PHYS512 A6

Vasily Piccone

November 2022

1

We begin question 1 by writing out the discrete convolution formula:

$$(f * g)[n] = \sum_{k=-\infty}^{\infty} f[k]g[n-k] \tag{1}$$

Given that we are using an array of length L:

$$(f * g)[n] = \sum_{k=0}^{L-1} f[k]g[n-k] \tag{2}$$

Our desired output is the array $f[n]$ shifted by some amount a ($f[n+a]$). To achieve this, $g[n-k] = 1$ when $k = a$ and be zero everywhere else.

$$f[n+a] = \sum_{k=0}^{L-1} f[k]\delta_{k,a} \tag{3}$$

The plot of the Gaussian centered at 5 and its shifted counterpart can be seen below. The Gaussian is plotted from 0 to 10. The Gaussian was shifted by half the length of the array (the array had 1001 entries, so I shifted the array by 500). We note from the plot that the shifted array wraps around.

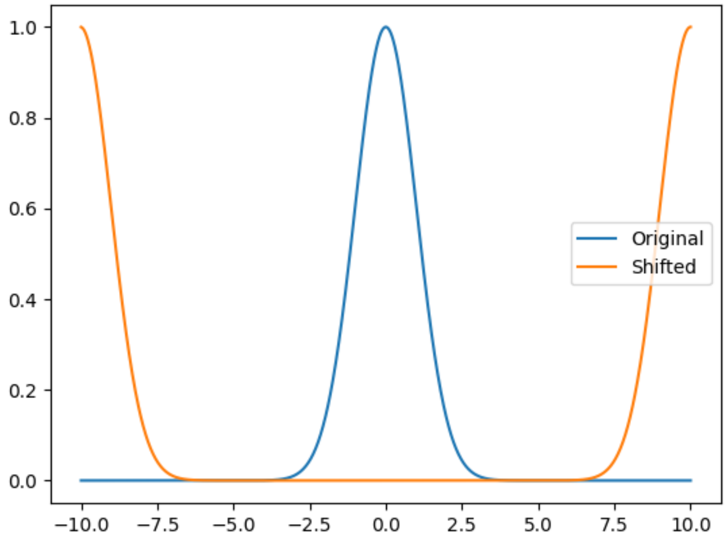

Figure 1: Plot of the Gaussian along with its shifted counterpart

2

2.1

The correlation function was implemented by using the numpy.fft functions as described in the assignment. The plot of the correlation of a Gaussian with itself can be seen below.

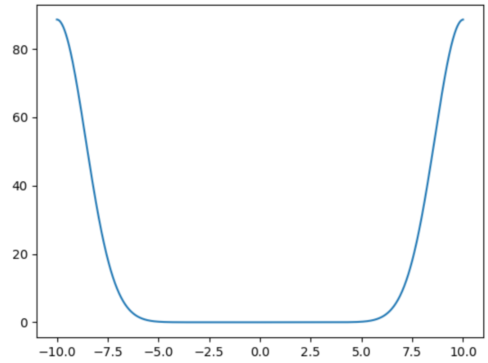

Figure 2: Plot of the correlation of a Gaussian with itself.

2.2

Unsurprisingly, the result of the correlation of a shifted Gaussian with itself is indeed the same as it was in question 2a. This is reasonable because if we shift the function being correlated with itself by some value a, its effectively the same thing as shifting the convolution by that same amount.

$$h[n] = \sum_{k=0}^{L-1} f[k]f[n-k] \rightarrow h[n+a] = \sum_{k=0}^{L-1} f[k]f[n+a-k] \tag{4}$$

The i^{th} elements of the arrays are therefore, equivalent, regardless of whether the function is shifted or not.

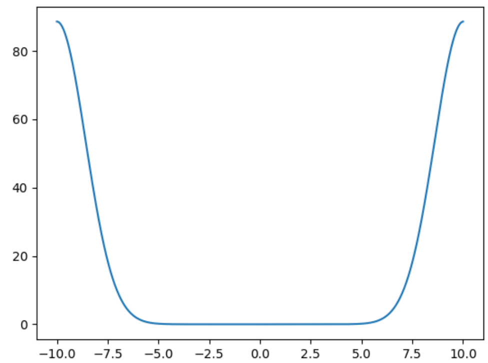

Figure 3: Plot of the correlation of a shifted Gaussian with itself.

3

We can avoid the circular nature of the discrete Fourier transform by ensuring the arrays are of the same length. This way, while we convolve them, we do not have the extra length of the longer array wrapping around. Below is the plot of two identical Gaussians, one with 1001 points and the other with 2001 points.

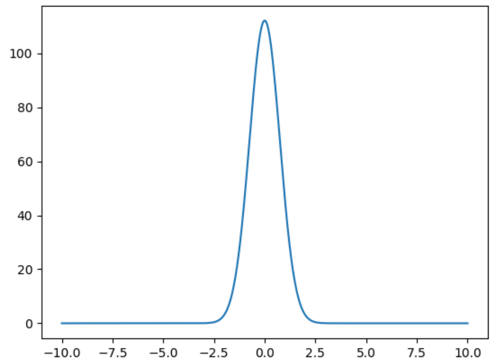

Figure 4: Plot of the wrap-around prevention correlation of a shifted Gaussian with itself.

