INFORMATION THEORY
Master of Logic, University of Amsterdam, 2016
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Practice problem set 6

This week's exercises deal with zero-error coding and channel capacity. You do not have to hand in these exercises, they are for practicing only. Problems marked with a ★ are generally a bit harder. If you have questions about any of the exercises, please post them in the discussion forum on Moodle, and try to help each other. We will also keep an eye on the forum.

Problem 1: Confusability graphs

For each of the channels below, give the corresponding confusability graph.

(a) $\mathcal{X} = \{1,2,3,4,5\}$, $\mathcal{Y} = \{a,b,c\}$, $P_{Y|X}(a|1) = P_{Y|X}(b|1) = P_{Y|X}(a|2) = P_{Y|X}(b|2) = \frac{1}{2}$, $P_{Y|X}(b|3) = \frac{1}{3}$, $P_{Y|X}(c|3) = \frac{2}{3}$, $P_{Y|X}(c|4) = P_{Y|X}(c|5) = 1$.

(b) $\mathcal{X} = \{1,2,3,4,5\}$, $\mathcal{Y} = \{a,b,c,d\}$, $P_{Y|X}(a|2) = P_{Y|X}(b|2) = P_{Y|X}(c|2) = P_{Y|X}(a|4) = P_{Y|X}(c|4) = P_{Y|X}(d|4) = \frac{1}{3}$, $P_{Y|X}(b|3) = P_{Y|X}(c|3) = \frac{1}{2}$, $P_{Y|X}(a|1) = P_{Y|X}(d|5) = 1$.

Problem 2

We are given the following joint distribution of $X \in \{1,2,3\}$ and $Y \in \{a,b,c\}$:

$$P_{XY}(1,a) = P_{XY}(2,b) = P_{XY}(3,c) = 1/6$$
$$P_{XY}(1,b) = P_{XY}(1,c) = P_{XY}(2,a) = P_{XY}(2,c) = P_{XY}(3,a) = P_{XY}(3,b) = 1/12.$$

Let $\hat{X}(Y)$ be an estimator for X (based on Y), and let $p_e = P[\hat{X} \neq X]$.

(a) Find an estimator $\hat{X}(Y)$ for which the probability of error p_e is as small as possible.

(b) Evaluate Fano's inequality for this problem, and compare.

Problem 3: Symmetric Channels

Consider the channel with transition matrix

$$P_{Y|X} = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.3 & 0.2 \\ 0.2 & 0.5 & 0.3 \end{bmatrix}.$$

Recall that in a transition matrix, the entry in the xth row and the yth column denotes the conditional probability $P_{Y|X}(y|x)$ that y is received when x has been sent.

A channel is said to be **symmetric** if the rows of the channel transition matrix $P_{Y|X}$ are permutations of each other, and the columns are permutations of each other. A channel is said to be **weakly symmetric** if every row of the transition matrix is a permutation of every other row and all the column sums $\sum_x P_{Y|X}(y|x)$ are equal.

For instance, the channel $P_{Y|X}$ above is symmetric, and the channel

$$Q_{Y|X} = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{bmatrix}$$

is weakly symmetric but not symmetric.

(a) Find the optimal input distribution and channel capacity of $Q_{Y|X}$.

(b) Give a general strategy how to compute the capacity of weakly symmetric channels. What is the optimal input distribution?

Problem 4: Multiple Channel Uses

Prove the lemma below stating that the capacity per transmission is not increased if we use a discrete memoryless channel many times. For inspiration, look again at the proof of the converse of Shannon's noisy-channel coding theorem.

Lemma 7.9.2 in [CT] Let Y^n be the result of passing X^n through a discrete memoryless channel of capacity C. Prove that for all P_{X^n}, it holds that $I(X^n; Y^n) \leq nC$.

Does your proof also work in case of coding with feedback (i.e. X_{i+1} is allowed to depend on X^i and Y^i)? If not, point out the steps in your proof where you use that there is no feedback.