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Analysis of a novel preconditioner for a class of p-level lower rank extracted systems

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SUMMARY

This paper proposes and studies the performance of a preconditioner suitable for solving a class of symmetric positive definite systems, $\hat{A}x = b$, which we call *p-level lower rank extracted systems* (*p-level LRES*), by the preconditioned conjugate gradient method. The study of these systems is motivated by the numerical approximation of integral equations with convolution kernels defined on arbitrary p-dimensional domains. This is in contrast to p-level Toeplitz systems which only apply to rectangular domains. The coefficient matrix, $\hat{A}$, is a principal submatrix of a p-level Toeplitz matrix, A, and the preconditioner for the preconditioned conjugate gradient algorithm is provided in terms of the inverse of a p-level circulant matrix constructed from the elements of A. The preconditioner is shown to yield clustering in the spectrum of the preconditioned matrix which leads to a substantial reduction in the computational cost of solving LRE systems. Copyright © 2006 John Wiley & Sons, Ltd.

KEY WORDS: preconditioner; conjugate gradient method; integral equations; convolution; non rectangular domains

0. INTRODUCTION

In this paper, we study the solution of a class of real symmetric positive definite linear systems, $\hat{A}x = b$, which we call *p-level lower rank extracted systems* (*p-level LRES*). They arise in the numerical approximation of convolution type integral equations defined on *arbitrary* p-dimensional domains. These equations appear in diverse scientific and engineering areas such as: in the solution of partial differential equations [1–3], in the solution of inverse problems in signal and image reconstruction [4, 5], in the analysis of the time-series data [6], in the modelling of tabular mining excavations [7], and in the modelling of elements in planar array antennae in the field of telecommunications [8]. The coefficient matrix, $\hat{A}$, is of

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the form $\hat{A} = \mathscr{L}^{\mathrm{T}} A \mathscr{L}$ where A is a p-level Toeplitz matrix and $\mathscr{L}$ (which we call *the extraction matrix*) is a submatrix of a permutation matrix; i.e. $\hat{A}$ is a principal submatrix of A. In the context of integral equations, A represents the convolution kernel and $\mathscr{L}$ gives a complete description of the domain of integration. For example, in two-dimensional integral equations, the kernel is represented by a two-level Toeplitz matrix, also called a block-Toeplitz Toeplitz-block (BTTB) matrix and the corresponding domain is described by $\mathscr{L}$. The one-level LRES have been studied and analysed in Reference [9]. It should be noted that the p-level Toeplitz systems, which represent convolution type integral equations on rectangular domains, can be considered a special case of p-level LRES. In this way, the Toeplitz systems and LRES have a very close relationship: on one hand the class of Toeplitz systems form a subclass of p-level LRES, while on the other hand, the embedding $\hat{A} = \mathscr{L}^{\mathrm{T}} A \mathscr{L}$ implies that each LRE system can be viewed as a subsystem of a Toeplitz system. While this analysis for p-level LRES applies to the special case of Toeplitz systems, the analysis that exists in the literature to assess the performance of the preconditioners for the Toeplitz systems cannot be applied directly to the p-level LRES.

Their close relation to Toeplitz systems makes it possible to exploit various techniques from the vast literature for Toeplitz systems to solve them. A comprehensive survey of methods to solve Toeplitz systems (especially iterative methods) has been presented in Reference [6]. Over the last decade, significant attention has been given to using the *preconditioned conjugate gradient method* (PCG) [10, 11] to solve Toeplitz systems. In this method, $PAx = Pb$ is solved instead of $Ax = b$. The matrix P is chosen so that the matrix PA has its spectrum clustered, which ensures better convergence rates. Several preconditioners based on circulant matrices have been proposed for BTTB systems [6, 12–15]. By contrast, not much attention has been given to p-level LRES in which the domains of integration are not always rectangular (or even connected). Preconditioners for elliptic partial differential equations with irregular domains (which have a specific sparse matrix structure) have been presented in References [1, 2] but these systems are distinct from LRES. The main contribution of this paper is that it proposes a solution strategy for a large class of p-level LRES which guarantees low computational expense in the $O(\mathbf{N}^{2-1/p} \log \mathbf{N})$ operations where $\mathbf{N}$ is the size of the coefficient matrix. This matches with the theoretical results on computational expense for circulant like preconditioners reported in Reference [16].

In this paper, we propose a preconditioner $\hat{P}$ for use with the PCG to solve the p-level LRES more efficiently. This preconditioner has been motivated by the one first introduced in Reference [7] for solving interacting crack problems that arise in modelling mining excavations. For interacting crack problems, there is a requirement to model a sequence of such sub-problems in which the interaction between sub-blocks at one step determines the extent of the sub-vectors at a subsequent step. One option is to set up a new system matrix for each new set of interacting sub-blocks. However, by treating each such subsystem as embedded in the larger system with system matrix A, we avoid this set-up process at each stage of the calculation and also derive a considerable computational advantage from the preconditioner. It is remarkable that the preconditioner constructed by using the encompassing p-level Toeplitz matrix yields such an efficient clustering of the eigenvalues associated with the multiple interacting sub-problems. Indeed, the extraction operators that we introduce to define the geometry of the interacting crack problem make it possible to capture the required information about the higher frequency modes associated with each of the subcracks/sub-excavations.

In the case of two-level Toeplitz systems, this preconditioner is very similar to the ones presented in Reference [14] to solve BTTB systems and, in Reference [15] to solve band Toeplitz systems. However, in Reference [15], a banded Toeplitz matrix is considered while in the LRE systems presented in this paper, the Toeplitz kernel is full. In the case of one-level Toeplitz systems, this preconditioner reduces to one of the preconditioners studied in Reference [17]. When compared to other iterative schemes for one-level Toeplitz systems, it has significantly better clustering characteristics and therefore, better convergence rates. In Reference [17], an elegant analysis of the performance of this preconditioner for one-level Toeplitz systems is presented. Furthermore, the elements of the preconditioner are shown to be approximations of the Fourier coefficients of the reciprocal of the generating function, a result which is not only theoretically interesting, but also provides scope for extensions to larger classes of systems.

In Section 1, we formulate the basic problem and introduce the circulant and preconditioner matrices. We establish some fundamental properties of the circulant matrices and their relation to the preconditioner. The main idea that we exploit in this paper is the same as the one used in Reference [17] to propose and analyse preconditioners for Toeplitz systems. More precisely, we show that the eigenvalues of the circulant matrix associated with the LRE system approximate its generating function, f, at certain points; and that the elements of the preconditioner are approximations of the Fourier coefficients of $1/f$. These properties are then used to establish the clustering and convergence properties of the preconditioner for the p-level LRES in Section 2. In Section 3, the results of some simulations are provided. We give simulation results of the application of the proposed preconditioner to several examples of p-level LRES (with different generating functions, different sizes of matrices and different shapes of domains) and study and quantify its performance. We also provide results of its performance for a p-level LRE matrix associated with a divergent series to study the robustness of this algorithm. Finally in Section 4, we present some concluding remarks.

