

# On Combinatorial Optimization Problems with Mobile Sites and Resources

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**Abstract**—This paper develops a framework to solve combinatorial resource allocation problems in a setting where the sites and resources are allowed to have dynamics. The formulation draws analogy from statistical physics to define the *Free Energy* function which is used as a measure of coverage function. A class of dynamics for the sites and resources is prescribed which guarantees coverage. This is done by casting the problem as a control problem in which we design the resource velocities to ensure that the time derivative of the Free Energy function is non positive.

## INTRODUCTION

There has been considerable research on problems that address deployment of static or mobile *resources* so that they *cover* a set of *sites* in a region. These problems are closely related to a class of combinatorial resource allocation problems that have been studied for a long time in various areas such as minimum distortion problem in data compression [1], facility location problems [2], optimal quadrature rules and discretization of partial differential equations [3], pattern recognition [4], drug discovery [5], neural networks [6], and clustering analysis [7]. In contrast these formulations are relatively recent in the control theory. Again these formulations have come up in various problems such as coarse quantization [8], [9], [10], coverage control, mobile sensing networks, and motion coordination algorithms [11], [12], [13], [14]. All these areas, either directly or not, bring together the concepts from information theory and control theory. These problems each with different and unrelated goals, in fact have some fundamental common attributes. The most important of them is that after disregarding the details, they aim to solve the same optimization problem - they try to obtain (1) an *optimal* partition of the underlying domain, and (2) an optimal assignment of values from a finite set to each cell of the partition.

These problems are typically computationally complex and time intensive. For example in the selection problem of drug discovery, choosing 30 representative compounds from an array of 1000 compounds will result in approximately  $3 \times 10^{25}$  possible partitions. This rules out any exhaustive search method over all partitions. It has been well documented (e.g. [15]) that most of these problems suffer from poor local minima that riddle the cost surface. This inherent non-convex nature of these problems calls for an efficient algorithm that does not get stuck in the local minima. The Deterministic Annealing (DA) algorithm developed in the data compression literature [17] is one such efficient algorithm. It offers two important features: (1) ability to avoid many poor local optima and (2) has a relatively faster convergence rate. It formulates an effective energy function called the *Free Energy* that is parameterized by a (pseudo)

temperature variable and this function is deterministically optimized at successively reduced temperatures.

In this paper, we consider a coverage problem that pertains to *mobile* sites and resources. We develop a notion of coverage and characterize a class of dynamics for sites and resources which guarantees coverage. In this process we give prescriptions for designing dynamics of resources so that they continue to cover a region when the sites in the region have known dynamics. This problem is a dynamic version of locational optimization problem where typically the elements of the region assumed to be static. The analysis and the algorithm developed here is based on fundamental principles of the DA algorithm. More precisely, We first justify that the Free Energy function proposed by the DA algorithm serves as a relevant measure (which in addition is easier to analyze) of the coverage function and then characterize the allowable dynamics for the sites and resources for which we can guarantee a non positive derivative of the Free Energy function.

## I. PROBLEM FORMULATION

This paper addresses a *coverage* problem over a region where its elements (referred to as *sites*) are in motion. This is in contrast to a typical locational optimization problem where sites are static. In this paper we motivate a measure for coverage by expanding on certain concepts from the static problem and the deterministic annealing algorithm used to solve it. This necessitates a brief discussion of the static problem and the deterministic annealing algorithm which we provide below.

### Static Problem:

In its prototypical form, the problem of selecting resource locations for the purpose of coverage of a set of sites can be described as:

Given a weight distribution  $p(x_i)$  of  $N$  sites  $x_i$  in a domain  $\Omega$ , find the best set of  $M$  resource locations  $r_j$  that solves the following minimization problem

$$\min_{r_j, 1 \leq j \leq M} \sum_{i=1}^N p(x_i) \left\{ \min_{1 \leq j \leq M} d(x_i, r_j) \right\} \quad (1)$$

Here  $d(x_i, r_j)$  represents an appropriate *distance* metric between the resource  $r_j$  from the site  $x_i$ ; and the weights  $p(x_i) > 0$  and their sum  $\sum_i p(x_i) = 1$ . In this paper we assume a specific metric  $d(x_i, r_j) = \|x_i - r_j\|^2$  and the domain  $\Omega \subset \mathbb{R}^2$ . In simpler terms,  $M$  resource locations are sought such that the weighted average squared distance of sites  $x_i$  in the domain to their nearest resource locations is minimized. This formulation in the context of facility

location problems, for example, can pertain to finding school locations ( $r_j$ ) so that weighted average distance of student ( $x_i$ ) to the nearest school is minimized.

