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On Hadamard and Kronecker Products Over Matrix of Matrices

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Abstract. In this paper, we define and study Hadmard and Kronecker products over matrix of matrices and their basic properties. Furthermore, we establish a connection the Hadamard product of matrix of matrices and the usual matrix of matrices multiplication. In addition, we show some application of the Kronecker product.

Keywords: *Hadamard (Schur) product, Kronecker sum, Kronecker product, matrix of matrices.*
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1 Introduction

Matrices and matrix operations play an important role in almost every branch of mathematics, computer graphics, communication, computational mathematics, natural and social sciences and engineering. The Hadamard and Kronecker products are studied and utilized widely in matrix theory, statistics [1, 2], physics[2], system theory and other areas; see, e.g. [3, 4, 5, 6, 7, 8, 9, 10, 11]. An equality connection between the Hadamard and Kronecker products look to be firstly used by e.g. [12, 13, 14]. In partitioned matrices, the Khatri-Rao product can be seen as a generalized Hadamard product which is discussed and used by many authors e.g. [15, 16]. Also Tracy-Singh product as a generalized Kronecker product is studied in [19, 20, 21]. Finally, the approach of this paper may not be practical conventional in all situations.

In the present paper, we define and study Hadamard and Kronecker product over the matrix of matrices (in a short way; MMs) which was presented newly by Kishka et al [22]. We now give a short overview of this paper. In section 2, we define and study Hadamard product over MMs and give some properties. Then we show the association between Hadamard and MMs product in section 3. Finally, in section 4, we introduce the Kronecker product and prove a number of its properties. In addition, we introduce the notation of the vector matrices (VMs)-operator from which applications can be submitted to Kronecker product.

Throughout this paper, the accompanying notations are utilized:

Let $\mathbb{K}$ be a field and $M_l(\mathbb{K})$ be the set of all $l \times l$ matrices defined on $\mathbb{K}$, I and O stand for the identity matrix and the zero matrix in $M_l(\mathbb{K})$, respectively. We denote by $\mathcal{M}_{m\times n}(M_l(\mathbb{K}))$ the set of all $m \times n$ MMs over $M_l(\mathbb{K})$, the elements of $\mathcal{M}_{m\times n}(M_l(\mathbb{K}))$ are denoted by $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{G}$,….

[22] Let $\mathcal{A} \in \mathcal{M}_{m\times n}(M_l(\mathbb{K}))$, then it can be written as a rectangular table of elements $A_{ij}; i = 1, 2, ..., m$ and $j = 1, 2, ..., n$, as follows:

$$\mathcal{A} = \begin{pmatrix} A_{11} & ... & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & ... & A_{mn} \end{pmatrix}.$$

Definition 1.1. *[22] Given a $m \times n$-MMs; $\mathcal{A} = (A_{ij})$. Its transpose is the $n \times m$-MMs $\mathcal{A}^t$, given by*

$$[\mathcal{A}^t]_{ji} = (A_{ij}) = [\mathcal{A}]_{ij}, \qquad i = 1, ..., m; j = 1, ..., n,$$

where $A_{ij} \in M_l\left(\mathbb{K}\right)$.

2 Hadamard Product

In this section, we give a definition of Hadamard product over MMs with some properties.

Definition 2.1. *Let $\mathcal{A} = (A_{ij})$ and $\mathcal{B} = (B_{ij}) \in \mathcal{M}_{m\times n}(M_l(\mathbb{K}))$, then*

$$\mathcal{A} \circ \mathcal{B} = (A_{ij}B_{ij}), i = 1, ..., m, j = 1, ..., n,$$

*where $A_{ij}, B_{ij} \in M_l(\mathbb{K})$, this product is called **Hadamard** or **Schur product** over MMs.*

Example 2.2. *Let $\mathcal{A}$ and $\mathcal{B} \in \mathcal{M}_2(M_2(\mathbb{K}))$ where*

$$\mathcal{A} = \begin{pmatrix} \begin{pmatrix} 2 & -3 \\ -1 & 1 \end{pmatrix} & \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix} \\ \begin{pmatrix} 2 & 6 \\ 2 & 4 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix},$$

and

$$\mathcal{B} = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 2 & -3 \\ -1 & 1 \end{pmatrix} & \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \end{pmatrix},$$

then

$$\mathcal{A} \circ \mathcal{B} = \begin{pmatrix} \begin{pmatrix} -1 & 1 \\ \frac{1}{3} & \frac{-2}{3} \end{pmatrix} & \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix} \\ \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} & \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \end{pmatrix}.$$

Right away we might investigate some fundamentals properties of the Hadamard product over MMs.

