Bachelorprojekt - Minimal surfaces

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1 Introduction

2 different types of curvature???

2.1 Normal curvature

ref: Oprea chap 2.2, Pressley chap 6.2

Let $D \subset \mathbb{R}^2$ be a subset of the real plane, and let a surface $M \subset \mathbb{R}^3$ be given by the parametrization

$$s(u,v) = (s^1(u,v), s^2(u,v), s^3(u,v)), \quad (u,v) \in D.$$

To understand the notion of normal curvature of a curve on M, we first need a definition the the tangent plane. We define the tangent plane at some point $p = s(u_0, v_0)$ on M to be the set of all tangent vectors at the point p of all curves on M, that pass through p.

Proposition 1 ref: pressley prop 4.4 s. 81 i pdf
The tangent plane at the point $p = s(u_0, v_0)$ of a surface M with parametrization $s : D \to \mathbb{R}^3$ is the subspace of $\mathbb{R}^3$ spanned by the tangent vectors s_u and s_v.

Proof:

For a curve γ on M given by the parametrization

$$\gamma(t) = s(u(t), v(t))$$

the tangent vector at some point on γ can be written as a linear combination of s_u and s_v since we have that (by the chain rule for two independent variables) se latex kode

$$\gamma' = s_u u' + s_v v'.$$

Let us now for some arbitrary constants $c_1, c_2 \in \mathbb{R}$ define a vector

$$v = c_1 s_u + c_2 s_v \in \text{span}\{s_u, s_v\}.$$

We can then define a curve

$$\gamma(t) = s(u_0 + c_1 t, v_0 + c_2 t)$$

for which we have

$$\gamma' = s_u(u_0 + c_1 t)' + s_v(v_0 + c_2 t)' = c_1 s_u + c_2 s_v = v.$$

It must therefore hold true that every vector in the span of s_u and s_v is the tangent vector at $p = s(u_0, v_0)$ for some curve on M. $\square$

We now define the unit normal vector to the tangent plane of M by the normalized cross product of the tangent vectors

$$\mathrm{N} = \frac{s_u \times s_v}{|s_u \times s_v|}.$$

Since s_u and s_v are the vectors that span the tangent plane, it is clear that taking their cross product will result in a vector that is orthogonal to both of them, and therefore orthogonal to the tangent plane.

If $\gamma(t) = s(u(t), v(t))$ is a unit speed curve on the surface M, we can define the normal curvature of γ by

$$\kappa_n = \gamma'' \cdot \mathrm{N}.$$

That is, the normal curvature is defined to be the projection of the acceleration of γ onto the normal direction.
Using again the chain rule along with ref. til bevis af 2.2.3 oprea, we get that

$$\begin{aligned}
\kappa_n &= -\gamma' \cdot \mathrm{N}' \\
&= -(u' s_u + v' s_v) \cdot (\mathrm{N}_u u' + \mathrm{N}_v v') \\
&= -(u')^2 s_u \cdot \mathrm{N}_u - u'v' s_v \cdot \mathrm{N}_u - u'v' s_u \cdot \mathrm{N}_v - (v')^2 s_v \cdot \mathrm{N}_v \\
&= -s_u \cdot \mathrm{N}_u (u')^2 - (s_v \cdot \mathrm{N}_u + s_u \cdot \mathrm{N}_v) u'v' - s_v \cdot \mathrm{N}_v (v')^2.
\end{aligned}$$

Letting L, M, N denote the coefficients of the second fundamental form reference til second fundamental form? eller lidt mere forklaring,

$$L = -s_u \cdot \mathrm{N}_u, \quad 2M = -(s_v \cdot \mathrm{N}_u + s_u \cdot \mathrm{N}_v), \quad N = -s_v \cdot \mathrm{N}_v,$$

we see that the normal curvature can be written

$$\kappa_n = L(u')^2 + 2Mu'v' + N(v')^2. \tag{1}$$

Using the above mentioned coefficients of the second fundamental form in addition to the coefficients of the first fundamental form, E, F, G , we can now define the following symmetric 2×2-matrices

$$\mathcal{F}_I = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \quad \mathcal{F}_{II} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

Using equation (1) and defining a vector $\Gamma = \begin{pmatrix} u' \\ v' \end{pmatrix}$, we see that the normal curvature is given by the follow matrix-vector product

$$\begin{aligned}
\kappa_n &= L(u')^2 + 2Mu'v' + N(v')^2 \\
&= (Lu' + Mv')u' + (Mu' + Nv')v' \\
&= \begin{pmatrix} u' \\ v' \end{pmatrix}^T \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} \\
&= \Gamma^T \mathcal{F}_{II} \Gamma.
\end{aligned}$$

2.2 Principal curvatures

Definition 2.1. *For a surface $M \subset \mathbb{R}^3$ with parametrization $s(u, v), (u, v) \in D$, we define the principal curvatures of the surface to be the two values of κ satisfying the quadratic equation*

$$\det(\mathcal{F}_{II} - \kappa \mathcal{F}_I) = 0. \tag{2}$$

We call these values κ_1 and κ_2 where $\kappa_1 \geq \kappa_2$.

Additionally, we define the Weingarten matrix of the surface by

$$\mathcal{W} = \mathcal{F}_I^{-1}\mathcal{F}_{II}.$$

By Definition 2.1 along with the fact that $\det(\mathcal{F}_I) \neq 0$ since $\mathcal{F}_I$ is invertible, (indsæt ref til at F1 er invertibel) we get that

$$\begin{aligned}
\det(\mathcal{F}_{II} - \kappa\mathcal{F}_I I) &= 0 \\
\det(\mathcal{F}_I(\mathcal{F}_I^{-1}\mathcal{F}_{II} - \kappa I)) &= 0 \\
\det(\mathcal{F}_I)\det(\mathcal{F}_I^{-1}\mathcal{F}_{II} - \kappa I) &= 0 \\
\det(\mathcal{F}_I^{-1}\mathcal{F}_{II} - \kappa I) &= 0,
\end{aligned}$$

i.e κ_1 and κ_2 are the eigenvalues of $\mathcal{W}$.

Now, from (2) we know that $\mathcal{F}_{II} - \kappa\mathcal{F}_I$ is a singular matrix if κ is one of the principal curvatures, and consequently there must then exist some vector $v = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ such that

$$(\mathcal{F}_{II} - \kappa\mathcal{F}_I)v = 0.$$

Definition 2.2. *If a vector $v = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ satisfies $(\mathcal{F}_{II} - \kappa\mathcal{F}_I)v = 0$, we define the principal vector corresponding to the principal curvature κ as the tangent vector $e = c_1 s_u + c_2 s_v$ of $s(u, v)$.*

The following theorem (usually called *Euler's theorem*) gives us a great connection between the normal curvature and the principal curvatures.

