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Practice problem set 1

This week's exercises deal with the basics of probability theory and entropy, as well as with different proof techniques. You do not have to hand in these exercises, they are for practicing only. Problems marked with a ★ are generally a bit harder. If you have questions about any of the exercises, please post them in the discussion forum on Moodle, and try to help each other. We will also keep an eye on the forum.

Problem 1: Inverse probabilities

What is the probability that two (or more) students in this exercise class have the same birthday? (Assume everybody was born in the same year.)

Problem 2: Events

Let $\mathcal{A}, \mathcal{B}$ be events (subsets of some sample space Ω). Prove the following identities:

(a) $P[\overline{\mathcal{A}}] = 1 - P[\mathcal{A}]$

(b) $P[\mathcal{A} \cup \mathcal{B}] = P[\mathcal{A}] + P[\mathcal{B}] - P[\mathcal{A}, \mathcal{B}]$

(c) $P[\mathcal{A}] = P[\mathcal{A}, \mathcal{B}] + P[\mathcal{A}, \overline{\mathcal{B}}]$

Problem 3: Proof by induction

(a) Prove by induction on n that for all $n \in \mathbb{N}_+$,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

(b) **Union bound** Prove the union bound for a finite number of events, which states that for arbitrary events $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$,

$$P\left(\bigcup_{i=1}^{n} \mathcal{A}_i\right) \leq \sum_{i=1}^{n} P(\mathcal{A}_i).$$

★ Can you find an exact formula for $P\left(\bigcup_{i=1}^{n} \mathcal{A}_i\right)$?

Problem 4

Let X be a random variable on the sample space $\mathcal{X}$, with associated distribution P_X.

(a) Verify that $(\mathcal{X}, \mathcal{P}(X), P_X)$ is a probability space.

(b) ...

Problem 5: Entropy of a deck of cards

(a) Compute the entropy of a perfectly shuffled (i.e. uniformly distributed over all possible orders) deck of 52 cards.

(b) Now suppose we have a perfectly shuffled big deck, consisting of two *identical* decks of 52 cards (104 cards in total). You cannot tell the difference between, for example, the ace of spades of one deck and the ace of spades of the other. Compute the entropy of the shuffled big deck.

Problem 6: Properties of entropy

Let X and Y be random variables.

(a) Prove that $H(X) = 0$ if and only if X is *constant*, i.e. there is an $x \in \mathcal{X}$ such that $P_X(x) = 1$, and $P_X(x') = 0$ for all $x' \neq x$.

(b) Prove that $H(X) = \log |\mathcal{X}|$ if and only if X is uniformly distributed.

(c) Prove that $H(XY) = H(X) + H(Y)$ if and only if X and Y are independent.