Chapter 1

Counting DAGs

1.1 Basic Facts and Definitions

I recall the definition of directed acyclic graphs informally here, in favour of providing a more precise definition tailored to this paper's use case later.

Definition 1.1.1 (Directed Acyclic Graph)**.** A directed graph $G = (V, E)$ with $E \subset V \times V$, is called acyclic, if there is no (finite) chain of vertices $(v_0, v_1, \ldots, v_n)$ with $v_i \in V, 1 < n, (v_i, v_{i+1}) \in E$, such that at least one of the vertices v_i has been visited twice:

$$G = (V, E \subset V \times V) \quad \text{dag}$$
$$:\leftrightarrow \forall n \geq 2 \forall (v_0, v_1, \ldots, v_n) \in V^{\times n} \colon (\forall i \colon (v_i, v_{i+1}) \in E \Rightarrow \forall i \neq j \colon v_i \neq v_j).$$

Do note in particular that despite the fact that this is an infinite schema of axioms, it is finite for each specific finite digraph. More specifically, it could be coded to check an arbitrary digraph for dag-ness. In this text I will not use this however, our digraphs will come as dags plain by their code-friendly presentation.

Proposition 1.1.2 (Sources and Sinks exist)**.** *Any finite dag has a vertex that has no outbound edges and a vertex that has no inbound edges.*

Proof. Assume for contradiction that there is no vertex with no outbound edge, then choose any one of them, and append any next vertex, if there's an edge pointing to it from your current vertex. Since there's no sink, this process can continue infinitely, since our graph is finite, eventually we repeat a vertex. This is a contradiction to acyclicity as described above.

The same argument applies to the digraph with inverted edges or just by walking the path backwards instead. □

Definition 1.1.3 (Topological Ordering)**.** If for a graph $G = (V, E)$ we have a graph homomorphism:

$$\varphi : G \to \Delta^{|G|} = (|G| = \{0, \ldots, |G|\}, \leq) \subset \mathbb{N}$$

which is injective on nodes, call φ a topological ordering of G.

More explicitly: If there exists a finite $n \in \mathbb{N}$ and a bijection $f \colon V \to \{0, \ldots, n\}$ such that $(i, j) \in E \Rightarrow fi \leq fj \in \mathbb{N}$ for all edges in E, then that bijection f induces a topological ordering φ like before.

Proposition 1.1.4 (Finite dags have topological orderings)**.** *A digraph is a dag iff it has a topological ordering (necessarily non unique for non-trivial cases).*

Proof. Assume a graph has a topological ordering, if it had a cycle, at least one edge would point backwards in that ordering, to contradiction. So the existence of a topological ordering implies the dag-ness of a (di)graph.

Assume now an abstract dag $G = (V, E)$, we want to construct a bijection $\varphi\colon V \to |V|$, such that $\varphi(i) \leq \varphi(j)$ whenever there is an edge $(i, j) \in E$ (recall: and no condition otherwise).

By the fact that sources and sinks exist, choose a source s. Since it has no inbound connections, set its label to 0. Now remove s from the graph G with all edges that pointed from s to G and find that subgraph is a dag, too, so it has a source, that can be labelled 1, and so forth up until all vertices have been exhausted. $\square$

Corollary 1.1.5 (Minimal topological orderings)**.** *Each dag can be given a minimal topological ordering, that is, a bijection of its vertex set to an interval* $\{0, \ldots, |V| - 1\}$ *that induces a graph homomorphism. (Corollary to the proof before.)*

1.2 The Category emb$\mathcal{DAG}$ of embedded DAGs

The section before discusses among other things that each finite directed acyclic graph can be embedded in the complete finite directed acyclic graph of its vertex order. Using that we reach the following definition for a combinatorial category of dags.