Theorem 2.3. *Let $\mathcal{A}$ and $\mathcal{B} \in \mathcal{M}_{m\times n}(M_l(\mathbb{K}))$, then $\mathcal{A} \circ \mathcal{B} = \mathcal{B} \circ \mathcal{A}$ if $M_l(\mathbb{K})$ is commutative ring.*

Proof. The proof follows directly from the fact that A_{ij} and B_{ij} are commutative. Let $\mathcal{A}$ and $\mathcal{B}$ be $m \times n$-MMs where $\mathcal{A} = (A_{ij})$ and $\mathcal{B} = (B_{ij})$, then

$$\mathcal{A} \circ \mathcal{B} = (A_{ij}B_{ij}) = (B_{ij}A_{ij}) = \mathcal{B} \circ \mathcal{A}.$$

□

Theorem 2.4. *The identity MMs under the Hadamard product is the $m \times n$ MMs with all entries equal to I, denoted $\mathcal{J}_{mn}$.*

Proof. Let $\mathcal{A} = (A_{ij}) \in \mathcal{M}_{m\times n}(M_l(\mathbb{K}))$, then $\mathcal{A} \circ \mathcal{J}_{mn} = (A_{ij}I_{ij}) = (A_{ij})$ and so $\mathcal{J}_{mn} \circ \mathcal{A} = (A_{ij})$. Therefore, $\mathcal{J}_{mn}$ as defined above is indeed the identity MMs under the Hadamard product. □

We have denoted the Hadamard identity as $\mathcal{J}_{mn}$ to avoid confusion with the "usual" identity MMs, $\mathcal{I}_n$.

Theorem 2.5. *Let $\mathcal{A} = (A_{ij}) \in \mathcal{M}_{m\times n}(M_l(\mathbb{K}))$, then $\mathcal{A}$ has a Hadamard inverse, denoted by $\hat{\mathcal{A}}$ if and only if A_{ij} are non-singular $\forall i = 1, 2, ..., m;\ j = 1, 2, ..., n$. Furthermore, $\hat{\mathcal{A}} = (A_{ij}^{-1})$.*

Proof. ($\Rightarrow$) Let $\mathcal{A} = (A_{ij}) \in \mathcal{M}_{m\times n}(M_l(\mathbb{K}))$ with Hadamard inverse $\hat{\mathcal{A}}$, then $\mathcal{A} \circ \hat{\mathcal{A}} = \left(A_{ij}A_{ij}^{-1}\right) = (I_{ij}) = \mathcal{J}_{mn}$, which is only possible when all entries of $\mathcal{A}$ are invertible. In other words A_{ij} are non-singular matrices $\forall i = 1, 2, ..., m;\ j = 1, 2, ..., n$.
($\Leftarrow$) Take any $\mathcal{A} = (A_{ij}) \in \mathcal{M}_{m\times n}(M_l(\mathbb{K}))$ such that A_{ij} are non-singular $\forall i = 1, 2, ..., m;\ j = 1, 2, ..., n$. Then there exists $\hat{\mathcal{A}} = (A_{ij}^{-1})$ such that $\mathcal{A} \circ \hat{\mathcal{A}} = \hat{\mathcal{A}} \circ \mathcal{A} = \mathcal{J}_{mn}$, and so $\mathcal{A}$ has an inverse $\hat{\mathcal{A}} = (A_{ij}^{-1})$. □

We have denoted the Hadamard inverse as $\hat{\mathcal{A}}$ to avoid confusion with the "usual" inverse MMs, $\mathcal{A}^{-1}$.

Theorem 2.6. *Let $\mathcal{A}$, $\mathcal{B} \in \mathcal{M}_{m\times n}(M_l(\mathbb{K}))$, then*

$$(\mathcal{A} \circ \mathcal{B})^t = \mathcal{A}^t \circ \mathcal{B}^t.$$

Proof. Suppose that $\mathcal{A}$, $\mathcal{B} \in \mathcal{M}_{m\times n}(M_l(\mathbb{K}))$ where $\mathcal{A} = (A_{ij})$ and $\mathcal{B} = (B_{ij})$, then

$$\begin{aligned} (\mathcal{A} \circ \mathcal{B})^t &= (A_{ij}B_{ij})^t \\ &= (A_{ji}B_{ji}) \\ &= \mathcal{A}^t \circ \mathcal{B}^t. \end{aligned}$$

□

Remark 2.7. *If $M_l(\mathbb{K})$ is commutative ring with unity, then $(\mathcal{A} \circ \mathcal{B})^t = \mathcal{B}^t \circ \mathcal{A}^t$.*

Theorem 2.8. *The set $\mathcal{M}_n(M_l(\mathbb{K}))$ of non-singular MMs is a group under Hadamard product.*

Proof. Let $\mathcal{A} = (A_{ij})$, $\mathcal{B} = (B_{ij})$ and $\mathcal{C} = (C_{ij}) \in \mathcal{M}_n(M_l(\mathbb{K}))$ be MMs then it is easy to see that $\mathrm{M}_n(M_l(\mathbb{K}))$ is a group. □

Theorem 2.9. *Suppose that $E \in M_l(\mathbb{K})$ and $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C} \in \mathcal{M}_n(M_l(\mathbb{K}))$, then*
(1) $\mathcal{C}\circ(\mathcal{A} + \mathcal{B}) = \mathcal{C} \circ \mathcal{A} + \mathcal{C} \circ \mathcal{B}$.
(2) $E(\mathcal{A} \circ \mathcal{B}) = (E\mathcal{A}) \circ \mathcal{B} = \mathcal{A} \circ (E\mathcal{B})$.
where $M_l(\mathbb{K})$ is commutative ring with unity.

