

Chapter 1

Counting DAGs

1.1 Basic Facts and Definitions

I recall the definition of directed acyclic graphs informally here, in favour of providing a more precise definition tailored to this paper's use case later.

Definition 1.1.1 (Directed Acyclic Graph). A directed graph $G = (V, E)$ with $E \subset V \times V$, is called acyclic, if there is no (finite) chain of vertices (v_0, v_1, \dots, v_n) with $v_i \in V, 1 < n, (v_i, v_{i+1}) \in E$, such that at least one of the vertices v_i has been visited twice:

$$G = (V, E \subset V \times V) \quad \text{dag} \\ : \Leftrightarrow \forall n \geq 2 \forall (v_0, v_1, \dots, v_n) \in V^{\times n} : (\forall i : (v_i, v_{i+1}) \in E \Rightarrow \forall i \neq j : v_i \neq v_j).$$

Do note in particular that despite the fact that this is an infinite schema of axioms, it is finite for each specific finite digraph. More specifically, it could be coded to check an arbitrary digraph for dag-ness. In this text I will not use this however, our digraphs will come as dags plain by their code-friendly presentation.

Proposition 1.1.2 (Sources and Sinks exist). *Any finite dag has a vertex that has no outbound edges and a vertex that has no inbound edges.*

Proof. Assume for contradiction that there is no vertex with no outbound edge, then choose any one of them, and append any next vertex, if there's an edge pointing to it from your current vertex. Since there's no sink, this process can continue infinitely, since our graph is finite, eventually we repeat a vertex. This is a contradiction to acyclicity as described above.

The same argument applies to the digraph with inverted edges or just by walking the path backwards instead. \square

Definition 1.1.3 (Topological Ordering). If for a graph $G = (V, E)$ we have a graph homomorphism:

$$\varphi : G \rightarrow \Delta^{|G|} = (|G| = \{0, \dots, |G|\}, \leq) \subset \mathbb{N}$$

which is injective on nodes, call φ a topological ordering of G .

More explicitly: If there exists a finite $n \in \mathbb{N}$ and a bijection $f : V \rightarrow \{0, \dots, n\}$ such that $(i, j) \in E \Rightarrow fi \leq fj \in \mathbb{N}$ for all edges in E , then that bijection f induces a topological ordering φ like before.

Proposition 1.1.4 (Finite dags have topological orderings). *A digraph is a dag iff it has a topological ordering (necessarily non unique for non-trivial cases).*

Proof. Assume a graph has a topological ordering, if it had a cycle, at least one edge would point backwards in that ordering, to contradiction. So the existence of a topological ordering implies the dag-ness of a (di)graph.

Assume now an abstract dag $G = (V, E)$, we want to construct a bijection $\varphi: V \rightarrow |V|$, such that $\varphi(i) \leq \varphi(j)$ whenever there is an edge $(i, j) \in E$ (recall: and no condition otherwise).

By the fact that sources and sinks exist, choose a source s . Since it has no inbound connections, set its label to 0. Now remove s from the graph G with all edges that pointed from s to G and find that subgraph is a dag, too, so it has a source, that can be labelled 1, and so forth up until all vertices have been exhausted. \square

Corollary 1.1.5 (Minimal topological orderings). *Each dag can be given a minimal topological ordering, that is, a bijection of its vertex set to an interval $\{0, \dots, |V| - 1\}$ that induces a graph homomorphism. (Corollary to the proof before.)*

1.2 The Category embDAG of embedded DAGs

The section before discusses among other things that each finite directed acyclic graph can be embedded in the complete finite directed acyclic graph of its vertex order. Using that we reach the following definition for a combinatorial category of dags.

Definition 1.2.1 (embDAG). Define the category embDAG as follows:

As objects we have a chosen natural number $n \in \mathbb{N}$, call it vertex count, together with a chosen subset $E \subset \{0, \dots, n - 1\} \times \{0, \dots, n - 1\}$, which satisfies:

$$\forall (i, j) \in E: 0 \leq i < j < n \quad (\text{top}),$$

call E the edge set of the object (n, E) .