Notation

- The bold symbols denote a finite sequence of numbers or a mathematical expression involving them. The length of the sequence and the expression is determined by the context in which they appear.
    - *Sums and products*: $\mathbf{m}=\mathbf{k}+\mathbf{l} \Longleftrightarrow m_j=k_j+l_j$, $\mathbf{m}=\mathbf{k}+l \Longleftrightarrow m_j=k_j+l$, $\mathbf{N}=N_0N_1\cdots N_{p-1}$, $\mathbf{2}=2^p$.
    - *Exponents*: $\mathbf{N}^{\boldsymbol{\alpha}}=N_0^{\alpha_0}N_1^{\alpha_1}\cdots N_{p-1}^{\alpha_{p-1}}$, $\mathbf{N^2}=N_0^2N_1^2\cdots N_{p-1}^2$, $\mathbf{N}^2=\mathbf{N^2}$.
    - *Index vector*: $b_{\mathbf{n}}=b_{n_0n_1\cdots n_{p-1}}$. However the symbol $\boldsymbol{\theta}_{\mathbf{2N}}$ is an exception and denotes the vector $((2\pi/2N_0)(2\pi/2N_1)\cdots(2\pi/2N_{p-1}))^{\mathrm{T}}$. If the length of the index is not clear from the context then the subscript denotes the last index value; e.g. $n_{\mathbf{j}_k}=n_{j_0\cdots j_k}$ and $n_{\mathbf{j}_k j_{k+1}}=n_{\mathbf{j}_{k+1}}$.
    - *Products between bold symbols* is given by the sum of their term wise products: $\mathbf{jk}\boldsymbol{\theta}_{\mathbf{2N}}=j_0k_0(2\pi/2N_0)+j_1k_1(2\pi/2N_1)+\cdots+j_{p-1}k_{p-1}(2\pi/2N_{p-1})$.
    - *Boolean operations*: $\mathbf{n}\leqslant\mathbf{N} \Longleftrightarrow n_j\leqslant N_j$, $\mathbf{n}=\mathbf{N}\Longleftrightarrow n_j=N_j$; $\mathbf{n}<M \Longleftrightarrow n_j<M$, $\mathbf{n}=M \Longleftrightarrow n_j=M$; $\mathbf{rs}<\mathbf{RS}\Longleftrightarrow r_js_j<R_jS_j$, etc.; for all j.
    - *Arithmetic operations*: $\sum_{\mathbf{n}<\mathbf{N}} a_{\mathbf{n}}=\sum_{n_0<N_0}\cdots\sum_{n_{p-1}<N_{p-1}} a_{n_0n_1\cdots n_{p-1}}$,

$$\lim_{\mathbf{n}\to\infty} a_{\mathbf{n}}=\lim_{\substack{n_j\to\infty\\ 0\leqslant j\leqslant p-1}} a_{n_on_1\cdots n_{p-1}}$$

- *p-sequences*: $\{a_{\mathbf{k}}\}$, $\mathbf{k}\in\mathbb{Z}^p$ is a sequence in $\mathbb{R}$, i.e. $\{a_{\mathbf{k}}\}=\{a_{k_0k_1\cdots k_{p-1}}\}$.
- *Arguments of functions*:

$$f(\mathbf{j}\boldsymbol{\theta}_{\mathbf{2N}}) \Longleftrightarrow f(j_0\theta_0, j_1\theta_1, \ldots, j_{p-1}\theta_{p-1})$$

$$\theta_j \triangleq 2\pi/2N_j$$

$$\begin{aligned} f(\boldsymbol{\theta}) &= \sum_{\mathbf{n}=-\infty}^{\infty} a_{\mathbf{n}} e^{i\mathbf{n}\boldsymbol{\theta}} \\ &\Longleftrightarrow f(\theta_0, \theta_1, \ldots, \theta_{p-1}) \\ &= \sum_{n_0<N_0} \cdots \sum_{<N_{p-1}} a_{n_0n_1\cdots n_{p-1}} \times e^{i(n_0\theta_0+\cdots+n_{p-1}\theta_{p-1})} \end{aligned}$$

- *p-block matrices*:
  - $A\in\mathbb{R}^{\mathbf{N}\times\mathbf{N}}$ ($\mathbf{N}=\prod_{k=0}^{p-1}N_k$) is called a p-block matrix if A has a nested block structure with N_0^2 subblocks (each, a $(\mathbf{N}/N_0)\times(\mathbf{N}/N_0)$ matrix) which we call one-level blocks; and each of these one-level blocks is itself a block matrix with N_1^2 subblocks (each, a $(\mathbf{N}/N_0N_1)\times(\mathbf{N}/N_0N_1)$ matrix) which we call two-level blocks; and so on. Note that the last such level is the pth level and each block in this level is a scalar. For consistency of certain notions, we define the matrix A itself to be a zero-level block.

    Example

$$A=\begin{pmatrix} A_0 & A_1 & A_2 \\ A_3 & A_4 & A_5 \\ A_6 & A_7 & A_8 \end{pmatrix}, \quad A_{j_0}=\begin{pmatrix} a_{j_00} & a_{j_01} \\ a_{j_02} & a_{j_03} \end{pmatrix}$$

    Here A is a two-block $\mathbf{N}\times\mathbf{N}$ matrix where $\mathbf{N}=N_0\times N_1=3\times 2$. There are N_0^2 $(=9)$ one-level blocks, A_{j_0}, $(0\leqslant j_0\leqslant 8)$. The dimension of each A_{j_0} is $(\mathbf{N}/N_0)\times(\mathbf{N}/N_0)$ $(=2\times 2)$. Similarly in each one-level block, there are N_1^2 $(=4)$ two-level blocks, $A_{j_0j_1}$ $(0\leqslant j_1\leqslant 3)$ each of size $(\mathbf{N}/N_0N_1)\times(\mathbf{N}/N_0N_1)$ $(=1\times 1)$ scalars $a_{j_0j_1}$ $(0\leqslant j_1\leqslant 3)$.

  - e_k^r with $1\leqslant r\leqslant p$, $0\leqslant k\leqslant N_r-1$ denotes a matrix given by $(0\ \cdots\ 0\ I\ 0\ \cdots\ 0)^{\mathrm{T}}$ where 0 and I are $\prod_{m=r}^{p-1}N_m\times\prod_{m=r}^{p-1}N_m$ matrices and I is the kth block. Note that if G is an $(r-1)$-level block, then $(e_k^r)^{\mathrm{T}}Ge_l^r$ is a r-level block for all $1\leqslant r\leqslant p$. In fact, it is the block that is both in the kth block-row and lth block-column of G. e_k^r and e_k will be used interchangeably whenever the level r can be determined by the context in which these matrices appear.
  - $[A]_{\mathbf{k}_{r-1},\mathbf{l}_{r-1}}$ denotes an r-level block given by $(\mathrm{e}_{k_{r-1}}^r)^{\mathrm{T}}\cdots(\mathrm{e}_{k_0}^1)^{\mathrm{T}}A\mathrm{e}_{l_0}^1\cdots\mathrm{e}_{l_{r-1}}^r$. Here, both $\mathbf{k}$ and $\mathbf{l}$ denote are sequences of length r. Note that $[A]_{k_{p-1},l_{p-1}}$ is a scalar and $\mathbf{k_{p-1}},\mathbf{l_{p-1}}$ are vectors of length p. In this case, we use the notation $[A]_{k_{p-1},l_{p-1}}$ interchangeably with $[A]_{\mathbf{k},\mathbf{l}}$.

- A and $A_{\mathbf{N}}$ are used interchangeably to denote matrices, the latter is used to emphasize the dimension of the matrix (and its structure). In particular, $I_{\mathbf{N}}$ denotes an $\mathbf{N}\times\mathbf{N}$ identity matrix.
- T is a symmetric p-block Toeplitz matrix if T is a p-block matrix and $[T]_{\mathbf{k},\mathbf{l}}=[T]_{|\mathbf{k}-\mathbf{l}|,\mathbf{0}}$. Thus, the matrix T of dimension $\mathbf{N}\times\mathbf{N}$ can be constructed if we know a finite p-sequence of length, $\mathbf{N}$, given by $\{[T]_{\mathbf{k},\mathbf{0}}\}_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}}$. This p-sequence we represent by $\phi_{\mathbf{N}}(T)$; i.e. $\phi_{\mathbf{N}}(T)\triangleq\{[T]_{\mathbf{k},\mathbf{0}}\}_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}-\mathbf{1}}$. Often, a bi-infinite p-sequence, $\phi(T)$, is given and the p-block Toeplitz matrix is formed by a contiguous subsequence of this sequence, i.e. $\phi_{\mathbf{N}}(T)\subset\phi(T)$. This bi-infinite p-sequence is called *the generating p-sequence* and the function (when it exists), $f(\boldsymbol{\theta})=\sum_{\mathbf{n}=-\infty}^{\infty}t_{\mathbf{n}}\mathrm{e}^{i\mathbf{n}\boldsymbol{\theta}}$ where $\phi(T)=\{t_{\mathbf{n}}\}_{\mathbf{n}=-\infty}^{\infty}$ is called *the generating function*.
- To every generating p-sequence, $\{t_{\mathbf{k}}\}$, we also associate a $\mathbf{2N}\times\mathbf{2N}$ p-block symmetric Toeplitz matrix, $C_{2\mathbf{N}}$, called the *p-block circulant matrix*. If $\phi_{\mathbf{N}}(C)=\{[C]_{\mathbf{k},\mathbf{0}}\}_{\mathbf{k}=\mathbf{0}}^{\mathbf{2N}-\mathbf{1}}\triangleq\{c_{\mathbf{k}}\}_{\mathbf{k}=\mathbf{0}}^{\mathbf{2N}-\mathbf{1}}$, then

$$c_{k_0k_1\cdots k_{p-1}}=t_{\psi_0(k_0)\psi_1(k_1)\cdots\psi_{p-1}(k_{p-1})}$$

  where $\psi_j(k_j)=\min(k_j,2N_j-k_j)$.

