

Chapter 1

2024-04-20 – The basic Hopf fibre bundles

1.1 Explicit construction

Definition 1.1.1. For $k = \mathbb{R}^n$ assume a multiplication $\mu: k \otimes_{\mathbb{R}} k \rightarrow k$ making $k^\times \supset \mathbb{S}(k) \cong \mathbb{S}^{n-1}$ into an H -space. Then we can construct a fibre bundle as follow: Consider the unit sphere in $\mathbb{S}(k \times k)$, which can be identified as $\mathbb{S}(\mathbb{R}^{2n}) = \mathbb{S}^{2n-1}$, and the one-point compactification of $\mathbb{S}^k = \mathbb{S}^{dim_{\mathbb{R}} k} = \mathbb{S}^n$.

Notice that for $\mathbb{S}(k^2)$ the components can be written as pairs of elements of k with norms each less than or equal to one, and norm-squares summing to one. Thus consider the map:

$$\begin{aligned} m_k: \quad \mathbb{S}^{2n-1} = \mathbb{S}(k \times k) &\rightarrow \quad \mathbb{S}^k \\ a, b &\mapsto a \cdot b^{-1}, \end{aligned}$$

where $(\bullet) \cdot b^{-1}$ has to be defined as ∞ for $b = 0$, (note that this implies $|a| = 1$, so $a \neq 0$.)

Proposition 1.1.2. *This is a fibre bundle for $k = \mathbb{R}, \mathbb{C}, \mathbb{H}$, with fibre $F = \mathbb{S}(k) = \mathbb{S}^0, \mathbb{S}^1, \mathbb{S}^3$.*

Proof. Consider the tiny case $m_{\mathbb{R}}: \mathbb{S}(\mathbb{R} \times \mathbb{R}) \rightarrow \mathbb{S}^{\mathbb{R}}$, specifically its fibre: $m_{\mathbb{R}}^{-1}(0 \in \mathbb{R}) = \{ (a, b) \in \mathbb{S}(\mathbb{R} \times \mathbb{R}) \mid |a|^2 + |b|^2 = 1 \wedge ab^{-1} = 0 \}$. Find $m_{\mathbb{R}}^{-1}(0) = \{ (a, b) \in \mathbb{S}(\mathbb{R} \times \mathbb{R}) \mid a = 0 \wedge |b| = 1 \} = \{0\} \times \{\pm 1\} \cong \mathbb{S}^0$. Similarly for $t \neq 0$ in \mathbb{R} consider: $ab^{-1} = t \Leftrightarrow a = tb \Leftrightarrow b = at^{-1}$. It follows $m_{\mathbb{R}}^{-1}t = \{ (\pm 1, \pm t^{-1}) \} \cong \mathbb{S}^0$ for each target in \mathbb{R} . Finally consider $m_{\mathbb{R}}^{-1}\infty = \{ (a, b) \in \mathbb{S}(\mathbb{R} \times \mathbb{R}) \mid |a|^2 + |b|^2 = 1 \wedge ab^{-1} = \infty \} = \{ (a, 0) \mid |a| = 1 \} \cong \mathbb{S}^0$.

Similarly find that $m_k^{-1}z \cong \mathbb{S}(k)$ by considering:

$$m_k^{-1}z = \{ (a, b) \mid |a|^2 + |b|^2 = 1 \wedge ab^{-1} = z \} = \{ (u, u^{-1}z^{-1}) \mid u \in \mathbb{S}(k) \}.$$

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