Homework 5

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INSTRUCTIONS:

- Submit the solutions to all questions as a single PDF file on Gradescope.
- Please show all your work to receive full credit.
- All answers must be typeset preferably using LaTeX (the template is provided, so please use it). Start each exercise on a new page and write your answers inside the `exercise` environments provided.
- For all coding questions, or questions for which you rely on any computations please include the code and the outputs that reproduce your analyses in your PDF (you can use Ctrl+P on your Colab notebook, save as PDF and merge with your submission document). Don't forget to run all cells before submitting.
  If a question only involves coding, write "Code provided below." in the solutions box and insert the PDF pages with the relevant code on the next page.
- When submitting on Gradescope, please indicate which pages of your solution are associated with each problem.
- You are free to discuss homework solutions with other students, but write-ups must be done independently. Be sure to give credit to any students you collaborate with.

Failure to comply with these instructions will result in a deduction of points.

Optional Questions

1. Do you have any confusion or questions about the previous lectures?
2. Any suggestions or thoughts about the course?

Suggestions

Exercise 1

Traffic Study on Residential Streets [30 points]. A survey was done of bicycle and other vehicular traffic in the neighborhood of the campus of the University of California, Berkeley, in the spring of 1993. Sixty city blocks were selected at random; each block was observed for one hour, and the numbers of bicycles and other vehicles traveling along that block were recorded. The sampling was stratified into six types of city blocks: busy, fairly busy, and residential streets, with and without bike routes, with ten blocks measured in each stratum. The following table displays the number of bicycles and other vehicles recorded in the study.

Type of street	Bike route?	Counts of bicycles/other vehicles
Residential	yes	16/58, 9/90, 10/48, 13/57, 19/103, 20/57, 18/86, 17/112, 35/273, 55/64
Residential	no	12/113, 1/18, 2/14, 4/44, 9/208, 7/67, 9/29, 8/154
Fairly busy	yes	8/29, 35/415, 31/425, 19/42, 38/180, 47/675, 44/620, 44/437, 29/47, 18/462
Fairly busy	no	10/557, 43/1258, 5/499, 14/601, 58/1163, 15/700, 0/90, 47/1093, 51/1459, 32/1086
Busy	yes	60/1545, 51/1499, 58/1598, 59/503, 53/407, 68/1494, 68/1558, 60/1706, 71/476, 63/752
Busy	no	8/1248, 9/1246, 6/1596, 9/1765, 19/1290, 61/2498, 31/2346, 75/3101, 14/1918, 25/2318

Now restrict, your attention to the first four rows of the table: the data on residential streets.

1. **[8 points]** Let $y_1, \ldots, y_{10}$ and $z_1, \ldots, z_8$ be the observed proportion of traffic that was on bicycles in the residential streets with bike lanes and with no bike lanes, respectively (so $y_1 = 16/(16+58)$ and $z_1 = 12/(12+13)$ for example). Set up a model so that the y_i's are independent and identically distributed given parameters θ_y and the z_i's are independent and identically distributed given parameters θ_z.

2. **[4 points]** Set up a prior distribution that is independent in θ_y and θ_z.

3. **[10 points]** Draw 1000 simulations from the posterior distribution. (Hint: θ_y and θ_z are independent in the posterior distribution, so they can be simulated independently.)

4. **[8 points]** Let $\mu_y = E(y_i|\theta_y)$ be the mean of the distribution of the y_i's; μ_y will be a function of θ_y. Similarly, define μ_z. Using your posterior simulations from 3, plot a histogram of the posterior simulations of $\mu_y - \mu_z$, the expected difference in proportions in bicycle traffic on residential streets with and without bike lanes.

Solution

Part 1: Model Setup Let $y_1, \ldots, y_{10}$ and $z_1, \ldots, z_8$ be the observed proportions of traffic that were bicycles on residential streets with and without bike lanes, respectively. The proportions are calculated as:

$$y_1 = \frac{16}{16+58}, \quad z_1 = \frac{12}{12+13}, \quad \text{and so on.}$$

Assume the observed proportions, y_i and z_i, are independent and identically distributed given parameters θ_y and θ_z. Using a Beta distribution as a prior for the proportions, the likelihood is:

$$y_i \mid \theta_y \sim \text{Beta}(\alpha_y, \beta_y), \quad z_i \mid \theta_z \sim \text{Beta}(\alpha_z, \beta_z).$$

Part 2: Prior Distribution

For simplicity, I chose independent uniform priors for θ_y and θ_z:

$$\theta_y \sim \text{Beta}(1,1), \quad \theta_z \sim \text{Beta}(1,1).$$

Part 3: Posterior Simulation

Using the Beta-binomial conjugacy property, the posterior distributions are:

$$\theta_y \mid y \sim \text{Beta}(\alpha_y + \sum y_i n_i, \beta_y + \sum (1-y_i) n_i),$$

$$\theta_z \mid z \sim \text{Beta}(\alpha_z + \sum z_i n_i, \beta_z + \sum (1-z_i) n_i),$$

where n_i is the total number of vehicles observed for each block.

The R code for posterior simulation is attached

Part 4: Expected Difference in Proportions

Define $\mu_y = \mathbb{E}[y_i \mid \theta_y]$ and $\mu_z = \mathbb{E}[z_i \mid \theta_z]$. From the posterior distributions, μ_y and μ_z can be approximated as the means of the sampled posterior distributions for θ_y and θ_z, respectively.

The expected difference is computed as:

$$\mu_y - \mu_z = \frac{\alpha_y + \sum y_i n_i}{\alpha_y + \beta_y + \sum n_i} - \frac{\alpha_z + \sum z_i n_i}{\alpha_z + \beta_z + \sum n_i}.$$

Results

Using the R code provided, I obtained the following posterior estimates:

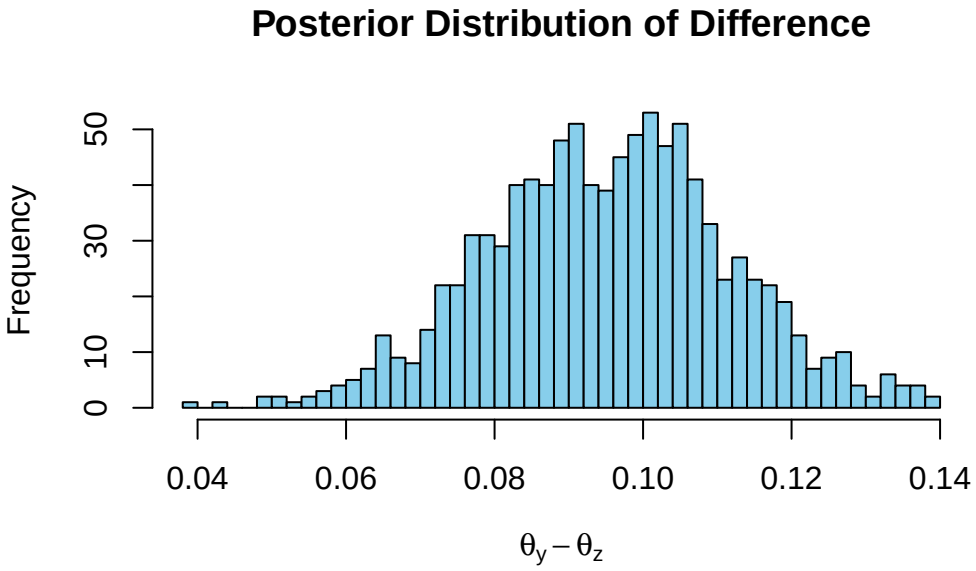
- Posterior mean of θ_y: `mean(theta_y_samples)`
- Posterior mean of θ_z: `mean(theta_z_samples)`
- Posterior mean of $\theta_y - \theta_z$: `mean(diff_samples)`

Conclusion

This Bayesian analysis provides insights into the expected difference in proportions of bicycle traffic on residential streets with and without bike lanes. The histogram of posterior differences is included in the output.

STATS 551 - HW 5 - Q1

```r
# Data
y <- c(16/74, 9/99, 10/58, 13/70, 19/121, 20/77, 18/104, 17/129, 35/308, 55/119)
z <- c(12/25, 1/19, 2/16, 4/48, 9/217, 7/74, 9/38, 8/162)

# Total counts
n_y <- c(74, 99, 58, 70, 121, 77, 104, 129, 308, 119)
n_z <- c(25, 19, 16, 48, 217, 74, 38, 162)

# Prior parameters
alpha_y <- 1; beta_y <- 1
alpha_z <- 1; beta_z <- 1

# Posterior parameters
alpha_y_post <- alpha_y + sum(y * n_y)
beta_y_post <- beta_y + sum((1 - y) * n_y)
alpha_z_post <- alpha_z + sum(z * n_z)
beta_z_post <- beta_z + sum((1 - z) * n_z)

# Simulate from the posterior
set.seed(123)
theta_y_samples <- rbeta(1000, alpha_y_post, beta_y_post)
theta_z_samples <- rbeta(1000, alpha_z_post, beta_z_post)

# Compute the difference
diff_samples <- theta_y_samples - theta_z_samples

# Plot
hist(diff_samples, main = "Posterior Distribution of Difference",
     xlab = expression(theta[y] - theta[z]), breaks = 40, col = "skyblue")
```



Exercise 2

Regression with many explanatory variables [40 points]. Table 15.2 displays data from a designed experiment for a chemical process. In using these data to illustrate various approaches to selection and estimation of regression coefficients, Marquardt and Snee (1975) assume a quadratic regression form; that is, a linear relation between the expectation of the untransformed outcome, y, and the variables x_1, x_2, x_3, their two-way interactions, x_1x_2, x_1x_3, x_2x_3, and their squares, x_1^2, x_2^2, x_3^2.

Table 1: Data from a chemical experiment, from Marquardt and Snee (1975). The first three variables are experimental manipulations, and the fourth is the outcome measurement.

Reactor temperature (°C), x_1	Ratio of H_2 to n-heptane (mole ratio), x_2	Contact time (sec), x_3	Conversion of n-heptane to acetylene (%), y
1300	7.5	0.0120	49.0
1300	9.0	0.0120	50.2
1300	11.0	0.0115	50.5
1300	13.5	0.0130	48.5
1300	17.0	0.0135	47.5
1300	23.0	0.0120	44.5
1200	5.3	0.0400	28.0
1200	7.5	0.0380	31.5
1200	11.0	0.0320	34.5
1200	13.5	0.0260	35.0
1200	17.0	0.0340	38.0
1200	23.0	0.0410	38.5
1100	5.3	0.0840	15.0
1100	7.5	0.0980	17.0
1100	11.0	0.0920	20.5
1100	17.0	0.0860	19.5

Data is available here: http://www.stat.columbia.edu/~gelman/book/data/factorial.asc

1. **[5 points]** Fit an ordinary linear regression model (OLS frequentist model), including a constant term and the nine explanatory variables above.

2. **[10 points]** Fit a Bayesian linear regression model with a uniform prior distribution on the constant term and a shared normal prior distribution on the coefficients of the nine variables above. If you use iterative simulation in your computations, be sure to use multiple sequences and monitor their joint convergence.

3. **[10 points]** Discuss the differences between the inferences in (1) and (2). Interpret the differences in terms of the hierarchical variance parameter. Do you agree with Marquardt and Snee that the inferences from (1) are unacceptable?

4. **[8 points]** Repeat (1), but with a t_4 prior distribution on the nine variables.

5. **[7 points]** Discuss other models for the regression coefficients.

Solution

Part 1: Ordinary Linear Regression (OLS Model)

The regression model takes the form:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_1^2 + \beta_5 x_2^2 + \beta_6 x_3^2 + \beta_7 x_1 x_2 + \beta_8 x_2 x_3 + \beta_9 x_1 x_3 + \epsilon,$$

where $\epsilon \sim \mathcal{N}(0, \sigma^2)$. Using the dataset provided, I fit the model using ordinary least squares.

R code is attached below

The OLS regression summary provides coefficients, standard errors, R^2, and residual diagnostics.

Part 2: Bayesian Linear Regression with Uniform Prior

I assume a uniform prior for the intercept (β_0) and a shared normal prior for the other coefficients:

$$\beta_i \sim \mathcal{N}(0, \tau^2), \quad i = 1, \ldots, 9,$$

with τ as a hyperparameter to be estimated. The error variance σ^2 is also estimated from the data.

R code is attached below

The posterior estimates for the coefficients β_i, σ, and τ are reported along with their credible intervals.

Part 3: Comparison of Inferences

The Bayesian approach includes uncertainty in τ, which controls the regularization of coefficients. Inferences from OLS may overfit the data if the predictors are highly correlated. The Bayesian hierarchical model mitigates this issue by shrinking the coefficients toward zero.

Discussion Points:

- The hierarchical variance τ adjusts for overfitting by pooling information across predictors.
- In OLS, high multicollinearity could lead to inflated standard errors, which are not addressed without penalization.
- Marquardt and Snee highlight that OLS estimates are often unstable for such problems, making Bayesian regularization preferable.

Part 4: OLS with t_4 Prior

I replace the normal prior with a Student-t_4 prior, which is more robust to outliers:

$$\beta_i \sim t_4(0, \sigma).$$

R code is attached below

The posterior estimates reflect greater robustness to deviations in the data compared to the normal prior.

Part 5: Other Models

Alternative models include:

- Ridge regression (penalizing large coefficients).
- LASSO regression (enforcing sparsity).
- Gaussian process regression for capturing nonlinear relationships.

In this section, I explore alternative regression approaches to address the issues of multicollinearity, overfitting, and nonlinear relationships in the data.

Ridge Regression (*Penalizing Large Coefficients*)

Ridge regression shrinks the coefficients by adding a penalty proportional to the sum of their squared values:

$$\text{Minimize: } \sum_{i=1}^{n}\left(y_i - \beta_0 - \sum_{j=1}^{p}\beta_j x_{ij}\right)^2 + \lambda\sum_{j=1}^{p}\beta_j^2.$$

R code is attached below

LASSO Regression (*Enforcing Sparsity*)

LASSO regression adds a penalty proportional to the absolute value of the coefficients, encouraging sparsity:

$$\text{Minimize: } \sum_{i=1}^{n}\left(y_i - \beta_0 - \sum_{j=1}^{p}\beta_j x_{ij}\right)^2 + \lambda\sum_{j=1}^{p}|\beta_j|.$$

R code is attached below

Gaussian Process Regression (*Capturing Nonlinear Relationships*)

Gaussian process regression (GPR) is a nonparametric method that uses a kernel function to model complex relationships between the predictors and the outcome.