Proof. Part 1.

$$\begin{aligned} [\mathcal{C}\circ(\mathcal{A} + \mathcal{B})]_{ij} &= [\mathcal{C}]_{ij}\,[\mathcal{A} + \mathcal{B}]_{ij} \\ &= [\mathcal{C}]_{ij}\left([\mathcal{A}]_{ij} + [\mathcal{B}]_{ij}\right) \\ &= [\mathcal{C}]_{ij}\,[\mathcal{A}]_{ij} + [\mathcal{C}]_{ij}\,[\mathcal{B}]_{ij} \\ &= [\mathcal{C} \circ \mathcal{A}]_{ij} + [\mathcal{C} \circ \mathcal{B}]_{ij} \\ &= [\mathcal{C} \circ \mathcal{A} + \mathcal{C} \circ \mathcal{B}]_{ij} \end{aligned}$$

Part 2.

$$\begin{aligned} [E(\mathcal{A} \circ \mathcal{B})]_{ij} &= E[\mathcal{A}]_{ij}[\mathcal{B}]_{ij} \\ &= [E\mathcal{A}]_{ij}[\mathcal{B}]_{ij} \\ &= [E\mathcal{A} \circ \mathcal{B}]_{ij} \\ &= [\mathcal{A}]_{ij}E[\mathcal{B}]_{ij} \\ &= [\mathcal{A}]_{ij}[E\mathcal{B}]_{ij} \\ &= [\mathcal{A} \circ E\mathcal{B}]_{ij}, \end{aligned}$$

since $M_l(\mathbb{K})$ is commutative ring with unity. □

Remark 2.10. *The above part 2 is also true if we replace E by $c \in \mathbb{K}$.*

3 Connection between Hadamard and MMs products

In this section, we show connection between Hadamard and MMs multiplications.

Let $\mathcal{A}=(A_{ij})\in\mathcal{M}_m(M_l(\mathbb{K}))$ and consider the set $S_{12...m}$ of m cyclic permutations of $12...m$ given by

$$S_{12...m}=\{123...(m-1)\,m,23...(m-1)\,m1,...,m12...(m-2)\,(m-1)\}. \tag{1}$$

If A_i is the i^{th} column of the $m\times m$ MMs $\mathcal{A}$, then we define

$$\mathcal{A}_{(s)}=(A_{s1}:A_{s2}:\ ...\ :A_{sm})\,, \tag{2}$$

i.e., $\mathcal{A}_{(s)}$ is the MMs $\mathcal{A}$ with permuted columns according to the permutation $s=s_1s_2...s_m$. It is easy to see that $\mathcal{A}_{(s)}=\mathcal{A}.\mathcal{I}_{(s)}$. Also, for $\mathcal{B}=(B_{ij})\in\mathcal{M}_m(M_l(\mathbb{K}))$, let us define

$$\mathcal{B}_{[S]}=\begin{pmatrix} B_{s_11} & B_{s_22} & \cdots & B_{s_mm}\\ B_{s_11} & B_{s_22} & \cdots & B_{s_mm}\\ \vdots & \vdots & \ddots & \vdots\\ B_{s_11} & B_{s_22} & \cdots & B_{s_mm}\end{pmatrix}. \tag{3}$$

With these notation, we have the following theorem.

Theorem 3.1. *Let $\mathcal{A}=(A_{ij})$ and $\mathcal{B}=(B_{ij})\in\mathcal{M}_m(M_l(\mathbb{K}))$, then*

$$\mathcal{A}.\mathcal{B}=\sum_s\mathcal{A}_{(s)}\circ\mathcal{B}_{[s]}=\sum_S\left(\mathcal{A}.\mathcal{I}_{(s)}\right)\circ\mathcal{B}_{[s]},$$

where $\mathcal{A}_{(S)}$ and $\mathcal{B}_{[S]}$ are defined as in (2) *and* (3) *, respectively, for a particular permutation $s\in S_{12...m}$.*

Proof. Suppose that $\mathcal{A}=(A_{ij})$ and $\mathcal{B}=(B_{ij})\in\mathcal{M}_m(M_l(\mathbb{K}))$, then

$$\begin{aligned}\mathcal{A}.\mathcal{B}&=\begin{pmatrix}\sum_{j=1}^m A_{1j}B_{j1} & \cdots & \sum_{j=1}^m A_{1j}B_{jm}\\ \sum_{j=1}^m A_{2j}B_{j1} & \cdots & \sum_{j=1}^m A_{2j}B_{jm}\\ \vdots & \ddots & \vdots\\ \sum_{j=1}^m A_{mj}B_{j1} & \cdots & \sum_{j=1}^m A_{mj}B_{jm}\end{pmatrix}\\ &=\left(\sum_{j=1}^m A_{ij}B_{j1}:\sum_{j=1}^m A_{ij}B_{j2}:...:\sum_{j=1}^m A_{ij}B_{jm}\right).\end{aligned}$$

