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Problem 1.1

$$f(x_1, x_2) = (x_1 - 1)^2 + 2(x_2 - 2)^2$$

$$g(x_1, x_2) = x_1^2 + x_2^2 - 1 \leq 0$$

Considering the above constraint, the penalty function and, consequently, f_p can be expressed in the following forms:

$$p(x_1, x_2, \mu) = \mu(max[0, g(x_1, x_2)])^2 = \mu(max[0, (x_1^2 + x_2^2 - 1)])^2$$

$$f_p(x_1, x_2, \mu) = \begin{cases} f(x_1, x_2) + p(x_1, x_2, \mu), & x_1^2 + x_2^2 \geq 1 \\ f(x_1, x_2), & x_1^2 + x_2^2 < 1 \end{cases}$$

$$f_p(x_1, x_2, \mu) = \begin{cases} (x_1 - 1)^2 + 2(x_2 - 2)^2 + \mu(x_1^2 + x_2^2 - 1), & x_1^2 + x_2^2 \geq 1 \\ (x_1 - 1)^2 + 2(x_2 - 2)^2, & x_1^2 + x_2^2 < 1 \end{cases}$$

Computing the gradient of f_p for all of the above cases and setting it to zero in order to find the stationary points:

$$gradf_p(x_1, x_2, \mu) = \frac{\partial f_p}{\partial x_1} + \frac{\partial f_p}{\partial x_2}$$

$$\frac{\partial f_p}{\partial x_1} = \begin{cases} 2(x_1 - 1) + 4\mu x_1(x_1^2 + x_2^2 - 1), & x_1^2 + x_2^2 \geq 1 \\ 2(x_1 - 1), & x_1^2 + x_2^2 < 1 \end{cases}$$

$$\frac{\partial f_p}{\partial x_2} = \begin{cases} 4(x_2 - 2) + 4\mu x_2(x_1^2 + x_2^2 - 1), & x_1^2 + x_2^2 \geq 1 \\ 4(x_2 - 2), & x_1^2 + x_2^2 < 1 \end{cases}$$

$$gradf_p(x_1, x_2, \mu) = 0 \Leftrightarrow \begin{cases} \frac{\partial f_p}{\partial x_1} = 0 \\ \frac{\partial f_p}{\partial x_2} = 0 \end{cases}$$

For $x_1^2 + x_2^2 \geq 1$:

$$\begin{cases} \frac{\partial f_p}{\partial x_1} = 0 \Leftrightarrow x_1 - 1 + 2\mu x_1(x_1^2 + x_2^2 - 1) = 0 \\ \frac{\partial f_p}{\partial x_2} = 0 \Leftrightarrow x_2 - 2 + \mu x_2(x_1^2 + x_2^2 - 1) = 0 \end{cases}$$

Setting $\mu = 0$, the above system of equations results in the following value:

$(x_1, x_2)^T = (1, 2)^T$,

which is the unconstrained minimum and starting minimum for gradient descend.

Problem 1.2 a

$$f(x_1, x_2) = 4x_1^2 - x_1x_2 + 4x_2^2 - 6x_2$$

$$gradf = (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}) = (8x_1 - x_2, -x_1 + 8x_2 - 6)$$

Stationary points of f in the interior of S:

$$gradf = 0 \Leftrightarrow \begin{cases} 8x_1 - x_2 = 0 \\ -x_1 + 8x_2 - 6 = 0 \end{cases} \qquad \Leftrightarrow (x_1, x_2)^T = (\frac{2}{21}, \frac{16}{21})^T$$

So, $f(\frac{2}{21}, \frac{16}{21}) = -\frac{16}{7} \approx -2.28$.

Stationary points of f on the open boundaries of S:

On the boundary where $x_1 = 0$ and $x_2 \in (0, 1)$:

$$f(0, x_2) = 4x_2^2 - 6x_2$$

$$\frac{df}{dx_2} = 0 \Leftrightarrow 8x_2 - 6 = 0 \Leftrightarrow x_2 = \frac{3}{4}$$

So, for $(\mathrm{x}_1, x_2)^T = (0, \frac{3}{4})^T$, $f(0, \frac{3}{4}) = -\frac{9}{4} = -2.25$.

On the boundary where $x_2 = 1$ and $x_1 \in (0, 1)$:

$$f(x_1, 1) = 4x_1^2 - x_1 - 2$$

$$\frac{df}{dx_1} = 0 \Leftrightarrow 8x_1 - 1 = 0 \Leftrightarrow x_1 = \frac{1}{8}$$

So, for $(\mathrm{x}_1, x_2)^T = (\frac{1}{8}, 1)^T$, $f\frac{1}{8}, 1) = -\frac{33}{16} \approx -2.06$.

On the boundary where $x_1 = x_2 = x$ and $x \in (0, 1)$:

$$f(x) = 7x^2 - 6x$$

$$\frac{df}{dx} = 0 \Leftrightarrow 14x - 6 = 0 \Leftrightarrow x = \frac{3}{7}$$

So, for $(\mathrm{x}_1, x_2)^T = (\frac{3}{7}, \frac{3}{7})^T$, $f(\frac{3}{7}, \frac{3}{7}) = -\frac{9}{7} \approx -1.28$.

