14/18 core points

EEN020 Computer Vision
Assignment 3
Epipolar Geometry

Vasiliki Kostara

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The Fundamental Matrix

3 *Theoretical Exercise 1*

Camera matrices $P_1 = \begin{bmatrix} I & \mathbf{0} \end{bmatrix}$ and $P_2 = [A|\mathbf{t}]$
The center of the first camera in homogeneous coordinates is $\mathbf{c_1} = \mathbf{0}$.Then, the epipole $\mathbf{e_2}$:

$$\mathbf{e_2} = P_2\mathbf{c_1} \Longleftrightarrow \mathbf{e_2} = \mathbf{t} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \quad \text{and} \quad [\mathbf{e_2}]_\times = 2\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

Then,

$$F = [\mathbf{e_2}]_\times A \Longleftrightarrow F = 2\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$$

The projection of the 3D-point $\mathbf{X}$ into P_1 in homogeneous coordinates is $\mathbf{x} = (0, 2, 1)$. Then the epipolar line will be:

$$\mathbf{l} \sim F\mathbf{x} \Longleftrightarrow \mathbf{l} \sim \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

In two dimensions the line equation is $l : y = x$. Therefore, the point $(1, 1)$ could be a projection of the same point $\mathbf{X}$ into $\mathbf{P_2}$.

3 *Theoretical Exercise 2*

With camera matrices $P_1 = \begin{bmatrix} I & \mathbf{0} \end{bmatrix}$ and $P_2 = \begin{bmatrix} A & \mathbf{t} \end{bmatrix}$ and camera centers $\mathbf{C_1} = \begin{bmatrix} \mathbf{c_1} \\ 1 \end{bmatrix}$ and $\mathbf{C_2} = \begin{bmatrix} \mathbf{c_2} \\ 1 \end{bmatrix}$

$$P_1\mathbf{C_1} = \mathbf{0} \Longleftrightarrow \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{c_1} \\ 1 \end{bmatrix} = \mathbf{0} \Longleftrightarrow \mathbf{c_1} = \mathbf{0}$$

$$P_2\mathbf{C_2} = \mathbf{0} \Longleftrightarrow \begin{bmatrix} A & \mathbf{t} \end{bmatrix} \begin{bmatrix} \mathbf{c_2} \\ 1 \end{bmatrix} = \mathbf{0} \Longleftrightarrow A\mathbf{c_2} + \mathbf{t} = \mathbf{0} \Longleftrightarrow$$
$$\begin{cases} y + z + 2 = 0 \\ 3x + 2y + 1 = 0 \\ 3z = 0 \end{cases} \Longleftrightarrow \begin{cases} x = 1 \\ y = -2 \\ z = 0 \end{cases} \Longleftrightarrow \mathbf{c_2} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

Epipoles

$$\mathbf{e_1} = P_1\mathbf{C_2} = \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{c_2} \\ 1 \end{bmatrix} = \mathbf{c_2} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

$$\mathbf{e_2} = P_2\mathbf{C_1} = \begin{bmatrix} A & \mathbf{t} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} = \mathbf{t} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$[\mathbf{e_2}]_\times = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -2 \\ -1 & 2 & 0 \end{pmatrix}$$

$$F = [\mathbf{e_2}]_\times A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -2 \\ -1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 3 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & -6 \\ 6 & 3 & -1 \end{pmatrix}$$

$$\mathbf{e_2}^T F = \begin{bmatrix} 2 & 1 & 0 \end{bmatrix} \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & -6 \\ 6 & 3 & -1 \end{pmatrix} = \begin{bmatrix} 0 & 0 & 6-6 \end{bmatrix} = \mathbf{0}^T$$

$$F\mathbf{e_1} = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & -6 \\ 6 & 3 & -1 \end{pmatrix} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 6-6 \end{bmatrix} = \mathbf{0}$$

12 *Theoretical Exercise 3*

With camera matrices $P_1 = \begin{bmatrix} I & \mathbf{0} \end{bmatrix}$ and $P_2 = \begin{bmatrix} A & \mathbf{t} \end{bmatrix}$ and camera centers $\mathbf{C_1} = \begin{bmatrix} \mathbf{c_1} \\ 1 \end{bmatrix}$ and $\mathbf{C_2} = \begin{bmatrix} \mathbf{c_2} \\ 1 \end{bmatrix}$ we have:

$$
\begin{aligned}
P_1\mathbf{C_1} = \mathbf{0} \Longleftrightarrow \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{c_1} \\ 1 \end{bmatrix} = \mathbf{0} \Longleftrightarrow \mathbf{c_1} = \mathbf{0} \\
\text{and} \\
P_2\mathbf{C_2} = \mathbf{0} \Longleftrightarrow \begin{bmatrix} A & \mathbf{t} \end{bmatrix} \begin{bmatrix} \mathbf{c_2} \\ 1 \end{bmatrix} = \mathbf{0} \Longleftrightarrow A\mathbf{c_2} + \mathbf{t} = \mathbf{0} \Longleftrightarrow \\
\mathbf{c_2} = -A^{-1}\mathbf{t}
\end{aligned}
$$

This gives us camera centers in the form of $\mathbf{C_1} = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}$ and $\mathbf{C_2} = \begin{bmatrix} -A^{-1}\mathbf{t} \\ 1 \end{bmatrix}$. Then, the epipoles will be given by:

$$
\begin{aligned}
\mathbf{e_1} = P_1\mathbf{C_2} = \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \begin{bmatrix} -A^{-1}\mathbf{t} \\ 1 \end{bmatrix} = -A^{-1}\mathbf{t} \\
\mathbf{e_2} = P_2\mathbf{C_1} = \begin{bmatrix} A & \mathbf{t} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} = \mathbf{t}
\end{aligned}
$$

For the fundamental matrix being $F = [\mathbf{e_2}]_\times A = [\mathbf{t}]_\times A$, we have:

$$
\begin{aligned}
F\mathbf{e_1} = [\mathbf{t}]_\times A\big(- A^{-1}\mathbf{t}\big) = -[\mathbf{t}]_\times \mathbf{t} = -\mathbf{t} \times \mathbf{t} = \mathbf{0} \\
\text{and} \\
\mathbf{e_2}^T F = \mathbf{t}^T[\mathbf{t}]_\times A = \mathbf{0}^T A = \mathbf{0}^T
\end{aligned}
$$

Lastly, due to the above equalities, we conclude that the nullspace of F is not trivial. This means that F is rank-deficient and, consequently, $det(F) = 0$.

