

## INFORMATION THEORY

Master of Logic, University of Amsterdam, 2017

TEACHER: Christian Schaffner, TA: Yfke Dulek, Alvaro Piedrafita

# Practice problem set 2

This week's exercises deal with entropy and source codes. You do not have to hand in these exercises, they are for practicing only. Problems marked with a ★ are generally a bit harder. If you have questions about any of the exercises, please post them in the [discussion forum on Moodle](#), and try to help each other. We will also keep an eye on the forum.

## Problem 1: Mutual information

Let  $X$ ,  $Y$  and  $Z$  be random variables such that  $I(X;Y) = 0$  and  $I(X;Z) = 0$ . Does it follow that  $I(Y;Z) = 0$ ? If so, prove it. If not, give a counterexample.

## Problem 2: Estimating entropy

([\[MacKay\]](#), Example 2.13:) A source produces a character  $x$  from alphabet  $\mathcal{A} = \{0, 1, 2, \dots, 9, \text{a}, \text{b}, \text{c}, \dots, \text{z}\}$ . With probability  $1/3$ ,  $x$  is a uniformly random numeral  $0, 1, 2, \dots, 9$ , with probability  $1/3$ ,  $x$  is a random vowel  $\text{a}, \text{e}, \text{i}, \text{o}, \text{u}$  and with probability  $1/3$ ,  $x$  is one of the 21 consonants. Estimate the entropy of  $X$ .

## Problem 3: Geometric distribution

The geometric( $p$ ) distribution of a random variable  $X$  is defined as the number of times one has to flip a Bernoulli( $p$ ) coin before it lands on heads:

$$P_X(k) = (1-p)^{k-1}p \quad \text{for } k = 1, 2, 3, \dots$$

Compute the entropy of the geometric distribution.

## Problem 4: An optimal code

Let  $X$  be a random variable.

- (a) Show that if there exists an  $n \in \mathbb{N}$  such that for all  $x \in \mathcal{X}$ ,  $P_X(x) = \frac{1}{2^n}$ , then there exists a source code whose expected length equals the entropy.
- (b) ([\[MacKay\]](#), Exercise 5.25:) Show that if for all  $x \in \mathcal{X}$ , there exists an  $n \in \mathbb{N}$  such that  $P_X(x) = \frac{1}{2^n}$ , then there exists a source code whose expected length equals the entropy.

## Problem 5: Stirling's Approximation

Let  $n \in \mathbb{N}$  and  $p \in [0, 1]$  such that  $np \in \mathbb{N}$ . Use the approximation  $\ln(n!) \approx n \ln(n)$  to prove that

$$\binom{n}{np} \approx 2^{n \cdot h(p)},$$

where  $h$  is the binary entropy function.

## ★ Problem 6: Unique decodability

Construct a binary symbol code (for a finite alphabet  $\mathcal{X}$  of your own choice) that is uniquely decodable, but for which there exists an *infinite* binary string that can be decoded in more than one way.