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Practice problem set 4

This week's exercises deal with entropy diagrams and stochastic processes. You do not have to hand in these exercises, they are for practicing only. Problems marked with a ★ are generally a bit harder. If you have questions about any of the exercises, please post them in the discussion forum on Moodle, and try to help each other. We will also keep an eye on the forum.

Problem 1: Entropy diagram

Show that the value

$$R(X;Y;Z) = I(X;Y) - I(X;Y|Z)$$

is invariant under permutations of its arguments.

Problem 2

For each statement below, specify a joint distribution P_{XYZ} of random variables X, Y, and Z (P_{XY} of X and Y in (a)) such that the following inequalities hold.

(a) There exists a y such that $H(X|Y=y) > H(X)$.

(b) $I(X;Y) > I(X;Y|Z)$

(c) $I(X;Y) < I(X;Y|Z)$

Note: the distributions have to be different from the examples from the lecture or the lecture notes.

Problem 3: Conditional mutual information

Consider a sequence of n binary random variables $X_1, X_2, ..., X_n$. Each sequence with an even number of 1's has probability $2^{-(n-1)}$ and each sequence with an odd number of 1's has probability 0. Find the mutual informations

$$I(X_1;X_2), I(X_2;X_3|X_1), ..., I(X_{n-1};X_n|X_1,...,X_{n-2}).$$

Problem 4: Markov chains with 4 random variables

Let $W \to X \to Y \to Z$ be a Markov chain with 4 RVs, i.e. it holds that $P_{Z|YXW} = P_{Z|Y}$ and both $W \leftrightarrow X \leftrightarrow Y$ and $X \leftrightarrow Y \leftrightarrow Z$ are Markov chains with three random variables as defined in the lecture.

(a) Show that $W \leftrightarrow (X,Y) \leftrightarrow Z$ is a Markov chain with three random variables $W, (XY), Z$.
Hint: Argue and use the fact that $P_{X|YZ} = P_{X|Y}$.

(b) Show that $Z \to Y \to X \to W$. Therefore, it is also justified to write $W \leftrightarrow X \leftrightarrow Y \leftrightarrow Z$, as in the case for three RVs.

★ Can you generalize the two properties above to Markov chains of $n > 4$ random variables?

Problem 5: Cesáro mean

Show that if $a_n \to a$ and $b_n = \frac{1}{n}\sum_{i=1}^{n} a_i$, then $b_n \to a$. This is Theorem 4.2.3 in Cover & Thomas.

Problem 6: Stationary processes

Let $..., X_{-1}, X_0, X_1, ...$ be a stationary (not necessarily Markov) stochastic process. Recall that this means that for any $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, it holds that $P_{X_1...X_n} = P_{X_{1+k},...,X_{n+k}}$. Which of the following statements are true? Prove or provide a counterexample.

(a) $H(X_n|X_0) = H(X_{-n}|X_0)$

(b) $H(X_n|X_0) \geq H(X_{n-1}|X_0)$

(c) $H(X_n|X_1, X_2, ..., X_{n-1}, X_{n+1})$ is nonincreasing in n.

(d) $H(X_n|X_1, X_2, ..., X_{n-1}, X_{n+1}, ..., X_{2n})$ is nonincreasing in n.