Asset Pricing Under Endogenous Expectations in a Multi-Agent Artificial Stock Market

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1 Abstract

This project attempts to replicate the core findings of "Asset Pricing Under Endogenous Expectations in an Artificial Stock Market" by W. Brian Arthur et al [1]. Using a computational model of an artificial stock market populated by heterogeneous, inductively rational agents, the study investigates how market behavior emerges from evolving expectations. Agents formulate and test multiple forecasting models to predict future prices and dividends, selecting those that perform best. Through simulation, we observe two distinct market regimes: a rational-expectations equilibrium where technical trading and excess volatility are absent, and a complex regime where rich psychological behaviors emerge. In the latter, technical trading becomes viable, bubbles and crashes occur, and trading volume and volatility display persistence and autocorrelation, consistent with empirical financial data. The results demonstrate that the dynamics of financial markets can be explained without assuming full rationality or noise traders—highlighting the endogenous and evolutionary nature of expectations in asset pricing.

2 Introduction

Financial markets present a fascinating paradox: while academic theories often depict them as examples of efficiency driven by rational agents, the daily experiences of traders frequently point to a far more complex, psychologically driven arena characterized by sentiment, herd effects, and emergent patterns. This constant dichotomy highlights the limitations of traditional models in completely capturing the full picture of market behavior, particularly phenomena like speculative bubbles, sudden crashes, and persistent volatility that defy simple rational explanations.

Addressing this gap requires frameworks capable of embracing heterogeneity and adaptive learning. A strong contribution in this direction is "Asset Pricing Under Endogenous Expectations in an Artificial Stock Market" by W. Brian Arthur et al [1]. Their work offers a compelling alternative by proposing that agents form expectations inductively, rather than through perfect, deductive rationality. In their model, heterogeneous agents continuously devise, test, and refine expectational models based on observed market performance. More importantly, the market itself is the collective outcome of these evolving, interacting expectations, creating a recursive system where beliefs are endogenous and co-evolve within an "environment of beliefs".

To investigate this theory, Arthur et al. developed the Santa Fe Artificial Stock Market (ASM), a computational laboratory. Their experiments demonstrated that such a system can organize itself into distinct behavioral regimes. With low rates of exploration for new forecasting rules, the market approximates a rational-expectations equilibrium. However, when agents explore more actively—a scenario arguably closer to real-world conditions—the ASM transitions into a "complex" or "rich psychological" regime. This state is characterized by emergent phenomena extremely similar to actual financial markets, including the viability of technical trading, the occurrence of temporary bubbles and crashes, fluctuating trading volumes, and statistical signatures like GARCH behavior in price volatility. The ability of their model to generate these features from simple agent interactions highlights its significance.

This report details a project focused on the analysis and implementation of the foundational model presented by Arthur et al. Our aim in this project is to replicate the original Santa Fe Artificial Stock Market model from scratch using Python, with the dual goals of validating the results presented in the original paper and deepening our understanding of how such multi-agent systems function. By re-implementing the model, we aim to observe whether the same distinct market regimes—rational and complex—emerge under similar conditions, and to critically examine the dynamics that drive them. In doing so, we hope to gain insights into how agents can continually improve their predictive models through adaptive learning, and how the choice of parameters—such as the rate of exploration—affect the equilibrium outcomes. This hands-on reconstruction also serves as an opportunity to better grasp the mechanisms of expectation formation, selection, and evolution within an artificial financial market.

3 Environment

3.1 Market

The artificial stock market is composed of 25 agents who interact exclusively through a central market mechanism. Unlike many multi-agent systems where agents may interact directly, here each agent interacts *only via the market*, making their interaction entirely mediated by prices and aggregate outcomes.

The market consists of a single risky asset—a stock—with price $p(t)$ at discrete time t. Each period lasts from t to $t+1$, and at the end of the period the stock pays a dividend $d(t+1)$. The dividend process is *exogenous*—i.e., it evolves independently of agents' actions or market outcomes—and follows an autoregressive process defined as:

$$d_{t+1} = \bar{d} + \rho(d_t - \bar{d}) + \varepsilon_t$$

where $\bar{d}$ is the long-run mean dividend, ρ is the persistence parameter, and $\varepsilon_t \sim \mathcal{N}(0, \sigma^2)$ is normally distributed noise.

