Chapter 1

2024-03-20 – Sets, Relations, Transitivity, Partial and Total Orders

1.1 The fundamental set categories

This paper makes liberal use of the language of (1-)categories, specifically adjunctions are numerous in these results. The unfamiliar reader is encouraged to look up their favourite adjunction anywhere, I can guarantee you have one, if you do not know it. Following along the adjunctions here inspired by any other adjunction, for example of the free-forgetful-type, helps clarify the roles of source- and target-categories in each adjunction here. For a more rigorous reference Saunders MacLane's "Categories for the Working Mathematician" will do nicely, there is no ultra-modern category tech here. However, for definiteness fix what categories are in this paper:

Definition 1.1.1. Consider a tuple $\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1, s, t, id, \circ)$ consisting of two sets $\mathcal{C}_0, \mathcal{C}_1$ in the sense recalled in the next definition and four set-maps $id\colon \mathcal{C}_0 \to \mathcal{C}_1, s, t\colon \mathcal{C}_1 \to \mathcal{C}_0, \circ \mathcal{C}_1 \times_{t,s} \mathcal{C}_1 \to \mathcal{C}_1$, where $\mathcal{C}_1 \times_{t,s} \mathcal{C}_1 := \{\ (c, d) \in \mathcal{C}_1 \times \mathcal{C}_1 \mid t(c) = s(d)\ \}$.

Call the above structure a 1-category, if the following compatibility conditions are satisfied:

$$\circ(\circ(f, g), h) = \circ(f, \circ(g, h)), \circ(1_\bullet, f) = f, \circ(f, 1_\bullet) = f.$$

No person would write composition like that though, so introduce the notation $d \circ c := \circ(c, d)$. Then the axioms become the more familiar:

$$(f \circ g) \circ h = f \circ (g \circ h), 1_\bullet \circ f = f, f \circ 1_\bullet = f.$$

Remark 1.1.2. Note that in particular from the perspective of this paper all categories are small, as in, the set of all objects and all morphisms is actually a set. Obviously the category of all sets or all relations do not satisfy having sets of objects like above, but still satisfy the relations. The distinction between small and large categories in this paper is however an easily noticeable contrast even without emphasis.

Remark 1.1.3. In light of the associativity of $\circ$ one could make an argument to denote n-fold composites as $\circ(f_n, \ldots, f_1)$ though.

Since a lot of the following arguments rely on very specific descriptions of the involved sets, it seems wise to recall a few of the set-theoretic notions here, also to fix notation. They are however classical, and strictly speaking a lot of the results of this paper can be recovered in far smaller models of set. It, however, seems opportune to state in proper generality for all sets all that can be done for such general sets.

Definition 1.1.4. A set for the purposes of this paper is a collection of elements $S = \{s | s \in S\}$ which uniquely identify the set, i.e. given another set $T = \{t | t \in S\}$, both are equal if and only if they have the same elements:

$$S = T \Leftrightarrow \forall s \in S \colon s \in T \wedge \forall t \in T \colon t \in S.$$

A map of sets $f \colon S \to T$ is informally a "formula" that for each element of S unambiguously assigns an element of T. For example the set of all maps between S, T sets can be realised as a set explicitly as follows:

$$Set(S, T) := \{ \ f \subset S \times T \mid \forall s \in S \exists! t \in T \colon (s, t) \in f \ \}.$$

Maps are the binary relations between the sets S and T such that each source element has a unique target element, not necessarily different for different $s \in S$. For f an element of $Set(S, T)$ hence write the guaranteed unique element of T that is in f-relation to it as its f-image, i.e. $f(s) = t :\Leftrightarrow (s, t) \in f$. Maps are hence equal if and only if they are equal as sets if and only if they agree on each source element. Maps can be composed in the familiar way, identities are realised as the diagonal for each set. The fact that maps compose associatively is a well-known fact of set theory just quoted here.

Call the category of sets thus "defined" Set.

Remark 1.1.5. Note that implicitly one has to fix an inaccesible cardinal here, and limit the cardinality of the included sets below that cardinal. Then one can accurately identify Set as a hierarchy of 1-categories of ever increasing largeness, where the definition of what is allowed to be considered a set extends constantly.

For many applications that is an exceptionally important point, for these applications it is not, hence suppress such considerations wherever convenient here.

Remark 1.1.6. The axiomatically minded reader is encouraged to take this definition as, assume the ZFC axioms, and a model of category theory of "small" categories in it, i.e. categories that have object- and morphism-sets constructed by these ZFC axioms of our universe. All categories of interest in this paper are small (hence also locally small).

Remark 1.1.7. Note that classically in set one can construct arbitrary unions, arbitrary disjoint unions, arbitrary products (by axiom of choice), subobjects, quotients, colimits, limits. Sets are a very convenient category in which to construct base objects, but usually not structured enough.

Definition 1.1.8. A binary relation R on two sets S, T is a subset of their product $R \subset S \times T$, denote $s \in S, t \in T$ to be in R relation as $sRt :\Leftrightarrow (s, t) \in R$.

A map of relations $f\colon R_0 \to R_1$ is a pair of maps $f_S\colon S_0 \to S_1$ and $f_T\colon T_0 \to T_1$ for $R_i \subset S_i \times T_i$, such that restricting f to elements of R_0 only yields elements of R_1, say the morphism respects the relation:

$$f \text{ relation morphism } :\Leftrightarrow \forall (s,t) \in R_0\colon (f_S(s), f_T(t)) \in R_1.$$

Remark 1.1.9. Since composition of heteregoneous relations restricts to composition of set maps, recall the definition for clarity.

Definition 1.1.10. Given two binary relations $R_0 \subset S \times T, R_1 \subset T \times U$ with compatible middle set T, define the composite relation $R_1 \circ R_0 \subset S \times U$ as follows:

$$(s,u) \in R_1 \circ R_0 :\Leftrightarrow \exists t \in T\colon sR_0t \wedge tR_1u.$$

I.e. the composite relation contains all such pairs such that in the middle step there exists a common relating element in the "forgotten set" T.

Remark 1.1.11. The only such general relations that feature in this content are the set maps defined as above. Note that composition of relation is just as associative as composition of set maps.

It is convenient for this paper to restrict to a more restrictive class of relations.

Definition 1.1.12. A homogeneous binary relation R on a set S is a subset of the product of S with itself $R \subset S \times S$. Call S the underlying set of R.

A map of homogenous binary relations $f\colon R_0 \to R_1$ is a map of the underlying sets $f\colon S \to T$ that respects the relation, i.e. $\forall s,t \in S : sR_0t \Rightarrow f(s)R_1f(t)$. Maps contain identities, they compose, and associatively, because that is true on Set.

Call the category of homogeneous binary relations thus defined Rel.

Note in addition that composition of relations restricts to homogeneous relations.

Remark 1.1.13. Do note that in particular each map of underlying sets induces a map of the empty relations.

Proposition 1.1.14. *Consider for each non-empty set the empty relation as follows $R = (\emptyset \subset S \times S)$, then for any other non empty relation there are not any maps into R. For any other relation, empty or not the set of maps from the empty relation to it is the same as a map of the underlying sets. In particular the relation $\emptyset$ on the underlying set $\emptyset$ is an initial object of the category of relations.*

Proof. If the relation is non-empty, there is no way to define a map from it into an empty set, in particular the underlying set map cannot induce a compatible map on relations. Hence follows the first claim.

