31/34 core points, great job!

EEN020 Computer Vision
Assignment 1
Elements of Projective Geometry

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Points in Homogeneous Coordinates

Theoretical Exercise 1

5 The given points can be transformed to homogeneous coordinates by dividing every coordinate x, y, z by z, with $z \neq 0$:

$$\mathbf{x_1} = \begin{bmatrix} 4 \\ -16 \\ 2 \end{bmatrix} \longrightarrow \mathbf{x_1} = \begin{bmatrix} 2 \\ -8 \\ 1 \end{bmatrix}$$

$$\mathbf{x_2} = \begin{bmatrix} -3 \\ 7 \\ -1 \end{bmatrix} \longrightarrow \mathbf{x_2} = \begin{bmatrix} 3 \\ -7 \\ 1 \end{bmatrix}$$

$$\mathbf{x_3} = \begin{bmatrix} 9\lambda \\ -3\lambda \\ 6\lambda \end{bmatrix}, \lambda \neq 0 \longrightarrow \mathbf{x_3} = \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Thus, the Cartesian coordinates of the above points in homogeneous coordinates in two dimensions are:

$$\mathbf{x_1} = \begin{bmatrix} 2 \\ -8 \end{bmatrix}, \quad \mathbf{x_2} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}, \quad \mathbf{x_3} = \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{bmatrix}$$

The points $\mathbf{x_4} = \begin{bmatrix} -6 \\ 3 \\ 0 \end{bmatrix}$ and $\mathbf{x_5} = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$ are points at infinity and do not have inhomogeneous coordinate representation. However, they both belong to the projective space $\mathbb{P}^2 = \mathbb{R}^3 - \mathbf{0}$ and they *are different* points.

Computer Exercise 1

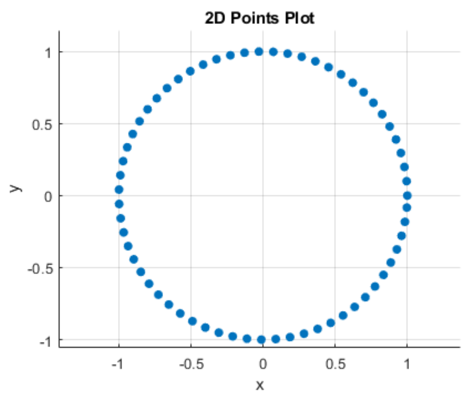

Figure 1: Plot of the points in $x2D$.

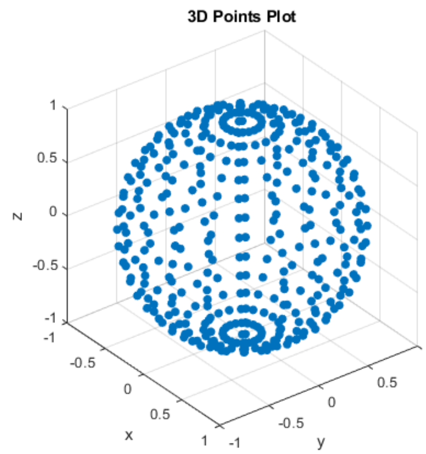

Figure 2: Plot of he points in $x3D$.

Lines

Theoretical Exercise 2

4

Let $\hat{\mathbf{e_x}}$, $\hat{\mathbf{e_y}}$ and $\hat{\mathbf{e_z}}$ be the standard unit vectors of the three dimensional Cartesian system.

The homogeneous coordinates of the intersection of the lines l_1 and l_2 are given by the position vector equal to $\mathbf{l_1} \times \mathbf{l_2}$:

$$\begin{aligned} \mathbf{l_1} \times \mathbf{l_2} &= \\ \begin{vmatrix} \hat{\mathbf{e_x}} & \hat{\mathbf{e_y}} & \hat{\mathbf{e_z}} \\ -1 & 1 & 1 \\ 6 & 3 & 1 \end{vmatrix} &= \\ \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} \hat{\mathbf{e_x}} - \begin{vmatrix} -1 & 1 \\ 6 & 1 \end{vmatrix} \hat{\mathbf{e_y}} + \begin{vmatrix} -1 & 1 \\ 6 & 3 \end{vmatrix} \hat{\mathbf{e_z}} &= \\ -2\hat{\mathbf{e_x}} + 7\hat{\mathbf{e_y}} - 9\hat{\mathbf{e_z}} &= \\ -9\left(\frac{2}{9}\hat{\mathbf{e_x}} - \frac{7}{9}\hat{\mathbf{e_y}} + \hat{\mathbf{e_z}}\right) \end{aligned}$$

