On Bounds, Winning Strategies, and Generalizing the Zeckendorf Game

Polymath REU[1]
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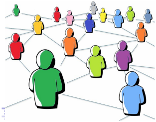

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Zeckendorf Decomposition

Fibonaccis: $F_1 = 1$, $F_2 = 2$, , $F_3 = 3$, $F_4 = 5$, $F_{n+2} = F_{n+1} + F_n$.

Zeckendorf's Theorem

Every positive integer can be written uniquely as a sum of one or more non-consecutive Fibonacci numbers.

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Example: $2020 = 1597 + 377 + 34 + 8 + 3 + 1 = F_{16} + F_{13} + F_8 + F_5 + F_3 + F_1$.

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Example: $2020 = 1597 + 377 + 34 + 8 + 3 + 1 = F_{16} + F_{13} + F_8 + F_5 + F_3 + F_1$.
Remark: We define Fibonacci sequence as 1,2,3,5... instead of the usual 1,1,2,3... as it allows us to have a unique representation of non-consecutive Fibonacci numbers. i.e. final decomposition won't have 0 or two 1's.

The Zeckendorf Game

This game is introduced in "The Zeckendorf Game" paper[1].

The Game

At the beginning of the game, there is an unordered list of n 1's. Let $F_1 = 1$, $F_2 = 2$, and $F_{i+1} = F_i + F_{i-1}$; therefore the initial list is $\{F_1^n\}$.
On each turn, a player can do one of the following moves until there's no more possible moves left:

1. $F_{i-1} \wedge F_i \rightarrow F_{i+1}$
2. If the list has two of the same Fibonacci number, $F_i \wedge F_i$ then
    a. if $i = 1$, $F_1 \wedge F_1 \rightarrow F_2$
    b. if $i = 2$, $F_2 \wedge F_2 \rightarrow F_1 \wedge F_3$
    c. if $i \geq 3$, $F_i \wedge F_i \rightarrow F_{i-2} \wedge F_{i+1}$

Remark: The game always terminates at the Zeckendorf decomposition!

Sample Game

- if $i = 1$,
$F_1 \wedge F_1 \rightarrow F_2$
- if $i = 2$,
$F_2 \wedge F_2 \rightarrow F_1 \wedge F_3$
- if $i \geq 3$,
$F_i \wedge F_i \rightarrow F_{i-2} \wedge F_{i+1}$
- $F_{i-1} \wedge F_i \rightarrow F_{i+1}$

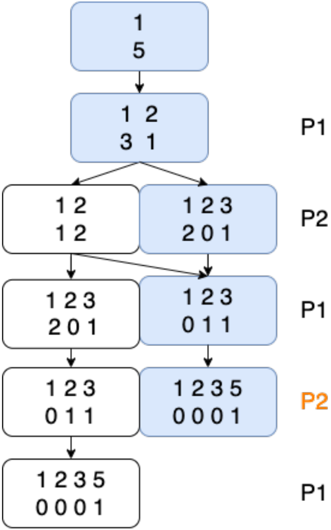

Figure: Tree Diagram for n=5

Games end

Theorem (Baird-Smith, P., Epstein, A., Flint, K., & Miller, S. J. (2018, May). *"The Zeckendorf Game"*.[1])

All games end in finitely many moves.

Proof: The sum of the square roots of the indices is a strict monovariant.

- Adding consecutive terms: $\left(\sqrt{k}+\sqrt{k}\right)-\sqrt{k+2}<0.$
- Splitting: $2\sqrt{k}-\left(\sqrt{k+1}+\sqrt{k+1}\right)<0.$
- Adding 1's: $2\sqrt{1}-\sqrt{2}<0.$
- Splitting 2's: $2\sqrt{2}-\left(\sqrt{3}+\sqrt{1}\right)<0.$

Bounds

Bounds on game length

Previous Result

Theorem (Baird-Smith, P., Epstein, A., Flint, K., & Miller, S. J. (2018). *"The Zeckendorf Game"*.[1])

Upper bound on length of game: $\log_\phi(\sqrt{5}n + 1/2)n$ (of size $n\,logn$)
Lower bound on length of game: $n - Z(n)$