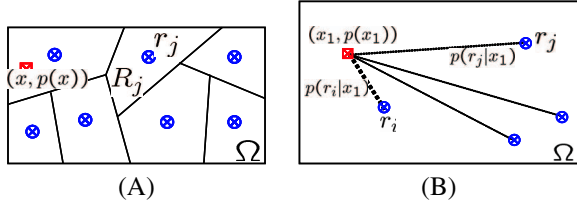


Fig. 1. (A) A schematic of the static problem in which the set of sites is partitioned into cells  $R_j$  and to each  $R_j$  a resource location  $r_j$  ascribed so that the coverage cost function is minimized (B) In DA, unlike Lloyd's algorithm, far off sites do affect the choice of  $r_j$  via the weighting parameters  $p(r_j|x_i)$ .

Alternatively, this problem can also be formulated as finding an *optimal* partition of the domain  $\Omega$  into  $M$  cells  $R_j$  (see Figure 1(A)) and assign to each cell  $R_j$  a resource location  $r_j$  such that the following cost function is minimized

$$\sum_{j=1}^M \sum_{x_i \in R_j} p(x_i) d(x_i, r_j).$$

These locational optimization problems are non convex and computationally complex. For example, the optimal allocation of 20 resources in a domain of 30 sites and a given weights would require search over 30 million partitions! This renders searches over all partitions practically impossible. Moreover, realistic objective functions have unpredictable surfaces with many local minima, and therefore require algorithms that are designed to avoid local minima. The DA algorithm [18] is suited for this purpose since it is specifically designed to avoid local minima.

#### Deterministic Annealing Algorithm

DA can be viewed as a modification of a popular algorithm called Lloyd's algorithm [1], [19]. Lloyd's algorithm is an iterative method which ensures that at each iteration, the partition of domain and the resource locations satisfy the following two necessary properties that the solution has: (1) Nearest Neighbor condition (Voronoi partitions): The partition of the domain is such that each site in the domain is associated to the nearest resource location. (2) Centroid condition: The resource locations are such that  $r_j$  is the centroid of the  $j$ th cell  $R_j$ . In this algorithm, the initial step consists of randomly choosing resource locations and then successively iterating between the steps of: (1) forming Voronoi partitions, and (2) choosing the centroid to be the resource location. It should be noted that the solution depends substantially on the initial allocation of resource locations as in the successive iterations the locations are influenced only by 'near' sites of the domain and are virtually independent of 'far' sites. Such algorithms get stuck at local minima as in their development the resource locations  $r_j$  are influenced only by sites  $x_i$  that are close to  $r_j$ . The DA algorithm does away with this local influence of domain sites by allowing each site  $x_i$  to be associated with every representative  $r_j$  through a weighting parameter  $p(r_j|x_i)$

(see Figure 1 (B)). The DA formulation includes a modified distortion term

$$D = \sum_{i=0}^{N-1} \sum_{j=0}^{M-1} d(x_i, r_j) p(x_i) p(r_j|x_i)$$

which is similar to the cost function in (1). It also includes an entropy term

$$H = - \sum_{i=0}^{N-1} \sum_{j=0}^{M-1} p(x_i) p(r_j|x_i) \log p(r_j|x_i) \quad (2)$$

which measures the randomness of distribution of the associated weights. This entropy is the highest when the influence on a resource location  $r_j$  by every site  $x_i$  is the same ( $p(r_j|x_i) = 1/N$ ). This algorithm solves the optimization problem:

$$\min_{r_j} \min_{p(r_j|x)} \underbrace{D - T_k H}_{:= \hat{F}}$$

at the  $k$ th iteration where  $T_k$  is a parameter called *temperature* which is chosen such that it tends to zero as  $k$  tends to infinity. At the asymptotics this problem reduces to (1). Clearly for large values of  $T_k$ , we mainly attempt to maximize the entropy. As  $T_k$  is lowered we trade entropy for the reduction in distortion, and as  $T_k$  approaches zero, we minimize  $D$  directly to obtain a hard (nearest neighbor) Lloyd-type solution. Minimizing the Free Energy term  $\hat{F}$  with respect to the association probabilities  $p(r_j|x_i)$  is straightforward and gives the *Gibbs* distribution

$$p(r_j|x_i) = \frac{e^{-d(x_i, r_j)/T_k}}{\sum_i e^{-d(x_i, r_j)/T_k}}$$

Substituting for  $p(r_j|x_i)$  in  $\hat{F}$  gives

$$F = -\frac{1}{\beta} \sum_i p(x_i) \log \sum_i e^{-\beta d(x_i, r_j)}, \quad (3)$$

where  $\beta = \frac{1}{T}$  is called the *annealing variable*. The function  $F$  is called *Free Energy* as this formulation has a close parallel in statistical physics [20]. This function plays an important role in defining the coverage function for the methodology proposed in this paper. Then minimizing  $F$  with respect to  $r_j$  results in implicit equations

$$r_j = \sum_i p(x_i|r_j) x_i, p(x_i|r_j) = \frac{p(x_i) p(r_j|x_i)}{\sum_i p(x_i) p(r_j|x_i)} \quad (4)$$

