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Two Degree of Freedom Robust Optimal Control Design using A Linear Matrix Inequality Optimization

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Abstract— **This paper proposes a new method for the control design for reference tracking in a two degree-of-freedom (2DOF) robust optimal control framework. The main contribution of this paper is formulation of 2DOF multi-objective optimal control problem in terms of linear matrix inequalities. The proposed method enables formulating and solving for a larger set of performance specifications than existing conventional 2DOF designs. This method also provides a platform for implementing mixed-norm optimization problems that model many tracking applications. The theoretical results are corroborated by experiments that apply the proposed control design for a tracking problem on a positioning system.**

I. INTRODUCTION

The design of 2DOF control systems, where the information from the reference and the measurement signals are independently fed to the controller, has been a topic of much research over last few decades. This research has resulted in many techniques, each with its own unique advantage. For instance, in [1], 2DOF control design is parameterized in terms of two independent stable rational parameters (one each for feedforward and feedback) and analyzed with respect to this parameterization. In [2], 2DOF control design is developed based on parameterization and optimization of the cost functional which include the tracking performance and plant saturation. In [3], the 2DOF control design, with extra emphasis on robustness, is developed based on the integration of the Glover and McFarlane loopshaping [4] and model-matching that makes the closed-loop responses close to a specified response. In addition, many techniques, such as in [5], [6], have been developed that address the case when the exogenous reference signal is known a priori.

In this paper, we propose a new optimal control method for designing 2DOF feedback laws. A 2DOF multi-objective optimal control problem based on $\mathcal{H}_\infty$ stacked sensitivity framework [7] and model-matching framework [8] is formulated. The main theoretical contribution of this paper is a formulation of this 2DOF multi-objective optimal control problem in terms of linear matrix inequalities (LMIs), which is solvable using standard convex optimization problem solving tools. The 2DOF optimal control problem is cast in such a way that it can be adapted into the LMI-based framework proposed in [9].

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The main advantage of this method is the flexibility in formulating the optimization problems. It facilitates solving of a certain class of mixed-norm optimization problems or pole-placement problems, which arise naturally in tracking problems. Another advantage is that it results in a relatively easily implementable controller (of lower order) since the optimal problem is cast in such a way that it avoids specification of certain extraneous weighting functions typically required in other methods.

This paper is organized as follows: Section II provides 2DOF control system framework and the general design objectives in context of tracking problems. In section III, the 2DOF optimal control problem is formulated. Design procedure by converting to LMI based convex optimization problem is explained and some discussion on this method is provided. In section IV, as a example of practical tracking problem, control design based on proposed method is demonstrated on positioning system in an atomic force microscope. Some discussions on 2DOF control and performance of 2DOF control on this system are described. We conclude with final observations in section V.

II. 2DOF CONTROL FRAMEWORK AND CONTROL OBJECTIVES

In contrast to the feedback-only scheme, where the controller acts only on the difference between the reference r and the output-measurement y_m, in 2DOF scheme, the controller acts independently on them. This scheme is implemented in multiple architectures and frequently used architectures are shown in Figure 1.

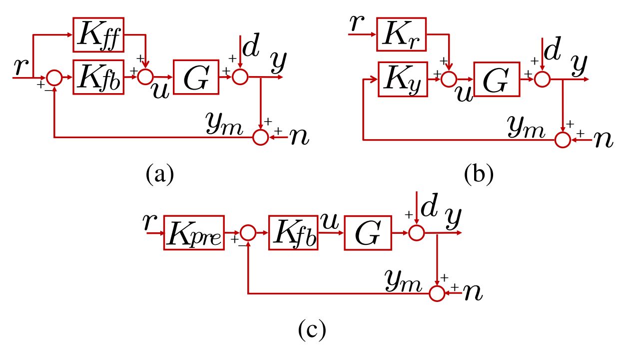


Fig. 1. 2DOF control architectures: (a) The feedforward-feedback scheme where the actuation signal $u = K_{ff}r + K_{fb}(r - y_m)$, (b) Another scheme where $u = K_r r + K_y y_m$, and (c) Prefilter architecture where $u = K_{fb}(K_{pre}r - y_m)$. The schemes (a) and (b) are equivalent as control designs in that one can be retrieved exactly in terms of the other. Practical implementable designs for controllers in (a) and (b) can easily be derived from control design in (c), however the vice-versa may require certain factorization procedures.