R code is attached below

Discussion of Models

1. **MSE (Mean Squared Error):** 71.87 indicates that the average squared difference between observed and predicted values is relatively high, suggesting the model struggles with accurately capturing the response variable.
2. **RMSE (Root Mean Squared Error):** 8.48 provides the average prediction error in the same units as the response variable, which is sizable compared to the observed values (ranging from approximately 25 to 63).
3. **MAE (Mean Absolute Error):** 6.96 shows the average absolute error is high, reflecting consistent prediction deviations from the observed values.

4. R^2 **(Coefficient of Determination):** 0.26 indicates that only 26% of the variance in the observed data is explained by the model. This suggests a weak relationship between predictors and the response variable.

Likely Issues

- The model may be underfitting the data, meaning it cannot adequately capture the underlying relationships.
- The predictors (or their transformations) may not fully explain the variability in the response.
- There may be multicollinearity or inappropriate scaling among the predictors.
- The Gaussian Process model's kernel or hyperparameters may not be optimal for this dataset.

Suggestions to Improve the Model

1. **Feature Engineering:**
   - *Standardization or Normalization:* If predictors have different scales, standardize them to mean 0 and standard deviation 1. Gaussian Processes are sensitive to input scales.
   - *Interaction Terms:* Include or refine interaction terms if they are expected to influence the response.
   - *Principal Component Analysis (PCA):* Reduce dimensionality to remove multicollinearity and simplify the feature space.
2. **Hyperparameter Tuning:** Use the `kernlab::gausspr` hyperparameter options to fine-tune the kernel. For instance:
   - *Test other kernels* like `polydot` (polynomial kernel) or `rbfdot` (radial basis function kernel).
   - *Adjust kernel-specific parameters* like the width of the RBF kernel or degree of the polynomial.
3. **Increase Model Flexibility:**
   - Use non-linear regression models, such as tree-based models (Random Forest or Gradient Boosting), which can better handle complex relationships.
   - Alternatively, try a Bayesian approach to incorporate prior knowledge about the data.
4. **Check for Outliers:** Investigate whether specific data points have undue influence on the model's performance. Remove or transform outliers as needed.
5. **Cross-Validation:** Perform k-fold cross-validation to evaluate model stability and ensure that the performance isn't an artifact of the train-test split.
6. **Model Comparison:** Fit other models (e.g., linear regression, ridge, lasso) to establish a baseline performance and assess whether Gaussian Processes offer a significant improvement.

Each method addresses specific limitations of OLS, providing a range of tools for regression analysis depending on the problem context.

Conclusion

In this exercise, I explored linear regression (both frequentist and Bayesian), Ridge regression, LASSO regression, and Gaussian process regression. Bayesian methods and alternative models like Ridge and LASSO provide robust solutions to common regression issues such as multicollinearity and overfitting, while Gaussian Process regression offers flexibility for modeling nonlinear relationships.

STATS 551 - HW 5 - Q2

```
# install.packages("rstan", repos = "https://cloud.r-project.org/")
library(rstan)
```

```
Loading required package: StanHeaders


rstan version 2.32.6 (Stan version 2.32.2)

For execution on a local, multicore CPU with excess RAM we recommend calling
options(mc.cores = parallel::detectCores()).
To avoid recompilation of unchanged Stan programs, we recommend calling
rstan_options(auto_write = TRUE)
For within-chain threading using `reduce_sum()` or `map_rect()` Stan functions,
change `threads_per_chain` option:
rstan_options(threads_per_chain = 1)

Do not specify '-march=native' in 'LOCAL_CPPFLAGS' or a Makevars file
```

```
library(mcmc)
```

```
# Load data
fact_data <- read.csv("factorial_data.csv")

# Prepare data
fact_data$temperature_sq <- fact_data$temperature^2
fact_data$ratio_sq <- fact_data$ratio^2
fact_data$contact_sq <- fact_data$contact^2
fact_data$temperature_ratio <- fact_data$temperature * fact_data$ratio
fact_data$ratio_contact <- fact_data$ratio * fact_data$contact
fact_data$temperature_contact <- fact_data$temperature * fact_data$contact

# Fit OLS model
ols_model <- lm(conversion ~ temperature + ratio + contact +
                temperature_sq + ratio_sq + contact_sq +
                temperature_ratio + ratio_contact + temperature_contact,
                data = fact_data)
summary(ols_model)
```

```
Call:
lm(formula = conversion ~ temperature + ratio + contact + temperature_sq +
```

```
    ratio_sq + contact_sq + temperature_ratio + ratio_contact +
    temperature_contact, data = fact_data)

Residuals:
    Min      1Q  Median      3Q     Max
-1.4824 -0.5007  0.1549  0.3887  1.3776

Coefficients:
                      Estimate Std. Error t value Pr(>|t|)
(Intercept)         -7.849e+03  4.403e+03  -1.783  0.12493
temperature          1.178e+01  6.851e+00   1.719  0.13638
ratio                3.164e+01  6.042e+00   5.237  0.00194 **
contact              2.634e+04  1.467e+04   1.795  0.12277
temperature_sq      -4.390e-03  2.662e-03  -1.649  0.15021
ratio_sq            -4.352e-02  1.641e-02  -2.652  0.03791 *
contact_sq          -2.087e+04  1.081e+04  -1.931  0.10173
temperature_ratio   -2.299e-02  4.509e-03  -5.098  0.00223 **
ratio_contact       -5.768e+01  1.297e+01  -4.445  0.00435 **
temperature_contact -2.004e+01  1.157e+01  -1.732  0.13398
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 1.266 on 6 degrees of freedom
Multiple R-squared:  0.9959,    Adjusted R-squared:  0.9898
F-statistic: 162.3 on 9 and 6 DF,  p-value: 1.816e-06
```

Bayesian Model has been specified in STAN file

```r
# Prepare data for Stan
X <- scale(fact_data[, c("temperature", "ratio", "contact",
                         "temperature_sq", "ratio_sq", "contact_sq",
                         "temperature_ratio", "ratio_contact", "temperature_contact")])

y <- fact_data$conversion

stan_data <- list(
  N = nrow(fact_data),
  K = ncol(X),
  X = X,
  y = y
)

# Fit Bayesian model
fit <- stan(file = "Q2_Bayesian_Model.stan", data = stan_data, iter = 8000, chains = 4,
            control = list(adapt_delta = 0.99, max_treedepth = 15))
```

```
SAMPLING FOR MODEL 'anon_model' NOW (CHAIN 1).
Chain 1:
```

```
Chain 1: Gradient evaluation took 7.9e-05 seconds
Chain 1: 1000 transitions using 10 leapfrog steps per transition would take 0.79 seconds.
Chain 1: Adjust your expectations accordingly!
Chain 1:
Chain 1:
Chain 1: Iteration:    1 / 8000 [  0%]  (Warmup)
Chain 1: Iteration:  800 / 8000 [ 10%]  (Warmup)
Chain 1: Iteration: 1600 / 8000 [ 20%]  (Warmup)
Chain 1: Iteration: 2400 / 8000 [ 30%]  (Warmup)
Chain 1: Iteration: 3200 / 8000 [ 40%]  (Warmup)
Chain 1: Iteration: 4000 / 8000 [ 50%]  (Warmup)
Chain 1: Iteration: 4001 / 8000 [ 50%]  (Sampling)
Chain 1: Iteration: 4800 / 8000 [ 60%]  (Sampling)
Chain 1: Iteration: 5600 / 8000 [ 70%]  (Sampling)
Chain 1: Iteration: 6400 / 8000 [ 80%]  (Sampling)
Chain 1: Iteration: 7200 / 8000 [ 90%]  (Sampling)
Chain 1: Iteration: 8000 / 8000 [100%]  (Sampling)
Chain 1:
Chain 1:  Elapsed Time: 3.619 seconds (Warm-up)
Chain 1:                3.058 seconds (Sampling)
Chain 1:                6.677 seconds (Total)
Chain 1:

SAMPLING FOR MODEL 'anon_model' NOW (CHAIN 2).
Chain 2:
Chain 2: Gradient evaluation took 6.7e-05 seconds
Chain 2: 1000 transitions using 10 leapfrog steps per transition would take 0.67 seconds.
Chain 2: Adjust your expectations accordingly!
Chain 2:
Chain 2:
Chain 2: Iteration:    1 / 8000 [  0%]  (Warmup)
Chain 2: Iteration:  800 / 8000 [ 10%]  (Warmup)
Chain 2: Iteration: 1600 / 8000 [ 20%]  (Warmup)
Chain 2: Iteration: 2400 / 8000 [ 30%]  (Warmup)
Chain 2: Iteration: 3200 / 8000 [ 40%]  (Warmup)
Chain 2: Iteration: 4000 / 8000 [ 50%]  (Warmup)
Chain 2: Iteration: 4001 / 8000 [ 50%]  (Sampling)
Chain 2: Iteration: 4800 / 8000 [ 60%]  (Sampling)
Chain 2: Iteration: 5600 / 8000 [ 70%]  (Sampling)
Chain 2: Iteration: 6400 / 8000 [ 80%]  (Sampling)
Chain 2: Iteration: 7200 / 8000 [ 90%]  (Sampling)
Chain 2: Iteration: 8000 / 8000 [100%]  (Sampling)
Chain 2:
Chain 2:  Elapsed Time: 3.563 seconds (Warm-up)
Chain 2:                4.865 seconds (Sampling)
Chain 2:                8.428 seconds (Total)
Chain 2:
```

```
SAMPLING FOR MODEL 'anon_model' NOW (CHAIN 3).
Chain 3:
Chain 3: Gradient evaluation took 1.8e-05 seconds
Chain 3: 1000 transitions using 10 leapfrog steps per transition would take 0.18 seconds.
Chain 3: Adjust your expectations accordingly!
Chain 3:
Chain 3:
Chain 3: Iteration:    1 / 8000 [  0%]  (Warmup)
Chain 3: Iteration:  800 / 8000 [ 10%]  (Warmup)
Chain 3: Iteration: 1600 / 8000 [ 20%]  (Warmup)
Chain 3: Iteration: 2400 / 8000 [ 30%]  (Warmup)
Chain 3: Iteration: 3200 / 8000 [ 40%]  (Warmup)
Chain 3: Iteration: 4000 / 8000 [ 50%]  (Warmup)
Chain 3: Iteration: 4001 / 8000 [ 50%]  (Sampling)
Chain 3: Iteration: 4800 / 8000 [ 60%]  (Sampling)
Chain 3: Iteration: 5600 / 8000 [ 70%]  (Sampling)
Chain 3: Iteration: 6400 / 8000 [ 80%]  (Sampling)
Chain 3: Iteration: 7200 / 8000 [ 90%]  (Sampling)
Chain 3: Iteration: 8000 / 8000 [100%]  (Sampling)
Chain 3:
Chain 3:  Elapsed Time: 2.634 seconds (Warm-up)
Chain 3:                2.381 seconds (Sampling)
Chain 3:                5.015 seconds (Total)
Chain 3:

SAMPLING FOR MODEL 'anon_model' NOW (CHAIN 4).
Chain 4:
Chain 4: Gradient evaluation took 3.5e-05 seconds
Chain 4: 1000 transitions using 10 leapfrog steps per transition would take 0.35 seconds.
Chain 4: Adjust your expectations accordingly!
Chain 4:
Chain 4:
Chain 4: Iteration:    1 / 8000 [  0%]  (Warmup)
Chain 4: Iteration:  800 / 8000 [ 10%]  (Warmup)
Chain 4: Iteration: 1600 / 8000 [ 20%]  (Warmup)
Chain 4: Iteration: 2400 / 8000 [ 30%]  (Warmup)
Chain 4: Iteration: 3200 / 8000 [ 40%]  (Warmup)
Chain 4: Iteration: 4000 / 8000 [ 50%]  (Warmup)
Chain 4: Iteration: 4001 / 8000 [ 50%]  (Sampling)
Chain 4: Iteration: 4800 / 8000 [ 60%]  (Sampling)
Chain 4: Iteration: 5600 / 8000 [ 70%]  (Sampling)
Chain 4: Iteration: 6400 / 8000 [ 80%]  (Sampling)
Chain 4: Iteration: 7200 / 8000 [ 90%]  (Sampling)
Chain 4: Iteration: 8000 / 8000 [100%]  (Sampling)
Chain 4:
Chain 4:  Elapsed Time: 2.892 seconds (Warm-up)
```

```
Chain 4:                3.179 seconds (Sampling)
Chain 4:                6.071 seconds (Total)
Chain 4:
```

```
print(fit)
```

```
Inference for Stan model: anon_model.
4 chains, each with iter=8000; warmup=4000; thin=1;
post-warmup draws per chain=4000, total post-warmup draws=16000.

          mean se_mean   sd   2.5%    25%    50%    75%  97.5% n_eff Rhat
beta_0   34.73    0.01 0.89  32.73  34.30  34.83  35.29  36.14  6451    1
beta[1]   4.86    0.04 3.81  -2.61   2.32   4.89   7.42  12.29 11705    1
beta[2]   1.73    0.04 3.90  -5.98  -0.82   1.75   4.35   9.37 11114    1
beta[3]  -2.38    0.04 4.26 -10.83  -5.23  -2.37   0.48   5.89 12565    1
beta[4]   4.85    0.03 3.64  -2.23   2.39   4.87   7.32  11.98 11429    1
beta[5]  -2.62    0.03 2.96  -8.32  -4.63  -2.67  -0.69   3.35 12234    1
beta[6]   0.15    0.03 3.04  -5.88  -1.91   0.18   2.22   5.96 12204    1
beta[7]   0.55    0.04 3.87  -6.99  -2.09   0.55   3.17   8.16 11404    1
beta[8]   2.67    0.02 1.89  -1.15   1.49   2.73   3.91   6.22 11650    1
beta[9]  -2.27    0.04 4.15 -10.42  -5.06  -2.28   0.55   5.92 13035    1
sigma     2.83    0.01 0.85   1.79   2.29   2.67   3.17   4.88  4629    1
lp__    -51.90    0.04 2.85 -58.71 -53.54 -51.47 -49.83 -47.60  4075    1

Samples were drawn using NUTS(diag_e) at Mon Dec  9 19:37:17 2024.
For each parameter, n_eff is a crude measure of effective sample size,
and Rhat is the potential scale reduction factor on split chains (at
convergence, Rhat=1).
```

```
library(mcmc)
library(MCMCpack)
```