Definition 1.2.1 (emb$\mathcal{DAG}$)**.** Define the category emb$\mathcal{DAG}$ as follows:
As objects we have a chosen natural number $n \in \mathbb{N}$, call it vertex count, together with a chosen subset $E \subset \{0, \ldots, n-1\} \times \{0, \ldots, n-1\}$, which satisfies:

$$\forall (i, j) \in E\colon 0 \leq i < j < n \quad \text{(top)},$$

call E the edge set of the object (n, E).

As morphisms we have all! set maps $f\colon \{0, \ldots, n-1\} \to \{0, \ldots, m-1\}$ for dags $(n, E), (m, F)$, such that: $\forall (i, j) \in E\colon (fi, fj) \in F$, i.e.

$$\mathrm{emb}\mathcal{DAG}((n, E), (m, F))$$

$$= \{f\colon \{0, 1, \ldots, n-1\} \to \{0, 1, \ldots, m-1\} \mid (i, j) \in E \Rightarrow (fi, fj) \in F \vee fi = fj\}.$$

Remark 1.2.2. For clarity, the induced graph of such an object has vertex set $\{0, \ldots, n-1\}$ and edges as defined by E.

Remark 1.2.3. By the discussion above we can see, that the topological ordering has in fact been incorporated into this definition, so that the induced graph associated to a pair (n, E) that satisfies the condition top, is necessarily a dag. In particular in what follows there is no need to check the acyclicity condition, it is actually a given fact in emb$\mathcal{DAG}$. Instead, in this category we need to keep compatible maps to $\mathbb{N}$ along.

Remark 1.2.4. One could instead choose the convention to consider dags with loops at each vertex, with would make the vertex set a recognisable subset of E, in which case the objects could be fully described by edge sets. Occassionally that might be a helpful extra perspective to consider, specifically when handling graph homomorphisms I jump between those conventions as convenient.

Remark 1.2.5. The category emb$\mathcal{DAG}$ in particular admits an upper bound on counting how many dags there are on a fixed number of nodes. Do note from vertex count 3 on, isomorphic dags can have different representations in emb$\mathcal{DAG}$, e.g. $\{0,1,2\},\{(0,1)\}$ and $\{0,1,2\},\{(0,2)\}$ are isomorphic, but each legitimately different objects in emb$\mathcal{DAG}$.

Example 1.2.6. The class of dags we need most instantly are complete graphs, which in accordance with topology I will call Δ^n:

$$V(\Delta^n) = \{0,1,\dots,n\}, E(\Delta^n) = \{(i,j) \mid i < j\}.$$

These are the edge count maximal objects in their vertex count, as in, there is no other dag on $n+1$ nodes with more or equally many edges to the Δ^n objects each in their degree.

Example 1.2.7. For some more non-trivial examples:

$$\mathbb{S}^0 = (\{0,1\},\{\}),$$

$$\mathbb{S}^1 = (\{0,1,2,3\},\{02,12,03,13\}),$$

$$\mathbb{S}^2 = (\{0,1,2,3,4,5\},\{02,12,03,13,04,14,24,34,05,15,25,35\}).$$

In a bit we will exhibit how these are actual spheres.

It is however visually convincing:

$$\mathbb{S}^0 = \; 0 \qquad 1,$$

$$\mathbb{S}^1 = \begin{tikzcd} & 2 & \\ 0 \arrow[ur] \arrow[dr] & & 1 \arrow[ul] \arrow[dl] \\ & 3, & \end{tikzcd}$$

$$\mathbb{S}^2 = \begin{tikzcd} & 4 & \\ 0 \arrow[ur] \arrow[rr] \arrow[d] \arrow[ddr] & & 2 \arrow[ul] \arrow[ddl] \\ 3 \arrow[uur] \arrow[dr] & & 1 \arrow[u] \arrow[ll] \arrow[uul] \arrow[dl] \\ & 5 & \end{tikzcd}$$

Definition 1.2.8 (Cone and Suspension C, Σ)**.** In addition we have a cone and suspension construction:

Let $G = (\{0, \ldots, n-1\}, E)$, construct CG with vertices $V(CG) = \{0, \ldots, n\}$ and edges: $E(CG) = E(G) \sqcup \{(i, n) | i \in \{0, \ldots, n-1\}\}$.

