

# Chapter 1

## 2024-04-20 – The basic Hopf fibre bundles

### 1.1 Explicit construction

**Definition 1.1.1.** For  $k = \mathbb{R}^n$  assume a multiplication  $\mu: k \otimes_{\mathbb{R}} k \rightarrow k$  making  $k^\times \supset \mathbb{S}(k) \cong \mathbb{S}^{n-1}$  into an  $H$ -space. Then we can construct a fibre bundle as follow: Consider the unit sphere in  $\mathbb{S}(k \times k)$ , which can be identified as  $\mathbb{S}(\mathbb{R}^{2n}) = \mathbb{S}^{2n-1}$ , and the one-point compactification of  $\mathbb{S}^k = \mathbb{S}^{dim_{\mathbb{R}} k} = \mathbb{S}^n$ .

Notice that for  $\mathbb{S}(k^2)$  the components can be written as pairs of elements of  $k$  with norms each less than or equal to one, and norm-squares summing to one. Thus consider the map:

$$\begin{aligned} m_k: \quad \mathbb{S}^{2n-1} = \mathbb{S}(k \times k) &\rightarrow \quad \mathbb{S}^k \\ a, b &\mapsto \quad a \cdot b^{-1}, \end{aligned}$$

where  $(\bullet) \cdot b^{-1}$  has to be defined as  $\infty$  for  $b = 0$ , (note that this implies  $|a| = 1$ , so  $a \neq 0$ .)

**Proposition 1.1.2.** *This is a fibre bundle for  $k = \mathbb{R}, \mathbb{C}, \mathbb{H}$ , with fibre  $F = \mathbb{S}(k) = \mathbb{S}^0, \mathbb{S}^1, \mathbb{S}^3$ .*

*Proof.* Consider first the case  $k = \mathbb{R}$ . The case  $k = \mathbb{R}$  can be identified as  $\mathbb{Z}/2 \rightarrow \mathbb{S}^1 \rightarrow \mathbb{S}^1$ , where the first circle is a unit sphere in  $\mathbb{R}^2$  and the second circle is the one-point-compactification of  $k = \mathbb{R}$ . Let  $t \in \mathbb{R}^+$  with  $|t| < 1$ , it defines a subset of  $\mathbb{S}(\mathbb{R} \times \mathbb{R})$  with

$$U_t := \{ (z, u) \in \mathbb{S}(\mathbb{R} \times \mathbb{R}) \mid u = t \wedge |z|^2 + |t|^2 = 1 \} \cong \{ z \in D\mathbb{R} \mid |z| = \sqrt{1 - |t|^2} \}.$$

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