As morphisms we have all! set maps $f: \{0, \dots, n - 1\} \rightarrow \{0, \dots, m - 1\}$ for dags $(n, E), (m, F)$, such that: $\forall (i, j) \in E: (fi, fj) \in F$, i.e.

$$\text{embDAG}((n, E), (m, F))$$

$$= \{f: \{0, 1, \dots, n - 1\} \rightarrow \{0, 1, \dots, m - 1\} \mid (i, j) \in E \Rightarrow (fi, fj) \in F \vee fi = fj\}.$$

Remark 1.2.2. For clarity, the induced graph of such an object has vertex set $\{0, \dots, n - 1\}$ and edges as defined by E .

Remark 1.2.3. By the discussion above we can see, that the topological ordering has in fact been incorporated into this definition, so that the induced graph associated to a pair (n, E) that satisfies the condition top, is necessarily a dag. In particular in what follows there is no need to check the acyclicity condition, it is actually a given fact in embDAG . Instead, in this category we need to keep compatible maps to \mathbb{N} along.

Remark 1.2.4. One could instead choose the convention to consider dags with loops at each vertex, which would make the vertex set a recognisable subset of E , in which case the objects could be fully described by edge sets. Occasionally that might be a helpful extra perspective to consider, specifically when handling graph homomorphisms I jump between those conventions as convenient.

Remark 1.2.5. The category embDAG in particular admits an upper bound on counting how many dags there are on a fixed number of nodes. Do note from vertex count 3 on, isomorphic dags can have different representations in embDAG , e.g. $\{0, 1, 2\}, \{(0, 1)\}$ and $\{0, 1, 2\}, \{(0, 2)\}$ are isomorphic, but each legitimately different objects in embDAG .

Example 1.2.6. The class of dags we need most instantly are complete graphs, which in accordance with topology I will call Δ^n :

$$V(\Delta^n) = \{0, 1, \dots, n\}, E(\Delta^n) = \{(i, j) \mid i < j\}.$$

These are the edge count maximal objects in their vertex count, as in, there is no other dag on $n+1$ nodes with more or equally many edges to the Δ^n objects each in their degree.

Example 1.2.7. For some more non-trivial examples:

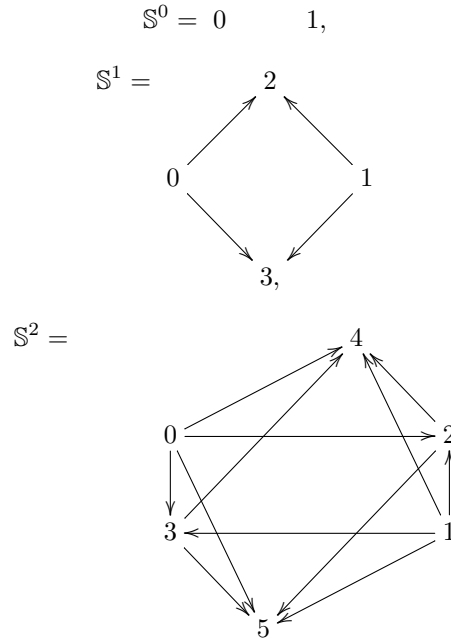
$$\mathbb{S}^0 = (\{0, 1\}, \{\}),$$

$$\mathbb{S}^1 = (\{0, 1, 2, 3\}, \{02, 12, 03, 13\}),$$

$$\mathbb{S}^2 = (\{0, 1, 2, 3, 4, 5\}, \{02, 12, 03, 13, 04, 14, 24, 34, 05, 15, 25, 35\}).$$

In a bit we will exhibit how these are actual spheres.

It is however visually convincing:



Definition 1.2.8 (Cone and Suspension C, Σ). In addition we have a cone and suspension construction:

Let $G = (\{0, \dots, n-1\}, E)$, construct CG with vertices $V(CG) = \{0, \dots, n\}$ and edges: $E(CG) = E(G) \sqcup \{(i, n) | i \in \{0, \dots, n-1\}\}$.