In this paper, we mainly explain the design in feedforward-feedback structure (where $u = K_{ff}r + K_{fb}(r-y)$, (a) in Figure 1) for convenience. G is the transfer function of the plant. The signal y represents its output, and the signal u represents the control input. The signal r represents the command or reference signal that the system needs to track, the disturbance signal d represents the *mechanical noise*- the effects of dynamics that are not incorporated in the model G, the signal n represents the sensor noise, the signal $y_m = y+n$ represents the noisy measurement, and the transfer functions K_{ff} and K_{fb} represent the feedforward and feedback control transfer function, respectively. The main objective for the design of the controllers K_{ff} and K_{fb} is to make the *tracking error* small for the frequency-bandwidth as high as possible in spite of the uncertainties.

The relevant closed-loop signals are given by

$$\begin{aligned} y &= SG(K_{ff}+K_{fb})r - Tn + Sd, \\ e &= S(1-GK_{ff})r + Tn - Sd, \\ u &= S(K_{ff}+K_{fb})r - SK_{fb}n - SK_{fb}d. \end{aligned} \tag{1}$$

where $S = (I+GK_{fb})^{-1}$, $T = I-S = (I+GK_{fb})^{-1}GK_{fb}$ and e represents tracking error. In the feedback-only design case, where $K_{ff} = 0$, the transfer function n to e and r to e cannot be simultaneously made small since $S+T = I$; whereas in 2DOF control design, this limitation is relaxed due to the extra freedom in design from K_{ff}. We use T_{yr} and S_{er} to denote the transfer function from r to y and from r to e, respectively, that is $S_{er} = S(I-GK_{ff})$, $T_{yr} = SG(K_{ff}+K_{fb})$.

The performance objectives in tracking problems are characterized in terms of error e in (1). Small tracking error can be achieved by designing that make S_{er}, T and S small in those frequency ranges where the frequency contents of r, n and d are dominant. Broadband tracking is measured by how small S_{er} is at operating frequency and its bandwidth ω_{BW} which is characterized by the frequency at which the magnitude $|S_{er}|$ crosses $1/\sqrt{2}$ [10]. The resolution of the closed-loop system is determined by the effect of noise from Tn term in (1). Therefore, the small roll-off frequency ω_T at which the magnitude $|T|$ crosses $1/\sqrt{2}$ and high roll-off rates for T guarantee good resolution. Robustness to modeling uncertainties is measured by $\|S\|_\infty$ which characterizes the effect of mechanical noise d on the output. Low (near 1) values of $\|S\|_\infty$ guarantee good gain margin and phase margin since $GM \geq \frac{\|S\|_\infty}{\|S\|_\infty - 1}$ and $PM \geq 2\sin^{-1}\frac{1}{\|S\|_\infty}$ [10].

III. Design Procedure for 2DOF Control

A. Problem formulation

In order to impose the performance objectives into optimal control setting, the configuration which is based on $\mathcal{H}_\infty$ stacked sensitivity framework [7] and model-matching framework [8] is considered as shown in Figure 2. Main idea in this setup is to shape the closed loop transfer functions S and T with weighting functions W_s and W_t to achieve robust stability, disturbance rejection and noise attenuation and to make the close-loop response close to a target reference response $T_{ref}r$. The regulated outputs are chosen to be $z_m = T_{ref}r - y$, the deviation from target response, $z_s = W_s(r-n-y)$, the weighted tracking error with noise, and $z_t = W_t y$, the weighted system output.

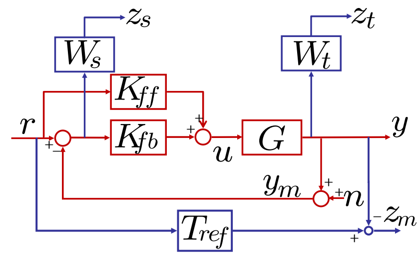


Fig. 2. 2DOF control design configuration. The signals z_m, z_s, and z_t represent the deviation from target response, the weighted tracking error with noise, and the weighted system output. The weights W_s and W_t are chosen to reflect the design specifications of robust stability, disturbance rejection and noise attenuation. The target response function T_{ref} is chosen to specify the performance of close loop response to reference signal. In this 2DOF framework the controller has access to both the reference r and the measured error $r-n-y$ signals.

