ON THE BOUNDARY VALUE PROBLEM FOR THE LAPLACE EQUATION WITH MIXED BOUNDARY CONDITIONS IN A HALF-BAND

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The paper proves the theorems of the existence and uniqueness of the solution of the problem for Laplace equations with mixed boundary conditions in the half-band, and integral representations for partial derivatives of the solution are obtained.

Consider the boundary value problem for the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

in the half-band $D = \{(x,y): \ 0 < x < \pi, y > 0\}$ in the function class $u(x,y) \in C(\overline{D}) \cap C^1(\overline{D} \cap \{y > 0\}) \cap C^2(D)$ with boundary conditions:

$$u(0,y) = 0, \ \frac{\partial u}{\partial x}(\pi, y) = 0, \ y > 0, \tag{2}$$

$$\lim_{y \to 0+0} \int_0^\pi \left[\frac{1}{k}\frac{\partial u}{\partial y}(x,y) - \frac{\partial u}{\partial x}(x,y) + \varphi(x) \right]^2 dx = 0, \ \varphi(x) \in L_2(0,\pi), \ k \in (-\infty,-1) \cup (0,+\infty), \tag{3}$$

$$u(x,y) \rightrightarrows 0, y \to +\infty. \tag{4}$$

A similar problem was considered as an auxiliary one in the study of the Tricomi-Neumann problem for the Lavrentiev-Bitsadze equation with boundary conditions of the second kind on the sides of the half-strip and a coefficient of $1/k$ at $u'_y(x,y), |k| > 1$, in the article [1]. The condition for gluing normal Frankl derivatives was set on the type change line . The case of $k = 1$ (a classical problem with a continuous gradient) was not considered and the uniqueness theorem for the auxiliary problem was not proved.

A.V. Bitsadze drew attention to the Tricomi problem with an elliptical part in the form of a half-strip in connection with the mathematical modeling of plane-parallel gas movements. In this case, the construction of a solution by a conformal mapping is reduced to a boundary value problem with respect to an analytical function in the upper half-plane [2, p. 327]. Based on the well-known Schwarz formula [2, p. 315] , A.V. Bitsadze wrote out the solution of this boundary value problem in quadratures.

In [4], an integral representation of the regular solution of the problem for the Laplace equation in a semicircle with a boundary condition of the first kind on a semicircle and two different boundary conditions such as an inclined derivative on two rectilinear sections of the boundary is obtained.

Theorem 1. *Let $k \in (-\infty,-1) \cup (0,+\infty)$, then the solution to the problem (1-4) exists, and it can be represented as a series*

$$u(x,y) = \sum_{n=0}^\infty A_n e^{-\left(n+\frac{1}{2}\right)y} \sin\left[\left(n+\frac{1}{2}\right)x\right], \tag{5}$$

where the coefficients A_n, $n = 0, 1, 2, \ldots$ *are determined from the decomposition*

$$\sum_{n=0}^{\infty} A_n \left(n + \frac{1}{2}\right) \sin\left[\left(n + \frac{1}{2}\right) x + \frac{\pi}{2} - \operatorname{arctg} \frac{1}{k}\right] = \frac{k}{\sqrt{1 + k^2}} \varphi(x). \tag{6}$$

Proof. Let's prove the existence of a solution to the problem $(1 - 4)$. By virtue of the main result of the [2] system $\{\sin[n + \beta/2] + \gamma/2\}_{n=1}^{\infty}$ forms the Riesz basis in the space $L_2(0, \pi)$ when $-3/2 < \gamma/\pi + \beta < 1/2$. In our case, $\gamma = \pi - 2\operatorname{arctg} 1/k$, $\beta = -1$, since indexing starts from zero in the original system. The specified inequalities are satisfied at $k \in (-\infty, -1) \cup (0, +\infty)$. Therefore, the system $\{\sin[(n + 1/2) x + \pi/2 - \operatorname{arctg} 1/k]\}_{n=0}^{\infty}$ forms the Riesz basis in $L_2(0, 1)$ and the two-sided Bessel inequality is valid:

$$C_1 \|\varphi\|_{L_2(0,\pi)} \leq \sum_{n=0}^{\infty} A_n^2 \left(n + \frac{1}{2}\right)^2 \leq C_2 \|\varphi\|_{L_2(0,\pi)}, \ 0 < C_1 < C_2,$$

where C_1, C_2 are independent of φ. Therefore, the series $\sum_{n=0}^{\infty} |A_n|$ converges and the series (5) converges uniformly. The function (5) is a solution to equation (1) and satisfies the boundary conditions (2) by construction. Condition (4) is satisfied because $\sum_{n=0}^{\infty} e^{-(n+1/2)y} = e^{-y/2}/(1 - e^{-y})$. Let's check the fulfillment of condition (3).