In the last step, the quantity inside the parentheses is normalized by $z = -9$ for homogeneous coordinates. Therefore, the last step shows that all the homogeneous coordinates ($\mathbf{x} \in \mathbb{P}^2$) can be expressed as:

$$\mathbf{x} = \begin{bmatrix} \frac{2}{9}\lambda \\ -\frac{7}{9}\lambda \\ \lambda \end{bmatrix}, \ \lambda \in \mathbb{R} - \{0\}$$

Thus, after dividing all the coordinates by λ, the corresponding point in $\mathbb{R}^2$ is:

$$\mathbf{x} = \begin{bmatrix} \frac{2}{9} \\ -\frac{7}{9} \end{bmatrix}$$

Similarly, the intersection of the lines l_3 and l_4 is given by:

$$\begin{aligned} \mathbf{l_3} \times \mathbf{l_4} &= \\ \begin{vmatrix} \hat{\mathbf{e_x}} & \hat{\mathbf{e_y}} & \hat{\mathbf{e_z}} \\ -3 & 0 & 1 \\ 5 & 0 & 4 \end{vmatrix} &= \\ \begin{vmatrix} 0 & 1 \\ 0 & 4 \end{vmatrix} \hat{\mathbf{e_x}} - \begin{vmatrix} -3 & 1 \\ 5 & 4 \end{vmatrix} \hat{\mathbf{e_y}} + \begin{vmatrix} -3 & 0 \\ 5 & 0 \end{vmatrix} \hat{\mathbf{e_z}} &= \\ 0\hat{\mathbf{e_x}} + 17\hat{\mathbf{e_y}} + 0\hat{\mathbf{e_z}} \end{aligned}$$

The coordinates of this point show that it is a point to infinity ($z = 0$). Moreover, the intersection of lines l_3 and l_4 shows that all the homogeneous coordinates, ($\mathbf{x} \in \mathbb{P}^2$), form the y axis. We exclude the element $\{0\}$, since $(0, 0, 0,) \notin \mathbb{P}^2$:

$$\mathbf{x} = \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix}, y \in \mathbb{R} - \{0\}$$

The line l that goes through the points $\mathbf{x_1} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{x_2} = \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix}$ with homogeneous coordinates is given by:

$$\begin{aligned} \mathbf{x_1} \times \mathbf{x_2} &= \\ \begin{vmatrix} x & y & z \\ -1 & 1 & 1 \\ 6 & 3 & 1 \end{vmatrix} &= \\ \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} x - \begin{vmatrix} -1 & 1 \\ 6 & 1 \end{vmatrix} y + \begin{vmatrix} -1 & 1 \\ 6 & 3 \end{vmatrix} z &= \\ -2x + 7y - 9z \end{aligned}$$

Thus, the line is given by the equation:

$$l : -2x + 7y - 9z = 0$$

which, in homogeneous coordinates ($z = 1$), is expressed as:

$$l : -2x + 7y - 9 = 0$$

Theoretical Exercise 3

3

the lines of matrix M consist of the transposed coefficient vectors of lines l_2 and l_1 respectively:

$$M = \begin{bmatrix} \mathbf{l_2^T} \\ \mathbf{l_1^T} \end{bmatrix}$$

In *Theoretical Exercise 2*, the coefficient vectors were switched:

$$A = \begin{bmatrix} \mathbf{l_1^T} \\ \mathbf{l_2^T} \end{bmatrix}$$

However, due to the definition of the nullspace, the resulting homogeneous coordinates of the intersection of l_1 and l_2 remain the same, since the intersection can be expressed both as $\mathbf{l_1} \times \mathbf{l_2}$ and $\mathbf{l_2} \times \mathbf{l_1}$. Consequently:

$$\begin{aligned} \mathbf{l_1} \times \mathbf{l_2} = \mathbf{l_2} \times \mathbf{l_1} = \mathbf{0} \implies \\ A\mathbf{x} = M\mathbf{x} = \mathbf{0} \implies \\ \mathcal{N}(A) = \mathcal{N}(M) \end{aligned}$$

As shown in *Theoretical Exercise 2*, all the points in the nullspace ($\mathcal{N}$) are the intersection point and every scalar multiplication of it, with the exception of zero:

$$\mathcal{N} = \left\{ \mathbf{x} \in \mathbb{R}^2 \,\middle|\, \mathbf{x} = \begin{bmatrix} \frac{2}{9}\lambda & -\frac{7}{9}\lambda \end{bmatrix}^T, \, \lambda \in \mathbb{R} - \{0\} \right\}$$

Computer Exercise 2

The line coefficients for pair points p_1, p_2 and p_3 are given respectively by the vectors:

$$\mathbf{l_1} = 10^5 \begin{bmatrix} -0.0018 \\ -0.0023 \\ 2.5923 \end{bmatrix}, \quad \mathbf{l_2} = 10^5 \begin{bmatrix} -0.0010 \\ -0.0021 \\ 2.3413 \end{bmatrix}, \quad \mathbf{l_3} = 10^5 \begin{bmatrix} 0.0005 \\ -0.0033 \\ 3.5582 \end{bmatrix}$$

By observation in Figure 3, the three lines l_1, l_2 and l_3 appear to be parallel in three dimensions.

