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Practice problem set 4

This week's exercises deal with entropy diagrams and stochastic processes. You do not have to hand in these exercises, they are for practicing only. Problems marked with a ★ are generally a bit harder. If you have questions about any of the exercises, please post them in the discussion forum on Moodle, and try to help each other. We will also keep an eye on the forum.

Problem 1: Entropy diagram

Show that the value

$$R(X;Y;Z) = I(X;Y) - I(X;Y|Z)$$

is invariant under permutations of its arguments.

Problem 2: Cesáro mean

Show that if $a_n \to a$ and $b_n = \frac{1}{n}\sum_{i=1}^{n} a_i$, then $b_n \to a$. This is Theorem 4.2.3 in Cover & Thomas.

Problem 3

For each statement below, specify a joint distribution P_{XYZ} of random variables X, Y, and Z (P_{XY} of X and Y in (a)) such that the following inequalities hold.

(a) There exists a y such that $H(X|Y = y) > H(X)$.

(b) $I(X;Y) > I(X;Y|Z)$

(c) $I(X;Y) < I(X;Y|Z)$

Note: the distributions have to be different from the examples from the lecture or the lecture notes.

Problem 4: Conditional mutual information

Consider a sequence of n binary random variables $X_1, X_2, ..., X_n$. Each sequence with an even number of 1's has probability $2^{-(n-1)}$ and each sequence with an odd number of 1's has probability 0. Find the mutual informations

$$I(X_1;X_2), I(X_2;X_3|X_1), ..., I(X_{n-1};X_n|X_1, ..., X_{n-2}).$$

Problem 5: Bernoulli process

Let $X_1, X_2, ...$ be distributed according to the Bernoulli(p) distribution. Consider the associated Markov chain $\{Y_i\}_{i=1}^{n}$, where Y_i is the number of 1's in the current run of 1's. For example, if $X^n = 101110...$, then $Y^n = 101230....$.

(a) Find the entropy rate of X^n.

(b) Find the entropy rate of Y^n

Problem 6: Stationary processes

Let $..., X_{-1}, X_0, X_1, ...$ be a stationary (not necessarily Markov) stochastic process. Which of the following statements are true? Prove or provide a counterexample.

(a) $H(X_n|X_0) = H(X_{-n}|X_0)$

(b) $H(X_n|X_0) \geq H(X_{n-1}|X_0)$

(c) $H(X_n|X_1, X_2, ..., X_{n-1}, X_{n+1})$ is nonincreasing in n.

(d) $H(X_n|X_1, X_2, ..., X_{n-1}, X_{n+1}, ..., X_{2n})$ is nonincreasing in n.

Problem 7: Branching process

A random process repeatedly flips a fair coin to choose between the two words `ab` and `abc`. A typical sample from this process is

`abcabcababababcabcabcabcabcabababcababababcabc...`

Let X_i denote the letter at the ith position.

(a) Draw the transition diagram for the process $X_1, X_2, X_3, ...$

(b) Is this process stationary?

(c) Compute the entropy rate of this process.