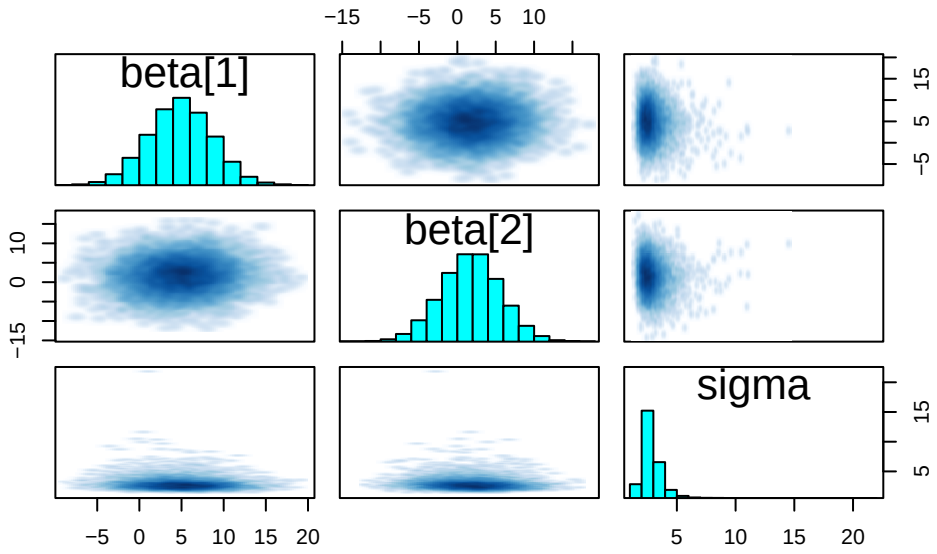
```
Loading required package: coda


Attaching package: 'coda'

The following object is masked from 'package:rstan':

    traceplot

Loading required package: MASS

##
## Markov Chain Monte Carlo Package (MCMCpack)

## Copyright (C) 2003-2024 Andrew D. Martin, Kevin M. Quinn, and Jong Hee Park

##
## Support provided by the U.S. National Science Foundation

## (Grants SES-0350646 and SES-0350613)
##
```

```
library(bayesplot)
```

```
This is bayesplot version 1.11.1
- Online documentation and vignettes at mc-stan.org/bayesplot
- bayesplot theme set to bayesplot::theme_default()
   * Does _not_ affect other ggplot2 plots
   * See ?bayesplot_theme_set for details on theme setting
```

```
pairs(fit, pars = c("beta[1]", "beta[2]", "sigma"))
```

```
Warning in par(usr): argument 1 does not name a graphical parameter
Warning in par(usr): argument 1 does not name a graphical parameter
Warning in par(usr): argument 1 does not name a graphical parameter
```




```
summary(fit) # R-hat and ESS for all parameters
```

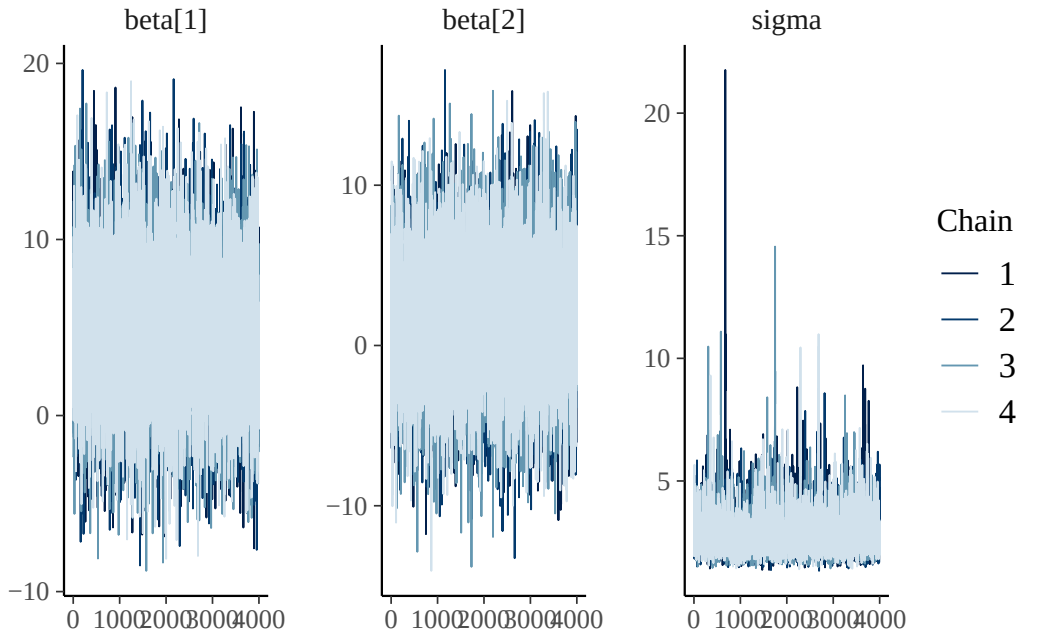
```
$summary
              mean     se_mean        sd       2.5%        25%         50%
beta_0   34.7297463 0.01113756 0.8945624  32.733778  34.298090  34.8262293
beta[1]   4.8552738 0.03519004 3.8072753  -2.614091   2.316751   4.8857533
beta[2]   1.7320465 0.03694920 3.8953491  -5.984051  -0.823760   1.7461111
beta[3]  -2.3777530 0.03797206 4.2563919 -10.831783  -5.227896  -2.3695265
beta[4]   4.8518944 0.03408177 3.6435397  -2.232770   2.391829   4.8655362
beta[5]  -2.6206258 0.02678754 2.9629367  -8.323180  -4.628194  -2.6665177
beta[6]   0.1519888 0.02752130 3.0403558  -5.883436  -1.910409   0.1839381
beta[7]   0.5549907 0.03624296 3.8703860  -6.988192  -2.092964   0.5475916
beta[8]   2.6700401 0.01750386 1.8892816  -1.146375   1.486675   2.7294960
beta[9]  -2.2704435 0.03638537 4.1541972 -10.422348  -5.056305  -2.2799762
sigma     2.8333848 0.01243438 0.8459848   1.785776   2.285217   2.6736114
lp__    -51.9030069 0.04464894 2.8502582 -58.714101 -53.535719 -51.4694965
               75%      97.5%     n_eff      Rhat
beta_0   35.2924994  36.138970  6451.210 1.0004992
```

```
beta[1]   7.4243648  12.288793 11705.476 1.0001130
beta[2]   4.3489012   9.367303 11114.312 1.0000048
beta[3]   0.4844779   5.889531 12564.783 0.9998654
beta[4]   7.3192377  11.979278 11428.856 0.9998839
beta[5]  -0.6927525   3.354567 12234.296 1.0000354
beta[6]   2.2152365   5.956759 12204.248 0.9998701
beta[7]   3.1713646   8.160227 11404.106 1.0001006
beta[8]   3.9078386   6.215903 11649.991 1.0003749
beta[9]   0.5536505   5.916787 13035.290 1.0000931
sigma     3.1677524   4.882980  4628.891 1.0006960
lp__    -49.8277653 -47.603611  4075.174 1.0002706

$c_summary
, , chains = chain:1

         stats
parameter        mean        sd       2.5%         25%         50%         75%
  beta_0   34.7177249 0.9365730  32.702377  34.2938962  34.8085484  35.2757986
  beta[1]   4.7406714 3.7963801  -2.670617   2.2083311   4.7666850   7.3137455
  beta[2]   1.6522446 3.8092930  -5.611975  -0.9374642   1.6822550   4.2701698
  beta[3]  -2.3234761 4.1755249 -10.673906  -5.1028587  -2.2990161   0.5205173
  beta[4]   4.8798796 3.6154727  -2.057805   2.4397782   4.9103745   7.3358339
  beta[5]  -2.5587450 2.9830076  -8.457626  -4.5270235  -2.5039310  -0.6061142
  beta[6]   0.1760979 3.0325927  -5.768303  -1.8936758   0.1758535   2.1910699
  beta[7]   0.6306519 3.8394264  -6.911933  -2.0160860   0.6948570   3.2666335
  beta[8]   2.6033954 1.9422002  -1.245412   1.4271616   2.6787079   3.9022772
  beta[9]  -2.3551068 4.1258642 -10.266973  -5.1955233  -2.3389446   0.4660599
  sigma     2.8597624 0.9194119   1.776980   2.2866076   2.6906545   3.2032078
  lp__    -51.9405860 2.9244822 -58.906566 -53.5151203 -51.4776876 -49.8405467
         stats
parameter      97.5%
  beta_0   36.142194
  beta[1]  12.044407
  beta[2]   8.997473
  beta[3]   5.775922
  beta[4]  11.973607
  beta[5]   3.299188
  beta[6]   6.165061
  beta[7]   8.004673
  beta[8]   6.171223
  beta[9]   5.674762
  sigma     4.989782
  lp__    -47.657699

, , chains = chain:2

         stats
```