Theorem 1 (Pressley [3], p. 137)**.** *Let γ be a curve on a surface M with parametrization s, and let κ_1 and κ_2 be the principal curvatures of s with corresponding non-zero principal vectors v_1 and v_2. The normal curvature of γ is then given by*

$$\kappa_n = \kappa_1\cos^2(\theta) + \kappa_2\sin^2(\theta),$$

where θ is the angle between γ' and v_1.

The way we have described the principal curvatures so far might not result in a very intuitive understanding of what the principal curvatures are. A nicer way they can be thought of, geometrically, is as follows

Corollary 1.1. *The principal curvatures at a point p of a surface are the maximum and minimum values of the normal curvature of all the curves on the surface that pass through p.*

Proof. We start with the case $\kappa_1 = \kappa_2$. By Theorem 1, we then clearly have that $\kappa_n = \kappa_1 = \kappa_2$ for every curve on the surface.

Now for the case of $\kappa_1 \neq \kappa_2$, we assume (by Definition 2.1) $\kappa_1 > \kappa_2$. From Theorem 1, we have

$$\begin{aligned}
\kappa_n &= \kappa_1\cos^2(\theta) + \kappa_2\sin^2(\theta) \\
&= \kappa_1(\cos^2(\theta) + \sin^2(\theta) - \sin^2(\theta)) + \kappa_2\sin^2(\theta) \\
&= \kappa_1(1 - \sin^2(\theta)) + \kappa_2\sin^2(\theta) \\
&= \kappa_1 - \kappa_1\sin^2(\theta) + \kappa_2\sin^2(\theta) \\
&= \kappa_1 - (\kappa_1 - \kappa_2)\sin^2(\theta).
\end{aligned}$$

This makes it clear that $\kappa_n \leq \kappa_1$ and that $\kappa_n = \kappa_1$ if and only if $\sin^2(\theta) = 0$ (i.e. the tangent vector γ' and the principal vector e_1 are parallel). Similar manipulations show that $\kappa_n \geq \kappa_2$ with $\kappa_n = \kappa_2$ if and only if γ' and e_2 are parallel. $\square$

2.3 Mean curvature and minimal surfaces

With the theory presented thus far, we can now define the most important measure of curvature of surfaces, related to the study of minimal surfaces. lav til en rigtig definition

Definition 2.3. *If κ_1 and κ_2 are the principal curvatures of a surface M, we define the mean curvature, H, of M by*

$$H = \frac{1}{2}(\kappa_1 + \kappa_2).$$

It is convenient to determine a formula for H in terms of the previously mentioned coefficients of the first and second fundamental forms. By (2), we have that

$$\begin{aligned} \det(\mathcal{F}_{II} - \kappa\mathcal{F}_I) &= 0 \\ \det\left(\begin{pmatrix} L-\kappa E & M-\kappa F \\ M-\kappa F & N-\kappa G \end{pmatrix}\right) &= 0 \\ (L-\kappa E)(N-\kappa G) - (M-\kappa F)(M-\kappa F) &= 0 \\ LN - \kappa GL - \kappa EN + \kappa^2 EG - M^2 + \kappa MF + \kappa MF - \kappa^2 F^2 &= 0 \\ (EG-F^2)\kappa^2 + (2MF - GL - EN)\kappa + LN - M^2 &= 0. \end{aligned}$$

We see that the above equation is a quadratic equation in terms of κ with coefficients $a = (EG - F^2)$, $b = (2MF - GL - EN)$, $c = LN - M^2$. For a general quadratic equation with roots x_1, x_2 we have that $x_1 + x_2 = -\frac{b}{a}$. Therefore, we get

$$H = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{GL + EN - 2MF}{2(EG - F^2)}. \tag{3}$$

Definition 2.4. *A surface M is a minimal surface if $H = 0$ at every point of the surface.*

Considering this important property of zero mean curvature for minimal surfaces with regard to Def. 2.3, we can gather directly that the principal curvatures κ_1 and κ_2 are always equal in size but with opposite signs, for minimal surfaces.

3 The catenoid (and other minimal surfaces?)

In 1744, Euler discovered the catenoid indsæt ref: `https://www.sciencedirect.com/topics/physics-and-astronomy/minimal-surface` - the first non trivial minimal surface (i.e. the first minimal surface that is not just a planar surface). The catenoid can be produced by taking two metal wires in the shape of rings and dipping them into liquid soap. Take the rings out, and poke a whole through the middle of them. You will then be left with a layer of soap film between the rings in the shape of a catenoid. While holding the rings (representing the boundary curves) very close together, the surface might resemble a short cylinder. If you pull the rings further apart however, the neck of the catenoid will begin to narrow until, at some point, it collapses and breaks into two disks.

The catenoid is a surface of revolution and is generated by rotating a catenary around an axis. Let a catenary in the (x,z)-plane be given by $x = c\cosh(\frac{z}{c})$, for some (non-zero) constant $c \in \mathbb{R}$. The constant c determines the distance from the z-axis to the point of the curve where it is closest to the z-axis (i.e. the "bottom" of the curve). The surface we get by rotation around the z-axis will then be parameterized by

$$s(u,v) = (c\cosh\left(\frac{u}{c}\right)\cos(v),\ c\cosh\left(\frac{u}{c}\right)\sin(v),\ u), \tag{4}$$

where $u \in \mathbb{R}, v \in [-\pi, \pi)$.

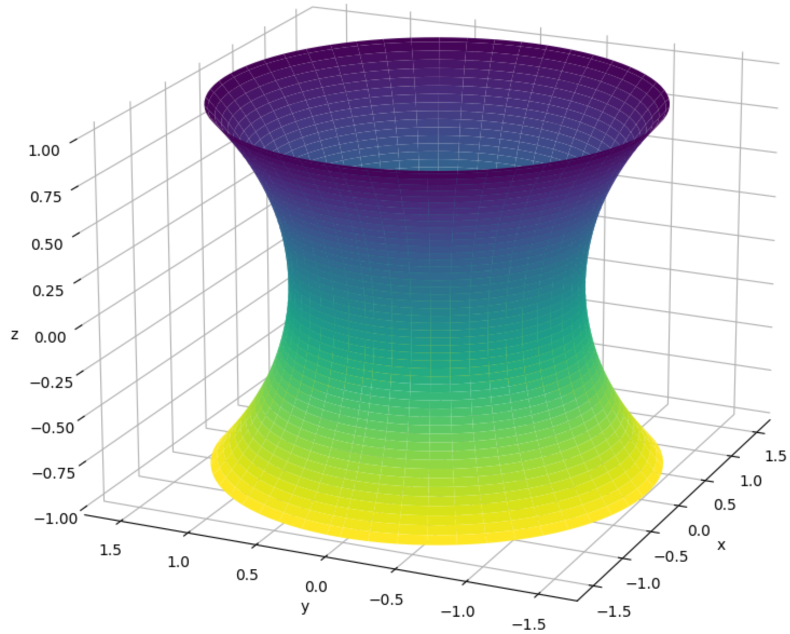

Figure 1: The catenoid with $c = 1$ and parameters $u \in [-1, 1], v \in [-\pi, \pi)$.

vis at katenoiden er en minimalflade (oprea. s. 76 pdf), og skriv om relationen mellem højde og cirkel radius (fra dit maple ark).