Let $G = (\{0, \dots, n-1\}, E)$, construct ΣG with vertices $V(\Sigma G) = \{0, \dots, n, n+1\}$ and edges: $E(\Sigma G) = E(G) \sqcup \{(i, n) | i \in \{0, \dots, n-1\}\} \sqcup \{(i, n+1) | i \in \{0, \dots, n-1\}\}$.

These are actual cones and suspensions as well.

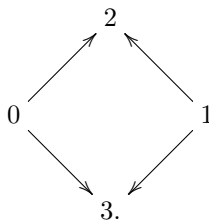
Example 1.2.9. For illustration consider the diagrams of CS^0 and ΣS^0 : Start with S^0 :

$$0 \quad 1$$

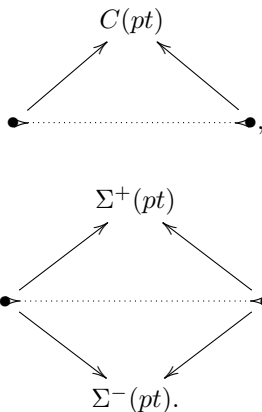
add a cone point (2) and edges to it

$$0 \longrightarrow 2 \longleftarrow 1.$$

For ΣS^0 add another point 3 and just not the edge (2, 3):



Remark 1.2.10. More generally the illustrations for C and Σ are quite helpful:



Example 1.2.11. Do note we have:

$$C\Delta^{n-1} = \Delta^n \text{ for } n \geq -1.$$

As well as:

$$\Sigma S^{n-1} = S^n$$

for the examples defined above.

Definition 1.2.12 (Simplices, Spheres, Disks $\Delta^n, \mathbb{S}^n, \mathbb{D}^n$). Define $\Delta^{-1} = (\emptyset, \emptyset)$, and inductively: $\Delta^n := C\Delta^{n-1}$ with C the cone functor. Define $\mathbb{S}^0 = (\{0, 1\}, \emptyset)$, and inductively: $\mathbb{S}^n := \Sigma\mathbb{S}^{n-1}$. For $i \geq 1$ define: $\mathbb{D}^i = C\mathbb{S}^{i-1}$.

Specifically for $i = 1$ denote $I := \mathbb{D}^1$ as our model for the unit interval in embDAG .

For vertex count 1 there are no edges to choose, so there is only one dag representing the one point graph in the category embDAG . For two vertices there is only one edge to choose, hence 2 graphs with vertex count 2, which are non-isomorphic.

Proposition 1.2.13 (Number of legal incidence matrices in embDAG). *For each degree $n \geq 0$, the count of distinct edge sets per degree can be calculated as:*

$$|\text{embDAG}_{0,n}| = 2^{\binom{n}{2}}.$$

Proof. For a fixed degree $n \in \mathbb{N}$, we need to check, how many $E \subset \{0, \dots, n-1\} \times \{0, \dots, n-1\}$ satisfy the condition top. In fact there is a maximal such E representing the complete dag Δ^{n-1} on n vertices, satisfying:

$$(i, j) \in E(\Delta^{n-1}) \Leftrightarrow i < j.$$

In particular each other such graph of vertex count n can be reached by deleting a uniquely determined set of edges from Δ^{n-1} .

So count the edges of Δ^{n-1} , since by definition they are each ordered, we can in fact count two-element subsets of $\{0, \dots, n-1\}$ and interpret them canonically as an ordered tuple in each case. Hence the edge count is:

$$|E\Delta^{n-1}| = \binom{n}{2}.$$

So the count of dags is identified as the number of (arbitrary) subsets of $E\Delta^{n-1}$:

$$|\text{embDAG}_n| = 2^{|E\Delta^{n-1}|} = 2^{\binom{n}{2}}.$$

□

Corollary 1.2.14. *Do note in particular we have the recurrence relation:*

$$|\text{embDAG}_{0,n}| = |\text{embDAG}_{0,n-1}| * 2^{n-1},$$

so in particular we can interpret the formula through the sum over all natural numbers up to a threshold:

$$|\text{embDAG}_{0,n}| = \prod_{i=0}^{n-1} 2^i = 2^{\sum_{i=0}^{n-1} i}.$$

Proof. By direct calculation with the above formula, we get:

$$\frac{|\text{embDAG}_{0,n}|}{|\text{embDAG}_{0,n-1}|} = 2^{\binom{n}{2} - \binom{n-1}{2}},$$

and by the relation

$$\binom{n}{2} = \binom{n-1}{1} + \binom{n-1}{2}$$

get

$$\frac{|\text{embDAG}_{0,n}|}{|\text{embDAG}_{0,n-1}|} = 2^{\binom{n-1}{1}} = 2^{n-1}$$