By definition of relation-maps such a map is a map of underlying sets, respecting relations. Since there are not any relations to respect in the empty relation, such a map is always trivially a relation-map, so all set maps induce a relation-map. □

Proposition 1.1.15. *The one-point set with relation equality $(\{*\}, \{(*,*)\})$ is a terminal object in Rel.*

Proof. As a corollary to the above proposition, the largest a set of relation morphisms can get, is, if the source relation is empty. In that case the map and the fact that it is a map of relations is equivalent to the underlying set map. For that case there exists only one underlying set map to any one-point set. Hence, since the terminal object displayed above has all relations between its elements, any set map to the one-point set defines a unique relation-map. Hence the resulting relation-map is unique. □

Definition 1.1.16. Every set S has two trivially associated homogeneous relations, the discrete one and the collapsible one. Fix notations in accordance with the upcoming topological interpretations:

$$S^\delta = \{\ (s, s) \mid s \in S\ \} \quad \Delta^S = \{\ (s, t) \mid s, t \in S\ \}.$$

In addition note that every relation on S has a unique morphism from S^δ, which is the identity on S, and a unique morphism to Δ^S, which is the identity on S.

Definition 1.1.17. Call a relation with any underlying set an empty relation if it has no elements. Call a relation discrete if it can be realised as S^δ by its underlying set. Call a relation collapsible if it can be reliased as Δ^S by its underlying set.

Remark 1.1.18. Note that interpreted in the context of sets and set maps the discrete relation is exactly the identity map of its underlying set. Collapsible relations are never maps for sets with at least two elements, but the empty set and one point set each induce the identity on themselves this way too.

Lemma 1.1.19. *Morphisms into collapsible relations are the same as the set maps of underlying sets, i.e. for U the forgetful functor as above, there is an adjunction:*

$$U : Rel \leftrightarrow Set : \Delta^\bullet.$$

With U the forgetful functor $Rel \to Set$, synonymously with "underlying set functor", as above.

Proof. The claim is $Rel(R \subset S \times S, \Delta^T) = Set(UR = S, T)$. Since Δ^T contains every possible pair, being a map of relations into it is an empty condition, so forgetting down to underlying sets retains all information and the argument works □

Corollary 1.1.20. *Since colimits are defined as left adjoints follows by the standard 1-category argument, colimits of relations are created by forgetting down to sets. I.e. take the colimit of the relations and their underlying sets componentwise in Set, that is a valid colimit in Rel.*

Lemma 1.1.21. *There is an isomorphism of categories: Sets and set maps are the same thing as the full subcategory of binary homogeneous discrete relations.*

Proof. Any set can be represented as its associated discrete relation, any map of sets trivially induces a map of discrete relations, because the condition of being a relation map is vacuously satisfied. Hence forgetting down the discrete relation recovers the same sets and maps.

Coming from a discrete relation, forgetting down to the underlying set forgets nothing about the maps, because they do not have to satisfy non-trivial relation-compatibility conditions on discrete relations. Hence they are uniquely described on their underlying sets and each such map induces a map of relations by passing to the discrete relations again, recovering the relations and their maps. □

Remark 1.1.22. It might seem brutal notation to denote by *Rel* such a stricter category of relations than the more general definition above. However, apart from maps the relations in this paper that structure the content are all homogeneous, so they deserve emphasis.

1.2 Reflexive Relations are complete and cocomplete

The category of (homogeneous binary) relations *Rel* a few interesting endofunctors, consider the first one of this paper.

Definition 1.2.1. For each relation $R \subset S \times S$ consider its reflexive closure:

$$refl \colon Rel \to Rel$$

with $refl(R) := \{\ (s,s) \mid s \in S\ \} \cup R$. Its naturally compatible with maps of (homogeneous) relations, hence a functor.

Call a relation R reflexive if it satisfies $relf(R) = R$, which is exactly the case if all diagonal elements of the underlying set were already part of R.

Call the full subcategory of reflexive relations *RRel*, i.e. *RRel* has objects relations that are $refl$-fixed points and all relation morphisms between them.

Remark 1.2.2. One can equivalently, since the subcategory is full, consider the reflexive closure the left-adjoint to the forgetful functor from reflexive relations to relations:

$$refl \colon Rel \leftrightarrow RRel \colon U.$$

That point does not clarify things substantially here, but might be helpful in other contexts.

Lemma 1.2.3. *Morphisms out of a discrete relation into any reflexive relation are the same as the set maps of the underlying sets. I.e. send each reflexive relation $RRel \ni R \subset S \times S$ to its underlying set $UR = S \in Set$, and similarly on maps, then there is an adjunction:*

$$(\bullet)^{\delta} \colon Set \leftrightarrow RRel \colon U.$$

Call U the forgetful functor $RRel \to Set$, synonymously with "underlying set functor".

Proof. Staring at $RRel(M^{\delta}, R \subset S \times S) = Set(M, S)$ long enough will convince the reader this is true. Explicitly: A map of homogeneous relations from a discrete relation only is a map of equal pairs (m, m), hence these are equal in S too, so forget down to the underlying set map. Since a morphism of relations is defined via the underlying set map with properties, which are vacuously satisfied for a discrete source object follows the above equality. Note however that only reflexive R's admit such maps in this way, thus the restriction to that subcategory. □

Corollary 1.2.4. *The adjunction as above restricts to the subcategory of reflexive relations.*

$$U\colon RRel \leftrightarrow Set\colon \Delta^{\bullet}.$$

Proof. The claim is $Set(UR, S) = RRel(R, \Delta^S)$, i.e. for a reflexive relation R the relation morphisms to a collapsible relation with underlying set S are exactly the same as set morphisms from R's underlying set. Since $RRel$ is a full subcategory of Rel this follows from the analogous lemma on Rel, since Δ^S is a reflexive relation for every set S, hence also contained in $RRel$. □

Remark 1.2.5. Note interestingly that making a relation irreflexive is not a functor without assuming a smaller class of maps. If $f\colon R_0 \to R_1$ being a map of relations depended on a relation xR_0y with $x \neq y$ and $fx = fy$ with fxR_1fy, then making R_1 irreflexive breaks the property of f being a morphism of relations. So any non-injective map of homogeneous relations does not induce a map on the associated irreflexive relations.

Remark 1.2.6. The adjunctions establish an interesting conflict of intuitions. There should clearly be more relations than there are plain sets, which is expressed by the fact that Δ^S allow for any kind of map into them, so recover all set maps of the underlying sets of all relations, but there are clearly more relations that are not Δ^S for any non-empty set S.

On the other hand the other adjunction involving the discrete relations $(\bullet)^{\delta}$ is a formalisation of the fact that a map of relations is uniquely determined by what it does on the underlying sets, everything else are conditions. Hence because every relation admits a map from the discrete relation on its vertex set with underlying set map the identity, see in particular that a map of relations is uniquely determined on the underlying set, and it follows from the adjunction.

Corollary 1.2.7. *By identifying sets with the category of discrete relations the above adjunctions can actually be interpreted as adjunctions to an inclusion $i\colon Set \hookrightarrow RRel$.*

Proof. This is basically a matter of notational preference, but important enough to emphasise. □

Finally note the result for which the adjunctions were set up:

Theorem 1.2.8. *The category of reflexive relations is complete and cocomplete, with colimits and limits computed componentwise for objects as well as morphisms.*

Proof. The plain existence of left- and right-adjoints to the forgetful functor to sets guarantees the above fact.

Consider for simplicity and explicity inductive limits and colimits: Given a system $(R_0 \to R_1 \to \ldots \to R_n \to R_{n+1} \to \ldots)$ of arbitrary maps of reflexive relations and their underlying sets, define the colimit underlying set as the colimit of the underlying sets, then the induced relation is defined exactly by being related in a finite step. This satisfies the universal property of the colimit in reflexive relations, hence is a reflexive relation representing that colimit.

Similarly given a system $(\ldots \to R_{n+1} \to R_n \to \ldots \to R_1 \to R_0)$ of arbitrary maps of reflexive relations and underlying sets, define the limit underlying set as the limit of underlying sets. Elements are in relation if they are so in each R_i step. On underlying sets the universal property is inherited from sets, and the extended relation is uniquely forced by that structure.

Arbitrary diagrams follow similarly in both cases, simply replace "in a finite step" with "in some representing summand for the quotient" in the colimit, the limit-case works verbatim. The limit and colimit relations are each well-defined because the system-maps are maps of relations. □

Each of the following corollaries follows directly from the theorem, but it is worth explicitly constructing some special cases of limits and colimits, which are of further use here.