We will now see that the catenoid indeed is a minimal surface. We first calculate the tangent vectors s_u and s_v,

$$\begin{aligned} s_u &= (\sinh(u)\cos(v),\ \sinh(u)\sin(v),\ 1) \\ s_v &= (-\cosh(u)\sin(v),\ \cosh(u)\cos(v),\ 0). \end{aligned}$$

We then get the unit normal vector

$$\begin{aligned} \mathrm{N} = \frac{s_u \times s_v}{|s_u \times s_v|} &= \frac{(-\cosh(u)\cos(v),\ -\cosh(u)\sin(v),\ \sinh(u)\cosh(u))}{\cosh(u)^2} \\ &= \Big(-\frac{\cos(v)}{\cosh(u)},\ -\frac{\sin(v)}{\cosh(u)},\ \frac{\sinh(u)}{\cosh(u)}\Big). \end{aligned}$$

This is everything we need to calculate the first and second fundamental form coefficients,

$$E = s_u \cdot s_u = \cosh(u)^2, \quad F = s_u \cdot s_v = 0, \quad G = s_v \cdot s_v = \cosh(u)^2$$

and (note: since $F = 0$, there is no need to calculate M)

$$L = -s_u \cdot \mathrm{N}_u = -1, \quad N = -s_v \cdot \mathrm{N}_v = 1.$$

By (3), we finally get

$$H = \frac{GL + EN - 2MF}{2(EG - F^2)} = \frac{-\cosh(u)^2 + \cosh(u)^2}{2\cosh(u)^4} = 0,$$

i.e. the mean curvature is 0 at every point of the catenoid. måske også vis at helicoiden er en minimalflade

If we assume that the two boundary curves of the catenoid that are produced by fixing the u-parameter at its maximum and minimum values, respectively, are parallel circles of equal radius, then it is easily shown that we have the following relation between the height of the catenoid and the circle radius

$$R = c\cosh\Big(\frac{h}{2c}\Big),$$

where R is the circle radius and h is the height of the catenoid. For instance, the catenoid in figure 1 has height $h = 2$, hence the circles have radii

$$R = 1 \cdot \cosh\Big(\frac{2}{2 \cdot 1}\Big) \approx 1.54.$$

måske relevanthttps://math.uchicago.edu/~may/REU2019/REUPapers/Zheng,SiqiClover.pdf https://mathworld.wolfram.com/MinimalSurfaceofRevolution.html

3.1 catenoid only surface of revolution proof...

Theorem 2. *If M is both a surface of revolution and a minimal surface, then M is either part of a catenoid or part of a plane.*

Proof. For simplicity, we assume that the axis of revolution is the z-axis, and that the generator curve γ lies in the x, z-plane. By these assumptions we have

$$\gamma(u) = (f(u), 0, g(u))$$

and the generated surface of revolution

$$s(u, v) = (f(u)\cos(v), f(u)\sin(v), g(u))$$

for some functions $f > 0$, g.
Assuming γ is unit speed, we get by calculation of the first and second fundamental form coefficients

$$E = 1, F = 0, G = f^2$$

and

$$L = f'g'' - f''g', M = 0, N = fg'.$$

Hence by equation (3), we have

$$H = \frac{1}{2}\Big(f'g'' - f''g' + \frac{g'}{f}\Big) \tag{5}$$

Now suppose that we restrict u to some open interval (α, β) for which we have $g'(u_0) \neq 0$ for all $u_0 \in (\alpha, \beta)$. Since γ is unit speed, we have

$$(f')^2 + (g')^2 = 1.$$

Differentiating both sides, we get

$$f'f'' + g'g'' = 0$$
$$\Downarrow$$
$$(f'g'' - f''g')g' = -(f')^2 f'' - f''(g')^2 = -f''((f')^2 + (g')^2) = -f''$$
$$\Downarrow$$
$$(f'g'' - f''g') = -\frac{f''}{g'}.$$

Equation (5) then becomes
$$H = \frac{1}{2}\Big(\frac{g'}{f} - \frac{f''}{g'}\Big).$$
Since M is minimal ($H = 0$), we must then have that
$$\begin{aligned}\frac{1}{2}\frac{f''}{g'} &= \frac{1}{2}\frac{g'}{f} \\ f'' &= \frac{1-(f')^2}{f} \\ ff'' &= 1-(f')^2.\end{aligned}$$
Now, we want to somehow solve this differential equation. If we define a new function $h = f'$, then we have by the chain rule that
$$f'' = \frac{dh}{du} = \frac{dh}{df}\frac{df}{du} = \frac{dh}{df}h.$$
We can then rewrite the differential equation as
$$fh\frac{dh}{df} = 1-h^2.$$
We note that $1-h^2 \neq 0$ since $g' \neq 0$ and $(g')^2 = 1-(f')^2$. Hence we can integrate both sides to get
$$\begin{aligned}\int \frac{h}{1-h^2}dh &= \int \frac{1}{f}df \\ \frac{1}{\sqrt{1-h^2}} &= af \\ h &= \frac{\sqrt{a^2f^2-1}}{af},\end{aligned}$$
where a is some non-zero constant. Since $h = \frac{df}{du}$, we get
$$\begin{aligned}\frac{df}{du} &= \frac{\sqrt{a^2f^2-1}}{af} \\ df\,af &= \sqrt{a^2f^2-1}du \\ \frac{af}{\sqrt{a^2f^2-1}}df &= du.\end{aligned}$$
We integrate once more to get
$$\begin{aligned}\int \frac{af}{\sqrt{a^2f^2-1}}df &= \int du \\ \frac{\sqrt{a^2f^2-1}}{a} &= u+b \\ a^2f^2 &= (u+b)^2a^2+1 \\ f &= \frac{\sqrt{(u+b)^2a^2+1}}{a},\end{aligned}$$
where b is a constant of integration that we by a simple coordinate transformation $u = u+b$ can assume to be 0, so we get
$$f = \frac{\sqrt{u^2a^2+1}}{a}.$$

