{-1}} \Big) \xi.$$

By step (7) in Algorithm 1, we have

$$\langle x^* - A_t, \widehat{\beta}_k \rangle \le \left(\|x^*\|_{V_k^{-1}} + \|A_t\|_{V_k^{-1}} \right) \xi.$$

According to Lemma 8 in Valko et al. (2014), for all the actions $x \in \mathbb{A}_k$ defined in Eq. (7),

$$||x||_{V_k^{-1}}^2 \le \frac{1}{t_k - t_{k-1}} \sum_{t=t_{k-1}+1}^{t_k} ||x_t||_{V_{t-1}^{-1}}^2.$$

Then using the elliptical potential lemma (Lemma 19.4 in Lattimore and Szepesvári (2020)), with probability at least $1 - Kp^d\delta$, we have

$$R_n \le 2 \sum_{k=0}^{K} (t_{k+1} - t_k) \left(\|x^*\|_{V_k^{-1}} + \|A_t\|_{V_k^{-1}} \right) \xi$$

$$\le 4 \sum_{k=0}^{K} (t_{k+1} - t_k) \sqrt{\frac{1}{t_k - t_{k-1}} \log \left(\frac{\det(V_k)}{\det(\Lambda)} \right)} \xi,$$

where $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_1, \lambda_2, \dots, \lambda_2)$. According to Lemma 5 in (Valko et al., 2014), we have

$$\log\left(\frac{\det(V_k)}{\det(\Lambda)}\right) \le k\log(1+\frac{n}{\lambda_1}) + \sum_{i=k+1}^{p^d}\log(1+\frac{t_i}{\lambda_2}),$$

where $\sum_{i=k+1}^{p^d} t_i \leq T$. With the choice of λ_2 ,

$$\log\left(\frac{\det(V_k)}{\det(\Lambda)}\right) \le k\log(1+\frac{n}{\lambda_1}) + \sum_{i=k+1}^{p^d} \frac{t_i}{\lambda_2} \le 2k\log(1+\frac{n}{\lambda_1}).$$

We know that $t_{k+1} - t_k = 2^{k-1}$ and $t_k - t_{k-1} = 2^{k-2}$. Then one can have

$$\sum_{k=0}^{K} (t_{k+1} - t_k) \frac{1}{\sqrt{t_k - t_{k-1}}} = \sum_{k=0}^{K} 2^{k/2} \le \sqrt{n}.$$

Overall, with probability at least $1 - Kp^d\delta$, we have

$$\begin{split} R_n &\leq 8\sqrt{2kn\log(1+\frac{n}{\lambda_1})} \Big(2\sqrt{14\log(2/\delta)} + \sqrt{\lambda_1} \|\boldsymbol{\beta}_{1:k}\|_2 + \sqrt{\lambda_2} \|\boldsymbol{\beta}_{(k+1):p^d}\|_2 \Big) \\ &= 8(2\sqrt{14\log(2\log(n)p^d/\delta)} + \sqrt{\lambda_1} \|\boldsymbol{\beta}_{1:k}\|_2) \sqrt{2kn\log(1+\frac{n}{\lambda_1})} + 8\sqrt{2}n \|\boldsymbol{\beta}_{(k+1):p^d}\|_2. \end{split}$$

This ends the proof.

S.3 Auxiliary Results

S.3.1 An equivalent formulation of tensor bandits

We write $\hat{\mathbf{U}}_{1\perp}, \dots, \hat{\mathbf{U}}_{d\perp}$ as the orthogonal basis of the complement subspaces of $\hat{\mathbf{U}}_1, \dots \hat{\mathbf{U}}_{d}$. By definitions, $[\hat{\mathbf{U}}_j \hat{\mathbf{U}}_{j\perp}]$ is an orthogonal matrix for all $j \in [d]$ such that

$$[\widehat{\mathbf{U}}_{j}\widehat{\mathbf{U}}_{j\perp}][\widehat{\mathbf{U}}_{j}\widehat{\mathbf{U}}_{j\perp}]^{\top} = [\widehat{\mathbf{U}}_{j}\widehat{\mathbf{U}}_{j\perp}]^{\top}[\widehat{\mathbf{U}}_{j}\widehat{\mathbf{U}}_{j\perp}] = \mathbb{I}_{d\times d}.$$