Clearly, the usual product of MMs $\mathcal{A}.\mathcal{B}$ decomposes uniquely as the sum of m MMs $\mathcal{C}_{(s)}$, where $s=s_1s_2...s_m\in S_{123...m}$. Such a decomposition can be constructed as follows:
The first MMs $\mathcal{C}_{(12...m)}$ is obtained by

$$\mathcal{C}_{12...m}=(A_{i1}B_{11}:A_{i2}B_{22}:\ ...\ :A_{im}B_{mm})=\mathcal{A}_{(12...m)}\circ\mathcal{B}_{[12...m]}.$$

The second MMs is selected according to the permutation $23...m1$, *i.e.*,

$$\mathcal{C}_{23...m1}=(A_{i2}B_{21}:A_{i3}B_{32}:\ ...\ :A_{i1}B_{1m})=\mathcal{A}_{(23...m1)}\circ\mathcal{B}_{[23...m1]}.$$

Following this procedure and taking the MMs according to the complete set of m cyclic permutations of $12...m$ in (1) , we obtain the $(m-1)^{th}$ MMs as

$$\mathcal{C}_{(m-1)m1...(m-3)(m-2)}=\mathcal{A}_{\left((m-1)m1...(m-3)(m-2)\right)}\circ\mathcal{B}_{\left[(m-1)m1...(m-3)(m-2)\right]}.$$

Finally, the MMs corresponding to the ultimate permutation $m1\cdots(m-2)(m-1)$ is formed by the remaining summands as

$$\mathcal{C}_{m1\cdots(m-2)(m-1)}=\mathcal{A}_{\left(m1\cdots(m-2)(m-1)\right)}\circ\mathcal{B}_{\left[m1\cdots(m-2)(m-1)\right]}.$$

Thus, we have

$$\begin{aligned}\mathcal{A}.\mathcal{B}=&\ \mathcal{A}_{(12...m)}\circ\mathcal{B}_{[12...m]}+\mathcal{A}_{(23...m1)}\circ\mathcal{B}_{[23...m1]}+....+\\ &\mathcal{A}_{\left((m-1)m1...(m-3)(m-2)\right)}\circ\mathcal{B}_{\left[(m-1)m1...(m-3)(m-2)\right]}+\mathcal{A}_{\left(m1\cdots(m-2)(m-1)\right)}\circ\mathcal{B}_{\left[m1\cdots(m-2)(m-1)\right]},\end{aligned}$$

which is the required result. □

Example 3.2. *Let* $\mathcal{A}=\begin{pmatrix} 2I & O \\ I & 3I \end{pmatrix}$ *and* $\mathcal{B}=\begin{pmatrix} 2I & I \\ I & O \end{pmatrix}\in\mathcal{M}_2(M_2(\mathbb{K}))$, *then*

$$\begin{aligned}\mathcal{A}.\mathcal{B} &= \begin{pmatrix} 4I & 2I \\ 5I & I \end{pmatrix}\\ &= \begin{pmatrix} 2I & O \\ I & 3I \end{pmatrix}\circ\begin{pmatrix} 2I & O \\ 2I & O \end{pmatrix}+\begin{pmatrix} O & 2I \\ 3I & I \end{pmatrix}\circ\begin{pmatrix} I & I \\ I & I \end{pmatrix}\\ &= \mathcal{A}_{(12)}\circ\mathcal{B}_{[12]}+\mathcal{A}_{(21)}\circ\mathcal{B}_{[21]}.\end{aligned}$$

Example 3.3. *Let* $\mathcal{A}=(A_{ij})$ *and* $\mathcal{B}=(B_{ij})\in\mathcal{M}_3(M_l(\mathbb{K}))$, *then*

$$\begin{aligned}\mathcal{A}.\mathcal{B} &= \begin{pmatrix} A_{11}B_{11}+A_{12}B_{21}+A_{13}B_{31} & A_{11}B_{12}+A_{12}B_{22}+A_{13}B_{32} & A_{11}B_{13}+A_{12}B_{23}+A_{13}B_{33} \\ A_{21}B_{11}+A_{22}B_{21}+A_{23}B_{31} & A_{21}B_{12}+A_{22}B_{22}+A_{23}B_{32} & A_{21}B_{13}+A_{22}B_{23}+A_{23}B_{33} \\ A_{31}B_{11}+A_{32}B_{21}+A_{33}B_{31} & A_{31}B_{12}+A_{32}B_{22}+A_{33}B_{32} & A_{31}B_{13}+A_{32}B_{23}+A_{33}B_{33} \end{pmatrix}\\ &= \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}\circ\begin{pmatrix} B_{11} & B_{22} & B_{33} \\ B_{11} & B_{22} & B_{33} \\ B_{11} & B_{22} & B_{33} \end{pmatrix}+\begin{pmatrix} A_{12} & A_{13} & A_{11} \\ A_{22} & A_{23} & A_{21} \\ A_{32} & A_{33} & A_{31} \end{pmatrix}\circ\begin{pmatrix} B_{21} & B_{32} & B_{13} \\ B_{21} & B_{32} & B_{13} \\ B_{21} & B_{32} & B_{13} \end{pmatrix}\\ &\quad+\begin{pmatrix} A_{13} & A_{11} & A_{12} \\ A_{23} & A_{21} & A_{22} \\ A_{33} & A_{31} & A_{32} \end{pmatrix}\circ\begin{pmatrix} B_{31} & B_{12} & B_{23} \\ B_{31} & B_{12} & B_{23} \\ B_{31} & B_{12} & B_{23} \end{pmatrix}\\ &= \mathcal{A}_{(123)}\circ\mathcal{B}_{[123]}+\mathcal{A}_{(231)}\circ\mathcal{B}_{[231]}+\mathcal{A}_{(312)}\circ\mathcal{B}_{[312]}.\end{aligned}$$