Using again the unit speed condition, we have then for g that

$$
\begin{aligned}
(g')^2 &= 1 - (f')^2 = 1 - h^2 = \frac{1}{a^2 f^2} \\
g' &= \pm \frac{1}{af} \\
g' &= \pm \frac{1}{\sqrt{u^2 a^2 + 1}}
\end{aligned}
$$

Integrating again, and using the logarithmic definition of $\sinh^{-1}$ and the rules for composition of hyperbolic and inverse hyperbolic functions [1], we get

$$
\begin{aligned}
g &= \frac{\ln(au + \sqrt{a^2u^2 + 1})}{a} + c \\
g &= \frac{1}{a}\sinh^{-1}(au) + c \\
a(g - c) &= \sinh^{-1}(au) \\
\frac{1}{a}\cosh(a(g - c)) &= \frac{\sqrt{1 + a^2u^2}}{a} \\
f &= \frac{1}{a}\cosh(a(g - c)).
\end{aligned}
$$

Assuming $c = 0$ (since we otherwise could just translate along the z-axis accordingly), we finally see that the generator curve of the surface is given by

$$x = \frac{1}{a}\cosh(az),$$

- a catenary. Hence the surface must be part of a catenoid.

□

Note: for the above proof we made the choice to assume $g'(u) \neq 0$ in the considered parameter interval. Hence why the surface shows to be part of a catenoid and not a plane. For the remaining argument as to why the surface must be part of a plane if not a catenoid, we refer to [Pressley [3], p. 210].

4 Harmonic functions

Definition ? A function $\psi(x, y),\ \psi : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ is said to be *harmonic* if its second order partial derivatives are continuous and it satisfies the Laplace equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

i.e.

$$\Delta\psi = 0,$$

where Δ is the Laplace-operator.

To understand the connection between minimal surfaces and harmonic functions, we first need to define what it means for a parametrization to be isothermal.

Definition ? For a parametrization $s(u, v), (u, v) \in D$, we say that $s(u, v)$ is isothermal if $F = 0$

and $E = G$, i.e. $s_u \cdot s_v = 0$ and $s_u \cdot s_u = s_v \cdot s_v$.

The notion of an isothermal parametrization lets us simplify the calculation of the mean curvature, H, of a surface. Recall that for the earlier calculation of the mean curvature of the catenoid, we found that $F = 0$, $E = G = \cosh(u)^2$. Hence the parametrization (4) is isothermal. It is easily seen that for an isothermal parametrization $s(u, v)$, the mean curvature formula (3) reduces to

$$H = \frac{GL + EN - 2MF}{2(EG - F^2)} = \frac{E(L + N)}{2E^2} = \frac{L + N}{2E}. \tag{6}$$

Theorem ? There exists a parametrization with isothermal coordinates for any minimal surface $M \subseteq \mathbb{R}^3$. referer til bevis: oprea s. 73 i bogen.

Theorem ? Let $s(u, v)$ be an isothermal parametrization of some surface M, then

$$\Delta s = (2EH)\mathrm{N}.$$

referer til bevis: oprea s. 75 i bogen

These two results are great because they now let us introduce the following corollary for determining whether or not a surface is a minimal surface, using harmonic functions.
Corollary ? Let a surface M be given by an isothermal parametrization
$s(u, v) = (s^1(u, v), s^2(u, v), s^3(u, v))$, then M is a minimal surface if and only if s^1, s^2 and s^3 are harmonic functions.
Proof:
The corollary states a bi-implication, hence we will need to prove the implication going both ways. Let us first assume that M is minimal surface. In that case, by the definition of a minimal surface we have that $H = 0$ which by theorem indsæt ref til theorem oven over implies that

$$\Delta s = (2EH)\mathrm{N} = 0 \cdot \mathrm{N} = 0$$

i.e. s^1, s^2, s^3 are harmonic. Let us now instead begin by assuming that s^1, s^2, s^3 are harmonic functions. By theorem indsæt samme ref and the definition of harmonic functions, we get that

$$\Delta s = 0 = (2EH)\mathrm{N}.$$

Since we have that $E = s_u \cdot s_u \neq 0$ and $\mathrm{N} = \frac{s_u \times s_v}{|s_u \times s_v|} \neq 0$, for the above equation to hold true, it is clear that we must have $H = 0$. Thus M must be a minimal surface.

5 Weierstrass-Enneper representation of minimal surfaces

Building on top of the theory for isothermal parametrizations and harmonic functions, we come to an important part of the theory behind minimal surfaces. Using Weierstrass-Enneper representations, we will soon be able to represent and construct minimal surfaces using just a couple of functions that fulfill certain requirements. To proceed, we first need some elementary complex analysis. We denote the complex plane as the set

$$\mathbb{C} = \{z = u + iv | u, v \in \mathbb{R}\},$$

where $i = \sqrt{-1}$.

Let us a define a function $f : \mathbb{C} \to \mathbb{C}$. We say that f is complex differentiable at some point $z_0 \in \mathbb{C}$ if the following limit exists

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}. \tag{7}$$

Definition ? Define a function $f : \mathbb{C} \to \mathbb{C}$ and let D be an open subset of the complex plane. We say that f is *holomorphic* in D if the limit (7) exists for all $z_0 \in D$.
Definition ? ref: complex analysis bog i onedrive s. 86 Define a function $g : \mathbb{C} \to \mathbb{C}$, an open subset D of the complex plane and some sequence of points $\{z_0, z_1, z_2, ...\}$ that does not have limit points in D. We say that g is *meromorphic* in D if g is holomorphic in $D - \{z_0, z_1, z_2, ...\}$ and g has poles at all points in $\{z_0, z_1, z_2, ...\}$.
Proposition ? f is holomorphic if and only if

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

indsæt ref: `https://math.berkeley.edu/~vvdatar/m185f16/notes/Lecture-8_Cauchy_Riemann.pdf`

Now say we have an isothermal parametrization with real coordinates $s(u,v) = (s^1(u,v), s^2(u,v), s^3(u,v))$ for a minimal surface M. We can then define the complex parameters $z = u + iv$ and $\bar{z} = u - iv$. Note that the partial derivatives with respect to complex variables are given by

$$\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\Big), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\Big). \tag{8}$$

