Chapter 1

VR complexes, Flags and DAGs

1.1 basics

Definition 1.1.1. DAG - vertices = $[n]$

Definition 1.1.2. VR - $VR_\varepsilon(X)$ wrt max norm

Proposition 1.1.3. $VR = Flag(sk_1(VR(X)))$

1.2 Each dag is a VR complex

Definition 1.2.1. A VR embedding of a directed acyclic graph $G = (VG, EG)$ is a collection of points $X = \{x_1, \dots, x_{|G|}\}$ and a fixed $\varepsilon > 0$, with a map $\iota_G \colon VG \to \mathbb{R}^2$ with image X, such that the Vietoris-Rips-Complex recovers the graph:

$$\begin{aligned} VR_\varepsilon(X) &= \{\ (x_i, x_j) \in X^2 \mid i \leq j, d(x_i, x_j) < \varepsilon\ \} \\ &\cong (VG, EG). \end{aligned}$$

Lemma 1.2.2. *For a disconnected graph $G = H_1 \sqcup H_2$ which has a VR embedding for each summand H_i, the sum G has a VR embedding as well.*

Proof. Given two such embeddings by the assumptions one only has to make sure no edges are introduced between H_1 and H_2. To do so consider their isomorphisms $VR_{\varepsilon_1}(i_1 H_1) \cong H_1$ and $VR_{\varepsilon_2}(i_2 H_2) \cong H_2$. One convinces oneself easily that by scaling just one of the embeddings, it can be arranged that $\varepsilon := \varepsilon_1 = \varepsilon_2$ without loss of generality. By embedding both (compact, hence bounded) images in $\mathbb{R}^2$ at more distance than ε, a VR complex with the same or smaller $\bar{\varepsilon}$ recovers the disjoint sum $G = VR_\varepsilon(i_1 H_1 \sqcup i_2 H2)$ for $\sqcup$ just a suggestive notation for the embedding of the summands at "great distance". $\square$

Lemma 1.2.3. *Every fully connected graph has a VR embedding.*

Proof. Count the vertices $G = \{1, \dots, n\}$ with $n \geq 0$ and embed them as $X = \{(i, 0) | 1 \leq i \leq n\}$. Then $VR_{n+\varepsilon}(X) \cong G$ for any $\varepsilon > 0$. $\square$

Lemma 1.2.4. *Consider a subgraph in a graph $M \subset G$ defined by a restricted vertex set VM and all induced edges of G. Additionally for each partition of $VG = V_1 \sqcup VM \sqcup V_2$, consider the induced subgraphs $G_1 = G[V_1 \sqcup VM]$ and $G_2 = G[V_2 \sqcup VM]$.*

If G_1 and G_2 each already see all of G's edges, i.e. if $EG_1 \cup EG_2 = EG$: Then G is the categorical pushout of G_1 and G_2 over M.

Proof. Given a map $G \to Z$ for any graph Z there clearly are uniquely induced maps restricted to G_1 and G_2, which are compatible over M.

Given a pair of maps $\varphi_i \colon G_i \to Z$ that agree on M, show that there is a unique map $\varphi \colon G \to Z$ inducing the φ_i.

By assumption on the vertex sets φ is clearly uniquely determined, so one has to show that the map induced by the map on vertex sets is in fact a dag map. This is true if and only if there are not any edges connecting V_1 and V_2, otherwise one can universally construct an example of a dag map on both subsets, agreeing on the intersection, that breaks that edge. By assumption on the edge-sets, this cannot happen, any edge in G is either part of G_1 or G_2 or both, hence of M. So all edges of $G \to Z$ in fact are edges in Z as well, hence get the unique induced map of the pushout. □

Lemma 1.2.5. *Let $G = VR_1(X)$ for some finite point set $X \subset \mathbb{R}^2$, then X can be perturbed in such a way that $EG = \{\ (v, w) \in X \times X \mid v = w \vee (\varepsilon_l < d(v, w) < \varepsilon_r \wedge d(v, w) = |v_x - w_x|)\ \}$. In words: The edge set are the pairs of nodes which are close enough, and all of them have their maximal distance to each other in the x-component.*

Proof. If G is arranged so that all distances are instead realised by y-components, just transpose $\tau \colon \mathbb{R}^2 \to \mathbb{R}^2$. The resulting image satisfies the x-component condition then. If G is arranged already satisfying the conditions, there is nothing to prove. Hence consider the set of all vertices such that all their distances to other vertices are realised in each case by the x-component, the sets of all vertices which realise each case by their y-component, and the set of vertices which are in neither other set, i.e. which have one vertex which realises their distance with x and another with y.

