Chapter 1

2024-04-20 – The basic Hopf fibre bundles

1.1 Explicit construction

Definition 1.1.1. For $k = \mathbb{R}^n$ assume a multiplication $\mu\colon k \otimes_{\mathbb{R}} k \to k$ making $k^\times \supset \mathbb{S}(k) \cong \mathbb{S}^{n-1}$ into an H-space. Then we can construct a fibre bundle as follow: Consider the unit sphere in $\mathbb{S}(k \times k)$, which can be identified as $\mathbb{S}(\mathbb{R}^{2n}) = \mathbb{S}^{2n-1}$, and the one-point compactification of $\mathbb{S}^k = \mathbb{S}^{dim_{\mathbb{R}} k} = \mathbb{S}^n$.

Notice that for $\mathbb{S}(k^2)$ the components can be written as pairs of elements of k with norms each less than or equal to one, and norm-squares summing to one. Thus consider the map:

$$\begin{aligned} m_k\colon \quad \mathbb{S}^{2n-1} = \mathbb{S}(k \times k) \to &\quad \mathbb{S}^k \\ a, b \mapsto &\quad a \cdot b^{-1}, \end{aligned}$$

where $(\bullet) \cdot b^{-1}$ has to be defined as ∞ for $b = 0$, (note that this implies $|a| = 1$, so $a \neq 0$.)

Proposition 1.1.2. *This is a fibre bundle for $k = \mathbb{R}, \mathbb{C}, \mathbb{H}$, with fibre $F = \mathbb{S}(k) = \mathbb{S}^0, \mathbb{S}^1, \mathbb{S}^3$.*

Proof. Consider first the case $k = \mathbb{R}$. The case $k = \mathbb{R}$ can be identified as $\mathbb{Z}/2 \to \mathbb{S}^1 \to \mathbb{S}^1$, where the first circle is a unit sphere in $\mathbb{R}^2$ and the second circle is the one-point-compactification of $k = \mathbb{R}$. Let $t \in \mathbb{R}^+$ with $|t| < 1$, it defines a subset of $\mathbb{S}(\mathbb{R} \times \mathbb{R})$ with

$$U_t := \{\ (x, y) \in \mathbb{S}(\mathbb{R} \times \mathbb{R}) \mid xy^{-1} = t \wedge |x|^2 + |y|^2 = 1\ \}.$$

With $|t| < 1$ and $z, t \in \mathbb{R}$ the fibre condition and the unit sphere condition assemble to give:

$$\begin{aligned} xy^{-1} &= t \\ |x|^2 + |y|^2 &= 1 \\ |t| &\leq 1 \end{aligned}$$

$$\begin{aligned} x &= ty \\ \Leftrightarrow |t|^2|y|^2 + |y|^2 &= 1 \\ |t| &\leq 1 \end{aligned}$$

$$\begin{aligned} x &= ty \\ \Leftrightarrow (|t|^2 + 1)|y|^2 &= 1 \\ |t| &\leq 1 \end{aligned}$$

$$\begin{aligned} x &= ty \\ \Leftrightarrow |y| &= \sqrt{\frac{1}{|t|^2 + 1}} \\ |t| &\leq 1. \end{aligned}$$

$\square$