Tricomi Problem for the Lavrentiev-Bitsadze Equation with a Semi-Strip in the Elliptic Part

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1 Problem Statement

The Tricomi problem for the Lavrentiev-Bitsadze equation is considered

$$(sgn(y))\frac{\partial^2 u}{\partial x^2}(x,y)+\frac{\partial^2 u}{\partial y^2}(x,y)=0 \tag{1}$$

in the domain $D=D^{+}\cup D^{-}$, where $D^{+}=\{(x,y):\ 0<x<\pi,\ 0<y<+\infty\}$, $D^{-}=\{(x,y):\ -y<x<y+\pi,\ -\pi/2<y<$ in the class of functions $u(x,y)\in C^2(D^+)\cap C^2(D^-)\cap C(\overline{D^+\cup D^-})$ with boundary conditions

$$u(0,y)=0,\ u(\pi,y)=0,\ 0<y<+\infty, \tag{2}$$

$$u(x,-x)=f(x),\ 0\le x\le \pi/2,\ f(0)=0, \tag{3}$$

$$u(x,y)\rightrightarrows 0,\ y\to+\infty \tag{4}$$

and the Frankl matching condition

$$\frac{1}{k}\frac{\partial u}{\partial y}(x,+0)=\frac{\partial u}{\partial y}(x,-0),\ 0<x<\pi, \tag{5}$$

where $k\in(-\infty,+\infty), k\neq 0$.

2 Main Results

Theorem 1. *The solution to the problem (1) - (5) is unique.*

Proof. Suppose there exist two solutions $u_1(x,y), u_2(x,y)$ to the problem (1)-(5). Then $u(x,y)=u_1(x,y)-u_2(x,y)$ is a solution to the problem (1)-(5) with the function $f(x)\equiv 0$. In this case, $u(x,y)=F(x+y)-F(0)$.

From this, it follows that the equality $\frac{\partial u}{\partial y}-\frac{\partial u}{\partial x}=0$ holds for all points x and y in the domain of hyperbolicity. Using the matching condition (5), we get

$$\frac{1}{k}\frac{\partial u}{\partial y}-\frac{\partial u}{\partial x}|_{y=0+0}=0. \tag{6}$$

As a result, we obtain the problem of finding a harmonic function $u(x,y)$ in the domain D^+ with boundary conditions (2),(4),(6).

By the Zaremba-Giraud principle and equality (6), the extremum cannot be attained on the interval $\{(x,y):\ 0<x<\pi,\ y=0\}$. On the closed lateral sides and at infinity, the extremum cannot be attained due to conditions (2) and (4). The theorem is proved.

It is known that the general solution in D^- of equation (1) has the form

$$u(x,y)=F(x+y)+f(\frac{x-y}{2})-F(0). \tag{7}$$

Differentiate equality (7):

$$\frac{\partial u}{\partial y}(x,y)-\frac{\partial u}{\partial x}(x,y)|_{y=0+0}=-f'\left(\frac{x}{2}\right),\ 0<x<\pi.$$

Using the matching condition (5), we arrive at the equality

$$\frac{1}{k}\frac{\partial u}{\partial y}(x,0+0)-\frac{\partial u}{\partial x}(x,0+0)=-f'\left(\frac{x}{2}\right),\ 0<x<\pi.$$

Then we obtain in the domain D^+ the auxiliary problem for the Laplace operator

$$\frac{\partial^2 u}{\partial x^2}(x,y)+\frac{\partial^2 u}{\partial y^2}(x,y)=0 \tag{8}$$

with boundary conditions

$$u(0,y)=0,\ u(\pi,y)=0,\ 0<y<+\infty, \tag{9}$$

$$\frac{1}{k}\frac{\partial u}{\partial y}(x,0+0)-\frac{\partial u}{\partial x}(x,0+0)=-f'\left(\frac{x}{2}\right), \tag{10}$$

$$u(x,y)\rightrightarrows 0,\ y\to+\infty \tag{11}$$

Theorem 2. *Let $|k|<1,\ k\neq 0,\ f(x)\in C[0,\pi/2]\cap C^2(0,\pi/2),\ f'(x)\in L_2(0,\pi/2)$. Then the solution to the problem (8)-(11) exists and can be represented as a series*

