Fourier Series

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Contents

***Abstract*—This manual provides a simple introduction to Fourier Series**

1 Periodic Function

Let

$$x(t) = A_0 \left|\sin\left(2\pi f_0 t\right)\right| \tag{1.1}$$

1.1 Plot $x(t)$.
Solution: The Python code `codes/1_1.py` plots $x(t)$ in Fig. (1.1).

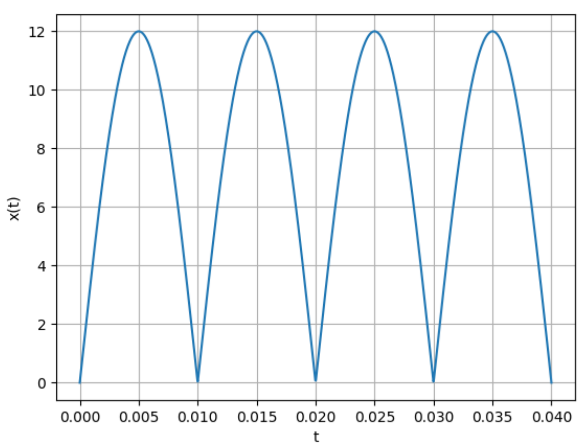

Fig. 1.1: $x(t)$

1.2 Show that $x(t)$ is periodic and find its period.
Solution: From Fig. (1.1), we see that $x(t)$ is periodic. Further,

$$x\left(t + \frac{1}{f_0}\right) = A_0 \left|\sin\left(2\pi f_0\left(t + \frac{1}{f_0}\right)\right)\right| \tag{1.2}$$

$$= A_0 \left|\sin\left(2\pi f_0 t + 2\pi\right)\right| \tag{1.3}$$

$$= A_0 \left|\sin\left(2\pi f_0 t\right)\right| \tag{1.4}$$

Hence the period of $x(t)$ is $\frac{1}{f_0}$.

2 Fourier Series

Consider $A_0 = 12$ and $f_0 = 50$ for all numerical calculations.

2.1 If

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\mathbf{j}2\pi k f_0 t} \tag{2.1}$$

show that

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\mathbf{j}2\pi k f_0 t}\, dt \tag{2.2}$$

Solution: We have for some $n \in \mathbb{Z}$,

$$x(t) e^{-\mathbf{j}2\pi n f_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{\mathbf{j}2\pi (k-n) f_0 t} \tag{2.3}$$

But we know from the periodicity of $e^{\mathbf{j}2\pi k f_0 t}$,

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{\mathbf{j}2\pi k f_0 t}\, dt = \frac{1}{f_0}\delta(k) \tag{2.4}$$

Thus,

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\mathbf{j}2\pi n f_0 t}\, dt = \frac{c_n}{f_0} \tag{2.5}$$

$$\implies c_n = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\mathbf{j}2\pi n f_0 t}\, dt \tag{2.6}$$

2.2 Find c_k for (1.1)

Solution: Using (2.2),

$$c_n = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left|\sin\left(2\pi f_0 t\right)\right| e^{-\mathbf{j}2\pi n f_0 t}\, dt \tag{2.7}$$

$$= f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left|\sin\left(2\pi f_0 t\right)\right| \cos\left(2\pi n f_0 t\right)\, dt$$
$$+ \mathbf{j} f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left|\sin\left(2\pi f_0 t\right)\right| \sin\left(2\pi n f_0 t\right)\, dt \tag{2.8}$$

$$= 2f_0 \int_0^{\frac{1}{2f_0}} A_0 \sin\left(2\pi f_0 t\right) \cos\left(2\pi n f_0 t\right)\, dt \tag{2.9}$$

$$= f_0 A_0 \int_0^{\frac{1}{2f_0}} \left(\sin\left(2\pi\left(n+1\right) f_0 t\right)\right)\, dt$$
$$- f_0 A_0 \int_0^{\frac{1}{2f_0}} \left(\sin\left(2\pi\left(n-1\right) f_0 t\right)\right)\, dt \tag{2.10}$$

$$= A_0 \frac{1+(-1)^n}{2\pi}\left(\frac{1}{n+1} - \frac{1}{n-1}\right) \tag{2.11}$$

$$= \begin{cases} \frac{2A_0}{\pi\left(1-n^2\right)} & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \tag{2.12}$$