Corollary 1.2.9. *Reflexive Relations have pushouts.*

Proof. Given three reflexive relations $(R_0 \subset S_0 \times S_0, R_1 \subset S_1 \times S_1, R_2 \subset S_2 \times S_2)$, with set maps $i_1 \colon S_0 \to S_1$ and $i_2 \colon S_0 \to S_2$ each inducing relation maps on the R_i. Then consider the pushout of underlying sets $S_1 \cup_{S_0} S_2$ with set maps $j_1 \colon S_1 \to S_1 \cup_{S_0} S_2$, $j_2 \colon S_2 \to S_1 \cup_{S_0} S_2$.

By construction of colimits in set an element $s \in S_1 \cup_{S_0} S_2$ can be represented by either elements of S_1 or S_2, with identifications over S_0. Hence set the quotient relation to be $[x]R[y] :\Leftrightarrow (x, y \in S_1, xR_1y) \vee (x, y \in S_2, xR_2y)$. It is clearly a relation, since it has pairs in the same underlying set, and it is reflexive, because all relations involved are.

It satisfies the universal property of the pushout, since it has the minimal set of relation pairs given the diagram, and the morphism from the common source R_0 ensures that each compatible pair of maps factors over the quotient. □

Remark 1.2.10. Be aware that pushouts of relations are a relatively brutal construction, consider the following few examples.

Let the relations R_0, R_1, R_2 be as follows: All are defined on the underlying set of two elements, denote it $S = \{0, 1\}$, the maps are the ones induced by the identity of S.

Let $R_0 = S^\delta$, and set $R_1 = S^\delta \cup \{(0, 1)\}$ and $R_2 = S^\delta \cup \{(1, 0)\}$. Then the pushout relation is defined on the same set S, contains the diagonal trivially, but also every possible pair, because $(0, 1)$ and $(1, 0)$ are introduced each from their summands. So pushing out the two relations that could be called $\leq$ and $\geq$ respectively, get the trivial collapsible relation on the underlying sets. This example generalises to arbitrary sets with at least two points.

Consider also explicitly quotients, that is any map of relations $R_0 \to R_1$ on arbitrary underlying sets $S_0 \to S_1$, and consider the unique map $R_0 \to pt$ to the terminal relation, the one-point-relation on the one-point set.

Then the resulting pushout has underlying set S_1/imS_0, and the relation recovers exactly the relations on $S_1 \setminus imS_0$ while the whole underlying set S_0 has been collapsed to a point in the quotient, which satisfies reflexivity.

Corollary 1.2.11. *Reflexive relations have coequalisers.*

Proof. Let $f, g \colon R_0 \to R_1$ be two morphisms of reflexive relations with underlying sets S_0, S_1 respectively. Consider the underlying quotient set $S = S_1/\sim$ with equivalence relation generated by $fx \sim gx \forall x \in S_0$. Induce a relation $R/\sim$ by the relation of S_1, i.e. $[x]R/\sim[y] :\Leftrightarrow \exists a, b \in S_1 : a \in [x], b \in [y], aR_1b$. Since R_1 is reflexive, so is $R/\sim$. The universal property is readily verified. □

Corollary 1.2.12. *Reflexive relations have pullbacks.*

Proof. Let $f\colon R_1 \to R_0$ and $g\colon R_2 \to R_0$ be two morphisms of relations with a common target and underlying sets S_i as before. Consider the pullback set $S = S_1 \times_{S_0} S_2 = \{ (s,t) \in S_1 \times S_2 \mid fs = gt \}$ and induce a relation R on S componentwise: $(s,t)R(u,v) :\Leftrightarrow sR_1u \wedge tR_2v$. The universal property is easy to prove. ☐

Corollary 1.2.13. *Reflexive relations have equalisers.*

Proof. Let $f, g\colon R_0 \to R_1$ be two parallel maps of relations on underlying sets S_i, and consider the set $S := \{ s \in S_0 \mid fs = gs \}$, and induce a relation by restricting R_0 to that subset. It is a again a reflexive relation and clearly the pair satisfies the universal property of an equaliser of f and g. ☐

1.3 Transitivity

So far our basic categories include $RRel, Rel, Set$ connected by a few free-forgetful adjunctions. Add another such category to the chain by considering transitive relations, for that introduce another endofunctor on relations.

Definition 1.3.1. For an arbitrary relation $R \subset S \times S$ with underlying set S consider the i-fold composition of R with itself:

$$R^1 := R, R^i := R \circ R^{i-1} \forall i > 0.$$

More explicitly $R^i = \{ (s,u) \in S \times S \mid \exists (t_1, \ldots, t_{i-1}) \forall j sRt_1, t_{i-1}Ru, t_jRt_{j+1} \}$ is the relation defined by all pairs that can be connected by a chain of relations of length $i-1$. Specifically the one-fold iterate is the relation itself with no intermediate connectors.

Since the iterates are all subsets of a common superset $S \times S$ it also makes unambiguous sense to consider their union:

$$trs(R) := \bigcup_{i \geq 1} R^i,$$

which then consists of the original relation R, as well as all pairs that were R-connectable by some finite sequence of consecutive R relations of S elements. Then call $trs(R)$ (with underlying set S) the transitive closure of R. It is a functor from relations to itself, call the full subcategory of transitive relations in all relations $TRel$, i.e. the category on objects relations R such that $trs(R) = R$ with all relation maps between them.

There is a reflexive variation on this by introducing the zero-th exponent.

Definition 1.3.2. For an arbitrary relation $R \subset S \times S$ with underlying set S consider the i-fold composition of R with itself:

$$R^0 := [S]^\delta, R^i := R \circ R^{i-1} \forall i > 0,$$

and define the unital transitive closure as the analogous union:

$$trs_1(R) := \bigcup_{i \geq 0} R^i,$$

it consists of all diagonal elements of the underlying set, hence $trs_1(R)$ is a reflexive relation for any relation R. Call the full subcategory of reflexive and transitive relations with all relation maps between them $TRRel$.

Lemma 1.3.3. *The transitive closure is both the left- and the right-adjoint in the free forgetful adjunction between the category of all relations and the category of transitive relations. It restricts to adjunctions on reflexive relations.*

$$trs\colon Rel \Leftrightarrow TRel\colon U,$$

$$U\colon TRel \Leftrightarrow Rel\colon trs,$$

and

$$trs\colon RRel \Leftrightarrow TRRel\colon U,$$

$$U\colon TRRel \Leftrightarrow RRel\colon trs.$$

Proof. The claim is $TRel(trs(R_0), R_1) = Rel(R_0, UR_1)$, as well as $Rel(U(R_0), R_1) = TRel(R_0, trs(R_1))$, and the according claim for reflexive relations. These are however all trivially satisfied, since the transitive and reflexive relation categories are defined as full subcategories of relations. In other words, there is nothing specifically reflexive or transitive a morphism of relations could satisfy or respect. □

Hence follows as before for reflexive relations that the category of transitive reflexive relations has all limits and colimits.

Lemma 1.3.4. *The category of transitive reflexive relations is complete and cocomplete, the limits and colimits can be computed just as in relations, hence pointwise.*

Proof. First consider limits, i.e. a candidate object $lim_I R_i$. The underlying set of $lim_I R_i$ clearly has to be the limit of the underlying I-diagram in set. When realised as subset of the product $\Pi_I S_i$, the induced relation is defined by two I-tuples in the S_i being in lim_I relation if and only if each of their components are. If each R_i is reflexive, so is the limit-relation, if each R_i is transitive, so is the limit relation. So pullbacks, equalisers, subobjects are all the underlying constructions of reflexive relations as above.