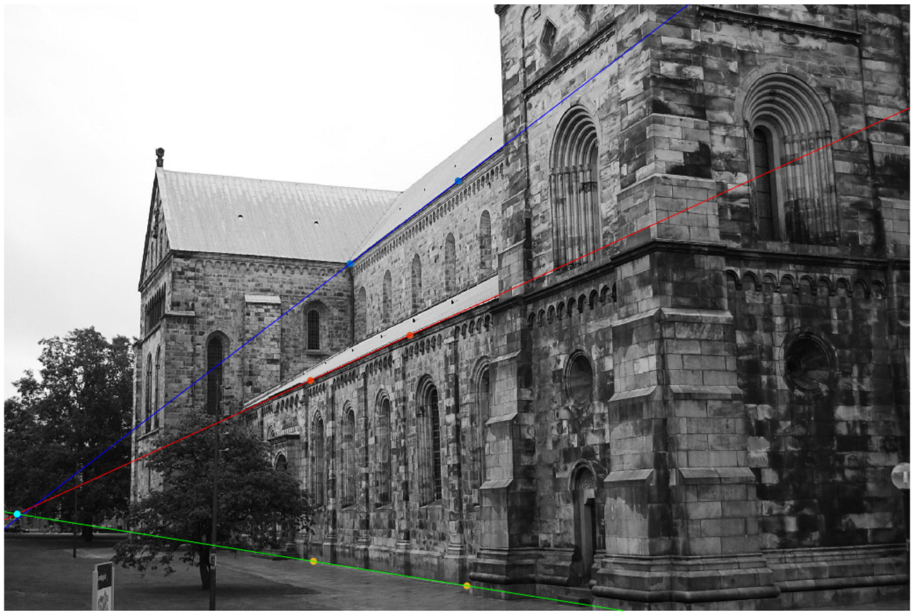

Figure 3: Full plot for computer exercise 2. l_1 : blue, l_2: red, l_3: green, intersection between l_2 and l_3, X: cyan

The distance between line l_1 and the intersection X between lines l_2 and l_3 in homogeneous coordinates is:

$$d = 8.2695 \text{ , where } X = \begin{bmatrix} 28.3 \\ 1089.8 \\ 1 \end{bmatrix}$$

If the three lines intersected at the exact same point ($d = 0$) in two dimensions, they would be perfectly parallel in three dimensions. In our case, the distance is not exactly close to zero. However, it is very small compared to the size of the image, since every dimension of the image is 3 orders of magnitude larger than the distance.

The reasons for this result vary. One possibility is that the way the points were obtained and the lines calculated is prone to errors. One other possibility is that having parallel lines in three dimensions is an ideal interpretation of the real properties of the object we're studying. Therefore, when the lines are extended to a scale that can be considered close to infinity, there will always be a very small distance close to zero between one line and the intersection of the other two.

Consequently, even if there were no errors in calculations, the three lines being parallel in three dimensions is a valid approximation.

Projective Transformations

5 ## *Theoretical Exercise 4*

The transformations $\mathbf{y_1}$ and $\mathbf{y_2}$ are:

$$\mathbf{y_1} \sim H\mathbf{x_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{y_2} \sim H\mathbf{x_2} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

The line l_1 containing $\mathbf{x_1}$ and $\mathbf{x_2}$ is:

$$\mathbf{x_1} \times \mathbf{x_2} = \begin{vmatrix} x & y & z \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} x - \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} y + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} z = -x - y + z$$

$$l_1 : -x - y + z = 0$$

with line vector

$$\mathbf{l_1} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

and the line l_2 containing $\mathbf{y_1}$ and $\mathbf{y_2}$ is:

$$\mathbf{y_1} \times \mathbf{y_2} = \begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} x - \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} y + \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} z = -2x + y + z$$

$$l_2 : -2x + y + z = 0$$

with line vector

$$\mathbf{l_2} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

The matrix $(H^{-1})^T$ is given by:

$$(H^{-1})^T = \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$

Lastly, we compute $(H^{-1})^T\mathbf{l_1}$:

$$
\begin{aligned}
(H^{-1})^T\mathbf{l_1} &= \frac{1}{2}\begin{bmatrix}-2\\1\\1\end{bmatrix} \implies \\
(H^{-1})^T\mathbf{l_1} &= \frac{1}{2}\mathbf{l_2} \implies \\
\mathbf{l_2} &= 2(H^{-1})^T\mathbf{l_1}
\end{aligned}
$$

Therefore, we arrive at the conclusion that $\mathbf{l_2} \sim (H^{-1})^T\mathbf{l_1}$, which shows that lines l_1 and l_2 are preserved.

