Pingala Series

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***Abstract*—This manual provides a simple introduction to Transforms**

1 JEE 2019

Let α,β are the roots of the equation

$$z^2 - z - 1 = 0 \tag{1.1}$$

and,

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1 \tag{1.2}$$

$$b_n = a_{n-1} + a_{n+1}, \quad n \geq 2, \quad b_1 = 1 \tag{1.3}$$

Verify the following using a python code.

1.1

$$\sum_{k=1}^{n} a_k = a_{n+2} - 1, \quad n \geq 1 \tag{1.4}$$

1.2

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{10}{89} \tag{1.5}$$

1.3

$$b_n = \alpha^n + \beta^n, \quad n \geq 1 \tag{1.6}$$

1.4

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{8}{89} \tag{1.7}$$

Solution: Download the python code for the problem from

```
$ wget https://raw.githubusercontent.com/
    Varunaditya1/Linear−System−and−Signal
    −Processing−EE3900/main/
    Pingala_Series/Codes/jee_2019.py
```

and run it using the command

```
python3 jee_2019.py
```

2 Pingala Series

2.1 The *one sided* Z-transform of $x(n)$ is defined as

$$X^+(z) = \sum_{n=0}^{\infty} x(n) z^{-n}, \quad z \in \mathbb{C} \tag{2.1}$$

2.2 The *Pingala* series is generated using the difference equation

$$x(n+2) = x(n+1) + x(n), \tag{2.2}$$

$$x(0) = x(1) = 1, n \geq 0 \tag{2.3}$$

Generate a stem plot for $x(n)$.

Solution: Download the python code for the stem plot from

```
$ wget https://raw.githubusercontent.com/
    Varunaditya1/Linear−System−and−Signal
    −Processing−EE3900/main/
    Pingala_Series/Codes/stem.py
```

and run it using the command

```
python3 stem.py
```

2.3 Find $X^+(z)$.

Solution: Applying the one-sided $z-$transformation to (2.3), will give us

$$\mathcal{Z}^+[x(n+2)] = \mathcal{Z}^+[x(n+1)] + \mathcal{Z}^+[x(n)] \tag{2.4}$$

$$\left(z^2 - z - 1\right) X^+(z) = z^2 \tag{2.5}$$

$$X^+(z) = \frac{1}{1 - z^{-1} - z^{-2}} \tag{2.6}$$

$$= \frac{1}{\left(1 - \alpha z^{-1}\right)\left(1 - \beta z^{-1}\right)}, \quad |z| > \alpha \tag{2.7}$$