Considering colimits is a bit more work because of the transitive closure involved. The most salient point about this adjunction is that reflexivity is inherited on quotients, but transitivity is not, so the transitive closure functor has to feature despite the apparent simplicity of the adjunction. Consider hence a candidate object $colim_I R_i$, with underlying set realised by $colim_I S_i$ as a quotient of the disjoint union of the S_i modulo the equivalence relation generated by the I-diagram. On that quotient induce the relation

$$[x]R[y] :\Leftrightarrow \exists i \in I, s, t \in S_i\colon [s] = [t] \in colim_I S_i \wedge sR_i t.$$

It is reflexive, because each contributing summand contributes the diagonal elements of its underlying set. However there is no reason for a colimit of transitive relations to be transitive itself and in fact it is not in proper generality. So take the transitive closure, which is the same as the unital closure on reflexive relations, $trs(colimR_i) = trs_1(colimR_i)$ of the colimit of the summand relations R_i, then this is evidently a transitive relation, which satisfies the universal property of the colimit in transitive relations. □

Example 1.3.5. Let us show an example where the pushout of two transitive relations would not be transitively closed without applying the closure functor. Consider the transitive and reflexive relations $1 < 3 < 5$ and $2 < 3 < 4$ with pushout over 3, also reflexive and transitive. Then the colimit relation R has all these relations $R = \{(1,1),(3,3),(5,5),(2,2),(4,4),(1,3),(1,5),(3,5),(2,3),(3,4),(2,4)\}$. Hence notice, the first relation introduced a relation $1 < 3$ and the second one introduced a relation $3 < 4$, but on its own R has no reason to also introduce $1 < 4$ in the colimit. Symmetrically in this example for $2 < 3$ and $3 < 5$, which needs an additional relation $2 < 5$.

Specifically the minimal failing example is the pushout of two "edges" $1 < 2$ and $2 < 3$, because $1 < 3$ only follows after transitively closing.

The example generalises into the following observation.

Lemma 1.3.6. *Note that for each set the discrete relation and the collapsible relation on it are reflexive and transitive relations. In the category of sets the following diagram is trivially a pushout for any two elements $s, t \in S$:*

$$\begin{tikzcd} S \setminus \{s,t\} \arrow[r] \arrow[d] & S \setminus \{s\} \arrow[d] \\ S \setminus \{t\} \arrow[r] & S. \end{tikzcd}$$

Analogously the following diagram is a pushout in set for each two $s, t \in S$:

$$\begin{tikzcd} \{s\} \arrow[r] \arrow[d] & \{s,t\} \arrow[d] \\ S \setminus \{t\} \arrow[r] & S. \end{tikzcd}$$

These induce the following four pushout diagrams in transitive reflexive relations by taking discrete and collapsible relations: For the first diagram get:

$$\begin{tikzcd} (S \setminus \{s,t\})^\delta \arrow[r] \arrow[d] & (S \setminus \{s\})^\delta \arrow[d] \\ (S \setminus \{t\})^\delta \arrow[r] & S^\delta, \end{tikzcd}$$

and

$$\begin{tikzcd} \Delta^{S \setminus \{s,t\}} \arrow[r] \arrow[d] & \Delta^{S \setminus \{s\}} \arrow[d] \\ \Delta^{S \setminus \{t\}} \arrow[r] & \Delta^S. \end{tikzcd}$$

For the second diagram get:

$$\begin{tikzcd} s^\delta \arrow[r] \arrow[d] & \{s,t\}^\delta \arrow[d] \\ (S \setminus \{t\})^\delta \arrow[r] & S^\delta, \end{tikzcd}$$

and

$$\begin{tikzcd} \Delta^s \arrow[r] \arrow[d] & \Delta^{\{s,t\}} \arrow[d] \\ \Delta^{S\setminus\{t\}} \arrow[r] & \Delta^S. \end{tikzcd}$$

Each of these are pushout diagrams in the category of transitive, reflexive relations.

Proof. The universal properties are all readily verified since the underlying maps all involve the above mentioned pushout diagrams in sets, and the induced relations are identified as the above objects.

The notable point is that without transitive closure the pushouts for $\Delta^\bullet$ would clearly not be transitively closed in general just like in the example above, hence the pushout in the category of reflexive and transitive relations of collapsible relations becomes another collapsible relation. □

1.4 Antisymmetry

The upshot of this section is that antisymmetry is the only colimit/quotient-troublesome relation of an order relation. It cannot be broken by limit constructions, but colimits of antisymmetric things do not need to be antisymmetric. In addition there is no natural way to introduce a functor identifying antisymmetric relations as the image of that functor like for reflexivity and transitivity. Essentially the fact that the condition induces for each non-trivial relation a non-trivial non-existence condition can break antisymmetry of pushouts.

Definition 1.4.1. A relation $R \subset S \times S$ is called antisymmetric, if for all pairs of elements $s, t \in S$ it follows $sRt \wedge tRs \to s = t$. I.e. there is only ever at most a relation in one direction and not the other.

Denote the full subcategory of all antisymmetric relations, i.e. on objects relations that are asymmetric and with morphisms all relation maps between them, as $ARel$. Denote the category of antisymmetric reflexive relations $ARRel$, the category of antisymmetric, transitive, reflexive relations $ARTRel$, or by the more common name partially ordered sets, or posets, $Poset$.

Example 1.4.2. Consider the two relations $\{(0,0),(1,1),(0,1)\}$ and $\{(0,0),(1,1),(1,0)\}$ with intersection the discrete relation $\{(0,0),(1,1)\}$. All three involved relations are antisymmetric, i.e. $\forall x, y \in S\colon xRy \wedge yRx \Rightarrow x = y$, but in the pushout there is the relation $(0,1)$ and the relation $(1,0)$ while $0 \neq 1$, violating antisymmetry.

This naturally leads to two concepts on general relations. The dual object of this is of more immediate use here, but the subobject concept seems easier to understand as the first definition.

Definition 1.4.3. For any relation $R \subset S \times S$ consider its maximal antisymmetric subrelation $R_<$ as follows, the underlying set $S_<$ is the subset of S, where R restrict to an antisymmetric relation:

$$S_< = \{\ s \in S \mid \forall t \in S\colon sRt \wedge tRs \to s = t\ \},$$

and the induced relation is the restriction of R.

Proposition 1.4.4. *The maximal antisymmetric subrelation of any relation is in fact antisymmetric.*

Proof. By definition of $S_<$ this follows for each pair from the condition in two redundant ways. $\square$

Example 1.4.5. Notice that the maximal antisymmetric subrelation is maximal in that it allows a natural description for arbitrary relations, there may well be antisymmetric relations properly between the whole relation and the maximal antisymmetric quotient as defined above, which are obtained by restricting the relation too, not just the underlying set. There seems to be no natural way to antisymmetrise a relation by picking a subrelation without additional structure on the underlying sets like a total order. Presumably the argument would have to involve strictifying the axiom of choice in the guise of the well-ordering theorem into choosing a well-ordering for each set in a way that becomes a functor after everything is picked. Those details are beyond the scope of this paper in this generality.

Remark 1.4.6. The maximal antisymmetric suboject is not needed for the constructions of limits, in particular equalisers, subobjects, etc., since the reflexive relation limits of an I-diagram with only antisymmetric relations and relation morphisms between them, are antisymmetric without any additional assumptions or constructions. Phrased differently, the maximal antisymmetric subobject of an antisymmetric relation is obviously the relation itself, hence limits of antisymmetric relations agree with their maximal antisymmetric subobjects too.

Proposition 1.4.7. *The (reflexive) relation limits of antisymmetric relations are antisymmetric.*

Proof. In fact reflexivity is not needed for the limit to exist. Note that the induced relation is defined as being in relation in each component. Hence follows from a relation xRy and yRx in the limit a relation in each component, so x and y are equal in each component, hence $x = y$. $\square$

Corollary 1.4.8. *The category of antisymmetric relations $ARel$ is complete, i.e. has all limits. The limit of reflexive antisymmetric relations is itself reflexive and antisymmetric, hence $ARRel$ is a complete subcategory of $ARel$. Finally posets are respected by limits too, since transitivity is respected on all components, so $Poset = ARTRel$ is a complete subcategory of $ARRel$.*

Remark 1.4.9. Dually to limits there is a bit more to do, for this consider the best approximation "from the right" of an relation to be an antisymmetric relation by compressing the underlying set.

