Unit 2

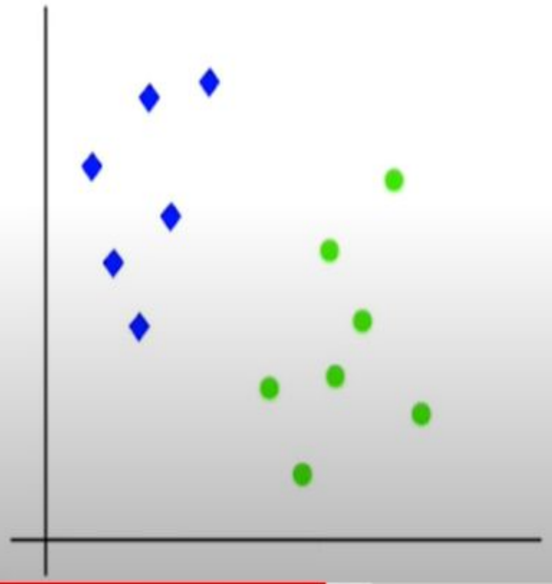
SVM,Types, Problems,Kernel functions

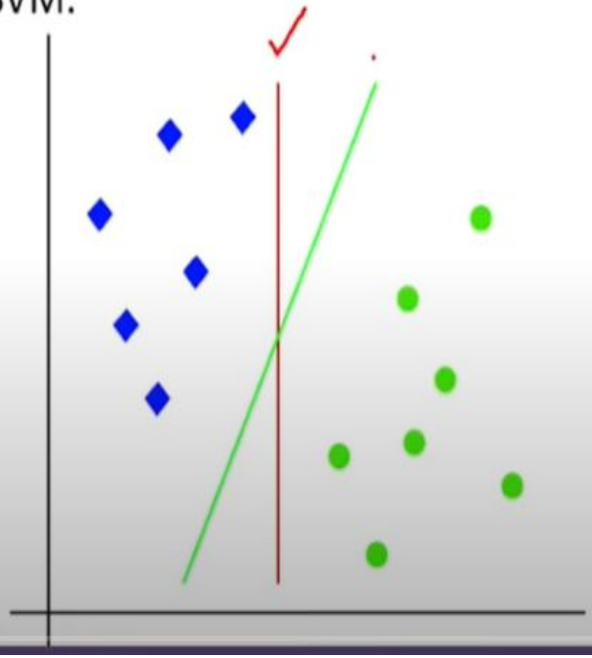
Support Vector Machine – Introduction

- Support Vector Machine or SVM is one of the most popular Supervised Learning algorithms, which is used for Classification as well as Regression problems.
- However, primarily, it is used for Classification problems in Machine Learning.
- The goal of the SVM algorithm is to create the best line or decision boundary that can segregate n-dimensional space into classes so that we can easily put the new data point in the correct category in the future.
- This best decision boundary is called a hyperplane.

Support Vector Machine – Hyperplane

- There can be multiple lines/decision boundaries to segregate the classes in n-dimensional space, but we need to find out the best decision boundary that helps to classify the data points.
- This best boundary is known as the hyperplane of SVM.





Support Vector Machine – Hyperplane

- The dimensions of the hyperplane depend on the features present in the dataset, which means if there are 2 features (as shown in image), then hyperplane will be a straight line.
- And if there are 3 features, then hyperplane will be a 2-dimension plane.
- We always create a hyperplane that has a maximum margin, which means the maximum distance between the data points.

Support Vector Machine – Types of SVM

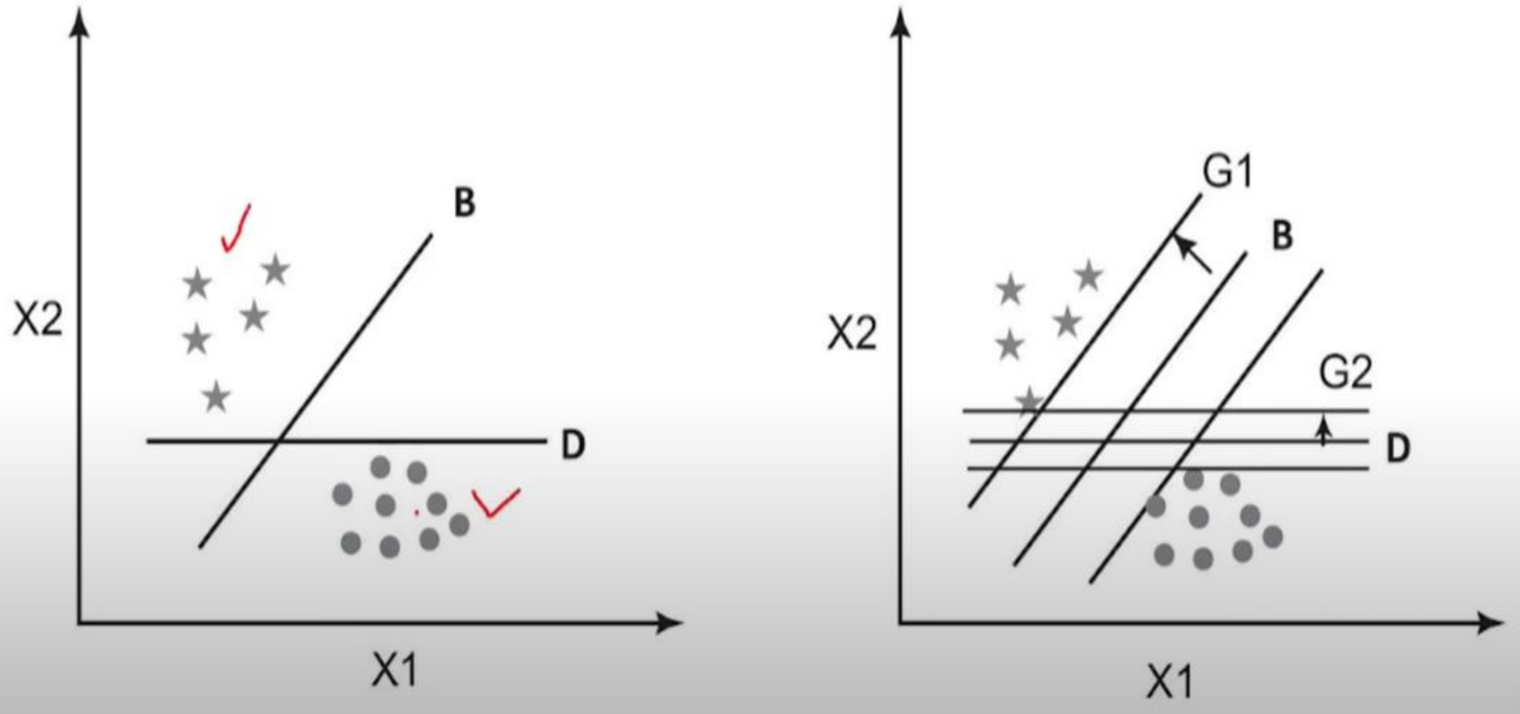
SVM can be of two types:

1. **Linear SVM:** Linear SVM is used for linearly separable data, which means if a dataset can be classified into two classes by using a single straight line, then such data is termed as linearly separable data, and classifier is used called as Linear SVM classifier.