1 *Theoretical Exercise 4*

$$\tilde{\mathbf{x}}_{\mathbf{2}}^T \tilde{F} \tilde{\mathbf{x}}_{\mathbf{1}} = 0 \Longleftrightarrow$$
$$\mathbf{x_2}^T (N_2^T \tilde{F} N_1^T) \mathbf{x_1} = 0$$

$$F = N_2^T \tilde{F} N_1$$

Computer Exercise 1

PART I

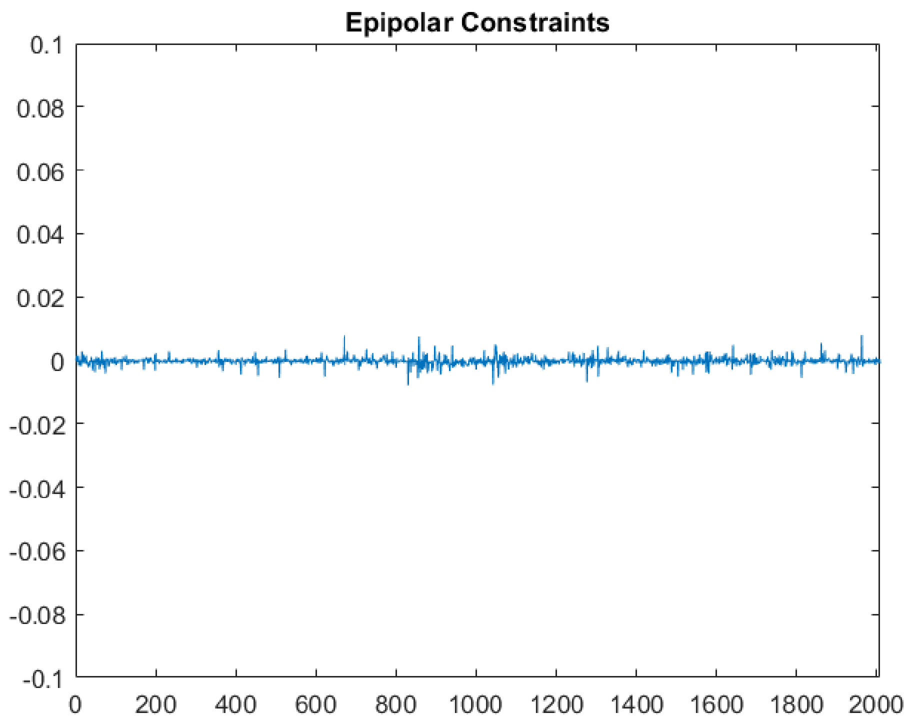
The fundamental matrix is

$$F_I = \begin{pmatrix} 0 & 0 & 0.0058 \\ 0 & 0 & -0.0267 \\ -0.0072 & 0.0263 & 1 \end{pmatrix}$$