Theorem (Li, R., Li, X., Miller, S. J., Mizgerd, C., Sun, C., Xia, D., & Zhou, Z. (2020).*"Deterministic Zeckendorf Games"*.[2])

Upper bound on length of game: $3n - 3Z(n) - IZ(n) + 1$

Notations

$Z(n)$: number of terms in Zeckendorf Decomposition. $Z(n) = \Theta(\log n)$
$IZ(n)$: sum of indices in Zeckendorf Decomposition. $IZ(n) = O(\log^2 n)$

Bounds on game length

Previous Result

Theorem (Li, R., Li, X., Miller, S. J., Mizgerd, C., Sun, C., Xia, D., & Zhou, Z. (2020).*"Deterministic Zeckendorf Games"*.[2])

The upper bound of the game is given by the sum of the three parts:

a. $\boldsymbol{MS_2} \leq n - 2Z(n) + 1$
b. $MC_1 + MC_2 \leq n - Z(n)$
c. $MC_3 + MC_4 + \cdots + MC_{i_{\max}(n)} + MS_3 + MS_4 + \cdots + MS_{i_{\max}(n)} \leq n - IZ(n)$

Notations

$\boldsymbol{MS_i}$: number of Split moves at F_i
$(F_2 \wedge F_2 \rightarrow F_1 \wedge F_3$ or $F_i \wedge F_i \rightarrow F_{i-2} \wedge F_{i+1})$
MC_i: number of Combine moves at F_i
i.e. $(F_1 \wedge F_1 \rightarrow F_2$ or $F_{i-1} \wedge F_i \rightarrow F_{i+1})$
$i_{\max}$: the largest index m such that $F_m \leq n$

Bounds on game length

New Result

Result:

New upper bound: $\frac{\sqrt{5}+3}{2}n - \frac{\sqrt{5}+1}{2}Z(n) - IZ(n) \approx 2.6n - 1.6\,Z(n) - IZ(n)$

- **Idea:** Tighten the bound of $\boldsymbol{MS_2}$

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- **Idea:** Tighten the bound of $\boldsymbol{MS_2}$

  1. Based on the fact that there is at most one F_2 at the end of the game, find relation between $\boldsymbol{MS_2}$ and other MC_i's and MS_i's.

     $$\text{Ex: } MS_2 \leq (MC_1 - MC_2 - MC_3 + MS_4)/2$$

  2. Expand this inequality by replacing any $MS_i (i \geq 3)$ terms on the right hand side with similar inequalities.

  3. Find patterns in the coefficients of MC_i's and MS_i's on the right hand side.

- **Idea Continued:** Tighten the bound of $\boldsymbol{MS_2}$

  iv. Evaluate the inequality for MS_2 based on the patterns in the coefficients and the previous bounds on MC_i's and MS_i's.

  v. Combine the new bound on MS_2 with the other two previous bounds to give a tighter game bound.

Winning Strategies

Winning strategies
Previous Results

Theorem (Baird-Smith, P., Epstein, A., Flint, K., & Miller, S. J. (2018). *"The Zeckendorf Game"*.[1])

For all $n > 2$, Player 2 has the winning strategy for 2 player game.

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  Remark: Non-constructive!

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- **Idea:** Suppose player m has the winning strategy $(1 \leq m \leq p)$. Then player m-1 can steal player m's winning strategy.
    1. Since for all $n \geq 5$, $p \geq 3$ games, any player m's winning path does not contain the following 3 consecutive steps. If it contains, player in step 2 can do $1+2=3$ instead and player $m-1$ can steal the winning strategy:
    Step $1: 1+1=2$ (Combine two 1s into one 2)
    Step $2: 1+1=2$ (Combine two 1s into one 2)
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  2. Then we construct other $m-1$ players' moves containing these 3 consecutive steps, which contradicts above, so player m has no winning strategy.

Corollary of Result 1:

In a game consisting of t teams and exactly k consecutive players each team. When n is significantly large, for any $t \geq 3, k = t - 1$, no team has winning strategy.