$$u(x,y)=\sum_{n=1}^{\infty}A_n e^{-ny}\sin nx, \tag{12}$$

where the condition (10) is understood in the integral sense

$$\int_0^{\pi}\left[\frac{1}{k}\frac{\partial u}{\partial y}(x,y)-\frac{\partial u}{\partial x}(x,y)+f'\left(\frac{x}{2}\right)\right]^2 dx\to 0,\ y\to 0+0,$$

and the coefficients A_n are determined from the equality

$$\sum_{n=1}^{\infty}nA_n\sin\left[nx+\arctan k\right]=\frac{k}{\sqrt{1+k^2}}f'\left(\frac{x}{2}\right) \tag{13}$$

Proof.

The system $\{\sin\left[nx+\arctan k\right]\}_{n=1}^{\infty}$ forms a Riesz basis in $L_2(0,\pi)$ for $k\in(-1,1)$ according to the main result of [2]. Therefore, the two-sided Bessel inequality holds

$$C_1\|f'\|^2_{L_2(0,\pi)}\leq\sum_{n=1}^{\infty}n^2A_n^2\leq C_2\|f'\|^2_{L_2(0,\pi)},\ 0<C_1<C_2,$$

where the constants C_1,C_2 do not depend on f'. Therefore, the series $\sum\limits_{n=1}^{\infty}|A_n|$ converges and the series (12) converges uniformly. The function (12) satisfies equation (8) with boundary conditions (9) by construction. Condition (11) is satisfied since $\sum\limits_{n=1}^{\infty}e^{-ny}=\frac{e^{-y}}{1-e^{-y}}=\frac{1}{e^y-1}$. Let's check the fulfillment of condition (10). Let

$$M(x)=\frac{1}{k}\frac{\partial u}{\partial y}-\frac{\partial u}{\partial x}+f'\left(\frac{x}{2}\right)$$

$$M(x)=-\frac{1}{k}\sum_{n=1}^{\infty}nA_ne^{-ny}\sin nx-\sum_{n=1}^{\infty}nA_ne^{-ny}\cos nx+f'\left(\frac{x}{2}\right)=$$
$$=-\sum_{n=1}^{\infty}nA_ne^{-ny}\left[\frac{1}{k}\sin nx+\cos nx\right]+f'\left(\frac{x}{2}\right)=$$

$$= -\frac{\sqrt{1+k^2}}{k}\sum_{n=1}^{\infty} nA_n e^{-ny}\left[\frac{1}{\sqrt{1+k^2}}\sin nx + \frac{k}{\sqrt{1+k^2}}\cos nx\right] + f'\left(\frac{x}{2}\right) =$$
$$= -\frac{\sqrt{1+k^2}}{k}\sum_{n=1}^{\infty} nA_n e^{-ny}\sin\left[nx + \operatorname{arctg} k\right] + f'\left(\frac{x}{2}\right) =$$
$$= \frac{\sqrt{1+k^2}}{k}\sum_{n=1}^{\infty} nA_n\left(1-e^{-ny}\right)\sin\left[nx + \operatorname{arctg} k\right].$$

Let's show that $\lim\limits_{y\to 0+0} I(y) = 0$.

$$I(y) = \int\limits_0^{\pi} M(x)^2 dx \le I_1(y) + I_2(y),$$

$$I_1(y) = \frac{2\sqrt{1+k^2}}{k}\int\limits_0^{\pi}\left[\sum_{n=1}^{m} nA_n \sin\left[nx + \operatorname{arctg} k\right]\left(1-e^{-ny}\right)\right]^2 dx$$

$$I_2(y) = \frac{2\sqrt{1+k^2}}{k}\int\limits_0^{\pi}\left[\sum_{n=m+1}^{+\infty} nA_n \sin\left[nx + \operatorname{arctg} k\right]\left(1-e^{-ny}\right)\right]^2 dx$$

Fix an arbitrary positive ε, then

$$I_2(y) \le C_3\sum_{n=m+1}^{\infty} n^2A_n^2(1-e^{-ny})^2 \le C_3\sum_{n=m+1}^{\infty} n^2A_n^2 < \frac{\varepsilon}{2}.$$

This is true if m is sufficiently large, since the series is convergent.

$$I_1(y) \le C_4\sum_{n=1}^{m} n^2A_n^2(1-e^{-ny})^2 < \frac{\varepsilon}{2}$$

This is true for $0 < y < \delta$, if δ is sufficiently small. The theorem is proved.