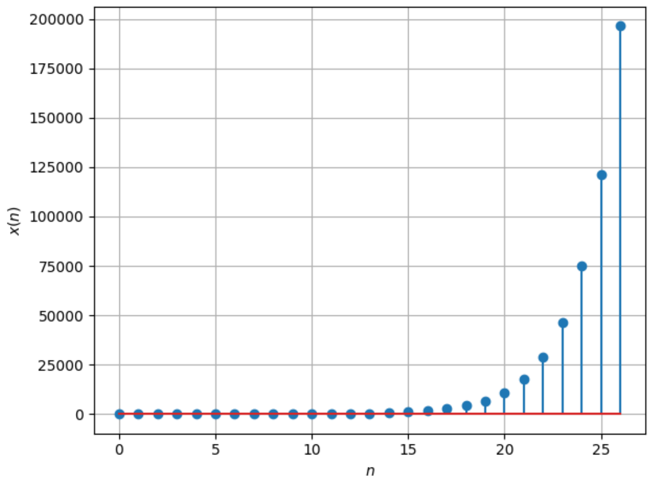

Fig. 2.2: Plot of $x(n)$

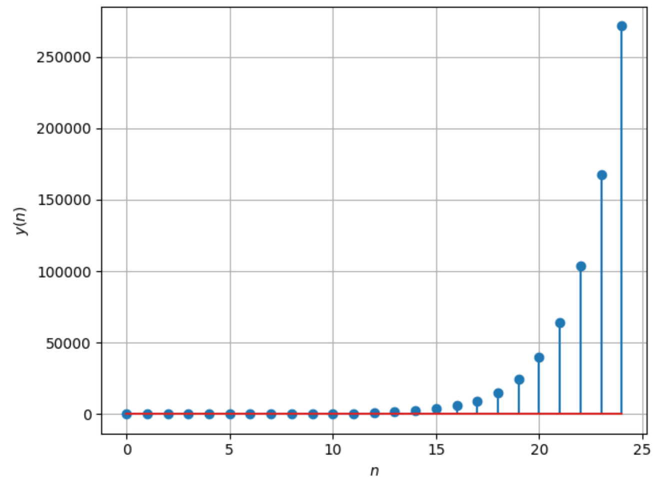

Fig. 2.5: Plot of $y(n)$

2.4 Find $x(n)$.
Solution: Expanding $X^{+}(z)$ from (2.7), gives

$$X^{+}(z) = \frac{1}{(\alpha-\beta)\,z^{-1}}\left[\frac{1}{1-\alpha z^{-1}} - \frac{1}{1-\beta z^{-1}}\right] \tag{2.8}$$

$$= \frac{1}{(\alpha-\beta)}\sum_{n=0}^{\infty}(\alpha^{n}-\beta^{n})\,z^{-n+1} \tag{2.9}$$

$$= \sum_{n=1}^{\infty}\frac{\alpha^{n}-\beta^{n}}{\alpha-\beta}z^{-n+1} \tag{2.10}$$

$$= \sum_{k=0}^{\infty}\frac{\alpha^{k+1}-\beta^{k+1}}{\alpha-\beta}z^{-k} \tag{2.11}$$

where $k := n+1$.
Therefore,

$$x(n) = \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}u(n) = a_{n+1}\,u(n) \tag{2.12}$$

2.5 Sketch

$$y(n) = x\,(n-1) + x\,(n+1)\,, \quad n \geq 0 \tag{2.13}$$

Solution: Download the Python code for sketch from

```
$ wget https://raw.githubusercontent.com/
    Varunaditya1/Linear−System−and−Signal
    −Processing−EE3900/main/
    Pingala_Series/Codes/y_z.py
```

and run it using the command

```
python3 y_z.py
```

2.6 Find $Y^{+}(z)$.
Solution: Applying the one-sided $z-$transformation to (2.13), will give us

$$\mathcal{Z}^{+}[y(n)] = \mathcal{Z}^{+}[x(n-1)] + \mathcal{Z}^{+}[x(n+1)] \tag{2.14}$$

$$= \frac{z+z^{-1}}{1-z^{-1}-z^{-2}} - z \tag{2.15}$$

as $x(n) = 0\ \forall\ n < 0$, therefore,

$$Y^{+}(z) = \frac{1+2z^{-1}}{1-z^{-1}-z^{-2}}, \quad |z| \geq \alpha \tag{2.16}$$

2.7 Find $y(n)$.
Solution: Using (2.7),

$$Y^{+}(z) = (1+2z^{-1})\sum_{n=0}^{\infty}x(n)z^{-n} \tag{2.17}$$

$$= \sum_{n=0}^{\infty}x(n)z^{-n} + \sum_{n=1}^{\infty}2x(n-1)z^{-n} \tag{2.18}$$

$$= x(0) + \sum_{n=1}^{\infty}(x(n)+2x(n-1))\,z^{-n} \tag{2.19}$$

Solving the above equation gives us

$$y(n) = \alpha^{n+1} + \beta^{n+1} \tag{2.20}$$

Thus, $y(0) = x(0) = 1$ and for $n \geq 1$, using the fact that α and β are the roots of the equation $z^2 - z - 1 = 0$,

$$y(n) = \frac{\left(\alpha^{n+1}-\beta^{n+1}\right) + (2\alpha^{n}+2\beta^{n})}{\alpha-\beta} \tag{2.21}$$

Thus, $y(n) = \alpha^{n+1} + \beta^{n+1}$ for $n \geq 0$ as $\alpha + \beta = 1$. Comparing (2.20) with the definition of b_n, we see that $y(n) = b_{n+1}$. Hence, $b_n = \alpha^n + \beta^n$.