Winning Strategies

New Results

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- **Idea:** the proof is similar to the last slide. The difference is that we use $3k$ consecutive players instead of 3 players, and the middle k players can all do the stealing move $1+2=3$.

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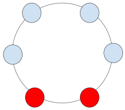
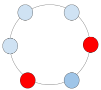


1. According to the corollary of result 1, in this situation, the 2-player alliance does not have a winning strategy. Therefore, the 4-player alliance has a winning strategy in this case.

Winning Strategies

New Results



Suppose the 4-player alliance starts with player a. From Player a's we can do the following:

1. Player $a : 1 + 1 = 2$
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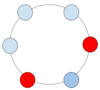
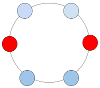
Player $a + 1$ can do 1+2=3 instead, so the 4-player alliance can steals the winning strategy, which contradicts.





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Player $a+1$ can do 1+2=3 instead, so the 4-player alliance can steals the winning strategy, which contradicts.

Then the situation is similar to 3-player case: when $n \geq 5, p \geq 3$, no player has winning strategy. From this previous result, we know that the 2-player alliance does not have a winning strategy. So the 4-player alliance will have a winning strategy.

Corollary of Result 2:

In a game of at least 7 players with 2 alliances, if one team has at least 5 players (big alliance) and the other has 2 players, then no matter what the players' positions are, the big alliance always has a winning strategy.

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  1. **If the two players in the small alliance are separated:** then the big alliance is divided into 2 parts (at least 5 players in total). By pigeonhole principle, at least one part will have at least 3 players (call it big part)

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  1. **If the two players in the small alliance are separated:** then the big alliance is divided into 2 parts (at least 5 players in total). By pigeonhole principle, at least one part will have at least 3 players (call it big part)
  2. Suppose the 3 consecutive players in the 4-player alliance starts with player a. Then from the first move of player a, they can do the following:
  3. Player $a: 1+1=2$ (Combine two 1s into one 2)
     Player $a+1: 1+1=2$ (Combine two 1s into one 2)
     Player $a+2: 2+2=1+3$ (Split two 2s into 1 and 3)

1. **Idea: If the two players of the small alliance are consecutive**
2. **If $p \geq 8$:** The proof is similar to previous result, and we can let the 6 consecutive players do the stealing strategy in one round.
3. **If $p = 7$** The proof is similar to previous results except that we need to do more rounds of 1+1=2 to ensure that there are enough 2s to do the stealing strategy.

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- **Idea:** As the team is sufficiently large, it can build up enough 2s given time. From there, the $2m$ players in a row can steal any winning strategy the opponents might have.
  1. Suppose the opponents have a winning strategy. Then the big alliance can build up at least m of 2s before the $2m$ players in a row go. Those players can then either take m of $1+1=2$ steps and m of $2+2=1+3$ steps or simply take m of $1+2=3$ steps instead.

Generalized Game

So far the game has only been played with constant coefficients, and results rely on Zeckendorf decomposition properties.

Recurrence Relation with non-constant coefficients:
(Dai, L., Ding, P., Luo, T., Zhang. Y, & Miller. S.J. (2020). *"Generalizing Zeckendorf's Theorem to Recurrence with Non-Constant Coefficient[3]"*)

1. The sequence with non-constant coefficients

$$\boxed{a_{n+1} = n\,a_n + a_{n-1}}$$

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   $$\boxed{a_{n+1} = n\,a_n + a_{n-1}}$$

   Sequence starts with 1, 2, 5, 17, 73, ...