Definition 1.4.10. For any relation $R \subset S \times S$ introduce its maximal antisymmetric quotient $R^<$ as follows, the underlying set $S^<$ is the underlying set S modulo the equivalence relation generated by pairs violating asymmetry:

$$S^< = S/(x \sim y \Leftrightarrow xRy \wedge yRx).$$

Induce a relation on the quotient as before on representatives. The maximal asymmetric quotient of R is reflexive, if R is, it trivially satisfies asymmetry.

For a transitive relation however that quotient is not necessarily transitive, so potentially one application of transitive closure is necessary in the relevant categories, where that is part of the colimit, hence quotient construction. The following insight provides the cornerstone of why colimits of posets look the way they do.

Lemma 1.4.11. *The maximal antisymmetric quotient is left-adjoint to the forgetful functor from asymmetric reflexive relations to reflexive relations:*

$$(\bullet)^{<} : RRel \Leftrightarrow ARRel : U.$$

Proof. The claim is $ARRel(R^{<}, A) = RRel(R, UA)$. I.e. any morphism from an arbitrary relation uniquely factors through the maximal antisymmetric quotient of R. This is easily verified. Specifically for the set of all morphisms from any relation to an antisymmetric relation to universally satisfy that equation find: A morphism that needs the compatibility condition xAy in an antisymmetric relation A coming from an arbitrary relation R, can be modified into a morphism that needs yAx satisfied. So the quotient introduces all the relations necessary to make $R^{<}$ a universal source object associated to any reflexive relation. □

This lemma gives the first fundamental result of this paper.

Theorem 1.4.12. *The category of antisymmetric reflexive relations ARRel and antisymmetric reflexive and transitive relations, i.e. posets, $ARTRel = Poset$ are both cocomplete.*

Proof. Consider an arbitrary I-diagram in $ARRel$, and the colimit in reflexive relations. Then the resulting relation is not necessarily antisymmetric, but reflexive automatically. Make it reflexive by passing to the maximal antisymmetric quotient, in the category with targets only antisymmetric relations, find that this satisfies the universal property of the colimit in $ARRel$.

For an arbitrary I-diagram in $ARTRel$ take the aforementioned colimit in $ARRel$, by the previous observations that colimit is not necessarily transitive on its own, so in addition take the transitive closure, and find that in the category $ARTRel = Poset$ this satisfies the universal property of the colimit. □

Remark 1.4.13. Reemphasise that fact: A limit of posets is just taken componentwise.

A colimit of posets is given as the componentwise setwise colimit of relations and underlying sets, then forcing asymmetry by an equivalence relation, and then extending with transitive closure. In particular, all small limits and colimits in $ARRel$ and $ARTRel$ exist, but colimits can potentially collapse away a lot of the underlying set.

1.5 Total orders

The applications in this paper do not need the full force of well ordering, total orders do just fine, hence totality is the final property of relations to investigate. To make the constructions natural it is necessary to restrict to specific models for sets and relations though.

Definition 1.5.1. Call a relation $R \subset S \times S$ with underlying set S total, if for each pair $s, t \in S$ at least one of sRt or tRs is satisfied. Call the category of total relations with all relation morphisms between them $ToRel$. Analogously define antisymmetric and transitive variants $AToRel$, $ToTRel$, and for both $AToTRel$.

Remark 1.5.2. Note that totality defined in this way trivially implies reflexivity, i.e. $ToRRel = ToRel$.

Lemma 1.5.3. *Limits of total relations considered as reflexive relations are total and preserve antisymmetry and transitivity.*

To avoid the set-theoretic naturality trouble one gets from the axiom of choice in categorical considerations, restrict to smaller categories of relations. The specific size of the sets seems incidental to the results, they are just dictated by the envisioned applications in computational machine code.

Definition 1.5.4. Consider the $\mathbb{N}$-totalised versions of the above categories of relations: An $\mathbb{N}$-totalised relation $R \subset S \times S$ is a relation, i.e. an object of Rel, together with an injective map of underlying sets $\iota_{\mathbb{N}} \colon S \to \mathbb{N}$, which induces a map of relations $(S, R) \to (\mathbb{N}, \leq_{\mathbb{N}})$, i.e. $\forall s, t \in S \colon sRt \to \iota(s) \leq_{\mathbb{N}} \iota(t)$.

Morphisms are!?

Call the category so obtained $Rel_{\mathbb{N}}$.

Proposition 1.5.5. *By the antisymmetry observations above get that for any $\mathbb{N}$-totalised relation the map $\iota_{\mathbb{N}}$ factors uniquely over the maximal antisymmetric quotient of the relation. Hence only antisymmetric relations can even have a totalising map $\iota_{\mathbb{N}}$.*

Proof. A relation is not antisymmetric if and only if there is a pair of unequal elements which are in both directions of the relation to each other, hence $\iota_{\mathbb{N}}$ cannot be injective while factoring over the quotient, because $(\mathbb{N}, \leq_{\mathbb{N}})$ is antisymmetric. □

Proposition 1.5.6. *A relation has a totalising map $\iota_{\mathbb{N}}$ if and only if its reflexive completion has one if and only if its transitive closure has one.*

Proof. Adding "reflexive elements" does not change anything about the map $\iota_{\mathbb{N}}$. Any relation-map can be extended / restricted that way to $\mathbb{N}$ because $\mathbb{N}$ is reflexive. Since $(\mathbb{N}, \leq_{\mathbb{N}})$ is transitive, $\iota_{\mathbb{N}}$ uniquely extends to the transitive closure, and defines a unique map on the original relation. □

The result that every directed acyclic graph has a topological ordering yields a very satisfying converse in this situation:

Theorem 1.5.7. *Let $R \subset S \times S$ be a reflexive, antisymmetric and transitive relation with finite underlying set S, then it has an injective relation preserving map $\iota_{\mathbb{N}} \colon (S, R) \to (\mathbb{N}, \leq_{\mathbb{N}})$.*

Proof. The relations with underlying set the empty set or a one-point set have obvious embeddings to $\mathbb{N}$, hence proceed by induction on the cardinality of S.

Let S have at least two but finitely many elements, claim there is a least one R-minimal element in S, i.e. an element $s \in S$ such that $\forall t \in S \colon tRs \to t = s$. Assume for contradiction that were not true, then each element has an incoming relation from a different element. Proceed inductively backwards along such a chain, which by finiteness of S must cycle eventually. By transitivity and antisymmetry get that the elements all cannot have been different, hence there is an R-minimal element in S. Dually the same is true for maximal elements in a reflexive, antisymmetric and transitive relation.

So consider an R-maximal element $s \in S$, by induction $S \setminus \{s\}$ is a set with fewer elements and an induced reflexive, antisymmetric and transitive relation, so get an injective relation preserving map $\iota_{\mathbb{N}} \colon S \setminus \{s\} \to \mathbb{N}$ and extend to S by setting $\iota_{\mathbb{N}}(s) = max(\iota_{\mathbb{N}}(S \setminus \{s\})) + 1$. This extends $\iota_{\mathbb{N}}$ to an injective map on all of S, which is relation preserving by construction. □

Remark 1.5.8. The finiteness assumption is quite fundamental in this theorem since clearly a countably infinite relation can be constructed that has neither a minimum or a maximum like $\mathbb{Z}$, so the argument breaks down, as does the result. If $\mathbb{Z}$ were possible to include injectively and order-preserving into $(\mathbb{N}, \leq_{\mathbb{N}})$ it would have to have a minimum, which it does not.

Given the definitions the following is a triviality, but that is in fact the intention of what the definition abstracted away.

