

Practice problem set 4

This week's exercises deal with entropy diagrams and stochastic processes. You do not have to hand in these exercises, they are for practicing only. Problems marked with a ★ are generally a bit harder. If you have questions about any of the exercises, please post them in the [discussion forum on Moodle](#), and try to help each other. We will also keep an eye on the forum.

Problem 1: Entropy diagram

Show that the value

$$R(X; Y; Z) = I(X; Y) - I(X; Y|Z)$$

is invariant under permutations of its arguments.

Problem 2: Cesàro mean

Show that if $a_n \rightarrow a$ and $b_n = \frac{1}{n} \sum_{i=1}^n a_i$, then $b_n \rightarrow a$. This is Theorem 4.2.3 in Cover & Thomas.

Problem 3

For each statement below, specify a joint distribution P_{XYZ} of random variables X , Y , and Z (P_{XY} of X and Y in (a)) such that the following inequalities hold.

- (a) There exists a y such that $H(X|Y=y) > H(X)$.
- (b) $I(X; Y) > I(X; Y|Z)$
- (c) $I(X; Y) < I(X; Y|Z)$

Note: the distributions have to be different from the examples from the lecture or the lecture notes.

Problem 4: Conditional mutual information

Consider a sequence of n binary random variables X_1, X_2, \dots, X_n . Each sequence with an even number of 1's has probability $2^{-(n-1)}$ and each sequence with an odd number of 1's has probability 0. Find the mutual informations

$$I(X_1; X_2), I(X_2; X_3|X_1), \dots, I(X_{n-1}; X_n|X_1, \dots, X_{n-2}).$$

Problem 5: Bernoulli process

Let X_1, X_2, \dots be distributed according to the Bernoulli(p) distribution. Consider the associated Markov chain $\{Y_i\}_{i=1}^n$, where Y_i is the number of 1's in the current run of 1's. For example, if $X^n = 101110\dots$, then $Y^n = 101230\dots$.

- (a) Find the entropy rate of X^n .
- (b) Find the entropy rate of Y^n

Problem 6: Stationary processes

Let $\dots, X_{-1}, X_0, X_1, \dots$ be a stationary (not necessarily Markov) stochastic process. Which of the following statements are true? Prove or provide a counterexample.

- (a) $H(X_n|X_0) = H(X_{-n}|X_0)$
- (b) $H(X_n|X_0) \geq H(X_{n-1}|X_0)$
- (c) $H(X_n|X_1, X_2, \dots, X_{n-1}, X_{n+1})$ is nonincreasing in n .
- (d) $H(X_n|X_1, X_2, \dots, X_{n-1}, X_{n+1}, \dots, X_{2n})$ is nonincreasing in n .

Problem 7: Branching process

A random process repeatedly flips a fair coin to choose between the two words ab and abc. A typical sample from this process is

abcbcabababcbcabcbcabcbcabababcbababcbabc...

Let X_i denote the letter at the i th position.

- (a) Draw the transition diagram for the process X_1, X_2, X_3, \dots
- (b) Is this process stationary?
- (c) Compute the entropy rate of this process.