Using (1), the closed-loop matrix transfer function from the exogenous inputs $w = [r \quad n]^T$ to the regulated outputs $z = [z_m \quad z_s \quad z_t]^T$ is given by

$$\begin{bmatrix} z_m \\ z_s \\ z_t \end{bmatrix} = \underbrace{\begin{bmatrix} T_{ref} - T_{yr} & T \\ W_s S_{er} & -W_s S \\ W_t T_{yr} & -W_t T \end{bmatrix}}_{=\Phi} \begin{bmatrix} r \\ n \end{bmatrix}. \tag{2}$$

The minimization of matrix Φ in (2) does not correspond to the design objective since Φ includes the other redundant terms T, $W_s S_{er}$ and $W_t T_{yr}$ which are not related to design objectives. Also algebraic limitations of $S+T = I$ and $S_{er} + T_{yr} = I$ prevent from reaching satisfactory solution since these constraints severely restrict the feasible space of controllers that make $\|\Phi\|$ small.

In our approach, specifications are directly imposed on certain transfer functions instead of posing the problem in terms of regulated variables. More specifically, we target the transfer functions $T_{ref} - T_{yr}$, $W_s S$ and $W_t T$. This is realizable by the multi-objective framework proposed in [9].

The cost objectives to be minimized are $T_{ref} - T_{yr}$ which is the transfer function from r to z_m and $-[W_s S \quad W_t T]^T$ which is the transfer function from n to $[z_s \quad z_t]^T$. The optimization problem is stated as

$$\min_{K \in \mathcal{K}} \rho \|T_{ref} - T_{yr}\|_{\alpha_1} + \left\| \begin{matrix} W_s S \\ W_t T \end{matrix} \right\|_{\alpha_2} \tag{3}$$

where $K = [K_{ff} \quad K_{fb}]$, $\mathcal{K}$ is the set of stabilizing controllers and the parameter ρ reflects the user defined relative emphasis between model matching and robust performance and $\|\cdot\|_{\alpha_i, i\in\{1,2\}}$ are norms (generally 2-norm or ∞-norm) that can possibly be different from each other. Figure 3 shows the general framework of multi-objective optimization in (3) corresponding to Figure 2.

B. Conversion to LMI optimization

This subsection describes the adaptation of the 2DOF control design in the framework developed in [9]. The

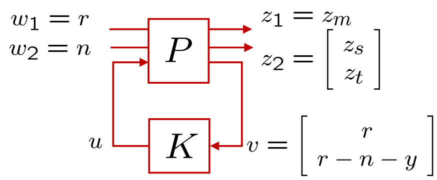


Fig. 3. General framework for 2DOF control in (3): The multi-objective optimization problem is to design a controller that minimizes the sum of ρ times the transfer function from w_1 to z_1 and the transfer function from w_2 to z_2. This is in contrast to stacked sensitivity framework that minimizes the transfer function from $w = [w_1 \ w_2]^T$ to $[z_1 \ z_2]^T$.

generalized matrix for the system P in Figure 3 is given by

$$\left[\begin{array}{c} z_m \\ z_s \\ z_t \\ \hline r \\ r-n-y \end{array}\right] = \underbrace{\left[\begin{array}{cc|c} T_{ref} & 0 & -G \\ W_s & -W_s & W_s G \\ 0 & 0 & W_t G \\ \hline I & 0 & 0 \\ I & -I & -G \end{array}\right]}_{=P} \left[\begin{array}{c} r \\ n \\ u \end{array}\right]. \quad (4)$$

We represent the state space realization of P by

$$P \equiv \left[\begin{array}{c|cc} A & B_w & B \\ \hline C_z & D_{zw} & D_z \\ C & D_w & 0 \end{array}\right] \quad (5)$$

and the controller by

$$K(s) = [K_{ff}(s) \quad K_{fb}(s)] \equiv \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array}\right]. \quad (6)$$

In terms of these realizations, the overall closed loop transfer Φ in (2) is then derived as

$$\begin{aligned} \Phi &= \left[\begin{array}{cc|c} A + BD_kC & BC_k & B_w + BD_kD_w \\ B_kC & A_k & B_kD_w \\ \hline C_z + D_zD_kC & D_zC_k & D_{zw} + D_zD_kDw \end{array}\right] \\ &\equiv \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right]. \end{aligned} \quad (7)$$

We represent the transfer functions from w_j to z_j in Figure 3 by Φ_j's ($j = 1, 2$). (The transfer function Φ_1=$T_{ref} - T_{yr}$ represents the model mismatch and Φ_2=$-[W_sS \ \ W_tT]^T$ reflects the robustness and performance objectives). Note that these are the transfer functions of interest to be minimized. Each of these transfer functions can be written in terms of the matrix transfer function Φ in (2) as $\Phi_j = L_j \Phi R_j$, where the matrices L_j and R_j are chosen as

$$\begin{aligned} L_1 &= \left[\begin{array}{ccc} I & 0 & 0 \end{array}\right], \ R_1 = \left[\begin{array}{c} I \\ 0 \end{array}\right], \\ L_2 &= \left[\begin{array}{ccc} 0 & I & 0 \\ 0 & 0 & I \end{array}\right] \text{and } R_2 = \left[\begin{array}{c} I \\ 0 \end{array}\right]. \end{aligned} \quad (8)$$