#### A. Some observations on the DA algorithm:

We make the following observations on the DA algorithm which play crucial role in defining and analyzing the combinatorial resource allocation problem in the dynamic setting:

- O1. The crux of the DA algorithm consists of minimizing  $F$  with respect to  $\{r_j\}$  starting at high values of  $T_k$  and tracking its minimum while lowering the values of  $T_k$ . Note that this algorithm reformulates the coverage problem stated in (1) as minimizing the Free Energy term  $F$ . Thus  $F$  can be regarded as a measure of coverage since it captures the essence of the formulation in (1).

- O2. The implicit equations for resource locations  $r_j = \sum_i p(x_i|r_j)x_i$  in (4) where  $p(x_i|r_j)$  denotes the posterior probability calculated using Bayes's rule convey clearly the centroid aspect of the result. So the DA algorithm when solved at a fixed temperature (or fixed  $\beta$  value) is characteristically the same as the Lloyd's algorithm. In this sense, the annealing variable parameterizes a class of Lloyd's-type algorithms.
- O3. The temperature variable can be used to parameterize the coarseness of clusters sought (see [18] for details) - i.e. when DA algorithm is ran by fixing temperature values ( $\beta$  is kept fixed), the algorithm with higher  $\beta$  values will yield finer clusters (each distinct resource location will be covering fewer sites) than when ran at lower values of  $\beta$ .
- O4. The implicit equations for resource locations  $r_j = \sum_i p(x_i|r_j)x_i$  are typically solved using the iterative scheme

$$r_j^+ = \sum_i p(x_i|r_j)x_i \text{ for all } j.$$

This is equivalent to the *Descent method* [21] as shown in [22]:

$$r^+ = r - \beta P_2^{-1} \left( \frac{\partial F}{\partial r} \right)^T,$$

where  $r = (r_1 \ r_2 \ \dots \ r_M)^T$  and  $P_2 = \text{diag}(p(r_1) \dots p(r_M))$ . Furthermore we can convert this difference equation into a differential equation

$$\dot{r} = u \quad (5)$$

where  $u = -\kappa\beta P_2^{-1} \left( \frac{\partial F}{\partial r} \right)^T$  where  $\kappa$  is a positive constant. The DA algorithm essentially solves this differential equation for a fixed  $\beta$  till it converges to a solution and then uses the solution as initial condition for the same differential equation but for a higher  $\beta$  value and repeats this process till some prespecified stopping condition is reached. This observation where we view the DA Algorithm as a series of differential equations forms one of the starting points for the formulation in the dynamic setting which is described in the next section.

### B. Dynamic Setup

The above observations form the main idea for defining and solving the dynamic coverage problem. In the dynamic set up, the domain sites  $x_i$  and the resource locations  $r_j$  are allowed to have dynamics and the goal is to design trajectories for  $r_j$  such that some notion of coverage is satisfied. More precisely, we consider  $N$  sites  $x_i = [\xi_i \ \eta_i]^T \in \mathbb{R}^2$  and  $M$  resource locations  $r_j = [\rho_j \ \omega_j]^T \in \mathbb{R}^2$  whose dynamics are given by

$$\begin{aligned} \dot{x} &= \phi(x, r), \ x(0) = x_0 \\ \dot{r} &= u, \ r(0) = r_0 \end{aligned} \Leftrightarrow \dot{\zeta} = f(t, \zeta) \quad (6)$$

where  $x = [\xi \ \eta]^T \in \mathbb{R}^{2N}$  and  $r = [\rho \ \omega]^T \in \mathbb{R}^{2M}$  ( $\zeta = [x \ r]^T$ ) represent the mobile sites and the resources and  $\phi \in \mathbb{R}^{2N}$  and  $u \in \mathbb{R}^{2M}$  ( $f = [\phi \ u]^T$ ) are the respective velocities. Therefore the problem is defined by

Given a weight distribution  $p(x_i)$  of  $N$  sites  $x_i$  in a domain  $\Omega$ , and the dynamics of the sites and resources described by (6), find the best velocity field  $u \in U$  that minimizes a coverage cost function  $C(x_0, r_0)$ .

