Circuits and Transforms

Varunaditya Singhal

Contents

***Abstract*—This manual provides a simple introduction to Transforms**

1 Definitions

1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \tag{1.1}$$

2. The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t) e^{-st}\, dt \tag{1.2}$$

2 Laplace Transform

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes $q_1\,\mu C$. Then S is switched to position Q. After a long time, the charge on the capacitor is $q_2\,\mu C$.

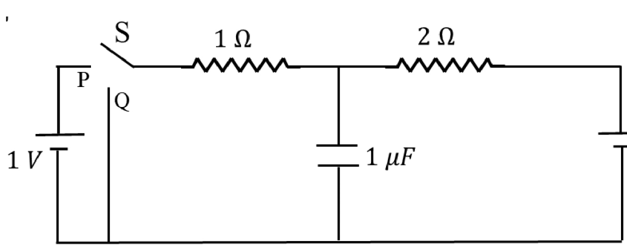

Fig. 2.1

2. Find q_1.
Solution: The equivalent circuit at steady-state when the switch is at P is shown alongside. Assuming the circuit to be grounded at G and the relative potential at point X to be V, we use KCL at X and get

$$\frac{V-1}{1} + \frac{V-2}{2} = 0 \tag{2.1}$$

$$\implies V = \frac{4}{3}\,\text{V} \tag{2.2}$$

Hence,

$$q_1 = CV = \frac{4}{3}\,\mu\text{C} \tag{2.3}$$

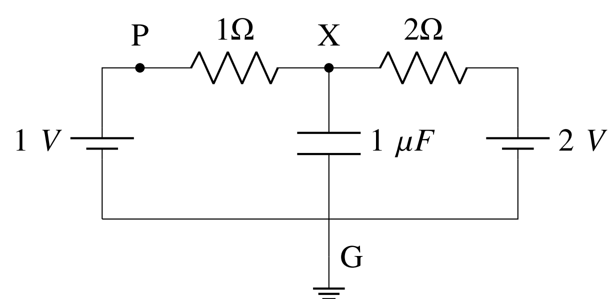

Fig. 2.2

3. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC.
Solution: We have,

$$u(t) \overset{\mathcal{L}}{\longleftrightarrow} \int_{0}^{\infty} u(t) e^{-st} dt \tag{2.4}$$

$$= \int_{0}^{0} \frac{1}{2} e^{-st} dt + \int_{0}^{\infty} e^{-st} dt \tag{2.5}$$

$$= \frac{1}{s}, \quad \Re(s) > 0 \tag{2.6}$$

4. Show that

$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad a > 0 \tag{2.7}$$

and find the ROC.
Solution: Note that by substituting $s := s + a$ in (2.6), and considering $a \in \mathbb{R}$,

$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \int_0^\infty u(t)e^{-(s+a)t}dt \tag{2.8}$$

$$= \frac{1}{s+a}, \quad \Re(s) > -a \tag{2.9}$$

5. Now consider the following resistive circuit transformed from Fig. 2.1 where

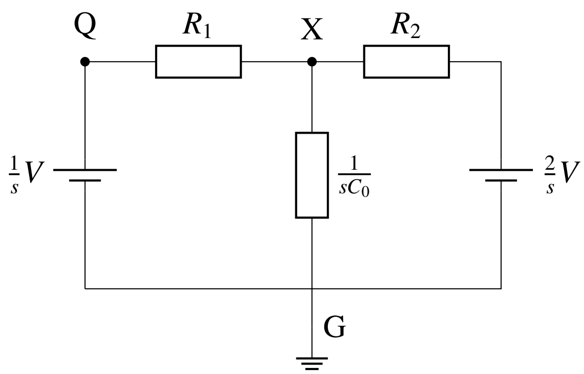

Fig. 2.3

$$u(t) \overset{\mathcal{L}}{\longleftrightarrow} V_1(s) \tag{2.10}$$

$$2u(t) \overset{\mathcal{L}}{\longleftrightarrow} V_2(s) \tag{2.11}$$

Find the voltage across the capacitor $V_{C_0}(s)$.
Solution: We see that

$$V_1(s) = \frac{1}{s}V_2(s) = \frac{2}{s} \tag{2.12}$$

Now, labelling points G and X as in Fig. 2.2, we use KCL at X.