Denote a rotated true reward tensor as

$$\mathcal{Y} = \mathcal{X} \times_1 [\widehat{\mathbf{U}}_1; \widehat{\mathbf{U}}_{1\perp}] \cdots \times_d [\widehat{\mathbf{U}}_d; \widehat{\mathbf{U}}_{d\perp}] \in \mathbb{R}^{p_1 \times \cdots \times p_d},$$

where \times_1 is the marginal multiplication defined in Eq. (2). Denote

$$\mathcal{E}_1 = [\widehat{\mathbf{U}}_1; \widehat{\mathbf{U}}_{1\perp}]^ op oldsymbol{e}_{i_{1t}} \circ \cdots \circ [\widehat{\mathbf{U}}_d; \widehat{\mathbf{U}}_{d\perp}]^ op oldsymbol{e}_{i_{dt}}, \mathcal{E}_2 = oldsymbol{e}_{i_{1t}} \circ \cdots \circ oldsymbol{e}_{i_{dt}}.$$

We want to prove

$$\langle \mathcal{Y}, \mathcal{E}_1 \rangle = \langle \mathcal{X}, \mathcal{E}_2 \rangle.$$

To see this, we use a fact of the Kronecker product (see details in Section 2.6 in (Kolda and Bader, 2009)). Let $\mathcal{Z}_1 \in \mathbb{R}^{I_1 \times \cdots I_N}$ and $A^{(n)} \in \mathbb{R}^{J_n \times I_n}$ for all $n \in [N]$. Then, for any $n \in [N]$, we have

$$\mathcal{Z}_2 = \mathcal{Z}_1 \times_1 A^{(1)} \cdots \times_N A^{(N)}$$

$$\Leftrightarrow \mathcal{M}_n(\mathcal{Z}_2) = A^{(n)} \mathcal{M}_n(\mathcal{Z}_1) \Big(A^{(N)} \otimes \ldots \otimes A^{(n+1)} \otimes A^{(n-1)} \otimes \cdots \otimes A^{(1)} \Big)^\top,$$

where $\mathcal{M}_n(\mathcal{Z})$ is the mode-*n* matricization and \otimes is a Kronecker product. Denote $H = [\widehat{\mathbf{U}}_2\widehat{\mathbf{U}}_{2\perp}] \otimes \cdots \otimes [\widehat{\mathbf{U}}_d; \widehat{\mathbf{U}}_{d\perp}]$. By a matricization of \mathcal{Y}, \mathcal{E} along the first mode, we have

$$\begin{split} \left\langle \mathcal{Y}, \mathcal{E}_{1} \right\rangle &= \left\langle \mathcal{M}_{1}(\mathcal{Y}), \mathcal{M}_{1}(\mathcal{E}_{1}) \right\rangle \\ &= \left\langle [\widehat{\mathbf{U}}_{1}; \widehat{\mathbf{U}}_{1\perp}] \mathcal{M}_{1}(\mathcal{X}) H^{\top}, [\widehat{\mathbf{U}}_{1}; \widehat{\mathbf{U}}_{1\perp}] \mathcal{M}_{1}(\mathcal{E}_{2}) H^{\top} \right\rangle \\ &= \operatorname{trace} \left(H \mathcal{M}_{1}(\mathcal{X})^{\top} [\widehat{\mathbf{U}}_{1}; \widehat{\mathbf{U}}_{1\perp}]^{\top} [\widehat{\mathbf{U}}_{1}; \widehat{\mathbf{U}}_{1\perp}] \mathcal{M}_{1}(\mathcal{E}_{2}) H^{\top} \right) \\ &= \operatorname{trace} \left(H \mathcal{M}_{1}(\mathcal{X})^{\top} \mathcal{M}_{1}(\mathcal{E}_{2}) H^{\top} \right) \\ &= \left\langle \mathcal{X} \times_{1} \mathbb{I}_{d \times d} \times_{2} [\widehat{\mathbf{U}}_{2} \widehat{\mathbf{U}}_{2\perp}] \cdots \times_{d} [\widehat{\mathbf{U}}_{d}; \widehat{\mathbf{U}}_{d\perp}], \boldsymbol{e}_{\mathbf{i}_{1t}} \circ \cdots \circ [\widehat{\mathbf{U}}_{d}; \widehat{\mathbf{U}}_{d\perp}]^{\top} \boldsymbol{e}_{\mathbf{i}_{dt}} \right\rangle. \end{split}$$

Recursively using the above arguments along each mode, we reach our conclusion.