In this paper, we propose a measure for coverage function and instead of solving an optimal control problem as defined above, we only aim at ensuring that the cost decreases along the trajectories. More specifically, we choose the Free Energy function  $F$  as a measure of the coverage (as justified in the previous section) and design  $u$  such that the time derivative of the Free Energy function  $\frac{dF}{dt} \leq 0$ . Since minimizing  $F$  is similar to minimizing the coverage cost function described in (1), this design aims at decreasing the cost function along the trajectories  $\zeta(t)$ . We make the following remarks about this formulation:

- R1. This formulation is generalization of the static problem when viewed from the point that the resource allocation solutions are usually iterative (as in the DA or Lloyd's algorithms) and furthermore can be viewed as differential equations as in (5). In the dynamic setting we augment this equation with the dynamics for the sites  $\dot{x} = \phi$ . It should be noted that the trajectory for resource locations enforced by equation (5) at a fixed  $\beta$  in the DA algorithm ensures that  $\frac{dF}{dt}$  is non positive since  $\frac{dF}{dt} = \frac{\partial F}{\partial r} u = -\beta \left( \frac{\partial F}{\partial r} \right)^T P_2^{-1} \left( \frac{\partial F}{\partial r} \right)^T \leq 0$ . This paper studies the conceptual generalization of this property of the DA algorithm.
- R2. There is a vital difference between the DA algorithm and the way the problem is formulated here - The problem dealt in this paper freezes the annealing variable  $\beta$  at a constant value and does not include the cooling rate (or annealing dynamics) in its formulation. In this sense it sacrifices the advantages of the annealing process and therefore is qualitatively similar to Lloyd-type algorithms. The issue of including the cooling rate is discussed later.
- R3. The cost function in the problem formulated has not been made specific. In our ongoing research, we are considering a cost function based on Free Energy function in which the distance function includes the velocities:  $d(x, r) = \|x - r\|^2 + \|\phi - u\|^2$ . In this setting the coverage goal is to solve an optimal control problem

$$\min_u \int_0^\infty F(x, r, \phi(x), u) dt.$$

This design is not discussed in this paper.

Inspite of complex nonlinear (non quadratic) structure of  $F = -\frac{1}{\beta} \sum_i p(x_i) \log \sum_j e^{-\beta((\xi_i - \rho_j)^2 + (\eta_i - \omega_j)^2)}$ , its derivative exhibits an algebraic structure that is similar to a derivative of a quadratic function, making it available for analysis and design. In fact, the partial derivative of  $F$  with respect to coordinates  $\zeta = [\xi \ \eta \ \rho \ \omega]^T$  shows remarkable structure - It is given by:

$$\left( \frac{\partial F}{\partial \zeta} \right)^T = 2 \underbrace{\begin{pmatrix} I_2 \otimes P_1 & -I_2 \otimes P_{12} \\ -I_2 \otimes P_{12}^T & I_2 \otimes P_2 \end{pmatrix}}_{\triangleq \Gamma} \zeta, \quad (7)$$

where  $\otimes$  represents matrix Kronecker product,  $I_2$  a  $2 \times 2$  identity matrix,  $P_1 = \text{diag}(p(x_i)) \in \mathbb{R}^{N \times N}$ ;  $P_{12} = [p(x_i, r_j)] \in \mathbb{R}^{N \times M}$ ,  $p(x_i, r_j) = p(x_i)p(r_j|x_i)$ ; and  $P_2 = \text{diag}(p(r_j)) \in \mathbb{R}^{M \times M}$  where  $p(r_j) = \sum_i p(x_i, r_j)$  (Note that each of the above matrices are completely determined by the given weights  $p(x_i)$  and the Gibbs distribution  $p(r_j|x_i)$  that comes in the DA algorithm). Therefore from (6) and (7)

$$\frac{dF}{dt} = \left( \frac{\partial F}{\partial \zeta} \right) \dot{\zeta} = 2\zeta^T \Gamma f(t, \zeta). \quad (8)$$

Here  $\Gamma = \Gamma(\zeta)$  is nonlinear and state dependent matrix but also possesses structure which makes analysis and design easy. Some of the properties of  $\Gamma$  are listed below:

P1:  $\Gamma$  is a symmetric positive semidefinite matrix for all  $\zeta$ .