□

Remark 1.2.15. In particular one can see how extremely fast that count grows in n , essentially as $O(\exp(n^2))$.

vertex count	degree	dag count	log2 dag count
1	0	1	0
2	1	2	1
3	2	8	3
4	3	64	6
5	4	1024	10
6	5	32768	15
7	6	2097152	21
8	7	268435456	28
9	8	687194476736	36
10	9	35184372088832	45
11	10	$\sim 3 * 10^{16}$	55
12	11	$\sim 7 * 10^{19}$	66
13	12	$\sim 3 * 10^{23}$	78

In degree 23 the count reaches $2^{276} \sim 10^{83}$, hence there are more such dags in degree 23 in embDAG , than there are atoms in the observable universe. At degree 14 there are already about 10^{28} dags, hence more than atoms in a human body.

Lemma 1.2.16. *There is a functor $\oplus: \text{embDAG} \times \text{embDAG} \rightarrow \text{embDAG}$, given on objects and vertices as*

$$\{0, \dots, n-1\} \oplus \{0, \dots, m-1\} := \{0, \dots, n-1, n, \dots, n+m-1\},$$

Given two dags $(n, E), (m, F)$, with the edge subsets:

$$(i, j) \in (E \oplus F) \Leftrightarrow (i, j) \in E \vee (i-n, j-n) \in F.$$

Hence for two graph morphisms $\varphi: (n, E) \rightarrow (n', E'), \psi: (m, F) \rightarrow (m', F')$:

$$(\varphi \oplus \psi)(k) := \begin{cases} \varphi(k) & 0 \leq k \leq n-1 \\ \psi(k-n) + n' & n \leq k \leq n+m-1 \end{cases}.$$

Proposition 1.2.17 (Pushouts exist!). *WRONG!!! 02 -> 012, 02 -> 0 doesn't have a pushout*

The category of dags has (finite) pushouts!

Proof. Assume given two dag-morphisms $\varphi_k: H \rightarrow G_k$, we want to construct an object $G_1 \cup_H G_2$ and two morphisms $\iota_k: G_k \rightarrow G_1 \cup_H G_2$, such that for each object Z and pair of morphisms $f_k: G_k \rightarrow Z$ such that $f_1 \circ \varphi_1 = f_2 \circ \varphi_2$, there is a unique map $F: G_1 \cup_H G_2 \rightarrow Z$, such that $F \circ \iota_k = f_k$.

Assume $H = (\{0, \dots, k-1\}, E(H))$, $G_1 = (\{0, \dots, l-1\}, E(G_1))$, $G_2 = (\{0, \dots, m-1\}, E(G_2))$, and hence $\varphi_k: H \rightarrow G_k$ each represented by vertex maps $\varphi_1: \{0, \dots, k-1\} \rightarrow \{0, \dots, l-1\}$ and $\varphi_2: \{0, \dots, k-1\} \rightarrow \{0, \dots, m-1\}$ respectively. □

Remark 1.2.18. At this point I am fixing a bijection

$$\omega_{n,m}: \{0, \dots, n-1\} \times \{0, \dots, m-1\} \rightarrow \{0, \dots, nm-1\}$$

which is appropriately symmetric and associative.

Two choices of prominence are the idea of going through a 2d table either by row or by column first, like:

$$\omega_{n,m}(i, j) = i \cdot n + j,$$

or

$$\bar{\omega}_{n,m}(i, j) = i + j \cdot m.$$

These specifically satisfy:

$$\bar{\omega}_{n,m}(i, j) = \omega_{m,n}(j, i).$$

There are actually at least countably infinitely many such choices, e.g. by choosing divisors of n and m . Here I mainly want to choose one, sometimes use the other when convenient, but calm the reader that the specific choice is quite inessential.