  Example
  Let $\phi(A)$ be given by a bi-infinite two-sequence $\{a_{j_0j_1}\}$ and using this sequence we construct the following matrix:

$$A=\begin{pmatrix}A_0 & A_1 & A_2\\ A_1 & A_0 & A_1\\ A_2 & A_1 & A_0\end{pmatrix},\quad A_{j_0}=\begin{pmatrix}a_{j_00} & a_{j_01} & a_{j_02}\\ a_{j_01} & a_{j_00} & a_{j_01}\\ a_{j_02} & a_{j_01} & a_{j_00}\end{pmatrix}$$

  Here A is a two-block Toeplitz matrix with $[A]_{k_{r-1},l_{r-1}}=[A]_{|k_{r-1}-l_{r-1}|,\mathbf{0}_{r-1}}$; for e.g. $[A]_{0,1}=[A]_{2,1}=A_1$. $\phi_{\mathbf{N}}(A)$ is the two-sequence given by $\{a_{j_0j_1}\}$ where $\mathbf{N}=3\times3$ and $\mathbf{0}\leqslant\mathbf{j}\leqslant\mathbf{N}-\mathbf{1}$. The associated p-block circulant matrix, $C_{2\mathbf{N}}$, is a symmetric block Toeplitz matrix which is completely determined by its first block row given by $(C_0\cdots C_{N_0}C_{N_0-1}\cdots C_1)$ and $C_{j_0}, 0\leqslant j_0\leqslant N_0-1$ is itself a symmetric Toeplitz matrix where its first row is given by $(a_{j_00}\cdots a_{j_0N_1}a_{j_0(N_1-1)}\cdots a_{j_01})$.

- The suffix, *p-block* in p-block matrices, p-block Toeplitz matrices, and p-block circulant matrices; the suffix, *p-* from p-sequences; and the suffix *p-level* from p-level LRES will be dropped in contexts whenever there is no loss of clarity in doing so.
- $\|\cdot\|$ represents the norm (maximum singular value) of the matrix, A.

1. PROBLEM FORMULATION AND SOLUTION

Problem setting: For any LRE system, $\hat{A}x=b$, with the domain given by $\hat{G}$, the coefficient matrix, $\hat{A}$ is completely determined by the kernel of the associated integral equation and by the geometry of the domain. Accordingly, to every LRE system, we associate two matrices:

1. An $\mathbf{N} \times \mathbf{N}$ Toeplitz matrix, A, corresponding to the p-level Toeplitz system representing an integral equation with the same kernel as the LRE system, but whose domain, G, being a p-cell (given by cross products of p intervals $[a_0, b_0] \times [a_1, b_1] \times \cdots \times [a_{p-1}, b_{p-1}]$) contains the domain $\hat{G}$ of the LRE system. This matrix A contains all the information about the kernel.
2. A matrix, $\mathscr{L}$, which is a submatrix of an $\mathbf{N} \times \mathbf{N}$ permutation matrix. This matrix $\mathscr{L}$ characterizes the geometry of the domain $\hat{G}$. The determination of its structure from the geometry of the domain $\hat{G}$ is given in Sections 2.1 and 2.2.

The coefficient matrix, $\hat{A}$ then satisfies the relation $\hat{A} = \mathscr{L}^{\mathrm{T}} A \mathscr{L}$. This constitutes the main difference between the LRE and Toeplitz systems. The Toeplitz systems are completely determined by the kernel while for the LRES, one also needs to the know the structure of the domain besides knowing the kernel.

The association of a p-level LRE system to a p-level Toeplitz matrix as described above can be done in multiple ways. More precisely, this is a many-to-one association since many LRES having the same kernel but different geometries can be associated with the same Toeplitz system. In this paper, we consider a sequence of p-block Toeplitz matrices, $\{A_{\mathbf{N}}\}$, and study the sequence, $\{\mathscr{A}_{\mathbf{N}}\}$, of sets of LRE coefficient matrices ($\hat{A}$) that can be associated with each $A_{\mathbf{N}}$. The sequence, $\{A_{\mathbf{N}}\}$, is formed from a generating p-sequence from the Wiener class, $\{a_{\mathbf{n}}\}$ which satisfies the following assumptions.

Assumptions 1

1. The sequence $\{a_{\mathbf{n}}\}$ is in $\ell_1(\mathbb{Z}^p)$, i.e. $\sum_{\mathbf{n}=-\infty}^{\infty} |a_{\mathbf{n}}| \triangleq c < \infty$.
2. The corresponding generating function, given by $f(\boldsymbol{\theta}) = \sum_{\mathbf{n}=-\infty}^{\infty} a_{\mathbf{n}} \mathrm{e}^{i\mathbf{n}\boldsymbol{\theta}}$ satisfies:
   (a) $f = \sum_{\mathbf{k}=-\infty}^{\infty} a_{\mathbf{k}} \mathrm{e}^{i\mathbf{k}\boldsymbol{\theta}}$ is real, positive and bounded away from 0, i.e $a_{n_o n_1 \cdots n_{p-1}} = a_{\pm n_o \pm n_1 \pm \cdots \pm n_{p-1}}$ for all $\mathbf{n}$ in $\mathbb{Z}^p$; and there is an $\eta > 0$ such that $f(\boldsymbol{\theta}) > \eta > 0$ for all $\boldsymbol{\theta}$ in $\mathbb{T}^p$, where $\mathbb{T} = [-\pi\ \pi]$,
   (b) $\sum \mathbf{k}^{2+2\delta} |a_{\mathbf{k}}|^2 < \infty$ for some $\delta > \min(1, \max(0, (p-2)/4))$.

Note that the Assumption 1-(2a) ensures that the generating function gives rise to symmetric positive definite matrices. It should be remarked that we aim to study a more general class of Toeplitz matrices with generating functions $f \in L_1(\mathbb{R})$. For simulations we have considered generating functions that do not necessarily satisfy Assumption 1-(2a).

Preconditioned conjugate gradient method (*PCG*): We solve the LRES using the PCGM [18]. In this method, a suitably chosen matrix $\hat{P}$ (called the preconditioner) is designed and the system $\hat{P}\hat{A}x = \hat{P}b$ is solved instead of $\hat{A}x = b$. Unlike simpler iterative methods, the convergence rate of the PCG depends on the distribution of all eigenvalues of $\hat{P}\hat{A}$, and not exclusively on its extremal eigenvalues. Moreover, the PCG convergence is fast when the eigenvalues are clustered and $\hat{P}$ is designed so as to achieve this property.