4 *Theoretical Exercise 5*

Let $\mathbf{y}$ be the transformation $\mathbf{y} = \lambda H\mathbf{x}\,, \quad \lambda \in \mathbb{R}$. We make the initial hypothesis that $\mathbf{x}$ belongs to l_1 and $\mathbf{y}$ belongs to l_2:

$$
\mathbf{l_1}^T\mathbf{x} = 0 \quad \text{and} \quad \mathbf{l_2}^T\mathbf{y} = 0
$$

By disregarding the trivial solution $\mathbf{x} = \mathbf{0}$ we can make the following implications:

$$
\begin{aligned}
\begin{cases}\mathbf{l_1}^T\mathbf{x} = \mathbf{0}\\ \mathbf{l_2}^T\mathbf{y} = \mathbf{0}\end{cases} &\iff \\
\mathbf{l_1}^T\mathbf{x} = \mathbf{l_2}^T\mathbf{y} &\iff \\
\mathbf{l_1}^T H^{-1}H\mathbf{x} = \lambda\mathbf{l_2}^T H\mathbf{x} &\iff \\
\mathbf{l_1}^T H^{-1}H\mathbf{x} - \lambda\mathbf{l_2}^T H\mathbf{x} = 0 &\iff \\
\left(\mathbf{l_1}^T H^{-1} - \lambda\mathbf{l_2}^T\right)H\mathbf{x} = 0 &\iff \\
\mathbf{l_1}^T H^{-1} - \lambda\mathbf{l_2}^T = \mathbf{0} &\iff \\
\mathbf{l_1}^T H^{-1} = \lambda\mathbf{l_2}^T &\iff \\
\left(\mathbf{l_1}^T H^{-1}\right)^T = \left(\lambda\mathbf{l_2}^T\right)^T &\iff \\
\left(H^{-1}\right)^T\mathbf{l_1} = \lambda\mathbf{l_2}\,, &\quad \lambda \in \mathbb{R}
\end{aligned}
$$

By showing that $\left(H^{-1}\right)^T\mathbf{l_1} \sim \mathbf{l_2}$ through the initial hypothesis, we arrive to the conclusion that every projective transformation H preserves lines.

5 *Theoretical Exercise 6*

(a) H_1, H_2, H_3

(b) H_1, H_2,

(c) H_2

(d) H_2

(e) H_2

(f) H_1, H_2, H_3, H_4

(g) H_1, H_2, H_4

Strict characterisation of the matrices:

- H_1: affine
- H_2: Eucledian
- H_3: projective
- H_4: transformation matrix that doesn't belong to a specific category

The Pinhole Camera

Theoretical Exercise 7

5 The projections of the points $\mathbf{X_1}$, $\mathbf{X_2}$ and $\mathbf{X_3}$ with camera matrix P are respectively:

$$\mathbf{x_1} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \qquad \mathbf{x_2} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \qquad \mathbf{x_3} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

The projection of $\mathbf{X_1}$ ($\mathbf{x_1}$) is a point at infinity.

The camera centre is $c = (0, 0, 1)$. The principal vector is $\hat{\mathbf{e_z}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Therefore, the principal axis is the z axis.

Computer Exercise 3

By observing the direction of the principal axes in Figure 4 and comparing the corresponding images in Figures 5 and 6, we can conclude that the projections of the point model coincide with the images successfully for the majority of the points.