2.3 Verify (2.1) using python.
Solution: The Python code `codes/2_3.py` verifies (2.13).

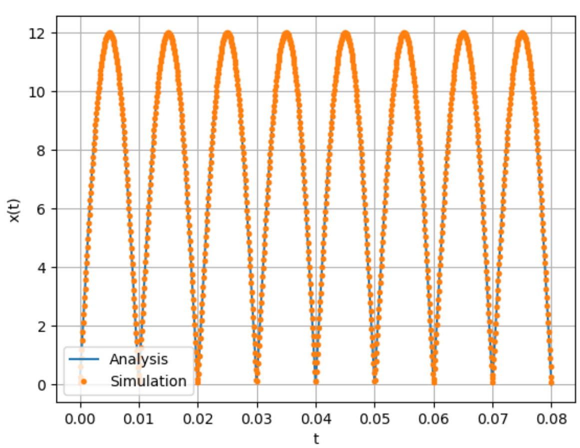

Fig. 2.3: Verification of (2.1).

2.4 Show that

$$x(t) = \sum_{k=0}^{\infty} \left(a_k \cos \mathbf{j}2\pi k f_0 t + b_k \sin \mathbf{j}2\pi k f_0 t\right) \tag{2.13}$$

and obtain the formulae for a_k and b_k.
Solution: From (2.1),

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\mathbf{j}2\pi k f_0 t} \tag{2.14}$$

$$= c_0 + \sum_{k=1}^{\infty} c_k e^{\mathbf{j}2\pi k f_0 t} + c_{-k} e^{-\mathbf{j}2\pi k f_0 t} \tag{2.15}$$

$$= c_0 + \sum_{k=1}^{\infty} \left(c_k + c_{-k}\right) \cos\left(2\pi k f_0 t\right)$$
$$+ \sum_{k=0}^{\infty} \left(c_k - c_{-k}\right) \sin\left(2\pi k f_0 t\right) \tag{2.16}$$

Hence, for $k \geq 0$,

$$a_k = \begin{cases} c_0 & k = 0 \\ c_k + c_{-k} & k > 0 \end{cases} \tag{2.17}$$

$$b_k = c_k - c_{-k} \tag{2.18}$$

2.5 Find a_k and b_k for (1.1)
Solution: From (2.1), we see that since $x(t)$ is even,

$$x(-t) = \sum_{k=-\infty}^{\infty} c_k e^{-\mathbf{j}2\pi k f_0 t} \tag{2.19}$$

$$= \sum_{k=-\infty}^{\infty} c_{-k} e^{\mathbf{j}2\pi k f_0 t} \tag{2.20}$$

$$= \sum_{k=-\infty}^{\infty} c_k e^{\mathbf{j}2\pi k f_0 t} \tag{2.21}$$

where we substitute $k \mapsto -k$ in (2.20). Hence, we see that $c_k = c_{-k}$. So, from (2.18) and for $k \geq 0$,

$$a_k = \begin{cases} \frac{2A_0}{\pi} & k = 0 \\ \frac{4A_0}{\pi\left(1-k^2\right)} & k > 0,\ k \text{ even} \\ 0 & \text{otherwise} \end{cases} \tag{2.22}$$

$$b_k = 0 \tag{2.23}$$

2.6 Verify (2.13) using python.
Solution: The Python code `codes/2_6.py` verifies (2.13).

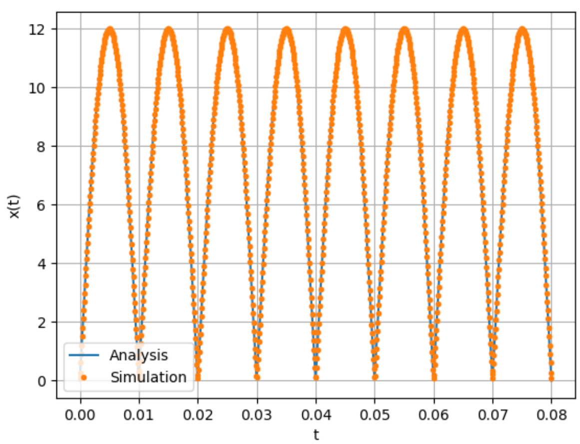

Fig. 2.6: Verification of (2.13).