Let $G = (\{0, \ldots, n-1\}, E)$, construct ΣG with vertices $V(\Sigma G) = \{0, \ldots, n, n+1\}$ and edges: $E(CG) = E(G) \sqcup \{(i, n) | i \in \{0, \ldots, n-1\}\} \sqcup \{(i, n+1) | i \in \{0, \ldots, n-1\}\}$.

These are actual cones and suspensions as well.

Example 1.2.9. For illustration consider the diagrams of $C\mathbb{S}^0$ and $\Sigma\mathbb{S}^0$: Start with $\mathbb{S}^0$:

$$0 \qquad 1$$

add a cone point (2) and edges to it

$$\begin{tikzcd} 0 \arrow[r] & 2 & 1 \arrow[l] \end{tikzcd} .$$

For $\Sigma\mathbb{S}^0$ add another point 3 and just not the edge $(2, 3)$:

$$\begin{tikzcd} & 2 & \\ 0 \arrow[ur] \arrow[dr] & & 1 \arrow[ul] \arrow[dl] \\ & 3. & \end{tikzcd}$$

Remark 1.2.10. More generally the illustrations for C and Σ are quite helpful:

$$\begin{tikzcd} & C(pt) & \\ \bullet \arrow[ur] \arrow[rr, dotted, dash] & & \bullet, \arrow[ul] \end{tikzcd}$$

$$\begin{tikzcd} & \Sigma^+(pt) & \\ \bullet \arrow[ur] \arrow[dr] \arrow[rr, dotted, dash] & & \bullet \arrow[ul] \arrow[dl] \\ & \Sigma^-(pt). & \end{tikzcd}$$

Example 1.2.11. Do note we have:

$$C\Delta^{n-1} = \Delta^n \text{ for } n \geq -1.$$

As well as:

$$\Sigma\mathbb{S}^{n-1} = \mathbb{S}^n$$

for the examples defined above.

Definition 1.2.12 (Simplices, Spheres, Disks $\Delta^n, \mathbb{S}^n, \mathbb{D}^n$)**.** Define $\Delta^-1 = (\emptyset, \emptyset)$, and inductively: $\Delta^n := C\Delta^{n-1}$ with C the cone functor. Define $\mathbb{S}^0 = (\{0,1\}, \emptyset)$, and inductively: $\mathbb{S}^n := \Sigma\mathbb{S}^{n-1}$. For $i \geq 1$ define: $\mathbb{D}^i = C\mathbb{S}^{i-1}$.

Specifically for $i = 1$ denote $I := \mathbb{D}^1$ as our model for the unit interval in emb$\mathcal{DAG}$.

For vertex count 1 there are no edges to choose, so there is only one dag representing the one point graph in the category emb$\mathcal{DAG}$. For two vertices there is only one edge to choose, hence 2 graphs with vertex count 2, which are non-isomorphic.

Proposition 1.2.13 (Number of legal incidence matrices in emb$\mathcal{DAG}$)**.** *For each degree $n \geq 0$, the count of distinct edge sets per degree can be calculated as:*

$$|\mathrm{emb}\mathcal{DAG}_{0,n}| = 2^{\binom{n}{2}}.$$

Proof. For a fixed degree $n \in \mathbb{N}$, we need to check, how many $E \subset \{0, \ldots, n-1\} \times \{0, \ldots, n-1\}$ satisfy the condition top. In fact there is a maximal such E representing the complete dag Δ^{n-1} on n vertices, satisfying:

$$(i,j) \in E(\Delta^{n-1}) \Leftrightarrow i < j.$$

In particular each other such graph of vertex count n can be reached by deleting a uniquely determined set of edges from Δ^{n-1}.