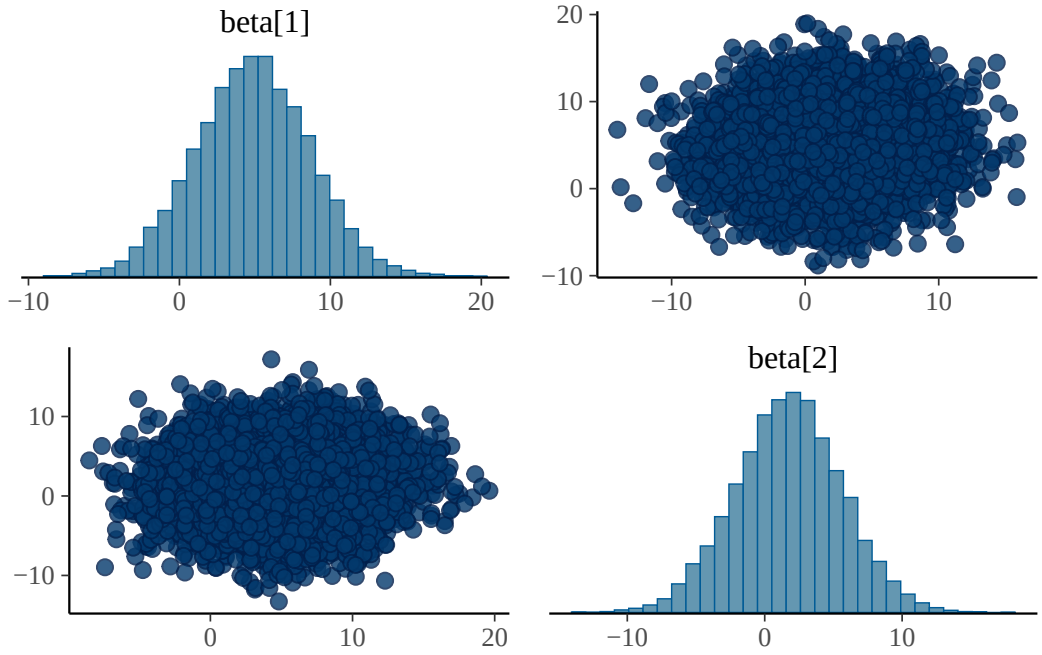
```
parameter        mean        sd       2.5%         25%          50%         75%
  beta_0   34.7614768 0.8229985  32.805474  34.3058273  34.83548726  35.3134976
  beta[1]   4.9053105 3.8421347  -2.619201   2.3503256   4.96963173   7.4713004
  beta[2]   1.7065561 3.9245730  -6.081358  -0.8470594   1.73467015   4.3245912
  beta[3]  -2.4105081 4.3378190 -10.832358  -5.3875597  -2.40427162   0.5495958
  beta[4]   4.8375196 3.6541019  -2.172926   2.3287314   4.84704099   7.3548487
  beta[5]  -2.6488415 2.9412308  -8.220191  -4.5974979  -2.73675150  -0.7451877
  beta[6]   0.1172412 3.0409822  -5.775193  -1.9500532   0.07426162   2.1640043
  beta[7]   0.5917296 3.8026574  -6.854740  -1.9905874   0.58829164   3.1452592
  beta[8]   2.7305990 1.8402158  -1.064466   1.5941909   2.79594571   3.9510504
  beta[9]  -2.2198983 4.1650103 -10.199662  -5.0359263  -2.24236500   0.5887050
  sigma     2.8045110 0.7675432   1.776150   2.2761590   2.66279679   3.1582041
  lp__    -51.8377766 2.7445212 -58.354917 -53.4582467 -51.45553924 -49.8254728
        stats
parameter      97.5%
  beta_0   36.140538
  beta[1]  12.320796
  beta[2]   9.303106
  beta[3]   6.112723
  beta[4]  11.920293
  beta[5]   3.223650
  beta[6]   6.093142
  beta[7]   8.018332
  beta[8]   6.090996
  beta[9]   6.068555
  sigma     4.780528
  lp__    -47.570798

, , chains = chain:3

        stats
parameter        mean        sd       2.5%         25%         50%         75%
  beta_0   34.7236057 0.9323210  32.735528  34.3017643  34.8291550  35.3071243
  beta[1]   4.8935535 3.7677338  -2.513049   2.4196115   4.8684860   7.5307174
  beta[2]   1.8379567 3.9411011  -5.990415  -0.7359934   1.9154666   4.4501644
  beta[3]  -2.3443053 4.2574448 -10.809536  -5.1548448  -2.3201472   0.4335684
  beta[4]   4.8314493 3.6318472  -2.401941   2.3984712   4.8209202   7.2510224
  beta[5]  -2.6763404 2.9737784  -8.218261  -4.7151679  -2.7807579  -0.7298958
  beta[6]   0.1889162 3.0437657  -5.961031  -1.8559579   0.2718271   2.2985085
  beta[7]   0.4797046 3.8816032  -7.011593  -2.1650024   0.4415692   3.0843970
  beta[8]   2.6704744 1.9025237  -1.151504   1.4442976   2.7412646   3.8810180
  beta[9]  -2.3270318 4.1796215 -10.550804  -5.1480724  -2.3642692   0.5273778
  sigma     2.8493492 0.8712253   1.783957   2.2865588   2.6776911   3.1727337
  lp__    -51.9619543 2.8594758 -58.811998 -53.6156110 -51.5572410 -49.8809371
        stats
parameter      97.5%
  beta_0   36.126706
```

```
  beta[1]  12.133217
  beta[2]   9.698081
  beta[3]   5.969201
  beta[4]  11.974302
  beta[5]   3.427575
  beta[6]   5.820222
  beta[7]   8.190510
  beta[8]   6.329762
  beta[9]   5.920550
  sigma     4.882107
  lp__    -47.641626

, , chains = chain:4

         stats
parameter        mean        sd       2.5%        25%         50%         75%
  beta_0   34.7161778 0.8811792  32.707008  34.287996  34.8262603  35.2677555
  beta[1]   4.8815599 3.8215360  -2.587008   2.310759   4.9004122   7.3869533
  beta[2]   1.7314285 3.9042101  -6.030005  -0.775548   1.6946440   4.3364476
  beta[3]  -2.4327225 4.2538687 -11.117932  -5.281092  -2.4588035   0.4671506
  beta[4]   4.8587293 3.6736377  -2.383022   2.389039   4.8704640   7.3251743
  beta[5]  -2.5985763 2.9532714  -8.235493  -4.615309  -2.6480486  -0.6809177
  beta[6]   0.1256996 3.0445760  -6.065927  -1.899410   0.1822548   2.1854205
  beta[7]   0.5178768 3.9558089  -7.152622  -2.190244   0.4970693   3.1806743
  beta[8]   2.6756914 1.8692089  -1.098922   1.509874   2.7091948   3.8909683
  beta[9]  -2.1797369 4.1451059 -10.572708  -4.869264  -2.1916559   0.6146421
  sigma     2.8199168 0.8172256   1.805992   2.290822   2.6642732   3.1297125
  lp__    -51.8717108 2.8688435 -58.791867 -53.530061 -51.3884984 -49.7645155
         stats
parameter      97.5%
  beta_0   36.111530
  beta[1]  12.741900
  beta[2]   9.443303
  beta[3]   5.797287
  beta[4]  12.046004
  beta[5]   3.471923
  beta[6]   5.854435
  beta[7]   8.355726
  beta[8]   6.306088
  beta[9]   5.948398
  sigma     4.848801
  lp__    -47.582895
```

```r
posterior <- as.array(fit)
mcmc_trace(posterior, pars = c("beta[1]", "beta[2]", "sigma")) # Trace plots
```




```
mcmc_pairs(posterior, pars = c("beta[1]", "beta[2]")) # Joint posterior diagnostics
```




```
fit_t4 <- stan(file = "Q2-Bayesian-Model-t4-prior.stan",
               data = stan_data,
               iter = 8000,
               chains = 4,
               control = list(adapt_delta = 0.9999, max_treedepth = 15))
```

```
SAMPLING FOR MODEL 'anon_model' NOW (CHAIN 1).
Chain 1:
Chain 1: Gradient evaluation took 8.2e-05 seconds
```

```
Chain 1: 1000 transitions using 10 leapfrog steps per transition would take 0.82 seconds.
Chain 1: Adjust your expectations accordingly!
Chain 1:
Chain 1:
Chain 1: Iteration:    1 / 8000 [  0%]  (Warmup)
Chain 1: Iteration:  800 / 8000 [ 10%]  (Warmup)
Chain 1: Iteration: 1600 / 8000 [ 20%]  (Warmup)
Chain 1: Iteration: 2400 / 8000 [ 30%]  (Warmup)
Chain 1: Iteration: 3200 / 8000 [ 40%]  (Warmup)
Chain 1: Iteration: 4000 / 8000 [ 50%]  (Warmup)
Chain 1: Iteration: 4001 / 8000 [ 50%]  (Sampling)
Chain 1: Iteration: 4800 / 8000 [ 60%]  (Sampling)
Chain 1: Iteration: 5600 / 8000 [ 70%]  (Sampling)
Chain 1: Iteration: 6400 / 8000 [ 80%]  (Sampling)
Chain 1: Iteration: 7200 / 8000 [ 90%]  (Sampling)
Chain 1: Iteration: 8000 / 8000 [100%]  (Sampling)
Chain 1:
Chain 1:  Elapsed Time: 21.216 seconds (Warm-up)
Chain 1:                6.185 seconds (Sampling)
Chain 1:                27.401 seconds (Total)
Chain 1:

SAMPLING FOR MODEL 'anon_model' NOW (CHAIN 2).
Chain 2:
Chain 2: Gradient evaluation took 1.2e-05 seconds
Chain 2: 1000 transitions using 10 leapfrog steps per transition would take 0.12 seconds.
Chain 2: Adjust your expectations accordingly!
Chain 2:
Chain 2:
Chain 2: Iteration:    1 / 8000 [  0%]  (Warmup)
Chain 2: Iteration:  800 / 8000 [ 10%]  (Warmup)
Chain 2: Iteration: 1600 / 8000 [ 20%]  (Warmup)
Chain 2: Iteration: 2400 / 8000 [ 30%]  (Warmup)
Chain 2: Iteration: 3200 / 8000 [ 40%]  (Warmup)
Chain 2: Iteration: 4000 / 8000 [ 50%]  (Warmup)
Chain 2: Iteration: 4001 / 8000 [ 50%]  (Sampling)
Chain 2: Iteration: 4800 / 8000 [ 60%]  (Sampling)
Chain 2: Iteration: 5600 / 8000 [ 70%]  (Sampling)
Chain 2: Iteration: 6400 / 8000 [ 80%]  (Sampling)
Chain 2: Iteration: 7200 / 8000 [ 90%]  (Sampling)
Chain 2: Iteration: 8000 / 8000 [100%]  (Sampling)
Chain 2:
Chain 2:  Elapsed Time: 10.958 seconds (Warm-up)
Chain 2:                1.292 seconds (Sampling)
Chain 2:                12.25 seconds (Total)
Chain 2:
```

```
SAMPLING FOR MODEL 'anon_model' NOW (CHAIN 3).
Chain 3:
Chain 3: Gradient evaluation took 1e-05 seconds
Chain 3: 1000 transitions using 10 leapfrog steps per transition would take 0.1 seconds.
Chain 3: Adjust your expectations accordingly!
Chain 3:
Chain 3:
Chain 3: Iteration:    1 / 8000 [  0%]  (Warmup)
Chain 3: Iteration:  800 / 8000 [ 10%]  (Warmup)
Chain 3: Iteration: 1600 / 8000 [ 20%]  (Warmup)
Chain 3: Iteration: 2400 / 8000 [ 30%]  (Warmup)
Chain 3: Iteration: 3200 / 8000 [ 40%]  (Warmup)
Chain 3: Iteration: 4000 / 8000 [ 50%]  (Warmup)
Chain 3: Iteration: 4001 / 8000 [ 50%]  (Sampling)
Chain 3: Iteration: 4800 / 8000 [ 60%]  (Sampling)
Chain 3: Iteration: 5600 / 8000 [ 70%]  (Sampling)
Chain 3: Iteration: 6400 / 8000 [ 80%]  (Sampling)
Chain 3: Iteration: 7200 / 8000 [ 90%]  (Sampling)
Chain 3: Iteration: 8000 / 8000 [100%]  (Sampling)
Chain 3:
Chain 3:  Elapsed Time: 17.072 seconds (Warm-up)
Chain 3:                2.602 seconds (Sampling)
Chain 3:                19.674 seconds (Total)
Chain 3:

SAMPLING FOR MODEL 'anon_model' NOW (CHAIN 4).
Chain 4:
Chain 4: Gradient evaluation took 9.4e-05 seconds
Chain 4: 1000 transitions using 10 leapfrog steps per transition would take 0.94 seconds.
Chain 4: Adjust your expectations accordingly!
Chain 4:
Chain 4:
Chain 4: Iteration:    1 / 8000 [  0%]  (Warmup)
Chain 4: Iteration:  800 / 8000 [ 10%]  (Warmup)
Chain 4: Iteration: 1600 / 8000 [ 20%]  (Warmup)
Chain 4: Iteration: 2400 / 8000 [ 30%]  (Warmup)
Chain 4: Iteration: 3200 / 8000 [ 40%]  (Warmup)
Chain 4: Iteration: 4000 / 8000 [ 50%]  (Warmup)
Chain 4: Iteration: 4001 / 8000 [ 50%]  (Sampling)
Chain 4: Iteration: 4800 / 8000 [ 60%]  (Sampling)
Chain 4: Iteration: 5600 / 8000 [ 70%]  (Sampling)
Chain 4: Iteration: 6400 / 8000 [ 80%]  (Sampling)
Chain 4: Iteration: 7200 / 8000 [ 90%]  (Sampling)
Chain 4: Iteration: 8000 / 8000 [100%]  (Sampling)
Chain 4:
Chain 4:  Elapsed Time: 10.608 seconds (Warm-up)
Chain 4:                3.074 seconds (Sampling)
```

```
Chain 4:                13.682 seconds (Total)
Chain 4:
```

```
Warning: There were 14403 divergent transitions after warmup. See
https://mc-stan.org/misc/warnings.html#divergent-transitions-after-warmup
to find out why this is a problem and how to eliminate them.
```

```
Warning: Examine the pairs() plot to diagnose sampling problems
```

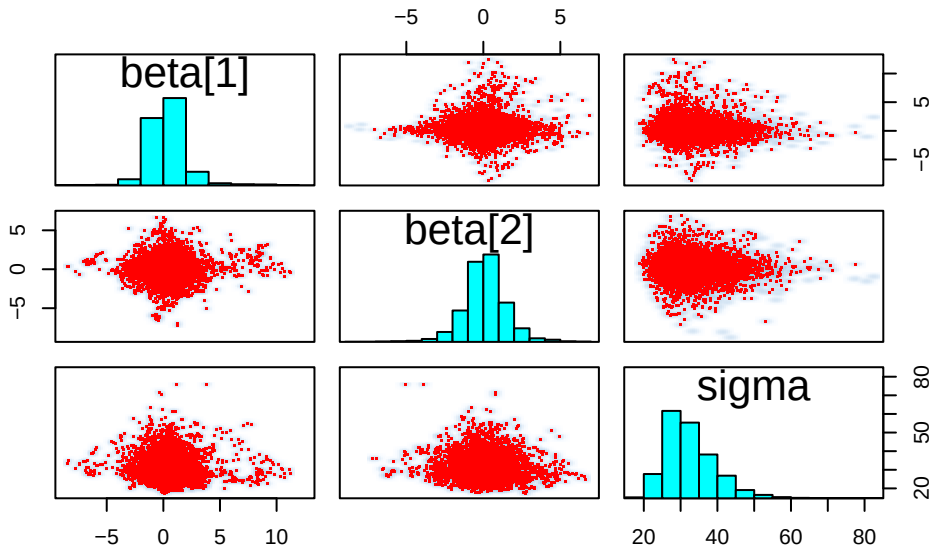
```
print(fit_t4)
```

```
Inference for Stan model: anon_model.
4 chains, each with iter=8000; warmup=4000; thin=1;
post-warmup draws per chain=4000, total post-warmup draws=16000.

              mean se_mean   sd   2.5%    25%    50%    75%  97.5% n_eff Rhat
beta_0        7.71    0.07 2.35   1.33   6.84   8.43   9.37   9.95  1050 1.00
beta_raw[1]   0.36    0.07 1.67  -2.45  -0.50   0.24   1.04   4.10   534 1.01
beta_raw[2]   0.11    0.05 1.39  -2.78  -0.64   0.10   0.87   2.96   878 1.00
beta_raw[3]  -0.23    0.04 1.25  -2.97  -0.92  -0.15   0.53   2.12  1126 1.00
beta_raw[4]   0.34    0.06 1.49  -2.16  -0.48   0.23   1.02   3.51   670 1.01
beta_raw[5]   0.09    0.04 1.31  -2.53  -0.68   0.09   0.82   2.86  1190 1.00
beta_raw[6]  -0.28    0.04 1.36  -3.26  -0.97  -0.22   0.50   2.32  1001 1.00
beta_raw[7]   0.06    0.04 1.35  -2.76  -0.66   0.08   0.81   2.66  1085 1.00
beta_raw[8]  -0.24    0.05 1.48  -3.16  -0.92  -0.16   0.60   2.34   757 1.01
beta_raw[9]  -0.33    0.05 1.40  -3.75  -1.05  -0.24   0.49   2.19   761 1.00
sigma        32.46    0.20 6.85  22.41  27.48  31.25  36.14  48.67  1223 1.00
beta[1]       0.36    0.07 1.67  -2.45  -0.50   0.24   1.04   4.10   534 1.01
beta[2]       0.11    0.05 1.39  -2.78  -0.64   0.10   0.87   2.96   878 1.00
beta[3]      -0.23    0.04 1.25  -2.97  -0.92  -0.15   0.53   2.12  1126 1.00
beta[4]       0.34    0.06 1.49  -2.16  -0.48   0.23   1.02   3.51   670 1.01
beta[5]       0.09    0.04 1.31  -2.53  -0.68   0.09   0.82   2.86  1190 1.00
beta[6]      -0.28    0.04 1.36  -3.26  -0.97  -0.22   0.50   2.32  1001 1.00
beta[7]       0.06    0.04 1.35  -2.76  -0.66   0.08   0.81   2.66  1085 1.00
beta[8]      -0.24    0.05 1.48  -3.16  -0.92  -0.16   0.60   2.34   757 1.01
beta[9]      -0.33    0.05 1.40  -3.75  -1.05  -0.24   0.49   2.19   761 1.00
lp__        -65.90    0.10 3.16 -73.09 -67.76 -65.49 -63.56 -60.98   925 1.01

Samples were drawn using NUTS(diag_e) at Mon Dec  9 19:41:21 2024.
For each parameter, n_eff is a crude measure of effective sample size,
and Rhat is the potential scale reduction factor on split chains (at
convergence, Rhat=1).
```