Theorem 3. *Let $k > 0$, then the solution to the problem (8) - (11) is unique*

Proof. Let's prove the uniqueness of the solution to this problem. Let $u(x,y)$ be a solution to the homogeneous problem. Introduce the notation $C_\varepsilon = (0,\varepsilon), C_R = (0,R), D_R = (\pi,R), D_\varepsilon = (\pi,\varepsilon)$. $\prod_{R\varepsilon}$ is the rectangle $C_\varepsilon C_R D_R D_\varepsilon$. The following relations hold:

$$0 = \iint\limits_{\Pi_{R\varepsilon}} (R-y)(u_{xx}+u_{yy})dxdy.$$

Note that

$$(R-y)(u_{xx}+u_{yy})u = \left((R-y)\,u_xu\right)_x + \left((R-y)\,u_yu\right)_y - (R-y)\left(u_x^2+u_y^2\right) + u_yu =$$
$$= (R-y)\left(u_{xx}u + u_x^2\right) + \left(-u_y + (R-y)\,u_{yy}u + (R-y)\,u_y^2\right) - (R-y)\left(u_x^2+u_y^2\right) + u_yu$$

Substitute this expression into the integral:

$$I = \iint\limits_{\Pi_{R\varepsilon}} \left((R-y)\,u_xu\right)_x dxdy + \iint\limits_{\Pi_{R\varepsilon}} \left((R-y)\,u_yu\right)_y dxdy - \iint\limits_{\Pi_{R\varepsilon}} (R-y)\left(u_x^2+u_y^2\right) + \iint\limits_{\Pi_{R\varepsilon}} u_yu dxdy.$$

Simplify these integrals:

$$\iint\limits_{\Pi_{R\varepsilon}} \left((R-y)\,u_xu\right)_x dxdy = \int\limits_{[\varepsilon,R]} \left[(R-y)u_xu\right]|_0^{\pi} dy = \int\limits_{[\varepsilon,R]} \left[(R-y)u_x(\pi,y)u(\pi,y) - (R-y)u_x(0,y)u(0,y)\right] dy = 0$$

since both integrands are zero due to condition (9)

$$\iint\limits_{\Pi_{R\varepsilon}} \left((R-y)\,u_y u\right)_y dxdy = \int\limits_{[0,\pi]} \left[(R-y)\,u_y u\right]\Big|_{\varepsilon}^{R} dx = \int\limits_{[0,\pi]} \left[0-(R-\varepsilon)\,u_y(x,\varepsilon)u(x,\varepsilon)\right] dx = -\int\limits_{C_\varepsilon D_\varepsilon} (R-\varepsilon)\,u_y u dx$$

$$\iint\limits_{\Pi_{R\varepsilon}} u_y u dxdy = \iint\limits_{\Pi_{R\varepsilon}} \left(\frac{u^2}{2}\right)'_y dxdy = \int\limits_{[0,\pi]} \left[\frac{u^2(x,R)}{2} - \frac{u^2(x,\varepsilon)}{2}\right] dx$$

In the end, we get

$$I = -\iint\limits_{\Pi_{R\varepsilon}} (R-y)\left(u_x^2+u_y^2\right) dxdy - \int\limits_{C_\varepsilon D_\varepsilon} (R-\varepsilon)\,u_y u dx - \int\limits_{C_\varepsilon D_\varepsilon} \frac{u^2}{2} dx + \int\limits_{C_R D_R} \frac{u^2}{2} dx$$