According to decomposition (6), condition (3) takes the form

$$I(y) = \int_0^\pi \left[\frac{1}{k} \frac{\partial u}{\partial y} - \frac{\partial u}{\partial x} + \varphi(x)\right]^2 dx,$$

consider the integrand in more detail:

$$M(x) = \frac{1}{k} \frac{\partial u}{\partial y} - \frac{\partial u}{\partial x} + \varphi(x) =$$

$$= \sum_{n=0}^{\infty} \left[A_n \left(n + \frac{1}{2}\right) e^{-\left(n + \frac{1}{2}\right)} \left(-\frac{1}{k} \sin\left[\left(n + \frac{1}{2}\right) x\right] - \cos\left[\left(n + \frac{1}{2}\right) x\right]\right)\right] + \varphi(x) =$$

$$= \sum_{n=0}^{\infty} \left[\frac{\sqrt{1 + k^2}}{k} A_n \left(n + \frac{1}{2}\right) e^{-\left(n + \frac{1}{2}\right)} \left(-\frac{1}{\sqrt{1 + k^2}} \sin\left[\left(n + \frac{1}{2}\right) x\right] - \frac{k}{\sqrt{1 + k^2}} \cos\left[\left(n + \frac{1}{2}\right) x\right]\right)\right] + \varphi(x) =$$

$$= \sum_{n=0}^{\infty} \left[\frac{\sqrt{1 + k^2}}{k} A_n \left(n + \frac{1}{2}\right) e^{-\left(n + \frac{1}{2}\right)} \left(-\sin\left[\operatorname{arctg} \frac{1}{k}\right] \sin\left[\left(n + \frac{1}{2}\right) x\right] - \cos\left[\operatorname{arctg} \frac{1}{k}\right] \cos\left[\left(n + \frac{1}{2}\right) x\right]\right)\right] + \varphi(x)$$

$$= \sum_{n=0}^{\infty} \left[\frac{\sqrt{1 + k^2}}{k} A_n \left(n + \frac{1}{2}\right) e^{-\left(n + \frac{1}{2}\right)} \left(-\cos\left[\left(n + \frac{1}{2}\right) x - \operatorname{arctg} \frac{1}{k}\right]\right)\right] + \varphi(x) =$$

$$= \frac{\sqrt{1 + k^2}}{k} \sum_{n=0}^{\infty} \left[A_n \left(n + \frac{1}{2}\right) \left(1 - e^{-\left(n + \frac{1}{2}\right)}\right) \cos\left[\left(n + \frac{1}{2}\right) x - \operatorname{arctg} \frac{1}{k}\right]\right].$$

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$$I(y) = \int_0^\pi [M(x)]^2\, dx.$$

Let's prove that $I(y) \to 0$ for $y \to 0 + 0$. Let's write down the inequality

$$I(y) \leq I_1(y) + I_2(y),\ \text{where}$$

$$I_1(y) = \frac{2\sqrt{1+k^2}}{k} \int_0^\pi \left[\sum_{n=0}^m A_n \left(n+\frac{1}{2}\right)\left(1 - e^{-\left(n+\frac{1}{2}\right)y}\right)\cos\left[\left(n+\frac{1}{2}\right)x - \operatorname{arctg}\frac{1}{k}\right]\right]^2 dx,$$

$$I_2(y) = \frac{2\sqrt{1+k^2}}{k} \int_0^\pi \left[\sum_{n=m+1}^\infty A_n \left(n+\frac{1}{2}\right)\left(1 - e^{-\left(n+\frac{1}{2}\right)y}\right)\cos\left[\left(n+\frac{1}{2}\right)x - \operatorname{arctg}\frac{1}{k}\right]\right]^2 dx.$$

Let's fix an arbitrary number $\varepsilon > 0$. By virtue of the left-hand side of the Bessel inequality, we have an estimate

$$I_2(y) = \frac{2\sqrt{1+k^2}}{k} \int_0^\pi \left[\sum_{n=m+1}^\infty A_n \left(n+\frac{1}{2}\right)\left(1 - e^{-\left(n+\frac{1}{2}\right)y}\right)\cos\left[\left(n+\frac{1}{2}\right)x - \operatorname{arctg}\frac{1}{k}\right]\right]^2 dx \leq$$

$$\leq C_3 \sum_{n=m+1}^\infty A_n^2 \left(n+\frac{1}{2}\right)^2 \left(1 - e^{-\left(n+\frac{1}{2}\right)y}\right)^2 \leq C_3 \sum_{n=m+1}^\infty A_n^2 \left(n+\frac{1}{2}\right)^2 < \frac{\varepsilon}{2},$$

if m is large enough.

