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New exponential synchronization criteria for time-varying delayed neural networks with discontinuous activations[☆]

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ABSTRACT

This paper investigates the problem of exponential synchronization of time-varying delayed neural networks with discontinuous neuron activations. Under the extended Filippov differential inclusion framework, by designing discontinuous state-feedback controller and using some analytic techniques, new testable algebraic criteria are obtained to realize two different kinds of global exponential synchronization of the drive–response system. Moreover, we give the estimated rate of exponential synchronization which depends on the delays and system parameters. The obtained results extend some previous works on synchronization of delayed neural networks not only with continuous activations but also with discontinuous activations. Finally, numerical examples are provided to show the correctness of our analysis via computer simulations. Our method and theoretical results have a leading significance in the design of synchronized neural network circuits involving discontinuous factors and time-varying delays.

1. Introduction

As far as we know, the non-Lipschitz or discontinuous neuron activations widely exist in many practical neural networks. Usually, the discontinuities of activations are caused by some interesting engineering tasks such as switching in electronic circuits, dry friction, systems oscillating under the effect of an earthquake and so on (see Cortés, 2008, Filippov, 1988, Forti & Nistri, 2003, Liu, Chen, Cao, & Lu, 2011 and Luo, 2009). Unfortunately, the additional difficulties will arise if discontinuities of activation are considered in the neural network dynamical systems. Actually, this kind of dynamical neuron system is usually described by the differential equation system possessing discontinuous right-hand side. It should be pointed out that many results in the classical theory of differential equation have been shown to be invalid since the given vector field is no longer continuous. In this case, the continuously differentiable solution is not guaranteed for the discontinuous neuron system. Moreover, it is necessary to reveal what changes will occur for different dynamic behaviors when discontinuous activations are introduced into the neural networks. In order to overcome these difficulties, Forti et al. first introduced the theory of differential inclusion given by Filippov to investigate the dynamical behaviors of neural networks with discontinuous activations (Forti & Nistri, 2003). Since then, neural networks with discontinuous activations have received a great deal of attention. Under the new framework named Filippov differential inclusion framework (Filippov, 1988), many excellent results on dynamical behaviors have been obtained for neural networks with discontinuous activations (Allegretto, Papini, & Forti, 2010; Cai, Huang, Guo, & Chen, 2012; Forti, Grazzini, Nistri, & Pancioni, 2006; Forti, Nistri, & Papini, 2005; Huang, Cai, Zhang, & Duan, 2013; Huang, Wang, & Zhou, 2009; Liu & Cao, 2009; Liu, Cao, & Yu, 2012; Liu et al., 2011; Lu & Chen, 2005, 2008; Papini & Taddei, 2005). However, most of existing papers are focused on the existence and convergence of equilibrium and periodic solution (or almost periodic solution) for neural network models with discontinuous activations. To the best of our knowledge, there is not much research concerning more complex dynamical behaviors such as chaos, bifurcation and synchronization for neuron systems with discontinuous activations.

On the other hand, the issues of chaos synchronization have been extensively studied for a rather long time since the pioneering work of Pecora and Carroll in 1990 (see Pecora & Carroll, 1990). It is worth mentioning that synchronization means the dynamics of nodes share the same time-spatial property and can be induced by coupling or by external forces. In fact, synchronization is a typical collective behavior which can be found in a wide variety of research fields such as biological systems, meteorology

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and secure communications (see Collins & Stewart, 1993, Duane, Webster, & Weiss, 1999, Liao & Huang, 1999 and Mirollo, Strogatz, & Williams, 1990). There are many types of synchronization including complete synchronization, anti-synchronization, phase synchronization, etc. Nowadays, chaos synchronization of neural networks has become a hot research topic owing to its theoretical significance (see, for example, Cao, Wang, & Sun, 2007, Hoppensteadt & Izhikevich, 2000, Lu, Ho, & Wang, 2009 and Yang, Cao, Long, & Rui, 2010). Recently, the interest of synchronization problem is shifting to the networks with discontinuous neuron activations despite the fact that the synchronization is not easy to be realized because of the discontinuous vector field. In Liu and Cao (2010), the complete synchronization was considered for the delayed neural networks with discontinuous activation functions via approximation approach. In Liu, Cao et al. (2012) and Liu et al. (2011), the quasi-synchronization criteria were obtained for discontinuous or switched networks. That is to say, the synchronization error can only be controlled within a small region around zero, but cannot approach zero with time. In Yang and Cao (2013), the authors investigated the exponential synchronization of delayed neural networks with discontinuous activations by constructing suitable Lyapunov functionals. Also, Liu et al. got some sufficient conditions on synchronization of linearly coupled dynamical neuron systems with non-Lipschitz right-hand sides (Liu, Lu, & Chen, 2012). But the synchronization criteria were expressed in integral inequalities and the discontinuous functions were weakened to be weak-QUAD or semi-QUAD. It should be noted that such synchronization criteria may be not easily verified in practice and there still lack new and efficacious methods for realizing synchronization control of discontinuous neural networks. Moreover, the new controller for synchronization should be designed. In addition, in many practical applications of neural networks, time delays between neuron signals are typical phenomena due to internal or external uncertainties. Because of the finite speed of signal propagation and the finite switch speed of neuron amplifiers, the time-delays in neurons are usually time variant and sometimes vary dramatically with time (Hou & Qian, 1998; Huang, Ho, & Lam, 2005). Therefore, it is necessary for us to investigate the synchronization problems for time-varying delayed dynamical neuron systems with discontinuous activations via the Filippov differential inclusion framework.

Notations: Let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}^n$ denote the n-dimensional Euclidean space. Given the column vectors $x = (x_1, x_2, \dots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$ and $y = (y_1, y_2, \dots, y_n)^{\mathrm{T}} \in \mathbb{R}^n$, where the superscript T denotes the transpose operator, $\langle x, y\rangle = x^{\mathrm{T}}y = \sum_{i=1}^{n} x_i y_i$ represents the scalar product of x, y, while $\|x\|$ denotes any vector norm in $\mathbb{R}^n$. Given a set $\mathbb{E} \subset \mathbb{R}^n$, by meas($\mathbb{E}$) we mean the Lebesgue measure of set $\mathbb{E}$ in $\mathbb{R}^n$ and $\overline{\mathrm{co}}[\mathbb{E}]$ denotes the closure of the convex hull of $\mathbb{E}$. If $z \in \mathbb{R}^n$ and $\delta > 0$, $\mathscr{B}(z, \delta) = \{z^* \in \mathbb{R}^n : \|z^* - z\| \le \delta\}$ denotes the ball of δ about z. Given the function $V : \mathbb{R}^n \to \mathbb{R}$, ∂V denotes Clarke's generalized gradient of V.

The remainder of this paper is outlined as follows. In Section 2, the model description and preliminaries including some necessary definitions and lemmas are stated. In Section 3, the main results and their rigorous proofs are given. Some new exponential synchronization criteria for time-varying delayed neural networks with discontinuous activations are proposed via introducing discontinuous state-feedback controller. In Section 4, two numerical examples are provided to illustrate the theoretical results. Finally, some conclusions are drawn in Section 5.

2. Model description and preliminaries

In this paper, we consider the time-varying delayed neural networks described by the following differential equations:

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t-\tau(t))) + I_i,$$
$$i = 1, 2, \dots, n, \tag{1}$$

where $x_i(t)$ denotes the state variable of the potential of the ith neuron at time t; $c_i > 0$ denotes the self-inhibition with which the ith neuron will reset its potential to the resting state in isolations when disconnected from the network; a_{ij} represents the connection strength of jth neuron on the ith neuron; $f_j(\cdot)$ denotes the activation function of jth neuron; I_i is the external input to the ith neuron; $\tau(t)$ denotes the time-varying transmission delay at time t and is a continuous function satisfying

$0 \le \tau(t) \le \tau$ (here τ is a nonnegative constant).

Throughout this paper, the discontinuous neuron activations in (1) are assumed to satisfy the following properties:

(H1) For each $i = 1, 2, \dots, n$, $f_i : \mathbb{R} \to \mathbb{R}$ is continuous except on a countable set of isolate points $\{\rho_k^i\}$, where there exist finite right and left limits, $f_i^+(\rho_k^i)$ and $f_i^-(\rho_k^i)$, respectively. Moreover, f_i has at most a finite number of discontinuities on any compact interval of $\mathbb{R}$.

(H2) For every $i = 1, 2, \dots, n$, there exist nonnegative constants L_i and p_i such that

$$\sup_{\xi_i \in \overline{\mathrm{co}}[f_i(u)],\, \eta_i \in \overline{\mathrm{co}}[f_i(v)]} |\xi_i - \eta_i| \le L_i|u - v| + p_i,$$
$$\forall u, v \in \mathbb{R}, \tag{$*$}$$

where

$$\overline{\mathrm{co}}[f_i(\theta)] = \left[\min\{f_i^-(\theta), f_i^+(\theta)\},\ \ \max\{f_i^-(\theta), f_i^+(\theta)\}\right]$$
for $\theta \in \mathbb{R}$.

Remark 1. In general, the constant p_i in the condition (H2) should not equal to zero due to the discontinuity of the function f_i. Therefore, there exists essential difference between the condition (H2) and the Lipschitz condition in the previous literature. Especially, if the discontinuous function f_i satisfies the condition (H1) and is monotonically non-decreasing, then the following condition (H3) is satisfied.