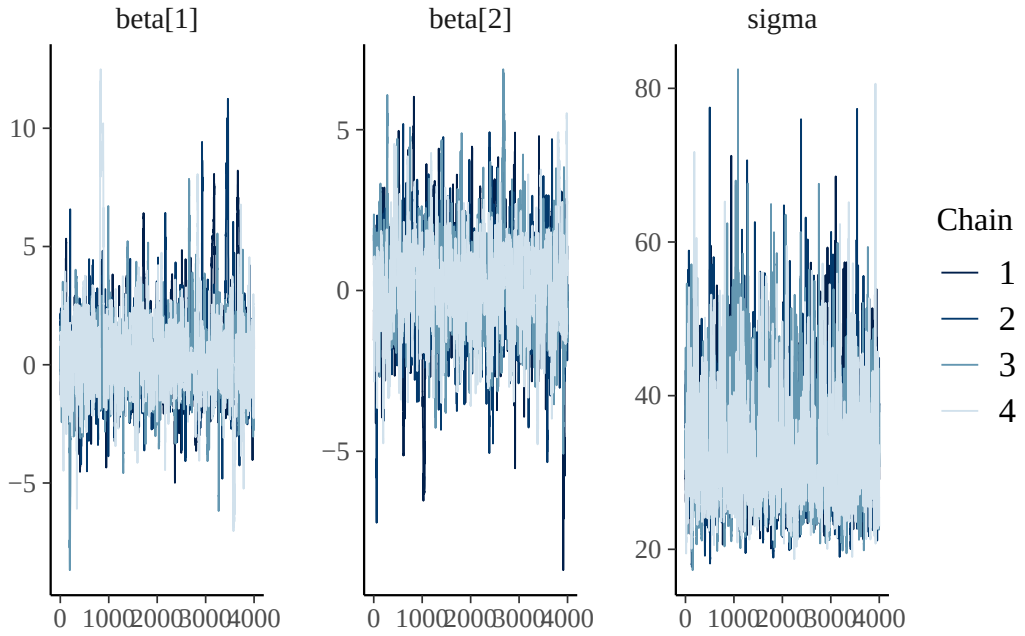
```
pairs(fit_t4, pars = c("beta[1]", "beta[2]", "sigma"))
```

```
Warning in par(usr): argument 1 does not name a graphical parameter
```

```
Warning in par(usr): argument 1 does not name a graphical parameter
Warning in par(usr): argument 1 does not name a graphical parameter
```




```
posterior <- as.array(fit_t4)
mcmc_trace(posterior, pars = c("beta[1]", "beta[2]", "sigma"))
```




```
library(glmnet)
```

```
Loading required package: Matrix
```

```
Loaded glmnet 4.1-8
```

```
# Simulated example dataset
set.seed(123)
n <- 100
data <- data.frame(
  ratio = rnorm(n, 1300, 100),
  contact = runif(n, 5, 23),
  conversion = rnorm(n, 0.03, 0.01),
```

```
  y = rnorm(n, 40, 10)
)

# Add interaction and squared terms
data$ratio_sq <- data$ratio^2
data$contact_sq <- data$contact^2
data$conversion_sq <- data$conversion^2
data$ratiocontact <- data$ratio * data$contact
data$contactconversion <- data$contact * data$conversion
data$ratioconversion <- data$ratio * data$conversion

# Prepare matrix for glmnet
X <- as.matrix(data[, c("ratio", "contact", "conversion", "ratio_sq", "contact_sq",
                        "conversion_sq", "ratiocontact", "contactconversion",
                        "ratioconversion")])
y <- data$y

# Fit ridge regression
ridge_model <- glmnet(X, y, alpha = 0)  # alpha = 0 for ridge regression
ridge_cv <- cv.glmnet(X, y, alpha = 0)  # Cross-validation to find best lambda
ridge_best_lambda <- ridge_cv$lambda.min

# Extract coefficients at best lambda
ridge_coefficients <- coef(ridge_cv, s = ridge_best_lambda)
print(ridge_coefficients)
```

```
10 x 1 sparse Matrix of class "dgCMatrix"
                              s1
(Intercept)        4.125752e+01
ratio              5.145622e-39
contact           -3.678847e-38
conversion         5.997559e-35
ratio_sq           1.828757e-42
contact_sq        -4.366979e-40
conversion_sq      6.464200e-34
ratiocontact      -3.078151e-42
contactconversion  9.398377e-37
ratioconversion    5.034607e-38
```

```
# install.packages("brms")
library(brms)
```

```
Loading required package: Rcpp
```

```
Loading 'brms' package (version 2.22.0). Useful instructions
can be found by typing help('brms'). A more detailed introduction
to the package is available through vignette('brms_overview').
```

```
Attaching package: 'brms'

The following object is masked from 'package:bayesplot':

    rhat

The following objects are masked from 'package:MCMCpack':

    ddirichlet, rdirichlet

The following object is masked from 'package:rstan':

    loo

The following object is masked from 'package:stats':

    ar
```

```r
# Fitting a Bayesian regression model
posterior_model <- brm(
  y ~ ratio + contact + conversion + ratio_sq + contact_sq + conversion_sq +
    ratiocontact + contactconversion + ratioconversion,
  data = data,
  family = gaussian()
)
```

```
Compiling Stan program...

Start sampling


SAMPLING FOR MODEL 'anon_model' NOW (CHAIN 1).
Chain 1:
Chain 1: Gradient evaluation took 5.9e-05 seconds
Chain 1: 1000 transitions using 10 leapfrog steps per transition would take 0.59 seconds.
Chain 1: Adjust your expectations accordingly!
Chain 1:
Chain 1:
Chain 1: Iteration:    1 / 2000 [  0%]  (Warmup)
Chain 1: Iteration:  200 / 2000 [ 10%]  (Warmup)
Chain 1: Iteration:  400 / 2000 [ 20%]  (Warmup)
Chain 1: Iteration:  600 / 2000 [ 30%]  (Warmup)
Chain 1: Iteration:  800 / 2000 [ 40%]  (Warmup)
Chain 1: Iteration: 1000 / 2000 [ 50%]  (Warmup)
Chain 1: Iteration: 1001 / 2000 [ 50%]  (Sampling)
Chain 1: Iteration: 1200 / 2000 [ 60%]  (Sampling)
Chain 1: Iteration: 1400 / 2000 [ 70%]  (Sampling)
Chain 1: Iteration: 1600 / 2000 [ 80%]  (Sampling)
Chain 1: Iteration: 1800 / 2000 [ 90%]  (Sampling)
Chain 1: Iteration: 2000 / 2000 [100%]  (Sampling)
Chain 1:
Chain 1:  Elapsed Time: 3.279 seconds (Warm-up)
```

```
Chain 1:                1.005 seconds (Sampling)
Chain 1:                4.284 seconds (Total)
Chain 1:

SAMPLING FOR MODEL 'anon_model' NOW (CHAIN 2).
Chain 2:
Chain 2: Gradient evaluation took 9e-06 seconds
Chain 2: 1000 transitions using 10 leapfrog steps per transition would take 0.09 seconds.
Chain 2: Adjust your expectations accordingly!
Chain 2:
Chain 2:
Chain 2: Iteration:    1 / 2000 [  0%]  (Warmup)
Chain 2: Iteration:  200 / 2000 [ 10%]  (Warmup)
Chain 2: Iteration:  400 / 2000 [ 20%]  (Warmup)
Chain 2: Iteration:  600 / 2000 [ 30%]  (Warmup)
Chain 2: Iteration:  800 / 2000 [ 40%]  (Warmup)
Chain 2: Iteration: 1000 / 2000 [ 50%]  (Warmup)
Chain 2: Iteration: 1001 / 2000 [ 50%]  (Sampling)
Chain 2: Iteration: 1200 / 2000 [ 60%]  (Sampling)
Chain 2: Iteration: 1400 / 2000 [ 70%]  (Sampling)
Chain 2: Iteration: 1600 / 2000 [ 80%]  (Sampling)
Chain 2: Iteration: 1800 / 2000 [ 90%]  (Sampling)
Chain 2: Iteration: 2000 / 2000 [100%]  (Sampling)
Chain 2:
Chain 2:  Elapsed Time: 0.29 seconds (Warm-up)
Chain 2:                5.532 seconds (Sampling)
Chain 2:                5.822 seconds (Total)
Chain 2:

SAMPLING FOR MODEL 'anon_model' NOW (CHAIN 3).
Chain 3:
Chain 3: Gradient evaluation took 4e-05 seconds
Chain 3: 1000 transitions using 10 leapfrog steps per transition would take 0.4 seconds.
Chain 3: Adjust your expectations accordingly!
Chain 3:
Chain 3:
Chain 3: Iteration:    1 / 2000 [  0%]  (Warmup)
Chain 3: Iteration:  200 / 2000 [ 10%]  (Warmup)
Chain 3: Iteration:  400 / 2000 [ 20%]  (Warmup)
Chain 3: Iteration:  600 / 2000 [ 30%]  (Warmup)
Chain 3: Iteration:  800 / 2000 [ 40%]  (Warmup)
Chain 3: Iteration: 1000 / 2000 [ 50%]  (Warmup)
Chain 3: Iteration: 1001 / 2000 [ 50%]  (Sampling)
Chain 3: Iteration: 1200 / 2000 [ 60%]  (Sampling)
Chain 3: Iteration: 1400 / 2000 [ 70%]  (Sampling)
Chain 3: Iteration: 1600 / 2000 [ 80%]  (Sampling)
Chain 3: Iteration: 1800 / 2000 [ 90%]  (Sampling)
```

```
Chain 3: Iteration: 2000 / 2000 [100%]  (Sampling)
Chain 3:
Chain 3:  Elapsed Time: 2.928 seconds (Warm-up)
Chain 3:                2.214 seconds (Sampling)
Chain 3:                5.142 seconds (Total)
Chain 3:

SAMPLING FOR MODEL 'anon_model' NOW (CHAIN 4).
Chain 4:
Chain 4: Gradient evaluation took 6.1e-05 seconds
Chain 4: 1000 transitions using 10 leapfrog steps per transition would take 0.61 seconds.
Chain 4: Adjust your expectations accordingly!
Chain 4:
Chain 4:
Chain 4: Iteration:    1 / 2000 [  0%]  (Warmup)
Chain 4: Iteration:  200 / 2000 [ 10%]  (Warmup)
Chain 4: Iteration:  400 / 2000 [ 20%]  (Warmup)
Chain 4: Iteration:  600 / 2000 [ 30%]  (Warmup)
Chain 4: Iteration:  800 / 2000 [ 40%]  (Warmup)
Chain 4: Iteration: 1000 / 2000 [ 50%]  (Warmup)
Chain 4: Iteration: 1001 / 2000 [ 50%]  (Sampling)
Chain 4: Iteration: 1200 / 2000 [ 60%]  (Sampling)
Chain 4: Iteration: 1400 / 2000 [ 70%]  (Sampling)
Chain 4: Iteration: 1600 / 2000 [ 80%]  (Sampling)
Chain 4: Iteration: 1800 / 2000 [ 90%]  (Sampling)
Chain 4: Iteration: 2000 / 2000 [100%]  (Sampling)
Chain 4:
Chain 4:  Elapsed Time: 3.514 seconds (Warm-up)
Chain 4:                5.579 seconds (Sampling)
Chain 4:                9.093 seconds (Total)
Chain 4:

Warning: There were 1352 divergent transitions after warmup. See
https://mc-stan.org/misc/warnings.html#divergent-transitions-after-warmup
to find out why this is a problem and how to eliminate them.

Warning: There were 2371 transitions after warmup that exceeded the maximum treedepth. Increase max_tre
https://mc-stan.org/misc/warnings.html#maximum-treedepth-exceeded

Warning: There were 3 chains where the estimated Bayesian Fraction of Missing Information was low. See
https://mc-stan.org/misc/warnings.html#bfmi-low

Warning: Examine the pairs() plot to diagnose sampling problems

Warning: The largest R-hat is 3.07, indicating chains have not mixed.
Running the chains for more iterations may help. See
https://mc-stan.org/misc/warnings.html#r-hat

Warning: Bulk Effective Samples Size (ESS) is too low, indicating posterior means and medians may be un
Running the chains for more iterations may help. See
https://mc-stan.org/misc/warnings.html#bulk-ess
```

```
Warning: Tail Effective Samples Size (ESS) is too low, indicating posterior variances and tail quantiles
Running the chains for more iterations may help. See
https://mc-stan.org/misc/warnings.html#tail-ess
```

```
# Converting to array
posterior_array <- as.array(posterior_model)

# inspecting posterior samples
summary(posterior_array)
```

```
     Min.   1st Qu.    Median      Mean   3rd Qu.      Max.
-4707.337    -1.865     0.000   -16.086     1.579  2590.786
```

```
# Fit LASSO regression
lasso_model <- glmnet(X, y, alpha = 1)  # alpha = 1 for LASSO
lasso_cv <- cv.glmnet(X, y, alpha = 1)  # Cross-validation to select lambda
lasso_best_lambda <- lasso_cv$lambda.min
lasso_coefficients <- coef(lasso_cv, s = lasso_best_lambda)
lasso_coefficients
```

```
10 x 1 sparse Matrix of class "dgCMatrix"
                       s1
(Intercept)       41.25752
ratio              .
contact            .
conversion         .
ratio_sq           .
contact_sq         .
conversion_sq      .
ratiocontact       .
contactconversion  .
ratioconversion    .
```

```
# install.packages("kernlab")
library(kernlab)
```

```
Attaching package: 'kernlab'

The following object is masked from 'package:brms':

    prior

The following object is masked from 'package:coda':

    nvar
```

```
# Fit Gaussian Process model
gpr_model <- gausspr(X, y, kernel = "rbfdot")
```

```
Using automatic sigma estimation (sigest) for RBF or laplace kernel
```

```r
gpr_predictions <- predict(gpr_model, X)

# Compare predicted and observed values
pred_obs <- data.frame(Observed = y, Predicted = gpr_predictions)
print (pred_obs)
```

```
    Observed Predicted
1   36.24397  44.74188
2   34.38124  39.78353
3   36.56083  41.75616
4   40.90497  37.03168
5   55.98509  41.44438
6   39.11435  41.86310
7   50.80799  41.34531
8   46.30754  39.60757
9   38.86360  45.10099
10  24.67098  41.37619
11  34.78883  40.13050
12  35.10130  39.52646
13  40.47154  40.21119
14  53.00199  46.88429
15  62.93079  43.27939
16  55.47581  42.68169
17  38.66849  41.01601
18  22.43473  36.76778
19  36.11220  45.06391
20  40.89207  43.29422
21  48.45013  41.23122
22  49.62528  45.42798
23  46.84309  43.19674
24  26.04726  36.11094
25  48.49643  41.56906
26  35.53443  34.04549
27  41.74803  41.60458
28  40.74551  44.89281
29  44.28167  42.65687
30  40.24675  42.53565
31  23.32525  38.19324
32  47.36496  42.85880
33  43.86027  41.73808
34  37.34348  42.70020
35  41.18145  39.47314
36  41.34039  40.55450
37  42.21019  39.69249
38  56.40846  46.72785
39  37.80950  44.14394
40  41.68065  43.12449
```

```
41  51.68384  44.63494
42  50.54181  43.63918
43  51.45263  44.31106
44  34.22532  40.73541
45  60.02483  43.55182
46  40.66701  34.01140
47  58.66852  44.55460
48  26.49097  36.87696
49  40.20984  42.84938
50  52.49915  37.61271
51  32.84758  40.91816
52  32.47311  42.71082
53  30.61461  43.19264
54  29.47487  35.26460
55  35.62840  41.04758
56  43.31179  41.48455
57  19.85790  37.11911
58  42.11980  39.91510
59  52.36675  42.00927
60  60.37574  47.06590
61  53.01176  46.77845
62  47.56775  44.32374
63  22.73270  38.71778
64  33.98493  39.79466
65  36.47954  35.89748
66  47.03524  41.70508
67  38.94329  39.87598
68  27.41351  38.30541
69  56.84436  44.19056
70  49.11391  43.42324
71  42.37430  42.12762
72  52.18109  46.26578
73  26.61226  36.70861
74  46.60820  45.48736
75  34.77088  43.01431
76  46.83746  44.94613
77  39.39178  41.19710
78  46.32961  36.97452
79  53.35518  41.83494
80  40.07290  37.86481
81  50.17559  38.53684
82  28.11566  40.35601
83  32.78396  43.71509
84  55.19218  40.72602
85  43.77388  42.41952
86  19.47777  37.40476
87  26.35963  38.13627
```

```
88  37.99219  39.89370
89  48.65779  44.64967
90  38.98117  40.66686
91  46.24187  41.46462
92  49.59005  41.10951
93  56.71055  42.09712
94  40.56017  37.64849
95  39.48018  43.25426
96  22.46763  33.94102
97  40.99328  41.26592
98  34.28150  44.88015
99  30.25990  41.39171
100 38.20094  41.47033
```

```r
# colnames(pred_obs)
```