Clearly, both u and v can be written in terms of z and $\bar{z}$, since $z + \bar{z} = 2u$ and $z - \bar{z} = 2iv$. Applying this coordinate transformation, we now write

$$s(z,\bar{z}) = (s^1(z,\bar{z}), s^2(z,\bar{z}), s^3(z,\bar{z})).$$

Let us now define a new parametrization based on $s(z,\bar{z})$ as follows.

$$\varphi = \frac{\partial s}{\partial z} = (s^1_z, s^2_z, s^3_z). \tag{9}$$

By (8), we get for each of the coordinate functions s^j, $j = 1, 2, 3$

$$\frac{\partial s^j}{\partial z} = \frac{1}{2}(s^j_u - i s^j_v).$$

We define the following operation on φ:

$$(\varphi)^2 = (s^1_z)^2 + (s^2_z)^2 + (s^3_z)^2,$$

which leads us to the next proposition.
Proposition ? Let $M \subset \mathbb{R}^3$ be the surface given by the parametrization $s(u,v)$ and define $\varphi = \frac{\partial s}{\partial z}$. Then $(\varphi)^2 = 0$ if and only if $s(u,v)$ is isothermal.
Proof: The proposition states a bi-implication, hence we will need to prove the implication going both ways. Firstly however, applying the rules of differentiation with respect to z on s, we see that

$$\begin{aligned}
(\varphi)^2 &= (s^1_z)^2 + (s^2_z)^2 + (s^3_z)^2 \\
&= (\frac{1}{2}(s^1_u - is^1_v))^2 + (\frac{1}{2}(s^2_u - is^2_v))^2 + (\frac{1}{2}(s^3_u - is^3_v))^2 \\
&= \frac{1}{4}((s^1_u)^2 - (s^1_v)^2 - i2s^1_u s^1_v) + \frac{1}{4}((s^2_u)^2 - (s^2_v)^2 - i2s^2_u s^2_v) + \frac{1}{4}((s^3_u)^2 - (s^3_v)^2 - i2s^3_u s^3_v) \\
&= \frac{1}{4}((s^1_u)^2 + (s^2_u)^2 + (s^3_u)^2) - ((s^1_v)^2 + (s^2_v)^2 + (s^3_v)^2) - i2(s^1_u s^1_v + s^2_u s^2_v + s^3_u s^3_v) \\
&= \frac{1}{4}(s_u \cdot s_u - s_v \cdot s_v - i2 s_u \cdot s_v) \\
&= \frac{1}{4}(E - G - i2F).
\end{aligned}$$

Now let us start by assuming that $s(u,v)$ is isothermal. Then it is easily seen from the above calculations that $(\varphi)^2 = \frac{1}{4}(E - G - i2F) = \frac{1}{4}(E - E - i2 \cdot 0) = 0$. Let us now instead begin by assuming that $(\varphi)^2 = 0$. From the above calculations, we must then have $\frac{1}{4}(E - G - i2F) = 0$. For the left hand side to equal 0, both the real and the imaginary part must equal 0. Clearly, we then have $F = 0$ and $E = G$ (since neither E or G can be negative). That is to say, $s(u,v)$ is isothermal.

We are nearly at the point where we can state the theorem that clarifies the connection between holomorphic functions and minimal surfaces. We need only a small lemma first that will come in handy soon.
Lemma ?

$$\frac{\partial \varphi}{\partial \bar{z}} = \frac{1}{4}\Delta s.$$

Proof:

$$\begin{aligned}\frac{\partial \varphi}{\partial \bar{z}} &= \frac{\partial}{\partial \bar{z}}\Big(\frac{\partial s}{\partial z}\Big) = \frac{\partial \varphi}{\partial \bar{z}}\Big(\frac{1}{2}(s_u - i s_v)\Big) \\ &= \frac{1}{4}(s_{uu} + i s_{uv} - i s_{vu} + s_{vv}) \\ &= \frac{1}{4}(s_{uu} + s_{vv}) \\ &= \frac{1}{4}\Delta s.\end{aligned}$$

Finally, we have everything we need to prove the following:
Theorem ?
Let $M \subset \mathbb{R}^3$ be the surface given by the parametrization $s(u,v)$ and define $\varphi = \frac{\partial s}{\partial z}$ for which we assume $(\varphi)^2 = 0$. Then M is a minimal surface if and only if φ^j is holomorphic for $j = 1,2,3$.
Proof:
The theorem states a bi-implication, hence we will need to prove the implication going both ways. Let us first assume that M is a minimal surface. Since $(\varphi)^2 = 0$, we know by proposition indsæt ref til ovenstående korollar that s is isothermal. Then by corollary indsæt ref til korollaren i harmonic functions sektionen, s^1, s^2 and s^3 are harmonic. If all of the coordinate functions of s are harmonic, s must satisfy Laplace's equation. Using Lemma indsæt ref til lemma ovenover, we then get

$$\Delta s = 0 \iff \frac{1}{4}\Delta s = 0 \iff \frac{\partial \varphi}{\partial \bar{z}} = 0$$

which by proposition indsæt ref til første proposition i afsnittet implies that the coordinate functions of φ are holomorphic. Let us instead begin by assuming that the coordinate functions are holomorphic. By the same lemma and proposition we then have

$$\frac{\partial \varphi}{\partial \bar{z}} = 0 \iff \frac{1}{4}\Delta s = 0 \iff \Delta s = 0.$$

If s satisfies Laplace's equation the coordinate functions of s must be harmonic and thus M is a minimal surface.

This theorem tells us that if we can determine three holomorphic functions φ^j for $j = 1,2,3$ such that $(\varphi)^2 = 0$, then the corresponding parametrization $s(z,\bar{z})$ will be an isothermal parametrization of a minimal surface. Say we find three such holomorphic functions. In order to actually obtain the parametrization $s(z,\bar{z})$, as it might become clear from (9), we need to integrate these functions.
Corollary ? ref: oprea cor. 3.6.3

$$s^j(z,\bar{z}) = c_j + 2\mathfrak{Re}\Big(\int \varphi^j dz\Big).$$

But how should we go about finding these three holomorphic functions that satisfy $(\varphi)^2 = 0$? A generalized method ref: oprea s. 95 is to choose a holomorphic function f and a meromorphic function g such that fg^2 is also holomorphic. With these functions we then define

$$\varphi^1 = \frac{1}{2}f(1-g^2), \quad \varphi^2 = \frac{i}{2}f(1+g^2), \quad \varphi^3 = fg. \tag{10}$$