(H3) For every $i = 1, 2, \dots, n$, there exist nonnegative constants L_i and p_i such that

$$\sup_{\xi_i \in \overline{\mathrm{co}}[f_i(u)],\, \eta_i \in \overline{\mathrm{co}}[f_i(v)]} (u - v)(\xi_i - \eta_i)$$
$$\le L_i(u - v)^2 + p_i|u - v|, \quad \forall u, v \in \mathbb{R},$$

where

$$\overline{\mathrm{co}}[f_i(\theta)] = \left[\min\{f_i^-(\theta), f_i^+(\theta)\}, \max\{f_i^-(\theta), f_i^+(\theta)\}\right]$$
for $\theta \in \mathbb{R}$.

Actually, if f_i satisfies the condition (H1) and is monotonically non-decreasing, then for $\forall \xi_i \in \overline{\mathrm{co}}[f_i(u)]$, $\eta_i \in \overline{\mathrm{co}}[f_i(v)]$, we have $(u - v)(\xi_i - \eta_i) \ge 0$ which implies $|u - v|\,|\xi_i - \eta_i| = (u - v)(\xi_i - \eta_i)$. Multiplying both sides of the inequality ($*$) by $|u - v|$, we obtain

$$\sup_{\xi_i \in \overline{\mathrm{co}}[f_i(u)],\, \eta_i \in \overline{\mathrm{co}}[f_i(v)]} |u - v|\,|\xi_i - \eta_i| \le L_i|u - v|^2 + p_i|u - v|,$$
$$\forall u, v \in \mathbb{R}.$$

That is to say, the condition (H3) holds. So the condition (H3) is a special case of (H2). For example, there are two classes of different situations illustrated in Fig. 1 when the discontinuous activation function $f_i(\theta)$ is discontinuous at $\theta = 0$ and satisfies (H2) and (H3), respectively. Here, we might as well take the two different cases of the discontinuous activation function $f_i(\theta)$ as follows:

$$\text{Case (a)}: \ f_i(\theta) = \begin{cases} \tanh(\theta) - 1, & \text{if } \theta \ge 0, \\ \tanh(\theta) + 1, & \text{if } \theta < 0. \end{cases}$$

$$\text{Case (b)}: \ f_i(\theta) = \begin{cases} \theta + 1, & \text{if } \theta \ge 0, \\ \theta - 1, & \text{if } \theta < 0. \end{cases}$$

Since neural network (1) is a delayed differential equation system possessing discontinuous right-hand side, the existence

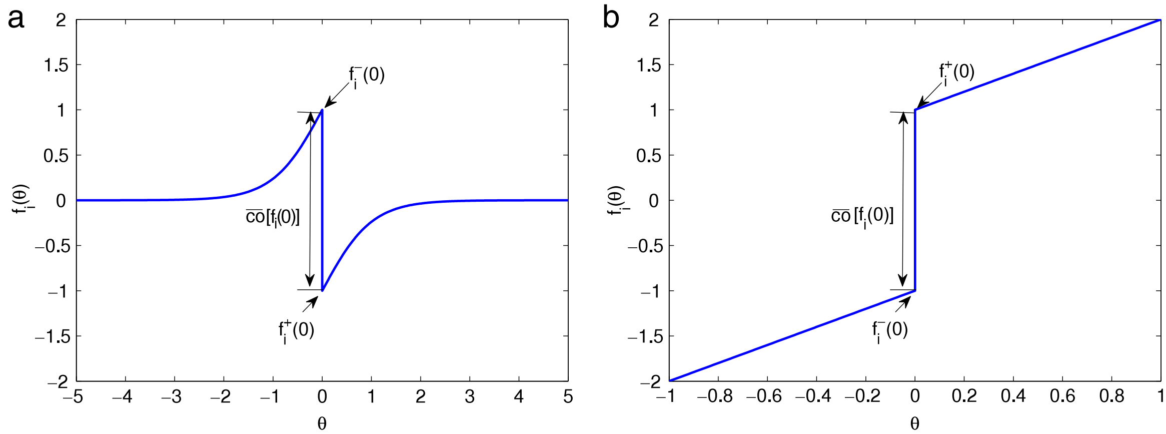

Fig. 1. Examples of discontinuous activation functions: (a) $f_i(\theta)$ is a non-monotonous function and satisfies the condition (H2); (b) $f_i(\theta)$ is a monotonically non-decreasing function and satisfies the condition (H3).

of a continuously differential solution (i.e., classical solution) is not guaranteed. In the following, we apply an extended Filippov-framework to discuss the so-called Filippov solution for delayed neural network (1) with discontinuous activation.

Definition 1. Suppose that to each point x of a set $\mathbb{E} \subset \mathbb{R}^n$ there corresponds a nonempty set $F(x) \subset \mathbb{R}^n$, then $x \mapsto F(x)$ is called a set-valued map from $\mathbb{E} \hookrightarrow \mathbb{R}^n$. Suppose $\mathbb{E} \subset \mathbb{R}^n$, then a set-valued map F with nonempty values is said to be upper semi-continuous (USC) at $x_0 \in \mathbb{E}$, if for any open set $\mathbb{N}$ containing $F(x_0)$, there exists a neighborhood $\mathbb{M}$ of x_0 such that $F(\mathbb{M}) \subset \mathbb{N}$.

Now we introduce the concept of Filippov solution by constructing the Filippov set-valued map (i.e., Filippov regularization Aubin & Cellina, 1984 and Filippov, 1988). Let τ be a given nonnegative real number and $C = C([-\tau, 0], \mathbb{R}^n)$ denotes the Banach space of continuous functions ϕ mapping the interval $[-\tau, 0]$ into $\mathbb{R}^n$ with the norm $\|\phi\|_C = \sup_{-\tau \le s \le 0} \|\phi(s)\|$. If for $b \in (0, +\infty]$, $x(t) : [-\tau, b) \to \mathbb{R}^n$ is continuous, then $x_t \in C$ is defined by $x_t(\theta) = x(t+\theta)$, $-\tau \le \theta \le 0$ for any $t \in [0, b)$. Consider the following non-autonomous delayed differential equation of the vector form:

$$\frac{dx}{dt} = f(t, x_t), \tag{2}$$

where $x_t(\cdot)$ denotes the history of the state from time $t - \tau$, up to the present time t; dx/dt denotes the time derivative of x and $f : \mathbb{R} \times C \to \mathbb{R}^n$ is measurable and essentially locally bounded. In this case, $f(t, x_t)$ is allowed to be discontinuous in x_t.

Let us consider the Filippov set-valued map $F : \mathbb{R} \times C \to 2^{\mathbb{R}^n}$ defined as follows

$$F(t, x_t) = \bigcap_{\delta>0} \bigcap_{\text{meas}(\mathbb{N})=0} \overline{co}[f(t, \mathcal{B}(x_t, \delta) \setminus \mathbb{N})]. \tag{3}$$

Here $\overline{co}[\mathbb{E}]$ is the closure of the convex hull of some set $\mathbb{E}$; intersection is taken over all sets $\mathbb{N}$ of Lebesgue measure zero and over all $\delta > 0$; $\mathcal{B}(x_t, \delta) := \{x_t^* \in C \mid \|x_t^* - x_t\|_C < \delta\}$; meas($\mathbb{N}$) denotes the Lebesgue measure of set $\mathbb{N}$.

Definition 2. A vector-valued function $x(t)$ defined on a non-degenerate interval $\mathbb{I} \subseteq \mathbb{R}$ is called a Filippov solution for delayed differential equation (2), if it is absolutely continuous on any compact subinterval $[t_1, t_2]$ of $\mathbb{I}$, and $x(t)$ satisfies the following delayed differential inclusion

$$\frac{dx}{dt} \in F(t, x_t), \quad \text{for a.e. } t \in \mathbb{I}. \tag{4}$$

Since the neuron activation $f_j(\cdot)$ in system (1) is defined as discontinuous function, we need to specify what is meant by a solution of the differential equation system (1) with a discontinuous right-hand side. For this purpose, by applying the above theories of set-valued maps and delayed differential inclusions, we extend the concept of the Filippov solution to the discontinuous and delayed neural network system (1) as follows:

Definition 3. A vector-valued function $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^{\mathrm{T}} : [-\tau, b) \to \mathbb{R}^n$, $b \in (0, +\infty]$, is a state solution of the delayed and discontinuous system (1) on $[-\tau, b)$ if

(i) x is continuous on $[-\tau, b)$ and absolutely continuous on any compact subinterval of $[0, b)$;
(ii) there exists a measurable function $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)^{\mathrm{T}} : [-\tau, b) \to \mathbb{R}^n$ such that $\gamma_j(t) \in \overline{co}[f_j(x_j(t))]$ for a.e. $t \in [-\tau, b)$ and

$$\frac{dx_i(t)}{dt} = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij}\gamma_j(t - \tau(t)) + I_i,$$
$$\text{for a.e. } t \in [0, b),\ i = 1, 2, \ldots, n. \tag{5}$$

Any function $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)^{\mathrm{T}}$ satisfying (5) is called an output solution associated with the state x. With this definition it turns out that the state $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^{\mathrm{T}}$ is a solution of (1) in the sense of Filippov since it satisfies

$$\frac{dx_i(t)}{dt} \in -c_i x_i(t) + \sum_{j=1}^{n} a_{ij}\overline{co}[f_j(x_j(t - \tau(t)))] + I_i,$$
$$\text{for a.e. } t \in [0, b),\ i = 1, 2, \ldots, n.$$

In this paper, the trajectory of the solution of neural network (1) is assumed to be chaotic. The next definition is the initial value problem (IVP) associated with (1) as follows.