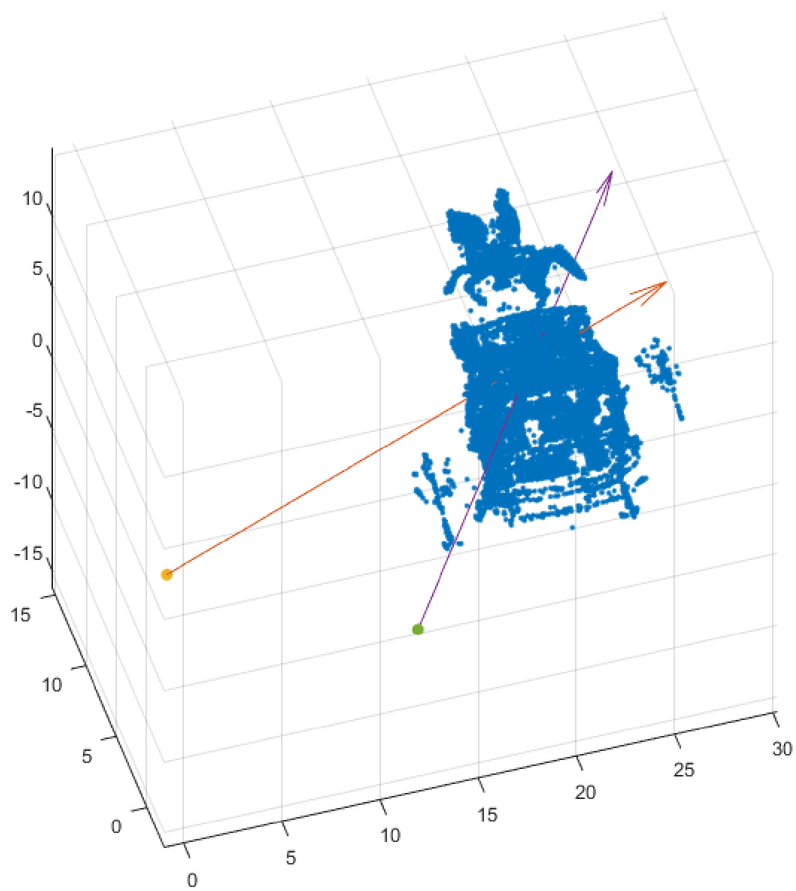

Figure 4: 3D Points U (blue), camera centers and axes of $\mathcal{P}_1$ (yellow, orange) and $\mathcal{P}_2$ (green, purple) respectively.

The center and the principal axis of $\mathcal{P}_1$ and $\mathcal{P}_2$ are respectively:

$$\mathbf{c_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{a_1} = \begin{bmatrix} 0.3129 \\ 0.9461 \\ 0.0837 \end{bmatrix} \qquad \text{and} \qquad \mathbf{c_2} = \begin{bmatrix} 6.6352 \\ 14.8460 \\ -15.0691 \end{bmatrix}, \quad \mathbf{a_2} = \begin{bmatrix} 0.0319 \\ 0.3402 \\ 0.9398 \end{bmatrix}$$

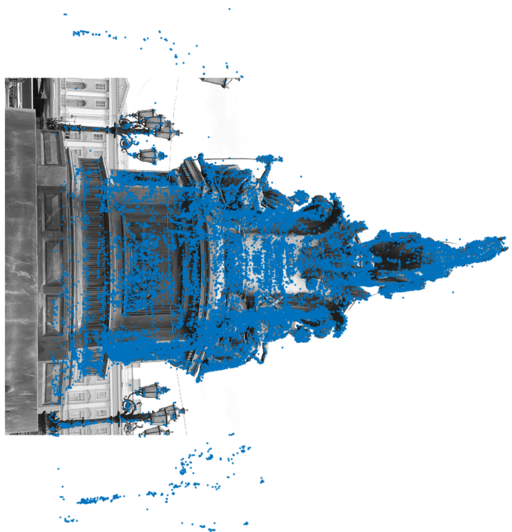

Figure 5: Image 1 and points $\mathcal{P}_1 \cdot U$

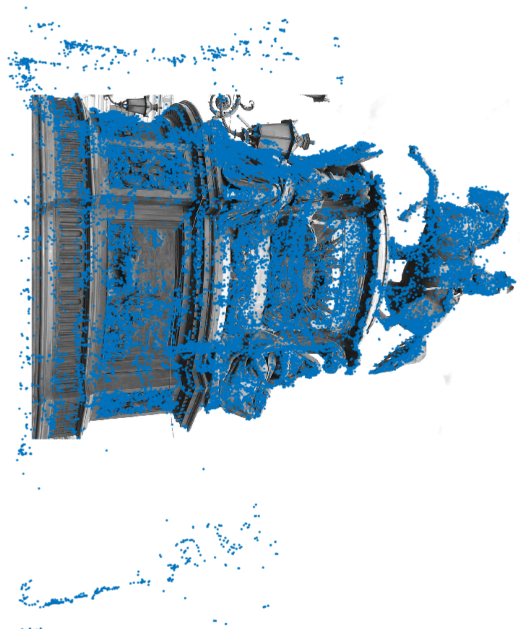

Figure 6: Image 2 and points $\mathcal{P}_2 \cdot U$

10 *Theoretical Exercise 8*

Let the hypothesis $\mathbf{U} \sim \begin{bmatrix} \mathbf{x} \\ s \end{bmatrix}$, $s \in \mathbb{R}$, be true. Then, every projection of $\mathbf{U}$ in $\mathcal{P}_1$ can be expressed as:

$$\begin{aligned} \mathcal{P}_1\mathbf{U} \sim \begin{bmatrix} \mathcal{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ s \end{bmatrix} = \\ = \mathcal{I}\,\mathbf{x} + s\,\mathbf{0} = \\ = \mathbf{x} \Longleftrightarrow \\ \mathbf{x} \sim \mathcal{P}_1\mathbf{U} \end{aligned}$$