Occasionally I need the inverse, let us pick the one of $\omega_{n,m}$:

$$\omega_{n,m}^{-1}(k) = (\lfloor k/n \rfloor, k \bmod n).$$

Example 1.2.19.

$\omega_{4,7}:$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	7	8	9	10	11	12	13
2	14	15	16	17	18	19	20
3	21	22	23	24	25	26	27

$\bar{\omega}_{4,7}:$	0	1	2	3	4	5	6
0	0	4	8	12	16	20	24
1	1	5	9	13	17	21	25
2	2	6	10	14	18	22	26
3	3	7	11	15	19	23	27

Lemma 1.2.20. *There is a functor $\otimes: \text{embDAG} \times \text{embDAG} \rightarrow \text{embDAG}$, given on objects and vertices as*

$$\{0, \dots, n-1\} \otimes \{0, \dots, m-1\} = \{0, \dots, nm-1\},$$

Given two dags $(n, E), (m, F)$ with the edge subsets:

$$(i, j) \in (E \otimes F) \Leftrightarrow ((\omega_{n,m}^{-1}(i))_1, (\omega_{n,m}^{-1}(j))_1) \in E \wedge ((\omega_{n,m}^{-1}(i))_2, (\omega_{n,m}^{-1}(j))_2) \in F,$$

with $(x, y)_1 = x$ and $(x, y)_2 = y$ in this equivalence.

Hence for two graph morphisms $\varphi: (n, E) \rightarrow (n', E'), \psi: (m, F) \rightarrow (m', F')$:

$$(\varphi \otimes \psi)(k) = \omega_{n',m'}(\varphi(\omega_{n,m}^{-1}(k)_1), \psi(\omega_{n,m}^{-1}(k)_2)).$$

Lemma 1.2.21. *For each pair $n, m \in \mathbb{N}$ we have natural bijections:*

$$\sigma_{n,m}: \{0, \dots, n+m-1\} \rightarrow \{0, \dots, m+n-1\}$$

and

$$\theta_{n,m}: \{0, \dots, nm-1\} \rightarrow \{0, \dots, mn-1\}.$$

They are given by:

$$\sigma_{n,m}(k) = \begin{cases} k+m & 0 \leq k \leq n-1 \Leftrightarrow m \leq k+m \leq n+m-1 \\ k-n & 0 \leq k-n \leq m-1 \Leftrightarrow n \leq k \leq n+m-1 \end{cases}$$

and:

$$\theta_{n,m}(k) = \lfloor k/n \rfloor + m \cdot (k \bmod n).$$

Both satisfy symmetry and associativity relations as follows: *Symmetries*

$$\sigma_{n,m} \circ \sigma_{m,n} = id_{\{0, \dots, m+n-1\}}$$

$$\sigma_{m,n} \circ \sigma_{n,m} = id_{\{0, \dots, n+m-1\}}$$

$$\theta_{n,m} \circ \theta_{m,n} = id_{\{0, \dots, mn-1\}}$$

$$\theta_{m,n} \circ \theta_{n,m} = id_{\{0, \dots, nm-1\}}$$

Associativity

$$\sigma_{n,m} \circ \sigma_{m,n} = id_{\{0, \dots, m+n-1\}}$$

$$\sigma_{m,n} \circ \sigma_{n,m} = id_{\{0, \dots, n+m-1\}}$$

$$\theta_{n,m} \circ \theta_{m,n} = id_{\{0, \dots, mn-1\}}$$

$$\theta_{m,n} \circ \theta_{n,m} = id_{\{0, \dots, nm-1\}}$$

Theorem 1.2.22. *The category embDAG is bipermutative with \oplus the disjoint union and symmetry isomorphisms $\sigma_{\bullet, \bullet}$, and \otimes the product with symmetry $\theta_{\bullet, \bullet}$. The category has strictly sum-neutral element the empty dag and strictly product-neutral element the one-point dag.*

The category has equalisers and coequalisers for each pair of morphisms, in particular finite pullbacks and pushouts.