A comparable expansion of $\mathcal{A}.\mathcal{B}$ when the MMs $\mathcal{A}$ and $\mathcal{B}$ are not square can be given by the following corollary.

Corollary 3.4. *Let* $\mathcal{A}=(A_{ij})\in\mathcal{M}_{m\times k}(M_l(\mathbb{K}))$ *and* $\mathcal{B}=(B_{ij})\in\mathcal{M}_{k\times m}(M_l(\mathbb{K}))$, *with* $m\leq k$, *then,*

$$\mathcal{A}.\mathcal{B}=\sum_s\mathcal{A}_{(s)}\circ\mathcal{B}_{[s]}=\sum_S\left(\mathcal{A}.\mathcal{I}_{(s)}\right)\circ\mathcal{B}_{[s]},$$

where the summation runs over all cyclic permutations $s=s_1s_2...s_m$ *(consisting of all the first* m *indices) of* $S_{12...m....k}$. *For a particular* $s=s_1s_2...s_m\in S_{12...m....k}$, $\mathcal{A}_{(s)}$ *is as given in (2) with* k *replaced by* m *and* $\mathcal{B}_{[s]}$ *is as in (3).*

Example 3.5. *Let us take*

$$\mathcal{A}=\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{pmatrix}\quad and\quad \mathcal{B}=\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{33} \end{pmatrix},$$

Here, $S_{123}=\{123,231,312\}$ *as usual, but the permutations that we consider have only the first two parts, i.e.,* $s=12,23,$ *or* 31. *Then,*

$$\begin{aligned}\mathcal{A}.\mathcal{B} &= \begin{pmatrix} A_{11}B_{11}+A_{12}B_{21}+A_{13}B_{31} & A_{11}B_{12}+A_{12}B_{22}+A_{13}B_{32} \\ A_{21}B_{11}+A_{22}B_{21}+A_{23}B_{31} & A_{21}B_{12}+A_{22}B_{22}+A_{23}B_{32} \end{pmatrix}\\ &= \mathcal{A}_{(12)}\circ\mathcal{B}_{[12]}+\mathcal{A}_{(23)}\circ\mathcal{B}_{[23]}+\mathcal{A}_{(31)}\circ\mathcal{B}_{[31]}\\ &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\circ\begin{pmatrix} B_{11} & B_{22} \\ B_{11} & B_{22} \end{pmatrix}+\begin{pmatrix} A_{12} & A_{13} \\ A_{22} & A_{23} \end{pmatrix}\circ\begin{pmatrix} B_{21} & B_{32} \\ B_{21} & B_{32} \end{pmatrix}\\ &\quad+\begin{pmatrix} A_{13} & A_{11} \\ A_{23} & A_{21} \end{pmatrix}\circ\begin{pmatrix} B_{31} & B_{12} \\ B_{31} & B_{12} \end{pmatrix}.\end{aligned}$$

4 Kronecker Product

In this section, we give a definition of Kronecker product over MMs, and some properties.

Definition 4.1. *Let $\mathcal{A} = (A_{ij}) \in \mathcal{M}_{n\times m}(M_l(\mathbb{K}))$ and $\mathcal{B} = (B_{ij}) \in \mathcal{M}_{p\times q}(M_l(\mathbb{K}))$, the $np \times mq$ MMs*

$$\mathcal{A}\otimes\mathcal{B} = \begin{pmatrix} A_{11}\mathcal{B} & A_{12}\mathcal{B} & ... & A_{1m}\mathcal{B} \\ A_{21}\mathcal{B} & A_{22}\mathcal{B} & ... & A_{2m}\mathcal{B} \\ \vdots & \vdots & \vdots & \vdots \\ A_{n1}\mathcal{B} & A_{n2}\mathcal{B} & ... & A_{nm}\mathcal{B} \end{pmatrix},$$

*is called **Kroncker product** of $\mathcal{A}$ and $\mathcal{B}$.*

In order to explore the variety of applications of the Kronecker product we introduce the notation of the VMs –operator.