S.3.2 Tensor completion algorithm and guarantee

For the sake of completeness, we state the tensor completion algorithm in (Xia et al., 2021). The goal is to estimate the true tensor $\mathcal{X} \in \mathbb{R}^{p_1 \times ... \times p_d}$ from

$$y_t = \langle \mathcal{X}, \mathcal{A}_t \rangle + \epsilon_t, t = 1, \dots, T,$$

where $A_t = e_{\mathbf{i}_{1t}} \circ \dots \circ e_{\mathbf{i}_{dt}}$. This is a standard tensor completion formulation with uniformly random missing data. The algorithm consists of two stages: spectral initialization and power iteration.

Spectral initialization. We first construct an unbiased estimator \mathcal{X}_{ini} for \mathcal{X} as follows:

$$\mathcal{X}_{\text{ini}} = \frac{p_1 \cdots p_d}{T} \sum_{t=1}^n y_t \mathcal{A}_t.$$

For each $j \in [d]$, we construct the following *U*-statistic:

$$\widehat{R}_j = \frac{(p_1 \cdots p_d)^2}{T(T-1)} \sum_{1 < t \neq t' < T} y_t y_{t'} \mathcal{M}_j(\mathcal{A}_t) \mathcal{M}_j(\mathcal{A}_t')^\top,$$

where \mathcal{M}_j is the mode-j matricization defined in Eq. (1). Compute the eigenvectors of $\{\widehat{R}_j\}_{j=1}^d$ with eigenvalues greater than δ , where δ is a tuning parameter, and denote them by $\{\widehat{\mathbf{U}}_j^{(0)}\}_{j=1}^d$.

Power iteration. Given $\{\widehat{\mathbf{U}}_{j}^{(l-1)}\}_{j=1}^{d}$, \mathcal{X}_{ini} can be denoised via projections to j-th mode. For $l=1,2,\ldots$, we alternatively update $\{\widehat{\mathbf{U}}_{j}^{(l-1)}\}_{j=1}^{d}$ as follows,

$$\widehat{\mathbf{U}}_{j}^{(l)} = \text{ first } r_{j} \text{ left singular vectors of } \mathcal{M}_{j} \Big(\mathcal{X}_{\text{ini}} \times_{j' < j} (\widehat{\mathbf{U}}_{j'}^{(l-1)})^{\top} \times_{j' > j} (\widehat{\mathbf{U}}_{j'}^{(l-1)})^{\top} \Big).$$

The iteration is stopped when either the increment is no more than the tolerance ε , i.e.,

$$\left\| \mathcal{X}_{\text{ini}} \times_{1} (\widehat{\mathbf{U}}_{1}^{(l)})^{\top} \cdots \times_{d} (\widehat{\mathbf{U}}_{d}^{(l)})^{\top} \right\|_{F} - \left\| \mathcal{X}_{\text{ini}} \times_{1} (\widehat{\mathbf{U}}_{1}^{(l-1)})^{\top} \cdots \times_{d} (\widehat{\mathbf{U}}_{d}^{(l-1)})^{\top} \right\|_{F} \le \varepsilon, \tag{S5}$$

or the maximum number of iterations is reached. With the final estimates $\hat{\mathbf{U}}_1, \dots, \hat{\mathbf{U}}_d$, it is natural to estimate \mathcal{S} and \mathcal{X} as

$$\widehat{\mathcal{S}} = \mathcal{X}_{\text{ini}} \times_1 \widehat{\mathbf{U}}_1^\top \cdots \times_d \widehat{\mathbf{U}}_d^\top, \widehat{\mathcal{X}} = \widehat{\mathcal{S}} \times_1 \widehat{\mathbf{U}}_1 \cdots \times_d \widehat{\mathbf{U}}_d.$$

Lemma S3. Suppose Assumption 1 holds. Suppose $\widehat{\mathcal{X}}_T$ is the low-rank tensor estimator constructed from T uniformly random samples by Algorithm 1 in Xia et al. (2021). Then for any $\alpha > 1$, if the number of samples $T \geq C_0 \alpha^3 r^{(d-2)/2} p^{d/2}$ for sufficiently large constant C_0 , the following holds with probability at least $1 - p^{-\alpha}$,

$$\frac{\|\widehat{\mathcal{X}}_T - \mathcal{X}\|_F}{\|\mathcal{X}\|_F} \le C_1 \sqrt{\frac{\alpha r p \log p}{T}},\tag{S6}$$

where C_1 is an absolute constant.

Lemma S3 is a direct application of Corollary 2 in Xia et al. (2021) with some constant terms ignored.