

Assignment-4

Adepu Vasisht

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Let X be a random variable having distribution function

$$F(x) = \begin{cases} 0, & x < 1 \\ \frac{a}{2}, & 1 \leq x < 2 \\ \frac{c}{6}, & 2 \leq x < 3 \\ 1, & x \geq 3 \end{cases}$$

where a and c are appropriate constants. Let $A_n = [1 + \frac{1}{n}, 3 - \frac{1}{n}]$, $n \geq 1$, and $A = \bigcup_{i=1}^{\infty} A_i$. If $\Pr(X \leq 1) = \frac{1}{2}$ and $E(X) = \frac{5}{3}$, then $\Pr(X \in A)$ equals to?

SOLUTION

A. Finding a and c

X is a discrete random variable because a and c are constants. So X can take only 3 values namely 1, 2 and 3. Since we know that

$$F(1) = \frac{a}{2} \quad (1)$$

$$\Pr(X \leq 1) = \frac{1}{2} \quad (2)$$

$$\implies \frac{a}{2} = \frac{1}{2} \quad (3)$$

We calculate the value of c using the given expectation value. The formula for a discrete variable is given by.

$$E(X) = \sum_{i=1}^3 \Pr(X = i) \cdot i$$

Since X is a discrete random variable we have $\Pr(X = 1) = \Pr(X \leq 1) = \frac{1}{2}$

And also since $F(x)$ is a cdf we have $F(2) = \Pr(X = 1) + \Pr(X = 2)$

$$\implies \Pr(X = 2) = \frac{c}{6} - \frac{1}{2}$$

Similarly we can say that $F(3) = F(2) + \Pr(X = 3)$

$$\implies \Pr(X = 3) = 1 - \frac{c}{6}$$

We have all the required probability values.

$$E(X) = \sum_{i=1}^3 \Pr(X = i) \cdot i \quad (4)$$

$$= \Pr(1) \cdot 1 + 2 \Pr(2) + 3 \Pr(3) \quad (5)$$

$$= \frac{1}{2} + 2 \left(\frac{c}{6} - \frac{1}{2} \right) + 3 \left(1 - \frac{c}{6} \right) \quad (6)$$

$$= \frac{1}{2} + 2 - \frac{c}{6} \quad (7)$$

$$\frac{5}{3} = \frac{5}{2} - \frac{c}{6} \quad (8)$$

$$\implies \frac{c}{6} = \frac{5}{2} - \frac{5}{3} \quad (9)$$

$$\frac{c}{6} = \frac{5}{6} \quad (10)$$

$$\therefore c = 5 \quad (11)$$

B. Finding the range of A

Given that $A_n = [1 + \frac{1}{n}, 3 - \frac{1}{n}]$, $n \geq 1$, and $A = \bigcup_{i=1}^{\infty} A_i$. We can see that for any i we have $A_i \subset A_{i+1}$ and we also know the property that if A is a subset of B then we have

$$A \cup B = B$$

By applying the above principal we get that $A = A_\infty$ and since when $n \rightarrow \infty \implies \frac{1}{n} \rightarrow 0$ but not equal to 0 so we have $A = A_\infty = (1, 3)$

C. Finding the probability

Since we know that X is a discrete random variable which can only take $X = 1, 2$ and 3 as values and $A = (1, 3)$ hence the only value X can take in A is 2 . We have $\Pr(X \in A) = \Pr(X = 2)$. And we know that

$$\Pr(X = 2) = \frac{c}{6} - \frac{1}{2} \quad (12)$$

$$\Pr(X = 2) = \frac{5}{6} - \frac{1}{2} \quad (\because c = 5) \quad (13)$$

$$\implies \Pr(X = 2) = \frac{1}{3} \quad (14)$$

Hence

$$\therefore \Pr(X \in A) = 0.33$$