Chapter 1

The skeletal 2-category of dags

1.1 The fundamental 2-category

In the spirit of module categories typically appearing in K-theory constructions, define a 1-category of directed acyclic graphs with very similar properties. Notice that a morphism of directed acyclic graphs which is fixed on vertices can either exist or not, but there is no freedom involved to allow two morphisms between that same source and target graph. The resulting category is hence a literal 1-category.

Definition 1.1.1. The skeletal category of finite embedded directed acyclic graphs, denoted *dag* has as objects the natural numbers.

The morphisms from n to n are directed acyclic graphs on the vertex set $\{1, \dots, n\}$ which respect $\leq_{\mathbb{N}}$, i.e. an edge in a directed acyclic graph is only ever non-decreasing with respect to the natural order of natural numbers. Interpreted differently a morphism in *dag* is a square matrix

$$A = (A_{ij}) 1 \leq i, j \leq n,$$

such that $A_{ii} = 1$ and $A_{ij} = 1 \Rightarrow 1 \leq i \leq j \leq n$. I.e. it is upper triangular, and each diagonal element is set to 1.

Set composition to be $(B \cdot A)_{ij} = min(max(k|B_{ik}A_{kj}))$. I.e. set each entry to be defined as: If there is some middle k connecting i in B to k, and then there is a j, such that k is connected in A to j, then i, j is an edge in $B \cdot A$.

Proposition 1.1.2. *This is in fact a (small) category.*

Proof. The existence of identities can be seen by setting id_n to be the matrix with diagonal elements 1 and rest 0, then follows $A \cdot id = id \cdot A = A$ for any A and id of compatible degree.

To check that $B \cdot A$ is well-defined, check if $B \cdot A$ in fact respects the $\leq_{\mathbb{N}}$ relation still. Given an edge $(i, j) \in B \cdot A$, there exists a vertex k, such that $(i, k) \in B$ and $(k, j) \in A$. Because both A and B are $\leq_{\mathbb{N}}$-compatibly directed, find $i \leq k$ and $k \leq j$, hence $i \leq j$, so (i, j) is a legal edge in $\leq_{\mathbb{N}}$.

For associativity note that the triple product $C \cdot B \cdot A$ has the natural "unbracketed" interpretation like above. It is a dag on the same vertex set as the three other graphs. It has edges the pairs of vertices x, y such that x has an edge to another vertex via C, that vertex has one to another in B and finally that one has one via A to y. There is nothing ineherently prioritised about the pairings, so it is associative. $\square$

Example 1.1.3. Note in particular that multiplying a dag with itself $A \cdot A \cdot \ldots \cdot A$ when interpreted with 1's on the diagonal as above, eventually yields the transitive closure of A by the above reasoning.

Proposition 1.1.4. *For $[n] = \{1, \ldots, n\}$ with $[0] = \emptyset$, consider the following bijections: The sum-bijection*

$$\sigma_{n,m} \colon [n] \sqcup [m] \to [n+m]$$

with $\sigma(k) = k$ for $k \in [n]$ and $\sigma(k) = k + n$ for $k[m]$. The product-bijection

$$\omega_{n,m} \colon [n] \times [m] \to [nm]$$

with $\omega_{n,m}(k,l) = (k-1)m + l$.

then? (relations listen, bip folgern fuer dag, H ausrechnen, fertig)

1.2 The fundamental bipermutative two element category

1.3 DAGs are a module category