Proposition 1.5.9. *For each $\mathbb{N}$-totalised relation there exists a unique embedding of the relation into $\mathbb{N}$ with underlying set map $\iota_{\mathbb{N}}$. The image as a subset with the induced $\leq_{\mathbb{N}}$ relation makes the relation-image it into a reflexive, antisymmetric, transitive and total relation. The (in this setting unique) total completion of the relation.*

Lemma 1.5.10. *There is a free-forgetful adjunction between $\mathbb{N}$-totalised ARTRel and ARToTRel:*

$$\iota_* \colon ARTRel_{\mathbb{N}} \leftrightarrow ARToTRel_{\mathbb{N}} \colon U.$$

Proof. The claim is $ARToTRel_{\mathbb{N}}(\iota_* R, T) = ARTRel_{\mathbb{N}}(R, UT)$. I.e. forgetting that a relation is total (when it is already antisymmetric, reflexive, transitive, and injectively mapped to $\mathbb{N}$) and considering all maps of posets that respect the $\mathbb{N}$-totalisation, is the same thing as replacing a totalised order by its totalisation, i.e.

$$\iota_*(R \subset S \times S, \iota_{\mathbb{N}} \colon (S, R) \to (\mathbb{N}, \leq_{\mathbb{N}})) = (\bar{R} \subset S \times S, \iota_{\mathbb{N}} \colon (S, \bar{R}) \to (\mathbb{N}, \leq_{\mathbb{N}}),$$

where $\bar{R}$ is the relation

$$x\bar{R}y :\Leftrightarrow \iota_{\mathbb{N}}(x) \leq_{\mathbb{N}} \iota_{\mathbb{N}}(y).$$

By definition of the $\mathbb{N}$-totalisation this extends the original relation. Hence get for any map from any antisymmetric, reflexive, transitive relation to any relation of the same type that is also total: The map has a unique extension to the totalisation of the source relation, if the target is a total order. □

Chapter 2

2024-03-18 – Directed Graphs, Simplicial Complexes and Simplicial Sets

2.1 DAGs

As a descriptive tool, graphs are invaluable in this paper, hence fix that definition first.

Definition 2.1.1. A directed acyclic graph $G = (V, E)$ is a totally ordered set of vertices V together with an edgelist E, i.e. a chosen subset of non-decreasing pairs of V, which includes all edge-touched vertices v as duplicate tuples (v, v) too:

$$EG \subset \{\ (v, w) \in V^{\times 2} \mid v \leq_V w\ \wedge (v, v) \in EG \wedge (w, w) \in EG\}.$$

A morphism of directed acyclic graphs $\varphi\colon G = (VG, EG) \to (VH, EH) = H$ is a map on vertices $\varphi\colon VG \to VH$, that can be restricted to a map on edges. Clearly $\varphi\colon VG \to VH$ induces a natural map $\varphi \times \varphi\colon VG \times VG \to VH \times VH$, which can be restricted to EG, if it is satisfied that $(\varphi \times \varphi)_*(EG) \subset EH$, call φ a morphism of directed acyclic graphs.

Remark 2.1.2. Since a finite directed graph is acyclic if and only if it admits a topological ordering, this recovers the classical notion on finite graphs. Taking colimits over finite subgraphs recovers the total notion.

Remark 2.1.3. Note however that this definition of morphisms is not the same definition as a lot of algebraic graph theory and other references that use homotopies on graphs have. Explicitly, the above definition of morphism allows to collapse edges onto vertices, and the graphs have no vertex loops. It is a minor detail, but it changes the terminal object, the product structure, etc.

Remark 2.1.4. The notion "directed acyclic graph" is a classic well-established wording, however, the "acyclic" clashes quite unfortunately with the topological applications of this paper, where acyclic has a lot stronger connotations.

As a consequence, in what follows they are only called *dag*s, even just graphs, when it is unambiguous. Undirected graphs, graphs with loops, multigraphs, all such variations are not subject of this paper anywhere, hence there should be no confusion.

2.2 Simplicial Complexes

Start by fixing the most bird's eye view of how a simplicial complex can be defined.

Note that choosing a total order on the vertices is usually not part of the definition. It is however part of every example the reader will ever see. It is of essential use in the identification of dags, complexes and simplicial sets, but plays a silent role in the definition.

Definition 2.2.1. A simplicial complex $X = (VX, \leq_{VX}, \sigma X)$ is a triple consisting of any set of vertices $VX \in Set$, together with a chosen total order $\leq_V X$ on the vertices, and σX a fixed subset of finite subsets of VX, which is closed under taking subsets. I.e. any subset of the vertices VX that is a subset of an element of σX is itself an element of σX:

$$\forall S \subset VX \colon (\exists T \in \sigma X \colon S \subset T) \Rightarrow S \in \sigma X.$$

Call an $(n+1)$-element subset (canonically ordered by restricting $\leq_{VX}$) $(v_0 <_{VX} \ldots <_{VX} v_n) = t \in \sigma X$ an n-simplex of X, and hence call σX the set of X's simplices. In addition assume for each $v \in VX$ that its associated 0-simplex is actually a simples $\{v\} \in \sigma X$.

A morphism of simplicial complexes $\varphi\colon X = (VX, \leq_{VX}, \sigma X) \to Y = (VY, \leq_{VY}, \sigma Y)$, is a set map $\varphi\colon VX \to VY$, which respects the chosen simplex sets:

$$\forall s \in \sigma X \colon s = (s_0 < \ldots < s_n) \Rightarrow \varphi_*(s) = \{\varphi(s_i)\}_{0 \leq i \leq n} \in \sigma Y,$$

where $\varphi_*(s) = \{\varphi(s_i)\}_{0 \leq i \leq n}$ is a subset of the VY vertices, which is potentially strictly smaller than s, if a vertex is repeated.

Remark 2.2.2. Note that assuming vertices are 0-simplices is not a loss of generality. Because for a pair $X = (VX, \sigma X)$, satisfying the other properties, the set $\bigcup \sigma X$ that is the union over all the finite subsets appearing in σX is a reduced vertex set satisfying that assumption.

In particular a morphism of simplicial complexes is uniquely determined by its restriction to either vertices or simplices.

Remark 2.2.3. Given a set map of vertices $\varphi_V \colon VX \to VY$, it induces a map of finite subsets by applying φ elementwise, hence covariantly: $\varphi_* \colon \mathcal{P}(VX) \to \mathcal{P}(VY)$.

For $\sigma X \subset \mathcal{P}(VX)$ and $\sigma Y \subset \mathcal{P}(VY)$, φ is a simplicial map if and only if, restricting to σX has its image in σY, i.e. φ_* induces a well-defined map of the simplices by restriction $\varphi_* \colon \sigma X \to \sigma Y$. Restricting to σX is not a condition, being able to restrict to σY is the salient point.

Remark 2.2.4. Since the content of this paper is about dags, simplicial complexes and simplicial sets, occasionally emphasise them apart as: dags, complexes, spaces.

2.3 Complexes and DAGs

Theorem 2.3.1. *Restricting a simplicial complex X to its vertices and 1-simplices defines a dag $sk_1 X = (VX, \{ e \in \sigma X \mid |e| \leq 2 \}$. Any directed acyclic graph $G = (VG, EG)$ in turn defines a simplicial complex, its flag complex $Flag(G) = (VG, \sigma(G))$ with σG the finite subsets of VG such that for each pair in that subset the edge is part of EG:*

$$Flag(G) := \{ \{(v_0 \leq_{VG} v_1 \leq_{VG} \ldots \leq_{VG} v_n)\} \in VG^* \mid \forall i \leq j : (v_i, v_j) \in EG \}.$$

Each of these are functors, i.e. maps of simplicial complexes induce dag-maps on the 1-skeleton and maps of dags induce simplicial complex maps on the flag complexes, respecting composition and identities covariantly.

Proof. Given a simplicial complex $(VX, \leq_{VX}, \sigma X)$ define the vertex set as just the same vertex set VX and the edgelist as the simplices of dimension less than or equal to 1. The total order on VX makes these simplices into non-decreasing tuples unambiguously. Clearly a simplicial map in particular induces a map of the 1-skeletons, since it at most reduces degrees, but never increases a simplex degree. Hence follows also the composition and identity property.