2. **Non-linear SVM:** Non-Linear SVM is used for non-linearly separated data, which means if a dataset cannot be classified by using a straight line, then such data is termed as non-linear data and classifier used is called as Non-linear SVM classifier.

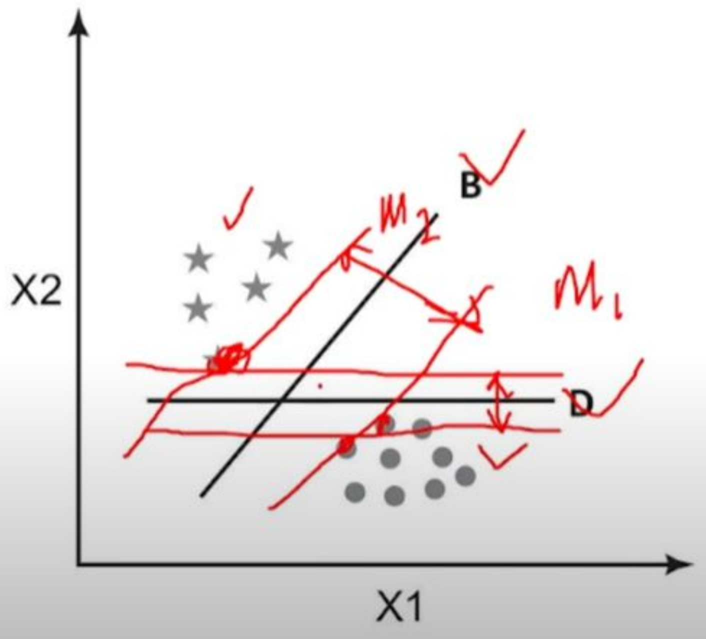
Support Vector Machine – Types of SVM

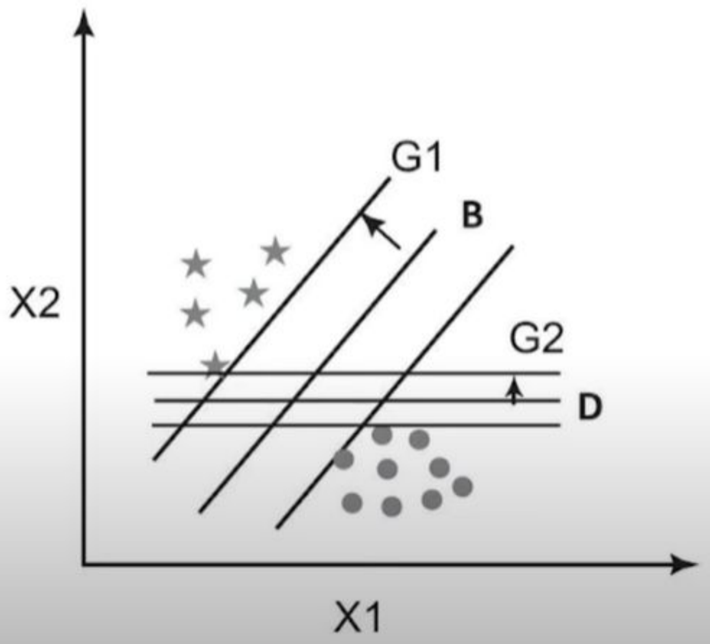
- **Linear SVM:**



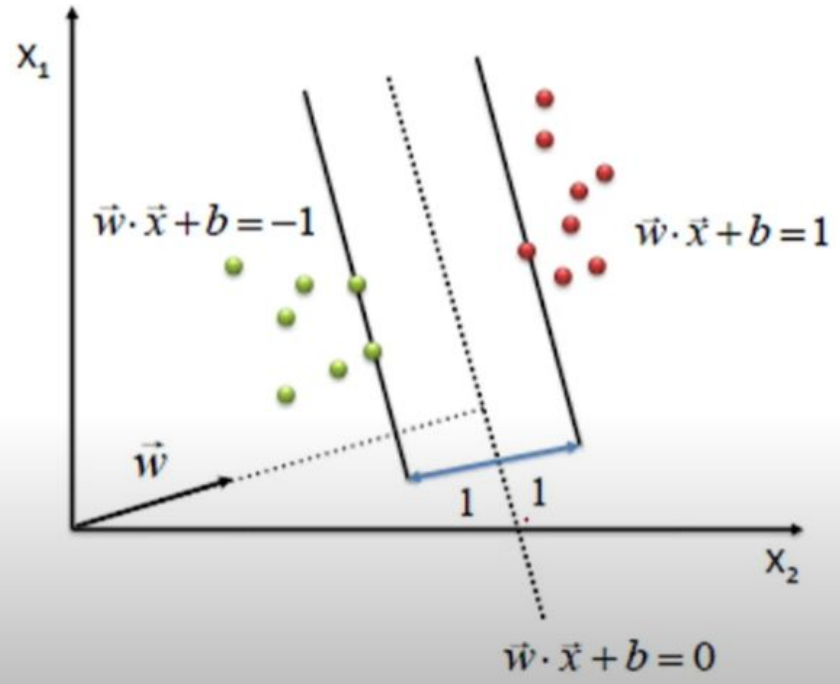
Support Vector Machine – Types of SVM

- **Linear SVM:**





Support Vector Machine – Types of SVM



$$\boxed{\max \frac{2}{\|w\|}}$$

