Chapter 1

2024-04-20 – The basic Hopf fibre bundles

1.1 Explicit construction

Definition 1.1.1. For $\daleth = \mathbb{R}^n$ assume a multiplication $\mu\colon \daleth \otimes_{\mathbb{R}} \daleth \to \daleth$ making $\daleth^\times \supset \mathbb{S}(\daleth) \cong \mathbb{S}^{n-1}$ into an H-space. Then we can construct a fibre bundle as follow: Consider the unit sphere in $\mathbb{S}(\daleth \times \daleth)$, which can be identified as $\mathbb{S}(\mathbb{R}^{2n}) = \mathbb{S}^{2n-1}$, and the one-point compactification of $\mathbb{S}^\daleth = \mathbb{S}^{dim_{\mathbb{R}} \daleth} = \mathbb{S}^n$.

Notice that for $\mathbb{S}(\daleth^2)$ the components can be written as pairs of elements of $\daleth$ with norms each less than or equal to one, and norm-squares summing to one. Thus consider the map:

$$
\begin{aligned}
m_\daleth\colon \quad \mathbb{S}^{2n-1} = \mathbb{S}(\daleth \times \daleth) \to & \quad \mathbb{S}^\daleth \\
a, b \mapsto & \quad a \cdot b^- 1.
\end{aligned}
$$

Proposition 1.1.2. *This is a fibre bundle for $k = \mathbb{R}, \mathbb{C}, \mathbb{H}$.*