# Derivation of Nonlinear Schrödinger Equation

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# Introduction

- The Newton's second law: $m\frac{d^2x}{dt^2} = F(x)$
- In quantum mechanics: no definite position or velocity/ a statistical or probabilistic interpretation of the state of the particle in terms of a wave function $u(x, t)$
- The Schrödinger equation (the analog of Newton's law of a quantum mechanical system):

$$i\frac{\partial u}{\partial t} + \Delta u + V(\mathrm{x})u = 0, \tag{1}$$

- the Nonlinear Schrrödinger (NLS) equations
  - Amplitude: slowly modulated in space and time,
  - $$i\frac{\partial u}{\partial t} + \Delta u + \gamma|u|^2 u = 0. \tag{2}$$

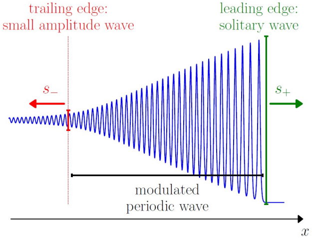
# Derivation of the Nonlinear Schrödinger equation

- A nonlinear wave equation $L(\partial_t, \nabla)u + G(u) = 0$
- Wave solutions $u = \epsilon\psi e^{i(\mathrm{k}\cdot\mathrm{X}-\omega t)}$
- The dispersion relation $\omega = \omega(\mathrm{k})$
- Cumulated nonlinear effects become significant on long time scales and large propagation distances.
- The complex amplitude depends on the slow variables $T = \epsilon t$ and $X = \epsilon x$
- $(i\partial_t - \omega(-i\partial_{\mathrm{X}}))\psi e^{i(\mathrm{k}\cdot\mathrm{X}-\omega t)} = 0$
- The derivatives $\partial_t$ and $\partial_{\mathrm{X}}$ are thus to be replaced by $\partial_t + \epsilon\partial_T$ and $\partial_{\mathrm{X}} + \epsilon\nabla$

# Derivation of the Nonlinear Schrödinger equation

- $$[i\partial_t + i\epsilon\partial_T - \omega(-i\partial_{\mathsf{X}} - i\epsilon\nabla)]\psi e^{i(\mathsf{k}\cdot\mathsf{X}-\omega t)} = 0. \tag{3}$$

- The frequency of the wave becomes dependent of the intensity and it leads to use the new frequency term $\Omega(-i\partial_{\mathsf{X}} - i\epsilon\nabla, \epsilon^2|\psi|^2)$ with $\Omega(-i\partial_{\mathsf{X}} - i\epsilon\nabla, 0) = \omega(-i\partial_{\mathsf{X}} - i\epsilon\nabla)$.
- $[\omega - i\epsilon\partial_T - \Omega(\mathsf{k} - i\epsilon\nabla, \epsilon^2|\psi|^2)]\psi e^{i(\mathsf{k}\cdot\mathsf{X}-\omega t)} = 0$
- Taylor expansion of the parameter $\Omega$

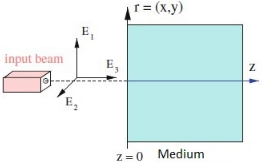



- $$i(\partial_T + \mathsf{v}_g\cdot\nabla)\psi + \epsilon\{\nabla\cdot(D\nabla\psi) + \gamma|\psi|^2\psi\} = 0, \tag{4}$$
- If we set $\xi = \mathsf{X} - T\mathsf{v}_g$ and $\tau = \epsilon T$, we get the NLS equation

$$i\frac{\partial\psi}{\partial\tau} + \nabla\cdot(D\nabla\psi) + \gamma|\psi|^2\psi = 0, \tag{5}$$

# The Example of Optical Waves

- The propagation of an electromagnetic wave is governed by Maxwell's equation
- Maxwell's equations in vacuum $\Delta \overrightarrow{E} = \frac{1}{c^2}\frac{\partial^2 \overrightarrow{E}}{\partial t^2}$
- In the absence of fluctuations, the solution $\overrightarrow{E} = (E_1, 0, 0), \quad E_1(x, y, z, t) = \mathcal{E} e^{i(k_0 z - \omega t)}$
- The $\overrightarrow{E} = (E_1, E_2, E_3)$ propagates in the $z$-direction, is localized in the transverse $(x, y)$-plane and is linearly-polarized in the $x$-direction.




- At "low" intensities of the electric field, the case of a linear polarization field $P_{lin} = \epsilon_0 \chi_0(\omega) E$
- In a linear dielectric, $n_0$ is the (linear) index of refraction $n_0^2 = 1 + \chi_0(\omega)$

# The Example of Optical Waves

- The propagation of continuous wave linearly-polarized laser beams in a linear dielectric is governed by the scalar Helmholtz equation

$$\Delta \mathrm{E}(z, y, z) + k_0^2 \mathrm{E} = 0, \quad k_0^2 = \frac{\omega^2}{c^2} n_0^2. \tag{6}$$

- The polarization field $\mathrm{P} = \mathrm{P}_{lin} + \mathrm{P}_{nl}$
- As E increases, the shift of the centers of the electrons orbits, hence the polarization field P, begins to have a nonlinear dependence on E.
- We can expand $\mathrm{P}_{nl}$ in a Taylor series in E $\mathrm{P}_{nl} = \chi_1(\omega)\mathrm{E}^2 + \chi_2(\omega)\mathrm{E}^3$
- The index of refraction is typically weakly nonlinear $n^2 = n_0^2 + 4N_1 n_0 |\mathrm{E}|^2$
- $k^2 = k_0^2(1 + \frac{4n_1}{n_0}|\mathrm{E}|^2)$
- We substitute $\mathrm{E} = e^{ik_0 z}\psi$ and we apply the paraxial approximation $\psi_{zz} \ll k_0 \psi_z$.

$$\Delta_\perp \psi + 2ik_0\psi_z + k_0^2 \frac{4n_1}{n_0}|\psi|^2\psi = 0. \tag{7}$$

# Conclusions

- NLS equation can be applied in many models.
  - Surface gravity waves
  - Nonlinear sine-Gordon equation

$$\text{NLS:} \quad i\partial_\tau A + \beta\partial_\xi^2 A + \gamma|A|^2 A = 0, \quad \beta = \frac{1}{2}\omega''(k), \quad \gamma = \frac{3}{2}\frac{\lambda}{\omega(k)}. \tag{8}$$

  - Bose-Einstein condensate (BEC) known as the Gross-Pitaevskii equation (GPE)

$$\text{NLS:} \quad i\hbar\frac{\partial\Psi(\mathrm{r},t)}{\partial t} = [-\frac{\hbar^2}{2m}\Delta + \mathrm{V}(\mathrm{r}) + G|\Psi(\mathrm{r},t)|^2]\Psi(\mathrm{r},t). \tag{9}$$