Given a dag $(VG, \leq_V G, EG)$ define the vertex set as just the same vertex set VG, and for each sequence of edges that are transitively closed in EG add a simplex of appropriate size in $\sigma Flag(G)$:

$$\sigma Flag(G) := \{ \{v_0 \leq_V Gv_1 \leq_V G \ldots \leq_V Gv_n\} \mid \forall i \leq j : (v_i, v_j) \in EG \},$$

where $\{v_0, v_1, \ldots, v_n\}$ is the set underlying the tuple $(v_0, \ldots, v_n)$, which could potentially have fewer elements than the tuple because our edgelist includes the vertices too (v, v). In particular it is trivially satisfied that $\sigma FlagG$ is closed with respect to subsets. The tuple $Flag(G) = (VG, \leq_V G, \sigma FlagG)$ hence defines simplicial complex.

A map of vertices that respects edges clearly induces a map of flag complexes, the identity induces the identity, composition is respected covariantly. □

Theorem 2.3.2. *Taking the 1-skeleton of a Flag complex of a dag naturally is exactly the same dag set wise. I.e.*

$$sk_1 \circ Flag = id_{DAG}.$$

Proof. Consider a dag (VG, EG), then its flag complex consists of simplices the vertices VG and flags defined by EG. The 1 skeleton of the flag complex consists of the vertices, which are VG, and the 1-flags, i.e. pairs of vertices (v_0, v_1) such that $(v_0, v_1) \in EG$, so that is exactly EG. □

Theorem 2.3.3. *The Flag complex of the 1-skeleton of a simplicial complex has a natural inclusion from the original complex. I.e.*

$$\exists \eta : Id_{sCx} \Rightarrow Flag \circ sk_1.$$

Proof. A map of simplicial complexes is uniquely determined by what it does on vertices. Do note that $Flag(sk_1(X))$ and X have the same vertices by definition, hence it remains to check, that the identity induces a simplicial map, i.e. respects simplices.

Given any simplex $s = (s_0 < \ldots < s_n) \in \sigma X$, note that all the subsets (s_i, s_j) are part of the 1-skeleton, hence part of the edges of the 1-skeleton. So by constructing the flag complex s is in particular an element of $\sigma Flag(sk_1(X))$. □

Example 2.3.4. Be very aware that this is usually a proper inclusion. Consider the simplicial complex $(\{0,1,2\},\{0,1,2,01,02,12\})$. Reducing to the 1-skeleton is basically the same pair, but extending via flag complex induces the simplex $\{012\}$ that was not there before.

In particular from a topological point of view this is the inclusion of a circle into a disc, so very much not an equivalence in any desirable sense.

There is however a big class of simplicial complexes for which the reverse direction is the identity too, for which the following definition is needed.

Definition 2.3.5. Given a simplicial complex $(VX, \leq_V X, \sigma X)$, define its subdivision $sd(X)$ as follows, the vertices are $V_{sd(X)} = \sigma X$, with total order induced by ordering the (finite!) tuples first by length, then order ordered tuples lexicographically according to $\leq_V X$, inducing a total order on $V_{sd(X)}$. The simplices are all chains of vertices $(v_0, v_1, \ldots, v_n) \in (\sigma X)^{n+1}$ such that $v_i \subset v_j \forall i \leq j$.

Subdivided simplicial complexes actually have a defining dag.

Proposition 2.3.6. *The subdivision of a simplicial complex is in fact the flag complex of a defining dag on vertices σX, totally ordered as above, and edges included according to subset relation, identity in particular.*

Proof. Note the ordering of vertices of $sd(X)$ implies that each edge of $sd(X)_1$ is in fact non-decreasing with respect to that length and lexicographic order. The vertices of that graph are the totally ordered set σX. It is easy to see that the flag complex of this graph is exactly the definition above. □

Hence because one side of the composition strictly equals the identity get that:

Proposition 2.3.7. *On the category of simplicial complexes which are at least once subdivided the flag complex of the 1-skeleton yiels the exact same complex $Flag \circ sk_1 \circ sd = id_{sCx} \circ sd$. I.e. the whole simplicial complex can be exactly setwise recovered from its vertices and edges.*

In fact the category of simplicial complexes which are equal to their flag complex is far bigger, one has to exclude only the above mentioned example in a universal manner.

Proposition 2.3.8. *Let $(VX, \leq_V X, \sigma X)$ be a simplicial complex. Assume additionally it satisfies on 1-simplices*

$$\forall u \leq_{VX} v \leq_{VX} w \in VX \colon (u,v),(v,w),(u,w) \in \sigma X \Rightarrow (u,v,w) \in \sigma X.$$

Call the simplex (u,v,w) a witness for the transitivity of u,v,w.

Then X is the flag complex of its 1-skeleton.

Proof. By induction the aforementioned condition in fact implies that every such flag of vertices with pairwise edges between them gives rise to a simplex in σX already. In particular, every simplex of σX is of that form. Hence X is the flag complex of its 1-skeleton, its edgelist. □

Corollary 2.3.9. *The subdivision satisfies the above condition, hence the identity* $Flag(sk_1(sd(\bullet))) = sd(\bullet)$ *follows as a corollary to the above proposition, too.*

Example 2.3.10. Note that there are basically two ways in which a simplicial complex can satisfy the condition. It is about either thin paths or transitive closures. Consider one extreme:

$$(0123, \{0, 1, 2, 3, 01, 12, 23, 03\}),$$

this is a path from 0 to 3 together with a direct shortcut edge 03. It is the flag complex of its 1-skeleton, since for no two pairs of composable edges $(u, v), (v, w)$ there is a third pair (u, w) which would necessitate a 2-simplex witnessing the transitivity (u, v, w).

On the other extreme consider a full simplex, i.e.

$$(012\ldots n, \mathcal{P}(\{0, 1, \ldots, n\})).$$

Then obviously for each triple of edges that should be transitive, there is a 2-simplex witnessing that transitivity, so the flag complex introduces no new simplices.

2.4 Spaces - Simplicial Sets

Introduce simplicial sets in a manner that is compatible with how dags and complexes are fixed.

Definition 2.4.1. Fix a pair of a sequence of totally ordered sets indexed over finite totally ordered sets $X = ((X_S)_{S \in Fin_{\leq}}, \{\phi^*\})$ together with for each map of finite totally ordered sets $\phi\colon (S, \leq_S) \to (T, \leq_T)$ a chosen "induced map" in the opposite direction $\phi^*\colon X_T \to X_S$.

If the chosen ϕ^* satisfy the contravariant functoriality condition

$$\phi^*\psi^* = (\psi \circ \phi)^* \quad \wedge \quad id_S^* = id_{X_S},$$

call the pair of sets and induced maps $X = ((X_S)_{S \in Fin_{\leq}}, \{\phi^*\})$ a simplicial set or a space, and the elements $x \in X_S$ the S-simplices of X.

A morphism of simplicial sets $f\colon X \to Y$ is a degreewise set-map $f_S\colon X_S \to Y_S$, i.e. a map of S-simplices for each finite totally ordered set S, which is compatible with the induced maps ϕ^* on X and Y, i.e. $\forall S, T \forall \phi_{\leq}\colon S \to T, f_S \circ \phi^* = \phi^* \circ f_T$.

Remark 2.4.2. Note that the choice of morphisms legitimises writing ϕ^* instead of the more explicitly descriptive ϕ^*_X and ϕ^*_Y.

Note also again, just like for dags and complexes, each set of S-simplices is chosen to be totally ordered, but it did not enter the definition in any other way. It serves to set up the correspondence between simplicial sets and simplicial complexes.

Definition 2.4.3. For a simplicial set $X = ((X_S)_S, \{\ \phi^*\colon X_T \to X_S \mid \phi \in Fin_{\leq}(S, T)\ \}_{S,T}$ and $x \in X_S$ any S-simplex and $i\colon \bar{S} \to S$ a proper injection of finite totally ordered sets, call such a i^*x a face of x. For $x \in X_S$ any S-simplex and $p\colon \bar{S} \to S$ a proper surjection call $p^*x \in X_{\bar{S}}$ a degeneracy of x.