In addition to the stock, there is a risk-free asset—a bank—that pays a constant return r per period. At each time step t, each agent i chooses how to allocate their wealth between holding shares of stock $h_i(t)$ and cash $M_i(t)$ in the bank. The agent's total wealth is given by:

$$w_i(t) = M_i(t) + h_i(t) \cdot p(t)$$

At the end of the period, the agent's wealth evolves as:

$$w_i(t+1) = (1+r)M_i(t) + h_i(t) \cdot p(t+1) + h_i(t) \cdot d(t+1)$$

Trading occurs through a *specialist* who matches supply and demand and sets the market-clearing price $p(t+1)$. Since the number of shares in the market is fixed, the specialist's challenge is to reconcile potentially mismatched buy and sell orders at each time step.

To handle price determination, we implement a *Walrasian auction* mechanism. In this approach, agents compute their own expectation of future returns based on the current market state and submit their demand for shares to the auctioneer. The auctioneer (or specialist) then adjusts the price iteratively until total demand equals the fixed supply. This method ensures market-clearing prices without direct negotiation between agents and reflects a core principle of general equilibrium theory.

The *information structure*—referred to as the "world"—is globally observable and identical for all agents. It consists of a 12-bit binary vector summarizing key market indicators:

- **Bits 1–6**: Thresholds for the price-dividend ratio, indicating whether $\frac{p \cdot r}{d}$ exceeds 0.25, 0.5, 0.75, 0.875, 1.0, and 1.125.
- **Bits 7–10**: Technical indicators, showing whether the current price is above its 5-, 10-, 100-, and 500-period moving averages.

- **Bits 11–12**: Fixed control bits, always set to 1 and 0 respectively, used to detect whether agents are acting on truly informative signals versus noise.

These 12 bits define the *market state* as perceived by agents and serve as the condition component in their predictive models. Agents selectively condition their forecasts on subsets of this state, allowing for the emergence of strategies that use fundamentals, technical analysis, or both, depending on which patterns prove useful over time.

3.2 Agents

Each agent in the model is designed to behave as an inductive learner, adapting to the environment by forming and refining forecasts of future market outcomes. Rather than applying a fixed, globally shared model of the market, agents rely on a personal collection of predictive rules—referred to as *predictors*—to estimate future values of price and dividend. The agent then makes investment decisions based on the most accurate of these predictors.

3.2.1 Predictors and Market Perception

Each agent possesses a fixed number M of predictors. A predictor is a conditional forecasting rule that operates in two parts:

1. A **condition array**, which is a 12-bit string that matches certain configurations of the observable market state (the "world").
2. A **forecasting rule**, which provides a linear prediction for the sum of the next period's price and dividend:

$$E[p_{t+1} + d_{t+1}] = a \cdot (p_t + d_t) + b$$

Each of the 12 bits in the condition array corresponds to a market descriptor (e.g., whether the price is above a moving average, or whether the price/dividend ratio exceeds a threshold). Each position can be set to 1 (true), 0 (false), or # ("don't care"), allowing flexible pattern recognition.

A predictor is said to be *active* when its condition array matches the current market state. At each time step, an agent identifies all active predictors and uses the most accurate one, i.e., the one with the lowest variance, to make a forecast. The forecast and its estimated variance are then used to compute the optimal stock demand using:

$$x_i(t) = \frac{E_i[p_{t+1} + d_{t+1}] - (1 + r)p_t}{\lambda \cdot \sigma^2_{i,j}}$$

where:

- $x_i(t)$: desired stock holdings for agent i,
- $E_i[p_{t+1} + d_{t+1}]$: agent's expectation of next period's payoff,
- $\sigma^2_{i,j}$: estimated forecast variance of agent's i selected predictor j,
- λ: the risk-aversion parameter set to 0.5 as per the paper,
- r: interest rate of the risk-free asset.

