Assignment-4

Adepu Vasisht

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Let X be a random variable having distribution function

$$F(x) = \begin{cases} 0, & x < 1 \\ \frac{a}{2}, & 1 \le x < 2 \\ \frac{c}{6}, & 2 \le x < 3 \\ 1, & x \ge 3 \end{cases}$$

where a and c are appropriate constants. Let $A_n = \left[1 + \frac{1}{n}, 3 - \frac{1}{n}\right]$, $n \ge 1$, and $A = \bigcup_{i=1}^{\infty} A_i$. If $\Pr(X \le 1) = \frac{1}{2}$ and $E(X) = \frac{5}{3}$, then $\Pr(X \in A)$ equals to?

Solution

A. *Finding a and c*

Clearly we can see that X is a discrete random variable since given a and c are constants. So X can take only 3 values namely 1,2 and 3. Since we know that

$$F(1) = \frac{a}{2} \tag{1}$$

$$\Pr(X \le 1) = \frac{1}{2} \tag{2}$$

$$\implies \frac{a}{2} = \frac{1}{2} \tag{3}$$

Now we calculate the value of c using the given expectation value. Since X is a discrete random variable the expectation value is given by

$$E(X) = \sum_{i=1}^{3} \Pr(X = i) \cdot i$$

Since we need to find the individual probabilities we do that calculation first. Now since X is a discrete random variable we have $\Pr(X = 1) = \Pr(X \le 1) = \frac{1}{2}$
And also since $F(x)$ is a cdf we have $F(2) = \Pr(X = 1) + \Pr(X = 2)$

$$\implies \Pr(X = 2) = \frac{c}{6} - \frac{1}{2}$$

Similarly we can say that $F(3) = F(2) + \Pr(X = 3)$

$$\implies \Pr(X = 3) = 1 - \frac{c}{6}$$

Now since we have all the probability values we can calculate the expectation value.

$$E(X) = \sum_{i=1}^{3} \Pr(X = i) \cdot i \tag{4}$$

$$= \Pr(1).1 + 2\Pr(2) + 3\Pr(3) \tag{5}$$

$$= \frac{1}{2} + 2\left(\frac{c}{6} - \frac{1}{2}\right) + 3\left([\right) 1 - \frac{c}{6}] \tag{6}$$

$$= \frac{1}{2} + 2 - \frac{c}{6} \tag{7}$$

$$\frac{5}{3} = \frac{5}{2} - \frac{c}{6} \tag{8}$$

$$\implies \frac{c}{6} = \frac{5}{2} - \frac{5}{3} \tag{9}$$

$$\frac{c}{6} = \frac{5}{6} \tag{10}$$

$$\therefore c = 5 \tag{11}$$

B. *Finding the range of A*

Given that $A_n = \left[1 + \frac{1}{n}, 3 - \frac{1}{n}\right]$, $n \ge 1$, and $A = \bigcup_{i=1}^{\infty} A_i$. We can see clearly that for any i we have $A_i \subset A_{i+1}$ and we also know the property that if A is a subset of B then we have

$$A \cup B = B$$

By applying the above principal we get that $A = A_\infty$ and since when $n \to \infty \implies \frac{1}{n} \to 0$ but not equal to 0 so we have $A = A_\infty = (1, 3)$

C. *Finding the probability*

We have to find the probability of $\Pr(X \in A)$ but we know that X is a discrete random variable which can only take $X = 1, 2$ and 3 as values and $A = (1, 3)$ and since the only value which X can take in A is 2. We have $\Pr(X \in A) = \Pr(X = 2)$. And we know that

$$\Pr(X = 2) = \frac{c}{6} - \frac{1}{2} \tag{12}$$

$$\Pr(X = 2) = \frac{5}{6} - \frac{1}{2} \quad (\because c = 5) \tag{13}$$

$$\implies \Pr(X = 2) = \frac{1}{3} \tag{14}$$

Hence

$$\therefore \Pr(X \in A) = 0.33$$