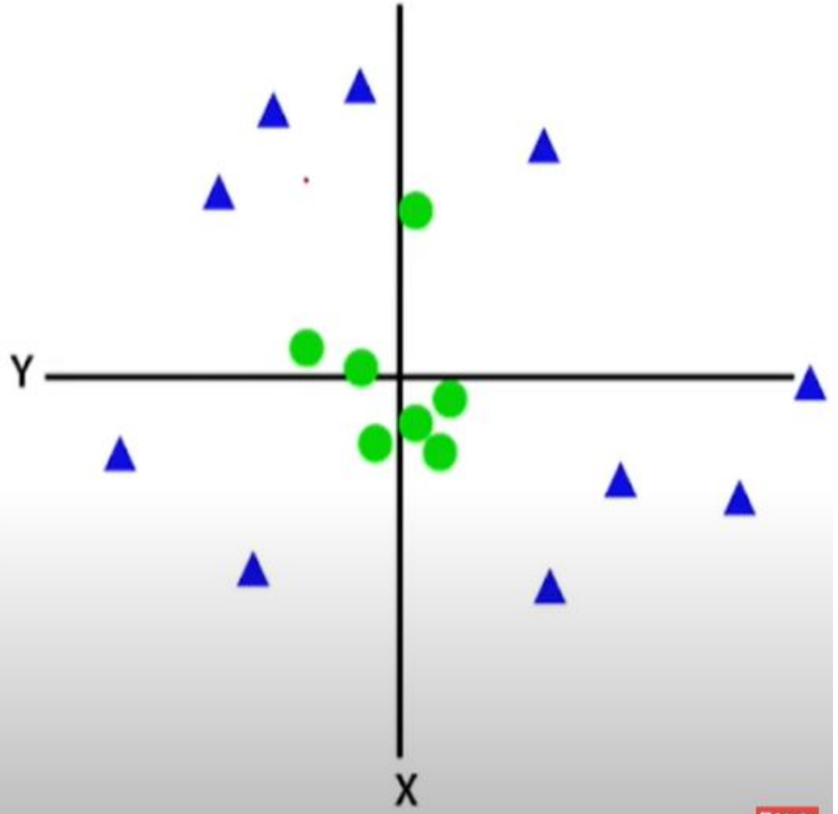
s.t.

$(w\cdot x+b)\geq 1, \forall x$ of class 1

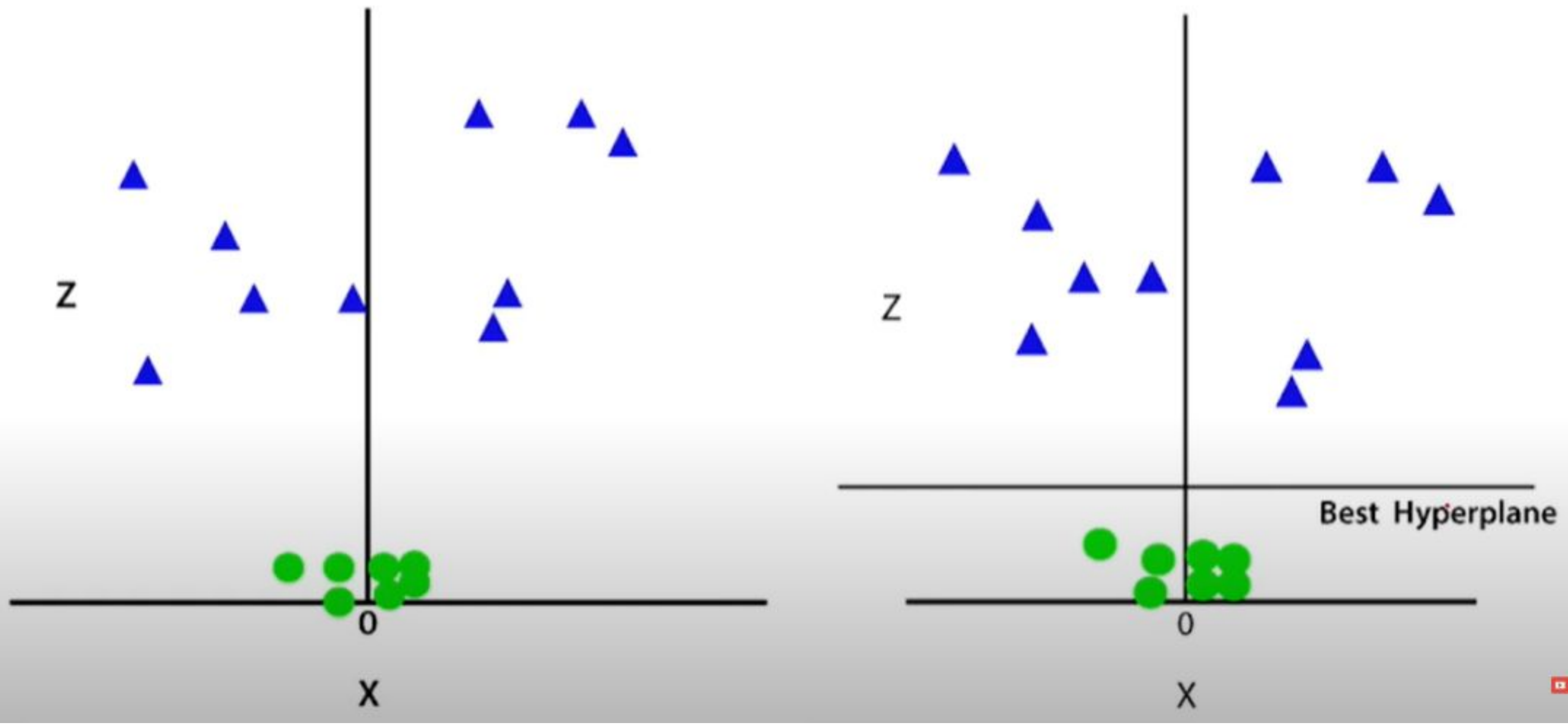
$(w\cdot x+b)\leq -1, \forall x$ of class 2

Support Vector Machine – Types of SVM

- **Non-Linear SVM:**
- If data is linearly arranged, then we can separate it by using a straight line, but for non-linear data, we cannot draw a single straight line.
- So to separate these data points, we need to add one more dimension.
- For linear data, we have used two dimensions x and y, so for non-linear data, we will add a third dimension z.
- It can be calculated as: $z=x^2+y^2$