3.2.2 Predictor Evaluation and Variance

Each predictor maintains a record of its variance over time, measured by the exponential moving average of squared forecast errors:

$$\sigma^2_{i,j}(t) = (1 - \theta) \cdot \sigma^2_{i,j}(t - 1) + \theta \cdot \left(p_{t+1} + d_{t+1} - (\hat{p}^{(j)}_{t+1} + \hat{d}^{(j)}_{t+1})\right)^2$$

where:

- $\sigma^2_{i,j}(t)$: variance estimate for predictor j of agent i,
- θ: learning rate ($\theta = \frac{1}{75}$ and $\theta = \frac{1}{150}$, in the Complex Regime and Rational Expectations Regime, respectively),
- $\hat{p}^{(j)}_{t+1} + \hat{d}^{(j)}_{t+1}$: forecast made by predictor j.

3.2.3 Evolution Through Genetic Algorithm

Every 250 periods in the Complex Regime or 1000 in the Rational Expectations Regime, each agent evolves their predictors using a genetic algorithm implemented as specified in [2], designed to improve the forecasting quality of their predictor pool. Each agent maintains a fixed number of predictors per asset, 100, and the algorithm proceeds as follows:

1. **Selection of Worst Predictors:** The predictors are sorted based on a fitness function, and the bottom 20% are selected for replacement. The fitness of predictor j for agent i is given by:

$$f_{i,j}(t) = M - \sigma^2_{i,j}(t) - C \cdot s_j$$

where:

- $\sigma^2_{i,j}(t)$ is the forecast variance of predictor j,
- s_j is the number of specific bits (0 or 1) in the predictor's 12-bit condition string (i.e., its specificity),
- $C = 0.005$ is a penalty coefficient discouraging overfitting.
- M is a constant whose value is irrelevant given tournament rankings.

2. **Parent Selection via Tournament:** For each new predictor to be generated, one or two parents are selected using **tournament selection of size 2** from the remaining top 80% predictors in the pool. This involves randomly sampling two predictors and selecting the one with the higher fitness.
3. **Crossover or Mutation:** Each new predictor is generated by one of two methods:
   - **Crossover:** With a probability of 0.3 in the Rational Regime or 0.1 in the Complex Regime, crossover is used to generate a new predictor. Two parents are selected as described in the previous point, and one of the following crossover methods is randomly chosen:
     (a) **Uniform Crossover:** One parameter (either a or b) is taken from one parent, and the other from the second parent.
     (b) **Linear Combination:** A weighted average of the parents' parameters is computed, with weights inversely proportional to their forecast variances:

$$a_{\text{child}} = \frac{w_1 a_1 + w_2 a_2}{w_1 + w_2}, \quad b_{\text{child}} = \frac{w_1 b_1 + w_2 b_2}{w_1 + w_2}$$

     where $w_1 = 1/\sigma_1^2$ and $w_2 = 1/\sigma_2^2$.
     (c) **Cloning:** One parent is chosen randomly and copied directly.

     For the condition string, each of the 12 bits is taken independently from either parent with equal probability. The forecast variance of the child is set to the average of the two parents' variances.
   - **Mutation:** With complementary probability (0.7 in the Rational Regime or 0.9 in the Complex Regime), a single parent is selected, and a new predictor is created by mutating this parent. Two kinds of mutations may occur:
     – **Parameter Mutation:** When a predictor undergoes mutation, one of three types of parameter mutation is selected at random:
       * **No Change (60% probability):** The predictor's parameters a and b remain unchanged.
       * **Additive Perturbation (20% probability):** Small values are added to the predictor's existing parameters:
         · a is perturbed by a value sampled uniformly from $[-0.0025, 0.0025]$,
         · b is perturbed by a value sampled uniformly from $[-0.0145, 0.0145]$.

         These ranges correspond to 0.05% of the full parameter domains and allow for local, fine-tuned exploration.
       * **Resampling (20% probability):** New values for the forecasting parameters are drawn independently from their original initialization ranges:

· a is sampled uniformly from the interval $[0.7, 1.2]$,
· b is sampled uniformly from the interval $[-10, 19.002]$.

This allows the predictor to jump to a new region in parameter space and encourages global exploration.

Note: The parameter values described above are not the result of tuning or experimentation in this implementation. They are directly inherited from the original specification in Arthur et al. [1] to maintain consistency with the baseline model.