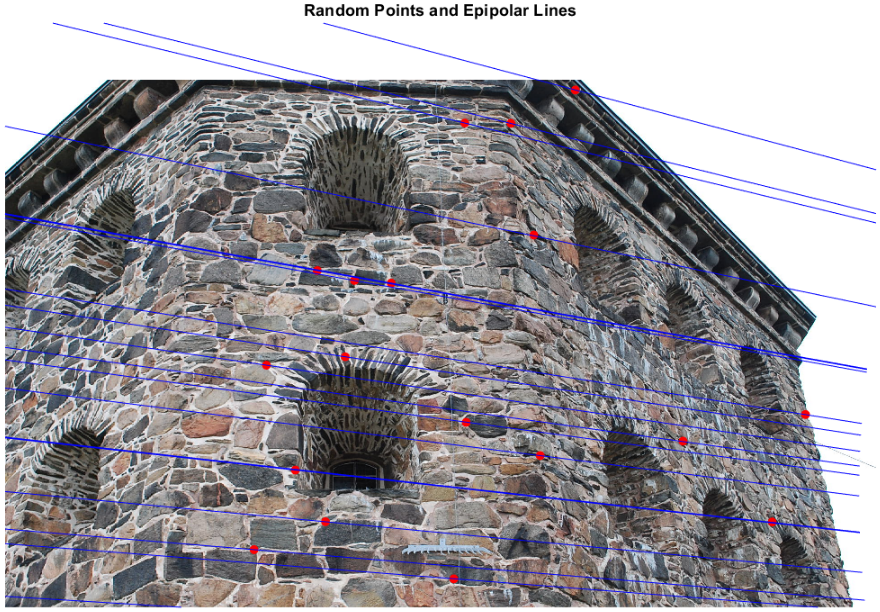
and the mean distance is

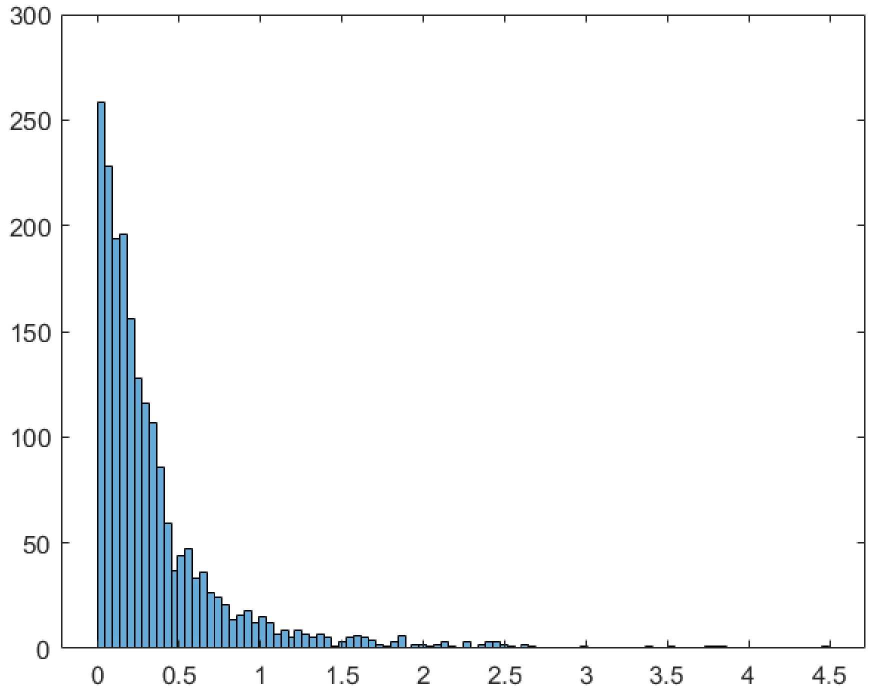
$$d_{MAE} = 0.3612$$

The epipolar constraints are roughly fulfilled, the epipolar lines match the random points and most distances are below 0.5, as seen in the figures below.


Epipolar Constraints
0.1
0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
-0.1
0
200
400
600
800
1000
1200
1400
1600
1800
2000

Random Points and Epipolar Lines





Distances Histogram

PART II

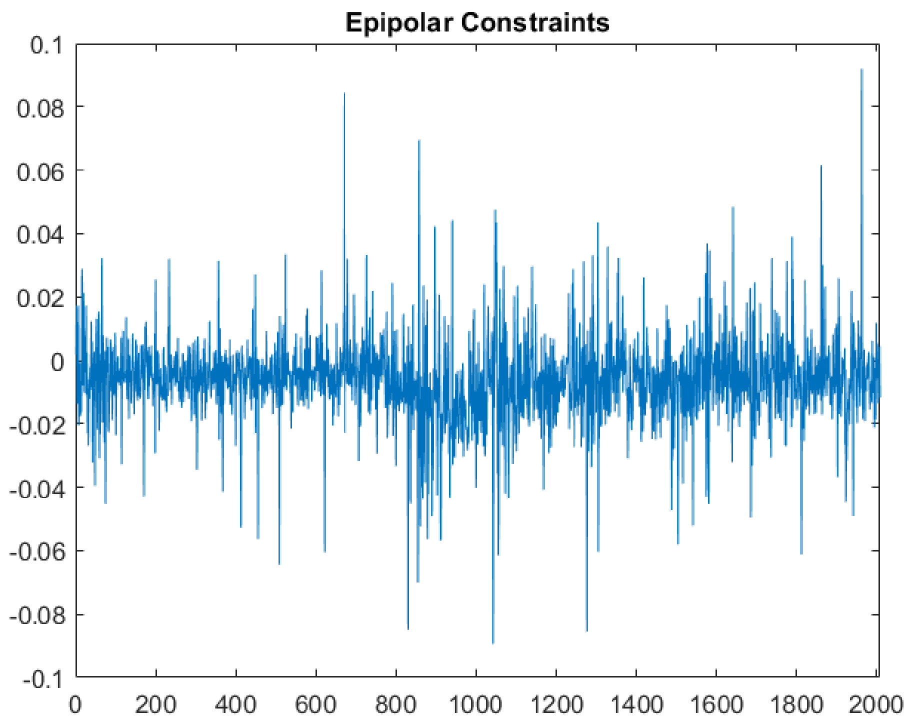
The fundamental matrix is

$$F_{II} = \begin{pmatrix} 0 & 0 & 0.0058 \\ 0 & 0 & -0.0266 \\ -0.0072 & 0.0262 & 1 \end{pmatrix}$$

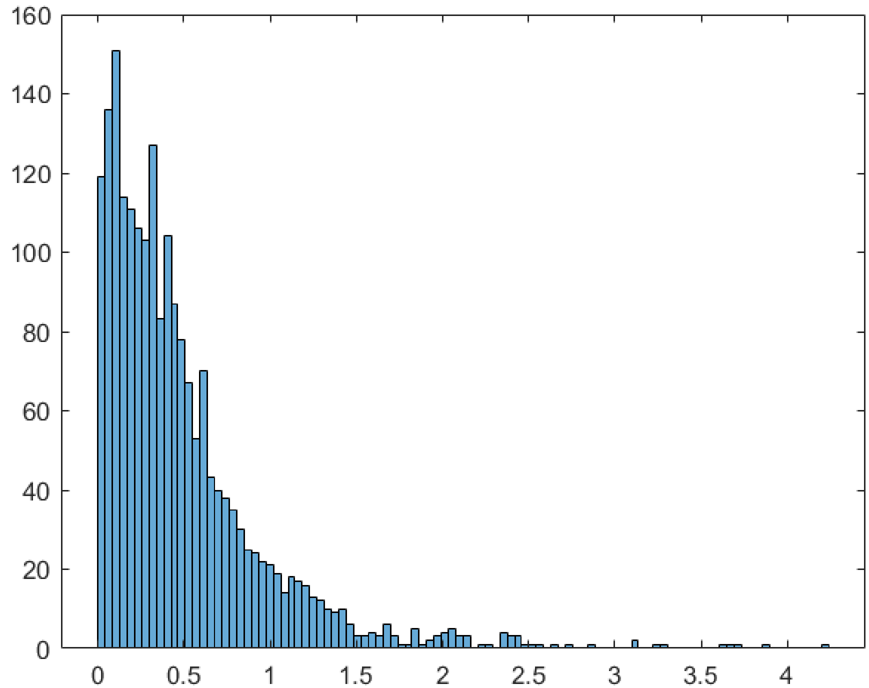
and the mean distance is

$$d_{MAE} = 0.4878$$

The epipolar constraints are small, but larger than in PART I. Still, the epipolar lines match the random points and the distances lie mostly below 0.5, as presented in the figures below.