There are combinatorial functors modelling cones C , as well as suspensions Σ .

Proof.

□

Theorem 1.2.23. *There is a functor $Fl: \text{embDAG} \rightarrow s\text{Set}$, which assigns to each dag $G = (V, E)$ as n -simplices the graph homomorphisms from Δ^n to G . Faces and degeneracies are induced by the relevant maps on Δ^n .*

The functor is strictly naturally sum preserving, that is $Fl(G \sqcup H) = Fl(G) \sqcup Fl(H)$ on objects as well as morphisms, also $F(\emptyset) = \emptyset$.

The functor is strictly naturally product preserving, that is $Fl(G \times H) = Fl(G) \times Fl(H)$ on objects as well as morphisms, also $F() = *$.*

In addition the functor satisfies:

$$(Fl \circ C) = (C \circ Fl)$$

for an appropriately chosen cone-functor C on simplicial sets, as well as

$$(Fl \circ \Sigma) = (\Sigma \circ Fl)$$

for a similar suspension-functor Σ on simplicial sets.

Proof. By the universal property of the product \otimes on embDAG we get on n -simplices:

$$\text{embDAG}(\Delta^n, G \times H) = \text{embDAG}(\Delta^n, G) \times \text{embDAG}(\Delta^n, H),$$

which exactly fits the description of the categorical product on simplicial sets. For sums, since Δ^n is a connected graph, on embDAG we get on n -simplices:

$$\text{embDAG}(\Delta^n, G \sqcup H) = \text{embDAG}(\Delta^n, G) \sqcup \text{embDAG}(\Delta^n, H),$$

which does not preclude that there might be more splittings of G or H , but it does prove the compatability with disjoint union of both categories. \square

Chapter 2

Combinatorial Homotopy of DAGs

2.1 Homotopic Maps

Reminder 2.1.1. Recall we have defined $I = C\mathbb{S}^0$, giving the dag: $0 \longleftarrow 2 \longrightarrow 1$.

Chapter 3

Restart DAGs 2024-03-05

3.1 Core Definition: The small Category of finite directed acyclic graphs.

Definition 3.1.1 (Category of DAGs). In this text THE category of finite directed acyclic graphs \mathcal{DAG} is defined as follows.

The objects are finite subsets of non-decreasing pairs of natural numbers:

$$\mathcal{DAG}_0 = \{ E \subset \mathbb{N} \times \mathbb{N} \mid \forall (x, y) \in E: x \leq y \wedge (x, x) \in E \wedge (y, y) \in E \}.$$

Say the edge set E describes a directed graph $G(E) = (V, E)$. Call the edges of type (x, x) vertices, i.e. "the vertex x ". Since the set E is finite, there is a maximal $n \in \mathbb{N}$ appearing in all the pairs of E , call that the height $height(E)$. Also there is a finite amount of vertices, i.e. edges (x, x) , in that edge set, call the number of vertices the degree $deg(E)$.

Morphisms are those maps of natural numbers that respect these subsets:

$$\mathcal{DAG}_1(G_1, G_2) = \{ \varphi \in \text{Set}(\mathbb{N}, \mathbb{N}) \mid \forall (e_0, e_1) \in G_1: (\varphi e_0, \varphi e_1) \in G_2 \},$$

(modulo an equivalence relation saying we don't care what φ is beyond G_1 's range.)

In particular given our definition of objects, it is acceptable to collapse an edge (x, y) onto a vertex with a map that satisfies $\varphi x = \varphi y$, but it is not acceptable to "break" an edge.

Remark 3.1.2. Do notice that this definition still massively overrepresents even each isomorphism type of directed acyclic graph. The initial object, the empty graph represented by the empty subset of $\mathbb{N} \times \mathbb{N}$, is fortunately unique in this representation. The terminal object, the one-point graph however can be represented by any one-point set $\{(n, n)\} \subset \mathbb{N} \times \mathbb{N}$, so there are countably many isomorphic copies of those.

More generally it is quite clear that any edge set is isomorphic to an edge set with equal height and degree.