2. **Existence and Uniqueness of Decompositions**: Every natural number could be written uniquely in the form of $x = \sum s_i a_i$ where $x \in \mathbb{N}$, $0 \le s_i \le i$, a_i is the ith term in the sequence, s_i is the coefficient, and when $s_i = i$, $s_{i-1} = 0$, $i \in \mathbb{N}$, $i \neq 1$.
   Example: $70 = 4 \times 17 + 0 \times 5 + 1 \times 2 + 0 \times 1$, $8 = 1 \times 5 + 1 \times 2 + 1 \times 1$

The Generalized Game

At the beginning of the game, there is an unordered list of i 1's. Let $a_1 = 1$, $a_2 = 2$, and $a_{n+1} = n\,a_n + a_{n-1}$; therefore the initial list is $\{a_1^i\}$ where $i \in \mathbb{N}$. On each turn, a player can do one of the following moves:

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1. Combining move:
   - i. $a_n^n \wedge a_{n-1} \to a_{n+1}$
   - ii. $1^2 \to 2$
2. Splitting move:
   - i. $a_n^{n+1} \to a_{n+1} \wedge a_{n-1}^{n-2} \wedge a_{n-2}$
     $\because a_n^{n+1} \to a_n^n \wedge a_{n-1}^{n-1} \wedge a_{n-2} \to (a_n^n \wedge a_{n-1}) \wedge a_{n-1}^{n-2} \wedge a_{n-2} \to a_{n+1} \wedge a_{n-1}^{n-2} \wedge a_{n-2}$
   - ii. $2^3 \to 1 \wedge 5$

- $a_n^n \wedge a_{n-1} \rightarrow a_{n+1}$
- $1^2 \rightarrow 2$
- $a_n^{n+1} \rightarrow a_{n+1} \wedge a_{n-1}^{n-2} \wedge a_{n-2}$
- $2^3 \rightarrow 1 \wedge 5$

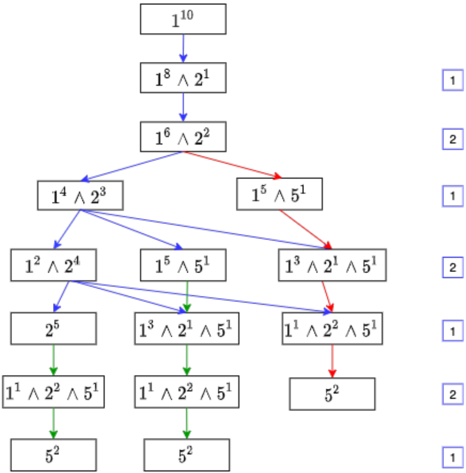

Figure: Game Tree for n=10

Result 1: Game is finite

Every game terminates within a finite number of moves at the unique decomposition given under the recurrence $a_{n+1} = n\,a_n + a_{n-1}$ discussed above.

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  Total number of terms is strictly decreasing mono-variant.

  Shown below are the number of terms over each move:

  1. Combining 1s: 2 terms $\rightarrow$ 1 term.
  2. Combining consecutive terms: $n+1$ terms $\rightarrow$ 1 term.
  3. Splitting move: $n+1$ terms $\rightarrow$ n term(s).

The game terminates at the unique decomposition described before, when there no more moves left.

Result 2: Game is playable (either player can win)

For any $n>5$, there are at least two games with different numbers of moves. Further, there is always a game with an odd number of moves and one with an even number of moves, indicating either can win.

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  1. The game starts with $i = 6 + j$, where $j \geq 0$, $i \in \mathbb{N}$.
     The following two sequences of moves, M_1 and M_2, result in the decomposition of 6:
     $M_1 = \{\{1 \wedge 1 \to 2\}, \{1 \wedge 1 \to 2\}, \{1 \wedge 2 \wedge 2 \to 6\}\}$, $|M_1| = 3$.
     $M_2 = \{\{1 \wedge 1 \to 2\}, \{1 \wedge 1 \to 2\}, \{1 \wedge 1 \to 2\}, \{2 \wedge 2 \wedge 2 \to 5 \wedge 1\}\}$, $|M_2| = 4$.

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  2. Let, M be the set of moves it takes to finish the rest of the game (after 6). i.e. it takes $|M| = k$ moves to finish the rest of the game.
     Regardless of what the k is, there are at least two sets with different number of moves, $M_1 \wedge M$ and $M_2 \wedge M$, that describe a complete game.

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  3. $|M_1 \wedge M| = 3 + k$, but $|M_2 \wedge M| = 2 + k$.