Epipolar Constraints
0.1
0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
-0.1
0
200
400
600
800
1000
1200
1400
1600
1800
2000

Random Points and Epipolar Lines



Distances Histogram

Theoretical Exercise 5

2

Let $\mathbf{X_1} = \begin{bmatrix} \mathcal{X}_1 \\ 1 \end{bmatrix}$ and $\mathbf{X_2} = \begin{bmatrix} \mathcal{X}_2 \\ 1 \end{bmatrix}$, where $\mathcal{X}_1 = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$ and $\mathcal{X}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$.

$$\begin{aligned} \mathbf{x_2}^T F \mathbf{x_1} &= \\ (P_2 \mathbf{X_2})^T F (P_1 \mathbf{X_1}) &= \\ \left([\mathbf{e_2}]_\times F \mathcal{X}_2 + \mathbf{e_2}\right)^T F \mathcal{X}_1 &= \\ \mathcal{X}_2^T F^T [\mathbf{e_2}]_\times^T F \mathcal{X}_1 + \mathbf{e_2}^T F \mathcal{X}_1 &= \\ -\mathcal{X}_2^T F^T [\mathbf{e_2}]_\times F \mathcal{X}_1 + \mathbf{0}^T \mathcal{X}_1 &= \\ -\mathcal{X}_2^T F^T [\mathbf{e_2}]_\times F \mathcal{X}_1 & \end{aligned}$$

Let $[\mathbf{e_2}]_\times = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}$. Then, we have:

$$\begin{aligned} F^T [\mathbf{e_2}]_\times F &= \\ \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 4 & 1 \end{pmatrix} \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 4 \\ 0 & 1 & 1 \end{pmatrix} &= \\ 2(x-z) \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix} & \end{aligned}$$

Continuing the above equation, we have:

$$\begin{aligned} \mathbf{x_2}^T F \mathbf{x_1} &= \\ -\mathcal{X}_2^T F^T [\mathbf{e_2}]_\times F \mathcal{X}_1 &= \\ -2(x-z) \begin{bmatrix} -1 & 2 & 0 \end{bmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix} \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} &= \\ -2(x-z) \begin{bmatrix} -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix} &= \\ -2(x-z)(-4+4+0) &= 0 \end{aligned}$$

The center of camera P_2 is in the nullspace of F, since $F\mathbf{e_1} = \mathbf{0}$ and $\mathbf{e_1} = \mathbf{c_2}$ (see *Theoretical Exercise 3*). Let the center be $\mathbf{C_2} = \begin{bmatrix} \mathbf{c_2} \\ 1 \end{bmatrix}$, where $\mathbf{c_2} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$:

$$\begin{aligned}
F\mathbf{e_1} &= \mathbf{0} \Leftrightarrow \\
F\mathbf{c_2} &= \mathbf{0} \Leftrightarrow \\
\begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 4 \\ 0 & 1 & 1 \end{pmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \mathbf{0} \Leftrightarrow \\
\begin{cases} y + z = 0 \\ 2x + 4z = 0 \\ y + z = 0 \end{cases} & \Leftrightarrow \\
\begin{cases} x = -2z \\ y = -z \end{cases} & \Leftrightarrow \\
\mathbf{c_2} &= \begin{bmatrix} -2z \\ -z \\ z \end{bmatrix}
\end{aligned}$$

If the center is the unit vector of the nullspace, with $||\mathbf{c_2}|| = |Z|\sqrt{6}$, we have:

$$\hat{\mathbf{c_2}} = \frac{\mathbf{c_2}}{||\mathbf{c_2}||} = \frac{z}{|z|\sqrt{6}} \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \Longleftrightarrow$$

$$\hat{\mathbf{c_2}} = \pm \frac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}, \quad \text{if } z > 0 \text{ or } z < 0 \text{ respectively.}$$

The Essential Matrix

12 *Theoretical Exercise 6*

If $[\mathbf{t}]_\times = USV^T$, then

$$[\mathbf{t}]_\times^T[\mathbf{t}]_\times = (USV^T)^T USV^T = VS^T U^T USV^T = VS^T SV^T = VS^2 V^T$$

Let the eigenvalues and the eigenvectors of $[\mathbf{t}]_\times^T[\mathbf{t}]_\times$ be λ and $\mathbf{u}$ respectively. Then, the eigenvalue equation will be:

$$\begin{aligned}
[\mathbf{t}]_\times^T[\mathbf{t}]_\times \mathbf{u} &= \lambda \mathbf{u} \Leftrightarrow \\
VS^2V^T\mathbf{u} &= \lambda \mathbf{u} \Leftrightarrow \\
V^TVS^2V^T\mathbf{u} &= \lambda V^T\mathbf{u} \Leftrightarrow \\
S^2(V^T\mathbf{u}) &= \lambda(V^T\mathbf{u}) \xLeftrightarrow{V^T\mathbf{u}=\mathbf{w}} \\
S^2\mathbf{w} &= \lambda\mathbf{w} \Leftrightarrow
\end{aligned}$$

Then, the eigenvalues of $[\mathbf{t}]_\times^T[\mathbf{t}]_\times$ can be found by:

$$det(S^2 - \lambda I) = 0 \Leftrightarrow \prod_{i=1}^{3}(s_{ii}^2 - \lambda_i) \Leftrightarrow \lambda_i = s_{ii}^2$$

Thus, the eigenvalues of $[\mathbf{t}]_\times^T[\mathbf{t}]_\times$ are the squared singular values.
By definition, an antisymmetric matrix is $-[\mathbf{t}]_\times = [\mathbf{t}]_\times^T$. Therefore, it implies the following:

$$-\mathbf{t} \times \mathbf{t} \times \mathbf{w} = -[\mathbf{t}]_\times[\mathbf{t}]_\times\mathbf{w} = [\mathbf{t}]_\times^T[\mathbf{t}]_\times\mathbf{w}$$

and, consequently:

$$\begin{aligned}
-\mathbf{t} \times \mathbf{t} \times \mathbf{w} &= \lambda\mathbf{w} \Leftrightarrow \\
[\mathbf{t}]_\times^T[\mathbf{t}]_\times\mathbf{w} &= \lambda\mathbf{w}
\end{aligned}$$

the above eigenvalue equation shows that $\mathbf{w}$ is an eigenvector of $[\mathbf{t}]_\times^T[\mathbf{t}]_\times$ with eigenvalue λ.