$$\frac{V - \frac{1}{s}}{R_1} + \frac{V - \frac{2}{s}}{R_2} + sC_0V = 0 \tag{2.13}$$

$$V\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right) = \frac{1}{s}\left(\frac{1}{R_1} + \frac{2}{R_2}\right) \tag{2.14}$$

$$V(s) = \frac{\frac{1}{R_1} + \frac{2}{R_2}}{s\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right)} \tag{2.15}$$

$$= \frac{\frac{1}{R_1} + \frac{2}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2}}\left(\frac{1}{s} - \frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + s}\right) \tag{2.16}$$

6. Find $v_{C_0}(t)$. Plot using python.
Solution: Taking the inverse Laplace transform in (2.16),

$$V(s) \overset{\mathcal{L}}{\longleftrightarrow} \frac{2R_1 + R_2}{R_1 + R_2}u(t)\left(1 - e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{t}{C_0}}\right) \tag{2.17}$$

$$= \frac{4}{3}\left(1 - e^{-\left(1.5\times10^6\right)t}\right)u(t) \tag{2.18}$$

The python code `codes/2_6.py` plots the graph below.

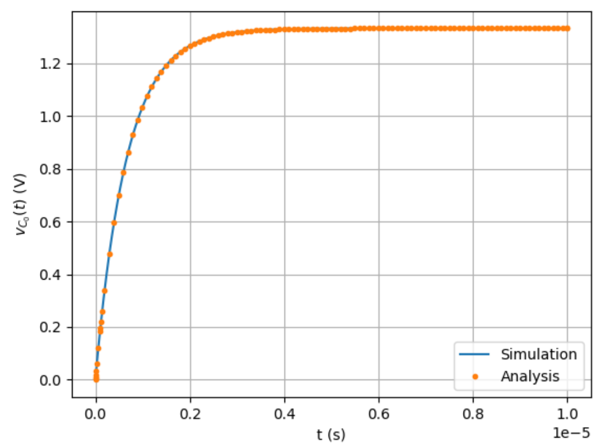

Fig. 2.4: $v_{C_0}(t)$ before the switch is flipped

7. Verify your result using ngspice.
Solution: The ngspice script `codes/2_7.cir` simulates the given circuit and the generated output is depicted in Fig. (2.4)

3 Initial Conditions

1. Find q_2 in Fig. 2.1.
Solution: The equivalent circuit at steady state when the switch is at Q is shown below.

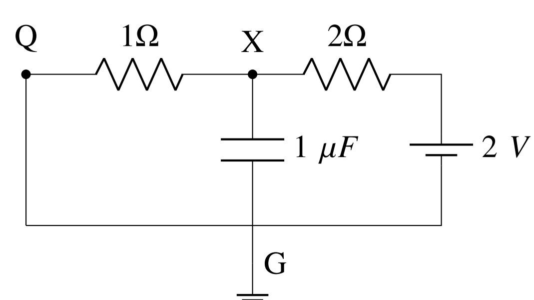

Fig. 3.1

Since capacitor behaves as an open circuit, we use KCL at X.

$$\frac{V-0}{1}+\frac{V-2}{2}=0 \implies V=\frac{2}{3}\,\text{V} \tag{3.1}$$

and hence, $q_2 = \frac{2}{3}\,\mu\text{C}$.

2. Draw the equivalent s-domain resistive circuit w hen S is switched to position Q. Use variables R_1, R_2, C_0 for the passive elements.

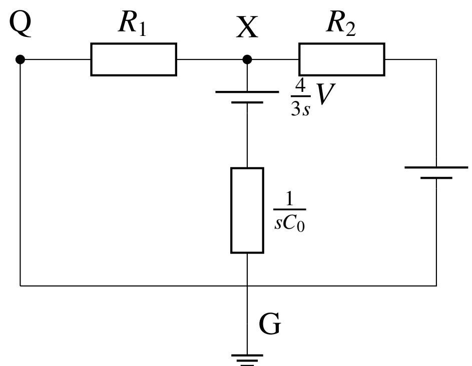

Fig. 3.2

3. $V_{C_0}(s) = ?$
Solution: Using KCL at node X in Fig. 3.2

$$\frac{V-0}{R_1}+\frac{V-\frac{2}{s}}{R_2}+sC_0\left(V-\frac{4}{3s}\right)=0 \tag{3.2}$$

$$\implies V_{C_0}(s)=\frac{\frac{2}{sR_2}+\frac{4C_0}{3}}{\frac{1}{R_1}+\frac{2}{R_2}+sC_0} \tag{3.3}$$