So count the edges of Δ^{n-1}, since by definition they are each ordered, we can in fact count two-element subsets of $\{0, \ldots, n-1\}$ and interpret them canonically as an ordered tuple in each case. Hence the edge count is:

$$\left|E\Delta^{n-1}\right| = \binom{n}{2}.$$

So the count of dags is identified as the number of (arbitrary) subsets of $E\Delta^{n-1}$:

$$|\mathrm{emb}\mathcal{DAG}_n| = 2^{\left|E\Delta^{n-1}\right|} = 2^{\binom{n}{2}}.$$

□

Corollary 1.2.14. *Do note in particular we have the recurrence relation:*

$$|\mathrm{emb}\mathcal{DAG}_{0,n}| = |\mathrm{emb}\mathcal{DAG}_{0,n-1}| * 2^{n-1},$$

so in particular we can interpret the formula through the sum over all natural numbers up to a threshold:

$$|\mathrm{emb}\mathcal{DAG}_{0,n}| = \prod_{i=0}^{n-1} 2^i = 2^{\sum_{i=0}^{n-1} i}.$$

Proof. By direct calculation with the above formula, we get:

$$\frac{|\mathrm{emb}\mathcal{DAG}_{0,n}|}{|\mathrm{emb}\mathcal{DAG}_{0,n-1}|} = 2^{\binom{n}{2} - \binom{n-1}{2}},$$

and by the relation

$$\binom{n}{2} = \binom{n-1}{1} + \binom{n-1}{2}$$

get

$$\frac{|\mathrm{emb}\mathcal{DAG}_{0,n}|}{|\mathrm{emb}\mathcal{DAG}_{0,n-1}|} = 2^{\binom{n-1}{1}} = 2^{n-1}$$

□

Remark 1.2.15. In particular one can see how extremely fast that count grows in n, essentially as $O(exp(n^2))$.

vertex count	degree	dag count	log2 dag count
1	0	1	0
2	1	2	1
3	2	8	3
4	3	64	6
5	4	1024	10
6	5	32768	15
7	6	2097152	21
8	7	268435456	28
9	8	687194476736	36
10	9	35184372088832	45
11	10	$\sim 3 * 10^{16}$	55
12	11	$\sim 7 * 10^{19}$	66
13	12	$\sim 3 * 10^{23}$	78

In degree 23 the count reaches $2^{276} \sim 10^{83}$, hence there are more such dags in degree 23 in emb$\mathcal{DAG}$, than there are atoms in the observable universe. At degree 14 there are already about 10^{28} dags, hence more than atoms in a human body.

Lemma 1.2.16. *There is a functor* $\oplus\colon \mathrm{emb}\mathcal{DAG} \times \mathrm{emb}\mathcal{DAG} \to \mathrm{emb}\mathcal{DAG}$*, given on objects and vertices as*

$$\{0,\dots,n-1\} \oplus \{0,\dots,m-1\} \colon = \{0,\dots,n-1,n,\dots,n+m-1\},$$

Given two dags $(n, E), (m, F)$*, with the edge subsets:*

$$(i,j) \in (E \oplus F) \Leftrightarrow (i,j) \in E \vee (i-n, j-n) \in F.$$

Hence for two graph morphisms $\varphi\colon (n,E) \to (n', E'), \psi\colon (m,F) \to (m',F')$*:*

$$(\varphi \oplus \psi)(k) \colon = \begin{cases} \varphi(k) & 0 \le k \le n-1 \\ \psi(k-n) + n' & n \le k \le n+m-1 \end{cases}.$$

Proposition 1.2.17 (Pushouts exist!)**.** *The category of dags has (finite) pushouts!*

Proof. Assume given two dag-morphisms $\varphi_k\colon H \to G_k$, we want to construct an object $G_1 \cup_H G_2$ and two morphisms $\iota_k\colon G_k \to G_1 \cup_H G_2$, such that for each object Z and pair of morphisms $f_k\colon G_k \to Z$ such that $f_1 \circ \varphi_1 = f_2 \circ \varphi_2$, there is a unique map $F\colon G_1 \cup_H G_2 \to Z$, such that $F \circ \iota_k = f_k$.