Support Vector Machine – Types of SVM



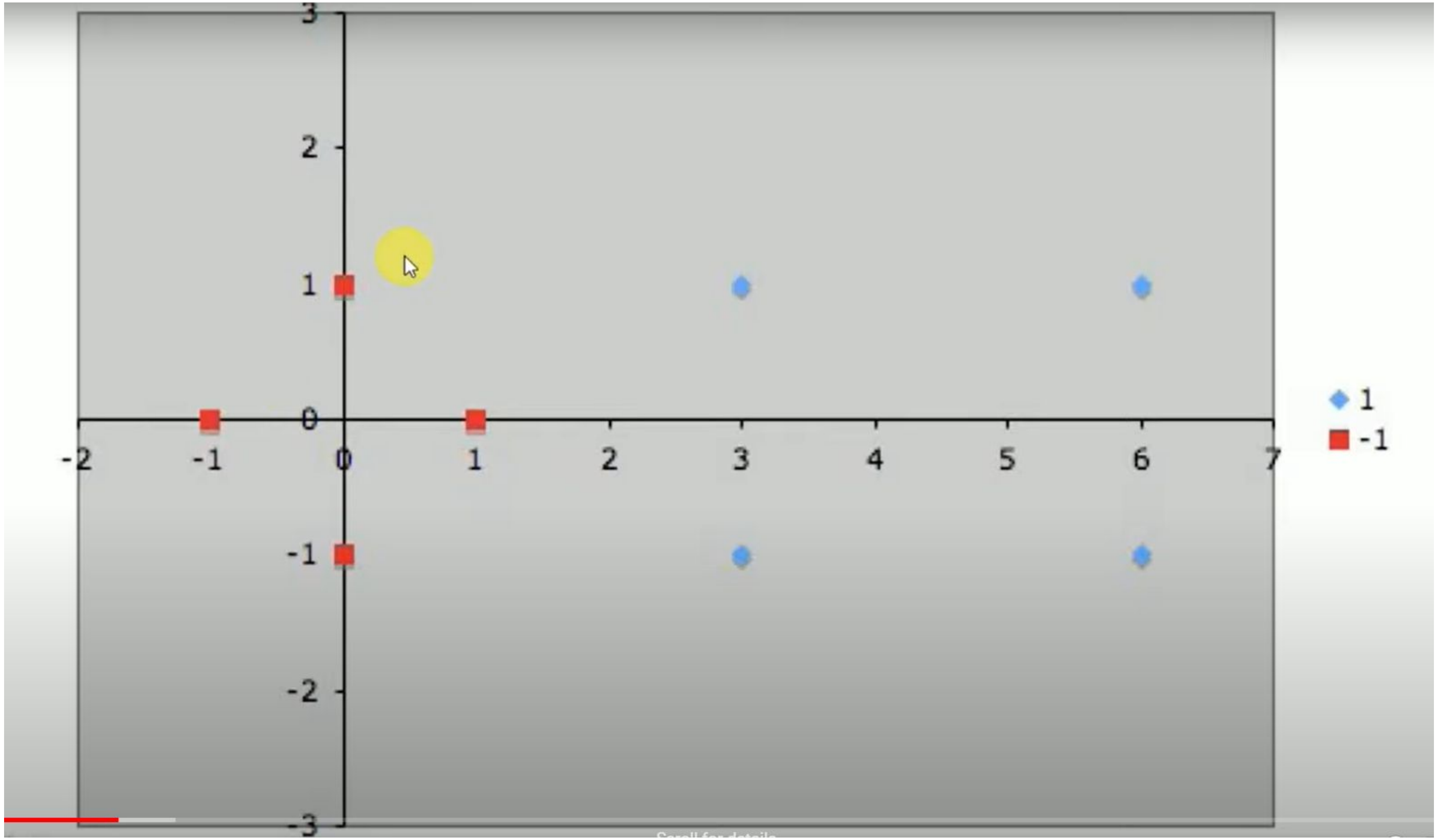
Example on Linear SVM

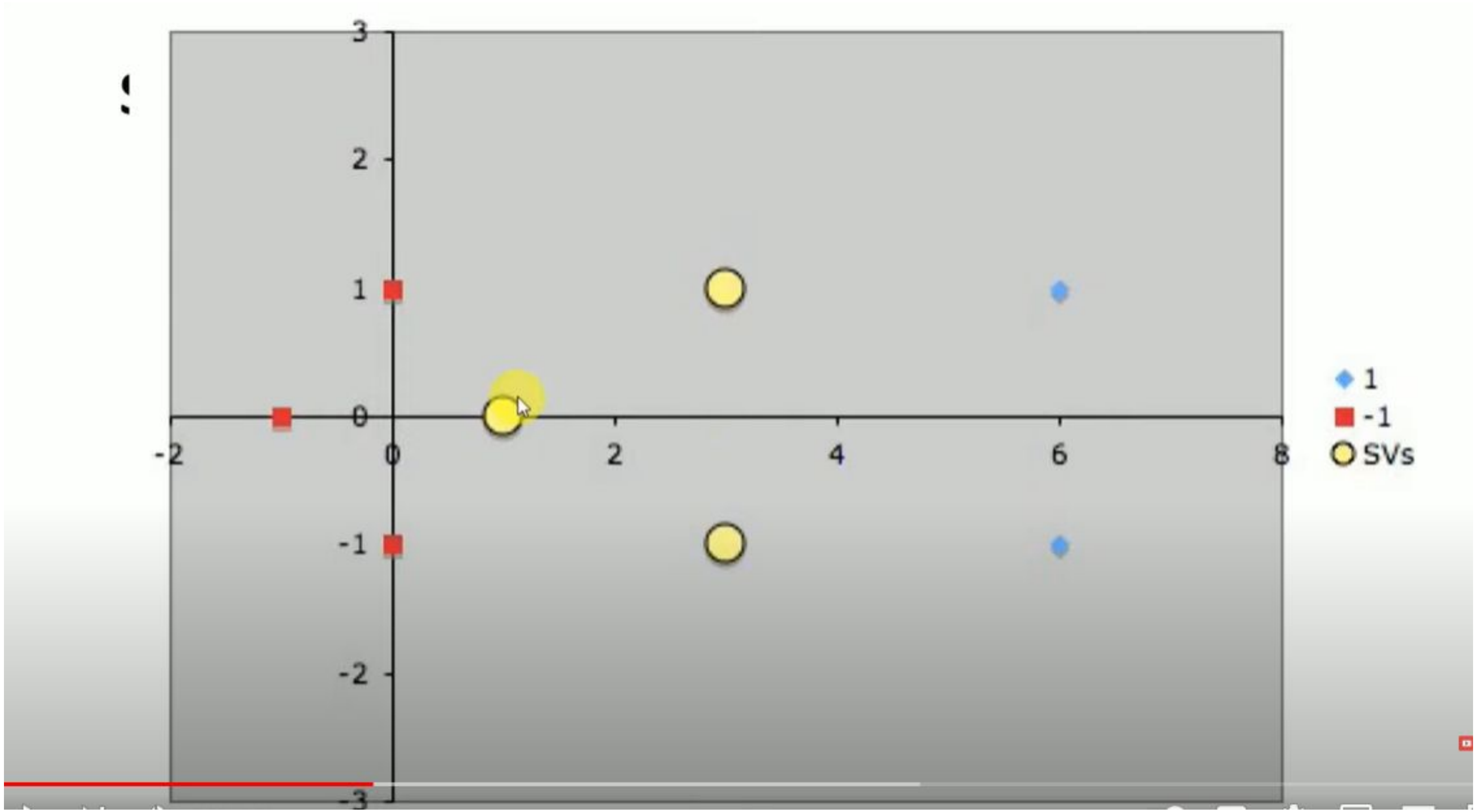
Suppose we are given the following positively labeled data points,

$$\left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 6 \\ 1 \end{pmatrix}, \begin{pmatrix} 6 \\ -1 \end{pmatrix} \right\}$$

and the following negatively labeled data points,

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\}$$


3
2
1
0
-1
-2
-3
-2
-1
0
1
2
3
4
5
6
7
1
-1



3
2
1
0
-1
-2
-3
-2
0
2
4
6
8
1
-1
SVs

Support Vector Machine - Linear Example Solved

- Each vector is augmented with a 1 as a bias input

- So, $s_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then $\widetilde{s_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$
- Similarly,