Definition 4 (*IVP*)**.** For any continuous function $\phi = (\phi_1, \phi_2, \ldots, \phi_n)^{\mathrm{T}} : [-\tau, 0] \to \mathbb{R}^n$ and any measurable selection $\psi = (\psi_1, \psi_2, \ldots, \psi_n)^{\mathrm{T}} : [-\tau, 0] \to \mathbb{R}^n$, such that $\psi_j(s) \in \overline{co}[f_j(\phi_j(s))]$ ($j = 1, 2, \ldots, n$) for a.e. $s \in [-\tau, 0]$, an absolute continuous function $x(t) = x(t, \phi, \psi)$ associated with a measurable function γ is said to be a solution of the Cauchy problem for system (1) on $[-\tau, b)$ (b might be $+\infty$) with initial value $(\phi(s), \psi(s))$, $s \in [-\tau, 0]$, if

$$\begin{cases} \dfrac{dx_i(t)}{dt} = -c_i x_i(t) + \displaystyle\sum_{j=1}^{n} a_{ij}\gamma_j(t - \tau(t)) + I_i, \\ \quad \text{for a.e. } t \in [0, b), \\ \gamma_j(t) \in \overline{co}[f_j(x_j(t))], \quad \text{for a.e. } t \in [0, b), \\ x(s) = \phi(s), \quad \forall s \in [-\tau, 0], \\ \gamma(s) = \psi(s), \quad \text{for a.e. } s \in [-\tau, 0]. \end{cases} \tag{6}$$

Remark 2. Suppose that the conditions (H1) and (H2) are satisfied, then the growth condition (8) in Theorem 1 of Liu et al. (2011) holds. Therefore, any IVP for (1) has at least one solution x on $[0, +\infty)$.

Consider the neural network model (1) as the driver system, the controlled response system can be described as follows:

$$\frac{dy_i(t)}{dt} = -c_i y_i(t) + \sum_{j=1}^{n} a_{ij} f_j(y_j(t-\tau(t))) + I_i + u_i(t), \quad i = 1, 2, \ldots, n, \tag{7}$$

where $u_i(t)$ is the controller to be designed for reaching synchronization of the drive–response system. The other parameters are the same as those defined in system (1).

According to Definition 3 and Remark 2, we can obtain the initial value problem (IVP) of response system (7) as follows:

$$\begin{cases} \dfrac{dy_i(t)}{dt} = -c_i y_i(t) + \sum_{j=1}^{n} a_{ij} \zeta_j(t-\tau(t)) + I_i + u_i(t), \\ \quad \text{for a.e. } t \in [0, +\infty), \\ \zeta_j(t) \in \overline{\mathrm{co}}[f_j(y_j(t))], \quad \text{for a.e. } t \in [0, +\infty), \\ y(s) = \nu(s), \quad \forall s \in [-\tau, 0], \\ \zeta(s) = \omega(s), \quad \text{for a.e. } s \in [-\tau, 0]. \end{cases} \tag{8}$$

For the sake of convenience, we denote

$$\|\phi - \nu\|_1 = \sup_{-\tau \le s \le 0} \sum_{i=1}^{n} |\phi_i(s) - \nu_i(s)|,$$

$$\|\phi - \nu\|_2 = \sup_{-\tau \le s \le 0} \sum_{i=1}^{n} (\phi_i(s) - \nu_i(s))^2.$$

Definition 5. The drive system (1) and the response system (7) with discontinuous neuron activations are said to be

- the first type globally exponentially synchronized if there exist positive constants $\mathscr{M} \ge 1$ and $\alpha > 0$ such that

$$\sum_{i=1}^{n} |y_i(t) - x_i(t)| \le \mathscr{M} \|\phi - \nu\|_1 \exp\{-\alpha t\} \quad \text{for all } t \ge 0;$$

- the second type globally exponentially synchronized if there exist positive constants $\mathscr{M} \ge 1$ and $\alpha > 0$ such that

$$\sum_{i=1}^{n} (y_i(t) - x_i(t))^2 \le \mathscr{M} \|\phi - \nu\|_2 \exp\{-\alpha t\} \quad \text{for all } t \ge 0.$$

Here, α is called the estimated rate of exponential synchronization.

Definition 6 (*Regular* (Clarke, 1983)). Given a locally Lipschitz function $V(x) : \mathbb{R}^n \to \mathbb{R}$, the usual right-sided directional derivative of V at x in the direction $\nu \in \mathbb{R}^n$ is defined as

$$D^+V(x, \nu) = \lim_{\hbar \to 0^+} \frac{V(x + \hbar\nu) - V(x)}{\hbar}$$

when this limit exists. The generalized directional derivative of V at x in the direction $\nu \in \mathbb{R}^n$ is defined as

$$\overline{D}_C V(x, \nu) = \limsup_{\hbar \to 0^+, z \to x} \frac{V(z + \hbar\nu) - V(z)}{\hbar}.$$

We say the function $V(x) : \mathbb{R}^n \to \mathbb{R}$ is regular at $x \in \mathbb{R}^n$, if for each $\nu \in \mathbb{R}^n$, the usual right-sided directional derivative of V at x in the direction $\nu \in \mathbb{R}^n$ exists, and $D^+V(x, \nu) = \overline{D}_C V(x, \nu)$. The function $V(x)$ is said to be regular in $\mathbb{R}^n$, if it is regular for any $x \in \mathbb{R}^n$.

Definition 7 (*C-regular* (Clarke, 1983)). Function $V(x) : \mathbb{R}^n \to \mathbb{R}$ is said to be C-regular, if and only if $V(x)$ is

(i) regular in $\mathbb{R}^n$;
(ii) positive definite, i.e., we have $V(x) > 0$ for $x \neq 0$, and $V(0) = 0$;
(iii) radially unbounded, that is, $V(x) \to +\infty$ as $\|x\| \to +\infty$.

Lemma 1 (*Chain Rule* (Clarke, 1983; Forti et al., 2006)). *If $V(x) : \mathbb{R}^n \to \mathbb{R}$ is C-regular, and $x(t) : [0, +\infty) \to \mathbb{R}^n$ is absolutely continuous on any compact subinterval of $[0, +\infty)$. Then, $x(t)$ and $V(x(t)) : [0, +\infty) \to \mathbb{R}$ are differential for almost all $t \in [0, +\infty)$ and*

$$\frac{dV(x(t))}{dt} = \left\langle \varsigma(t), \frac{dx(t)}{dt} \right\rangle, \quad \forall \varsigma(t) \in \partial V(x(t)).$$

3. Main results

In this section, we consider the global exponential synchronization of time-varying delayed neural networks with discontinuous activations by using discontinuous state-feedback controller. Based on extended Filippov-framework and some analytic techniques, we propose a series of new criteria for synchronization which are different from those of the existing literature. Now let us define the synchronization error between the drive and the response as follows

$$e_i(t) = y_i(t) - x_i(t), \quad i = 1, 2, \ldots, n.$$

Then, from (6) and (8), we can obtain the following synchronization error system

$$\frac{de_i(t)}{dt} = -c_i e_i(t) + \sum_{j=1}^{n} a_{ij} \beta_j(t-\tau(t)) + u_i(t), \quad \text{for a.e. } t \in [0, +\infty), \ i = 1, 2, \ldots, n, \tag{9}$$

where $\beta_j(t-\tau(t)) = \zeta_j(t-\tau(t)) - \gamma_j(t-\tau(t))$. In order to realize synchronization goal, we choose the following discontinuous state-feedback controller

$$u_i(t) = -\kappa_i e_i(t) - \eta_i \mathrm{sign}(e_i(t)), \tag{10}$$

where κ_i and η_i are control gains to be determined.

Theorem 1. *Under the assumptions* (H1) *and* (H2), *suppose further that*

(H4) $\eta_i \ge \sum_{j=1}^{n} |a_{ij}| p_j$, *and* $\mathscr{A} < \mathscr{C}$, *where* $\mathscr{A} = \max_{1 \le j \le n} \left(\sum_{i=1}^{n} |a_{ij}| L_j \right)$, $\mathscr{C} = \min_{1 \le i \le n} \{c_i + \kappa_i\}$.