Proposed preconditioner. We prescribe the preconditioner for the coefficient matrix, $\hat{A}$, of an LRE system in the following way. We first form the matrices $A_{\mathbf{N}}$ and $\mathscr{L}$ as in previous section and then construct a $\mathbf{2N} \times \mathbf{2N}$ circulant matrix $C_{\mathbf{2N}}$ (see notation for this construction). Since $\hat{A}$ is a principal submatrix of $A_{\mathbf{N}}$, given by $\mathscr{L}^{\mathrm{T}} A_{\mathbf{N}} \mathscr{L}$, it is also a principal submatrix of $C_{\mathbf{2N}}$; i.e. $\hat{A} = \bar{\mathscr{L}}^{\mathrm{T}} C_{\mathbf{2N}} \bar{\mathscr{L}}$. Its structure is completely determined by the geometry of the domain of the LRE system. The preconditioner, $\hat{P}$ is then defined by $\hat{P} = \bar{\mathscr{L}}^{\mathrm{T}} C_{\mathbf{2N}}^{-1} \bar{\mathscr{L}}$. In the case of Toeplitz systems, $\mathscr{L}$ is equal to an $\mathbf{N} \times \mathbf{N}$ Identity matrix and hence the corresponding matrix, $\bar{\mathscr{L}}$, has a rank of $\mathbf{N}$ which is greater than any other LRE system associated with $A_{\mathbf{N}}$. Hence the name LRE matrices. This prescription of $\hat{P}$ in terms of $C_{\mathbf{2N}}$ is a generalization of the preconditioner proposed for one-level systems in Reference [9]. Therefore, it inherits the numerical and algorithmic advantages (see Reference [9]) of using this preconditioner in the PCG algorithm.

In this paper, our aim is to show that the sequence $\mathscr{C}_N$ of sets of preconditioned LRE matrices $(\hat{P}\hat{A})$ corresponding to the sequence of Toeplitz matrices $\{A_N\}$ have spectra clustered around 1. One of the important features of our prescription is that it is given in terms of circulant matrices whose structure is exploited to establish the clustering. We now present some of the important properties of these matrices.

Proposition 1.1

1. (a) The circulant matrix $C_{\mathbf{2N}}$ is diagonalizable, i.e. $C_{\mathbf{2N}} = F_{\mathbf{N}}^H \Lambda_{\mathbf{2N}} F_{\mathbf{N}}$ where $F_{\mathbf{N}} = F_0 \otimes F_1 \otimes \cdots \otimes F_{p-1}$, $F_j^H = F_j^{-1}$ and $[F_j]_{kl} = (1/\sqrt{2N_j})\mathrm{e}^{\mathrm{i}2\pi kl/2N_j}$ with $0 \leqslant k, l \leqslant 2N_j - 1$ for all $0 \leqslant j \leqslant p-1$.
   (b) $\Lambda_{\mathbf{2N}}$ is a diagonal matrix and its diagonal elements, $\lambda_{\mathbf{j}}$, are given by $\lambda_{\mathbf{j}} = \sum_{\mathbf{n}=-(\mathbf{N}-1)}^{\mathbf{N}} a_{\mathbf{n}} \mathrm{e}^{i\mathbf{jn}\theta_{\mathbf{2N}}}$, $\mathbf{0} \leqslant \mathbf{j} \leqslant \mathbf{2N} - \mathbf{1}$.
   (c) $\lambda_{n_0 \cdots n_j \cdots n_{p-1}} = \lambda_{n_0 \cdots (2N_j - n_j) \cdots n_{p-1}}$ for $0 < j < p-1$ and $\mathbf{0} \leqslant \mathbf{n} \leqslant \mathbf{2N} - \mathbf{1}$.
2. There exists an $\mathbf{N}_0$ in $\mathbb{N}^p$ and a c_0 in $\mathbb{R}^+$ such that $C_{\mathbf{2N}}$ is positive definite and $\|C_{\mathbf{2N}}^{-1}\| < c_0$ for all $\mathbf{N} > \mathbf{N}_0$ and $\mathbf{k}$ in $\mathbb{Z}^p$.
3. $C_{\mathbf{2N}}^{-1}$ is a p-block symmetric circulant matrix, and the associated sequence, $\phi_{\mathbf{N}}(C_{\mathbf{2N}}^{-1}) \triangleq \{\xi_{\mathbf{n}}\}$ is given by $\xi_{\mathbf{n}} = (1/\mathbf{2N}) \sum_{\mathbf{k}=-(\mathbf{N}-\mathbf{1})}^{\mathbf{N}} (1/\lambda_{\mathbf{k}}) \mathrm{e}^{i\mathbf{kn}\boldsymbol{\theta}_{\mathbf{2N}}}$, for all $\mathbf{n} \in \mathbb{Z}^p$.

These results can be easily verified by simple algebraic manipulations.

Remark
Since, $(C_{\mathbf{2N}})^{-1}$ is a circulant matrix and $\phi_{\mathbf{N}}(C_{\mathbf{2N}}^{-1}) = \{\xi_{\mathbf{n}}\}$, the p-level Toeplitz matrix, $P_{\mathbf{N}}$, for which $\phi_{\mathbf{N}}(C_{\mathbf{2N}}^{-1}) = \{\xi_{\mathbf{n}}\}$, forms the preconditioner of the p-level Toeplitz matrix, $A_{\mathbf{N}}$. Furthermore, for any coefficient matrix of an LRE system given by $\hat{A} = \mathscr{L}^{\mathrm{T}} A_{\mathbf{N}} \mathscr{L}$, its preconditioner has the same structure and is given by $\hat{P} = \mathscr{L}^{\mathrm{T}} P_{\mathbf{N}} \mathscr{L}$.

1.1. *Relation to Fourier coefficients of* $1/f$

Note that $\lambda_{\mathbf{j}} = \sum_{\mathbf{k}=-(\mathbf{N}-\mathbf{1})}^{\mathbf{N}} a_{\mathbf{k}} \mathrm{e}^{i\mathbf{jk}\boldsymbol{\theta}_{\mathbf{2N}}}$ is an approximation for $f(\boldsymbol{\theta})$ at $\mathbf{j}\boldsymbol{\theta}_{\mathbf{2N}}$; and $\xi_{\mathbf{n}} = (1/\mathbf{2N}) \sum_{\mathbf{k}=-(\mathbf{N}-\mathbf{1})}^{\mathbf{N}} (1/\lambda_{\mathbf{k}}) \mathrm{e}^{i\mathbf{kn}\boldsymbol{\theta}_{\mathbf{2N}}}$ is a Riemann sum approximation of the multiple integral $(1/2\pi) \int_{-\pi}^{\pi} (1/f(\boldsymbol{\theta})) \mathrm{e}^{i\mathbf{j}\theta} \, \mathrm{d}\theta$, which is the $\mathbf{j}$th Fourier coefficient of $1/f$. This suggests that the elements of $\hat{P}$ are approximations of the Fourier coefficients of $g(\theta) \triangleq 1/f(\boldsymbol{\theta})$. We establish this in the following proposition.

Proposition 1.2

1. There exists a sequence $\{\gamma_{\mathbf{k}}\}$ in $\ell_1(\mathbb{Z}^p)$ such that $g(\theta)=\sum_{\mathbf{k}=-\infty}^{\infty}\gamma_{\mathbf{k}}\mathrm{e}^{\mathbf{ik}\theta}$ for all $\boldsymbol{\theta}$ in $\mathbb{T}^p$.
2. For every $\varepsilon>0$, there exist $\mathbf{M}$ in $\mathbb{N}^p$ such that
   (a) $\sum_{k_j=N_j}^{\infty}\sum_{\mathbf{k}\backslash k_j=\mathbf{1}}^{\infty}|a_{\mathbf{k}}|\leqslant\varepsilon$ and $\sum_{k_j=N_j}^{\infty}\sum_{\mathbf{k}\backslash k_j=\mathbf{1}}^{\infty}|\gamma_{\mathbf{k}}|\leqslant\varepsilon$ for all $\mathbf{N}>\mathbf{M}$,
   (b) $\sum_{\mathbf{k}=-(\mathbf{N}-1)}^{\mathbf{N}}|\gamma_{\mathbf{k}}-\xi_{\mathbf{k}}^{\mathbf{N}}|\leqslant\varepsilon$ for all $\mathbf{N}>\mathbf{M}$.