In the second term, we are dealing with a finite number of elements, therefore:

$$I_1(y) = \frac{2\sqrt{1+k^2}}{k} \int_0^\pi \left[\sum_{n=0}^m A_n \left(n+\frac{1}{2}\right)\left(1 - e^{-\left(n+\frac{1}{2}\right)y}\right)\cos\left[\left(n+\frac{1}{2}\right)x - \operatorname{arctg}\frac{1}{k}\right]\right]^2 dx \leq$$

$$\leq C_4 \sum_{n=0}^m A_n^2 \left(n+\frac{1}{2}\right)^2 \left(1 - e^{-\left(n+\frac{1}{2}\right)y}\right)^2 < \frac{\varepsilon}{2}$$

at $0 < y < \delta$, if δ is small enough. Condition (3) is fulfilled. The theorem has been proved.

Theorem 2. *Let $k > 0$, then the solution of the problem (1-4) is unique*

Proof. Let's prove the uniqueness of the solution to this problem. Let $u(x, y)$ be the solution of a homogeneous problem. Let's introduce the notation $C_\varepsilon = (0, \varepsilon), C_R = (0, R), D_R = (\pi, R), D_\varepsilon = (\pi, \varepsilon)$. $\prod_{R\varepsilon}$ is a rectangle $C_\varepsilon C_R D_R D_\varepsilon$. The following ratios are valid:

$$0 = \iint_{\prod_{R\varepsilon}} (R - y)(u_{xx} + u_{yy})\,dx\,dy.$$

Note that

$$(R - y)(u_{xx} + u_{yy})u = ((R - y)\,u_x u)_x + ((R - y)\,u_y u)_y - (R - y)\left(u_x^2 + u_y^2\right) + u_y u =$$

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$$= (R - y)\left(u_{xx}u + u_x^2\right) + \left(-u_y + (R - y)u_{yy}u + (R - y)u_y^2\right) - (R - y)\left(u_x^2 + u_y^2\right) + u_y u$$

Substitute this expression into the integral:

$$I = \iint\limits_{\Pi_{R\varepsilon}} \left((R - y)u_x u\right)_x dxdy + \iint\limits_{\Pi_{R\varepsilon}} \left((R - y)u_y u\right)_y dxdy - \iint\limits_{\Pi_{R\varepsilon}} (R - y)\left(u_x^2 + u_y^2\right) + \iint\limits_{\Pi_{R\varepsilon}} u_y u dxdy.$$

Let's simplify these integrals:

$$\iint\limits_{\Pi_{R\varepsilon}} \left((R - y)u_x u\right)_x dxdy = \int\limits_{[\varepsilon,R]} \left[(R - y)u_x u\right]\big|_0^\pi dy = \int\limits_{[\varepsilon,R]} \left[(R - y)u_x(\pi, y)u(\pi, y) - (R - y)u_x(0, y)u(0, y)\right] dy = 0$$

because both integrands are equal to zero due to condition (2)

$$\iint\limits_{\Pi_{R\varepsilon}} \left((R - y)u_y u\right)_y dxdy = \int\limits_{[0,\pi]} \left[(R - y)u_y u\right]\big|_\varepsilon^R dx = \int\limits_{[0,\pi]} \left[0 - (R - \varepsilon)u_y(x, \varepsilon)u(x, \varepsilon)\right] dx = -\int\limits_{C_\varepsilon D_\varepsilon} (R - \varepsilon)u_y u dx$$

$$\iint\limits_{\Pi_{R\varepsilon}} u_y u dxdy = \iint\limits_{\Pi_{R\varepsilon}} \left(\frac{u^2}{2}\right)_y' dxdy = \int\limits_{[0,\pi]} \left[\frac{u^2(x, R)}{2} - \frac{u^2(x, \varepsilon)}{2}\right] dx$$