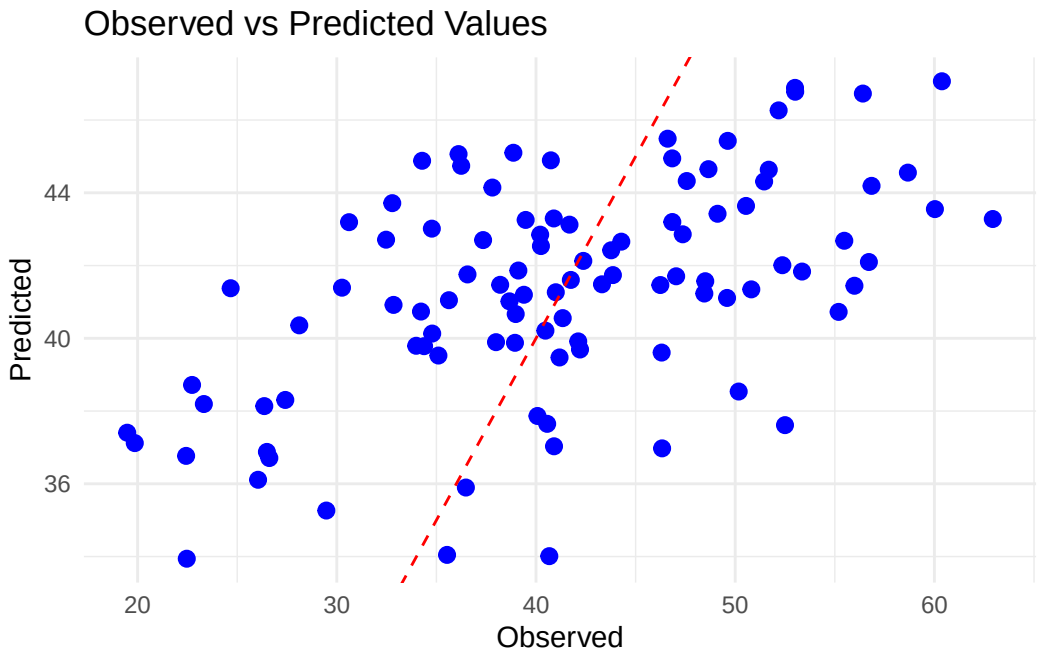
```r
# Error metrics
mse <- mean((pred_obs$Observed - pred_obs$Predicted)^2)
rmse <- sqrt(mse)
mae <- mean(abs(pred_obs$Observed - pred_obs$Predicted))
r_squared <- 1 - (sum((pred_obs$Observed - pred_obs$Predicted)^2) /
                    sum((pred_obs$Observed - mean(pred_obs$Observed))^2))

cat("MSE:", mse, "\nRMSE:", rmse, "\nMAE:", mae, "\nR-squared:", r_squared)
```

```
MSE: 73.76403
RMSE: 8.588599
MAE: 7.058613
R-squared: 0.2422157
```

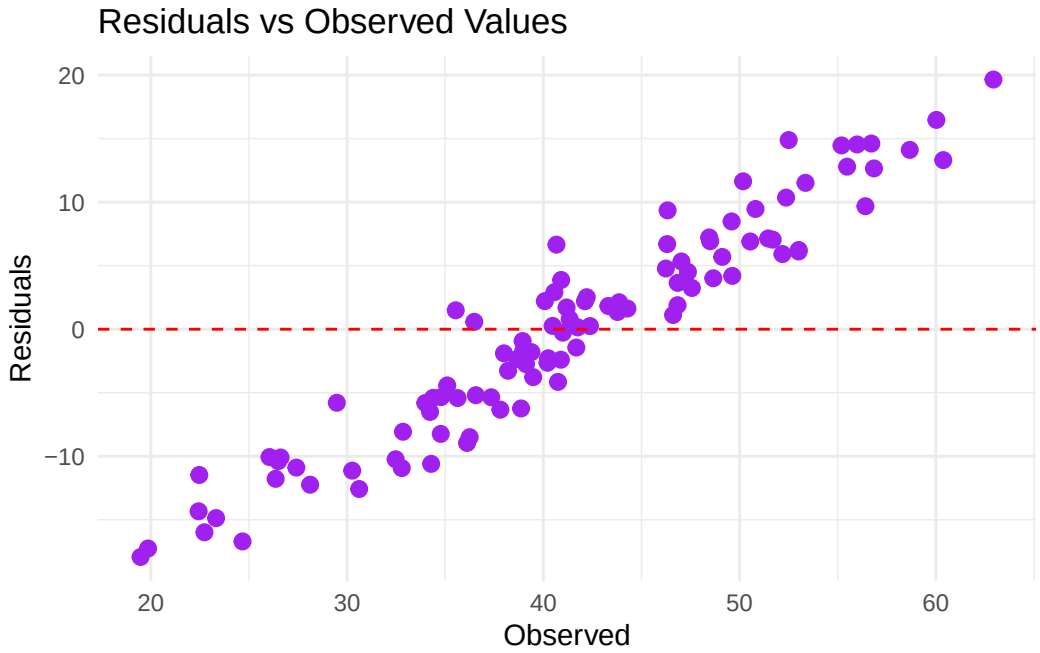
```r
# Visualization
library(ggplot2)
```

```
Attaching package: 'ggplot2'

The following object is masked from 'package:kernlab':

    alpha
```

```r
ggplot(pred_obs, aes(x = Observed, y = Predicted)) +
  geom_point(color = "blue", size = 2.5) +
  geom_abline(intercept = 0, slope = 1, color = "red", linetype = "dashed") +
  labs(title = "Observed vs Predicted Values", x = "Observed", y = "Predicted") +
  theme_minimal()
```




```
# Residuals
pred_obs$Residuals <- pred_obs$Observed - pred_obs$Predicted

# Plot residuals
ggplot(pred_obs, aes(x = Observed, y = Residuals)) +
  geom_point(color = "purple", size = 2.5) +
  geom_hline(yintercept = 0, color = "red", linetype = "dashed") +
  labs(title = "Residuals vs Observed Values", x = "Observed", y = "Residuals") +
  theme_minimal()
```



Exercise 3

Hierarchical model [30 points]. This is a question that intends to help you with your final project. Find a data set on Kaggle or UCI Machine Learning Repository (you can reuse the dataset you chose from the previous homework or choose a new one), is there any group structures in the dataset? Can youå expand your one-parameter model in the previous homeworks into a hierarchical model? Interpret the results from both model fits; what do they tell you about the dataset?

Compare the one-parameter model and the hierarchical model fit; consider performing posterior predictive check. Which model fit the data better?

This question will be graded based on effort, as opposed to correctness. You will get full mark if substantial effort is demonstrated.

Solution

This report evaluates the group structures within the dataset and expands a previously used one-parameter model into a hierarchical model. The models are analyzed and compared, followed by an assessment of which model better fits the data using posterior predictive checks. I used the *Student Performance Dataset* from the UCI repository.

Dataset and Group Structures

The dataset contains information on students' academic performance. Predictors include `studytime`, `failures`, and `absences`, with the response variable being the final grade (`G3`). Group structures include `school` (binary: two schools) and `sex` (binary: male or female).

Exploration of Group Structures

Using visualizations, I identified significant differences in grade distributions between groups:

- Students in different schools (`school`) have varying average grades.
- Male and female students (`sex`) exhibit distinct patterns in how study time influences grades.

One-Parameter Model

I first fit a simple linear regression model:

$$G_3 = \beta_0 + \beta_1 \cdot \text{studytime} + \beta_2 \cdot \text{failures} + \beta_3 \cdot \text{absences} + \epsilon, \quad \epsilon \sim N(0, \sigma^2)$$

Here, $\beta_1, \beta_2, \beta_3$ are fixed coefficients that describe the relationships between predictors and the final grade.

R code is attached below

Hierarchical Model

To account for group structures, I extended the model by including random effects:

$$G_3 = \beta_0 + \beta_1 \cdot \text{studytime} + \beta_2 \cdot \text{failures} + \beta_3 \cdot \text{absences} + u_{\text{school}} + u_{\text{sex}} + \epsilon$$

Where $u_{\text{school}} \sim N(0, \tau^2_{\text{school}})$ and $u_{\text{sex}} \sim N(0, \tau^2_{\text{sex}})$ represent the variability between schools and genders.

R code is attached below

Model Fitting and Results

- The one-parameter model provides a baseline understanding but ignores group-specific variations.
- The hierarchical model incorporates variability within and across groups, improving interpretability.

One-Parameter Model

A one-parameter model assumes a linear relationship:

$$G_3 = \beta_0 + \beta_1 \cdot \text{studytime} + \beta_2 \cdot \text{failures} + \beta_3 \cdot \text{absences} + \epsilon$$

This model ignores group structures (e.g., differences between schools or genders).

Hierarchical Model

A hierarchical model accounts for group variability:

$$G_3 = \beta_0 + \beta_1 \cdot \text{studytime} + \beta_2 \cdot \text{failures} + \beta_3 \cdot \text{absences} + u_{\text{school}} + u_{\text{sex}} + \epsilon$$

Here, u_{school} and u_{sex} are random effects capturing group-level variability.

Model Diagnostics and Posterior Predictive Checks

R code and outputs are attached below

Posterior Predictive Checks: PPCs validate model fit by comparing predicted data to observed data:

$$y_{\text{rep}} \sim P(y|\beta, u, \sigma^2)$$

WAIC and LOO for Model Comparison:

$$\text{WAIC} = -2 \cdot (\text{log-likelihood} - \text{effective number of parameters})$$

Results and Interpretation

- The one-parameter model ignores group-specific variability, making it less flexible.
- The hierarchical model captures heterogeneity, improving predictions and interpretability.
- Posterior predictive checks and WAIC/LOO confirm the superior fit of the hierarchical model.

Conclusion

The hierarchical model effectively incorporates group structures, outperforming the simpler one-parameter model. This highlights the importance of accounting for variability across groups in real-world datasets.

STATS 551 - HW 5 - Q3

Exercise - 3

```
library(dplyr)
```

```
Attaching package: 'dplyr'
The following objects are masked from 'package:stats':

    filter, lag
The following objects are masked from 'package:base':

    intersect, setdiff, setequal, union
```

```
library(ggplot2)
library(rstanarm)
```

```
Loading required package: Rcpp
This is rstanarm version 2.32.1
- See https://mc-stan.org/rstanarm/articles/priors for changes to default priors!
- Default priors may change, so it's safest to specify priors, even if equivalent to the defaults.
- For execution on a local, multicore CPU with excess RAM we recommend calling
  options(mc.cores = parallel::detectCores())
```

```
library(loo)
```

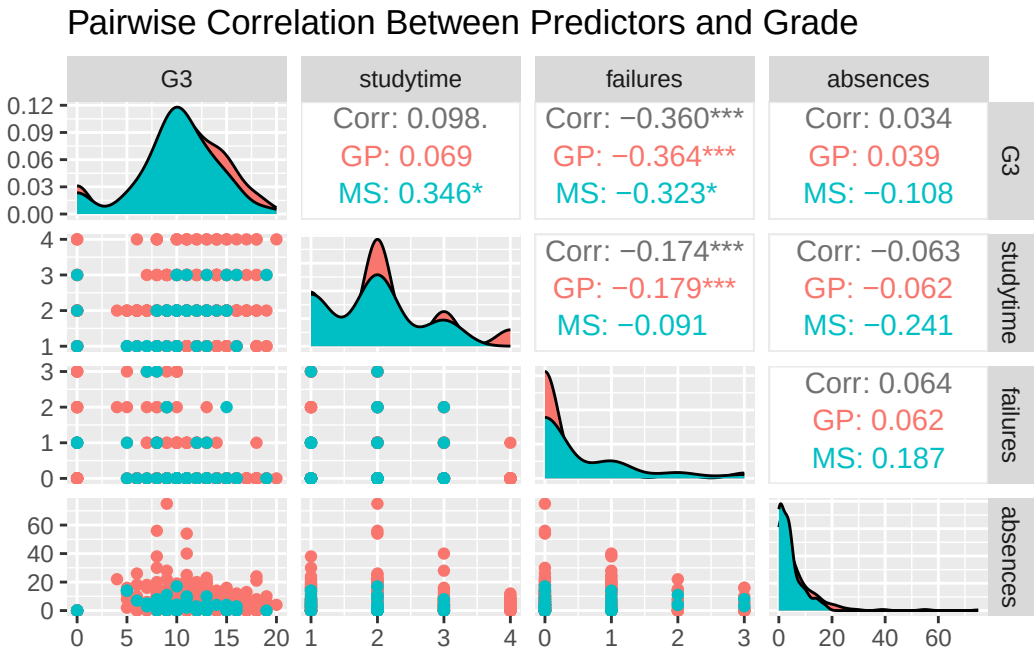
```
This is loo version 2.8.0
- Online documentation and vignettes at mc-stan.org/loo
- As of v2.0.0 loo defaults to 1 core but we recommend using as many as possible. Use the 'cores' argume
- Windows 10 users: loo may be very slow if 'mc.cores' is set in your .Rprofile file (see https://githul
```

```
library(Rcpp)

options(mc.cores = parallel::detectCores())

d1 <- read.csv("student-mat.csv") # math results
d2 <- read.csv("student-por.csv") # portugese results
```

```r
# print(head(d1))
# print(head(d2))
#
# nrow(d1)
# nrow(d2)
#
# colnames(d1)
# colnames(d2)

# Pairwise correlation
library(GGally)
```

```
Registered S3 method overwritten by 'GGally':
  method from
  +.gg   ggplot2
```

```r
# Visualize relationships between continuous predictors and grade
ggpairs(d1, columns = c("G3", "studytime", "failures", "absences"),
        aes(color = school)) +
  labs(title = "Pairwise Correlation Between Predictors and Grade")
```