– **Condition String Mutation:** Each bit in the 12-bit condition string is mutated independently with probability 0.03. The mutation logic is:
  * A '0' bit is replaced by '1' or '#' with probabilities $\frac{1}{3}$ and $\frac{2}{3}$, respectively.
  * A '1' bit is replaced by '0' or '#' with probabilities $\frac{1}{3}$ and $\frac{2}{3}$, respectively.
  * A '#' bit is replaced by '0' or '1', each with probability $\frac{1}{2}$.

  If any bit is mutated, the forecast variance of the new predictor is reset to the **mean variance** of the agent's current predictor pool to reflect uncertainty about its reliability. If no bits are mutated but the new predictor's variance is significantly below typical values (more than one standard deviation below the default), its variance is reset to the **median variance**.

4. **Replacement:** The newly generated predictor replaces one of the predictors initially selected for removal. All new predictors enter with adjusted variances, and their influence grows only if they prove accurate in future forecasting tasks.

This evolutionary process allows agents to adaptively explore and refine their forecasting strategies. The choice between crossover and mutation—and the associated probabilities—plays a critical role in shaping the diversity and responsiveness of the predictor population. In the rational regime, a higher crossover rate encourages gradual refinement of existing strategies. In contrast, the complex regime's higher mutation rate fosters exploration and behavioral diversity, giving rise to more dynamic and psychologically rich market dynamics. This mechanism forms the core of the agents' learning behavior, enabling them to react to feedback and evolve strategies in a decentralized, inductive manner.

3.2.4 Agent Learning and Behavior

Through this mechanism, agents behave like decentralized statisticians. They experiment with hypotheses (predictors), act on the most promising ones, and discard those that fail. The constant competition and adaptation among predictors leads to the emergence of complex behaviors at the market level, including trend following, mean reversion, and even speculative bubbles—depending on how agents' expectations co-evolve within the system.

4 Results

In the original study by Arthur et al. [1], the artificial stock market exhibited two distinct behavioral regimes, depending on the rate at which agents explored new forecasting strategies via the genetic algorithm. These regimes are:

4.1 Rational Expectations Regime

4.1.1 Intuition Behind Rational Expectations Equilibrium

In financial theory, a rational expectations equilibrium (REE) occurs when all agents form beliefs about the future that are consistent with the actual outcomes produced by the market. That is, agents' expectations are model-consistent: they do not systematically mispredict future prices or dividends.

This does not require agents to be omniscient or perfect in every forecast, but it does mean that on average, their expectations are accurate given the underlying structure of the economy.

From a game-theoretic perspective, REE can be seen as a Nash equilibrium of a forecasting game: each agent chooses a strategy (a forecasting model) assuming others do the same, and the resulting market prices confirm those forecasts. No agent has an incentive to unilaterally switch forecasting models, because the current one already best responds to the collective outcome.

In this model, a rational agent is one whose forecasts of future prices and dividends are statistically consistent with the structure of the market. Since the dividend process is exogenously defined and known to follow a stationary AR(1) process, the rational forecasting rule depends entirely on the current dividend and price levels. Thus, under rational expectations, agents do not rely on technical indicators or speculative trends, but rather form expectations using the dividend series alone — which fully determines the asset's fundamental value in equilibrium.

4.1.2 Emergence of the Rational Expectations Regime

This regime emerges when the **rate of exploration**—that is, the frequency of genetic algorithm updates and mutation/crossover activity—is set to a low level. In this scenario, agents are slow to adopt new predictors and primarily refine existing ones. As a result, heterogeneous expectations gradually converge toward a shared, rational expectation. The market stabilizes around the homogeneous rational expectations equilibrium (h.r.e.e), characterized by:

- Low trading volume,
- No persistent deviations from fundamental value,
- Absence of technical trading or bubbles,

4.2 Complex or Rich Psychological Regime

This regime arises when the rate of exploration is moderately increased—e.g., by invoking the genetic algorithm more frequently and setting a higher learning rate θ. Under these conditions, agents continuously experiment with new predictors, allowing for persistent heterogeneity of beliefs and expectations. The market no longer converges to equilibrium but instead exhibits:

- High trading volume,
- Emergence of technical trading patterns,
- Bubbles and crashes,
- Evolving, non-stationary dynamics.

