Chapter 1

2024-04-20 – The basic Hopf fibre bundles

1.1 Explicit construction

Definition 1.1.1. For $k = \mathbb{R}^n$ assume a multiplication $\mu\colon k \otimes_{\mathbb{R}} k \to k$ making $k^\times \supset \mathbb{S}(k) \cong \mathbb{S}^{n-1}$ into an H-space. Then we can construct a fibre bundle as follow: Consider the unit sphere in $\mathbb{S}(k \times k)$, which can be identified as $\mathbb{S}(\mathbb{R}^{2n}) = \mathbb{S}^{2n-1}$, and the one-point compactification of $\mathbb{S}^k = \mathbb{S}^{dim_{\mathbb{R}} k} = \mathbb{S}^n$.

Notice that for $\mathbb{S}(k^2)$ the components can be written as pairs of elements of k with norms each less than or equal to one, and norm-squares summing to one. Thus consider the map:

$$\begin{aligned} m_k\colon \quad \mathbb{S}^{2n-1} = \mathbb{S}(k \times k) \to &\quad \mathbb{S}^k \\ a, b \mapsto &\quad a \cdot b^{-1}, \end{aligned}$$

where $(\bullet) \cdot b^{-1}$ has to be defined as ∞ for $b = 0$, (note that this implies $|a| = 1$, so $a \neq 0$.)

Proposition 1.1.2. *This is a fibre bundle for* $k = \mathbb{R}, \mathbb{C}, \mathbb{H}$, *with fibre* $F = \mathbb{S}(k) = \mathbb{S}^0, \mathbb{S}^1, \mathbb{S}^3$.

Proof. Consider the tiny case $m_{\mathbb{R}}\colon \mathbb{S}(\mathbb{R}\times\mathbb{R}) \to \mathbb{S}^{\mathbb{R}}$, specifically its fibre: $m_{\mathbb{R}}^{-1}(0 \in \mathbb{R}) = \{\ (a, b) \in \mathbb{S}(\mathbb{R}\times\mathbb{R}) \mid |a|^2 + |b|^2 = 1 \wedge ab^{-1} = 0\ \}$. Find $m_{\mathbb{R}}^{-1}(0) = \{\ (a, b) \in \mathbb{S}(\mathbb{R}\times\mathbb{R}) \mid a = 0 \wedge |b| = 1\ \} = \{0\} \times \{\pm 1\} \cong \mathbb{S}^0$. Similarly for $t \neq 0$ in $\mathbb{R}$ consider: $ab^{-1} = t \Leftrightarrow a = tb \Leftrightarrow b = at^{-1}$. It follows $m_{\mathbb{R}}^{-1}t = \{\ (\pm 1, \pm t^{-1})\ \} \cong \mathbb{S}^0$ for each target in $\mathbb{R}$. Finally consider $m_{\mathbb{R}}^{-1}\infty = \{\ (a, b) \in \mathbb{S}(\mathbb{R} \times \mathbb{R}) \mid |a|^2 + |b|^2 = 1\ \wedge\ ab^{-1} = \infty\ \} = \{\ (a, 0)\ \mathbb{S}(\mathbb{R} \times 0)\} \cong \mathbb{S}^0$. ∎