INFORMATION THEORY
Master of Logic, University of Amsterdam, 2016
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Practice problem set 1

This week's exercises deal with the basics of probability theory and entropy, as well as with different proof techniques. You do not have to hand in these exercises, they are for practicing only. Problems marked with a ★ are generally a bit harder. If you have questions about any of the exercises, please post them in the discussion forum on Moodle, and try to help each other. We will also keep an eye on the forum.

Problem 1: Two dice

Consider an experiment where we throw two fair six-sided dice: a red one and a blue one.

(a) What is the probability space $(\Omega, \mathcal{F}, P)$ for this experiment? What would be the probability space if the dice were both red (i.e. indistinguishable)?

(b) Let X be the random variable that describes the sum of the two outcomes. Describe its range $\mathcal{X}$ and distribution P_X. What is $P_X(7) = P(X = 7)$?

(c) Let Y be the random variable that describes the *parity* of the sum, i.e. $\mathcal{Y} = \{\text{even}, \text{odd}\}$. What is $P_{X|Y}(7|\text{odd})$? And $P_{X|Y}(7|\text{even})$?

(d) Verify that for an arbitrary random variable X, $(\mathcal{X}, \mathcal{P}(X), P_X)$ is a probability space.

Problem 2: Inverse probabilities

What is the probability that two (or more) students in this exercise class have the same birthday? (Assume everybody was born in the same year.)

Problem 3: Events

Let $\mathcal{A}, \mathcal{B}$ be events (subsets of some sample space Ω). Prove the following identities:

(a) $P[\overline{\mathcal{A}}] = 1 - P[\mathcal{A}]$

(b) $P[\mathcal{A} \cup \mathcal{B}] = P[\mathcal{A}] + P[\mathcal{B}] - P[\mathcal{A}, \mathcal{B}]$

(c) $P[\mathcal{A}] = P[\mathcal{A}, \mathcal{B}] + P[\mathcal{A}, \overline{\mathcal{B}}]$

Problem 4: Proof by induction

(a) Prove by induction on n that for all $n \in \mathbb{N}_+$,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

(b) **Union bound** Prove the union bound for a finite number of events, which states that for arbitrary events $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$,

$$P\left(\bigcup_{i=1}^{n} \mathcal{A}_i\right) \leq \sum_{i=1}^{n} P(\mathcal{A}_i).$$

★ Can you find an exact formula for $P\left(\bigcup_{i=1}^{n} \mathcal{A}_i\right)$?

Problem 5: Properties of entropy

Let X and Y be random variables.

(a) Prove that $H(X) = 0$ if and only if X is *constant*, i.e. there is an $x_0 \in \mathcal{X}$ such that $P_X(x_0) = 1$, and $P_X(x') = 0$ for all $x' \neq x_0$.

(b) Prove that $H(XY) = H(X) + H(Y)$ if and only if X and Y are independent.

(c) Prove that $H(X) = \log |\mathcal{X}|$ if X is uniformly distributed.

★ Prove that X is uniformly distributed if $H(X) = \log |\mathcal{X}|$.

Problem 6: Entropy of a deck of cards

(a) Compute the entropy of a perfectly shuffled deck of 52 cards (i.e. the set of cards is uniformly distributed over all possible orders).

(b) Now suppose we have a perfectly shuffled big deck, consisting of two *identical* decks of 52 cards (104 cards in total). You cannot tell the difference between, for example, the ace of spades of one deck and the ace of spades of the other. Compute the entropy of the shuffled big deck.