Definition 4.2. *For any $\mathcal{A} \in \mathcal{M}_{m\times n}(M_l(\mathbb{K}))$, the VMs-operator is defined as*

$$VMs\,(\mathcal{A}) = (A_{11}, ..., A_{m1}, ..., A_{1n}, ..., A_{mn})\,,$$

i.e., the entries of $\mathcal{A}$ are stacked column wise forming a vector of matrices of length mn.

Example 4.3. *Let*

$$\mathcal{A} = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \\ \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix},$$

and

$$\mathcal{B} = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix},$$

then

$$\mathcal{A}\otimes\mathcal{B} = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} & \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} & \begin{pmatrix} 4 & 2 \\ -2 & 4 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} & \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \\ \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} & \begin{pmatrix} 4 & 2 \\ -2 & 4 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \\ \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} & \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix}.$$

4.1 Properties of the Kronecker Product

Theorem 4.4. *Let $\mathcal{A} \in \mathcal{M}_{m\times n}(M_l(\mathbb{K}))$, $\mathcal{B} \in \mathcal{M}_{r\times s}(M_l(\mathbb{K}))$, $\mathcal{C} \in \mathcal{M}_{n\times p}(M_l(\mathbb{K}))$ and $\mathcal{D} \in \mathcal{M}_{s\times t}(M_l(\mathbb{K}))$, then*

$$(\mathcal{A}\otimes\mathcal{B})\,(\mathcal{C}\otimes\mathcal{D}) = \mathcal{AC}\otimes\mathcal{BD}.$$

Proof. Simply verify that

$$\begin{aligned} (\mathcal{A}\otimes\mathcal{B})\,(\mathcal{C}\otimes\mathcal{D}) &= \begin{pmatrix} A_{11}\mathcal{B} & ... & A_{1n}\mathcal{B} \\ \vdots & \ddots & \vdots \\ A_{m1}\mathcal{B} & ... & A_{mn}\mathcal{B} \end{pmatrix} \begin{pmatrix} C_{11}\mathcal{D} & ... & C_{1p}\mathcal{D} \\ \vdots & \ddots & \vdots \\ C_{n1}\mathcal{D} & ... & C_{np}\mathcal{D} \end{pmatrix} \\ &= \begin{pmatrix} {}_{k=1}^{n}A_{1k}C_{k1}\mathcal{BD} & ... & {}_{k=1}^{n}A_{1k}C_{kp}\mathcal{BD} \\ \vdots & \ddots & \vdots \\ {}_{k=1}^{n}A_{mk}C_{k1}\mathcal{BD} & ... & {}_{k=1}^{n}A_{mk}C_{kp}\mathcal{BD} \end{pmatrix} \\ &= \mathcal{AC}\otimes\mathcal{BD}. \end{aligned}$$

□

Theorem 4.5. *Let $\mathcal{A} \in \mathcal{M}_{m\times n}(M_l(\mathbb{K}))$ and $\mathcal{B} \in \mathcal{M}_{r\times s}(M_l(\mathbb{K}))$, then $(\mathcal{A}\otimes\mathcal{B})^t = \mathcal{A}^t \otimes \mathcal{B}^t$.*

Proof. Suppose that $\mathcal{A} = (A_{ij})$, $\mathcal{B} = (B_{kl})$ and simply verify using the definition kronecker product over MMs

$$\begin{aligned} (\mathcal{A}\otimes\mathcal{B})^t &= \left\{ \begin{pmatrix} A_{11}\mathcal{B} & ... & A_{1n}\mathcal{B} \\ \vdots & \ddots & \vdots \\ A_{m1}\mathcal{B} & ... & A_{mn}\mathcal{B} \end{pmatrix} \right\}^t \\ &= \begin{pmatrix} A_{11}\mathcal{B}^t & ... & A_{m1}\mathcal{B}^t \\ \vdots & \ddots & \vdots \\ A_{1n}\mathcal{B}^t & ... & A_{nm}\mathcal{B}^t \end{pmatrix} \\ &= \mathcal{A}^t \otimes \mathcal{B}^t. \end{aligned}$$

□

Corollary 4.6. *Let $\mathcal{A} \in \mathcal{M}_{m\times n}(M_l(\mathbb{K}))$ and $\mathcal{B} \in \mathcal{M}_{r\times s}(M_l(\mathbb{K}))$ be MMs symmetric, then $\mathcal{A}\otimes\mathcal{B}$ is symmetric.*

Proof. Suppose that $\mathcal{A} = (A_{ij})$ and $\mathcal{B} = (B_{kl})$ are symmetric MMs and $A_{ij}, B_{kl} \in M_l(\mathbb{K})$, then

$$\begin{aligned} (\mathcal{A}\otimes\mathcal{B})^t &= \mathcal{A}^t \otimes \mathcal{B}^t \\ &= \mathcal{A}\otimes\mathcal{B}. \end{aligned}$$