- $s_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, then $\widetilde{s_2} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$ and $s_3 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, then $\widetilde{s_3} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$

Support Vector Machine - Linear Example Solved

$$\begin{aligned}
\alpha_1 \tilde{s_1} \cdot \tilde{s_1} + \alpha_2 \tilde{s_2} \cdot \tilde{s_1} + \alpha_3 \tilde{s_3} \cdot \tilde{s_1} &= -1 \\
\alpha_1 \tilde{s_1} \cdot \tilde{s_2} + \alpha_2 \tilde{s_2} \cdot \tilde{s_2} + \alpha_3 \tilde{s_3} \cdot \tilde{s_2} &= +1 \\
\alpha_1 \tilde{s_1} \cdot \tilde{s_3} + \alpha_2 \tilde{s_2} \cdot \tilde{s_3} + \alpha_3 \tilde{s_3} \cdot \tilde{s_3} &= +1
\end{aligned}$$

$$\alpha_1 \begin{pmatrix}1\\0\\1\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix} + \alpha_2 \begin{pmatrix}3\\1\\1\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix} + \alpha_3 \begin{pmatrix}3\\-1\\1\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix} = -1$$

$$\alpha_1 \begin{pmatrix}1\\0\\1\end{pmatrix}\begin{pmatrix}3\\1\\1\end{pmatrix} + \alpha_2 \begin{pmatrix}3\\1\\1\end{pmatrix}\begin{pmatrix}3\\1\\1\end{pmatrix} + \alpha_3 \begin{pmatrix}3\\-1\\1\end{pmatrix}\begin{pmatrix}3\\1\\1\end{pmatrix} = 1$$

$$\alpha_1 \begin{pmatrix}1\\0\\1\end{pmatrix}\begin{pmatrix}3\\-1\\1\end{pmatrix} + \alpha_2 \begin{pmatrix}3\\1\\1\end{pmatrix}\begin{pmatrix}3\\-1\\1\end{pmatrix} + \alpha_3 \begin{pmatrix}3\\-1\\1\end{pmatrix}\begin{pmatrix}3\\-1\\1\end{pmatrix} = 1$$

Support Vector Machine - Linear Example Solved

$$\alpha_1 \tilde{s_1} \cdot \tilde{s_1} + \alpha_2 \tilde{s_2} \cdot \tilde{s_1} + \alpha_3 \tilde{s_3} \cdot \tilde{s_1} = -1$$
$$\alpha_1 \tilde{s_1} \cdot \tilde{s_2} + \alpha_2 \tilde{s_2} \cdot \tilde{s_2} + \alpha_3 \tilde{s_3} \cdot \tilde{s_2} = +1$$
$$\alpha_1 \tilde{s_1} \cdot \tilde{s_3} + \alpha_2 \tilde{s_2} \cdot \tilde{s_3} + \alpha_3 \tilde{s_3} \cdot \tilde{s_3} = +1$$

$$\alpha_1(1+0+1)+\alpha_2(3+0+1)+\alpha_3(3+0+1) = -1$$
$$\alpha_1(3+0+1)+\alpha_2(9+1+1)+\alpha_3(9-1+1) = 1$$
$$\alpha_1(3+0+1)+\alpha_2(9-1+1)+\alpha_3(9+1+1) = 1$$

$$\alpha_1 \begin{pmatrix}1\\0\\1\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix} + \alpha_2 \begin{pmatrix}3\\1\\1\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix} + \alpha_3 \begin{pmatrix}3\\-1\\1\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix} = -1$$

$$\alpha_1 \begin{pmatrix}1\\0\\1\end{pmatrix}\begin{pmatrix}3\\1\\1\end{pmatrix} + \alpha_2 \begin{pmatrix}3\\1\\1\end{pmatrix}\begin{pmatrix}3\\1\\1\end{pmatrix} + \alpha_3 \begin{pmatrix}3\\-1\\1\end{pmatrix}\begin{pmatrix}3\\1\\1\end{pmatrix} = 1$$

$$\alpha_1 \begin{pmatrix}1\\0\\1\end{pmatrix}\begin{pmatrix}3\\-1\\1\end{pmatrix} + \alpha_2 \begin{pmatrix}3\\1\\1\end{pmatrix}\begin{pmatrix}3\\-1\\1\end{pmatrix} + \alpha_3 \begin{pmatrix}3\\-1\\1\end{pmatrix}\begin{pmatrix}3\\-1\\1\end{pmatrix} = 1$$