Let $B_j = B_w R_j$, $C_j = L_j C_z$, $D_j = L_j D_{zw} R_j$, $E_j = L_j D_z$, $H_j = D_w R_j$ ($j = 1, 2$) then,

$$\begin{aligned} \Phi_j &= \left[\begin{array}{c|c} \bar{A} & \bar{B}R_j \\ \hline L_jC & L_jDR_j \end{array}\right] \\ &= \left[\begin{array}{cc|c} A + BD_kC & BC_k & B_j + BD_kH_j \\ B_kC & A_k & B_kH_j \\ \hline C_j + E_jD_kC & E_jC_k & D_j + E_jD_kH_j \end{array}\right] \\ &\equiv \left[\begin{array}{c|c} \bar{A} & \bar{B_j} \\ \hline \bar{C_j} & \bar{D_j} \end{array}\right] \text{ for } j = 1, 2. \end{aligned} \quad (9)$$

Now, we can impose the performance condition (or design specification) for each Φ_j separately. The condition for $\mathcal{H}_\infty$ performance $\|\Phi_j\|_\infty < \gamma_j$ can be imposed by the bounded-real lemma [11], [9]. Equivalent problem to these performance conditions with Hurwitz $\bar{A}$ is determining $P_j > 0$ that satisfy the following matrix inequality

$$\left[\begin{array}{ccc} \bar{A}^T P_j + P_j \bar{A} & P_j \bar{B_j} & \bar{C_j}^T \\ \bar{B_j}^T P_j & -\gamma_j I & \bar{D_j}^T \\ \bar{C_j} & \bar{D_j} & -\gamma_j I \end{array}\right] < 0 \quad (10)$$

for $j \in \{1, 2\}$. The condition for $\mathcal{H}_2$ performance $\|\Phi_j\|_2 < \gamma_j$ is more complicated. By the positive-real lemma [12], [9], it is equivalent to find $P_j > 0$ and auxiliary parameter Q that satisfy the following matrix inequalities

$$\begin{aligned} &\left[\begin{array}{cc} \bar{A}^T P_j + P_j \bar{A} & P_j \bar{B_j} \\ \bar{B_j}^T P_j & -I \end{array}\right] < 0, \\ &\left[\begin{array}{cc} P_j & C_j^T \\ C_j & Q \end{array}\right] > 0, \quad Tr(Q) < \gamma_j \end{aligned} \quad (11)$$

for $j \in \{1, 2\}$.

However, the matrix inequalities in (10,11) are not linear in terms of the actual design variables (A_k, B_k, C_k, D_k). However they can be converted into LMIs if we impose the conservatism

$$P_1 = P_2 \equiv P \quad (12)$$

and change the variables through an appropriate transformation. This assumption brings the conservatism in design but it recovers linearity of variables.

A short sketch of construction of the transformation of variables that makes (10,11) a LMI in design variables is given below (The detailed proof can be found in [9]). We decompose the unknown positive definite matrix P and its inverse as

$$P = \left[\begin{array}{cc} S & N \\ N^T & ? \end{array}\right], \ P^{-1} = \left[\begin{array}{cc} R & M^T \\ M^T & ? \end{array}\right] \quad (13)$$

where the terms shown as ? are unimportant. The transformation is defined in terms of the following matrices

$$\Pi_1 = \left[\begin{array}{cc} R & I \\ M^T & 0 \end{array}\right], \ \Pi_2 = \left[\begin{array}{cc} I & S \\ 0 & N^T \end{array}\right]. \quad (14)$$

Note that the decomposition of P in (13) and that $PP^{-1} = P^{-1}P = I$ imply that $MN^T = I - RS$, $P\Pi_1 = \Pi_2$.

The new (transformed) variables are defined as

$$\begin{aligned} \hat{A} = &\ NA_kM^T + NB_kCR + SBC_kM^T \\ &+ S(A + BD_kC)R, \\ \hat{B} = &\ NB_k + SBD_k, \\ \hat{C} = &\ C_kM^T + D_kCR, \quad \hat{D} = \ D_k. \end{aligned} \quad (15)$$

By congruence transformation of block diagonal matrix which consists of Π_1 and I, (10,11) can be transformed into the LMIs with variables of $\hat{A}, \hat{B}, \hat{C}, \hat{D}, R, S$.

