Assignment-4

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GATE 2021(ST) Q.No 16

Let X be a random variable having distribution function

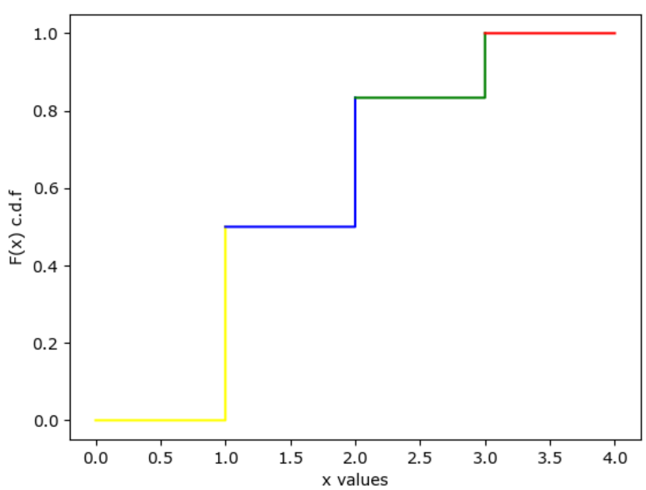
$$F(x) = \begin{cases} 0, & x < 1 \\ \frac{a}{2}, & 1 \le x < 2 \\ \frac{c}{6}, & 2 \le x < 3 \\ 1, & x \ge 3 \end{cases}$$

where a and c are appropriate constants. Let $A_n = \left[1 + \frac{1}{n}, 3 - \frac{1}{n}\right]$, $n \ge 1$, and $A = \bigcup_{i=1}^{\infty} A_i$. If $\Pr\left(X \le 1\right) = \frac{1}{2}$ and $E\left(X\right) = \frac{5}{3}$, then $\Pr\left(X \in A\right)$ equals to?

Solution

A. *Proving that X is a discrete variable*

To prove the above statement we plot $F\left(x\right)$ which is the c.d.f of our random variable X by taking some arbitrary values for a and c as it is mentioned that a and c are appropirate constants. The graph will be as shown in the figure below.



As we can see the graph is step-wise which means X is a discrete random variable, moreover we can see that turns take place for $x = 1, 2, 3$ so X can take only these values.

B. *Finding a and c*

So X can take only 3 values namely 1,2 and 3. Since we know that

$$F\left(1\right) = \frac{a}{2} \tag{1}$$

$$\Pr\left(X \le 1\right) = \frac{1}{2} \tag{2}$$

$$\implies \frac{a}{2} = \frac{1}{2} \tag{3}$$

We calculate the value of c using the given expectation value. The formula for a discrete variable is given by.

$$E\left(X\right) = \sum_{i=1}^{3} \Pr\left(X = i\right) \cdot i$$

Since X is a discrete random variable we have $\Pr\left(X = 1\right) = \Pr\left(X \le 1\right) = \frac{1}{2}$
And also since $F\left(x\right)$ is a cdf we have $F\left(2\right) = \Pr\left(X = 1\right) + \Pr\left(X = 2\right)$

$$\implies \Pr\left(X = 2\right) = \frac{c}{6} - \frac{1}{2}$$

Similarly we can say that $F\left(3\right) = F\left(2\right) + \Pr\left(X = 3\right)$

$$\implies \Pr\left(X = 3\right) = 1 - \frac{c}{6}$$

We have all the required probability values.

$$E(X) = \sum_{i=1}^{3} \Pr(X = i) \cdot i \tag{4}$$

$$= \Pr(1).1 + 2\Pr(2) + 3\Pr(3) \tag{5}$$

$$= \frac{1}{2} + 2\left(\frac{c}{6} - \frac{1}{2}\right) + 3\left(1 - \frac{c}{6}\right) \tag{6}$$

$$\frac{5}{3} = \frac{5}{2} - \frac{c}{6} \tag{7}$$

$$\implies \frac{c}{6} = \frac{5}{6} \tag{8}$$

$$\therefore c = 5 \tag{9}$$

C. *Finding the range of A*

Given that $A_n = \left[1 + \frac{1}{n}, 3 - \frac{1}{n}\right]$, $n \geq 1$, and $A = \bigcup_{i=1}^{\infty} A_i$. We can see that for any i we have $A_i \subset A_{i+1}$ and we also know the property that if A is a subset of B then we have

$$A \cup B = B$$

By applying the above principal we get that $A = A_\infty$ and since when $n \to \infty \implies \frac{1}{n} \to 0$ but not equal to 0 so we have $A = A_\infty = (1, 3)$

D. *Finding the probability*

We know that X can only take $X = 1, 2 \,\text{and}\, 3$ as values and $A = (1, 3)$ hence the only value X can take in A is 2. We have $\Pr(X \in A) = \Pr(X = 2)$. And we know that

$$\Pr(X = 2) = F(2) - F(1) \quad (\because X \text{is discrete}) \tag{10}$$

$$\Pr(X = 2) = \frac{5}{6} - \frac{1}{2} \quad (\because c = 5 \,\text{and}\, a = 1) \tag{11}$$

$$\implies \Pr(X = 2) = \frac{1}{3} = 0.33 \tag{12}$$