Pingala Series

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Abstract—**This manual provides a simple introduction to Transforms**

1 JEE 2019

Let α, β are the roots of the equation

$$z^2 - z - 1 = 0 \qquad (1.1)$$

and,

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1 \qquad (1.2)$$

$$b_n = a_{n-1} + a_{n+1}, \quad n \geq 2, \quad b_1 = 1 \qquad (1.3)$$

Verify the following using a python code.
1.1

$$\sum_{k=1}^{n} a_k = a_{n+2} - 1, \quad n \geq 1 \qquad (1.4)$$

1.2

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{10}{89} \qquad (1.5)$$

1.3

$$b_n = \alpha^n + \beta^n, \quad n \geq 1 \qquad (1.6)$$

1.4

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{8}{89} \qquad (1.7)$$

Solution: Download the python code for the problem from

$ wget https://raw.githubusercontent.com/kn−vardhan/EE3900−Digital−Signal−Processing/main/pingala/codes/jee_2019.py

and run it using the command

python3 jee_2019.py

2 PINGALA SERIES

2.1 The *one sided* Z-transform of $x(n)$ is defined as

$$X^+(z) = \sum_{n=0}^{\infty} x(n)z^{-n}, \quad z \in \mathbb{C} \qquad (2.1)$$

2.2 The *Pingala* series is generated using the difference equation

$$x(n+2) = x(n+1) + x(n), \qquad (2.2)$$

$$x(0) = x(1) = 1, n \geq 0 \qquad (2.3)$$

Generate a stem plot for $x(n)$.
Solution: Download the python code for the stem plot from

$ wget https://raw.githubusercontent.com/kn−vardhan/EE3900−Digital−Signal−Processing/main/pingala/codes/stem.py

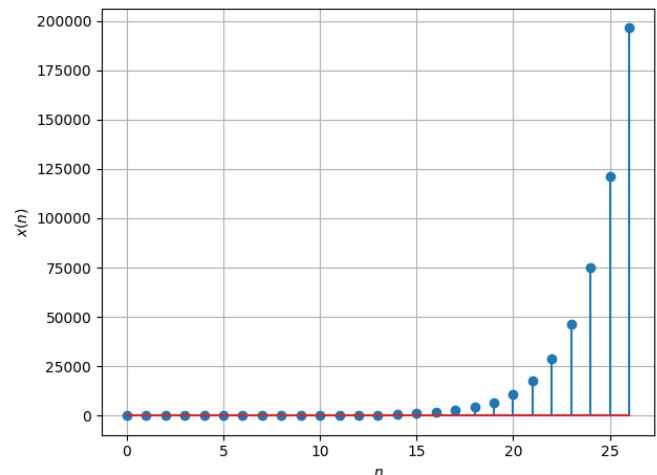and run it using the command

python3 stem.py

Fig. 2.2: Plot of $x(n)$

2.3 Find $X^+(z)$.

Solution: Applying the one-sided z−transformation to (2.3), will give us

$$\mathcal{Z}^+[x(n+2)] = \mathcal{Z}^+[x(n+1)] + \mathcal{Z}^+[x(n)]$$
$$(2.4)$$

$$\left(z^2 - z - 1\right)X^+(z) = z^2 \qquad (2.5)$$

$$X^+(z) = \frac{1}{1 - z^{-1} - z^{-2}} \qquad (2.6)$$

$$= \frac{1}{(1 - \alpha z^{-1})(1 - \beta z^{-1})}, \qquad |z| > \alpha \qquad (2.7)$$