For instance, if we impose $\mathcal{H}_\infty$ performance on Φ_2 which is natural choice for robustness, then congruence transform of $diag(\Pi_1, I, I)$ will change (10) into

$$\begin{bmatrix} \mathcal{Q}(AR+B\hat{C}) & (*) & (*) & (*) \\ \hat{A}+(A+B\hat{D}C)^T & \mathcal{Q}(SA+\hat{B}C) & (*) & (*) \\ (B_2+B\hat{D}H_2)^T & B_2^T S + H_2^T \hat{B}^T & -\gamma_2 I & (*) \\ C_2 R + E_2 \hat{C} & C_2 + E_2 \hat{D} C & D_2 & -\gamma_2 I \end{bmatrix} < 0 \tag{16}$$

and $\Pi_1^T P \Pi_1$ becomes

$$\begin{bmatrix} R & (*) \\ I & S \end{bmatrix} > 0. \tag{17}$$

Here, $(*)$ can be inferred by symmetry and the operation $\mathcal{Q}(Z)$ denotes $Z + Z^T$. Note that (16,17) are linear in $\hat{A}, \hat{B}, \hat{C}, \hat{D}, R, S$. Similar results can be derived for $\mathcal{H}_2$ performance case.

Now, we can solve the multi-objective optimization problem in (3). It has a solution if the following minimization problem has a solution $(\hat{A}, \hat{B}, \hat{C}, \hat{D}, R, S, \gamma_1, \gamma_2)$.

$$\begin{array}{l} \min \quad \rho\gamma_1 + \gamma_2 \\ \text{subject to LMIs corresponding to the performance} \\ \text{specifications on } \Phi_1 \text{ and } \Phi_2. \end{array} \tag{18}$$

For example, if we impose $\mathcal{H}_\infty$ performance on both Φ_1 and Φ_2 (i.e. $\alpha_1 = \alpha_2 = \infty$) then the minimization problem is

$$\begin{array}{l} \min \quad \rho\gamma_1 + \gamma_2 \\ \text{subject to} \\ \begin{bmatrix} \mathcal{Q}(AR+B\hat{C}) & (*) & (*) & (*) \\ \hat{A}+(A+B\hat{D}C)^T & \mathcal{Q}(SA+\hat{B}C) & (*) & (*) \\ (B_1+B\hat{D}H_1)^T & B_1^T S + H_1^T \hat{B}^T & -\gamma_1 I & (*) \\ C_1 R + E_1 \hat{C} & C_1 + E_1 \hat{D} C & D_1 & -\gamma_1 I \end{bmatrix} < 0 \\ \begin{bmatrix} \mathcal{Q}(AR+B\hat{C}) & (*) & (*) & (*) \\ \hat{A}+(A+B\hat{D}C)^T & \mathcal{Q}(SA+\hat{B}C) & (*) & (*) \\ (B_2+B\hat{D}H_2)^T & B_2^T S + H_2^T \hat{B}^T & -\gamma_2 I & (*) \\ C_2 R + E_2 \hat{C} & C_2 + E_2 \hat{D} C & D_2 & -\gamma_2 I \end{bmatrix} < 0 \\ \begin{bmatrix} R & (*) \\ I & S \end{bmatrix} > 0. \end{array} \tag{19}$$

This is well defined convex optimization problem with LMIs in parameters $(\hat{A}, \hat{B}, \hat{C}, \hat{D}, R, S, \gamma_1, \gamma_2)$ and can be solved by standard convex optimization problem solving tools.

The final step is reconstruction of (A_k, B_k, C_k, D_k) from $(\hat{A}, \hat{B}, \hat{C}, \hat{D}, R, S)$ obtained from the previous convex problem. This is done using the equations in (16). For full order control design, M and N are invertible and (A_k, B_k, C_k, D_k) can be uniquely determined as

$$\begin{aligned} A_k &= N^{-1}(\hat{A} - (\hat{B} - SB\hat{D})CR \\ &\quad -SB(\hat{C} - \hat{D}CR) - S(A + B\hat{D}C)R)M^{-T} \\ B_k &= N^{-1}(\hat{B} - SB\hat{D}) \\ C_k &= (\hat{C} - \hat{D}CR)M^{-T} \\ D_k &= \hat{D}. \end{aligned} \tag{20}$$

C. Discussion of This Methodology

One of the principal advantages of the LMI-based approach is that it allows *simultaneous* incorporation of different metrics of performance constraints such as $\mathcal{H}_\infty$, $\mathcal{H}_2$ as well as design requirements such as specifications on passivity, asymptotic tracking and pole locations [9], [12]. This generality of approach, specially incorporation of mixed norm optimization problem, proves very useful in practical applications. The measure on robustness is well characterized by infinity norm, but the natural metrics on objectives can vary depending on the applications and specific requirements. For instance, the metric on model mismatch can be given in terms of either $\mathcal{H}_\infty$ norm or $\mathcal{H}_2$ norm depending on whether we are interested in the 'worst-case' or average performance.