Here the notation $\mathbf{k}\setminus k_j$ denotes summation over all indices except k_j, and $\xi_{\mathbf{k}}^{\mathbf{N}}$ has been used instead of $\xi_{\mathbf{k}}$ to emphasize the dimension, $\mathbf{N}$.

Proof

(1) This is obtained by generalizing Theorem 18.21 in Reference [19, pp. 367–368].

(2a) Let $0<\tilde{\delta}<\delta$. From the Cauchy–Schwarz inequality, we have

$$\sum\frac{|\mathbf{k}^{1+\delta}a_{\mathbf{k}}|}{|\mathbf{k}^{1/2+\tilde{\delta}}|}\leqslant\left(\sum|\mathbf{k}^{2+2\delta}|a_{\mathbf{k}}|^2|\right)^{1/2}\left(\sum\left|\frac{1}{\mathbf{k}^{1+2\tilde{\delta}}}\right|\right)^{1/2}<\infty\Rightarrow\sum|\mathbf{k}^{(1/2)+\sigma}a_{\mathbf{k}}|<\infty$$

where $\tilde{\sigma}=\delta-\tilde{\delta}>0$. Therefore, for every $\varepsilon>0$, there exists an $\mathbf{M}$ in $\mathbb{N}^p$ such that for $\mathbf{N}>\mathbf{M}$:

$$\sum_{k_j=N_j}^{\infty}\sum_{\mathbf{k}\backslash k_j=\mathbf{1}}^{\infty}|\mathbf{k}^{(1/2)+\sigma}a_{\mathbf{k}}|<\varepsilon\Rightarrow\sum_{k_j=N_j}^{\infty}\sum_{\mathbf{k}\backslash k_j=\mathbf{1}}^{\infty}|a_{\mathbf{k}}|\leqslant\frac{\varepsilon}{N_j^{(1/2)+\tilde{\sigma}}}$$

From Lemma A.1 in Appendix A, we have that $\sum\mathbf{k}^{\mathbf{2}+2\delta}|a_{\mathbf{k}}|^2<\infty\Rightarrow\sum\mathbf{k}^{\mathbf{2}+2\hat{\delta}}|\gamma_{\mathbf{k}}|^2<\infty$ for some $\hat{\delta}>0$. Then we can similarly show that $\sum_{k_j=N_j}^{\infty}\sum_{\mathbf{k}\backslash k_j=\mathbf{1}}^{\infty}|\gamma_{\mathbf{k}}|\leqslant\varepsilon/N_j^{(1/2)+\hat{\sigma}}$.

(2b) We define a sequence $\{h_j^{\mathbf{N}}\}$ by $h_{\mathbf{j}}^{\mathbf{N}}=g(\mathbf{j}\boldsymbol{\theta}_{2\mathbf{N}})-\sum_{\mathbf{k}=-(\mathbf{N}-1)}^{\mathbf{N}}\xi_{\mathbf{k}}^{\mathbf{N}}\mathrm{e}^{i\mathbf{jk}\boldsymbol{\theta}_{2\mathbf{N}}}$ for all $-(\mathbf{N}-\mathbf{1})\leqslant j\leqslant\mathbf{N}$. But the sum on the right side of this equation is an approximation of $g(\mathbf{j}\boldsymbol{\theta}_{2\mathbf{N}})$ and can be simplified as

$$\begin{aligned}\sum_{\mathbf{k}=-(\mathbf{N}-\mathbf{1})}^{\mathbf{N}}\xi_{\mathbf{k}}^{\mathbf{N}}\mathrm{e}^{i\mathbf{jk}\boldsymbol{\theta}_{2\mathbf{N}}}&=\frac{1}{\mathbf{2N}}\sum_{\mathbf{k}=-(\mathbf{N}-\mathbf{1})}^{\mathbf{N}}\sum_{\mathbf{l}=-(\mathbf{N}-\mathbf{1})}^{\mathbf{N}}\frac{1}{\lambda_{\mathbf{l}}^{\mathbf{N}}}\mathrm{e}^{i\mathbf{kl}\boldsymbol{\theta}_{2\mathbf{N}}}\mathrm{e}^{i\mathbf{jk}\boldsymbol{\theta}_{2\mathbf{N}}}\\&=\sum_{\mathbf{l}=-(\mathbf{N}-\mathbf{1})}^{\mathbf{N}}\frac{1}{\lambda_{\mathbf{l}}^{\mathbf{N}}}\prod_{v=0}^{p-1}\delta_{l_v-j_v}=\frac{1}{\lambda_{\mathbf{j}}^{\mathbf{N}}}\end{aligned}$$

for all $-(\mathbf{N}-\mathbf{1})\leqslant\mathbf{j}\leqslant\mathbf{N}$. Now, $\lambda_j^{\mathbf{N}}=\sum_{\mathbf{k}=-(\mathbf{N}-1)}^{\mathbf{N}}a_{\mathbf{k}}\mathrm{e}^{i\mathbf{jk}\boldsymbol{\theta}_{2\mathbf{N}}}$ is an approximation of $f(\mathbf{j}\boldsymbol{\theta}_{\mathbf{N}})$; and an estimate of $h_{\mathbf{j}}^{\mathbf{N}}=g(\mathbf{j}\boldsymbol{\theta}_{2\mathbf{N}})-1/\lambda_{\mathbf{j}}^{\mathbf{N}}$ can be found by exploiting this approximation as follows:

$$\begin{aligned}h_{\mathbf{j}}^{\mathbf{N}}&=\underbrace{g(\mathbf{j}\boldsymbol{\theta}_{2\mathbf{N}})}_{=\frac{1}{f(\mathbf{j}\boldsymbol{\theta}_{2\mathbf{N}})}}-\frac{1}{\lambda_{\mathbf{j}}^{\mathbf{N}}}=-\frac{\overbrace{\sum_{\mathbf{k}=-\infty}^{\infty}a_{\mathbf{k}}\mathrm{e}^{i\mathbf{jk}\boldsymbol{\theta}_{2\mathbf{N}}}-\sum_{\mathbf{k}=-(\mathbf{N}-\mathbf{1})}^{\mathbf{N}}a_{\mathbf{k}}\mathrm{e}^{i\mathbf{jk}\boldsymbol{\theta}_{2\mathbf{N}}}}^{\triangleq R_{\mathbf{j}}^{a}}}{\lambda_{\mathbf{j}}^{\mathbf{N}}f(\mathbf{j}\theta_{\mathbf{N}})}\\&\Rightarrow|h_{\mathbf{j}}^{\mathbf{N}}|\leqslant\underbrace{c_0\|g\|_\infty}_{\triangleq c_2}|R_{\mathbf{j}}^{a}|\end{aligned}\tag{1}$$

where $R^a_{\mathbf{j}}$ is the approximation error of the truncated Fourier series of f at $\mathbf{j}\boldsymbol{\theta}_{2\mathbf{N}}$. Also, $h^{\mathbf{N}}_{\mathbf{j}}$ can be rewritten as

$$h^{\mathbf{N}}_{\mathbf{j}} = \underbrace{\sum_{\mathbf{k}=-(\mathbf{N}-\mathbf{1})}^{\mathbf{N}} \overbrace{(\gamma_{\mathbf{k}} - \xi^{\mathbf{N}}_{\mathbf{k}})}^{\triangleq\, \mu^{\mathbf{N}}_{\mathbf{k}}} \mathrm{e}^{i\mathbf{j}\mathbf{k}\boldsymbol{\theta}_{2\mathbf{N}}}}_{\eta^{\mathbf{N}}_{\mathbf{j}}} + R^{\gamma}_{\mathbf{j}}$$