So $\mathcal{A}\otimes\mathcal{B}$ is symmetric. □

Theorem 4.7. *If $\mathcal{A}$ and $\mathcal{B}$ are nonsingular, then $(\mathcal{A}\otimes\mathcal{B})^{-1} = \mathcal{A}^{-1}\otimes\mathcal{B}^{-1}$.*

Proof. Using the above Theorem 9, simply note that $(\mathcal{A}\otimes\mathcal{B})\left(\mathcal{A}^{-1}\otimes\mathcal{B}^{-1}\right) = \mathcal{I}\otimes\mathcal{I} = \mathcal{I}$. □

Theorem 4.8. *Suppose that $E \in M_l(\mathbb{K})$, $\mathcal{A} \in \mathcal{M}_{m\times n}(M_l(\mathbb{K}))$ and $\mathcal{B} \in \mathcal{M}_{r\times s}(M_l(\mathbb{K}))$, then*

$$(E\mathcal{A})\otimes\mathcal{B} = \mathcal{A}\otimes(E\mathcal{B}) = E(\mathcal{A}\otimes\mathcal{B}),$$

where $M_l(\mathbb{K})$ is commutative ring with unity.

Proof. Since

$$\begin{aligned} (E\mathcal{A})\otimes\mathcal{B} &= \begin{pmatrix} EA_{11}\mathcal{B} & ... & EA_{1n}\mathcal{B} \\ \vdots & \ddots & \vdots \\ EA_{m1}\mathcal{B} & ... & EA_{mn}\mathcal{B} \end{pmatrix} \\ &= \begin{pmatrix} A_{11}E\mathcal{B} & ... & A_{1n}E\mathcal{B} \\ \vdots & \ddots & \vdots \\ A_{m1}E\mathcal{B} & ... & A_{mn}E\mathcal{B} \end{pmatrix} \\ &= \mathcal{A}\otimes(E\mathcal{B}) \\ &= E\begin{pmatrix} A_{11}\mathcal{B} & ... & A_{1n}\mathcal{B} \\ \vdots & \ddots & \vdots \\ A_{m1}\mathcal{B} & ... & A_{mn}\mathcal{B} \end{pmatrix}. \end{aligned}$$

□

Remark 4.9. *The above Theorem 13 is also true if we replace E by $c \in \mathbb{K}$.*

Remark 4.10. *For all $\mathcal{A}$ and $\mathcal{B}$ we have; $(\mathcal{A}\otimes\mathcal{B}) \neq \mathcal{B}\otimes\mathcal{A}$ which can be justified by the following example.*

Example 4.11. *Let*

$$\mathcal{A}=\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \\ \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix},$$

and

$$\mathcal{B}=\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix},$$

then

$$\begin{aligned} (\mathcal{A}\otimes\mathcal{B}) &= \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\mathcal{B} & \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\mathcal{B} \\ \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\mathcal{B} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\mathcal{B} \end{pmatrix} \\ &\neq \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\mathcal{A} & \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\mathcal{A} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\mathcal{A} & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\mathcal{A} \end{pmatrix} \\ &= \mathcal{B}\otimes\mathcal{A}. \end{aligned}$$

Corollary 4.12. *Let $\mathcal{A}\in\mathcal{M}_m(M_l(\mathbb{K}))$ and $\mathcal{B}\in\mathcal{M}_n(M_l(\mathbb{K}))$, then*
(1) $Tr\,(\mathcal{A}\otimes\mathcal{B}) = Tr\,(\mathcal{A})\,Tr\,(\mathcal{B}) = Tr\,(\mathcal{B}\otimes\mathcal{A})$
(2) $\det(\mathcal{A}\otimes\mathcal{B}) = [\det(\mathcal{A})]^m\,[\det(\mathcal{B})]^n = \det(\mathcal{B}\otimes\mathcal{A})$

Definition 4.13. *Let $\mathcal{A}\in\mathcal{M}_m(M_l(\mathbb{K}))$ and $\mathcal{B}\in\mathcal{M}_n(M_l(\mathbb{K}))$, then the Kronecker sum(or tensor sum) of $\mathcal{A}$ and $\mathcal{B}$, denoted $\mathcal{A}\oplus\mathcal{B}$ is the $mn\times mn$ MMs $(\mathcal{I}_n\otimes\mathcal{A})+(\mathcal{B}\otimes\mathcal{I}_m)$.*

Example 4.14. *Let*

$$\mathcal{A}=\begin{pmatrix} I & 2I & 3I \\ 3I & 2I & I \\ I & I & 4I \end{pmatrix} \text{ and } \mathcal{B}=\begin{pmatrix} 2I & I \\ 2I & 3I \end{pmatrix},$$

then

$$\begin{aligned} \mathcal{A}\oplus\mathcal{B} &= (\mathcal{I}_2\otimes\mathcal{A})+(\mathcal{B}\otimes\mathcal{I}_3) \\ &= \begin{pmatrix} 3I & 2I & 3I & I & O & O \\ 3I & 4I & I & O & I & O \\ I & I & 6I & O & O & I \\ 2I & O & O & 4I & 2I & 3I \\ O & 2I & O & 3I & 5I & I \\ O & O & 2I & I & I & 7I \end{pmatrix}. \end{aligned}$$