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  1. Since for all $n \geq 16$, $p \geq 4$ games, any player m's winning path does not contain the following 4 consecutive steps.
     Step $1 : 1+1=2$
     Step $2 : 1+1=2$
     Step $3 : 1+1=2$
     Step $4 : 2+2+2=1+5$
     Because, if it does, then player in step 3 can do $2+2+2=1+5$ instead, and player $m-1$ can steal the winning strategy.

Result 3:

For all $p \geq 4$, $n \geq 16$ multi-player game, no player has winning strategy

- **Idea:** Suppose player m has the winning strategy $(1 \leq m \leq p)$. Then player m-1 can steal player m's winning strategy.
  1. Since for all $n \geq 16$, $p \geq 4$ games, any player m's winning path does not contain the following 4 consecutive steps.
     Step $1 : 1 + 1 = 2$
     Step $2 : 1 + 1 = 2$
     Step $3 : 1 + 1 = 2$
     Step $4 : 2 + 2 + 2 = 1 + 5$
     Because, if it does, then player in step 3 can do $2 + 2 + 2 = 1 + 5$ instead, and player $m - 1$ can steal the winning strategy.
  2. Thus, we can construct the moves of $m - 1$ player containing these 4 consecutive steps, which contradicts above, so player m has no winning strategy.

Future Directions

Future Direction

Zeckendorf Game:

1. Construct the winning strategy for the 2nd player (in a 2 player game).
2. Construction of alliances with winning strategy in multiplayer game ($p > 2$).
3. Further tighten the bound.

Generalized Game:

1. Existence of winning strategy in the two, three player game.
2. Conjecture: For any $n \geq 7$, player 1 always has a winning strategy.
3. Find lower and upper bounds for the length of the game.

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Result 4 for Generalized Game:

For $p = 3$, $n \geq 5$ 3-player Game, player 2 never has a winning strategy

- **Idea:** Suppose player 2 has the winning strategy. Then we can do the following from the beginning of the game:
  1. Step $1 : 1 + 1 = 2$ (player 1 has to do so as the first step of the game)
     Step $2 : 1 + 1 = 2$ (player 2 has to do so as the second step of the game)
     Step $3 : 1 + 1 = 2$ (we can let player 3 do so)
     Step $4 : 2 + 2 + 2 = 1 + 5$ (we can let player 1 do so)
  2. Then we can let player 3 in step 3 can do $1 + 2 + 2 = 5$ instead, so now player 1 steals the winning strategy, which contradicts.

Corollary of Result 1:

In a game consisting of t teams and exactly k consecutive players each team. When n is significantly large, for any $t \geq 3, k = t-1$, no team has winning strategy

- **Idea:** Suppose team m has the winning strategy $(1 \leq m \leq t)$. Then team m-1 can steal team m's winning strategy.
  1. Since for any $t \geq 3, k = t-1$, any team m's winning path doesn't contain the following $3k$ consecutive steps (unless one of the middle k players is in team m). If it contains, the middle k players listed below can all do $F_1 \wedge F_2 \to F3$ instead and team $m-1$ can steal the winning strategy:
     First k steps all do : $1+1=2$ (Combine two 1s into one 2)
     Middle k steps all do : $1+1=2$ (Combine two 1s into one 2)
     Last k steps all do : $2+2=1+3$ (Split two 2s into 1 and 3)
  2. Then we construct these $3k$ steps for other $m-1$ teams and we get contradiction.

1. **Idea: If the two players of the small alliance are consecutive**
2. **If $p \geq 8$:** Then the big alliance will have at least 6 consecutive players. If the big alliance starts with player a, then we can do the following.
3. Player $a : 1+1=2$
   Player $a+1 : 1+1=2$
   Player $a+2 : 1+1=2$
4. Player $a+3 : 1+1=2$
   Player $a+4 : 2+2=1+3$
   Player $a+5 : 2+2=1+3$
   So player $a+1$ and $a+2$ can both do the stealing move $1+2=3$, and then the big alliance can always steal the winning strategy.
5. **If $p=7$** The proof is similar to previous results except that we need to do more rounds of 1+1=2 to ensure that there are enough 2s to do the stealing strategy.