This approach also allows for interchanging the cost and constraints in an optimization problem. For instance, one can solve a problem where the cost function is maximization of robustness for given specifications of model mismatch and noise attenuation, That is, one can solve for problems of the form

$$\min_{K \in \mathcal{K}} \|S\|_\infty$$
$$\text{subject to} \|T_{ref} - T_{yr}\|_\infty < m_1 \text{ and } \|T\|_\infty < m_2.$$

Thus this approach makes possible addressing a variety of applications with diverse objectives.

It should be emphasized that this approach imposes a technical condition (12) to solve the optimization problem which is widely assumed in much mixed norm optimization research. However, this condition adds conservatism to the solution in the sense that it minimizes over a subset of $\mathcal{K}$ instead of the entire $\mathcal{K}$. An area of active research is on relaxing this technical condition that is applied currently in this approach that restricts the feasible set of controllers [13], [14].

In summary, 2DOF optimal control problem combining $\mathcal{H}_\infty$ stacked sensitivity synthesis and model-matching approach is cast into the LMI-based framework proposed in [9]. By adapting the LMI-based framework, the versatile advantages of LMI approach follows simultaneously such as mixed norm optimization or constrained optimization.

IV. Example: Positioning on Atomic Force Microscope

To demonstrate the described methodology, 2DOF control on the positioning system in Atomic Force Microscope(AFM) is designed and implemented. Since AFMs requires high bandwidth, robust, high resolution positioning, the proposed methodology aptly addresses these requirements. The detail description on this positioning system can be found in [15].

A. 2DOF Control Design and Experiment

The identified 7th order transfer function $G_{xx}(s)$ through experimental data is given as

$$\begin{aligned} G_{xx}(s) &= \frac{-1122.3157(s-1.152\times10^4)(s+543)}{(s+390.3)(s^2+470.9s+8.352\times10^6)} \\ &\quad \times \frac{(s^2+587.2s+8.628\times10^6)(s^2+226.5s+1.407\times10^7)}{(s^2+689.4s+1.315\times10^7)(s^2+4950s+2.44\times10^7)}. \end{aligned} \tag{21}$$

This modeling uncertainty from using low order model is accounted for by imposing the requirement of making the closed loop system robust to it on the control design.

The controller resulting from the design outlined above is applied to G_{xx} with $T_{ref} = \frac{1}{0.0003s+1}$, $W_s = \frac{0.5s+394.8}{s+0.3948}$ and $W_t = \frac{100s+9.475\times10^4}{s+1.184\times10^5}$ which reflect the performance objectives of high bandwidth, high resolution and robustness to modeling errors. High bandwidth requires that $|S_{er}|$ is small for wide operating frequency. It is imposed by T_{ref} function since matching $T_{yr} = 1 - S_{er}$ with T_{ref} shapes S_{er} indirectly. High resolution requires the roll-off frequency of ω_T to be small, which is reflected in W_t. The robustness requirement is reflected in W_s which enforces $\|S\|_\infty \leq 2$. ∞-norm is chosen for both model-matching and robust objectives and the weight of $\rho = 20$ is chosen for the multi-objective minimization.

$$\min_{K\in\mathcal{K}} \rho \left\|T_{ref} - T_{yr}\right\|_\infty + \left\| \begin{matrix} W_s S \\ W_t T \end{matrix} \right\|_\infty \tag{22}$$

The resulting 2DOF controller from this design is given by

$$\begin{aligned} K_{ff} &= \frac{2.746\times10^{14}(s+1.184\times10^5)(s+0.7883)(s+390.3)}{(s+3.131\times10^{11})(s+1.296\times10^8)(s+7817)(s+3333)} \\ &\times \frac{(s^2+470.9s+8.352\times10^6)(s^2+689.4s+1.315\times10^7)}{(s+543)(s+0.789)(s^2+587.2s+8.628\times10^6)} \\ &\times \frac{(s^2+4950s+2.44\times10^7)}{(s^2+226.5s+1.407\times10^6)} \\ K_{fb} &= \frac{6.9497\times10^{13}(s+1.184\times10^5)(s+3333)(s+390.3)}{(s+3.131\times10^{11})(s+1.296\times10^8)(s+7817)(s+3333)} \\ &\times \frac{(s^2+470.9s+8.352\times10^6)(s^2+689.4s+1.315\times10^7)}{(s+543)(s+0.789)(s^2+587.2s+8.628\times10^6)} \\ &\times \frac{(s^2+4950s+2.44\times10^7)}{(s^2+226.5s+1.407\times10^6)}. \end{aligned} \tag{23}$$

The corresponding closed loop transfer functions are shown in Figure 4.