Remark 4.15. *In general, $\mathcal{A}\oplus\mathcal{B}\neq\mathcal{B}\oplus\mathcal{A}$.*

Lemma 4.16. *The Kronecker product over MMs is associative, i.e.,*

$$(\mathcal{A}\otimes\mathcal{B})\otimes\mathcal{C}=\mathcal{A}\otimes(\mathcal{B}\otimes\mathcal{C}),$$

where $\mathcal{A}\in\mathcal{M}_{m\times n}(M_l(\mathbb{K}))$, $\mathcal{B}\in\mathcal{M}_{r\times s}(M_l(\mathbb{K}))$ and $\mathcal{C}\in\mathcal{M}_{s\times t}(M_l(\mathbb{K}))$.

Lemma 4.17. *The Kronecker product over MMs is right–distributive, i.e.,*

$$(\mathcal{A}+\mathcal{B})\otimes\mathcal{C}=\mathcal{A}\otimes\mathcal{C}+\mathcal{B}\otimes\mathcal{C},$$

where $\mathcal{A}$, $\mathcal{B}\in\mathcal{M}_{m\times n}(M_l(\mathbb{K}))$ and $\mathcal{C}\in\mathcal{M}_{s\times t}(M_l(\mathbb{K}))$.

Lemma 4.18. *The Kronecker product over MMs is left–distributive, i.e.,*

$$\mathcal{A}\otimes(\mathcal{B}+\mathcal{C})=\mathcal{A}\otimes\mathcal{B}+\mathcal{A}\otimes\mathcal{C},$$

where $\mathcal{A}\in\mathcal{M}_{m\times n}(M_l(\mathbb{K}))$, $\mathcal{B}$ and $\mathcal{C}\in\mathcal{M}_{s\times t}(M_l(\mathbb{K}))$.

4.2 Application

The Kronecker product can be used to present linear matrix equations in which the unknowns are MMs. Examples for such equations are:

$$\begin{aligned} \mathcal{A}\mathcal{X} &= \mathcal{B}, \\ \mathcal{A}\mathcal{X} + \mathcal{X}\mathcal{B} &= \mathcal{C}, \\ \mathcal{A}\mathcal{X}\mathcal{B} &= \mathcal{C}. \end{aligned}$$

These equations are equivalent to the following systems of equations:

$$\begin{aligned} (\mathcal{I} \otimes \mathcal{A})\, \mathrm{VMs}\,(\mathcal{X}) &= \mathrm{VMs}\,(\mathcal{B})\,, \\ \left[(\mathcal{I} \otimes \mathcal{A}) + \left(\mathcal{B}^t \otimes \mathcal{I}\right)\right] \mathrm{VMs}\,(\mathcal{X}) &= \mathrm{VMs}\,(\mathcal{C})\,, \\ \left(\mathcal{B}^t \otimes \mathcal{A}\right) \mathrm{VMs}\,(\mathcal{X}) &= \mathrm{VMs}\,(\mathcal{C})\,. \end{aligned}$$

4.3 Relation between Hadamard and Kronecker product

It is clear that the Hadamard product of two MMs is the principal sub-MMs of the Kronecker product of the two MMs. This relation can be expressed in an equation as follows.

Lemma 4.19. *For $\mathcal{A}$ and $\mathcal{B} \in \mathcal{M}_{m\times n}(M_l(\mathbb{K}))$, then we have*

$$\mathcal{A} \circ \mathcal{B} = \mathcal{Z}_1^t\,(\mathcal{A} \otimes \mathcal{B})\,\mathcal{Z}_2,$$

where $\mathcal{Z}_1$ is the selection MMs of order $m^2 \times m$ and $\mathcal{Z}_2$ is the selection MMs of order $n^2 \times n$.

Example 4.20. *Let*

$$\mathcal{A} = \begin{pmatrix} I & 2I \\ O & I \end{pmatrix},\ \mathcal{B} = \begin{pmatrix} 2I & I \\ I & O \end{pmatrix},$$

where $\mathcal{A}$ and $\mathcal{B} \in \mathcal{M}_2(M_2(\mathbb{K}))$, and the selection MMs

$$\mathcal{Z}_1 = \mathcal{Z}_2 = \begin{pmatrix} I & O & O & O \\ O & O & O & I \end{pmatrix},$$

then

$$\mathcal{A} \circ \mathcal{B} = \mathcal{Z}_1^t\,(\mathcal{A} \otimes \mathcal{B})\,\mathcal{Z}_2.$$

5 Conclusion

In this work, we have introduced Hadamard and Kronecker product over a new algebraic structure which is called MMs and some of their properties. It is worth to mention that the new concepts are applicable in many different branches of mathematics such as group theory, statistics, combinatorics including graphs, other discrete structures, and functional analysis. Further research on this topic is now under investigation and will be reported in forthcoming works.

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