4

4.1

We can easily show the relation provided in question 4a) by setting the base of the exponent to some value $\gamma = \exp(2\pi ik/N)$. The resulting expression is a simple geometric series.

$$\sum_{x=0}^{N-1} \exp(-2\pi ikx/N) = \sum_{x=0}^{N-1} \gamma^{-x} = \frac{1-\gamma^N}{1-\gamma} = \frac{1-\exp(-2\pi ik)}{1-\exp(-2\pi ik/N)} \tag{5}$$

4.2

If we directly evaluate this limit, we get an indeterminate form, as seen below.

$$\lim_{k\to 0} \frac{1-\exp(-2\pi ik)}{1-\exp(-2\pi ik/N)} = \frac{1-\exp(0)}{1-\exp(0)} = \frac{1-1}{1-1} = \frac{0}{0} \tag{6}$$

We cleverly apply L'Hopital's rule and re-evalulate the limit, indeed getting N as k approaches zero.

$$\lim_{k\to 0} \frac{\frac{d}{dk}(1-\exp(-2\pi ik))}{\frac{d}{dk}(1-\exp(-2\pi ik/N))} = \lim_{k\to 0} \frac{2N\pi i\exp(-2\pi ik)}{2\pi i\exp(-2\pi ik/N)} = N \tag{7}$$

For an integer value of k that is not a multiple of N, we note that $\exp(-2\pi ik) = 1$, since $k \in \mathbb{Z}$. Thus the numerator vanishes, and the denominator remains non-zero.

$$\frac{1-\exp(-2\pi ik)}{1-\exp(-2\pi ik/N)} = \frac{1-1}{1-\exp(-2\pi ik/N)} = \frac{0}{(1-\exp(-2\pi ik/N))} = 0 \tag{8}$$

4.3

I attempted to take the analytic Fourier transform by leveraging the result we found in question 4a) by re-arranging a sine wave in terms of complex exponentials ($sin(2\pi kn) = \frac{1}{2j}(\exp(2\pi jk_1 n) - \exp(-2\pi jk_1 n)$.

$$X[n] = \sum_{n=0}^{L-1} x[n]\exp(-2\pi ikn/N) = \frac{1}{2j}\sum_{n=0}^{L-1}\bigg(\exp(2\pi jk_1 n) - \exp(-2\pi jk_1 n)\bigg)\exp(-2\pi ikn/N) \tag{9}$$

$$\frac{1}{2j}\sum_{n=0}^{L-1}\bigg(\exp(2\pi jk_1 n) - \exp(-2\pi jk_1 n)\bigg)\exp(-2\pi ikn/N) = \frac{1}{2j}\sum_{n=0}^{L-1}\bigg(\exp(-2\pi ik'n/N) - \exp(-2\pi ik''n/N)\bigg) \tag{10}$$

We achieve this expression by setting $k' = -Nk_1 + k$ and $k'' = Nk_1 - k$. We can then easily find the result of this sum by leveraging the expression we derived in the first part of this question.

$$X[n] = \frac{1}{2j}\left(\frac{1-\exp(-2\pi ik')}{1-\exp(-2\pi ik'/N)} - \frac{1-\exp(-2\pi ik'')}{1-\exp(-2\pi ik''/N)}\right) \tag{11}$$

This result, plotted with the np.fft.fft Fourier transform can be seen in Figure 5. Instead, I computed the discrete Fourier transform as one would in an introductory signals and systems class. The algorithm used can be found here.

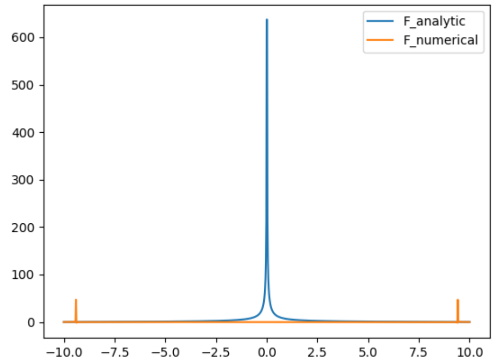

Figure 5: Plot of the first attempt at an ”analytic” fft in comparison with the numpy fft.

Therefore, I compared this ”pseudo-analytic” Fourier transform to numpy’s fft.rfft function. I had to remove every even entry from the pseudo-analytic Fourier transform to get the arrray sizes to agree for plotting purposes. The comparison plots can be seen below. We note that we are not very close to the delta function we would get from analytically computing the Fourier transform. We do get a spike around the $k = 6.5$ value, however we get considerable signal afterwards.

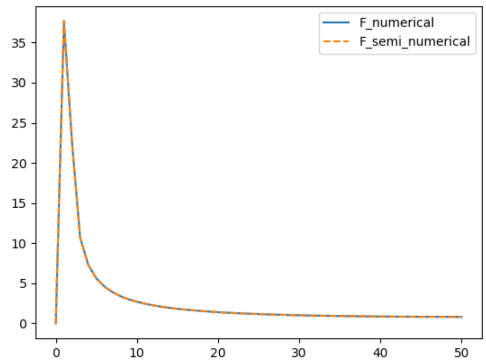

Figure 6: Plot of the ”pseudo-analytic” fft in comparison with the numpy fft.

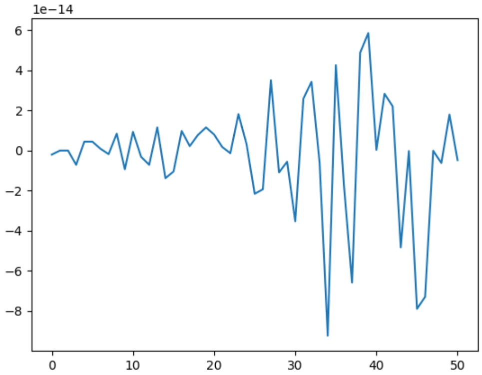

Figure 7: Plot of the difference of the "pseudo-analytic" fft and the numpy fft.

5

Unfortunately, due to thesis conflicts and issues with other assignments, I did not have the time to complete the remainder of this assignment. I will better manage my time moving forward to prevent this from re-occuring. Thank you for your understanding.