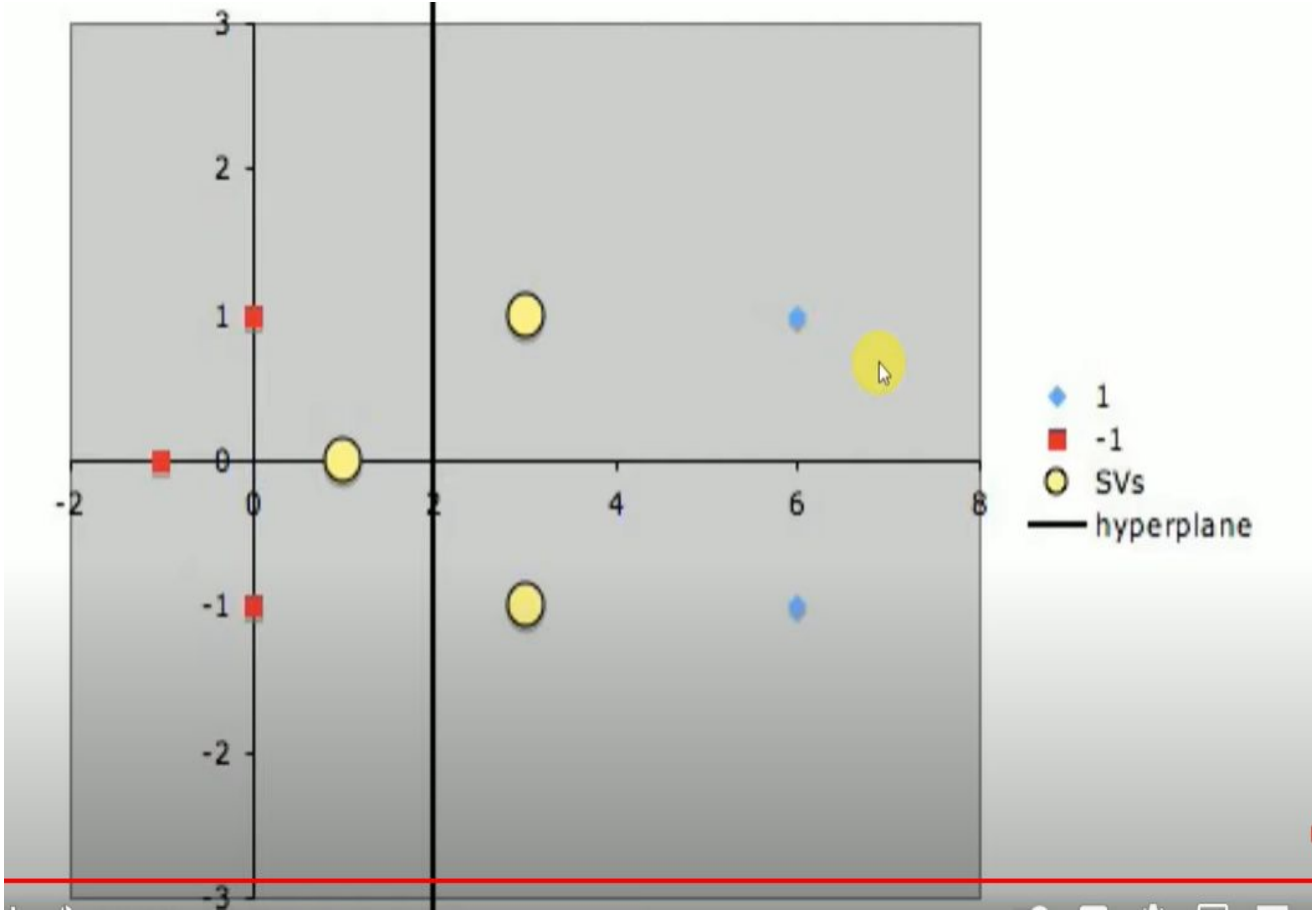
$$2\alpha_1 + 4\alpha_2 + 4\alpha_3 = -1$$
$$4\alpha_1 + 11\alpha_2 + 9\alpha_3 = 1$$
$$4\alpha_1 + 9\alpha_2 + 11\alpha_3 = 1$$

$$\alpha_1 = -3.5$$
$$\alpha_2 = 0.75$$
$$\alpha_3 = 0.75$$

Support Vector Machine - Linear Example Solved

$$
\begin{aligned}
\tilde{w} &= \sum_i \alpha_i \tilde{s}_i \\
&= -3.5 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 0.75 \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + 0.75 \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}
\end{aligned}
$$

- Finally, remembering that our vectors are augmented with a bias.
- We can equate the last entry in $\tilde{w}$ as the hyperplane offset b and write the separating
- Hyperplane equation $\boldsymbol{y} = \boldsymbol{wx} + \boldsymbol{b}$
- with $\boldsymbol{w} = \binom{1}{0}$ and $\boldsymbol{b} = -2$.



3
2
1
0
-1
-2
-3
-2
0
2
4
6
8
1
-1
SVs
hyperplane

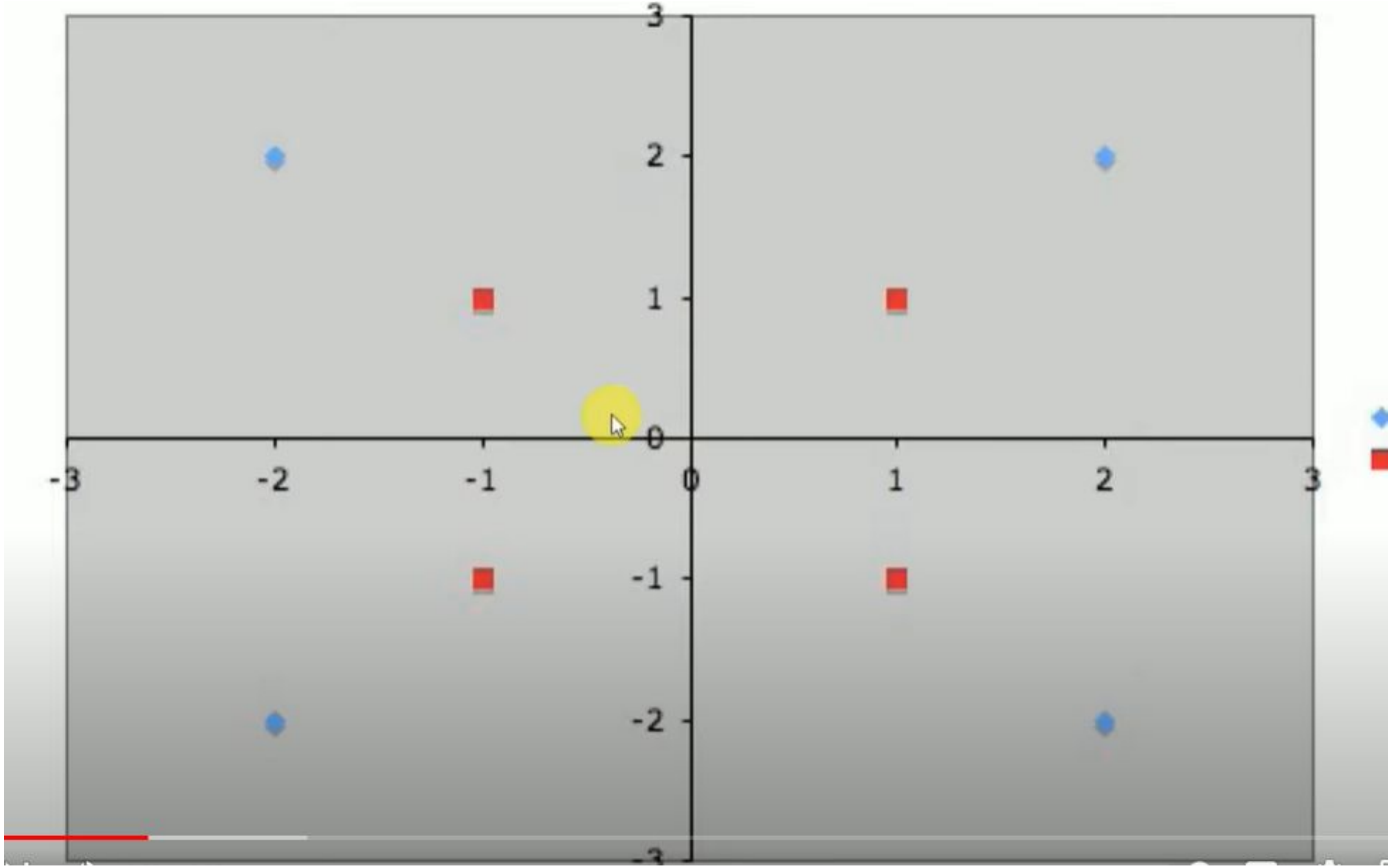
Support Vector Machine – Non Linear Example Solved