4.3 Simulations

4.3.1 Simulation Parameters

To investigate whether these two regimes also appear in our implementation, we ran two simulations per regime under these parameter conditions. The key differences in setup were:

- **Rational regime:**
  – Genetic algorithm invoked every 1000 periods,
  – Crossover probability: 0.3,
  – Accuracy update rate $\theta = \frac{1}{150}$.
- **Complex regime:**
  – Genetic algorithm invoked every 250 periods,
  – Crossover probability: 0.1,
  – Accuracy update rate $\theta = \frac{1}{75}$.

The results are shown in Appendix A.

4.3.2 Discussion

The results we got differ in some aspects from the ones presented in the original paper. The most significant source of divergence is that our results are based on only two runs with 250,000 steps, whereas the statistics in the original paper are the mean of 25 runs.

As shown in Figure 1, the behavior of price evolution in our implementation still exhibits the two distinct regimes described in the original study. In the rational regime (Figure 1a), prices remain relatively stable and closely track the asset's fundamental value. Volatility is low, but our results still show some speculative bubbles and crashes—features largely absent from the original paper. In contrast, the complex regime (Figure 1b) exhibits markedly different dynamics: prices become more volatile, with frequent and pronounced bubbles and crashes. These fluctuations emerge endogenously from the agents' ongoing exploration and adaptation, rather than from any exogenous shocks. This behavior aligns with the original study's findings and confirms that greater predictor turnover—caused by shorter evolutionary cycles—leads to persistent heterogeneity in agent expectations and more psychologically rich market dynamics. Notably, our Complex Regime also shows prolonged overvaluation and sharp corrections, further resembling the fat-tailed behavior seen in real financial markets.

Figure 2 replicates the figure shown in the original paper with the theoretical r.e.e fundamental price as well as the difference to the complex regime price plot in a short 200 step time frame. Our results differ since the complex regime prices exhibited lower price volatility than the benchmark Homogeneous Rational Expectations (HREE) price in this short time frame. The dominant force appears to be the evolution of mean-reversion or contrarian forecasting rules among the agents with occasional irrational and explosive behavior as observed in (Figure 1b).

Figure 3 shows the average number of technical-trading bits(bits 6 through 10, which compare the current price against moving averages) utilized by active predictors over time in each regime. This is one area where our results diverge more significantly from those in [2]. In the original paper, technical indicators gained relevance over time in the complex regime, suggesting a self-reinforcing ecological niche for trend-following and other technical strategies. By contrast, our results show a steady decline in the number of active technical bits across both regimes. In the rational regime (Figure 3a), this trend reflects the system's natural convergence to equilibrium, where fundamental predictors dominate. In the complex regime (Figure 3b), although the decline stabilizes at a slightly higher level, technical bits never regain prominence.

This lack of emergent technical trading likely results from two compounding factors. First, the volatile environment of our simulations creates an evolutionary disadvantage for specific, narrowly-triggered predictors. When a technical predictor fires during an extreme price spike, its forecast error—and resulting estimated variance—can be catastrophically large. These predictors are then systematically eliminated by the genetic algorithm, leading to selection bias toward generalist predictors that are active more frequently and less exposed to tail risks. Second, technical trading strategies require a co-evolving population of complementary strategies to become profitable. This fragile niche may not have formed in our simulation due to stochastic variability early in the run. Without this mutual reinforcement, technical predictors fail to demonstrate fitness and are outcompeted by simpler alternatives.

Table 1 summarizes key statistical properties of asset returns and trading activity observed over a 250,000-period simulation in both the rational and complex regimes. As expected, the standard deviation of returns is significantly higher in the complex regime (6.42) compared to the rational regime (0.66), reflecting the greater volatility and market instability caused by ongoing predictor turnover and agent heterogeneity. The complex regime also shows a higher trading volume, consistent with increased experimentation and divergence in beliefs across agents. However, Skewness, kurtosis, and return volatility in our results—particularly in the Complex Regime—are several orders of magnitude larger than in the original paper. This is not necessarily a contradiction: averaging across 2 runs might only represent one path through a highly stochastic system and is more susceptible to outlier events.