We arrive at the given fact, that $\mathbf{x}$ is the projection of $\mathbf{U}$ in $\mathcal{P}_1$. Therefore, the initial hypothesis is also true: $\mathbf{U} \sim \begin{bmatrix} \mathbf{x} \\ s \end{bmatrix}$. The collection of points U is a set of lines with equations $x + y + z + s = 0$, $s \in \mathbb{R}$. Moreover, if x and $\mathcal{P}_1$ are known, it is not enough to determine the value of s, due to the similarity $\mathbf{x} \sim \mathcal{P}_1\mathbf{U}$ not being an equality. Considering a geometrical approach, s determines the "height" of each line with respect to the origin. This is proven algebraically by the following implications:

If $\mathbf{U}$ belongs to the plane $\mathbf{\Pi}$, then:

$$\begin{aligned} \mathbf{\Pi}^T\mathbf{U} = 0 \Longleftrightarrow \\ \begin{bmatrix} \pi^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ s \end{bmatrix} = 0 \Longleftrightarrow \\ \pi^T\mathbf{x} + s = 0 \Longleftrightarrow \\ s = -\pi^T\mathbf{x} \end{aligned}$$

In order to verify that the homography $\mathcal{H}$ maps $\mathbf{x}$ to $\mathbf{y}$ first we find the projection:

$$\begin{aligned} \mathbf{y} \sim \mathcal{P}_2\mathbf{U} = \begin{bmatrix} \mathcal{R} & \mathbf{t} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ s \end{bmatrix} = \\ = \mathcal{R}\mathbf{x} + s\mathbf{t} \Longleftrightarrow \\ \mathbf{y} \sim \mathcal{R}\mathbf{x} + s\mathbf{t} \end{aligned}$$

Then, we find what the homography $\mathcal{H}$ maps $\mathbf{x}$ to as follows:

$$\begin{aligned}
\mathcal{H}\mathbf{x} &= \big(\mathcal{R} - \mathbf{t}\pi^T\big)\mathbf{x} = \\
&= \mathcal{R}\mathbf{x} + \mathbf{t}\big(- \pi^T\mathbf{x}\big) = \\
&= \mathcal{R}\mathbf{x} + s\mathbf{t} \iff \\
\mathcal{H}\mathbf{x} &\sim \mathbf{y}
\end{aligned}$$

Thus, the homography $\mathcal{H}$ maps $\mathbf{x}$ to $\mathbf{y}$.

Computer Exercise 4

The origin of the original image in Figure 7 is at $(0, 0)$.

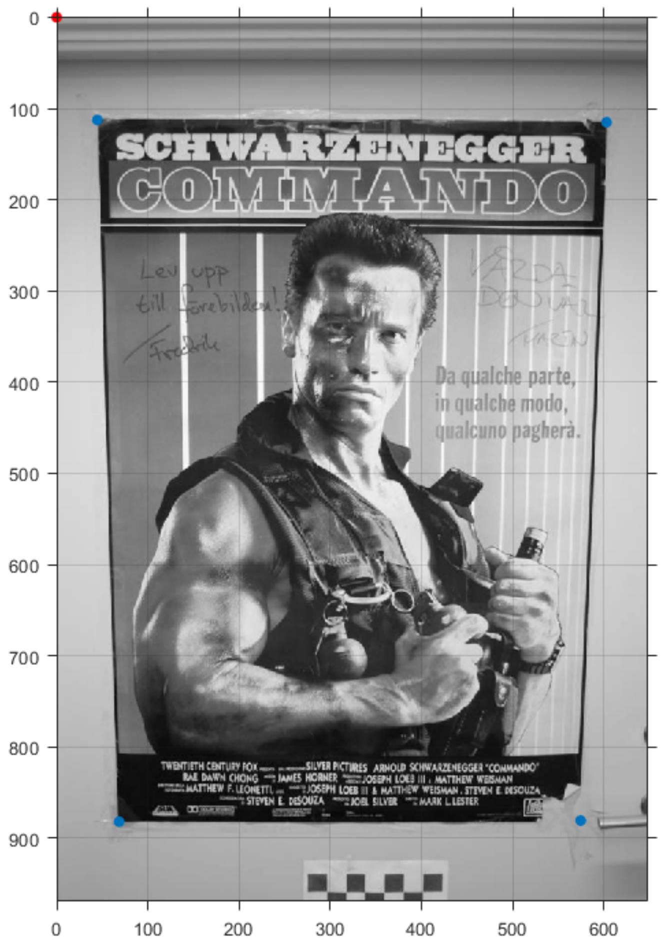

Figure 7: Original image, origin (red) and corners (blue).

The origin in Figure 8 is at $(0, 0)$, but the image corners are transferred. Also the scale is much smaller.