Add and subtract $\int\limits_{C_\varepsilon D_\varepsilon} k\,(R-\varepsilon)\,u_x u dx$, then

$$I = -\iint\limits_{\Pi_{R\varepsilon}} (R-y)\left(u_x^2+u_y^2\right) dxdy - \int\limits_{C_\varepsilon D_\varepsilon} (R-\varepsilon)\left(u_y - ku_x\right) u dx - \int\limits_{C_\varepsilon D_\varepsilon} (R-\varepsilon)\,k u_x u dx - \int\limits_{C_\varepsilon D_\varepsilon} \frac{u^2}{2} dx + \int\limits_{C_R D_R} \frac{u^2}{2} dx.$$

From this, it follows that

$$\iint\limits_{\Pi_{R\varepsilon}} (R-y)\left(u_x^2+u_y^2\right) dxdy + \frac{1}{2}\int\limits_{C_\varepsilon D_\varepsilon} u^2 dx + k\frac{R-\varepsilon}{2}u^2(\pi,\varepsilon) = \int\limits_{C_\varepsilon D_\varepsilon} (R-\varepsilon)\left(u_y-ku_x\right) u dx + \frac{1}{2}\int\limits_{C_R D_R} u^2 dx \le$$

$$\le \{\text{Cauchy-Bunyakovsky Inequality}\} \le (R-\varepsilon)\left[\int\limits_{C_\varepsilon D_\varepsilon} \left(u_y-ku_x\right)^2 dx\right]^{\frac{1}{2}} \left[\int\limits_{C_\varepsilon D_\varepsilon} u^2 dx\right]^{\frac{1}{2}} + \frac{1}{2}\int\limits_{C_R D_R} u^2 dx = M$$

Consider the following inequality: $(2ar-b)^2 \ge 0 \Rightarrow ra^2r^2 - 4abr + b^2 \ge 0 \Rightarrow ab \le ra^2 + b/(4r)$. Let $a = \left[(R-\varepsilon)\int\limits_{C_\varepsilon D_\varepsilon} \left(u_y-ku_x\right)^2 dx\right]^{\frac{1}{2}}$, $b = \left[(R-\varepsilon)\int\limits_{C_\varepsilon D_\varepsilon} u^2 dx\right]^{\frac{1}{2}}$, $r = R-\varepsilon$, then

$$M \le (R-\varepsilon)^2 \int\limits_{C_\varepsilon D_\varepsilon} \left(u_y-ku_x\right)^2 dx + \frac{1}{4}\int\limits_{C_\varepsilon D_\varepsilon} u^2 dx + \frac{1}{2}\int\limits_{C_R D_R} u^2 dx.$$

Regroup

$$\iint\limits_{\Pi_{R\varepsilon}} (R-y)\left(u_x^2+u_y^2\right) dxdy + \frac{1}{4}\int\limits_{C_\varepsilon D_\varepsilon} u^2 dx + k\frac{R-\varepsilon}{2}u^2(\pi,\varepsilon) \le (R-\varepsilon)^2 \int\limits_{C_\varepsilon D_\varepsilon} \left(u_y-ku_x\right)^2 dx + \frac{1}{2}\int\limits_{C_R D_R} u^2 dx.$$

Due to (10), the equality holds

$$\lim_{\varepsilon\to 0+0} \int\limits_{C_\varepsilon D_\varepsilon} \left(u_y-ku_x\right)^2 dx = 0,$$

from which it follows that

$$\lim_{\varepsilon\to 0+0} \iint\limits_{\Pi_{R\varepsilon}} (R-y)\left(u_x^2+u_y^2\right) dxdy + \frac{1}{4}\int\limits_0^\pi u^2(x,0)dx + k\frac{R}{2}u^2(\pi,0) \le \frac{1}{2}\int\limits_{C_R D_R} u^2 dx.$$

Now let $R\to\infty$, then $\int\limits_{C_R D_R} u^2 dx \to 0$, and in the left part all terms are non-negative, hence $u(x,y)\equiv 0$ in $\overline{D}$. The theorem is proved.