Finally, it is important to acknowledge the impact of inevitable implementation differences. Replicating a model originally written in the 1990s involves many subtle sources of divergence: differences in random number generation algorithms, floating-point precision, or undocumented edge-case logic (such as tie-breaking rules in tournament selection or variance initialization for new predictors) can all alter the simulation trajectory. While our implementation closely follows the mechanisms described in the paper, these kinds of low-level discrepancies are unavoidable and can contribute both to the observed statistical differences and to deviations in the model's emergent behavior.

5 Conclusions

Our replication of the artificial stock market model described by [1] successfully reproduces its core qualitative findings, including the emergence of two distinct behavioral regimes. The Rational Expectations Regime exhibits price stability and convergence to fundamentals, while the Complex Regime produces richer dynamics characterized by volatility, bubbles, and crashes. These results support the original study's conclusion that increased evolutionary activity among agents fosters persistent heterogeneity and more realistic market behavior.

However, our results also diverge in important ways. Quantitatively, we observe significantly higher skewness, kurtosis, and return volatility—especially in the Complex Regime—largely due to the use of only two long simulation runs instead of averaging over many. We also observe a systematic decline, rather than reinforcement, of technical trading strategies, which we attribute to both the fragility of their emergence and evolutionary disadvantages in volatile environments.

Finally, we believe that the discrepancies between our results and those reported in the original paper [1] may stem from our incorporation of missing implementation details, which we supplemented with clarifications from [2]. In particular, differences in how the mutation mechanism of the genetic algorithm and the initialization of variance in new predictors were implemented—both of which are critical for observing convergence toward different regimes—may have contributed significantly to these divergences.

References

[1] W. Brian Arthur, John H. Holland, Blake LeBaron, Richard Palmer, and Paul Tayler. Asset pricing under endogenous expectations in an artificial stock market. *Economica*, 65(1):1–28, 1997.

[2] Blake LeBaron, W. Brian Arthur, and Richard Palmer. Time series properties of an artificial stock market. *Journal of Economic Dynamics and Control*, 23(9-10):1487–1516, 1999. Accepted 20 November 1998.