3 Power of the Z transform

3.1 Show that

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(n) = x(n) * u(n-1) \tag{3.1}$$

Solution: From (2.12), and also as $x(n) = 0 \; \forall \; n < 0$, we get,

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(k) \tag{3.2}$$

$$= \sum_{k=-\infty}^{n-1} x(k) \tag{3.3}$$

$$= \sum_{k=-\infty}^{\infty} x(k)u(n-1-k) \tag{3.4}$$

$$= x(n) * u(n-1) \tag{3.5}$$

3.2 Show that

$$a_{n+2} - 1, \quad n \geq 1 \tag{3.6}$$

can be expressed as

$$[x(n+1) - 1]\, u(n) \tag{3.7}$$

Solution: From (2.12), we can see that,

$$a_{n+2} - 1 = [x(n+1) - 1], \quad n \geq 0 \tag{3.8}$$

Now, applying the property of $u(n)$ gives us,

$$a_{n+1} - 1 = [x(n+1) - 1]\, u(n), \quad n \geq 0 \tag{3.9}$$

3.3 Show that

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} = \frac{1}{10} X^+(10) \tag{3.10}$$

Solution:

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{a_{k+1}}{10^k} \tag{3.11}$$

$$= \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} \tag{3.12}$$

$$= \frac{1}{10} X^+(z) \tag{3.13}$$

$$= \frac{1}{10} \times \frac{100}{89} = \frac{10}{89} \tag{3.14}$$

3.4 Show that

$$\alpha^n + \beta^n, \quad n \geq 1 \tag{3.15}$$

can be expressed as

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right) u(n) \tag{3.16}$$

and find $W(z)$.

Solution: Putting $n = k + 1$ in (3.15) and using the definition of $u(n)$,

$$\alpha^n + \beta^n = \left(\alpha^{k+1} + \beta^{k+1}\right) u(k) \tag{3.17}$$

Hence, (3.15) can be expressed as

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right) u(n) = y(n) \tag{3.18}$$

Therefore,

$$W(z) = Y(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \tag{3.19}$$

3.5 Show that

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} = \frac{1}{10} Y^+(10) \tag{3.20}$$

Solution:

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{b_{k+1}}{10^k} \tag{3.21}$$

$$= \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} \tag{3.22}$$

$$= \frac{1}{10} Y^+(z) \tag{3.23}$$

$$= \frac{1}{10} \times \frac{120}{89} = \frac{12}{89} \tag{3.24}$$

3.6 Solve the JEE 2019 problem.

Solution: We know that

$$\sum_{k=1}^{n} a_k = x(n) * u(n-1) \tag{3.25}$$

But

$$x(n) * u(n-1) \overset{\mathcal{Z}}{\rightleftharpoons} X(z) z^{-1} U(z) \tag{3.26}$$

$$= \frac{z^{-1}}{\left(1 - z^{-1} - z^{-2}\right)\left(1 - z^{-1}\right)} \tag{3.27}$$

$$= z \left[\frac{1}{1 - z^{-1} - z^{-2}} - \frac{1}{1 - z^{-1}} \right] \tag{3.28}$$

Thus, we get

$$x(n) * u(n-1) \overset{\mathcal{Z}}{\rightleftharpoons} z \sum_{n=0}^{\infty} (x(n) - 1) z^{-n} \tag{3.29}$$

$$= \sum_{n=0}^{\infty} (x(n) - 1) z^{-n+1} \tag{3.30}$$

$$= \sum_{n=0}^{\infty} (x(n+1) - 1) z^{-n} \tag{3.31}$$

$$\tag{3.32}$$

From (2.12), we get

$$\sum_{k=1}^{n} a_k = a_{n+2} - 1 \tag{3.33}$$

We have already found the relevant answers of the options from (2.7), (3.3), and (3.5).
Therefore,
Correct Option(s) - $(a), (b), (c)$
Incorrect Option(s) - (d).