However, it is evidently a small category, since all its objects and morphisms are explicitly subsets of $\mathbb{N} \times \mathbb{N}$ and $\text{Set}(\mathbb{N}, \mathbb{N})$ respectively. Hence the whole category has cardinality at most $\text{Set}(\mathbb{N}, \mathbb{N}) = \aleph_1$. I shall make an effort to keep everything this small to keep it accessible to machine computation.

Example 3.1.3. The complete graph on n nodes or equivalently the n simplex in a simplicial sense Δ^n can be represented as the object:

$$\Delta^n = \{ (i, j) \in \mathbb{N} \times \mathbb{N} \mid 0 \leq i \leq j \leq n \}.$$

In particular each graph whose maximal vertex is n can be embedded in such a Δ^n , even uniquely given our model choices.

The simplicially experienced would expect me to write down $\partial\Delta^n$ now, this is a bit more effort in this category with very stiff objects and morphisms, from $n \geq 2$ on one needs one subdivision step to represent $\partial\Delta^n$ as a dag.

However, we can conveniently describe the horns:

$$\Lambda_k^n = \{ (i, j) \in \mathbb{N} \times \mathbb{N} \mid 0 \leq k \leq n \wedge ((0 \leq i \leq j \leq k) \vee (k \leq i \leq j \leq n)) \},$$

which have the evident inclusions:

$$j: \Lambda_k^n \rightarrow \Delta^n,$$

given by the identity on vertices, and since the horn condition is stronger than the complete graph condition, this is a dag map.

3.2 The Core Construction: Subdivision

TODO: hier ist harter off-by-one trouble drin .. fuegt Sd n basispunkt hinzu? sollte ich alles punktieren?

Lemma 3.2.1 (The Power Set Map). *For $P_{fin}(\mathbb{N}) = \{ S \subset \mathbb{N} \mid |S| < \infty \}$ the finite subsets of natural numbers we can define a map:*

$$b: P_{fin}(\mathbb{N}) \rightarrow \mathbb{N}$$

by assigning to a finite set its binary encoding:

$$\{n_0, n_1, \dots, n_k\} \mapsto \sum_{0 \leq i \leq k} 2^{n_i} (-1?).$$

It is evidently injective, one can recover each n_i as the binary digit indices where there is a 1. It is also surjective, each natural number can be written as a finite binary word. Call the inverse $\pi: \mathbb{N} \rightarrow P_{fin}(\mathbb{N})$, after all its target is the power set of its domain :) So for each $n \in \mathbb{N}$ we have a unique corresponding finite subset $\pi(n) \in P_{fin}(\mathbb{N})$. The map has another very pleasant property, it is monotonic. Specifically, a proper subset inclusion corresponds to adding 1s in the binary representation, so the number increases.

Definition 3.2.2 (Subdivision of DAGs). Let $E(G) \subset \mathbb{N} \times \mathbb{N}$ describe a directed graph G .

Define the edge set

$$E(Sd(G)) = \{ (n, m) = (\sum_i 2^{s_i}, \sum_j 2^{t_j}) \mid$$

$$\{s_i\}_i \subset \{t_j\}_j \forall i, j: (s_i, t_j) \in E(G) \wedge (s_i, s_j) \in E(G) \wedge (t_i, t_j) \in E(G) \}.$$

In words: The vertices are the numbers $n = \sum_i 2^{s_i}$ such that each non-decreasing pair (s_i, s_j) is an edge in $E(G)$, call those legal vertices. Between each such pair of legal vertices $n = \sum_i 2^{s_i}, m = \sum_j 2^{t_j}$, set $S = \{s_i\}_i, T = \{t_j\}_j$. Add an edge (n, m) if and only if $S \subset T$.

Remark 3.2.3. Do note that since $S \subset T$ is an inclusion of legal vertices, each "new" edge has already been "tested" by T .

Example 3.2.4 (Subdivision of Δ^n). Recall Δ^n is represented by the edge set $\{(i, j) | 0 \leq i \leq j \leq n\}$.

TODO: erlaubte quotienten ... Δ^n zweimal subdividen und dann rand weg ist erlaubt und gibt \mathbb{S}^n .