$$\Rightarrow |\eta^{\mathbf{N}}_{\mathbf{j}}| \leqslant |h^{\mathbf{N}}_{\mathbf{j}}| + |R^{\gamma}_{\mathbf{j}}| \Rightarrow |\eta^{\mathbf{N}}_{\mathbf{j}}| \leqslant 2c_2 \sum_{\mathbf{k}=\mathbf{N}}^{\infty} |a_{\mathbf{k}}| + 2 \sum_{\mathbf{k}=\mathbf{N}}^{\infty} |\gamma_{\mathbf{k}}| \tag{2}$$

where $R^{\gamma}_{\mathbf{j}}$ is the approximation error of the truncated Fourier series of g at $\mathbf{j}\boldsymbol{\theta}_{2\mathbf{N}}$ and the last inequality is obtained using inequality (1). Note that the sequence $\eta^{\mathbf{N}} \triangleq \{\eta^{\mathbf{N}}_{\mathbf{j}}\}$, $-(\mathbf{N}-\mathbf{1}) \leqslant \mathbf{j} \leqslant \mathbf{N}$ is a Discrete Fourier Series obtained from the sequence $\mu^{\mathbf{N}} \triangleq \{\mu^{\mathbf{N}}_{\mathbf{k}}\}$, $-(\mathbf{N}-\mathbf{1}) \leqslant \mathbf{k} \leqslant \mathbf{N}$. Therefore, the coefficients of these two series satisfy the Plancherel (Parseval) relation [10, 20, 21],

$$\|\mu^{\mathbf{N}}\|^2 = \frac{1}{\mathbf{2N}} \|\eta^{\mathbf{N}}\|^2 \Rightarrow \sum_{\mathbf{k}=-(\mathbf{N}-\mathbf{1})}^{\mathbf{N}} (\gamma_{\mathbf{k}} - \xi^{\mathbf{N}}_{\mathbf{k}})^2 \leqslant \left(\mathbf{2}c_2 \sum_{\mathbf{k}=\mathbf{N}}^{\infty} |a_{\mathbf{k}}| + \mathbf{2} \sum_{\mathbf{k}=\mathbf{N}}^{\infty} |\gamma_{\mathbf{k}}|\right)^2 \tag{3}$$

In the proof for (2a), we have shown that $\sum |\mathbf{k}^{(1/2)+\sigma} a_{\mathbf{k}}| < \infty$ and $\sum |\mathbf{k}^{(1/2)+\sigma} \gamma_{\mathbf{k}}| < \infty$ where $\sigma = \min\{\tilde{\sigma}, \hat{\sigma}\}$. Therefore for every $\varepsilon > 0$ there exists an $\mathbf{M}$ in $\mathbb{N}^p$ such that $\sum_{\mathbf{k}=\mathbf{N}}^{\infty} |a_{\mathbf{k}}| \leqslant \varepsilon / \mathbf{N}^{(1/2)+\sigma}$ and $\sum_{\mathbf{k}=\mathbf{N}}^{\infty} |\gamma_{\mathbf{k}}| \leqslant \varepsilon / \mathbf{N}^{(1/2)+\sigma}$. Therefore, inequality (3) becomes

$$\sum_{\mathbf{k}=-(\mathbf{N}-\mathbf{1})}^{\mathbf{N}} (\gamma_{\mathbf{k}} - \xi^{\mathbf{N}}_{\mathbf{k}})^2 \leqslant (\mathbf{2}c_2 + \mathbf{2})^2 \frac{\varepsilon^2}{\mathbf{N}^{1+2\sigma}} \quad \text{for all } \mathbf{N} > \mathbf{M}$$

$$\Rightarrow \sum_{\mathbf{k}=-(\mathbf{N}-\mathbf{1})}^{\mathbf{N}} |\gamma_{\mathbf{k}} - \xi^{\mathbf{N}}_{\mathbf{k}}| \leqslant (\mathbf{2}c_2 + \mathbf{2}) \frac{\varepsilon}{\mathbf{N}^{\sigma}} \quad \text{for all } \mathbf{N} > \mathbf{M}$$

where the last inequality is obtained by a simple application of the Cauchy–Schwarz inequality. □

2. p-LEVEL LOWER RANK EXTRACTED SYSTEMS

Now, we present the clustering characteristics of the spectra of the preconditioned matrices in LRES. A similar analysis for p-level Toeplitz systems has been presented in Appendix B, since some of the analysis in this section uses some results for Toeplitz systems, and, they also form an important class of problems in their own right. Moreover, Toeplitz systems are simpler systems to analyse than LRES, and therefore studying them makes it is easier to understand certain concepts in the analysis of LRES. The analysis of one-level Toeplitz system and the corresponding LRE system has been presented in Reference [9]. For the sake of clearer exposition and easier understanding of these concepts and the relation of LRES

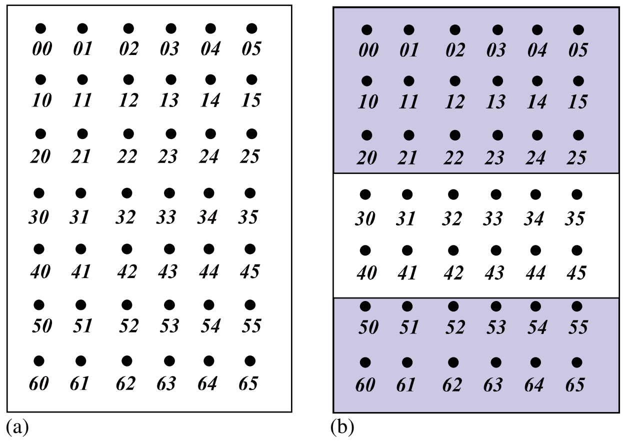


Figure 1. (a) Rectangular domain of the Toeplitz system; and (b) striped domain of the LRE system.

to corresponding Toeplitz systems, we present first the two-level case and then generalize to higher dimensions.

2.1. Two-level LRE systems

We first derive the structure of LRES from the following set of equations:

$$\sum_{n_0=0}^{N_0-1}\sum_{n_1=0}^{N_1-1}\bar{A}_{|k_0-n_0|,|k_1-n_1|}x_{n_0n_1}=b_{k_0k_1}\quad 0\leqslant k_0\leqslant N_0-1,\quad 0\leqslant k_1\leqslant N_1-1$$

where $\bar{A}$, X (with elements $x_{n_0n_1}$) and B (with elements $b_{k_0k_1}$), are $N_0\times N_1$ matrices. We view the indices $(n_0\ n_1)$ as co-ordinates of points in a grid on a rectangular plane (see Figure 1(a)). In the same way, $x_{n_0n_1}$ can be thought as a numerical representation of a field x on the rectangular plane. As the grid specified by the co-ordinates $(n_0\ n_1)$ represents a rectangle, we say that the underlying domain in the system given above is rectangular. The equations in this system can be rearranged to give the two-level system

$$\begin{pmatrix} A_0 & A_1 & \cdots & A_{N_0-1}\\ A_1 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \vdots\\ A_{N_0-1} & \cdots & \cdots & A_0\end{pmatrix}x=b$$

where A_j is a $N_1\times N_1$ symmetric Toeplitz matrix constructed from jth row of $\bar{A}$; x and b are vectors obtained by stacking columns of the matrices X and B one below the other, respectively. In Appendix B, we have presented the analysis of this system.