Theorem 4. *Let $|k| < 1$, $k \neq 0$ and $u(x,y)$ be a solution to the problem (8) - (11), then u_x, u_y can be represented as*

$$u_y(x,y) = -\frac{2k}{\pi\sqrt{1+k^2}}\mathrm{Im}\ \left(\frac{1-e^{iz}}{1+e^{iz}}\right)^{\gamma/\pi} e^{iz}\int\limits_0^{\pi} \frac{1}{(\mathrm{tg}\, t/2)^{\gamma/\pi}} \frac{\sin t}{\left(1-e^{i(z+t)}\right)\left(1-e^{i(z-t)}\right)} f'(\frac{t}{2})dt,$$

$$u_x(x,y) = \frac{2k}{\pi\sqrt{1+k^2}}\mathrm{Re}\ \left(\frac{1-e^{iz}}{1+e^{iz}}\right)^{\gamma/\pi} e^{iz}\int\limits_0^{\pi} \frac{1}{(\mathrm{tg}\, t/2)^{\gamma/\pi}} \frac{\sin t}{\left(1-e^{i(z+t)}\right)\left(1-e^{i(z-t)}\right)} f'(\frac{t}{2})dt,$$

where $\gamma = 2\,\mathrm{arctg}\, k$, $z = x+iy$.

Proof. Consider equality (12). The system of sines $\{\sin[nx + \mathrm{arctg}\, k]\}_{n=1}^{\infty}$ forms a basis in $L_2(0,\pi)$ for $k \in (-1,1)$. Therefore, for the coefficients nA_n, the following representation holds [2]:

$$nA_n = \int\limits_0^{\pi} h_n(t)F(t)dt,$$

where

$$F(x) = \frac{k}{\sqrt{1+k^2}}f'(\frac{x}{2})\ , h_n(t) = \frac{2}{\pi}\frac{(2\cos t/2)^{\beta}}{(\mathrm{tg}\, t/2)^{\gamma/\pi}}\sum_{k=1}^{n}\sin kt B_{n-k},\ B_l = \sum_{m=0}^{l} C_{\gamma/\pi}^{l-m}C_{-\gamma/\pi-l}^{m}(-1)^{l-m},\ C_l^n = \frac{l(l-1)\dots(l-n+1)}{n!}.$$

Let $u(x,y)$ be a solution to the problem $(8)-(11)$, then

$$u(x,y) = \sum_{n=1}^{\infty} A_n e^{-ny}\sin[nx]$$

and accordingly

$$u_y(x,y) = -\sum_{n=1}^{\infty} nA_n e^{-ny}\sin[nx] =$$

$$= -\sum_{n=1}^{\infty}\int\limits_0^{\pi} F(t)h_n(t)e^{-ny}\sin[nx]dt,$$

or

$$u_y(x,y) = -Im\ \sum_{n=1}^{\infty}\int\limits_0^{\pi} F(t)h_n(t)e^{-ny}e^{inx}dt =$$

$$= -Im\ \sum_{n=1}^{\infty}\int\limits_0^{\pi} F(t)h_n(t)e^{inz}dt$$

Swap the order of integration and summation

$$u_y(x,y) = -\mathrm{Im}\int\limits_0^{\pi} F(t)\sum_{n=1}^{\infty} h_n(t)e^{inz}dt.$$

Introduce new notation:

$$I(t,z) = \sum_{n=1}^{\infty} h_n(t)e^{inz}$$

$$I(t,z) = \frac{2}{\pi}\frac{(2\cos t/2)^{\beta}}{(\mathrm{tg}\, t/2)^{\gamma/\pi}}\sum_{n=1}^{\infty}\sum_{k=1}^{n}\sin kt B_{n-k}e^{inz} = \frac{2}{\pi}\frac{(2\cos t/2)^{\beta}}{(\mathrm{tg}\, t/2)^{\gamma/\pi}}\sum_{k=1}^{\infty}\sin kt\sum_{n=k}^{\infty} e^{inz}B_{n-k}$$

and new index $m = n - k$

$$I(t,z) = \frac{2}{\pi}\frac{(2\cos t/2)^{\beta}}{(\operatorname{tg} t/2)^{\gamma/\pi}}\sum_{k=1}^{\infty}\sin kt\sum_{m=0}^{\infty}e^{i(m+k)z}B_m = \frac{2}{\pi}\frac{(2\cos t/2)^{\beta}}{(\operatorname{tg} t/2)^{\gamma/\pi}}\sum_{k=1}^{\infty}e^{ikz}\sin kt\sum_{m=0}^{\infty}e^{imz}B_m,$$

$$\sum_{k=1}^{\infty}e^{ikz}\sin kt = \frac{e^{iz}\sin t}{\left(1-e^{i(z+t)}\right)\left(1-e^{i(z-t)}\right)}$$