*Proof:* From the definition of  $\Gamma$  in equation (7) it is clear that it is symmetric. From their definitions, note that the elements  $p(x_i)$ ,  $p(x_i, r_j)$  and  $p(r_j)$  of  $\Gamma$  have properties that are similar to probability mass functions; i.e. they satisfy  $\sum_i p(x_i) = 1$ ,  $\sum_i p(x_i, r_j) = p(r_j)$ ,  $\sum_j p(x_i, r_j) = p(x_i)$  and  $\sum_j p(r_j) = 1$ . These translate to  $P_{12}e_M = [p(x_1), \dots, p(x_N)]^T \triangleq p_x$  where the notation  $e_M = [1 \ 1 \dots 1]^T$  represents a vector of length  $M$ ,  $P_{12}^T e_N = [p(r_1), \dots, p(r_M)]^T \triangleq p_r$ ,  $e_N^T p_x = 1$  and  $e_M^T p_r = 1$ . This implies that  $\Gamma$  is diagonally dominant since the sum of absolute values of elements of the  $i^{\text{th}}$  row of  $P_{12}$  which is given by  $\sum_j |p(x_i, r_j)| = \sum_j p(x_i, r_j)$  adds to  $p(x_i)$  which is the diagonal element in the  $i^{\text{th}}$  row of  $P_1$ . Similarly  $j^{\text{th}}$  row of  $P_{12}^T$  adds up to  $p(r_j)$  which is the diagonal element in the  $j^{\text{th}}$  row of  $P_2$ . Thus absolute sum of off diagonal terms of each row of  $\Gamma$  is equal to the diagonal term in that row. i.e.  $\sum_{j \neq i} |\Gamma_{ij}| = \Gamma_{ii}$ . Therefore from Geršgorin theorem (Theorem 6.1.1 in [23]), we have that all the eigenvalues of  $\Gamma$  are located in the discs with centers at the diagonal elements and radii the absolute sum of the off diagonal elements in the corresponding row; i.e.,

$$\begin{aligned} \lambda_i(\Gamma) &\in \cup_{i=1} \{z_i \in \mathbb{C} \text{ s.t. } |z - \Gamma_{ii}| \leq \sum_{j \neq i} |\Gamma_{ij}| = \Gamma_{ii}\}, \\ \Rightarrow 0 &\leq \lambda_i(\Gamma) \leq 2\Gamma_{ii}, \quad 1 \leq i \leq 2N + 2M. \end{aligned}$$

Therefore  $\gamma$  is a symmetric matrix with all its eigenvalues nonnegative. This implies  $\Gamma$  is positive semidefinite. ■

P2:  $\Gamma$  can be decomposed as  $\alpha(I - T)$  where  $\alpha > 0$ ,  $I$  is the identity matrix and  $T$  is a symmetric doubly stochastic matrix with its spectral radius  $\rho(T) = 1$ .

*Proof:* Note that every element of  $P_1, P_2$  and  $P_{12}$  is nonnegative which implies that every off diagonal term in  $\Gamma$  is non positive; i.e.

$$\begin{aligned} \Gamma &\in Z_{2N+2M} \triangleq \{Q = [q_{ij}] \in \mathbb{R}^{(2N+2M) \times (2N+2M)} \\ &\text{s.t. } q_{ij} \leq 0 \text{ for } i \neq j\}. \end{aligned}$$

This implies that there exists a  $\bar{T} = [\bar{t}_{ij}] \in \mathbb{R}^{(2N+2M) \times (2N+2M)}$  and  $\alpha > 0$  in  $\mathbb{R}$  such that  $\Gamma$  is equal to  $\alpha I - \bar{T}$  where the spectral radius  $\lambda_{\max}(\bar{T}) \leq \alpha$  and  $\bar{t}_{ij} \geq 0$  for all  $i, j$  (This follows from Lemma 2.5.2.1 in [23]). Choose  $T = \frac{1}{\alpha} \bar{T}$ . This implies

$\Gamma = \alpha(I - T)$ ,  $\lambda_{\max}(T) \leq 1$  and  $T$  is element wise nonnegative. Also  $T$  is symmetric since  $\Gamma$  is symmetric. Furthermore

$$\begin{aligned} &\Gamma e_{2N+2M} \\ &= \begin{pmatrix} I_2 \otimes P_1 & -I_2 \otimes P_{12} \\ -I_2 \otimes P_{12}^T & I_2 \otimes P_2 \end{pmatrix} \begin{pmatrix} e_2 \otimes e_N \\ e_2 \otimes e_M \end{pmatrix} \\ &= \begin{pmatrix} e_2 \otimes P_1 e_N - e_2 \otimes P_{12} e_M \\ -e_2 \otimes P_{12}^T e_N + e_2 \otimes P_2 e_M \end{pmatrix} \\ &= \begin{pmatrix} e_2 \otimes (p_x - p_x) \\ e_2 \otimes (p_r - p_r) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

which implies  $T e_{2N+2M} = e_{2N+2M}$  and since  $\Gamma$  is symmetric,  $T$  is symmetric. Therefore  $e_{2N+2M}^T T = e_{2N+2M}^T$ . This implies  $T$  is a symmetric doubly stochastic matrix with  $|\lambda_{\max}(T)| = 1$ . ■