We then have

$$\begin{aligned}(\varphi)^2 &= (\frac{1}{2}f(1-g^2))^2 + (\frac{i}{2}f(1+g^2))^2 + (fg)^2 \\ &= \frac{1}{4}f^2(1-g^2)^2 - \frac{1}{4}f^2(1+g^2)^2 + f^2g^2 \\ &= \frac{1}{4}f^2(1+g^4-2g^2) - \frac{1}{4}f^2(1+g^4+2g^2) + f^2g^2 \\ &= -\frac{2}{4}f^2g^2 - \frac{2}{4}f^2g^2 + f^2g^2 \\ &= 0.\end{aligned}$$

This together with corollary indsæt ref til cor. ovenover gives us the Weierstrass-Enneper representation:

Theorem ? Let D be an open subset of the complex plane. If f is a holomorphic function on D and g is a meromorphic function on D such that fg^2 is a holomorphic function on D, then $s(z,\bar{z}) = (s^1(z,\bar{z}), s^2(z,\bar{z}), s^3(z,\bar{z}))$ with

$$\begin{aligned}s^1(z,\bar{z}) &= \mathfrak{Re}\Big(\int f(1-g^2)dz\Big) \\ s^2(z,\bar{z}) &= \mathfrak{Re}\Big(\int if(1+g^2)dz\Big) \\ s^3(z,\bar{z}) &= \mathfrak{Re}\Big(\int 2fg\,dz\Big)\end{aligned}$$

defines a minimal surface.

5.1 Weierstrass-Enneper representation examples

Let us take a look at some famous examples of minimal surfaces using the Weierstrass-Enneper representation.

We have already encountered the catenoid (figure 1), and shown that it has the isothermal parametrization

$$s(u,v) = (c\cosh\Big(\frac{u}{c}\Big)\cos(v),\ c\cosh\Big(\frac{u}{c}\Big)\sin(v),\ u), \tag{11}$$

where $u \in \mathbb{R}, v \in [-\pi, \pi)$.

Now say we get the great idea to define the following functions

$$\begin{aligned}f(z) &= -\mathrm{e}^{-z} \\ g(z) &= -\mathrm{e}^{z}.\end{aligned}$$

By (10), we then get the three holomorphic functions

$$\varphi^1 = \frac{\mathrm{e}^z - \mathrm{e}^{-z}}{2}, \quad \varphi^2 = -\frac{i}{2}(\mathrm{e}^z + \mathrm{e}^{-z}), \quad \varphi^3 = \mathrm{e}^z\mathrm{e}^{-z}.$$

Using Theorem indsæt ref til theorem lige over, we have

$$s^1(z,\bar{z}) = 2\mathfrak{Re}\Big(\int \varphi^1 dz\Big) = \mathfrak{Re}\Big(\int \mathrm{e}^z - \mathrm{e}^{-z} dz\Big) = \mathfrak{Re}(\mathrm{e}^z + \mathrm{e}^{-z}).$$

Applying then the coordinate transformation $z = u + iv$ to real coordinates, we have

$$\begin{aligned} s^1(u,v) &= \mathfrak{Re}(\mathrm{e}^{u+iv} + \mathrm{e}^{-u-iv}) \\ &= \mathfrak{Re}(\mathrm{e}^u \mathrm{e}^{iv} + \mathrm{e}^{-u}\mathrm{e}^{-iv} \\ &= \mathfrak{Re}(\mathrm{e}^u(\cos(v) + i\sin(v)) + \mathrm{e}^{-u}(\cos(v) - i\sin(v))) \\ &= \mathrm{e}^u\cos(v) + \mathrm{e}^{-u}\cos(v) \\ &= (\mathrm{e}^u + \mathrm{e}^{-u})\cos(v) \\ &= 2\cosh(u)\cos(v). \end{aligned}$$

indsæt ref til `https://en.wikipedia.org/wiki/Hyperbolic_functions` på sidste linje i alignen
Similarly for φ^2 we get

$$s^2(z,\bar{z}) = 2\mathfrak{Re}\Big(\int \varphi^2 dz\Big) = \mathfrak{Re}\Big(\int -i(\mathrm{e}^z + \mathrm{e}^{-z})dz\Big) = \mathfrak{Re}(-i(\mathrm{e}^z - \mathrm{e}^{-z})),$$

and therefore

$$\begin{aligned} s^2(u,v) &= \mathfrak{Re}(-i(\mathrm{e}^{u+iv} - \mathrm{e}^{-u-iv})) \\ &= \mathfrak{Re}(-i(\mathrm{e}^u\mathrm{e}^{iv} - \mathrm{e}^{-u}\mathrm{e}^{-iv})) \\ &= \mathfrak{Re}(-i(\mathrm{e}^u(\cos(v) + i\cos(v)) - \mathrm{e}^{-u}(\cos(v) - i\sin(v)))) \\ &= \mathrm{e}^u\sin(v) + \mathrm{e}^{-u}\sin(v) \\ &= (\mathrm{e}^u + \mathrm{e}^{-u})\sin(v) \\ &= 2\cosh(u)\sin(v). \end{aligned}$$

And for φ^3

$$s^3(z,\bar{z}) = 2\mathfrak{Re}\Big(\int \varphi^3 dz\Big) = \mathfrak{Re}\Big(\int 2\mathrm{e}^z\mathrm{e}^{-z} dz\Big) = \mathfrak{Re}(2z),$$

and therefore

$$\begin{aligned} s^3(u,v) &= \mathfrak{Re}(2u + i2v) \\ &= 2u. \end{aligned}$$

It now becomes clear that

$$s(u,v) = (2\cosh(u)\cos(v),\ 2\cosh(u)\sin(v),\ 2u)$$

is an isothermal parametrization of the form which defines a catenoid. Hence the catenoid is represented by the functions $f(z) = -\mathrm{e}^{-z}$ and $g(z) = -\mathrm{e}^z$. lav samme udregning med helicoid og plot den se dine udregninger i maple doku "weierstrassudregninger"

Say we instead choose the functions

$$\begin{aligned} f(z) &= \frac{-i}{2}\mathrm{e}^{-z} \\ g(z) &= -\mathrm{e}^z. \end{aligned}$$

We then get following holomorphic functions

$$\varphi^1 = -\frac{i}{4}(\mathrm{e}^{-z} - \mathrm{e}^{z}), \quad \varphi^2 = \frac{\mathrm{e}^{z} + \mathrm{e}^{-z}}{4}, \quad \varphi^3 = \frac{i}{2}\mathrm{e}^{z}\mathrm{e}^{-z}.$$