Then the discontinuous neural networks (1) *and* (7) *can achieve the first type global exponential synchronization under the state-feedback controller* (10), *where the estimated rate μ^* of exponential synchronization is the unique root of the following algebra equation*

$$\mu = \mathscr{C} - \mathscr{A} \exp\{\mu\tau\}. \tag{11}$$

Proof. Substituting the state-feedback controller (10) into the synchronization error system (9), we have

$$\frac{de_i(t)}{dt} = -(c_i + \kappa_i) e_i(t) + \sum_{j=1}^{n} a_{ij} \beta_j(t-\tau(t)) - \eta_i \mathrm{sign}(e_i(t)), \quad \text{for a.e. } t \in [0, +\infty). \tag{12}$$

Consider the following positive radially unbounded auxiliary function for the system (12):

$$V_i(t) = |e_i(t)|. \tag{13}$$

It is easy to see that $V_i(t)$ is C-regular. Notice that the function $|e_i|$ is locally Lipschitz continuous in e_i on $\mathbb{R}$. According to the definition of Clarke's generalized gradient of function $|e_i(t)|$ at $e_i(t)$, we have

$$\partial(|e_i(t)|) = \overline{\mathrm{co}}[\mathrm{sign}(e_i(t))] = \begin{cases} \{-1\}, & \text{if } e_i(t) < 0, \\ [-1, 1], & \text{if } e_i(t) = 0, \\ \{1\}, & \text{if } e_i(t) > 0. \end{cases}$$

That is, for any $\varsigma_i(t) \in \partial(|e_i(t)|)$, we have $\varsigma_i(t) = \mathrm{sign}(e_i(t))$, if $e_i(t) \neq 0$; while $\varsigma_i(t)$ can be arbitrarily chosen in $[-1, 1]$, if $e_i(t) = 0$. Especially, we choose $\varsigma_i(t) = \mathrm{sign}(e_i(t))$. Obviously, it can be seen that $\varsigma_i(t)e_i(t) = |e_i(t)|$. By the chain rule in Lemma 1, computing the time derivative of $V_i(t)$ along the trajectories of error system (12), we can obtain

$$\begin{aligned} \frac{\mathrm{d}|e_i(t)|}{\mathrm{d}t} &= \frac{\mathrm{d}e_i(t)}{\mathrm{d}t}\varsigma_i(t) \\ &= \left[-(c_i+\kappa_i)e_i(t) + \sum_{j=1}^n a_{ij}\beta_j(t-\tau(t)) \right. \\ &\quad \left. - \eta_i \mathrm{sign}(e_i(t))\right] \mathrm{sign}(e_i(t)) \\ &= -(c_i+\kappa_i)|e_i(t)| + \sum_{j=1}^n a_{ij}\beta_j(t-\tau(t))\mathrm{sign}(e_i(t)) \\ &\quad - \eta_i|\mathrm{sign}(e_i(t))|, \quad \text{for a.e. } t \in [0, +\infty). \end{aligned} \tag{14}$$

Applying the variation-of-constants formula to system (14), we can obtain that

$$\begin{aligned} |e_i(t)| &= |e_i(0)| \exp\{-(c_i+\kappa_i)t\} \\ &\quad + \int_0^t \exp\{-(c_i+\kappa_i)(t-s)\} \left[\sum_{j=1}^n a_{ij}\beta_j(s-\tau(s)) \right. \\ &\quad \left. \times \mathrm{sign}(e_i(s)) - \eta_i|\mathrm{sign}(e_i(s))|\right] \mathrm{d}s. \end{aligned} \tag{15}$$

Here, the symbol "$\int$" denotes the Lebesgue integration. Therefore, under the conditions (H2) and (H4), we can derive from (15) that

$$\begin{aligned} |e_i(t)| &\le |e_i(0)| \exp\{-(c_i+\kappa_i)t\} \\ &\quad + \int_0^t \exp\{-(c_i+\kappa_i)(t-s)\} \\ &\quad \times \left[\sum_{j=1}^n |a_{ij}|\,|\beta_j(s-\tau(s))| - \eta_i\right] |\mathrm{sign}(e_i(s))|\mathrm{d}s \\ &\le |e_i(0)| \exp\{-(c_i+\kappa_i)t\} \\ &\quad + \int_0^t \exp\{-(c_i+\kappa_i)(t-s)\} \left[\sum_{j=1}^n |a_{ij}| \right. \\ &\quad \left. \times (L_j|e_j(s-\tau(s))| + p_j) - \eta_i\right] |\mathrm{sign}(e_i(s))|\mathrm{d}s \\ &\le |e_i(0)| \exp\{-(c_i+\kappa_i)t\} + \int_0^t \exp\{-(c_i+\kappa_i)(t-s)\} \\ &\quad \times \sum_{j=1}^n |a_{ij}|L_j|e_j(s-\tau(s))|\mathrm{d}s \\ &\le |e_i(0)| \exp\{-\mathscr{C}t\} + \int_0^t \exp\{-\mathscr{C}(t-s)\} \\ &\quad \times \sum_{j=1}^n |a_{ij}|L_j|e_j(s-\tau(s))|\mathrm{d}s, \end{aligned} \tag{16}$$

where $\mathscr{C} = \min_{1\le i\le n}\{c_i+\kappa_i\}$. Summing up both sides of (16), we have

$$\begin{aligned} \sum_{i=1}^n |e_i(t)| &\le \sum_{i=1}^n |e_i(0)| \exp\{-\mathscr{C}t\} + \int_0^t \exp\{-\mathscr{C}(t-s)\} \\ &\quad \times \sum_{i=1}^n \sum_{j=1}^n |a_{ij}|L_j|e_j(s-\tau(s))|\mathrm{d}s \\ &= \sum_{i=1}^n |e_i(0)| \exp\{-\mathscr{C}t\} + \int_0^t \exp\{-\mathscr{C}(t-s)\} \\ &\quad \times \sum_{j=1}^n \sum_{i=1}^n |a_{ij}|L_j|e_j(s-\tau(s))|\mathrm{d}s \\ &\le \sum_{i=1}^n |e_i(0)| \exp\{-\mathscr{C}t\} + \max_{1\le j\le n}\left(\sum_{i=1}^n |a_{ij}|L_j\right) \\ &\quad \times \int_0^t \exp\{-\mathscr{C}(t-s)\} \sum_{j=1}^n |e_j(s-\tau(s))|\mathrm{d}s. \end{aligned} \tag{17}$$

For the sake of simplification, we set

$$e^{\mathrm{sum}}(t) = \sum_{i=1}^n |e_i(t)| = \sum_{i=1}^n |y_i(t) - x_i(t)|. \tag{18}$$

Then, from (17) and (18), we can obtain

$$\begin{aligned} e^{\mathrm{sum}}(t) &\le e^{\mathrm{sum}}(0) \exp\{-\mathscr{C}t\} \\ &\quad + \mathscr{A} \int_0^t \exp\{-\mathscr{C}(t-s)\} e^{\mathrm{sum}}(s-\tau(s))\mathrm{d}s, \end{aligned} \tag{19}$$

where $\mathscr{A} = \max_{1\le j\le n}\left(\sum_{i=1}^n |a_{ij}|L_j\right)$. Obviously, $e^{\mathrm{sum}}(s-\tau(s))$ is Riemann integrable due to its continuity. Notice that any Riemann integrable functions are Lebesgue integrable. So if we replace the Lebesgue integration in (19) with Riemann integration, the inequality (19) is still true. For convenience, we also use "$\int$" to denote the Riemann integration in the ensuing discussion. Now, let us denote

$$\mathscr{N} = \sup_{-\tau\le s\le 0} \sum_{i=1}^n |e_i(s)| = \|\phi - \nu\|_1.$$

Define the function $W(t)$ on $[-\tau, +\infty)$ as follows:

$$W(t) = \begin{cases} \mathscr{N} \exp\{-\mathscr{C}t\} + \mathscr{A} \displaystyle\int_0^t \exp\{-\mathscr{C}(t-s)\} \\ \quad \times e^{\mathrm{sum}}(s-\tau(s))\mathrm{d}s, & \text{if } t > 0, \\ \mathscr{N}, & \text{if } -\tau \le t \le 0. \end{cases} \tag{20}$$

It is clear that $W(t)$ is a continuous differential function on $[-\tau, +\infty)$.

On the other hand, we set

$$U(t) = \begin{cases} e^{\mathrm{sum}}(t), & \text{if } t > 0, \\ \mathscr{N}, & \text{if } -\tau \le t \le 0. \end{cases} \tag{21}$$

Obviously, for any $t \ge -\tau$, we obtain from (19)–(21) that

$$U(t) \le W(t). \tag{22}$$

For $t > 0$, calculating the time derivative of $W(t)$, we have

$$\begin{aligned} \frac{\mathrm{d}W(t)}{\mathrm{d}t} &= -\mathscr{C}\mathscr{N}\exp\{-\mathscr{C}t\} - \mathscr{C}\mathscr{A}\int_0^t \exp\{-\mathscr{C}(t-s)\} \\ &\quad \times e^{\mathrm{sum}}(s-\tau(s))\mathrm{d}s + \mathscr{A}e^{\mathrm{sum}}(t-\tau(t)) \\ &= -\mathscr{C}W(t) + \mathscr{A}e^{\mathrm{sum}}(t-\tau(t)) \\ &\le -\mathscr{C}W(t) + \mathscr{A}W(t-\tau(t)) \\ &\le -\mathscr{C}W(t) + \mathscr{A}\sup_{t-\tau(t)\le s\le t} W(s). \end{aligned} \tag{23}$$

Set

$$H(t)=\mathcal{N}\exp\{-\mu^{*}t\},\quad \text{for } t\geq -\tau, \tag{24}$$

where $\mu^{*}=\mu^{*}(\mathscr{C},\mathscr{A},\tau)$ is the unique root of the following algebra equation:

$$\mu=\mathscr{C}-\mathscr{A}\exp\{\mu\tau\}.$$

Actually, it is noted that $g(\mu)=\mu-\mathscr{C}+\mathscr{A}\exp\{\mu\tau\}$ is a strictly increasing function with respect to μ on $\mathbb{R}$ because $\frac{\mathrm{d}g(\mu)}{\mathrm{d}\mu}=1+\mathscr{A}\tau\exp\{\mu\tau\}>0$. Meanwhile, we can obtain from assumption (H4) that $g(0)=\mathscr{A}-\mathscr{C}<0$ and $g(\mathscr{C})=\mathscr{A}\exp\{\mathscr{C}\tau\}>0$. This shows that the equation $g(\mu)=0$ possesses a unique root $\mu=\mu^{*}$ on the interval $[0,\mathscr{C}]$.