We exploit these properties to characterize a class of dynamics  $f(t, \zeta)$  for which  $\frac{dF}{dt}$  is indeed non positive. First (in Theorem 1) we look at dynamics of the form  $A\zeta$  where  $A$  can be state dependent. Then (in Theorem 2) we look at the control Lyapunov function (CLF) based methods to derive the control laws (resource velocities) that guarantee  $\dot{F} < 0$  for velocity profiles of the sites ( $\phi(t)$ ) that satisfy some conditions.

*Theorem 1:* [Linear Dynamics] The dynamics  $\dot{\zeta} = A\zeta$ , where  $\zeta \in \mathbb{R}^{2N+2M}$  denotes locations of sites and resources, ensures that the derivative of the corresponding Free Energy function is non positive if the matrix

$$\begin{pmatrix} H(A) & \\ & H(A) - A^T T H^{-1}(A) T A \end{pmatrix}$$

is negative semidefinite for all  $\zeta$  where  $H(A) \triangleq \frac{1}{2}(A + A^T)$  and  $T$  is the symmetric doubly stochastic matrix defined in the decomposition of  $\Gamma$  described in Property P2.

*Proof:* From (8) we have  $\frac{dF}{dt} = 2\zeta^T \Gamma A \zeta$ . If we decompose  $\Gamma$  as  $\alpha(I - T)$  as in property P2, we obtain

$$\begin{aligned} \frac{dF}{dt} &= 2\alpha \zeta^T (I - T) A \zeta \\ &= \alpha \begin{pmatrix} \zeta \\ \zeta \end{pmatrix}^T \begin{pmatrix} H(A) & T A \\ A^T T & H(A) \end{pmatrix} \begin{pmatrix} \zeta \\ \zeta \end{pmatrix}. \end{aligned}$$

therefore  $\frac{dF}{dt} \leq 0$  if

$$R(A) \triangleq \begin{pmatrix} H(A) & T A \\ A^T T & I \end{pmatrix} \leq 0.$$

The assertion in the theorem directly follows from the fact that  $R(A)$  can be factorized as

$$\begin{bmatrix} I & 0 \\ A^T T H^{-1}(A) & I \end{bmatrix} \begin{pmatrix} H(A) & \\ & H(A) - A^T T H^{-1}(A) T A \end{pmatrix} \begin{bmatrix} I & 0 \\ A^T T H^{-1}(A) & I \end{pmatrix}^T.$$

**Remark:** Note that the matrix inequality is equivalent to  $H(A) \leq 0$  and  $H(A) - A^T T H^{-1}(A) T A < 0$ . The first inequality represents a notion of stability for the dynamics of  $\zeta$ .

*Corollary 1:*  $\frac{dF}{dt} \leq 0$  if  $H(A) \leq 0$  and the spectral radius  $\rho(A^T T H^{-1}(A) T A H^{-1}(A)) \leq 1$ .

*Proof:* This result is direct consequence of Theorem 7.7.7 in [23]. ■

*Corollary 2:*  $\frac{dF}{dt} \leq 0$  for all negative definite  $A$  that satisfy either  $\rho(T A T A^{-1}) \leq 1$  or  $A - T A T$  is negative semidefinite.

*Proof:* In this case  $H(A) = A < 0$  and therefore  $\rho(A^T T H^{-1}(A) T A H^{-1}(A)) = \rho(A T A^{-1} T A A^{-1}) = \rho(A T A^{-1} T) = \rho(T A T A^{-1}) \leq 1$ . Therefore from Corollary 1 the first part of assertion is true. Also for negative definite  $A$ ,  $\rho(T A T A^{-1}) \leq 1$  is equivalent to  $A - T A T$  being negative semidefinite (Theorem 7.7.3 in [23]). The result then follows from corollary 1. ■

**Corollary 3:**  $\frac{dF}{dt} \leq 0$  for all negative definite  $A$  that commute with  $T$ :  $T(\zeta)A(\zeta) = A(\zeta)T(\zeta)$ .

