Definition 0.0.1. Consider the category of commutative rigs $Crig$. An object is a crig, that is "ring without negatives", i.e. a tuple $(A, +, \cdot)$ with A a set, maps $+\colon A \times A \to A$, $\cdot\colon A \times A \to A$, which are associative and distributive in the usual way as for rings. However $(A, +)$ and $(A, \cdot)$ are only required to be an associative and commutative composition, neutral elements not required.

A morphism of crigs $\varphi\colon A \to B$ is a set map $A \to B$, which is additive and multiplicative morphism with respect to $+$ and $\cdot$.

This observation is a triviality, but a very useful one in absence of zero and unit requirements.

Proposition 0.0.2. *A map of crigs maps idempotents to idempotents with respect to both addition and multiplication. I.e.*

$$\forall e \in A, \varphi\colon A \to B\colon e^2 = e \Rightarrow f(e)^2 = f(e).$$

Lemma 0.0.3. *If a crig A has a zero element, i.e. $(A, +, 0)$ is a commutative monoid, then the existence of a surjective crig morphism $\varphi\colon A \to B$ for an arbitrary crig B, implies B has a zero too.*

Proof. Surjectivity implies that for each test case $0 + b$ there is a preimage $\bar{b}$ witnessing $0 + \bar{b} = \bar{b}$ and implying $0 + b = b$ in B. $\square$

Cancellation properties eventually obtain zero and one as consequences rather than choices in crigs.

Definition 0.0.4. An associative and commutative composition $(A, +)$ is said to **satisfy cancellation**, if

$$\forall a, b \in A\colon \exists k \in A\colon a + k = b + k \Rightarrow a = b.$$

That is, no two elements share an additive translate by the same element. This can be true multiplicatively in a crig, too, for the monoid $(A \setminus \{0\}, \cdot, 1)$, because $0a = 0\ \forall a \in A$.

A commutative monoid $(A, +, 0)$ with the cancellation property we call **strongly non-negative**, if

$$\forall a, b \in A\colon a + b = 0 \Rightarrow a = b = 0.$$

That is, for each element in A its negative is not already an element of A, unless the element is the zero element. This seems to not be too useful multiplicatively.

Lemma 0.0.5. *The trivial crigs $\emptyset, 0$ both satisfy cancellation vacuously. The trivial crig 0 satisfies strong non-negativity with respect to both addition and multiplication vacuously.*

In any cancellation crig A we have $0 = 0 \cdot 1$, hence $0 = 0 \cdot a$ for all $a \in A$, by $(A, \cdot, 1)$ being a monoid.

Proof. The first two observations are logical trivialities, for the last statement: Assume $A \supset \{0, 1, a, b\}$ not all of them necessarily distinct, but not all of them equal by assumption that one can choose two distinct elements in A. Note that it follows by distributivity in a crig:

$$0 \cdot 1 = (0 + 0) \cdot 1 = 0 \cdot 1 + 0 \cdot 1$$

which by cancellation and existence of 0 implies:

$$0 = 0 \cdot 1$$

since both have the same image after translation with $0 \cdot 1$. Since 1 is a unit we can write any a as $1 \cdot a$ and by associativity all is proved. $\square$

Remark 0.0.6. One can stumble into the strongly non-negative property in various ways. It could also be considered a very strong form of torsion-freeness. In case of $\mathbb{N}$ both perspectives coincide, so it does not matter for our study.

I do not expect this property to be of much use beyond $\mathbb{N}$ and its polynomial crigs, but bipermutative rig categories with objects $\mathbb{N}$ come up easily in K-theory constructions, so that case is quite enough motivation to abstract away into an important property like this.

Proposition 0.0.7. *There is a chain of crigs of natural numbers $\mathbb{N}_{\geq 2} \subset \mathbb{N}_{\geq 1} \subset \mathbb{N}_{\geq 0} =: \mathbb{N}_0 \subset \mathbb{Z}$, and there is no crig map in the reverse direction for any of these inclusions apart from the zero map in the last step.*

Proof. $\square$

Remark 0.0.8. Since the existence of negatives from the onset of a problem is the exception rather than the rule in this text, we shall rarely refer to groups as such. Specifically only commutative groups feature as completions or other monoid quotients which just happen to have additive inverses.

Example 0.0.9. THE example of study in this text are the natural numbers $\mathbb{N}$. Note that the definition left a tiny amount of wiggle room by not requiring a zero element, hence the set of positive natural numbers $\mathbb{N}_{>0} = \{1, \ldots, n, n+1, \ldots\}$ has a unique natural crig morphism to every other crig. It is given by unitality and induction over the fact that each natural number greater or equal to 2 can be written as a sum of two smaller numbers. In particular the natural numbers including zero $\mathbb{N} = \mathbb{N}_{\not 0}$ have a unique crig morphism $\mathbb{N}_{>0} \to \mathbb{N}_0$, but there is no crig map in the reverse direction: Any image of zero $f0$ would have to satisfy $f(0) + f(0) = f(0+0) = f(0)$, but this is not possible in positive natural numbers.

The specific argument identifies a useful property the natural numbers but many other crigs enjoy too, cancellation.

Proposition 0.0.10. *Each crig A has an associated crig of polynomials in one variable $A[X]$, satisfying the usual universal property: There is a crig inclusion $A \to A[X]$, and for each crig morphism $\varphi \colon A \to B$ with target crig B and each target element $b \in B$ there exists a unique extension of φ given by "evaluating X at b":*

$$\begin{tikzcd} A \arrow[r] \arrow[dr, "\varphi"'] & A[X] \arrow[d, "ev_b"] \\ & B. \end{tikzcd}$$

Proof. $\square$