Now we will investigate if $\mathbf{t}$ and an orthogonal vector to it $\mathbf{w}_\perp$ are eigenvectors of $[\mathbf{t}]_\times^T[\mathbf{t}]_\times$.

$$\begin{aligned}
[\mathbf{t}]_\times^T[\mathbf{t}]_\times\mathbf{t} &= \\
-\mathbf{t}\times\mathbf{t}\times\mathbf{t} = -\mathbf{t}\times\mathbf{0} &= \\
&= \mathbf{0}
\end{aligned}$$

and

$$\begin{aligned}
[\mathbf{t}]_\times^T[\mathbf{t}]_\times\mathbf{w}_\perp &= \\
= -\mathbf{t}\times\mathbf{t}\times\mathbf{w}_\perp &= \\
= (-\mathbf{t}\cdot\mathbf{w}_\perp)\cdot\mathbf{t} - (-\mathbf{t}\cdot\mathbf{t})\cdot\mathbf{w}_\perp &= \\
= \mathbf{0} + (\mathbf{t}\cdot\mathbf{t})\cdot\mathbf{w}_\perp &= \\
&= ||\mathbf{t}||^2\mathbf{w}_\perp
\end{aligned}$$

The above equations combined with the respective eigenvalue equations yield:

$$[\mathbf{t}]_\times^T[\mathbf{t}]_\times\mathbf{t} = \lambda\mathbf{t} \Leftrightarrow \lambda\mathbf{t} = \mathbf{0} \xLeftrightarrow{\mathbf{t}\neq\mathbf{0}} \lambda = 0$$

and

$$[\mathbf{t}]_\times^T[\mathbf{t}]_\times\mathbf{w}_\perp = \lambda\mathbf{w}_\perp \Leftrightarrow ||\mathbf{t}||^2\mathbf{w}_\perp = \lambda\mathbf{w}_\perp \xLeftrightarrow{\mathbf{w}_\perp\neq\mathbf{0}} \lambda = ||\mathbf{t}||^2$$

All the eigenvalues are $\lambda = 0$ with multiplicity 1 and $\lambda = ||\mathbf{t}||^2$ with multiplicity 2.

The singular values of $[\mathbf{t}]_\times$ are s_i, $i = 1, 2, 3$. Due to all that is proven above, the squared singular values are $s_i^2 = 0$ with multiplicity 1 and $s_i^2 = ||\mathbf{t}||^2$ with multiplicity 2. Therefore, the singular values of $[\mathbf{t}]_\times$ are 0, $||\mathbf{t}||$ and $||\mathbf{t}||$.

Lastly, for an essential matrix in the form of $E = [\mathbf{t}]_\times R$ we have the following:

$$E\,E^T = [\mathbf{t}]_\times RR^T[\mathbf{t}]_\times^T = [\mathbf{t}]_\times\,[\mathbf{t}]_\times^T = [\mathbf{t}]_\times^T[\mathbf{t}]_\times$$

This shows that the rotation matrix R does not affect the singular values of $[\mathbf{t}]_\times$, since it's orthogonal. Then, we can conclude that the singular values of E are the singular values of $[\mathbf{t}]_\times$, which proves the goal of this exercise.