2.4 Find $x(n)$.

Solution: Expanding $X^+(z)$ from (2.7), gives

$$X^+(z) = \frac{1}{(\alpha - \beta)z^{-1}}\left[\frac{1}{1 - \alpha z^{-1}} - \frac{1}{1 - \beta z^{-1}}\right]$$
$$(2.8)$$

$$= \frac{1}{(\alpha - \beta)}\sum_{n=0}^{\infty}(\alpha^n - \beta^n)z^{-n+1} \qquad (2.9)$$

$$= \sum_{n=1}^{\infty}\frac{\alpha^n - \beta^n}{\alpha - \beta}z^{-n+1} \qquad (2.10)$$

$$= \sum_{k=0}^{\infty}\frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta}z^{-k} \qquad (2.11)$$

where $k := n + 1$.
Therefore,

$$x(n) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}u(n) = a_{n+1}\,u(n) \qquad (2.12)$$

2.5 Sketch

$$y(n) = x(n-1) + x(n+1), \qquad n \geq 0 \qquad (2.13)$$

Solution: Download the Python code for sketch from

$ wget https://raw.githubusercontent.com/kn−
vardhan/EE3900−Digital−Signal−
Processing/main/pingala/codes/y_z.py
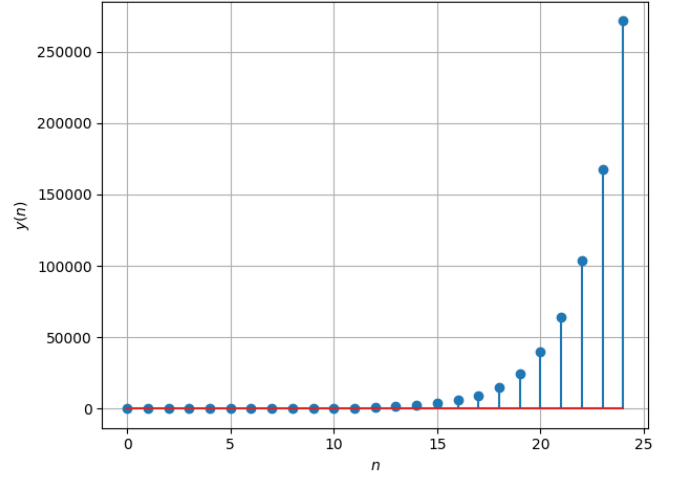
and run it using the command

python3 y_z.py

Fig. 2.5: Plot of $y(n)$

2.6 Find $Y^+(z)$.

Solution: Applying the one-sided z−transformation to (2.13), will give us

$$\mathcal{Z}^+[y(n)] = \mathcal{Z}^+[x(n-1)] + \mathcal{Z}^+[x(n+1)]$$
$$(2.14)$$

$$= \frac{z + z^{-1}}{1 - z^{-1} - z^{-2}} - z \qquad (2.15)$$

as $x(n) = 0 \; \forall \; n < 0$, therefore,

$$Y^+(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}}, \qquad |z| \geq \alpha \qquad (2.16)$$

2.7 Find $y(n)$.

Solution: Using (2.7),

$$Y^+(z) = (1 + 2z^{-1})\sum_{n=0}^{\infty}x(n)z^{-n} \qquad (2.17)$$

$$= \sum_{n=0}^{\infty}x(n)z^{-n} + \sum_{n=1}^{\infty}2x(n-1)z^{-n} \qquad (2.18)$$

$$= x(0) + \sum_{n=1}^{\infty}(x(n) + 2x(n-1))\,z^{-n} \qquad (2.19)$$

Solving the above equation gives us

$$y(n) = \alpha^{n+1} + \beta^{n+1} \qquad (2.20)$$

Thus, $y(0) = x(0) = 1$ and for $n \geq 1$, using the fact that α and β are the roots of the equation $z^2 - z - 1 = 0$,

$$y(n) = \frac{\left(\alpha^{n+1} - \beta^{n+1}\right) + (2\alpha^n + 2\beta^n)}{\alpha - \beta} \qquad (2.21)$$

Thus, $y(n) = \alpha^{n+1} + \beta^{n+1}$ for $n \geq 0$ as $\alpha + \beta = 1$. Comparing (2.20) with the definition of b_n, we see that $y(n) = b_{n+1}$. Hence, $b_n = \alpha^n + \beta^n$.