Assume $H = (\{0,\dots,k-1\}, E(H)), G_1 = (\{0,\dots,l-1\}, E(G_1)), G_2 = (\{0,\dots,m-1\}, E(G_2))$, and hence $\varphi_k\colon H \to G_k$ each represented by vertex maps $\varphi_1\colon \{0,\dots,k-1\} \to \{0,\dots,l-1\}$ and $\varphi_2\colon \{0,\dots,k-1\} \to \{0,\dots,m-1\}$ respectively. □

Remark 1.2.18. At this point I am fixing a bijection

$$\omega_{n,m}\colon \{0,\dots,n-1\}\times\{0,\dots,m-1\}\to\{0,\dots,nm-1\}$$

which is appropriately symmetric and associative.

Two choices of prominence are the idea of going through a 2d table either by row or by column first, like:

$$\omega_{n,m}(i,j)=i\cdot n+j,$$

or

$$\bar{\omega}_{n,m}(i,j)=i+j\cdot m.$$

These specifically satisfy:

$$\bar{\omega}_{n,m}(i,j)=\omega_{m,n}(j,i).$$

There are actually at least countably infinitely many such choices, e.g. by choosing divisors of n and m. Here I mainly want to choose one, sometimes use the other when convenient, but calm the reader that the specific choice is quite inessential.

Occassionally I need the inverse, let us pick the one of $\omega_{n,m}$:

$$\omega_{n,m}^{-1}(k)=(\lfloor k/n\rfloor, k \bmod n).$$

Example 1.2.19.

$\omega_{4,7}$:	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	7	8	9	10	11	12	13
2	14	15	16	17	18	19	20
3	21	22	23	24	25	26	27

$\bar{\omega}_{4,7}$:	0	1	2	3	4	5	6
0	0	4	8	12	16	20	24
1	1	5	9	13	17	21	25
2	2	6	10	14	18	22	26
3	3	7	11	15	19	23	27

Lemma 1.2.20. *There is a functor* $\otimes\colon \mathrm{emb}\mathcal{DAG}\times\mathrm{emb}\mathcal{DAG}\to\mathrm{emb}\mathcal{DAG}$*, given on objects and vertices as*

$$\{0,\dots,n-1\}\otimes\{0,\dots,m-1\}\colon=\{0,\dots,nm-1\},$$

Given two dags $(n,E),(m,F)$ *with the edge subsets:*

$$(i,j)\in(E\otimes F)\Leftrightarrow((\omega_{n,m}^{-1})(i)_1,(\omega_{n,m}^{-1})(j)_1)\in E\wedge((\omega_{n,m}^{-1})(i)_2,(\omega_{n,m}^{-1})(j)_2)\in F,$$

with $(x,y)_1=x$ *and* $(x,y)_2=y$ *in this equivalence.*

Hence for two graph morphisms $\varphi\colon(n,E)\to(n',E'),\psi\colon(m,F)\to(m',F')$*:*

$$(\varphi\otimes\psi)(k)=\omega_{n',m'}(\varphi(\omega_{n,m}^{-1}(k)_1),\psi(\omega_{n,m}^{-1}(k)_2)).$$

Lemma 1.2.21. *For each pair $n, m \in \mathbb{N}$ we have natural bijections:*

$$\sigma_{n,m} \colon \{0, \dots, n+m-1\} \to \{0, \dots, m+n-1\}$$

and

$$\theta_{n,m} \colon \{0, \dots, nm-1\} \to \{0, \dots, mn-1\}.$$

They are given by:

$$\sigma_{n,m}(k) \colon = \begin{cases} k+m & 0 \leq k \leq n-1 \Leftrightarrow m \leq k+m \leq n+m-1 \\ k-n & 0 \leq k-n \leq m-1 \Leftrightarrow n \leq k \leq n+m-1 \end{cases}$$