4. $v_{C_0}(t) = ?$ Plot using python.
Solution: From (3.3),

$$V_{C_0}(s)=\frac{4}{3}\left(\frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+s}\right) + \frac{2}{R_2\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}\left(\frac{1}{s}-\frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+s}\right) \tag{3.4}$$

Taking an inverse Laplace Transform,

$$v_{C_0}(t)=\frac{4}{3}e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}u(t) + \frac{2}{R_2\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}\left(1-e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}\right)u(t) \tag{3.5}$$

Substituting values gives

$$v_{C_0}(t)=\frac{2}{3}\left(1+e^{-\left(1.5\times10^{6}\right)t}\right)u(t) \tag{3.6}$$

The Python code `codes/3_4.py` plots the graph below.

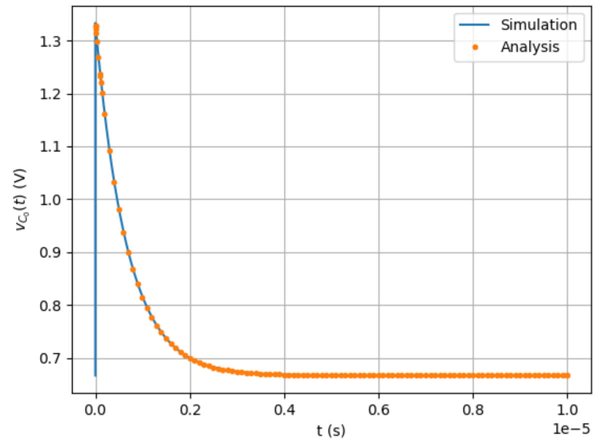

Fig. 3.3: $v_{C_0}(t)$ after the switch is flipped

5. Verify your result using ngspice.
Solution: The ngspice script `codes/3_5.cir` simulates the given circuit and the generated output is depicted in Fig. (3.3).

6. Find $v_{C_0}(0-), v_{C_0}(0+)$ and $v_{C_0}(\infty)$.
Solution: From the initial conditions,

$$v_{C_0}(0-)=\frac{q_1}{C}=\frac{4}{3}\,\text{V} \tag{3.7}$$

Using (3.6),

$$v_{C_0}(0+)=\lim_{t\to0+}v_{C_0}(t)=\frac{4}{3}\,\text{V} \tag{3.8}$$

$$v_{C_0}(\infty)=\lim_{t\to\infty}v_{C_0}(t)=\frac{2}{3}\,\text{V} \tag{3.9}$$

7. Obtain the Fig. in problem 3.2 using the equivalent differential equations.
Solution: The equivalent circuit in the t-domain is shown below.

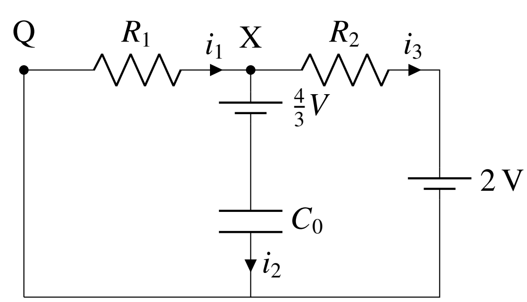

Fig. 3.4

From KCL and KVL,

$$i_1 = i_2 + i_3 \tag{3.10}$$

$$i_1 R_1 + \frac{4}{3} + \frac{1}{C_0}\int_0^t i_2 dt = 0 \tag{3.11}$$

$$\frac{4}{3} + \frac{1}{C_0}\int_0^t i_2 dt - i_3 R_2 - 2 = 0 \tag{3.12}$$

$$\tag{3.13}$$

Taking Laplace Transforms on both sides and using the properties of Laplace Transforms,

$$I_1 = I_2 + I_3 \tag{3.14}$$

$$I_1 R_1 + \frac{4}{3} + \frac{1}{sC_0} I_2 = 0 \tag{3.15}$$

$$\frac{4}{3} + \frac{1}{sC_0} I_2 - I_3 R_2 - 2 = 0 \tag{3.16}$$

$$\tag{3.17}$$

where $i(t) \xleftrightarrow{\mathcal{L}} I(s)$. Note that the capacitor is equivalent to a resistive element of resistance $R_C = \frac{1}{sC_0}$ in the s-domain. Equations (3.14) - (3.16) precisely describe Fig. 3.2.