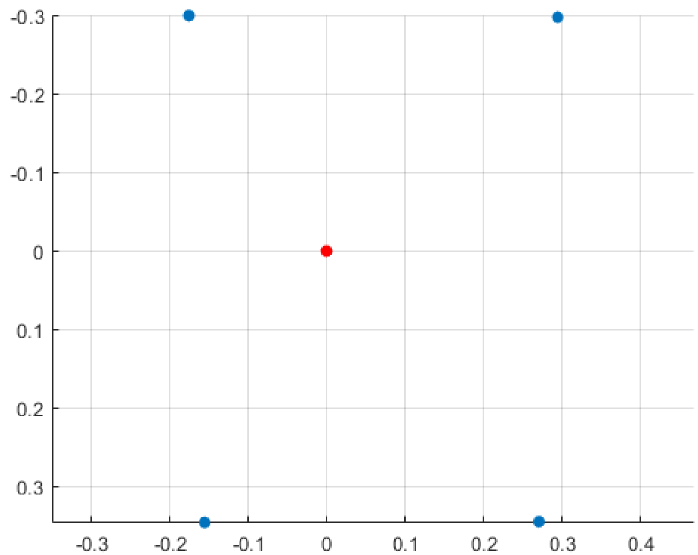

Figure 8: Origin (red) and corners (blue) of calibrated $\mathcal{P}_1$.

By observing Figure 9, both principal axes are directed towards the image. The overall plot looks reasonable, showing the different perspective of each camera. The second camera with center $\mathbf{c_2}$ has matrix:

$$\mathcal{P}_2 = \begin{bmatrix} \mathcal{R} & \mathbf{t} \end{bmatrix} \text{, where } \mathbf{t} = -\mathcal{R}\mathbf{c_2} = \begin{bmatrix} \sqrt{3} \\ 0 \\ 1 \end{bmatrix}$$

The center and the principal axis of $\mathcal{P}_1$ and $\mathcal{P}_2$ are respectively:

$$\mathbf{c_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{a_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{c_2} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{a_2} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{\sqrt{3}}{2} \end{bmatrix}$$

The flattened 3D points in the plane v that project onto the corner points are:

$$U_{flat} = \begin{pmatrix} -0.5175 & 0.8510 & 0.8471 & -0.4965 \\ -0.8837 & -0.8613 & 1.0798 & 1.1035 \\ 2.9457 & 2.8914 & 3.1323 & 3.1914 \\ 1. & 1. & 1. & 1. \end{pmatrix}$$

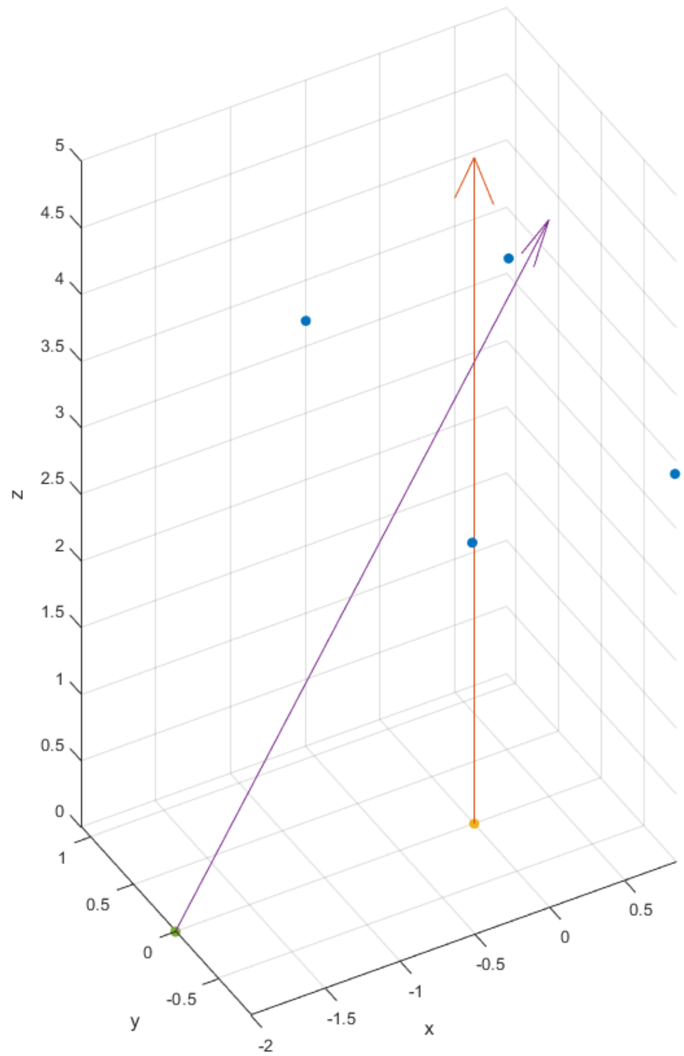

Figure 9: 3D visualisation of the image corners (blue), the centers and principal axes of $\mathcal{P}_1$ (orange) and $\mathcal{P}_2$ (purple) scaled by 5.