Computer Exercise 2

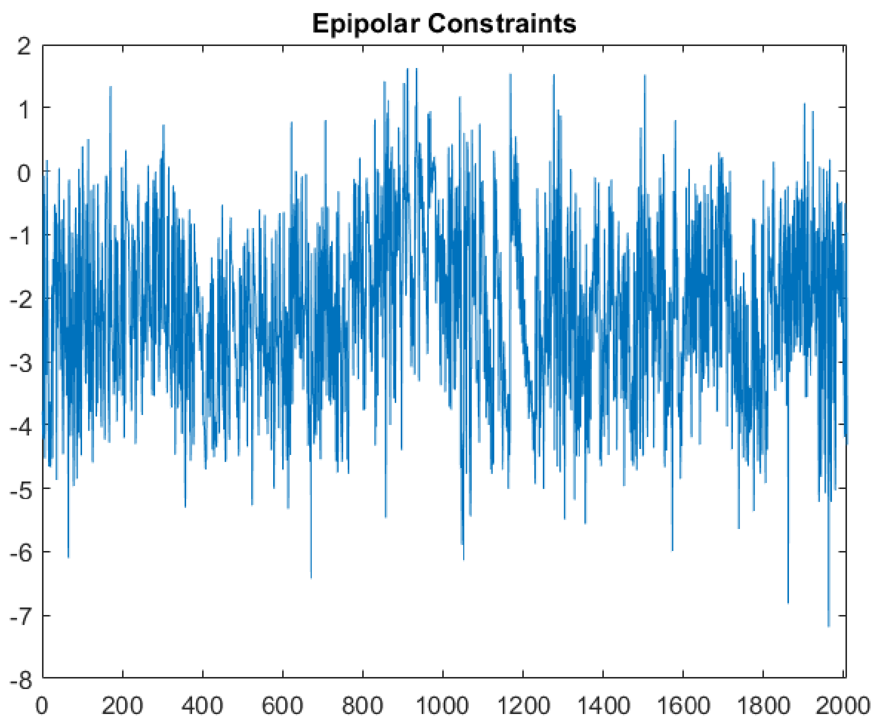
The essential matrix is

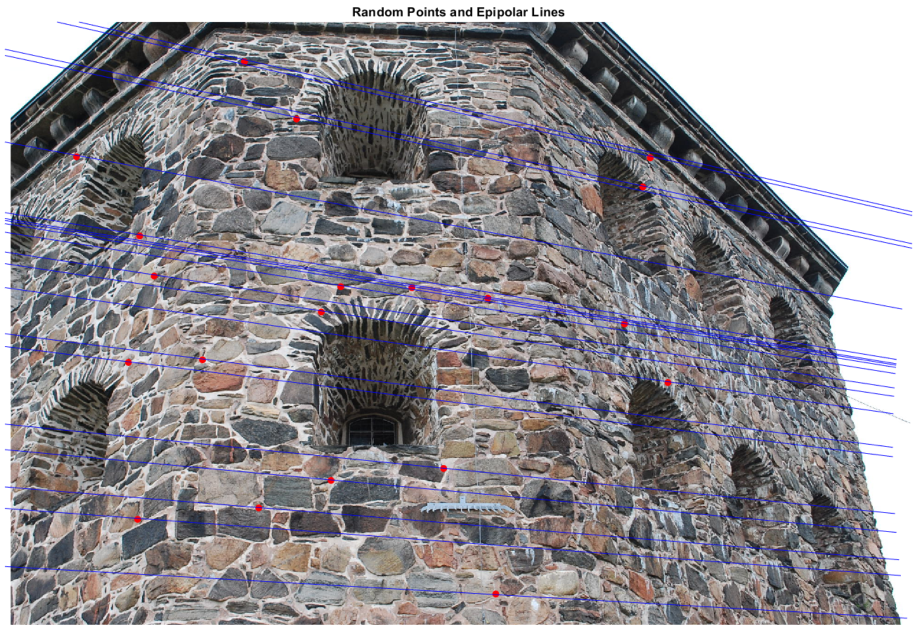
$$E = 10^3 \begin{pmatrix} -0.0089 & -1.0058 & 0.3771 \\ 1.2525 & 0.0784 & -2.4482 \\ -0.4728 & 2.5502 & 0.0010 \end{pmatrix}$$

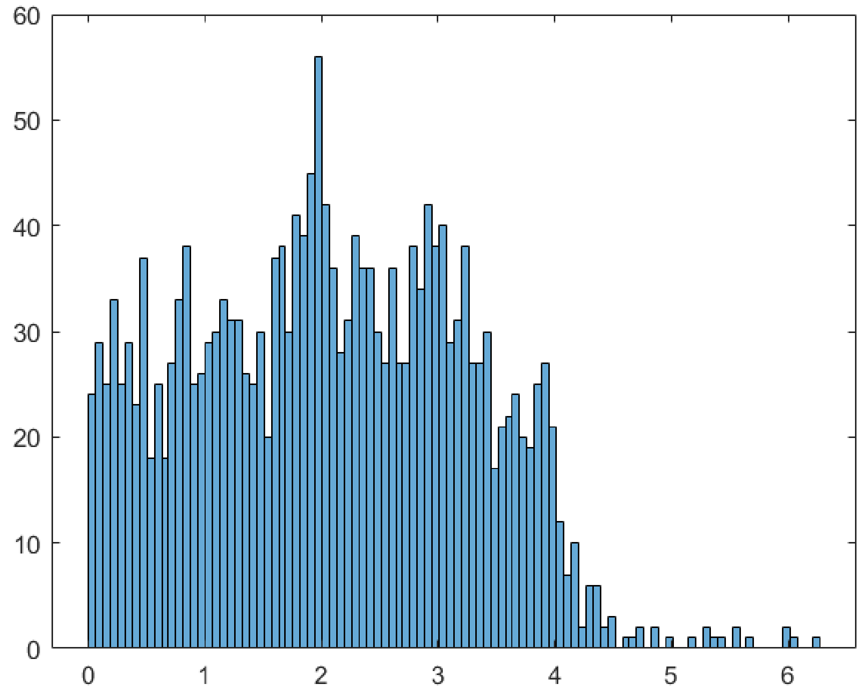
and the mean distance is

$$d_{MAE} = 2.0838$$

In comparison with *Computer Exercise 1*, the results are less optimal. The epipolar constraints are not fulfilled, even after enforcing $det(E) = 0$. Even though the epipolar lines roughly match the random points, the distances lie between 0 and 4.







Distances Histogram

5 Theoretical Exercise 7

$$UV^T = \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix}$$

$$det(UV^T) = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & -1/\sqrt{2} \\ 1 & 0 \end{vmatrix} + \frac{1}{\sqrt{2}} \begin{vmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \end{vmatrix} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = 1$$

$$E = U\mathrm{diag}([110])V^T = \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} =$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{x_2}^T E \mathbf{x_1} =$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -3 & 1 \end{bmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} =$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -3 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} =$$

$$= \frac{1}{\sqrt{2}}(3 - 3 + 0) = 0$$

By observation, we confirm that $\mathbf{X}(s) = \begin{bmatrix} 2 \\ 0 \\ 1 \\ s \end{bmatrix} = \begin{bmatrix} \mathbf{x_1} \\ s \end{bmatrix}$ and by projecting $\mathbf{X}(s)$ into P_1, we have $P_1\mathbf{X}(s) = \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x_1} \\ s \end{bmatrix} = \mathbf{x_1}$.

Let $A_1 = UWV^T$ and $A_2 = UW^TV^T$. Analytically, A_1 and A_2 are:

$$A_1 = \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} =$$
$$= \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ -1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} =$$
$$= \begin{pmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix}$$

Then, for each case of P_2 we calculate the projection $P_2\mathbf{X}(s)$ and compare it with $\mathbf{x_2}$:

- $P_2 = \begin{bmatrix} A_1 & \mathbf{u_3} \end{bmatrix}$

$$P_2\mathbf{X}(s) = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \\ s \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ s \end{bmatrix}$$

- $P_2 = \begin{bmatrix} A_1 & -\mathbf{u_3} \end{bmatrix}$

$$P_2\mathbf{X}(s) = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \\ s \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ -s \end{bmatrix}$$

- $P_2 = \begin{bmatrix} A_2 & \mathbf{u_3} \end{bmatrix}$

$$P_2\mathbf{X}(s) = \begin{pmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \\ s \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} \\ s \end{bmatrix} \sim x_2 = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} \text{ for } s = -\frac{1}{\sqrt{2}}$$