As a result, we get

$$I = -\iint\limits_{\Pi_{R\varepsilon}} (R - y)\left(u_x^2 + u_y^2\right) dxdy - \int\limits_{C_\varepsilon D_\varepsilon} (R - \varepsilon)u_y u dx - \int\limits_{C_\varepsilon D_\varepsilon} \frac{u^2}{2} dx + \int\limits_{C_R D_R} \frac{u^2}{2} dx$$

Add and subtract $\int\limits_{C_\varepsilon D_\varepsilon} k(R - \varepsilon)u_x u dx$, then

$$I = -\iint\limits_{\Pi_{R\varepsilon}} (R - y)\left(u_x^2 + u_y^2\right) dxdy - \int\limits_{C_\varepsilon D_\varepsilon} (R - \varepsilon)(u_y - ku_x)u dx - \int\limits_{C_\varepsilon D_\varepsilon} (R - \varepsilon)ku_x u dx - \int\limits_{C_\varepsilon D_\varepsilon} \frac{u^2}{2} dx + \int\limits_{C_R D_R} \frac{u^2}{2} dx.$$

It follows from this

$$\iint\limits_{\Pi_{R\varepsilon}} (R - y)\left(u_x^2 + u_y^2\right) dxdy + \frac{1}{2}\int\limits_{C_\varepsilon D_\varepsilon} u^2 dx + k\frac{R - \varepsilon}{2}u^2(\pi, \varepsilon) = \int\limits_{C_\varepsilon D_\varepsilon} (R - \varepsilon)(u_y - ku_x)u dx + \frac{1}{2}\int\limits_{C_R D_R} u^2 dx \le$$

$$\le \{\text{Cauchy-Bunyakovsky inequality}\} \le (R - \varepsilon)\left[\int\limits_{C_\varepsilon D_\varepsilon} (u_y - ku_x)^2 dx\right]^{\frac{1}{2}} \left[\int\limits_{C_\varepsilon D_\varepsilon} u^2 dx\right]^{\frac{1}{2}} + \frac{1}{2}\int\limits_{C_R D_R} u^2 dx = M$$

Consider the following inequality: $(2ar - b)^2 \ge 0 \Rightarrow ra^2 r^2 - 4abr + b^2 \ge 0 \Rightarrow ab \le ra^2 + b/(4r)$.

Take $a = \left[(R - \varepsilon)\int\limits_{C_\varepsilon D_\varepsilon} (u_y - ku_x)^2 dx\right]^{\frac{1}{2}}$, $b = \left[(R - \varepsilon)\int\limits_{C_\varepsilon D_\varepsilon} u^2 dx\right]^{\frac{1}{2}}$, $r = R - \varepsilon$, then

$$M \le (R - \varepsilon)^2 \int\limits_{C_\varepsilon D_\varepsilon} (u_y - ku_x)^2 dx + \frac{1}{4}\int\limits_{C_\varepsilon D_\varepsilon} u^2 dx + \frac{1}{2}\int\limits_{C_R D_R} u^2 dx.$$

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Let's regroup

$$\iint\limits_{\Pi_{R\varepsilon}} (R - y)\left(u_x^2 + u_y^2\right) dxdy + \frac{1}{4} \int\limits_{C_\varepsilon D_\varepsilon} u^2 dx + k\frac{R - \varepsilon}{2} u^2(\pi, \varepsilon) \leq (R - \varepsilon)^2 \int\limits_{C_\varepsilon D_\varepsilon} (u_y - ku_x)^2 dx + \frac{1}{2} \int\limits_{C_R D_R} u^2 dx.$$

By virtue of (3) there is equality

$$\lim_{\varepsilon \to 0+0} \int\limits_{C_\varepsilon D_\varepsilon} (u_y - ku_x)^2 dx = 0,$$

where does the ratio come from

$$\lim_{\varepsilon \to 0+0} \iint\limits_{\Pi_{R\varepsilon}} (R - y)\left(u_x^2 + u_y^2\right) dxdy + \frac{1}{4} \int\limits_0^\pi u^2(x, 0)dx + k\frac{R}{2} u^2(\pi, 0) \leq \frac{1}{2} \int\limits_{C_R D_R} u^2 dx.$$

Now let's aim $R \to \infty$, then $\int\limits_{C_R D_R} u^2 dx \to 0$, and in the left part all the terms are non-negative,n hence $u(x, y) \equiv 0$ in $\overline{D}$. The theorem has been proved.

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paper, the theorems of the existence and uniqueness of the solution of the Laplace equation with mixed boundary conditions in the half-band are proved, and integral representations for partial derivatives of the solution are obtained.

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