GATE 2021 AG -Q.2

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Question

If x is an integer with x>1, the solution of

$$\lim_{x\to\infty}\left(\frac{1}{x^2}+\frac{2}{x^2}+\frac{3}{x^2}+\cdots+\frac{x-1}{x^2}+\frac{1}{x}\right)$$

a) Zero
b) 0.5
c) 1.0
d) ∞ $(GATE\ 2021\ AG\ Q.2)$

Solution

Parameter	Description	Value
p(n)	$(n+1)^{th}$ term of the sequence	$\frac{n+1}{x^2}$
q(n)	sum of $(n+1)$ terms of the sequence	-

TABLE 4
Table of parameters

From table ,

$$P(z)=\frac{1}{x^2}\sum_{n=-\infty}^{n=\infty}(n+1)\,u(n)\,z^{-n} \tag{1}$$

$$nu(n)\overset{\mathcal{Z}}{\longleftrightarrow}\frac{z^{-1}}{(1-z^{-1})^2},|z|>1 \tag{2}$$

$$u(n)\overset{\mathcal{Z}}{\longleftrightarrow}\frac{1}{(1-z^{-1})},|z|>1 \tag{3}$$

From (1)

$$P(z)=\frac{1}{x^2}\frac{1}{(1-z^{-1})^2},|z|>1 \tag{4}$$

$$q(n)=p(n)*u(n) \tag{5}$$

$$\implies Q(z)=P(z)\,U(z) \tag{6}$$

$$=\left(\frac{1}{x^2}\frac{1}{(1-z^{-1})^2}\right)\left(\frac{1}{1-z^{-1}}\right) \tag{7}$$

$$=\frac{1}{x^2}\frac{1}{(1-z^{-1})^3},\quad |z|>1 \tag{8}$$

Using Contour Integration to find the inverse Z-transform,

$$q(x-1)=\frac{1}{2\pi j}\oint_C Q(z)\,z^{x-2}\,dz \tag{9}$$

$$=\frac{1}{2\pi j}\oint_C\frac{z^{x-2}}{x^2(1-z^{-1})^3}\,dz \tag{10}$$

From Cauchy's residue theorem,

$$\oint_C f(z)\,dz=2\pi i\sum R_j \tag{11}$$

$$R=\frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left((z-a)^m f(z)\right) \tag{12}$$

We can observe that the pole is repeated 3 times and thus $m=3$,

$$R=\frac{1}{(2)!}\lim_{z\to1}\frac{d^2}{dz^2}\left((z-1)^3\frac{z^{x+1}}{x^2(z-1)^3}\right) \tag{13}$$

$$R=\frac{1}{(2)!}\lim_{z\to1}\frac{d^2}{dz^2}\frac{z^{x+1}}{x^2} \tag{14}$$

$$R=\frac{1}{2}\frac{(x+1)(x)}{x^2} \tag{15}$$

$$\lim_{x\to\infty}q(x-1)=\frac{1}{2} \tag{16}$$

Hence , option (B) is correct.