Consider the second series:

$$\sum_{l=0}^{\infty}e^{ilz}B_l = \sum_{l=0}^{\infty}e^{ilz}\sum_{m=0}^{l}C_{\gamma/\pi}^{l-m}C_{-\gamma/\pi-\beta}^{m}(-1)^{l-m} = \sum_{m=0}^{\infty}\sum_{l=m}^{\infty}e^{ilz}C_{\gamma/\pi}^{l-m}C_{-\gamma/\pi-\beta}^{m}(-1)^{l-m} = |k = l-m| =$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}e^{i(m+k)z}C_{\gamma/\pi}^{k}C_{-\gamma/\pi-\beta}^{m}(-1)^{k} = \sum_{m=0}^{\infty}e^{imz}C_{-\gamma/\pi-\beta}^{m}\sum_{k=0}^{\infty}C_{\gamma/\pi}^{k}(-1)^{k}e^{ikz} = (1+e^{iz})^{-\gamma/\pi-\beta}(1-e^{iz})^{\gamma/\pi}$$

since in our case $\beta = 0,\ \gamma = 2\operatorname{arctg} k$. Finally, we obtain the formula

$$u_y(x,y) = -\operatorname{Im}\int_0^{\pi}F(t)I(t,z)dt =$$

$$= -\operatorname{Im}\int_0^{\pi}\frac{2}{\pi}\frac{(2\cos t/2)^{\beta}}{(\operatorname{tg} t/2)^{\gamma/\pi}}\frac{e^{iz}\sin t}{\left(1-e^{i(z+t)}\right)\left(1-e^{i(z-t)}\right)}(1+e^{iz})^{-\gamma/\pi-\beta}(1-e^{iz})^{\gamma/\pi}F(t)dt =$$

$$= -\frac{2}{\pi}\operatorname{Im}\left(\frac{1-e^{iz}}{1+e^{iz}}\right)^{\gamma/\pi}e^{iz}\int_0^{\pi}\frac{1}{(\operatorname{tg} t/2)^{\gamma/\pi}}\frac{\sin t}{\left(1-e^{i(z+t)}\right)\left(1-e^{i(z-t)}\right)}F(t)dt =$$

$$= -\frac{2k}{\pi\sqrt{1+k^2}}\operatorname{Im}\left(\frac{1-e^{iz}}{1+e^{iz}}\right)^{\gamma/\pi}e^{iz}\int_0^{\pi}\frac{1}{(\operatorname{tg} t/2)^{\gamma/\pi}}\frac{\sin t}{\left(1-e^{i(z+t)}\right)\left(1-e^{i(z-t)}\right)}f'(\frac{t}{2})dt$$

i.e., the representation:

$$u_y(x,y) = -\frac{2k}{\pi\sqrt{1+k^2}}\operatorname{Im}\left(\frac{1-e^{iz}}{1+e^{iz}}\right)^{\gamma/\pi}e^{iz}\int_0^{\pi}\frac{1}{(\operatorname{tg} t/2)^{\gamma/\pi}}\frac{\sin t}{\left(1-e^{i(z+t)}\right)\left(1-e^{i(z-t)}\right)}f'(\frac{t}{2})dt.$$

By similar reasoning, we obtain the representation

$$u_x(x,y) = \frac{2k}{\pi\sqrt{1+k^2}}\operatorname{Re}\left(\frac{1-e^{iz}}{1+e^{iz}}\right)^{\gamma/\pi}e^{iz}\int_0^{\pi}\frac{1}{(\operatorname{tg} t/2)^{\gamma/\pi}}\frac{\sin t}{\left(1-e^{i(z+t)}\right)\left(1-e^{i(z-t)}\right)}f'(\frac{t}{2})dt.$$

The theorem is proved.

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