- $P_2 = \begin{bmatrix} A_2 & -\mathbf{u_3} \end{bmatrix}$

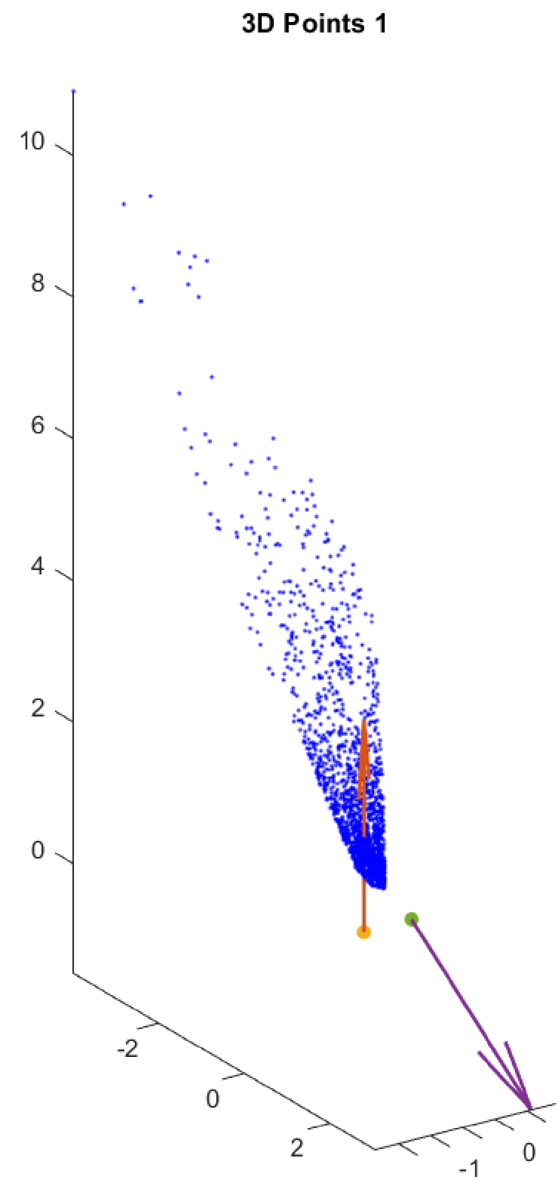
$$P_2\mathbf{X}(s) = \begin{pmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \\ s \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} \\ -s \end{bmatrix} \sim x_2 = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} \text{ for } s = \frac{1}{\sqrt{2}}$$

It is easy to prove that $det(A_2) = 1$ and, since A_2 is of odd dimension, $det(-A_2) = -1$. Hence, only $P_2\mathbf{X}(s)$ that result in $+s$ can be projected in front of the camera and $P_2\mathbf{X}(s)$ resulting in $-s$ are projected behind the camera. From the previous results we see that this stands for cameras with $\mathbf{t} = +\mathbf{u_3}$, which is $P_2 = \begin{bmatrix} UW^TV^T & \mathbf{u_3} \end{bmatrix}$.