A Results

(a) Rational Regime

(b) Complex Regime

Figure 1: Price Evolution

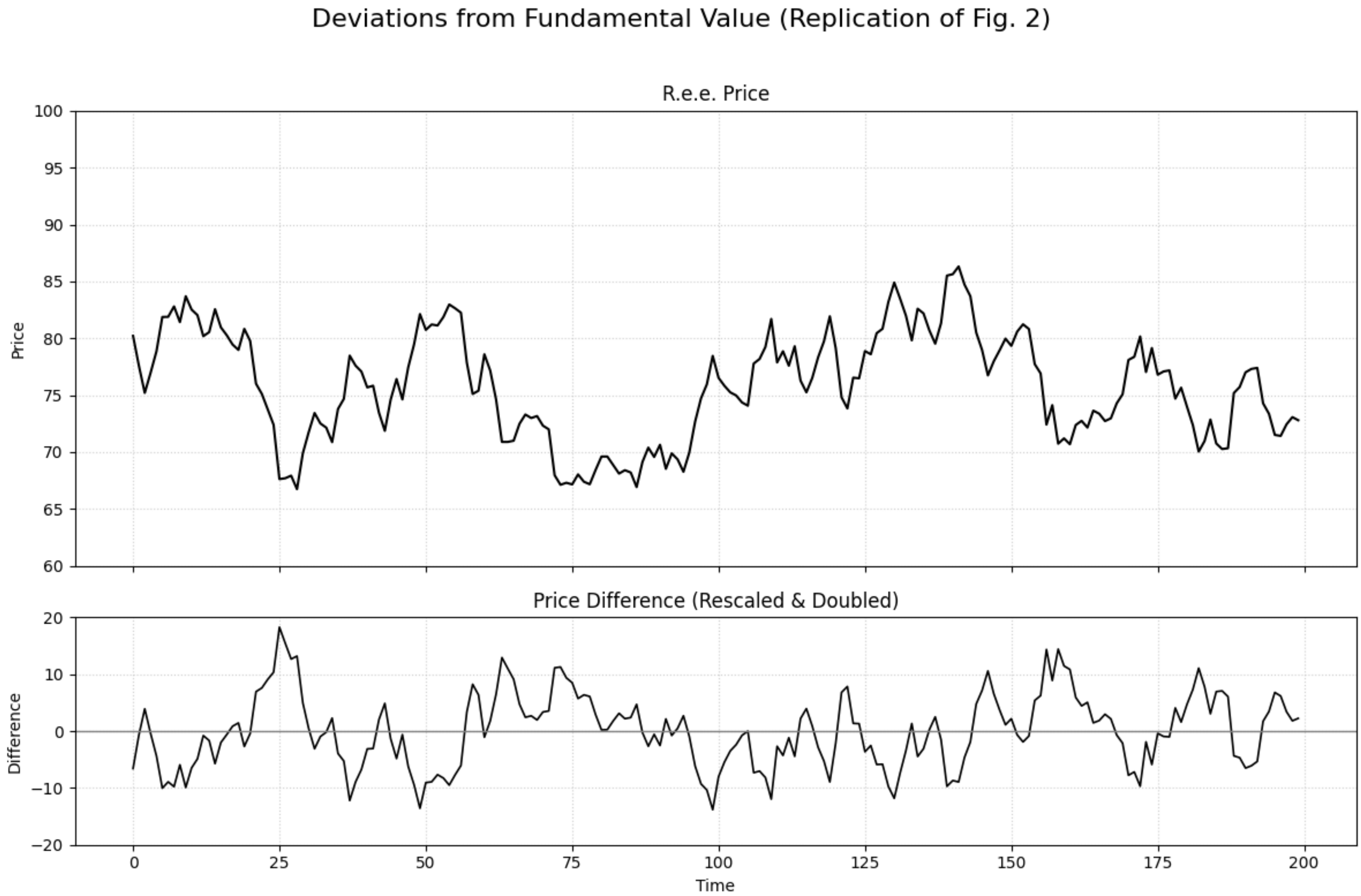
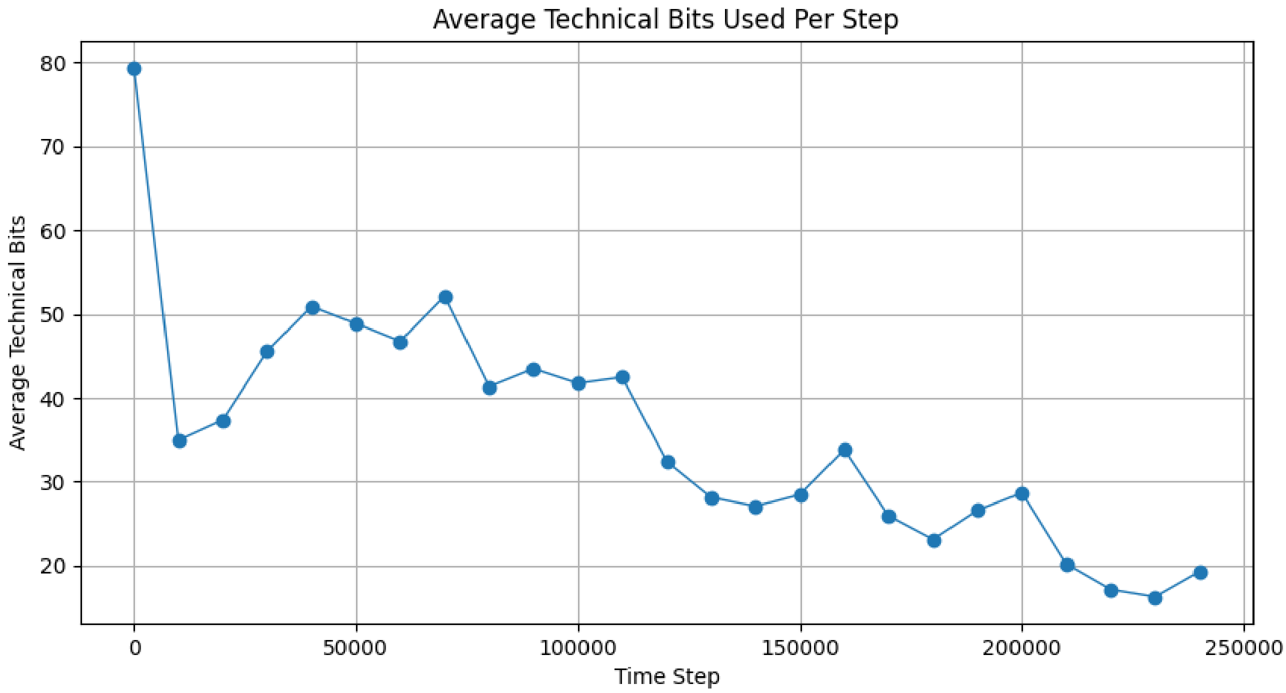
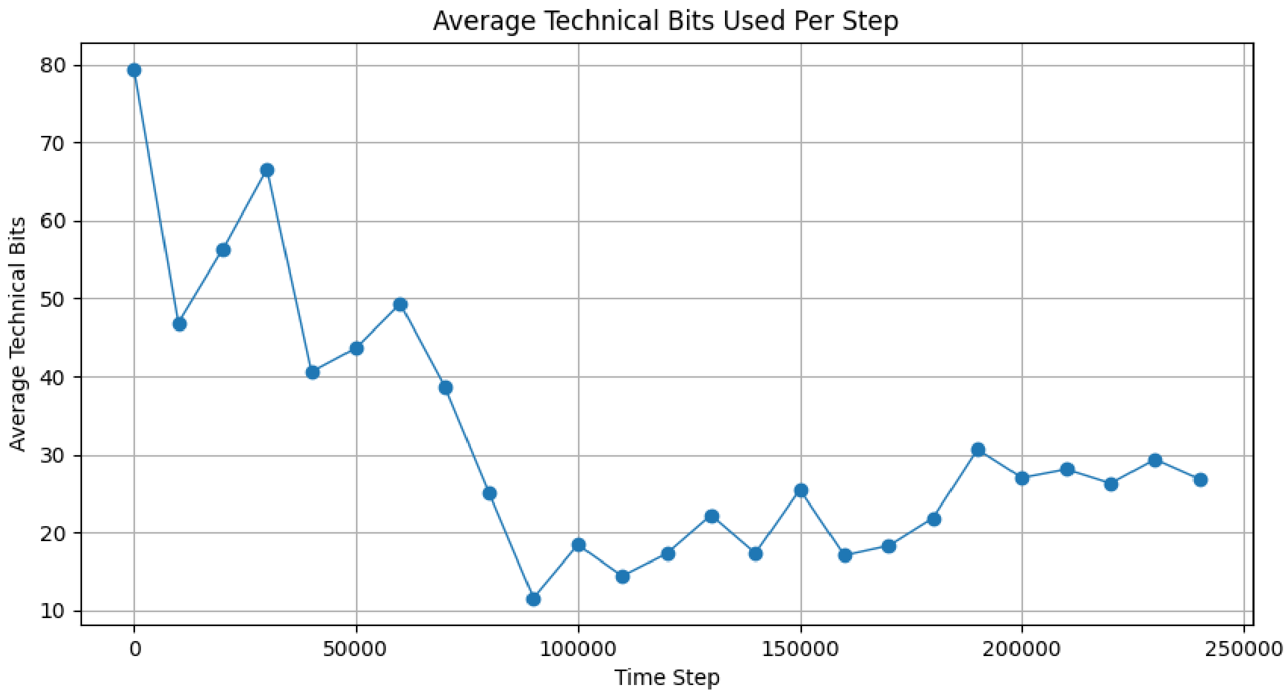


Figure 2: Deviations of the price series in the complex regime from fundamental value



(a) Rational Regime



(b) Complex Regime

Figure 3: Number of technical-trading bits that become set as the market evolves

	Rational Regime	**Complex Regime**
Mean	0.000017	-0.000031
Std. Dev.	0.655629	6.419677
Skewness	-4.616627	24.927930
Kurtosis	872.477578	3897.772458
Vol. Traded	26674.772917	35518.647083

Table 1: Statistics in the two regimes collected for 1 experiment with 250,000 periods