Using Theorem indsæt ref, we get

$$s^1(z,\bar{z}) = 2\mathfrak{Re}\Big(\int \varphi^1 dz\Big) = \mathfrak{Re}\Big(\int -\frac{i}{2}(\mathrm{e}^{-z} - \mathrm{e}^{z})dz\Big) = \mathfrak{Re}(-\frac{i}{2}(-\mathrm{e}^{-z} - \mathrm{e}^{z}))$$

and when transforming to real coordinates,

$$\begin{aligned} s^1(u,v) &= \mathfrak{Re}(-\frac{i}{2}(-\mathrm{e}^{-u-iv} - \mathrm{e}^{u+iv})) \\ &= \mathfrak{Re}(-\frac{i}{2}(-\mathrm{e}^{-u}\mathrm{e}^{-iv} - \mathrm{e}^{u}\mathrm{e}^{iv})) \\ &= \mathfrak{Re}(-\frac{i}{2}(-\mathrm{e}^{-u}(\cos(v) - i\sin(v)) - \mathrm{e}^{u}(\cos(v) + i\sin(v)))) \\ &= \frac{\mathrm{e}^{-u} - \mathrm{e}^{u}}{2}\sin(v). \end{aligned}$$

For φ^2 we get

$$s^2(z,\bar{z}) = 2\mathfrak{Re}\Big(\int \varphi^2 dz\Big) = \mathfrak{Re}\Big(\int \frac{\mathrm{e}^{z} + \mathrm{e}^{-z}}{2} dz\Big) = \mathfrak{Re}(\frac{\mathrm{e}^{z} - \mathrm{e}^{-z}}{2}),$$

and in real coordinates

$$\begin{aligned} s^2(u,v) &= \mathfrak{Re}\Big(-\frac{\mathrm{e}^{-u-iv}}{2} + \frac{\mathrm{e}^{u+iv}}{2}\Big) \\ &= \mathfrak{Re}\Big(-\frac{\mathrm{e}^{-u}\mathrm{e}^{-iv}}{2} + \frac{\mathrm{e}^{u}\mathrm{e}^{iv}}{2}\Big) \\ &= \mathfrak{Re}\Big(-\frac{\mathrm{e}^{-u}(\cos(v) - i\sin(v))}{2} + \frac{\mathrm{e}^{u}(\cos(v) + i\cos(v))}{2}\Big) \\ &= \frac{\mathrm{e}^{u} - \mathrm{e}^{-u}}{2}\cos(v). \end{aligned}$$

And finally for φ^3 we get

$$s^3(z,\bar{z}) = 2\mathfrak{Re}\Big(\int \varphi^3 dz\Big) = \mathfrak{Re}\Big(\int i\mathrm{e}^{z}\mathrm{e}^{-z} dz\Big) = \mathfrak{Re}(iz),$$

and in real coordinates

$$\begin{aligned} s^3(u,v) &= \mathfrak{Re}(i(u+iv)) \\ &= -v. \end{aligned}$$

All together we get the parametrization

$$s(u,v) = (\frac{\mathrm{e}^{-u} - \mathrm{e}^{u}}{2}\sin(v), \frac{\mathrm{e}^{u} - \mathrm{e}^{-u}}{2}\cos(v), -v). \tag{12}$$

The surface defined by this parametrization is shown in figure 2.

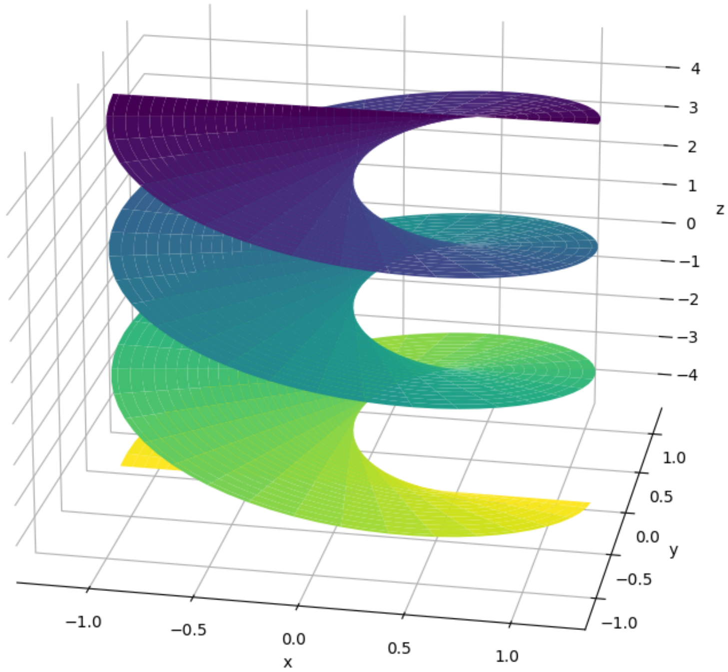

Figure 2: The 3D-surface defined by (12) with parameters $u \in [-1, 1], v \in [-\frac{3}{2}\pi, \frac{3}{2}\pi]$
.

As is apparent on figure 2, the parametrization (12) defines a helicoid.

6 The Gauss map of minimal surfaces

Recall that we for a surface M parametrized by $s(u, v)$ can define the unit normal vector

$$\mathrm{N}_p = \frac{s_u \times s_v}{|s_u \times s_v|}(p)$$

at every point $p \in M$.

Definition 6.1. *The Gauss map of a surface M is the map from M to the unit sphere (denoted by S^2) $G : M \to S^2$, defined by*

$$G(p) = \mathrm{N}_p.$$

More practically, for the parametrization $s(u, v)$ we write

$$G(s(u, v)) = \mathrm{N}(u, v),$$

which might help one realize that the Gauss map is simply a function that associates to some surface M a subset of the unit sphere.

Now, say we have some curve γ on M, and that we take the Gauss map along this curve. This way we get a new curve on S^2. A tangent vector of this new curve can then be obtained by the induced linear map on tangent vectors of G. [Oprea [2], p. 87]. We denote this map G_*. We have for the tangent vectors s_u and s_v, that $G_*(s_u) = \mathrm{N}_u$ and $G_*(s_v) = \mathrm{N}_v$ i.e. the tangent vector of the new curve is the image of a tangent vector on the original curve γ. Figure 3 illustrates the Gauss map of a helicoid. As we have seen, the helicoid is a minimal surface. Interestingly, there are some compelling results relating to the Gauss map of minimal surfaces.