Next, let $\lambda>1$ be a constant. For $-\tau\leq t\leq 0$, it follows from (20) and (24) that

$$W(t)=\mathcal{N}=H(t)\exp\{\mu^{*}t\}\leq H(t)\leq\lambda H(t).$$

That is

$$W(t)\leq\lambda H(t),\quad \text{for } -\tau\leq t\leq 0. \tag{25}$$

We claim that the inequality (25) still holds for all $t>0$. If this is not true, then there exists a real number $t^{*}\in(0,+\infty)$ such that

$$W(t)\leq\lambda H(t)\quad \text{for } -\tau\leq t<t^{*},\quad \text{and}$$
$$W(t^{*})=\lambda H(t^{*}),\qquad \left.\frac{\mathrm{d}W(t)}{\mathrm{d}t}\right|_{t=t^{*}}>\lambda\left.\frac{\mathrm{d}H(t)}{\mathrm{d}t}\right|_{t=t^{*}}. \tag{26}$$

It follows from (23) and (24) that

$$\begin{aligned}\left.\frac{\mathrm{d}W(t)}{\mathrm{d}t}\right|_{t=t^{*}}&\leq-\mathscr{C}W(t^{*})+\mathscr{A}\sup_{t^{*}-\tau(t^{*})\leq s\leq t^{*}}W(s)\\&=-\mathscr{C}\lambda H(t^{*})+\mathscr{A}\sup_{t^{*}-\tau(t^{*})\leq s\leq t^{*}}W(s)\\&\leq-\mathscr{C}\lambda H(t^{*})+\mathscr{A}\lambda\sup_{t^{*}-\tau(t^{*})\leq s\leq t^{*}}H(s)\\&=-\mathscr{C}\lambda H(t^{*})+\mathscr{A}\lambda\sup_{t^{*}-\tau(t^{*})\leq s\leq t^{*}}\mathcal{N}\exp\{-\mu^{*}s\}\\&\leq-\mathscr{C}\lambda H(t^{*})+\mathscr{A}\lambda\mathcal{N}\exp\{-\mu^{*}(t^{*}-\tau(t^{*}))\}\\&=-\mathscr{C}\lambda H(t^{*})+\mathscr{A}\lambda\mathcal{N}\exp\{-\mu^{*}t^{*}\}\exp\{\mu^{*}\tau(t^{*})\}\\&\leq-\mathscr{C}\lambda H(t^{*})+\mathscr{A}\lambda H(t^{*})\exp\{\mu^{*}\tau\}\\&=-\lambda\left(\mathscr{C}-\mathscr{A}\exp\{\mu^{*}\tau\}\right)H(t^{*})\\&=-\lambda\mu^{*}H(t^{*})\\&=\lambda\left.\frac{\mathrm{d}H(t)}{\mathrm{d}t}\right|_{t=t^{*}}.\end{aligned} \tag{27}$$

That is

$$\left.\frac{\mathrm{d}W(t)}{\mathrm{d}t}\right|_{t=t^{*}}\leq\lambda\left.\frac{\mathrm{d}H(t)}{\mathrm{d}t}\right|_{t=t^{*}}.$$

In view of (26), this is a contradiction. Therefore, we have

$$W(t)\leq\lambda H(t),\quad \text{for } t\geq 0. \tag{28}$$

Consequently, we can derive from (22), (24) and (28) that

$$U(t)\leq W(t)\leq\lambda H(t)=\lambda\mathcal{N}\exp\{-\mu^{*}t\},\quad \text{for } t\geq 0.$$

This means that

$$e^{\mathrm{sum}}(t)\leq\lambda\mathcal{N}\exp\{-\mu^{*}t\},\quad \text{for } t\geq 0.$$

Let $\lambda\to 1$, we can obtain that

$$e^{\mathrm{sum}}(t)\leq\mathcal{N}\exp\{-\mu^{*}t\},\quad \text{for } t\geq 0.$$

That is

$$\sum_{i=1}^{n}|y_i(t)-x_i(t)|\leq\|\phi-\nu\|_1\exp\{-\mu^{*}t\},\quad \text{for all } t\geq 0.$$

According to Definition 5, the discontinuous and delayed neural networks (1) and (7) achieve the first type global exponential synchronization under the state-feedback controller (10). The proof is complete.

Theorem 2. *Under the assumptions* (H1) *and* (H2), *suppose further that*

(H5) $\eta_i\geq\sum_{j=1}^{n}|a_{ij}|p_j$, *and* $\hat{\mathscr{A}}+\mathscr{B}<2\mathscr{C}$, *where*

$$\hat{\mathscr{A}}=\max_{1\leq j\leq n}\left(\sum_{i=1}^{n}|a_{ij}|L_j^2\right),\qquad \mathscr{B}=\max_{1\leq i\leq n}\left(\sum_{j=1}^{n}|a_{ij}|\right),$$
$$\mathscr{C}=\min_{1\leq i\leq n}\{c_i+\kappa_i\}.$$

Then the discontinuous neural networks (1) *and* (7) *can achieve the second type global exponential synchronization under the state-feedback controller* (10), *where the estimated rate* μ^{*} *of exponential synchronization is the unique root of the following algebra equation*

$$\mu=2\mathscr{C}-\mathscr{B}-\hat{\mathscr{A}}\exp\{\mu\tau\}. \tag{29}$$

Proof. Multiplying both sides of (12) by $2e_i(t)$, we have

$$\begin{aligned}\frac{\mathrm{d}e_i^2(t)}{\mathrm{d}t}&=-2(c_i+\kappa_i)e_i^2(t)+2\sum_{j=1}^{n}a_{ij}e_i(t)\beta_j(t-\tau(t))\\&\quad-2\eta_i|e_i(t)|,\quad \text{for a.e. } t\in[0,+\infty).\end{aligned} \tag{30}$$

Applying the variation-of-constants formula to system (30), we can obtain that

$$\begin{aligned}e_i^2(t)&=e_i^2(0)\exp\{-2(c_i+\kappa_i)t\}+2\int_0^t\exp\{-2(c_i+\kappa_i)(t-s)\}\\&\quad\times\left[\sum_{j=1}^{n}a_{ij}e_i(s)\beta_j(s-\tau(s))-\eta_i|e_i(s)|\right]\mathrm{d}s.\end{aligned} \tag{31}$$

Similar to (15), the symbol "$\int$" denotes the Lebesgue integration. Thus, taking the conditions (H2) and (H5) into account, by using the element inequality $2ab\leq a^2+b^2$, we can get from (31) that

$$\begin{aligned}e_i^2(t)&\leq e_i^2(0)\exp\{-2(c_i+\kappa_i)t\}+2\int_0^t\exp\{-2(c_i+\kappa_i)(t-s)\}\\&\quad\times\left[\sum_{j=1}^{n}|a_{ij}|\,|e_i(s)|\,|\beta_j(s-\tau(s))|-\eta_i|e_i(s)|\right]\mathrm{d}s\\&\leq e_i^2(0)\exp\{-2(c_i+\kappa_i)t\}\\&\quad+2\int_0^t\exp\{-2(c_i+\kappa_i)(t-s)\}\left[\sum_{j=1}^{n}|a_{ij}|\,|e_i(s)|\right.\\&\quad\left.\times(L_j|e_j(s-\tau(s))|+p_j)-\eta_i|e_i(s)|\right]\mathrm{d}s\\&\leq e_i^2(0)\exp\{-2(c_i+\kappa_i)t\}+2\int_0^t\exp\{-2(c_i+\kappa_i)(t-s)\}\\&\quad\times\sum_{j=1}^{n}|a_{ij}|\,|e_i(s)|L_j|e_j(s-\tau(s))|\mathrm{d}s\\&\leq e_i^2(0)\exp\{-2(c_i+\kappa_i)t\}+\sum_{j=1}^{n}|a_{ij}|\end{aligned}$$

$$
\begin{aligned}
&\times \int_0^t \exp\{-2(c_i+\kappa_i)(t-s)\}e_i^2(s)\mathrm{d}s \\
&+ \int_0^t \exp\{-2(c_i+\kappa_i)(t-s)\} \\
&\times \sum_{j=1}^n |a_{ij}|L_j^2 e_j^2(s-\tau(s))\mathrm{d}s \\
\le\ & e_i^2(0)\exp\{-2\mathscr{C}t\} \\
&+ \sum_{j=1}^n |a_{ij}| \int_0^t \exp\{-2\mathscr{C}(t-s)\}e_i^2(s)\mathrm{d}s \\
&+ \int_0^t \exp\{-2\mathscr{C}(t-s)\}\sum_{j=1}^n |a_{ij}|L_j^2 e_j^2(s-\tau(s))\mathrm{d}s, \qquad (32)
\end{aligned}
$$