3 POWER OF THE Z TRANSFORM

3.1 Show that

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(n) = x(n) * u(n-1) \qquad (3.1)$$

Solution: From (2.12), and also as $x(n) = 0 \; \forall \; n < 0$, we get,

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(k) \qquad (3.2)$$

$$= \sum_{k=-\infty}^{n-1} x(k) \qquad (3.3)$$

$$= \sum_{k=-\infty}^{\infty} x(k)u(n-1-k) \qquad (3.4)$$

$$= x(n) * u(n-1) \qquad (3.5)$$

3.2 Show that

$$a_{n+2} - 1, \quad n \geq 1 \qquad (3.6)$$

can be expressed as

$$[x(n+1) - 1]u(n) \qquad (3.7)$$

Solution: From (2.12), we can see that,

$$a_{n+2} - 1 = [x(n+1) - 1], \quad n \geq 0 \qquad (3.8)$$

Now, applying the property of $u(n)$ gives us,

$$a_{n+1} - 1 = [x(n+1) - 1]u(n), \quad n \geq 0 \quad (3.9)$$

3.3 Show that

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} = \frac{1}{10} X^+(10) \quad (3.10)$$

Solution:

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{a_{k+1}}{10^k} \qquad (3.11)$$

$$= \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} \qquad (3.12)$$

$$= \frac{1}{10} X^+(z) \qquad (3.13)$$

$$= \frac{1}{10} \times \frac{100}{89} = \frac{10}{89} \qquad (3.14)$$

3.4 Show that

$$\alpha^n + \beta^n, \quad n \geq 1 \qquad (3.15)$$

can be expressed as

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right) u(n) \qquad (3.16)$$

and find $W(z)$.
Solution: Putting $n = k + 1$ in (3.15) and using the definition of $u(n)$,

$$\alpha^n + \beta^n = \left(\alpha^{k+1} + \beta^{k+1}\right) u(k) \qquad (3.17)$$

Hence, (3.15) can be expressed as

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right) u(n) = y(n) \qquad (3.18)$$

Therefore,

$$W(z) = Y(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \qquad (3.19)$$

3.5 Show that

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} = \frac{1}{10} Y^+(10) \quad (3.20)$$

Solution:

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{b_{k+1}}{10^k} \qquad (3.21)$$

$$= \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} \qquad (3.22)$$

$$= \frac{1}{10} Y^+(z) \qquad (3.23)$$

$$= \frac{1}{10} \times \frac{120}{89} = \frac{12}{89} \qquad (3.24)$$

3.6 Solve the JEE 2019 problem.
Solution: We know that

$$\sum_{k=1}^{n} a_k = x(n) * u(n-1) \qquad (3.25)$$

But

$$x(n) * u(n-1) \overset{\mathcal{Z}}{\rightleftharpoons} X(z)z^{-1}U(z) \qquad (3.26)$$

$$= \frac{z^{-1}}{(1 - z^{-1} - z^{-2})(1 - z^{-1})} \qquad (3.27)$$

$$= z \left[\frac{1}{1 - z^{-1} - z^{-2}} - \frac{1}{1 - z^{-1}} \right] \qquad (3.28)$$

Thus, we get

$$x(n) * u(n-1) \overset{Z}{\rightleftharpoons} z \sum_{n=0}^{\infty} (x(n)-1) z^{-n} \quad (3.29)$$

$$= \sum_{n=0}^{\infty} (x(n)-1) z^{-n+1} \quad (3.30)$$

$$= \sum_{n=0}^{\infty} (x(n+1)-1) z^{-n} \quad (3.31)$$

$$\quad (3.32)$$

From (2.12), we get

$$\sum_{k=1}^{n} a_k = a_{n+2} - 1 \quad (3.33)$$

We have already found the relevant answers of the options from (2.7), (3.3), and (3.5).
Therefore,
Correct Option(s) - $(a), (b), (c)$
Incorrect Option(s) - (d).