In Figure 10, the normalized 2D corner points coincide with the 3D points, which is expected by the way we define the homography and the 3D points. The proof of this is given in *theoretical exercise 8*, where $\mathbf{x}$ are the normalized corners, U are the 3D corners projected on the plane $\mathbf{\Pi}$, $\mathcal{P}_2$ is the second camera matrix and $\mathcal{H}$ is the homography. Therefore, the homography is computed to be:

$$\mathcal{H} = \begin{pmatrix} -0.8899 & -0.0708 & 0.0709 \\ 0 & 1. & 0 \\ 0.5138 & -0.0409 & 1.1956 \end{pmatrix}$$

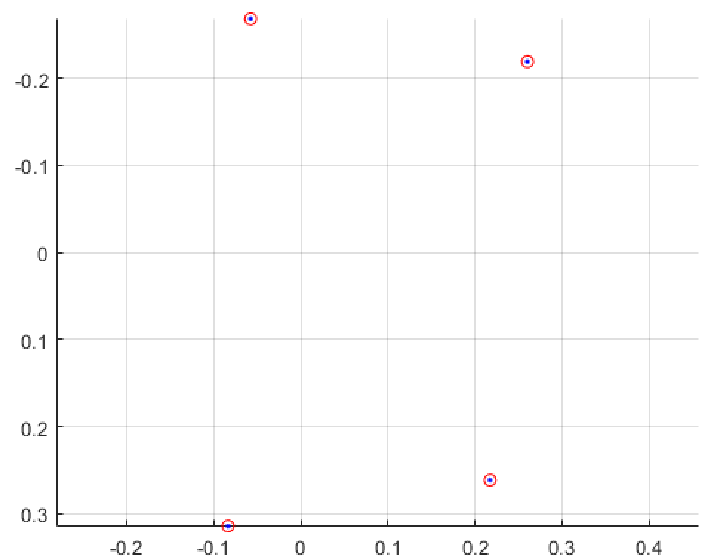

Figure 10: Corners after homography (blue dots) and U points after $\mathcal{P}_2$ projection (red circles).

Figure 11 shows the final projection of the image and the corner points of the poster. The corner points are in the correct positions, their shape and ratio matches that of the poster.

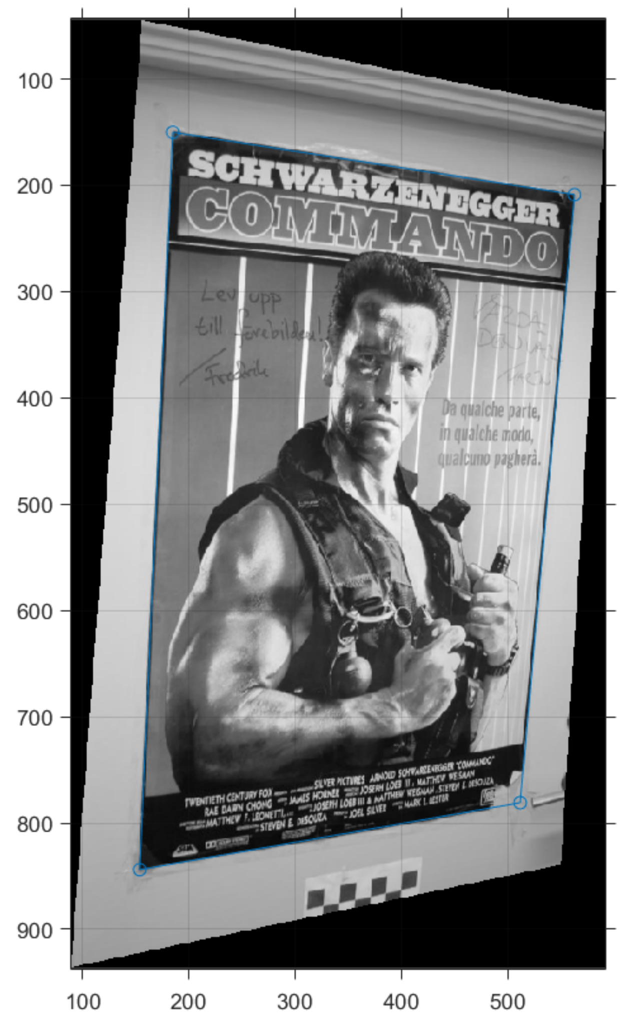

Figure 11: Projected image and corner points (blue).