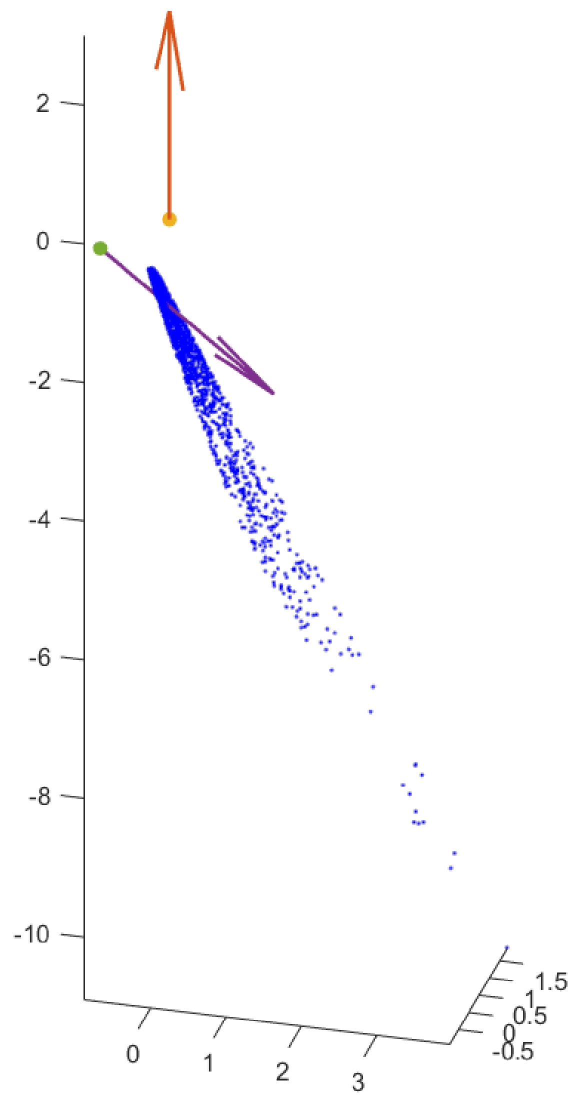
Computer Exercise 3

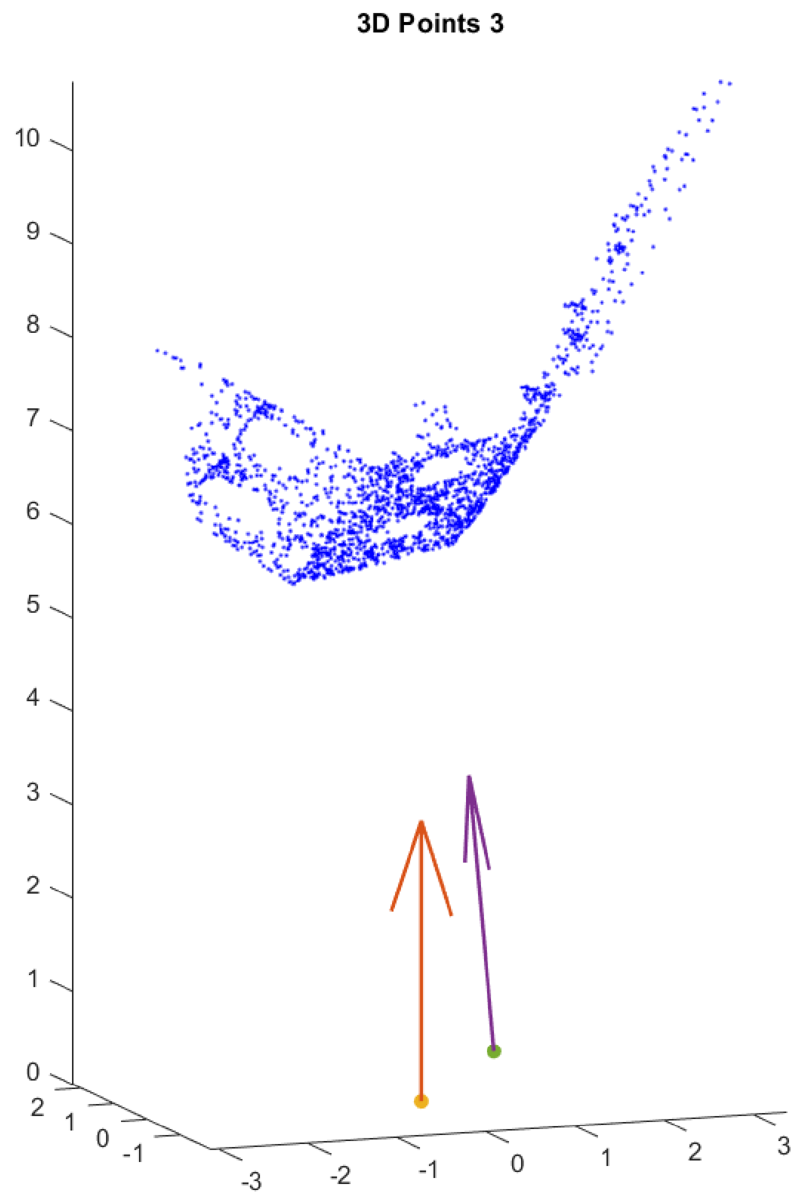
The correct 3D representation is given in the figure '3D Points 3' with the second camera matrix being $P_2 = \begin{bmatrix} UW^TV^T & \mathbf{u_3} \end{bmatrix}$.

The projection in the last figure looks reasonable. The errors look small and the projections are well aligned with the image points.

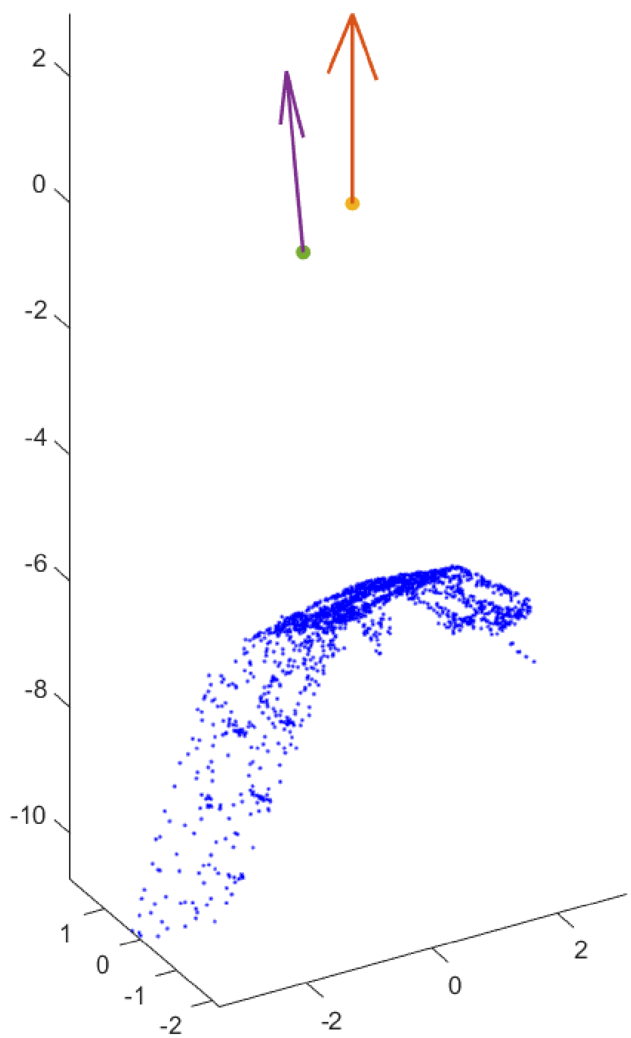

3D Points 1
10
8
6
4
2
0
-2
0
2
-1
0

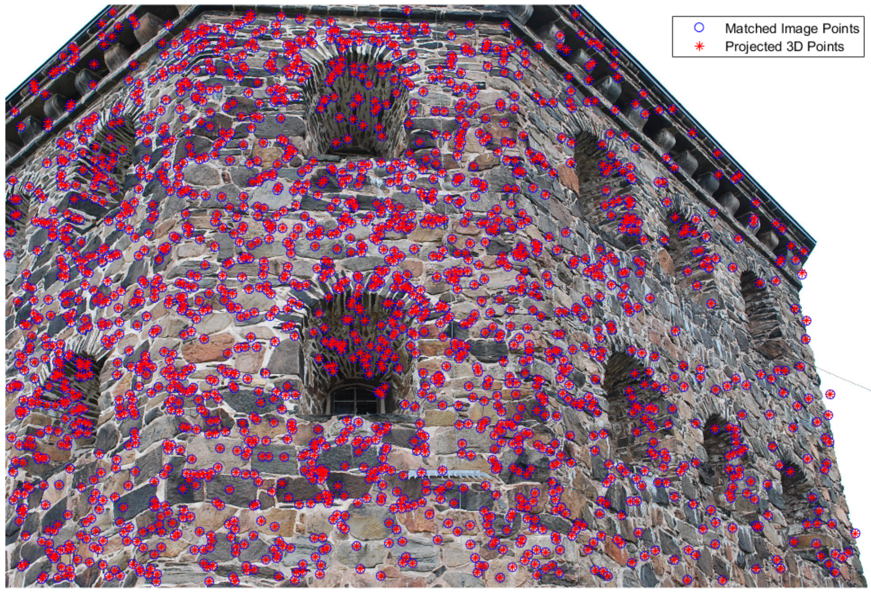
3D Points 2




3D Points 3
10
9
8
7
6
5
4
3
2
1
0
2
1
0
-1
-3
-2
-1
0
1
2
3

3D Points 4




Matched Image Points
Projected 3D Points