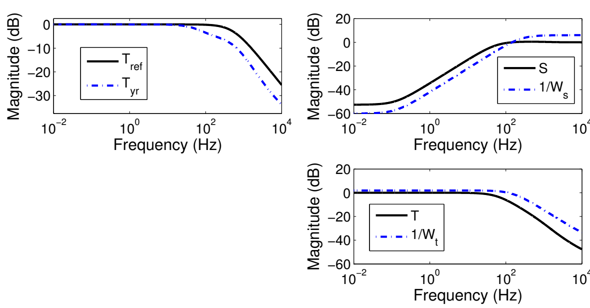
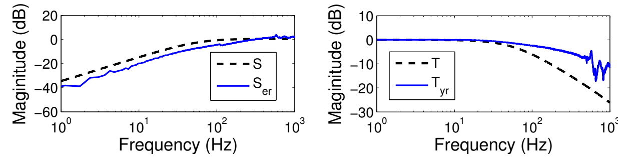


Fig. 4. Multi-objective synthesis of 2DOF control using LMI. Minimization results corresponding weight $\rho = 20$ are $\left\|T_{ref} - T_{yr}\right\|_\infty = 0.526$ and $\left\|[W_s S\ W_t T]^T\right\|_\infty = 2.48$.

The 2DOF control laws obtained from multi-objective synthesis were implemented. Figure 5(a) shows the experimentally obtained $\|S_{er}\|$ which gives the tracking performance (ω_{BW}= 161 Hz) with good robustness to modeling uncertainties ($\|S\|_\infty$=1.06) and Figure 5(b) shows the experimentally obtained $T_{yr}(s)$ with $\|T\|$ which gives the roll-off frequency ω_T=57.5 Hz.

B. Discussion on Roles of Feedforward Control

Since $S = (1 + GK_{fb})^{-1}$ is completely determined by K_{fb}, the feedback component is large in frequencies where



(a) (b)

Fig. 5. Magnitude of $S_{er}(s)$ and $T_{yr}(s)$(solid) obtained from experiment with S and T(dashed).

S need to be small. Because both $S_{er} = S(1 - GK_{ff})$ and S cannot be made small over the entire frequency (in order to allow for noise attenuation), the feedforward control K_{ff} plays an 'active' role in making S_{er} small beyond the frequency where S is not small (say greater than $1/\sqrt{2}$). The importance of feedforward control wanes as S_{er} becomes small beyond the resonance frequency of the positioning system. The bode plot of controller $K(s) = [K_{ff}(s)\ K_{fb}(s)]$ is shown in Figure 6.

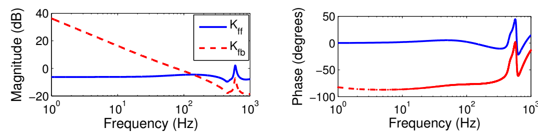


Fig. 6. Bode plot $K(s) = [K_{ff}(s)\ \ K_{fb}(s)]$. In low frequency, feedback control is dominant so that closed loop system has the robustness and high resolution while in the frequency where S cannot be small for T being small, feedforward control becomes important so that the system bandwidth can be increased.

This inherent division of roles between the feedforward and the feedback laws by the optimal control solution can be interpreted as follows. The feedback controller, that solely determines the robustness of the system, is also capable of providing high gains at low frequencies and thus is dominant in low frequencies. At higher frequencies, the fundamental algebraic limitations such as the algebraic Bode-integral law [16] restrict the values of the sensitivity function S and therefore the efficacy of the feedback component K_{fb}. In these frequencies, the feedforward dominates over the feedback component and is able to achieve higher bandwidths. Beyond structural resonance frequencies, the positioning system requires high values for actuation which becomes impractical due to saturation limits on the control signals. This is in consonance with that robustness and other properties of the feedback loop can be decoupled from closed loop reference tracking in 2DOF setup [17], [1].