Suppose we are given the following positively labeled data points,

$$\left\{ \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \end{pmatrix} \right\}$$

and the following negatively labeled data points,

$$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$



3
2
1
0
-1
-2
-3
-3
-2
-1
0
1
2
3

Support Vector Machine – Non Linear Example Solved

- Our goal, again, is to discover a separating hyperplane that accurately discriminates the two classes.
- Of course, it is obvious that no such hyperplane exists in the input space
- Therefore, we must use a nonlinear SVM (that is, we need to convert data from one feature space to another.

$$\Phi_1\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{cases} \begin{pmatrix} 4 - x_2 + |x_1 - x_2| \\ 4 - x_1 + |x_1 - x_2| \end{pmatrix} & \text{if } \sqrt{x_1^2 + x_2^2} > 2 \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & \text{otherwise} \end{cases}$$

Support Vector Machine – Non Linear Example Solved

$$\Phi_1\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{cases} \begin{pmatrix} 4 - x_2 + |x_1 - x_2| \\ 4 - x_1 + |x_1 - x_2| \end{pmatrix} & \text{if } \sqrt{x_1^2 + x_2^2} > 2 \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & \text{otherwise} \end{cases}$$

Positive Examples

$$\left\{\begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \end{pmatrix}\right\}$$

Support Vector Machine – Non Linear Example Solved

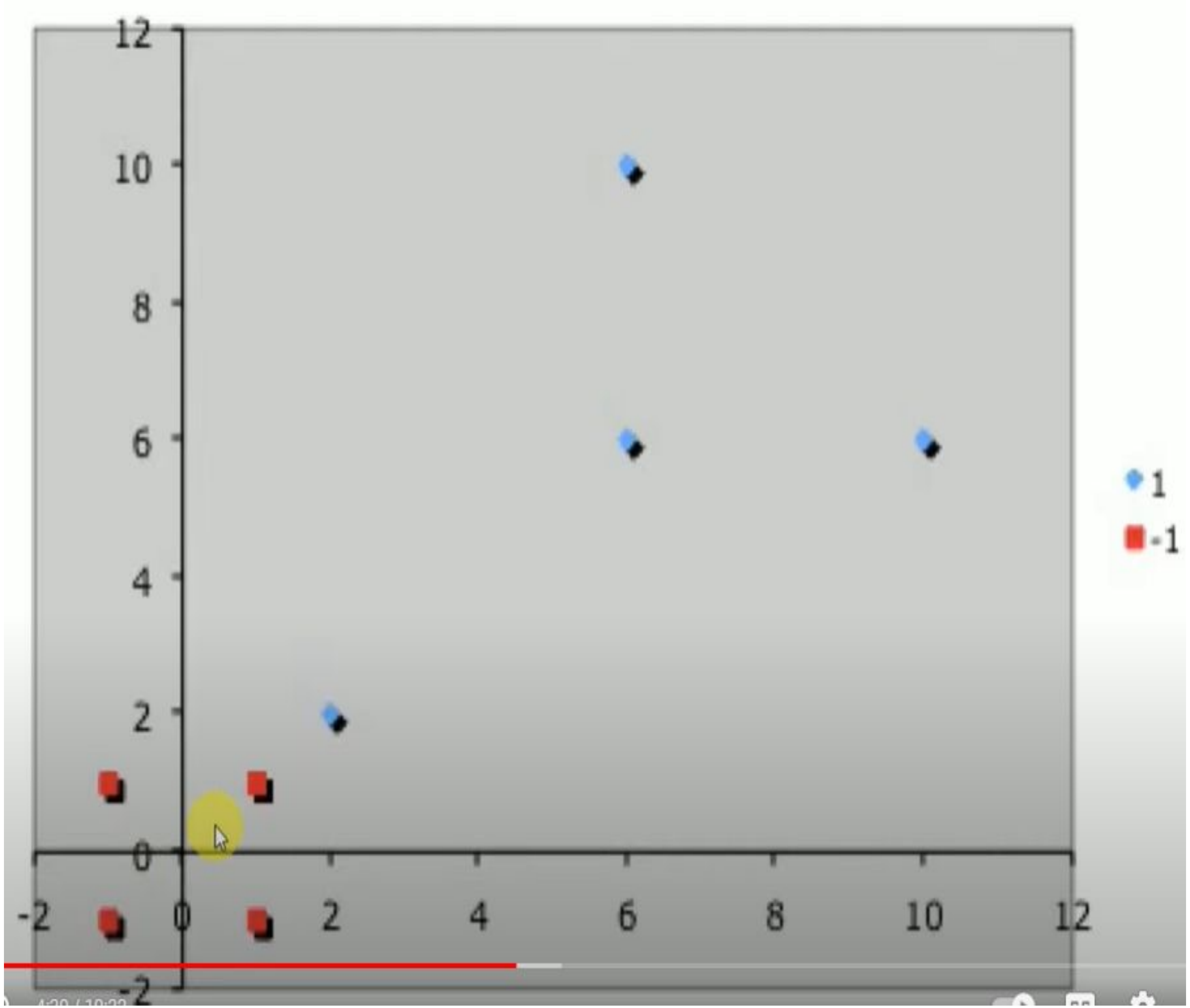
$$\Phi_1\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{cases} \begin{pmatrix} 4 - x_2 + |x_1 - x_2| \\ 4 - x_1 + |x_1 - x_2| \end{pmatrix} & \text{if } \sqrt{x_1^2 + x_2^2} > 2 \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & \text{otherwise} \end{cases}$$