and:

$$\theta_{n,m}(k) \colon = \lfloor k/n \rfloor + m \cdot (k \mathrm{mod} n).$$

Both satisfy symmetry and associativity relations as follows: Symmetries

$$\begin{aligned} \sigma_{n,m} \circ \sigma_{m,n} &= id_{\{0,\dots,m+n-1\}} \\ \sigma_{m,n} \circ \sigma_{n,m} &= id_{\{0,\dots,n+m-1\}} \\ \theta_{n,m} \circ \theta_{m,n} &= id_{\{0,\dots,mn-1\}} \\ \theta_{m,n} \circ \theta_{n,m} &= id_{\{0,\dots,nm-1\}} \end{aligned}$$

Associativity

$$\begin{aligned} \sigma_{n,m} \circ \sigma_{m,n} &= id_{\{0,\dots,m+n-1\}} \\ \sigma_{m,n} \circ \sigma_{n,m} &= id_{\{0,\dots,n+m-1\}} \\ \theta_{n,m} \circ \theta_{m,n} &= id_{\{0,\dots,mn-1\}} \\ \theta_{m,n} \circ \theta_{n,m} &= id_{\{0,\dots,nm-1\}} \end{aligned}$$

Theorem 1.2.22. *The category* emb$\mathcal{DAG}$ *is bipermutative with $\oplus$ the disjoint union and symmetry isomorphisms $\sigma_{\bullet,\bullet}$, and $\otimes$ the product with symmetry $\theta_{\bullet,\bullet}$. The category has strictly sum-neutral element the empty dag and strictly product-neutral element the one-point dag.*

The category has equalisers and coequalisers for each pair of morphisms, in particular finite pullbacks and pushouts.

There are combinatorial functors modelling cones C, as well as suspensions Σ.

Proof. □

Theorem 1.2.23. *There is a functor $Fl \colon$* emb$\mathcal{DAG}$ *$\to sSet$, which assigns to each dag $G = (V, E)$ as n-simplices the graph homomorphisms from Δ^n to G. Faces and degeneracies are induced by the relevant maps on Δ^n.*

The functor is strictly naturally sum preserving, that is $Fl(G \sqcup H) = Fl(G) \sqcup Fl(H)$ on objects as well as morphisms, also $F(\emptyset) = \emptyset$.

The functor is strictly naturally product preserving, that is $Fl(G \times H) = Fl(G) \times Fl(H)$ on objects as well as morphisms, also $F(*) = *$.

In addition the functor satisfies:

$$(Fl \circ C) = (C \circ Fl)$$

for an appropriately chosen cone-functor C on simplicial sets, as well as

$$(Fl \circ \Sigma) = (\Sigma \circ Fl)$$

for a similar suspension-functor Σ on simplicial sets.

Proof. By the universal property of the product $\otimes$ on emb$\mathcal{DAG}$ we get on n-simplices:

$$\mathrm{emb}\mathcal{DAG}(\Delta^n, G \times H) = \mathrm{emb}\mathcal{DAG}(\Delta^n, G) \times \mathrm{emb}\mathcal{DAG}(\Delta^n, H),$$

which exactly fits the description of the categorical product on simplicial sets. For sums, since Δ^n is a connected graph, on emb$\mathcal{DAG}$ we get on n-simplices:

$$\mathrm{emb}\mathcal{DAG}(\Delta^n, G \sqcup H) = \mathrm{emb}\mathcal{DAG}(\Delta^n, G) \sqcup \mathrm{emb}\mathcal{DAG}(\Delta^n, H),$$

which does not preclude that there might be more splittings of G or H, but it does prove the compatability with disjoint union of both categories. $\square$

Chapter 2

Combinatorial Homotopy of DAGs

2.1 Homotopic Maps

Reminder 2.1.1. Recall we have defined $I = C\mathbb{S}^0$, giving the dag: $0 \longleftarrow 2 \longrightarrow 1$.