where $\mathscr{C} = \min_{1\le i\le n}\{c_i+\kappa_i\}$. Making sum on both sides of the above inequality (32), we can obtain

$$
\begin{aligned}
\sum_{i=1}^n e_i^2(t) \le\ & \sum_{i=1}^n e_i^2(0)\exp\{-2\mathscr{C}t\} \\
&+ \sum_{i=1}^n\sum_{j=1}^n |a_{ij}| \int_0^t \exp\{-2\mathscr{C}(t-s)\}e_i^2(s)\mathrm{d}s \\
&+ \int_0^t \exp\{-2\mathscr{C}(t-s)\} \\
&\times \sum_{i=1}^n\sum_{j=1}^n |a_{ij}|L_j^2 e_j^2(s-\tau(s))\mathrm{d}s \\
\le\ & \sum_{i=1}^n e_i^2(0)\exp\{-2\mathscr{C}t\} + \max_{1\le i\le n}\left(\sum_{j=1}^n |a_{ij}|\right) \\
&\times \int_0^t \exp\{-2\mathscr{C}(t-s)\}\sum_{i=1}^n e_i^2(s)\mathrm{d}s \\
&+ \max_{1\le j\le n}\left(\sum_{i=1}^n |a_{ij}|L_j^2\right)\int_0^t \exp\{-2\mathscr{C}(t-s)\} \\
&\times \sum_{j=1}^n e_j^2(s-\tau(s))\mathrm{d}s. \qquad (33)
\end{aligned}
$$

For the simplification, we set

$$\hat{e}^{\mathrm{sum}}(t) = \sum_{i=1}^n e_i^2(t) = \sum_{i=1}^n (y_i(t)-x_i(t))^2. \qquad (34)$$

Then, from (33) and (34), we have

$$
\begin{aligned}
\hat{e}^{\mathrm{sum}}(t) \le\ & \hat{e}^{\mathrm{sum}}(0)\exp\{-2\mathscr{C}t\} \\
&+ \mathscr{B}\int_0^t \exp\{-2\mathscr{C}(t-s)\}\hat{e}^{\mathrm{sum}}(s)\mathrm{d}s \\
&+ \hat{\mathscr{A}}\int_0^t \exp\{-2\mathscr{C}(t-s)\}\hat{e}^{\mathrm{sum}}(s-\tau(s))\mathrm{d}s, \qquad (35)
\end{aligned}
$$

where $\mathscr{B} = \max_{1\le i\le n}\left(\sum_{j=1}^n |a_{ij}|\right)$ and $\hat{\mathscr{A}} = \max_{1\le j\le n}\left(\sum_{i=1}^n |a_{ij}| L_j^2\right)$. Similarly, $\hat{e}^{\mathrm{sum}}(s)$ and $\hat{e}^{\mathrm{sum}}(s-\tau(s))$ are Riemann integrable due to their continuities. And if we replace the Lebesgue integration in (35) with Riemann integration, the above inequality (35) is still true. In the following, we still use "$\int$" to denote the Riemann integration. For convenience, we let

$$\hat{\mathscr{N}} = \sup_{-\tau\le s\le 0}\sum_{i=1}^n e_i^2(s) = \|\phi-\nu\|_2.$$

Define the following function $\hat{W}(t)$ on $[-\tau,+\infty)$

$$
\hat{W}(t) = \begin{cases}
\hat{\mathscr{N}}\exp\{-2\mathscr{C}t\} + \mathscr{B}\displaystyle\int_0^t \exp\{-2\mathscr{C}(t-s)\}\hat{e}^{\mathrm{sum}}(s)\mathrm{d}s \\
\quad + \hat{\mathscr{A}}\displaystyle\int_0^t \exp\{-2\mathscr{C}(t-s)\}\hat{e}^{\mathrm{sum}}(s-\tau(s))\mathrm{d}s, \\
\quad \text{if } t>0, \\
\hat{\mathscr{N}}, \quad \text{if } -\tau\le t\le 0.
\end{cases} \qquad (36)
$$

Obviously, the function $\hat{W}(t)$ is continuous differential on $[-\tau,+\infty)$.

On the other hand, let us set

$$
\hat{U}(t) = \begin{cases}
\hat{e}^{\mathrm{sum}}(t), & \text{if } t>0, \\
\hat{\mathscr{N}}, & \text{if } -\tau\le t\le 0.
\end{cases} \qquad (37)
$$

Clearly, for any $t\ge -\tau$, it follows from (35)–(37) that

$$\hat{U}(t) \le \hat{W}(t). \qquad (38)$$

For $t>0$, calculating the time derivative of $\hat{W}(t)$, we can obtain

$$
\begin{aligned}
\frac{\mathrm{d}\hat{W}(t)}{\mathrm{d}t} &= -2\mathscr{C}\hat{W}(t) + \mathscr{B}\hat{e}^{\mathrm{sum}}(t) + \hat{\mathscr{A}}\hat{e}^{\mathrm{sum}}(t-\tau(t)) \\
&\le -2\mathscr{C}\hat{W}(t) + \mathscr{B}\hat{W}(t) + \hat{\mathscr{A}}\hat{W}(t-\tau(t)) \\
&\le (-2\mathscr{C}+\mathscr{B})\hat{W}(t) + \hat{\mathscr{A}}\sup_{t-\tau(t)\le s\le t}\hat{W}(s). \qquad (39)
\end{aligned}
$$

Let us define $\hat{g}(\mu) = \mu - 2\mathscr{C} + \mathscr{B} + \hat{\mathscr{A}}\exp\{\mu\tau\}$. Note that $\hat{g}(\mu)$ is a strictly increasing function with respect to μ on $\mathbb{R}$ due to $\frac{\mathrm{d}\hat{g}(\mu)}{\mathrm{d}\mu} = 1 + \hat{\mathscr{A}}\tau\exp\{\mu\tau\} > 0$. And we can get from assumption (H5) that $\hat{g}(0) = \hat{\mathscr{A}}+\mathscr{B}-2\mathscr{C} < 0$ and $\hat{g}(2\mathscr{C}-\mathscr{B}) = \hat{\mathscr{A}}\exp\{(2\mathscr{C}-\mathscr{B})\tau\} > 0$. This yields that the equation $\hat{g}(\mu) = 0$ possesses a unique root $\mu=\mu^*$ on the interval $[0, 2\mathscr{C}-\mathscr{B}]$. Set

$$\hat{H}(t) = \hat{\mathscr{N}}\exp\{-\mu^* t\}, \quad \text{for } t\ge -\tau, \qquad (40)$$

where $\mu^* = \mu^*(\mathscr{C}, \hat{\mathscr{A}}, \mathscr{B}, \tau)$ is the unique root of the following algebra equation:

$$\mu = 2\mathscr{C} - \mathscr{B} - \hat{\mathscr{A}}\exp\{\mu\tau\}.$$

Now, let $\lambda > 1$ be a constant. For $-\tau\le t\le 0$, we can obtain from (36) and (40) that

$$\hat{W}(t) = \hat{\mathscr{N}} = \hat{H}(t)\exp\{\mu^* t\} \le \hat{H}(t) \le \lambda\hat{H}(t).$$

That is

$$\hat{W}(t) \le \lambda\hat{H}(t), \quad \text{for } -\tau\le t\le 0. \qquad (41)$$

By using a similar argument as that in the proof of Theorem 1, we can prove that

$$\hat{W}(t) \le \lambda\hat{H}(t), \quad \text{for } t>0. \qquad (42)$$

Therefore, we can obtain from (38), (40) and (42) that

$$\hat{U}(t) \le \hat{W}(t) \le \lambda\hat{H}(t) = \lambda\hat{\mathscr{N}}\exp\{-\mu^* t\}, \quad \text{for } t\ge 0.$$

This shows that

$$\hat{e}^{\mathrm{sum}}(t) \le \lambda\hat{\mathscr{N}}\exp\{-\mu^* t\}, \quad \text{for } t\ge 0.$$

Let $\lambda\to 1$, we have

$$\hat{e}^{\mathrm{sum}}(t) \le \hat{\mathscr{N}}\exp\{-\mu^* t\}, \quad \text{for } t\ge 0.$$

That is

$$\sum_{i=1}^n (y_i(t)-x_i(t))^2 \le \|\phi-\nu\|_2\exp\{-\mu^* t\}, \quad \text{for all } t\ge 0.$$

By virtue of Definition 5, the discontinuous and delayed neural networks (1) and (7) achieve the second type global exponential synchronization under the state-feedback controller (10). The proof is complete.

Remark 3. In this section, we apply a new method to neural network with time-varying delay and discontinuous activation. That is, we only use some analytic techniques to study the exponential synchronization under discontinuous state-feedback controller. However, in the existing literature, the Lyapunov–Krasovskii functional method is usually utilized to deal with the synchronization problems (e.g. Liu & Cao, 2010 and Yang & Cao, 2013). Unfortunately, the suitable Lyapunov–Krasovskii functional is difficult to be constructed because its structure might be very complex. So there is not much research concerning synchronization for delayed neural networks with discontinuous activations. Moreover, the new exponential synchronization criteria in Theorems 1 and 2 are simple and can be easily verified.