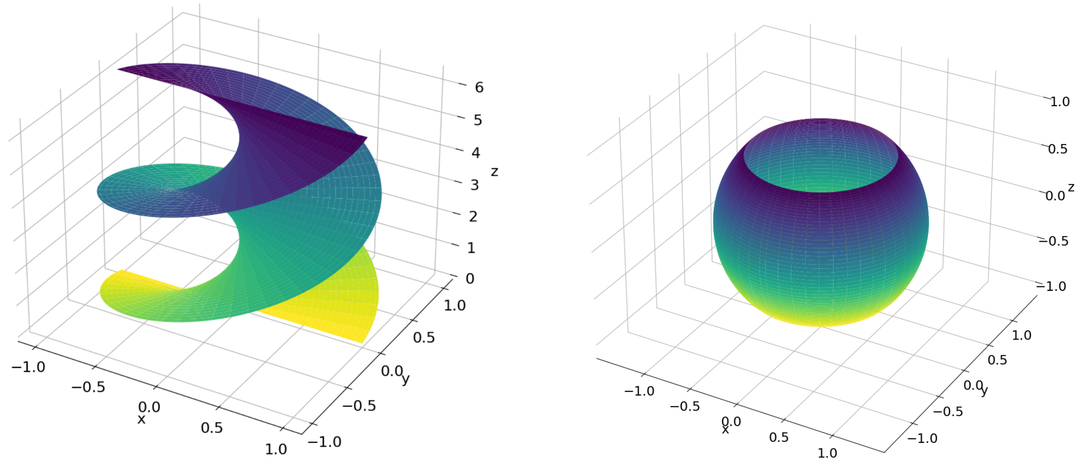

Figure 3: A helicoid (left) and the image of the Gauss map of the helicoid (right).

Firstly however, we need to introduce the concept of conformal linear maps.

Definition 6.2. *A linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ is said to be conformal if for some fixed $c > 0$*

$$T(v_1) \cdot T(v_2) = c^2 v_1 \cdot v_2$$

for all $v_1, v_2 \in \mathbb{R}^2$.

Theorem 3 (Oprea [2], p. 87)**.** *Say $\{v_1, v_2\}$ constitutes a basis of $\mathbb{R}^2$. Then T is conformal if and only if*

$$|T(v_1)| = c|v_1|, \quad |T(v_2)| = c|v_2|$$

and

$$T(v_1) \cdot T(v_2) = c^2 v_1 \cdot v_2,$$

for $c > 0$.

It should be noted for the following theorems that we say a map T is conformal if its induced linear map on tangent vectors T_* is conformal for every point in the domain of T. This is important since we have only defined what conformal means for linear maps on $\mathbb{R}^2$. In our context, the domain of the map in question - the Gauss map, is obviously some surface $M \subset \mathbb{R}^3$. Consequently, the map G_* is defined at a point of the surface M, at which the tangent plane is the domain of G_*.

At each point of M (and on the corresponding tangent plane), c is therefore a constant, but in the general case when considering all of M, we write $c(u, v)$.

Proposition 3.1. *Let a minimal surface M be defined by an isothermal parametrization $s(u, v)$. Then the Gauss map of M is conformal.*

Proof. For the proof we will make use of the following formulas [Oprea [2], p. 73]

$$G_*(s_u) = \mathrm{N}_u = -\frac{L}{E} s_u - \frac{M}{G} s_v$$
$$G_*(s_v) = \mathrm{N}_v = -\frac{M}{E} s_u - \frac{N}{G} s_v$$

(Note that G here is a second fundamental form coefficient and not the Gauss map).

The tangent vectors s_u and s_v together constitute a basis for the tangent plane and some point on M. Hence we will need to show that the conditions of Theorem 3 are satisfied for the induced linear map on tangent vectors G_*.

$s(u,v)$ is isothermal, so we have $s_u \cdot s_u = E = G$, $s_u \cdot s_v = F = 0$ and $H = \frac{L+N}{2E}$. We then get

$$\begin{aligned}
|\mathrm{N}_u|^2 &= \Big(\Big| -\frac{L}{E} \cdot s_u - \frac{M}{G} \cdot s_v \Big|\Big)\Big(\Big| -\frac{L}{E} \cdot s_u - \frac{M}{G} \cdot s_v \Big|\Big) \\
&= \Big| \frac{L \cdot s_u}{E} + \frac{M \cdot s_v}{E} \Big|^2 \\
&= \frac{1}{E^2}\Big(L^2 \cdot s_u^2 + M^2 \cdot s_v^2 + 2LM \cdot s_u \cdot s_v\Big) \\
&= \frac{1}{E^2}\Big(L^2 \cdot E + M^2 \cdot E) \\
&= \frac{L^2 + M^2}{E}.
\end{aligned}$$

By a similar calculation we get that

$$|\mathrm{N}_v|^2 = \frac{M^2 + N^2}{E},$$

and

$$\begin{aligned}
\mathrm{N}_u \cdot \mathrm{N}_v &= \Big(-\frac{L}{E} \cdot s_u - \frac{M}{G} \cdot s_v\Big)\Big(-\frac{M}{E} \cdot s_u - \frac{N}{G} \cdot s_v\Big) \\
&= \frac{(L \cdot s_u + M \cdot s_v)(M \cdot s_u + N \cdot s_v)}{E^2} \\
&= \frac{LME + MNE}{E^2} \\
&= \frac{M(L+N)}{E}.
\end{aligned}$$

Recall the mean curvature formula for isothermal parametrizations (6). For minimal surfaces, we have $H = 0$, which can only hold if $L = -N$.

We therefore have

$$|\mathrm{N}_u|^2 = |\mathrm{N}_v|^2 = \frac{L^2 + M^2}{E}$$

and

$$\mathrm{N}_u \mathrm{N}_v = 0,$$

which finally gives us

$$|G_*(s_u)| = |\mathrm{N}_u| = \frac{\sqrt{L^2 + M^2}}{E}\sqrt{E} = \frac{\sqrt{L^2 + M^2}}{E}|s_u|,$$

and

$$|G_*(s_v)| = |\mathrm{N}_v| = \frac{\sqrt{L^2 + M^2}}{E}\sqrt{E} = \frac{\sqrt{L^2 + M^2}}{E}|s_v|.$$

By Theorem 3, the Gauss map is conformal and has $c(u,v) = \frac{\sqrt{L^2+M^2}}{E}$. $\square$

Proposition 3.2.

References

[1] *Inverse hyperbolic functions*. 2024. URL: `https://en.wikipedia.org/wiki/Inverse_hyperbolic_functions`.

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[3] Andrew Pressley. *Elemental Differential Geometry*. springer-verlag london limited, 2001.