*Proof:* In this case  $H(A) \leq A \leq 0$  and therefore  $\rho(A^T T H^{-1}(A) T A H^{-1}(A)) = \rho(T^2) \leq 1$ . The result then follows from corollary 1. ■

**Remark:** Theorem 1 characterizes the possible velocity fields  $A\zeta$  for which  $\frac{dF}{dt} \leq 0$ . The structure on the  $A$  matrix imposed by the subsequent corollaries seem restrictive but the choice of resource dynamics provides for additional freedom which makes it easy to design for matrices that satisfy these structural constraints; i.e. the velocity  $u$  in (6) is a design variable which provides extra freedom in determining the structure of  $A$ .

**Theorem 2:** [Arbitrary Dynamics] For the dynamics  $\dot{\zeta} = \begin{pmatrix} \dot{x} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \phi \\ u \end{pmatrix}$ , where  $\zeta \in \mathbb{R}^{2N+2M}$  denotes locations of sites and resources, resource velocities  $u$  can be designed which will ensure that the derivative of the corresponding Free Energy function is non positive if

$$x^T S \phi < 0 \text{ whenever } r = Wx,$$

where  $P_1 Q_1 = P_{12} = Q_2 P_2$ ,  $S = I_2 \otimes ((I - Q_2 Q_1^T) P_1)$  and  $W = I_2 \otimes Q_2^T$ .

*Proof:* By introducing new variables  $\bar{r} \triangleq r - Wx$  and  $\bar{u} = (I_2 \otimes P_2)(u - W\phi)$ , the derivative of the Free Energy can be rewritten as

$$\frac{dF}{dt} = \begin{pmatrix} x \\ r \end{pmatrix}^T \Gamma \begin{pmatrix} \phi \\ u \end{pmatrix} = x^T S \phi + \bar{r}^T \bar{u}.$$

Here  $\bar{u}$  can always be designed to make  $\frac{dF}{dt} \leq 0$  as long as  $\bar{r}^T \bar{u} \neq 0$ . Moreover the condition in the Theorem guarantees that  $x^T S \phi \leq 0$  whenever  $r = Wx$ . Thus  $\dot{F} < 0$  with input  $u = (I_2 \otimes P_2)^{-1} \bar{u} + W\phi$  where

$$\bar{u} = \begin{cases} - \left( c_0 + \frac{a + \sqrt{a^T a + (b^T b)^2}}{b^2} \right) b & \text{if } b \neq 0 \\ 0 & \text{if } b = 0 \end{cases}$$

for any  $c_0 > 0$ . ■

**Remark:** This result can be further made stronger by easily extending this analysis for more general dynamics where the control  $u$  is of the form  $u = h(\zeta) + G(\zeta)v$ .

## II. ANALYSIS AND ONGOING WORK

This paper presents a formulation and preliminary analysis for the combinatorial resource allocation problem in a dynamic setting. This analysis raises many questions and stimulates further research - some remarks in this regard are given below.

### On the distributed aspect of the design

The design procedure developed in the paper is based on the DA algorithm which is a global algorithm in the sense that every site  $x_i$  affects the determination of the resource location  $r_j$ . However, the relative contribution by a site  $x_i$  on a resource  $r_j$  is measured by  $p(r_j|x_i)$  which decays exponentially with the distance  $d(x_i, r_j)$ . This suggests that the algorithm can be made local by appropriately discarding the contributions of ‘far-off’ sites with ‘negligible’ effects. This would require a characterization of *effective* radii around each resource which would determine the extent of truncation. In the proposed formulation, the annealing variable  $\beta$  is kept fixed (unlike the DA algorithm). Therefore this method can be thought of as a generalization of Voronoi cell-nearest neighbor algorithms (such as the Lloyd’s algorithm) which is parameterized by the annealing variable  $\beta$ . Higher  $\beta$  implies more dependence on immediate site neighbors of resource allocation  $r_j$  (In fact as  $\beta$  approaches infinity, the proposed algorithm approaches the Lloyd’s algorithm). On the other hand, formulations with lower  $\beta$  would imply more dependence on ‘far-off’ sites and thus better at avoiding local minima. Thus the variable  $\beta$  can be used to formulate problems to take into account the trade-off between the low computational cost of the distributed structure and the requirement of avoiding local minima.