C. Comparison with Feedback-only Design

For comparison with usual feedback-only 1DOF design, the feedback-only controller was designed by S/T stacked sensitivity optimal control synthesis such that the closed loop function has similar $|T|$ and $|S|$ as shown in Figure 7 which means almost identical resolution and robustness to model uncertainty.

The result of $\mathcal{H}_\infty$ synthesis yielded the 9th order following control laws.

$$\begin{aligned} K_{1D} &= \frac{6.0151\times10^{13}(s+2.531\times10^5)(s+390.3)}{(s+1.471\times10^{13})(s+3.292\times10^6)(s+1.203\times10^4)} \\ &\times \frac{(s^2+470.9s+8.352\times10^6)(s^2+689.4s+1.315\times10^7)}{(s+543)(s+0.8733)(s^2+587.2s+8.628\times10^6)} \\ &\times \frac{(s^2+4950s+2.44\times10^7)}{(s^2+226.6s+1.407\times10^6)} \end{aligned} \tag{24}$$

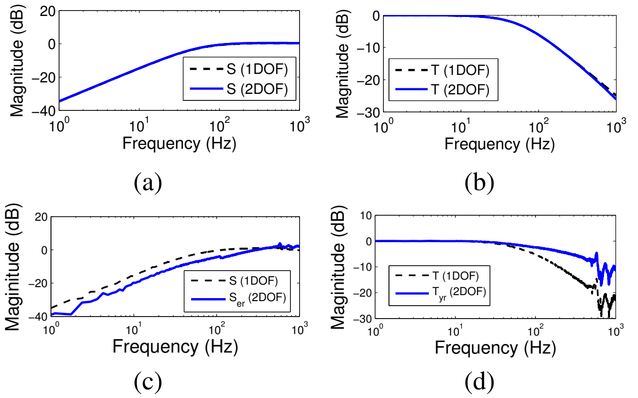


Fig. 7. Comparison of 1DOF(feedback-only) $\mathcal{H}_\infty$ mixed synthesis and 2DOF multi-objective synthesis control design. (a),(b) $|S|$ and $|T|$ from 1DOF control design(dashed) with $|S|$ and $|T|$ from 2DOF control design(solid). (c),(d) $|S|$ and $|T|$ from 1DOF control design(dashed) with $|S_{er}|$ and $|T_{yr}|$ from 2DOF control design(solid) obtained from experiment. The 2DOF multi-objective synthesis control achieves about 216% improvement in the tracking bandwidth.

The experiment results in $\|S\|_\infty$=1.17, ω_T=63.7 Hz and ω_{BW}= 51.0 Hz. There is an improvement of 216% in bandwidth for nearly the same values of resolution and robustness when compared to the optimal feedback-only design.

D. Discussion on the Limitation on Feedback-only Design

There are fundamental limitations on feedback-only design. The algebraic limitation $S + T = I$ already mentioned in section II motivates search for control designs that achieve a trade-off between the bandwidth and resolution requirements. Another limitation is that for any G with phase margin less than 90 degrees, which is true for most practical systems, the closed-loop bandwidth ω_{BW} cannot be larger than ω_T[10] which means the feedback control to achieve noise attenuation over target reference frequency range is impossible. The main limitations on the performance specifications can be described in terms of the Bode integral law $\int_0^\infty \log|S(j\omega)|d\omega = 0$ for stable system G with relative degrees of $KG \geq 2$. (More strict limitation for the systems with non-minimum phase zeros [16], [18]. This trade-off between the bandwidth, resolution, and robustness to modeling uncertainties is described in detail in [19], [15].

However, in 2DOF framework, some fundamental limitations that constrain the feedback-only designs do not hold. Since the bandwidth is determined by S_{er} not S the tradeoffs mentioned above can be relaxed. The tracking bandwidth ω_{BW}=161 Hz is higher than the roll-off frequency ω_T=57.5 Hz, which as discussed above, is impossible in feedback-only designs.

V. CONCLUSIONS

This paper proposed design methodology for 2DOF robust optimal control. To achieve the multiple objectives of performance enhancement and robustness, the 2DOF problem was posed as a LMI based convex optimization problem. In this approach, a large class of performance objectives can be achieved by employing different norms on different objectives and specifying constraints that reflect design constraints such as pole-locations, passivity, and other performance constraints.

The design study for positioning system on AFMs is described and its implementation simultaneously achieves noise attenuation, high tracking bandwidth and robustness to model uncertainties. Relative roles of feedforward and feedback components are analyzed, and analytical as well as experimental comparisons are made between the 1DOF and 2DOF systems.

The advantages of this methodology in terms of practical consequences as well as its flexibility over other competing 2DOF designs have been outlined.

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