For the specific finite totally ordered sets $[n] := \{0, \dots, n\} \subset (\mathbb{N}, \leq_{\mathbb{N}})$ ordered as subsets of the natural numbers, simply call the $\{0, \dots, n\}$-simplices n-simplices, and write: $X_n := X_{\{0,\dots,n\}} = X_{[n]}$.

Remark 2.4.4. Note that a simplicial set is uniquely determined by its n-simplices given for each $n \in \mathbb{N}$, since each finite totally ordered set has a unique isomorphism to a set of the form $[n]$. Hence the induced maps of the simplicial provide the rest of the structure on arbitrary finite totally ordered sets. In particular, it is natural to assume the total order on n-simplices is the same as on S-simplices for all other n-element sets S identified along the canonical isomorphism $(S, \leq_S) \to ([n], \leq_{\mathbb{N}})$. Hence a simplicial set can be defined only by defining its n-simplices.

2.5 Complexes are Spaces - Simplicial Complexes are special Simplicial Sets

2.5.1 Simplicial Complexes are Simplicial Sets

This is how the total orders enter, with these definitions a simplicial complex is just a special type of simplicial set.

Lemma 2.5.1. *Each simplicial complex X has a unique naturally associated simplicial set sX, in particular, morphisms of simplicial complexes naturally induce morphisms of their associated simplicial sets.*

Proof. Let $X = (VX, \leq_V X, \sigma X \subset \mathcal{P}(VX))$ be a simplicial complex, i.e. VX the set of vertices, totally ordered by $\leq_V X$ and σ_X a subset-closed subset of the finite subsets of X-vertices VX.

Define a simplicial set sX associated to X as follows on n-simplices only:

$$sX_n := \{ \ s \in VX^{\times(n+1)} \mid \forall i \leq j \colon s_i \leq s_j \wedge \{s_i\}_{0 \leq i \leq n} \in \sigma X \ \}.$$

I.e. n-simplices are VX-ordered $n+1$-tuples s of VX, such that the set with duplicates removed $\{s_i\}_{0 \leq i \leq n}$ is part of the original σX simplices of X.

To define the induced maps for X, given a map of finite totally ordered sets $\phi\colon [m] \to [n]$, and an element $s \in sX_n = \{ \ s \in X^{\times(n+1)} \mid \forall i \leq j \colon s_i \leq s_j \wedge \{s_i\}_{0 \leq i \leq n} \in \sigma X \ \}$. By applying the set-map ϕ on indices $s = (s_0 < \ldots < s_n)$ define $\phi^*(s) = \{s_{\phi(i)}\}$, i.e. the subset of σX defined by taking the ϕ image of s according to its tuple/$\leq_V X$-indices.

It easily follows that $id^* = id$ and $(\psi\varphi)^* = \varphi^*\psi^*$, hence sX is a simplicial set. Furthermore a morphism of simplicial complexes evidently induces a compatible morphism of these simplicial sets. The identity if started with the identity, and respecting compositions covariantly. ☐

2.5.2 Simplicial Complexes are a specific subcategory of Simplicial Sets

The fun part is to go back! For that as a first step identify which simplicial sets come from simplicial complexes.

Note that the condition on simplicial sets basically recalls the well-known notion that a morphism of simplicial complexes is uniquely determined by what it does on vertices, which is part of our definition too.

Theorem 2.5.2. *Let* $X = ((X_S)_S, \{ \ \phi^* \colon X_T \to X_S \mid \phi \in Fin_{\leq}(S,T) \ \}_{S,T}$ *be a simplicial set.*

If the simplices of X *satisfy that having all faces equal already implies equality for each pair of simplices, then* X *is a simplicial set constructed like above. I.e. if for all* $x, y \in X_S$ *and for all proper injections* $i \colon T \to S$ *the simplices* x *and* y *satisfy* $i^*x = i^*y$*, then* X *is naturally isomorphic to a naturally associated simplicial complex* $\bar{X}$ *with associated simplicial set* $s\bar{X} \cong X$*.*

Proof. For X a simplicial set define the vertex set $VX = X_0 = X_{\{0\}}$ the 0-simplices of X. Induce the total order from the total order on 0 simplices.

For $j_i \colon 0 \to n$ the (trivially order preserving) map that sends 0 to i, defined for each $0 \leq i \leq n$, note that the uniqueness condition means, one can uniquely describe an n-simplex $x \in X_n$ with the set of its vertices $(j_0^*x, j_1^*x, \ldots, j_n^*x)$. Note however that not every such tuple of vertices is a simplex in X obviously.

So define $\sigma\bar{X} := \{ \ \{(j_0^*x, j_1^*x, \ldots, j_n^*x)\} \in VX^{\times n} \mid \forall n \in \mathbb{N}, x \in X_n. \ \}$. I.e. define the simplicial complex $\bar{X}$ to be given by the 0-simplices of X as vertices, totally ordered as in X, and higher simplices are exactly the subsets of vertices that can be written as induced by j_i tuples like above. Since the set $\{j_i\}$ is a generator in the category of finite totally ordered sets, it actually implies $i^*x = i^*y$ if $(j_0^*x, j_1^*x, \ldots, j_n^*x) = (j_0^*y, j_1^*y, \ldots, j_n^*y)$, because each injection can be reconstructed from including points one by one inductively.

Clearly the subset condition on $\sigma\bar{X}$ is satisfied, by considering restriction of a simplex along inclusions $[n-1] \to [n]$, i.e. faces.

Consider the simplicial set that $\sigma\bar{X}$ generates according to the lemma immediately above. Clearly on vertices the identity is a natural map on level 0, and note that the condition implies exactly that one can uniquely reconstruct any n-simplex from its vertex-tuple like above, so the map of simplicial sets

$$j_* \colon X \to s\bar{X}$$

defined by sending an n-simplex x to $(j_0^*x, j_1^*x, \ldots, j_n^*x)$, is evidently compatible with the respective induced maps from finite ordered sets, because set maps are defined pointwise. It is injective in each degree by assumption and trivially surjective by construction of $\sigma\bar{X}$. Hence X is naturally isomorphic to the simplicial set of the simplicial complex $\bar{X}$, thus follows the claim. □

Corollary 2.5.3. *This implies in particular that for a simplicial set, which is a simplicial complex in the sense of the above theorem, all its simplicial subobjects are automatically simplicial complexes too, because the condition gets easier to satisfy on subsets.*

Remark 2.5.4. Note that the correspondence over the maps j_i allows to assume without loss of generality that simplices of a simplicial complex are given as such tuples of vertices. For general simplicial sets a priori this is not a requirement and also not helpful, because simplicial sets are precisely about allowing multiple simplices with the same faces, hence they can usually not be represented like this. In particular it follows in the category of simplicial sets too:

Theorem 2.5.5. *Any morphism from the simplicial set associated to a simplicial complex to any other simplicial set is uniquely determined by what it does on vertices / 0-simplices.*

2.6 Simplicial Sets can be resolved by a pair of Simplicial Complexes

Theorem 2.6.1. *Let* $X = ((X_S)_S, \{\ \phi^* \colon X_T \to X_S \mid \phi \in Fin_{\leq}(S,T)\ \}_{S,T}$ *be a simplicial set. Then there is an inclusion of simplicial complexes* $A \to Z$*, which as a quotient in simplicial sets exactly yields* X*.*

Proof. By (if necessary transfinite) induction, assume finitely many non-degenerate simplices in X. I.e. consider each simplicial set as the colimit over its subsets with finitely many non-degenerate simplices, the finite steps that follow still apply consistently.

□

2.7

Theorem 2.7.1.

$$sFlag(sk_1(sd^2(X))) = sd^2 X$$

for all simplicial SETS X*.*