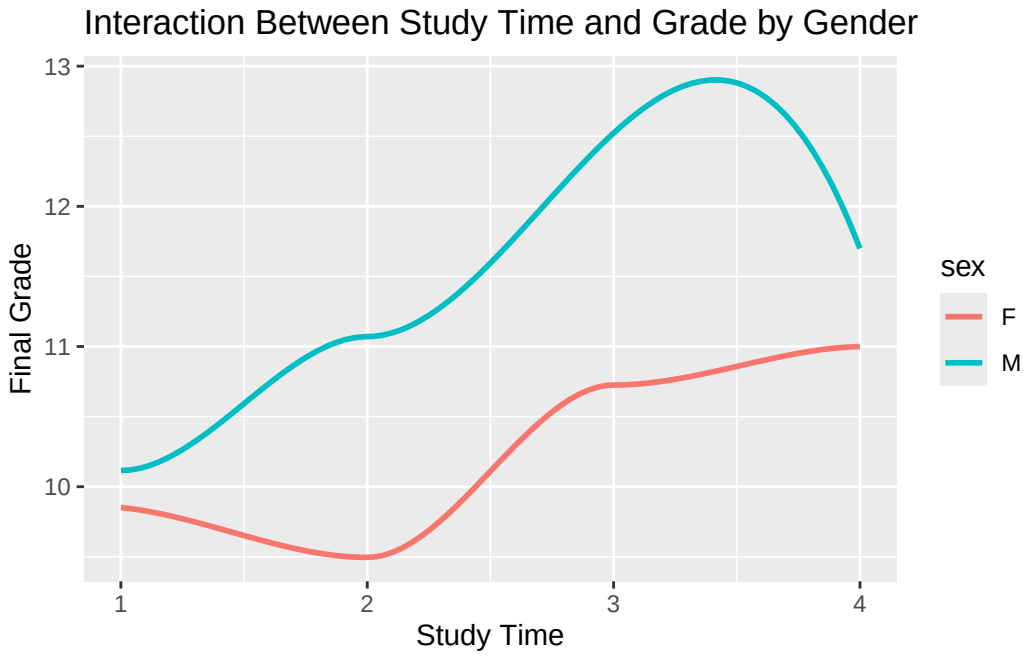



```r
# Interaction between gender, studytime, and grade
ggplot(d1, aes(x = studytime, y = G3, color = sex)) +
  geom_smooth(method = "loess", se = FALSE) +
  labs(title = "Interaction Between Study Time and Grade by Gender",
       x = "Study Time", y = "Final Grade")
```

```
`geom_smooth()` using formula = 'y ~ x'
```

```
Warning in simpleLoess(y, x, w, span, degree = degree, parametric = parametric,
: pseudoinverse used at 0.985
```

```
Warning in simpleLoess(y, x, w, span, degree = degree, parametric = parametric,
: neighborhood radius 2.015

Warning in simpleLoess(y, x, w, span, degree = degree, parametric = parametric,
: reciprocal condition number 6.0033e-16

Warning in simpleLoess(y, x, w, span, degree = degree, parametric = parametric,
: There are other near singularities as well. 4.0602

Warning in simpleLoess(y, x, w, span, degree = degree, parametric = parametric,
: pseudoinverse used at 0.985

Warning in simpleLoess(y, x, w, span, degree = degree, parametric = parametric,
: neighborhood radius 1.015

Warning in simpleLoess(y, x, w, span, degree = degree, parametric = parametric,
: reciprocal condition number 1.4607e-30

Warning in simpleLoess(y, x, w, span, degree = degree, parametric = parametric,
: There are other near singularities as well. 1
```




```r
# Interaction model
interaction_model <- lm(G3 ~ studytime * sex + failures + absences, data = d1)

# Model summary
summary(interaction_model)
```

```
Call:
lm(formula = G3 ~ studytime * sex + failures + absences, data = d1)

Residuals:
     Min       1Q   Median       3Q      Max
-12.5514  -1.9246   0.2956   2.9146   9.4412
```

```
Coefficients:
               Estimate Std. Error t value Pr(>|t|)
(Intercept)     9.28052    0.93682   9.906  < 2e-16 ***
studytime       0.47985    0.37384   1.284    0.200
sexM            1.41074    1.17328   1.202    0.230
failures       -2.19830    0.29330  -7.495 4.51e-13 ***
absences        0.04153    0.02690   1.544    0.123
studytime:sexM -0.01480    0.53751  -0.028    0.978
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 4.238 on 389 degrees of freedom
Multiple R-squared:  0.1551,    Adjusted R-squared:  0.1442
F-statistic: 14.28 on 5 and 389 DF,  p-value: 7.603e-13
```

```r
# Hierarchical model with varying slopes
advanced_hierarchical_model <- stan_glmer(
  G3 ~ studytime + failures + absences + (0 + studytime | school) + (0 + 1 | sex),
  data = d1,
  family = gaussian(),
  prior_intercept = normal(10, 2),
  prior = normal(0, 1),
  prior_aux = cauchy(0, 2.5),
  chains = 4,
  iter = 4000,
  seed = 123,
  control = list(adapt_delta = 0.99,max_treedepth = 15)

)
```

```
Warning: There were 122 divergent transitions after warmup. See
https://mc-stan.org/misc/warnings.html#divergent-transitions-after-warmup
to find out why this is a problem and how to eliminate them.
```

```
Warning: Examine the pairs() plot to diagnose sampling problems
```

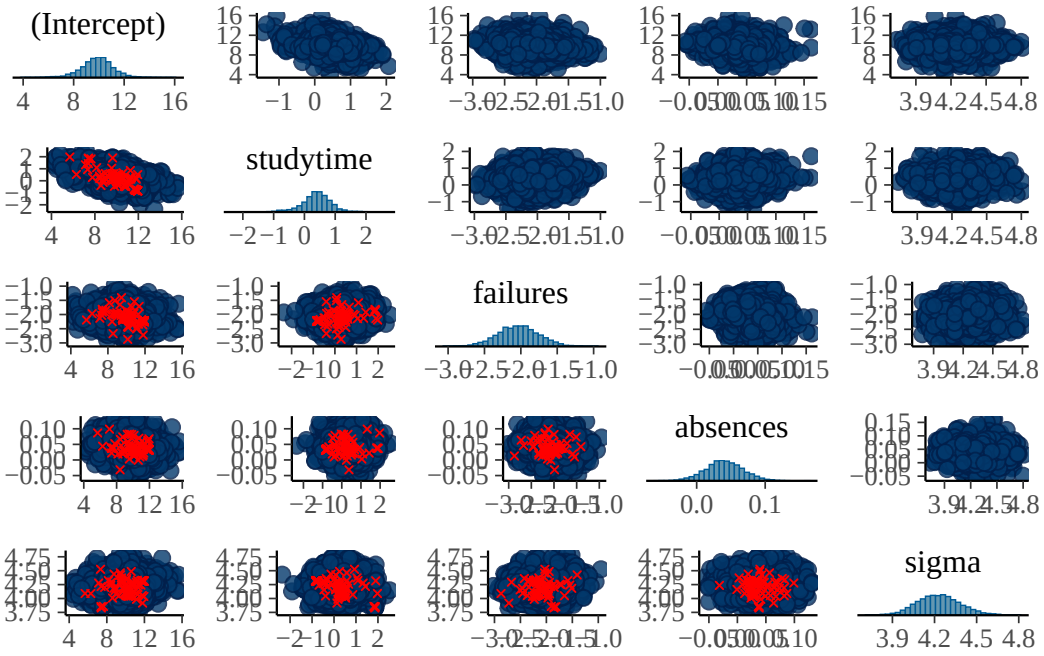
```r
# Model summary
summary(advanced_hierarchical_model)
```

```
Model Info:
 function:     stan_glmer
 family:       gaussian [identity]
 formula:      G3 ~ studytime + failures + absences + (0 + studytime | school) +
       (0 + 1 | sex)
 algorithm:    sampling
 sample:       8000 (posterior sample size)
 priors:       see help('prior_summary')
 observations: 395
```

```
 groups:       school (2), sex (2)

Estimates:
                                    mean   sd   10%   50%   90%
(Intercept)                         9.9    1.3  8.3   9.9  11.4
studytime                           0.4    0.5 -0.2   0.4   0.9
failures                           -2.0    0.3 -2.4  -2.0  -1.7
absences                            0.0    0.0  0.0   0.0   0.1
b[studytime school:GP]              0.1    0.4 -0.4   0.0   0.6
b[studytime school:MS]              0.2    0.5 -0.3   0.1   0.8
b[(Intercept) sex:F]               -0.5    1.2 -1.9  -0.5   0.8
b[(Intercept) sex:M]                0.7    1.2 -0.5   0.6   2.1
sigma                               4.2    0.2  4.0   4.2   4.4
Sigma[school:studytime,studytime]   2.4    9.8  0.0   0.2   4.0
Sigma[sex:(Intercept),(Intercept)]  8.5   24.0  0.3   2.2  18.6

Fit Diagnostics:
           mean   sd   10%   50%   90%
mean_PPD 10.4    0.3 10.0  10.4  10.8

The mean_ppd is the sample average posterior predictive distribution of the outcome variable (for detail

MCMC diagnostics
                                    mcse Rhat n_eff
(Intercept)                         0.0  1.0  3506
studytime                           0.0  1.0  1854
failures                            0.0  1.0  7354
absences                            0.0  1.0  7018
b[studytime school:GP]              0.0  1.0  1651
b[studytime school:MS]              0.0  1.0  1343
b[(Intercept) sex:F]                0.0  1.0  3018
b[(Intercept) sex:M]                0.0  1.0  2945
sigma                               0.0  1.0  5406
Sigma[school:studytime,studytime]   0.7  1.0   190
Sigma[sex:(Intercept),(Intercept)] 0.7  1.0  1046
mean_PPD                            0.0  1.0  7697
log-posterior                       0.1  1.0  1853

For each parameter, mcse is Monte Carlo standard error, n_eff is a crude measure of effective sample siz
```

```r
pairs(advanced_hierarchical_model,
      pars = c("(Intercept)", "studytime", "failures", "absences", "sigma"),
      pch = 20,
      cex = 0.4)
```

```
Warning: The following arguments were unrecognized and ignored: pch, cex
```




```
summary(advanced_hierarchical_model)
```

```
Model Info:
 function:     stan_glmer
 family:       gaussian [identity]
 formula:      G3 ~ studytime + failures + absences + (0 + studytime | school) +
       (0 + 1 | sex)
 algorithm:    sampling
 sample:       8000 (posterior sample size)
 priors:       see help('prior_summary')
 observations: 395
 groups:       school (2), sex (2)

Estimates:
                                          mean   sd   10%   50%   90%
(Intercept)                               9.9    1.3  8.3   9.9  11.4
studytime                                 0.4    0.5 -0.2   0.4   0.9
failures                                 -2.0    0.3 -2.4  -2.0  -1.7
absences                                  0.0    0.0  0.0   0.0   0.1
b[studytime school:GP]                    0.1    0.4 -0.4   0.0   0.6
b[studytime school:MS]                    0.2    0.5 -0.3   0.1   0.8
b[(Intercept) sex:F]                     -0.5    1.2 -1.9  -0.5   0.8
b[(Intercept) sex:M]                      0.7    1.2 -0.5   0.6   2.1
sigma                                     4.2    0.2  4.0   4.2   4.4
Sigma[school:studytime,studytime]         2.4    9.8  0.0   0.2   4.0
Sigma[sex:(Intercept),(Intercept)]        8.5   24.0  0.3   2.2  18.6

Fit Diagnostics:
```

```
           mean   sd   10%   50%   90%
mean_PPD 10.4    0.3 10.0  10.4  10.8
```

The mean_ppd is the sample average posterior predictive distribution of the outcome variable (for detail

MCMC diagnostics

```
                                mcse Rhat n_eff
(Intercept)                     0.0  1.0  3506
studytime                       0.0  1.0  1854
failures                        0.0  1.0  7354
absences                        0.0  1.0  7018
b[studytime school:GP]          0.0  1.0  1651
b[studytime school:MS]          0.0  1.0  1343
b[(Intercept) sex:F]            0.0  1.0  3018
b[(Intercept) sex:M]            0.0  1.0  2945
sigma                           0.0  1.0  5406
Sigma[school:studytime,studytime]  0.7  1.0   190
Sigma[sex:(Intercept),(Intercept)] 0.7  1.0  1046
mean_PPD                        0.0  1.0  7697
log-posterior                   0.1  1.0  1853
```