Positive Examples

$$\left\{\begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \end{pmatrix}\right\} \rightarrow \left\{\begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 10 \\ 6 \end{pmatrix}, \begin{pmatrix} 6 \\ 6 \end{pmatrix}, \begin{pmatrix} 6 \\ 10 \end{pmatrix}\right\}$$

Negative Examples

$$\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\} \rightarrow \left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$$


12
10
8
6
4
2
0
-2
-2
0
2
4
6
8
10
12
1
-1

Support Vector Machine – Non Linear Example Solved

- Now we can easily identify the support vectors,

$$\left\{ s_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, s_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\}$$

- Each vector is augmented with a 1 as a bias input

$$\tilde{s_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \tilde{s_2} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

Support Vector Machine – Non Linear Example Solved

$$\alpha_1 \tilde{s_1} \cdot \tilde{s_1} + \alpha_2 \tilde{s_2} \cdot \tilde{s_1} = -1$$
$$\alpha_1 \tilde{s_1} \cdot \tilde{s_2} + \alpha_2 \tilde{s_2} \cdot \tilde{s_2} = +1$$

$$\alpha_1(1+1+1) + \alpha_2(2+2+1) = -1$$
$$\alpha_1(2+2+1) + \alpha_2(4+4+1) = 1$$

$$\alpha_1 \begin{pmatrix}1\\1\\1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} + \alpha_2 \begin{pmatrix}2\\2\\1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} = -1$$

$$\alpha_1 \begin{pmatrix}1\\1\\1\end{pmatrix}\begin{pmatrix}2\\2\\1\end{pmatrix} + \alpha_2 \begin{pmatrix}2\\2\\1\end{pmatrix}\begin{pmatrix}2\\2\\1\end{pmatrix} = 1$$

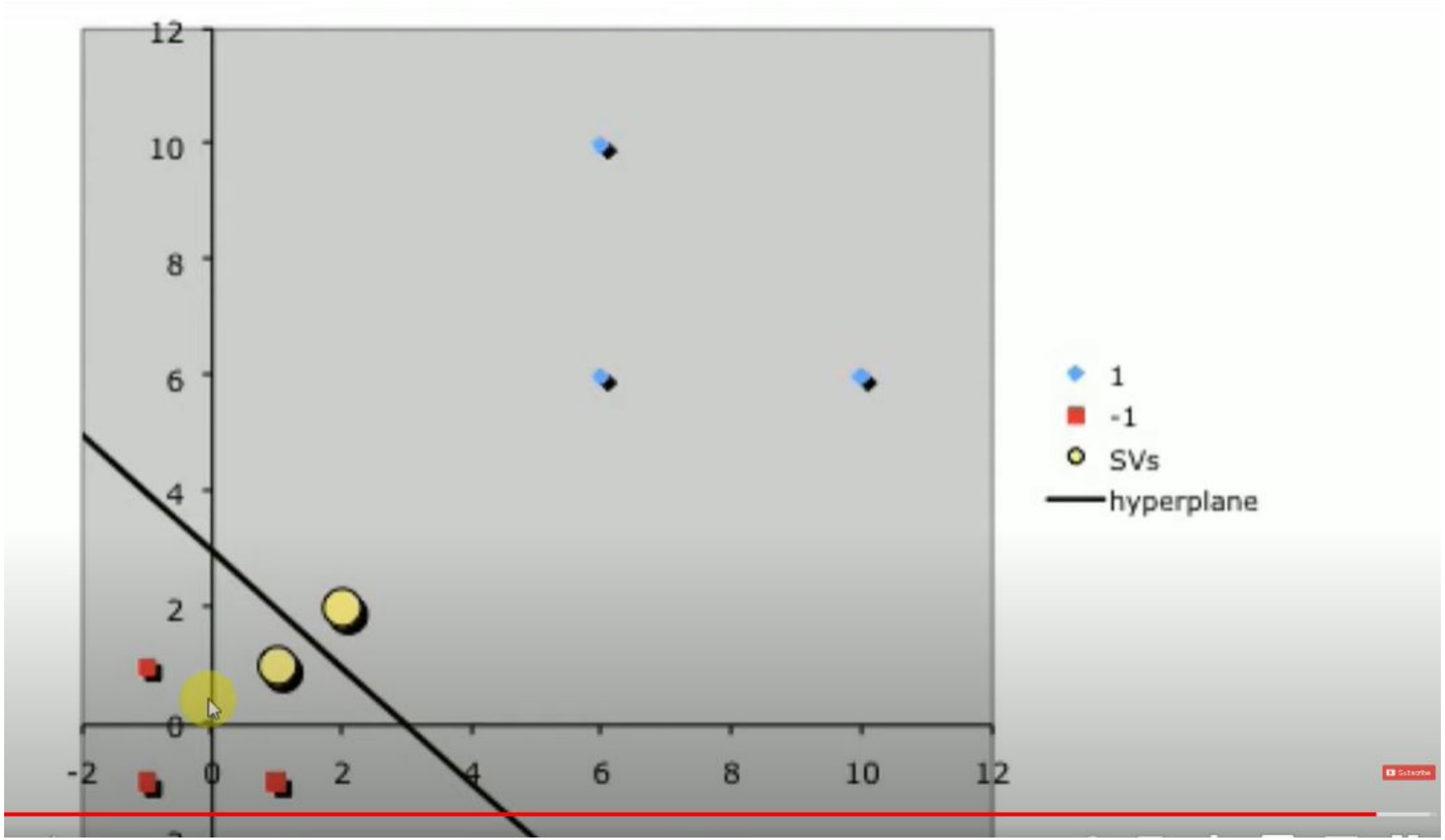
$$3\alpha_1 + 5\alpha_2 = -1$$
$$5\alpha_1 + 9\alpha_2 = 1$$

Support Vector Machine – Non Linear Example Solved

$$\tilde{w} = \sum_i \alpha_i \tilde{s}_i = -7\begin{pmatrix}1\\1\\1\end{pmatrix} + 4\begin{pmatrix}2\\2\\1\end{pmatrix}$$

$$= \begin{pmatrix}1\\1\\-3\end{pmatrix}$$

- Finally, remembering that our vectors are augmented with a bias.
- We can equate the last entry in $\tilde{w}$ as the hyperplane offset b and write the separating
- Hyperplane equation $\boldsymbol{y} = \boldsymbol{wx} + \boldsymbol{b}$
- with $\boldsymbol{w} = \binom{1}{1}$ and $\boldsymbol{b} = -\mathbf{3}$.


12
10
8
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4
2
0
-2
0
2
4
6
8
10
12
1
-1
SVs
hyperplane