Remark 4. To the best of our knowledge, there are some results on stability analysis of neural networks by using analytic techniques (e.g., Chen, Cao, & Huang, 2005). However, the neuron activation functions in Chen et al. (2005) are continuous. In this case, the analytic techniques of Chen et al. (2005) are invalid for dealing with discontinuous dynamical systems since the given vector field is no longer continuous. Based on the theory of differential inclusions introduced by Filippov, we have extended the approach to discontinuous systems. So, our approach is novel. On the other hand, only a few papers have investigated the synchronization problems of discontinuous neural networks via new analytic techniques. Future research work should be devoted to find more better tool and technique to study the synchronization and stability issues of discontinuous neuron dynamical systems.

Remark 5. Due to the discontinuity of activation function, the synchronization control is usually not easy to be realized under classical controllers such as linear state-feedback controller $u_i(t) = -\kappa_i e_i(t)$. In Liu, Cao et al. (2012) and Liu et al. (2011), the authors investigated the synchronization issues for delayed neural networks with discontinuous activations. Nevertheless, only the quasi-synchronization can be realized. Thus, by comparison, we find that the results obtained in this paper are better. In fact, we can see from above results that the discontinuous state-feedback controller given in (10) plays a key role to achieve exponential synchronization control.

Remark 6. For discontinuous state-feedback controller (10), if there is coupled control input between different neurons, the synchronization can still be achieved by similar discussion. For the sake of simplicity, we only consider that u_j ($j \neq i$) is not feedback to neuron i and takes as a control input for neuron i. In further research work, we will consider this problem. Moreover, the designing of new controller for synchronization of discontinuous networks is an interesting and challenging topic.

4. Examples and simulations

In this section, we present two numerical examples to check the synchronization criteria given in the previous sections.

Example 1. Consider a two-dimensional delayed neural network system as follows:

$$\begin{cases} \dfrac{dx_1(t)}{dt} = -c_1x_1(t) + \sum_{j=1}^{2} a_{1j}f_j(x_j(t-\tau(t))) + I_1, \\ \dfrac{dx_2(t)}{dt} = -c_2x_2(t) + \sum_{j=1}^{2} a_{2j}f_j(x_j(t-\tau(t))) + I_2, \end{cases} \tag{43}$$

where $c_1 = 1, c_2 = 1.2, a_{11} = -1.5, a_{12} = a_{21} = -0.5, a_{22} = -3$, $I_1 = I_2 = 0$ and $\tau(t) = 1$. The discontinuous activation functions are described by

$$f_i(x_i) = \begin{cases} \tanh(x_i) - 0.1, & x_i \geq 0,\ i = 1, 2, \\ \tanh(x_i) + 0.1, & x_i < 0,\ i = 1, 2. \end{cases}$$

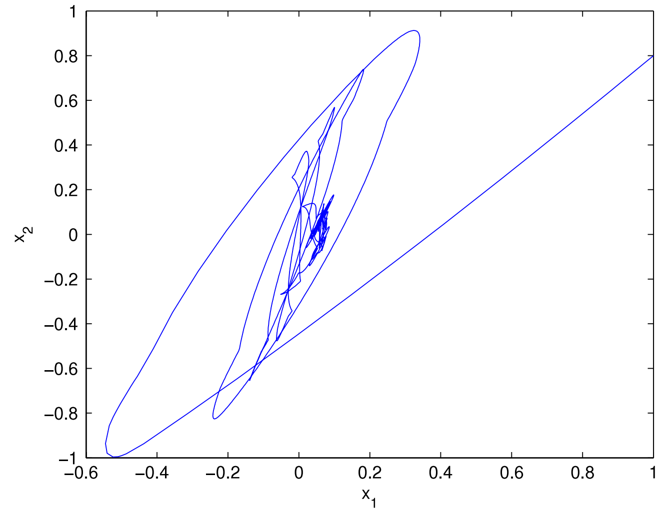

Fig. 2. Phase trajectories of neural network system (43) with initial value $\phi(t) = (1, 0.8)^{\mathrm{T}}$, $t \in [-1, 0]$.

Fig. 2 shows chaotic-like trajectory of system (43) with initial value $\phi(t) = (1, 0.8)^{\mathrm{T}}$, $t \in [-1, 0]$.

Obviously, the discontinuous activation function $f_i(x_i)$ is non-monotonic and satisfies the condition (H1). Meanwhile, 0 is a discontinuous point of the activation function $f_i(\cdot)$ and $\overline{co}[f_i(0)] = [f_i^+(0), f_i^-(0)] = [-0.1, 0.1]$. We can choose $L_1 = L_2 = 1$ and $p_1 = p_2 = 0.2$ such that the condition (H2) holds. Take $k_1 = k_2 = 3$, $\eta_1 = 0.4$ and $\eta_2 = 0.7$. By some simple computations, we have

$$\sum_{j=1}^{2} |a_{1j}|p_j = 0.4, \qquad \sum_{j=1}^{2} |a_{2j}|p_j = 0.7,$$

$$\mathscr{A} = \max_{1\leq j\leq 2}\left(\sum_{i=1}^{2} |a_{ij}|L_j\right) = 3.5, \qquad \mathscr{C} = \min_{1\leq i\leq 2}\{c_i + \kappa_i\} = 4.2.$$

It is easy to check that $\eta_i \geq \sum_{j=1}^{2} |a_{ij}|p_j (i = 1, 2)$ and $\mathscr{A} < \mathscr{C}$. This shows that the condition (H4) is satisfied. According to Theorem 1, the system (43) can achieve the first type global exponential synchronization with the corresponding response system under the state-feedback controller (10).

On the other hand, we can easily calculate that

$$\hat{\mathscr{A}} = \max_{1\leq j\leq 2}\left(\sum_{i=1}^{2} |a_{ij}|L_j^2\right) = 3.5,$$

$$\mathscr{B} = \max_{1\leq i\leq 2}\left(\sum_{j=1}^{2} |a_{ij}|\right) = 3.5, \qquad 2\mathscr{C} = 2\min_{1\leq i\leq 2}\{c_i + \kappa_i\} = 8.4.$$

Clearly, $\hat{\mathscr{A}} + \mathscr{B} < 2\mathscr{C}$ and (H5) is also satisfied. Therefore, by Theorem 2, the system (43) can also achieve the second type global exponential synchronization with the corresponding response system under the state-feedback controller (10).

In the numerical simulations, we take all the initial conditions as $[\phi(t), \psi(t)] = [(1, 0.8)^{\mathrm{T}}, (f_1(1), f_2(0.8))^{\mathrm{T}}]$, for $t \in [-1, 0]$ and $[\nu(t), \omega(t)] = [(-1, -0.5)^{\mathrm{T}}, (f_1(-1), f_2(-0.5))^{\mathrm{T}}]$ for $t \in [-1, 0]$. Figs. 3 and 4 show that the system (43) can also achieve global exponential synchronization with the corresponding response system under the state-feedback controller (10). These numerical simulations confirm the effectiveness of the theoretical results.

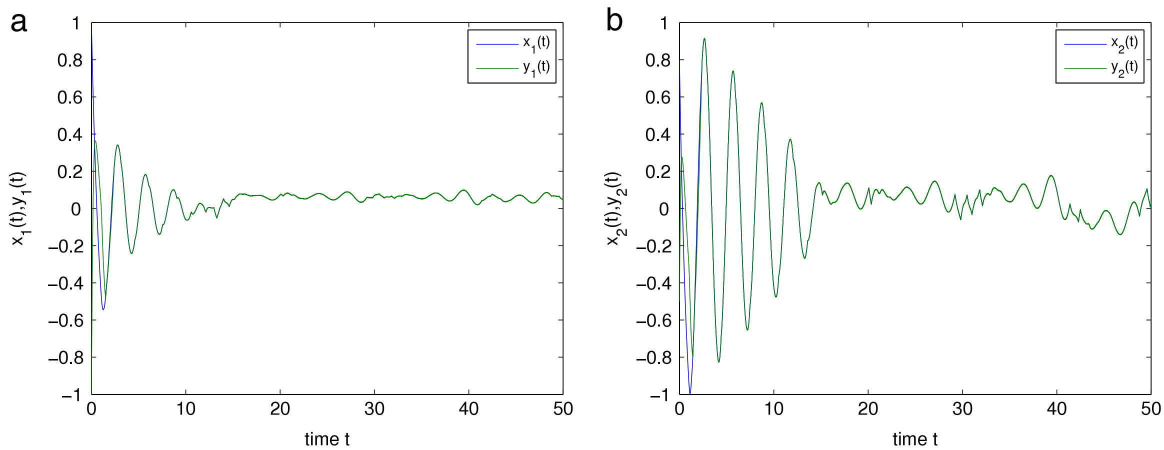

Fig. 3. (a) Time evolution of variables $x_1(t)$ and $y_1(t)$ for neural network (43) (drive system) and corresponding response system; (b) Time evolution of variables $x_2(t)$ and $y_2(t)$ for neural network (43) (drive system) and corresponding response system.