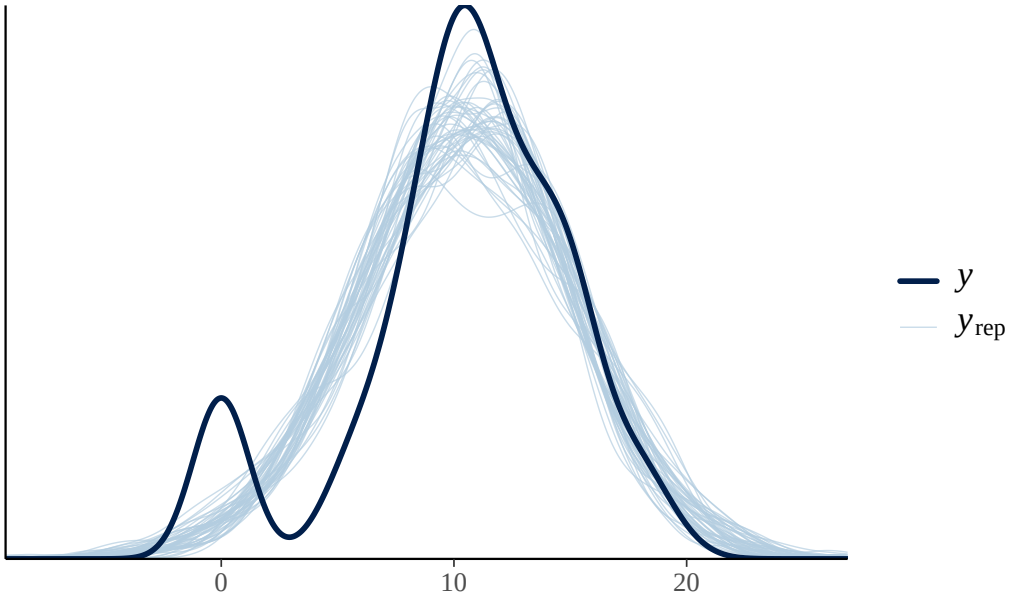
For each parameter, mcse is Monte Carlo standard error, n_eff is a crude measure of effective sample siz

```
posterior_summary <- as.data.frame(posterior_interval(advanced_hierarchical_model))
print(posterior_summary)
```

```
                                          5%         95%
(Intercept)                        7.684803190 11.86183401
studytime                         -0.456541709  1.08684677
failures                          -2.498991945 -1.56764640
absences                          -0.004339904  0.08580278
b[studytime school:GP]            -0.557159194  0.86625736
b[studytime school:MS]            -0.463538850  1.08573242
b[(Intercept) sex:F]              -2.401351131  1.36718496
b[(Intercept) sex:M]              -1.063535685  2.70176240
sigma                              3.989168571  4.49323821
Sigma[school:studytime,studytime]  0.001263832  8.78707645
Sigma[sex:(Intercept),(Intercept)] 0.170298758 33.10170253
```

```
# Global PPC
pp_check(advanced_hierarchical_model) +
  ggtitle("Global Posterior Predictive Check for Hierarchical Model")
```

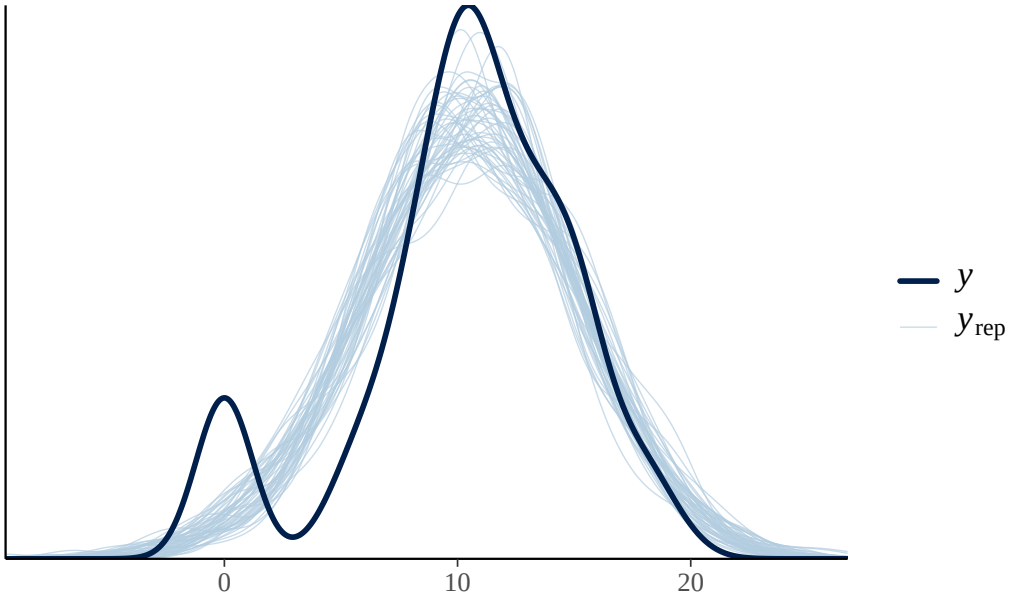
Global Posterior Predictive Check for Hierarchical Model




```
# Group-level PPC
pp_check(advanced_hierarchical_model, group = "school") +
  ggtitle("Posterior Predictive Check by School")
```

```
Warning: The following arguments were unrecognized and ignored: group
```

Posterior Predictive Check by School




```
# library(shinystan)
# launch_shinystan(advanced_hierarchical_model)
```

```
# posterior <- as.matrix(advanced_hierarchical_model)
# divergences <- sum(attr(posterior, "sampler_params")$divergent__)
# print(divergences)
```

```r
# Install and load rstanarm if needed
if (!requireNamespace("rstanarm", quietly = TRUE)) install.packages("rstanarm")
library(rstanarm)

# Bayesian interaction model
bayesian_interaction_model <- stan_glm(
  G3 ~ studytime * sex + failures + absences,
  data = d1,
  family = gaussian(),
  prior_intercept = normal(10, 5),
  prior = normal(0, 2.5),
  prior_aux = cauchy(0, 2),
  chains = 4,
  iter = 2000,
  seed = 123
)
```

```r
# Compute WAIC for the Bayesian models
waic_one_param <- waic(bayesian_interaction_model)
```

```
Warning:
1 (0.3%) p_waic estimates greater than 0.4. We recommend trying loo instead.
```

```r
waic_hierarchical <- waic(advanced_hierarchical_model)

# Print WAIC results
print(waic_one_param)
```

```
Computed from 4000 by 395 log-likelihood matrix.

          Estimate   SE
elpd_waic  -1135.0 15.7
p_waic         7.2  0.8
waic        2270.0 31.4

1 (0.3%) p_waic estimates greater than 0.4. We recommend trying loo instead.
```

```r
print(waic_hierarchical)
```

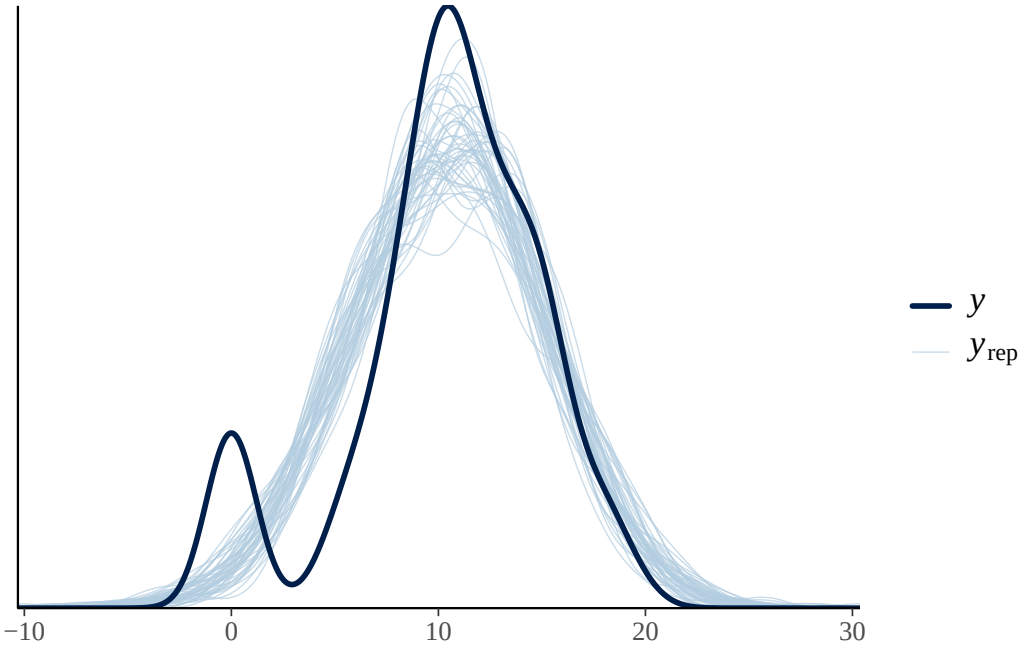
```
Computed from 8000 by 395 log-likelihood matrix.

          Estimate   SE
elpd_waic  -1134.9 15.7
p_waic         7.1  0.7
waic        2269.9 31.4
```

```r
# Compute LOO for both models
loo_one_param <- loo(bayesian_interaction_model)
```

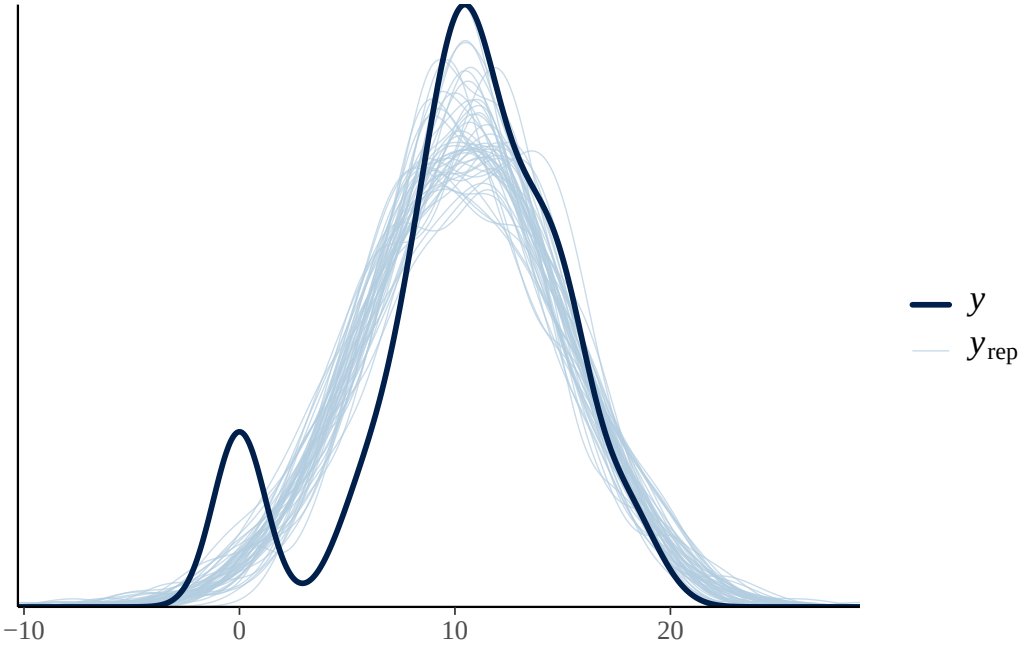
```r
loo_hierarchical <- loo(advanced_hierarchical_model)

# Compare models using loo_compare
loo_comparison <- loo_compare(loo_one_param, loo_hierarchical)

# Print LOO comparison results
print(loo_comparison)
```

```
                           elpd_diff se_diff
advanced_hierarchical_model  0.0       0.0
bayesian_interaction_model  -0.1       0.8
```

```r
# Posterior predictive checks for the interaction model
pp_check(bayesian_interaction_model)
```




```r
# Posterior predictive checks for the hierarchical model
pp_check(advanced_hierarchical_model)
```




```
# Extract posterior samples for random effects
library(bayesplot)
```

```
This is bayesplot version 1.11.1
```

```
- Online documentation and vignettes at mc-stan.org/bayesplot
```

```
- bayesplot theme set to bayesplot::theme_default()
```

```
   * Does _not_ affect other ggplot2 plots
```

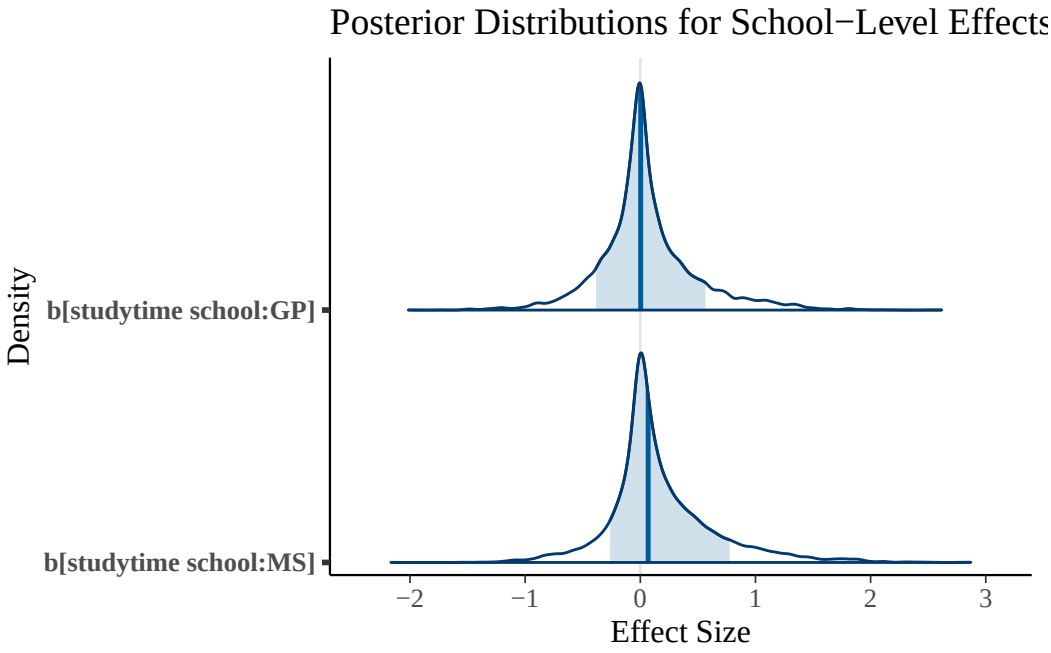
```
   * See ?bayesplot_theme_set for details on theme setting
```

```
posterior_samples <- as.data.frame(advanced_hierarchical_model)

colnames(posterior_samples) # List all parameter names
```

```
 [1] "(Intercept)"                          "studytime"
 [3] "failures"                             "absences"
 [5] "b[studytime school:GP]"               "b[studytime school:MS]"
 [7] "b[(Intercept) sex:F]"                 "b[(Intercept) sex:M]"
 [9] "sigma"                                "Sigma[school:studytime,studytime]"
[11] "Sigma[sex:(Intercept),(Intercept)]"
```

```
# Visualize posterior for group-level intercepts (schools)
mcmc_areas(
  posterior_samples,
  pars = c("b[studytime school:GP]", "b[studytime school:MS]"),
  prob = 0.8
) +
  labs(
    title = "Posterior Distributions for School-Level Effects",
    x = "Effect Size",
```

```
    y = "Density"
  )
```


Posterior Distributions for School−Level Effects
Density
b[studytime school:GP]
b[studytime school:MS]
−2
−1
0
1
2
3
Effect Size