### On the incorporation of annealing dynamics into the design procedure

One of the main contributions of this paper that helped in the generalization of the DA algorithm to the dynamic set up is viewing it as a set of differential equations. This viewpoint makes it amenable to system theoretic tools. The coverage problem can be solved better if we include the dynamics of the annealing variable into the design (Also see remark R2 in section I-B). This issue can be addressed in two ways - One way is to use Theorem 2 and make the proposed algorithm into an iterative algorithm by adding annealing effects. First the initial conditions for resources can be found by running a static DA and then annealing can be incorporated by running the control law suggested by Theorem 2 till  $\dot{F} = 0$  and then repeating again with a new (higher) value for the annealing variable  $\beta$ . Another way is by augmenting the site-resource dynamics in (6) with the annealing dynamics (or the cooling law)  $\dot{\beta} = \gamma$ . The  $\frac{dF}{dt}$  in this case becomes  $\zeta^T \Gamma \zeta + \frac{\gamma}{\beta^2} H$  where  $H$  is the positive entropy function described in (2). Thus this captures the fact that inclusion of the entropy term opposes the tendency of solution to get stuck to the local minima and that its effect decreases as  $\beta$  increases. The DA algorithm can be more accurately obtained by choosing  $\gamma > 0$ . This inclusion of annealing dynamics raises the question of separation of time scales between site, resource and annealing dynamics. Intuitively the dynamics of sites  $x$  should be slower than resources  $r$  (to be able to approximate the dynamic problem as a series of static ones); however the dynamics of  $\beta$  is more complex - In fact in the static case it has periods of slow dynamics followed by fast dynamics. This issue about the design of  $\gamma$  needs to be resolved and is a part of our ongoing research.

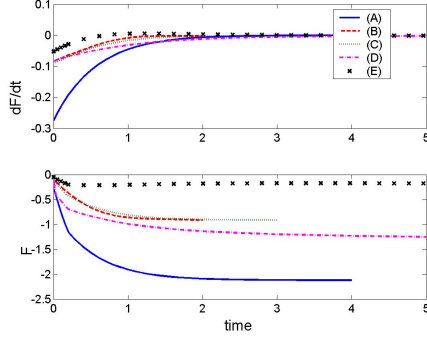


Fig. 2. The time derivative of the coverage measure  $F$  is non positive as predicted by Theorem 1. The dynamics considered are (A)  $\dot{\zeta} = -\zeta$ , (B)  $\dot{\zeta} = -D_1\zeta$  where  $D_2$  is a positive definite diagonal matrix, (C)  $\dot{\zeta} = (\Gamma^2 + \Gamma + I)\zeta$  and (D)  $\dot{\zeta} = S\zeta$  where  $S$  is a negative definite matrix. (E)  $\frac{dF}{dt}$  is not non positive for all  $t$  when the dynamics  $\dot{\zeta} = A\zeta$  does not satisfy Theorem 1.

### III. SIMULATIONS

The preliminary simulations show a lot of promise and validate the theoretical results. The simulations were done for different velocity fields  $\dot{\zeta} = A\zeta$ . The results are shown in Figure 2. The simulations were done for  $N = 10$  and  $M = 3$ ; and the matrix  $P_1$  and vector  $\zeta_0$  were created randomly using the ‘rand’ function in Matlab. We considered the dynamics governed by (A)  $\dot{\zeta} = -\zeta$ , (B)  $\dot{\zeta} = -D_1\zeta$  where  $D_2$  is a positive definite diagonal matrix, (C)  $\dot{\zeta} = (\Gamma^2 + \Gamma + I)\zeta$  and (D)  $\dot{\zeta} = S\zeta$  where  $S$  is a symmetric matrix. These dynamics were created so as to satisfy one or more corollaries of Theorem 1. The results show that the derivative of  $F$  is negative in all these cases and consequently  $F$  which signifies coverage is a decreasing function. The dynamics in (C) is given for the case where  $A(\zeta)$  is a function of  $\Gamma$  and therefore commutes with the matrix  $T$  described in property P2. Practically this can correspond to a scenario where there is a coordinated effort by the sites and resources to move in such a way that there is a better coverage. The simulation results presented here are for dynamics given for *both* the sites and resources. The simulations for synthesis of the control  $u$  that ensure  $A(\zeta)$  satisfies Theorem 1 is not presented here. The required synthesis is still being done and is part of our ongoing research. (E) shows a case where  $\dot{\zeta} = -D_2\zeta$  where  $D_2$  is a positive definite diagonal matrix but does not satisfy the conditions of Theorem 1. We see that  $dF/dt$  is indeed not nonpositive for all  $t$  but still  $F$  shows a decreasing trend over a large interval. Theorem 2 provides for design of resource trajectories for possible time varying site-velocity profiles. This design is being implemented and is pa of ongoing work.

### IV. CONCLUSIONS

In this paper we have developed a framework to solve combinatorial resource allocation problems in a dynamic setting. In this process we have used the Free Energy function as a measure of coverage. We have characterized a class of dynamics for the sites and resources which guarantees coverage in this set up. We have also casted this problem as

a control problem in which we design the resource velocities to ensure that the time derivative of the Free Energy function is non positive. Simulation results validate the theoretical results. These are preliminary results in the direction of solving locational problems in dynamic setting.

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