Stationary points of f on the limits(corners) of the boundaries of S:

$$f(0,0) = 0$$
$$f(0,1) = -2$$
$$f(1,1) = 1$$

Summarizing, the global minimum occurs in the interior of S and it's:

$$f(\frac{2}{21}, \frac{16}{21}) = -\frac{16}{7}$$

Problem 1.2 b

$$f(x_1,x_2) = 15 + 2x_1 + 3x_2$$
$$h(x_1,x_2) = x_1^2 + x_1x_2 + x_2^2 - 21 = 0$$

The given function, f, and constraint, h, above are used to form
the Lagrangian multipliers function, L, as follows:

$$L(x_1,x_2,\lambda) = f(x_1,x_2) + \lambda h(x_1,x_2) \Leftrightarrow$$
$$\Leftrightarrow L(x_1,x_2) = 15 + 1x_1 + 3x_2 + \lambda(x_1^2 + x_1x_2 + x_2^2 - 21) =$$
$$= \lambda x_1^2 + 2x_1 + \lambda x_2^2 + 3x_2 + \lambda x_1x_2 + 15 - 21\lambda$$

After calculating the gradient of L and setting it to zero (in order to find its stationary points),
the result is the system of three equations, (1), (2) and (3), as shown below:

$$gradL = (\frac{\partial L}{\partial x_1}, \frac{\partial L}{\partial x_2}, \frac{\partial L}{\partial \lambda}) = (2\lambda x_1 + \lambda x_2 + 2, \lambda x_1 + 2\lambda x_2 + 3, x_1^2 + x_1x_2 + x_2^2 - 21)$$

$$gradL = 0 \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} (1) : 2\lambda x_1 + \lambda x_2 + 2 = 0 \\ (2) : \lambda x_1 + 2\lambda x_2 + 3 = 0 \\ (3) : x_1^2 + x_1x_2 + x_2^2 - 21 = 0 \end{cases}$$

Solving equation (1) for x_2 yields:

$(4) : x_2 = -\frac{2}{\lambda} - 2x_1$.

Combining equations (2),(4) and solving for x_1 yields:

$(5) : x_1 = -\frac{1}{3\lambda}$

and equations (4), (5) for x_2 respectively yields:

$(6) : x_2 = -\frac{4}{3\lambda}$.

Combining equations (5), (6) with equation (3) yields the two following possible values of λ:

$\lambda = \pm\frac{1}{3}$.

Setting $\lambda = \frac{1}{3}$, the system of equations (1) and (2) yields $(x_1, x_2)^T = (-1, -4)^T$, for which the objective function is evaluated as $f(-1, -4) = 1$.

Respectively, for $\lambda = -\frac{1}{3}$, the system of equations (1) and (2) yields $(x_1, x_2)^T = (1, 4)^T$, for which the objective function is evaluated as $f(1, 4) = 29$.

As is obvious, $1 < 29$, and, thus, the minimum value of f is $f(x_1, x_2) = 1$, which occurs for $\lambda = \frac{1}{3}$ at the vectral coordinates $(x_1, x_2) = (-1, -4)$.

Problem 1.3

(c) - The theoretical minimum of the function $\mathrm{g}(\mathrm{x}_1, x_2)$:

$g(x_1, x_2) = (\frac{3}{2} - x_1 + x_1x_2)^2 + (\frac{9}{4} - x_1 + x_1x_2^2)^2 + (\frac{21}{8} - x_1 + x_1x_2^3)^2$

It is noticeable that g is the sum of squared quantities.
Thus, g can only either be zero or positive and, consequently,
its minima should occur at its lowest value, which is zero. For g = 0,
all three of its squared quantities must be simultaneously zero as well:

$$\begin{cases} (1) : \frac{3}{2} - x_1 + x_1x_2 = 0 \\ (2) : \frac{9}{4} - x_1 + x_1x_2^2 = 0 \\ (3) : \frac{21}{8} - x_1 + x_1x_2^3 = 0 \end{cases}$$

Solving equation (1) for x_2 yields:

$x_2 = 1 - \frac{3}{2x_1} = \frac{2x_1 - 3}{2x_1}, (4)$

Combining equations (2), (4) yields the value of x_1:

$x_1 = 3, (5)$

Then, combining equations (4), (5) yields the value of x_2:

$x_2 = \frac{1}{2} = 0.5, (6)$

Lastly, to calculate if equations (5),(6) verify equation (3):

$\frac{21}{8} - 3 + 3(\frac{1}{2})^3 =$

$= \frac{21}{8} - \frac{24}{8} + \frac{3}{8} =$

$= 0$ true!

Thus, $g(x_1, x_2)$ has only one global minimum at $(x_1, x_2)^T = (3, 0.5)^T$ and that minimum is zero.

So, if $fitness = \frac{1}{g(x_1,x_2)+1}$, the fitness value for the above global minimum should be one:

$fitness(g = 0) = 1$

(a) - The fitness for (x_1, x_2) in ten runs:

The table below presents the values of x(1), x(2) and their respective fitness values

for each of ten runs, as calculated by running RunSingle.m:

run	x(1)	x(2)	g[x(1),x(2)]	fitness
1	3.28	0.58	0.0179	0.982
2	3.18	0.55	0.0051	0.995
3	3.05	0.51	0.0004	0.999
4	2.81	0.45	0.0069	0.993
5	2.93	0.48	0.0008	0.999
6	3.67	0.63	0.0377	0.964
7	3.03	0.51	0.0001	0.999
8	3.12	0.53	0.0022	0.998
9	3.61	0.62	0.0346	0.966
10	2.97	0.49	0.0001	0.999

As is observed, the values calculated using Matlab do not fluctuate notably around the theoretical ones.

(b) - The different mutation probabilities and their respective medians

The table below presents the values of PMut and Median for each of ten runs, as calculated by running RunBatch.m:

PMut	Median
0.00	0.902
0.02	0.987
0.10	0.999
0.20	0.998
0.30	0.998
0.40	0.998
0.50	0.998
0.60	0.998
0.70	0.998
0.80	0.998
0.90	0.998
0.99	0.959
1.00	0.924

The closest median fitness value to the theoretical value occurs for mutation probability of 0.1. Even so, it is also obvious that choosing any value of PMut between 0.1 and 0.9 should yield satisfactory results for the median fitness.