Example 2. Consider the following three-dimensional delayed neural network system:

$$\begin{cases} \dfrac{dx_1(t)}{dt} = -x_1(t) - 0.5f_1(x_1(t-1)) + 0.2f_2(x_2(t-1)), \\ \dfrac{dx_2(t)}{dt} = -x_2(t) + 0.5f_1(x_1(t-1)) - 1.5f_2(x_2(t-1)), \\ \dfrac{dx_3(t)}{dt} = -x_3(t) - 2f_3(x_3(t-1)). \end{cases} \quad (44)$$

The discontinuous activation functions are described by

$$f_i(x_i) = \begin{cases} x_i + 0.2, & x_i \geq 0,\ i = 1, 2, 3, \\ x_i - 0.1, & x_i < 0,\ i = 1, 2, 3. \end{cases}$$

Fig. 5 shows chaotic-like trajectory of system (44) with initial value $\phi(t) = (1, 3, 5)^{\mathrm{T}},\ t \in [-1, 0]$.

We first observe that $c_1 = c_2 = c_3 = 1$, $a_{11} = -0.5$, $a_{12} = 0.2$, $a_{21} = 0.5$, $a_{22} = -1.5$, $a_{33} = -2$, $a_{13} = a_{23} = a_{31} = a_{32} = I_1 = I_2 = I_3 = 0$ and $\tau(t) = 1$. It is clear that the discontinuous activation function $f_i(x_i)$ satisfies the conditions (H1) and (H2) with $L_1 = L_2 = L_3 = 1$ and $p_1 = p_2 = p_3 = 0.3$. Take $k_1 = k_2 = k_3 = 2.5$, $\eta_1 = 1$, $\eta_2 = 1.7$ and $\eta_3 = 2$. By simple calculation, we can obtain

$$\sum_{j=1}^{3} |a_{1j}|p_j = 0.21, \qquad \sum_{j=1}^{3} |a_{2j}|p_j = \sum_{j=1}^{3} |a_{3j}|p_j = 0.6,$$

$$\mathscr{A} = \max_{1\leq j\leq 3}\left(\sum_{i=1}^{3} |a_{ij}|L_j\right) = 2, \qquad \mathscr{C} = \min_{1\leq i\leq 3}\{c_i + \kappa_i\} = 3.5.$$

Obviously, $\eta_i \geq \sum_{j=1}^{3} |a_{ij}|p_j (i = 1, 2, 3)$ and $\mathscr{A} < \mathscr{C}$. So the condition (H4) in Theorem 1 is satisfied and this implies the system (44) can achieve the first type global exponential synchronization with the corresponding response system under the state-feedback controller (10).

In addition, it is not difficult to calculate that

$$\hat{\mathscr{A}} = \max_{1\leq j\leq 3}\left(\sum_{i=1}^{3} |a_{ij}|L_j^2\right) = 2,$$

$$\mathscr{B} = \max_{1\leq i\leq 3}\left(\sum_{j=1}^{3} |a_{ij}|\right) = 2, \qquad 2\mathscr{C} = 2\min_{1\leq i\leq 3}\{c_i + \kappa_i\} = 7.$$

It is straightforward to check that $\hat{\mathscr{A}} + \mathscr{B} < 2\mathscr{C}$. Thus, all the conditions in Theorem 2 hold. Then the system (44) can also achieve the second type global exponential synchronization with the corresponding response system under the state-feedback controller (10). Consider the IVP of system (44) with initial conditions $[\phi(t), \psi(t)] = [(1, 3, 5)^{\mathrm{T}}, (f_1(1), f_2(3), f_3(5))^{\mathrm{T}}]$, for $t \in [-1, 0]$ and $[\nu(t), \omega(t)] = [(2.5, 2, 3)^{\mathrm{T}}, (f_1(2.5), f_2(2), f_3(3))^{\mathrm{T}}]$ for $t \in [-1, 0]$. Fig. 6 presents the trajectory of every error state which approaches to zero quickly as time goes. The above numerical simulations fit the theoretical results perfectly.

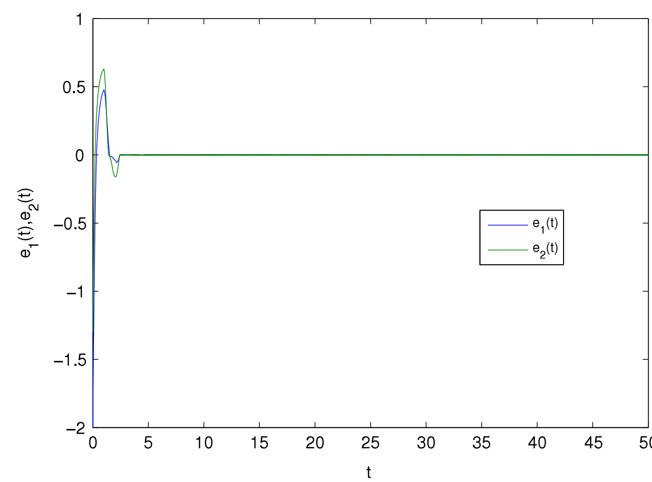

Fig. 4. Time response of synchronization error between drive system (43) and corresponding response system under the state-feedback controller (10).

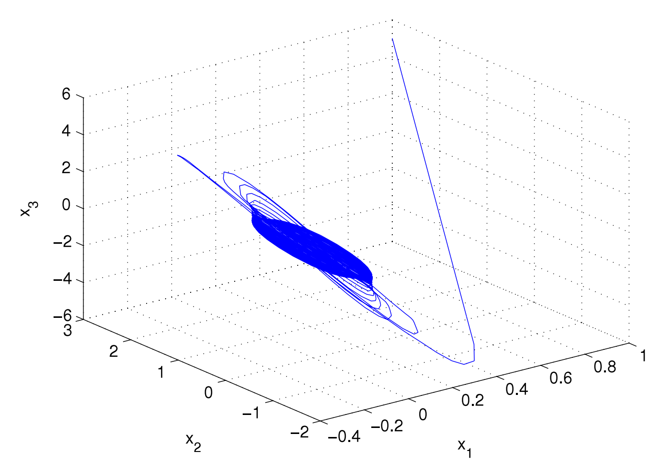

Fig. 5. Phase trajectories of neural network system (44) with initial value $\phi(t) = (1, 3, 5)^{\mathrm{T}}, t \in [-1, 0]$.

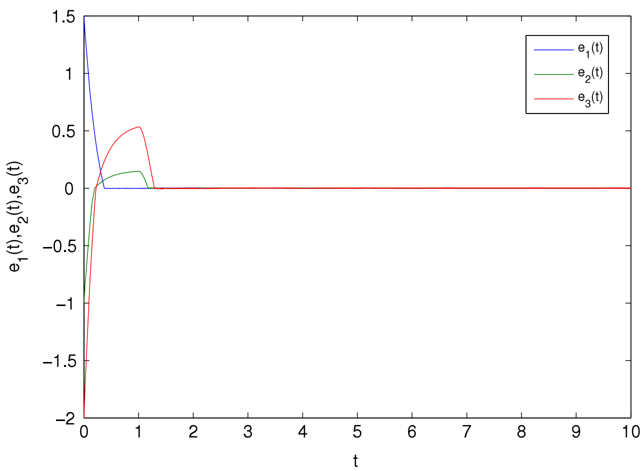

Fig. 6. Time response of synchronization error between drive system (44) and corresponding response system under the state-feedback controller (10).

Remark 7. Due to the special discontinuous switching features of activation functions and the lack of effective analysis methods, the synchronization between the drive system and the response system is difficult to be realized. Under the linear state feedback controller $u_i(t) = -\kappa_i e_i(t)$, only the quasi-synchronization can be achieved in the existing literature (see Liu, Cao et al., 2012 and Liu et al., 2011). That is to say, the synchronization error can only be controlled within a small region around zero, but cannot approach zero with time. However, in this paper, the synchronization error states can approach to zero quickly as time goes under the discontinuous state-feedback controller (10). From the proofs of the main results and numerical simulations, we can see that the discontinuous part $-\eta_i \text{sign}(e_i(t))$ in the controller (10) plays a key role in dealing with the state differences between the drive system and the response system.

5. Conclusions

In this paper, we have introduced an extended Filippov-framework to handle the synchronization of the time-varying delayed neural networks with discontinuous activations. By Filippov regularization, the concept of Filippov solution and the initial value problem for discontinuous and delayed neural networks have been given. In order to achieve synchronization control between the drive system and response system, a discontinuous state-feedback controller has been designed. By employing some new analytic techniques, several sufficient conditions have been derived to ensure the first and second type global exponential synchronization for the drive–response system. As is well known, in the design of neural network circuit for synchronization, it is often desired that the trajectories of the error states converge to zero in an exponential rate for guaranteeing fast response. Therefore, we have also provided the estimated rates of exponential synchronization which are determined by some algebra equations. Moreover, it has been shown that the estimated rates of exponential synchronization are dependent on the neural system parameters and delays. Our method and results are novel since there are few works on synchronization of the time-varying delayed neural networks with discontinuous activations and it is not required to construct suitable Lyapunov–Krasovskii functional. However, only quasi-synchronization can be achieved for neural networks with discontinuous activations in the earlier literature and there still lack effective analysis tools. In short, the analysis method in this paper may open up a new view for the design and application of delayed neural networks with discontinuous activations and other classes of switching networks such as memristor-based neural network.

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