□

Corollary 1.2.6. *A VR embedding of a graph can always be modified to have its vertices embedded in strictly y-coordinate-increasing order.*

Proof. This is clear for the empty graph, the one-point graph, any graph on two points. So assume the graph has at least 3 nodes which are already without loss of generality enumerated points in $\mathbb{R}^2$: $VG = X = \{\ (x_i, y_i) \in \mathbb{R}^2 \mid 1 \le i \le n\ \}$ with edgelist $EG = \{\ (v, w) \in X \times X \mid d(v, w) < 2\ \}$. I.e. the vertex set is a set of points in the 2d plane with an edge between two points if and only if their distance is less than 2.

Choose the distance $d(v, w) = \|v - w\|_\infty$, which is also no loss of generality since $\mathbb{R}^2$ is a finite-dimensional vector space. This results in the edge list becoming:

$$EG = \{\ ((v_x, v_y), (w_x, w_y)) \mid 0 \le |v_x - w_x| < 2 \wedge 0 \le |v_y - w_y| < 2\ \}.$$

Since the graph is finite there is also a non-zero minimum distance of the nodes

$$\forall v, w \in X \colon v \neq w \Rightarrow d(v, w) > \varepsilon.$$

By rescaling again assume without loss of generality the edge list is:

$$\begin{aligned} EG = \{ & ((v_x, v_y), (w_x, w_y)) \mid \\ & 0 \leq |v_x - w_x| < K \\ & \wedge 0 \leq |v_y - w_y| < K \\ & \wedge 1 < min(|v_x - w_x|, |v_y - w_y|) \ \} \cup \{(v, v) | v \in X\} \end{aligned}$$

for some natural number $K \in \mathbb{N}$.

If the index maximal node in X is already the highest y-coordinate in the embedding, inductively rearrange the nodes below it, and the claim follows.

Hence assume the maximal node is not y-maximally embedded, and fix that. Cover G as $G \setminus y \cup \bar{y}$ for $\bar{y}$ the graph on vertices X with edges

$$E\bar{y} = \{ \ (a, b) \in X \mid b = y \wedge a \neq y \wedge 0 \leq \|a - y\| < K \wedge 1 < min(|a_x - y_x|, |a_y - y_y|) < K \ \}.$$

□

Corollary 1.2.7. *A VR embedding of a graph can be modified to have the vertices embedded in any arbitrary order.*

Proof. Relabel the graph according to the vertex permutation and direct it according to the new labels. By the result above it can be embedded in increasing order, but the permutation is arbitrary, hence follows the result. □

Remark 1.2.8. The crucial point to inductively construct VR embeddings for all graphs is the following lemma.

Lemma 1.2.9. *Given a VR embedding of a graph $G \to \mathbb{R}^2$, and a binary partition of the vertices $VG = V_1 \sqcup V_2$, the embedding can be perturbed so that the vertices satisfy:*

$$\forall v \in V_1 \forall w \in V_2 \colon d(v, w) > \varepsilon,$$

for some $\varepsilon > 0$.

Proof. Assume without loss of generality the graph is a VR complex with radius 1: $VR_1(X)$ with □

Theorem 1.2.10. *Each dag G with finitely many vertices is a $sk_1(VR(X))$.*

Proof. I.e. it is to show that for each finite dag G there is a $2d$ point configuration of finitely many points with $sk_1(VR(X)) = G$. If that is to be true, assume wlog $VG = X$. By compactness assume wlog $X = \{ \ (x_i, y_i) \in [0, n] \times [0, 1] \mid 1 \leq i \leq |X| = n, \ \}$ such that $min_i(x_i) = x_0 = 0$ and $max_i(x_i) = x_n = n$.

For $G = (X, EG)$ with $|X| \leq 3$ it is easy to see all dags are VR complexes: The empty graph comes from the empty set, the one-point graph from any one-point set, the disconnected graph on two points can be realised with X satisfying the above with $\varepsilon < \frac{1}{2}$ and the connected graph on two points can be realised with $\varepsilon > \frac{1}{2}$. For $|X| = 3$ the maximally connected case is a triangle, so arrange $X = \{(0, 0), (0.5, 0), (1, 0)\}$, then for $\varepsilon > 1$ the complex $VR_\varepsilon(X)$ is fully connected on two vertices. If there is any edge not connected in the original graph on 3 vertices, one can see how to push two $\mathbb{R}^2$ points apart so that they stay connected to a third point, but not to each other. If there is another edge missing, push the third point closer to one extremal point, so it leaves the range of the other point. If there are not any edges in the original graph..

Continue by induction, consider the $x \in X$ with the maximal $pr_0(x)$-value, call it $x = (x_0, x_1)$.

$\square$