Now, if the underlying domain in a system is not rectangular, (see Figure 1(b)) but consists of subdomains (for instance, the domain in Figure 1(b) is the union of the shaded areas), then the corresponding equations are written as the following LRE system:

$$\underbrace{\begin{pmatrix} L_0^{\mathrm{T}} A_0 L_0 & L_0^{\mathrm{T}} A_1 L_1 & \cdots & L_0^{\mathrm{T}} A_{N_0-1} L_{N_0-1} \\ L_1^{\mathrm{T}} A_1 L_0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ L_{N_0-1}^{\mathrm{T}} A_{N_0-1} L_0 & \cdots & \cdots & L_{N_0-1}^{\mathrm{T}} A_0 L_{N_0-1} \end{pmatrix}}_{\triangleq \hat{A}} \hat{x} = \hat{b} \tag{4}$$

where L_i are extracting matrices and contain the information about which co-ordinates in the ith column of the rectangular grid are in the shaded part. For instance, in the above example with the striped domain, all L_i are identical and are given by

$$L_i = L = \begin{pmatrix} I_3 & 0 \\ 0 & 0 \\ 0 & I_2 \end{pmatrix}$$

Note that $\hat{A} = \mathscr{L}^{\mathrm{T}} A_{\mathbf{N}} \mathscr{L}$, where $\mathscr{L} = \mathrm{diag}(L_0, L_1, \ldots, L_{N_0-1})$. As mentioned earlier, the preconditioner is given the same structure as $\hat{A}$ and is given by $\hat{P} = \mathscr{L}^{\mathrm{T}} P_{\mathbf{N}} \mathscr{L}$. For this example $\mathbf{N} = N_0 \times N_1 = 7 \times 6$, $A_{\mathbf{N}}, P_{\mathbf{N}} \in \mathbb{R}^{42 \times 42}$, and $\mathscr{L} \in \mathbb{R}^{42 \times 35}$ and therefore $\hat{A}$ and $\hat{P}$ in $\mathbb{R}^{35 \times 35}$.

Striped domain: The striped domain we mentioned above is fundamental and more complex domains can be analysed in terms of this domain. In this case, the coefficient matrix, $\hat{A} = \mathscr{L}^{\mathrm{T}} A_{\mathbf{N}} \mathscr{L}$ where $\mathscr{L} = I_{N_0} \otimes L$ and L is an $N_1 \times K_1$ extracting matrix given by

$$L = \begin{array}{c} \begin{pmatrix} I_{m_0} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & I_{m_1} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \end{array} \begin{array}{l} \}r_0 \\ \}r_1 \\ \}r_2 \\ \vdots \end{array}$$

where L has n_m block-columns (with $\sum_{i=0}^{n_m-1} m_i = K_1$) and n_r block-rows. In this matrix, the ith block-column has only one Identity matrix (I_{m_i}) (with all other entries in this block-column being 0); and every alternate block-row is a zero block. We impose another condition on the structure of L in the following way. Let $\varepsilon > 0$ and $\mathbf{M} = \mathbf{M}(\varepsilon)$ in $\mathbb{N}^2$ be such that $\sum_{k_0=-\infty}^{\infty} \sum_{k_1=M_1}^{\infty} |a_{\mathbf{k}}| \leqslant \varepsilon$, $\sum_{k_0=-\infty}^{\infty} \sum_{k_1=M_1}^{\infty} |\gamma_{\mathbf{k}}| \leqslant \varepsilon$, and $\sum_{\mathbf{k}=-\mathbf{N}}^{\mathbf{N}} |\gamma_{\mathbf{k}} - \xi_{\mathbf{k}}^{\mathbf{N}}| \leqslant \varepsilon$ for all $\mathbf{N} > \mathbf{M}$ (this is possible by Proposition 1.2). Then we impose the condition that $r_i > M_1$ for all $0 \leqslant i \leqslant n_r - 1$.

In this way, to each Toeplitz matrix, $A_{\mathbf{N}}$, and a given $\varepsilon > 0$, we can associate a class of striped domains which satisfy the above conditions. We represent the set of these domains by $\mathscr{G}_{\mathbf{N}}^{\mathrm{sd}}(\varepsilon)$. Also to each domain $G \in \mathscr{G}_{\mathbf{N}}^{\mathrm{sd}}(\varepsilon)$, there corresponds an LRE system with the coefficient matrix, $\hat{A}_G = \mathscr{L}^{\mathrm{T}} A_{\mathbf{N}} \mathscr{L}$, and its preconditioner $\hat{P}_G = \mathscr{L}^{\mathrm{T}} P_{\mathbf{N}} \mathscr{L}$. We represent the set of preconditioned LRE coefficient matrices $(\hat{P}_G \hat{A}_G)$ by $\mathscr{C}_{\mathbf{N}}^{\mathrm{sd}}(\varepsilon)$, i.e.

$$\mathscr{C}_{\mathbf{N}}^{\mathrm{sd}}(\varepsilon) = \{\hat{P}_G \hat{A}_G | G \in \mathscr{G}_{\mathbf{N}}^{\mathrm{sd}}(\varepsilon)\}$$

Lemma 2.1
Under Assumptions 1, for every $\varepsilon > 0$, there exists an $\mathbf{M}$ in $\mathbb{N}^2$ such that for each $\hat{P}\hat{A} \in \mathscr{C}_{\mathbf{N}}^{\mathrm{sd}}(\varepsilon)$ and $\mathbf{N} > \mathbf{M}$, there exists $\mathbf{v} \in \mathbb{N}^2$ and a matrix D whose rank is at most $v_0 N_1 + v_1 N_0$ and which satisfies $\|I - \hat{P}\hat{A} - D\| \leqslant \varepsilon$. The constant, $\mathbf{v}$ depends only on the geometrical parameters of the underlying domain and ε.

We first introduce some notation in order to present the proof of this lemma more efficiently. Certain submatrices of type-$\mathscr{L}$ matrices are important in our analysis, which we represent here by

$D_L^x(N, m, n)$ is a $m \times n$ matrix given by

$$D_L^x(N,m,n) = (I_m \ 0 \ \cdots \ \cdots \ \cdots \ 0) \begin{pmatrix} \cdots & 0 & x_N & \cdots & x_2 & x_1 \\ \cdots & 0 & \ddots & \ddots & \ddots & x_2 \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & \ddots & \ddots & x_N \\ & & & & 0 & 0 \\ & & & & \vdots & \vdots \end{pmatrix} \begin{pmatrix} \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ I_n \end{pmatrix}$$

$D_R^x(N, m, n)$ is a $m \times n$ matrix given by

$$D_R^x(N,m,n) = (0 \ \cdots \ \cdots \ \cdots \ 0 \ I_m) \begin{pmatrix} \vdots & \vdots & & & & \\ 0 & 0 & & & & \\ x_N & \ddots & \ddots & & & \\ \vdots & \ddots & \ddots & \ddots & & \\ x_2 & \ddots & \ddots & \ddots & 0 & \cdots \\ x_1 & x_2 & \cdots & x_N & 0 & \cdots \end{pmatrix} \begin{pmatrix} I_n \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{pmatrix}$$

Proof of Lemma 2.1
Let $\tilde{\varepsilon} > 0$ and $\mathbf{M}$ and $\mathbf{N}$ in $\mathbb{N}^2$ be such that $\sum_{k_0=-\infty}^{\infty} \sum_{k_1=M_1}^{\infty} |a_{\mathbf{k}}| \leqslant \tilde{\varepsilon}$, $\sum_{k_0=-\infty}^{\infty} \sum_{k_1=M_1}^{\infty} |\gamma_{\mathbf{k}}| \leqslant \tilde{\varepsilon}$ and $\sum_{\mathbf{k}=-\mathbf{N}}^{\mathbf{N}} |\gamma_{\mathbf{k}} - \xi_{\mathbf{k}}^{\mathbf{N}}| \leqslant \tilde{\varepsilon}$ (this is possible by Proposition 1.2). Also, let $G \in \mathscr{G}_{\mathbf{N}}^{\mathrm{sd}}(\varepsilon)$ which

specifies the structure of L (and therefore $\mathscr{L}$) as described earlier in this section. Let $\tilde{L}$ be such that $LL^{\mathrm{T}}+\tilde{L}\tilde{L}^{\mathrm{T}}=I$ (since L is an extracting matrix, this can be done). Similarly we define, $\tilde{\mathscr{L}} \triangleq I_{N_0}\otimes \tilde{L}$ which satisfies $\mathscr{L}\mathscr{L}^{\mathrm{T}}+\tilde{\mathscr{L}}\tilde{\mathscr{L}}^{\mathrm{T}}=I$. We split P into a sum of three two-level matrices D^{ξ}, F^{ξ} and E^{ξ} in the following way:

$$\begin{pmatrix} P_0 & P_1 & \cdots & P_{N_0-1} \\ P_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ P_{N_0-1} & \cdots & \cdots & P_0 \end{pmatrix} = \underbrace{\begin{pmatrix} D_0^{\xi} & D_1^{\xi} & \cdots & D_{N_0-1}^{\xi} \\ D_1^{\xi} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ D_{N_0-1}^{\xi} & \cdots & \cdots & D_0^{\xi} \end{pmatrix}}_{\triangleq D^{\xi}} + \underbrace{\begin{pmatrix} F_0^{\xi} & F_1^{\xi} & \cdots & F_{N_0-1}^{\xi} \\ F_1^{\xi} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ F_{N_0-1}^{\xi} & \cdots & \cdots & F_0^{\xi} \end{pmatrix}}_{\triangleq F^{\xi}}$$

$$+ \underbrace{\begin{pmatrix} E_0^{\xi} & E_1^{\xi} & \cdots & E_{N_0-1}^{\xi} \\ E_1^{\xi} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ E_{N_0-1}^{\xi} & \cdots & \cdots & E_0^{\xi} \end{pmatrix}}_{\triangleq E^{\xi}}$$

where D_j^{ξ} is a block diagonal matrix with its kth block being the $r_k \times r_k$ diagonal block of the P_j (concomitant with the structure of L):

$$F_j^{\xi} = \begin{pmatrix} 0 & R_{01}^{\xi j} & 0 & \cdots & \cdots & 0 \\ L_{10}^{\xi j} & 0 & R_{12}^{\xi j} & \ddots & \ddots & \vdots \\ 0 & \ddots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & R_{n_r-2,n_r-1}^{\xi j} \\ 0 & \cdots & \cdots & 0 & L_{n_r-1,n_r-2}^{\xi j} & 0 \end{pmatrix}$$

where ξ^j is the first column of the Toeplitz matrix, P_j, and, $R_{kl}^{\xi j}=D_R^{\xi j}(M_1,r_k,r_l)$ and $L_{kl}^{\xi j}=D_L^{\xi j}(M_1,r_k,r_l)$; and $E_j^{\xi}=P_j-D_j^{\xi}-F_j^{\xi}$ which is a Toeplitz matrix with a central band (leading diagonals) of zeros for all $0\leqslant j\leqslant N_0-1$. From Lemmas A.2 and A.3 and Proposition 1.2, the norm of the matrix, E^{ξ}, can be estimated by

$$\|E^{\xi}\|\leqslant 2\sum_{k_0=0}^{N_0-1}\sum_{k_1=M_1}^{N_1-1}|\xi_{\mathbf{k}}^{\mathbf{N}}|\leqslant 2\sum_{k_0=0}^{N_0-1}\sum_{k_1=M_1}^{N_1-1}|\xi_{\mathbf{k}}^{\mathbf{N}}-\gamma_{\mathbf{k}}|+2\sum_{k_0=0}^{N_0-1}\sum_{k_1=M_1}^{N_1-1}|\gamma_{\mathbf{k}}|\leqslant 4\tilde{\varepsilon}$$

Note that $\mathscr{L}^{\mathrm{T}}D^{\xi}\tilde{\mathscr{L}}=0$ and $\mathscr{L}^{\mathrm{T}}F^{\xi}\mathscr{L}=0$. Similarly, the matrix, A, can be split as $A=D^{a}+F^{a}+E^{a}$ with $\|E^{a}\|\leqslant 4\tilde{\varepsilon}$. Therefore,

$$\begin{aligned}
\mathscr{L}^{\mathrm{T}}P\tilde{\mathscr{L}}\tilde{\mathscr{L}}^{\mathrm{T}}A\mathscr{L} &= \mathscr{L}^{\mathrm{T}}(D^{\xi}+F^{\xi}+E^{\xi})\tilde{\mathscr{L}}\tilde{\mathscr{L}}^{\mathrm{T}}(D^{a}+F^{a}+E^{a})\mathscr{L} \\
&= \mathscr{L}^{\mathrm{T}}F^{\xi}\tilde{\mathscr{L}}\tilde{\mathscr{L}}^{\mathrm{T}}F^{a}\mathscr{L}+\underbrace{\mathscr{L}^{\mathrm{T}}E^{\xi}\tilde{\mathscr{L}}\tilde{\mathscr{L}}^{\mathrm{T}}(F^{a}+E^{a})\mathscr{L}+\mathscr{L}^{\mathrm{T}}F^{\xi}\tilde{\mathscr{L}}\tilde{\mathscr{L}}^{\mathrm{T}}E^{a}\mathscr{L}}_{\bar{E}} \\
&= \mathscr{L}^{\mathrm{T}}F^{\xi}\tilde{\mathscr{L}}\tilde{\mathscr{L}}^{\mathrm{T}}F^{a}\mathscr{L}+\underbrace{\mathscr{L}^{\mathrm{T}}F^{\xi}\mathscr{L}\mathscr{L}^{\mathrm{T}}F^{a}\mathscr{L}}_{=0}+\bar{E} \\
&= \mathscr{L}^{\mathrm{T}}\underbrace{F^{\xi}F^{a}}_{\bar{D}}\mathscr{L}+\bar{E}
\end{aligned}$$

where $\bar{D}$ is a matrix with rank at most $2n_rM_1N_0$ (this is true due to the structures of F^{ξ} and F^{a}); and from the above estimates on $\|E^{\xi}\|$ and $\|E^{a}\|$ and Proposition 1.1, we have $\|\bar{E}\|\leqslant 4(c+c_0)\tilde{\varepsilon}$. Therefore

$$\begin{aligned}
\hat{P}\hat{A} &= \mathscr{L}^{\mathrm{T}}P\mathscr{L}\mathscr{L}^{\mathrm{T}}A\mathscr{L}=\mathscr{L}^{\mathrm{T}}PA\mathscr{L}-\mathscr{L}^{\mathrm{T}}P\tilde{\mathscr{L}}\tilde{\mathscr{L}}^{\mathrm{T}}A\mathscr{L} \\
&= \mathscr{L}^{\mathrm{T}}I\mathscr{L}-\underbrace{\mathscr{L}^{\mathrm{T}}(\bar{D}+\tilde{D})\mathscr{L}}_{\triangleq D}-\underbrace{(\bar{E}+\mathscr{L}^{\mathrm{T}}\tilde{E}\mathscr{L})}_{\triangleq E}
\end{aligned}$$

where the product PA is written as $PA=I-(\tilde{D}+\tilde{E})$ using Proposition B.1 with $\tilde{D}$ having rank at most $\sum_{j=0}^{1}M_j(\mathbf{N}/N_j)$ and $\|\tilde{E}\|\leqslant\tilde{\varepsilon}$. This implies that D has rank at most $\overbrace{2M_0}^{\triangleq v_0}N_1+\overbrace{2(n_m+1)M_1}^{\triangleq v_1}N_0$. Also $\|E\|\leqslant(4(c+c_0)+1)\tilde{\varepsilon}\triangleq\varepsilon$. As $\tilde{\varepsilon}>0$ and $G\in\mathscr{G}^{\mathrm{sd}}_{\mathbf{N}}(\varepsilon)$ were chosen arbitrarily, we have proved the lemma. □

Other domains in two dimensions: In the case of striped domains, we imposed structure on only one dimension, that is, the width of the stripes could vary (along the vertical direction) but their lengths were identical (in the horizontal direction). This translated to the fact that all the extracting matrices, L_i were identical. In the following analysis, we retain the structure in the vertical direction but no longer require that all L_i are identical. This imposes conditions on the horizontal directions of the grid similar to ones we imposed on the vertical direction. We assume there are n_c block-columns of widths $c_0,c_1,\ldots,c_{n_c-1}$ (see Figure 2) such that:

1. Each block-column has identical columns in the grid, i.e. the extracting matrices L_i are identical for all i corresponding to the same block-column (we represent the block-column by a superscript; e.g. $L_i=L^j$ for all i which correspond to the jth block-column).
2. Every alternate block-column is not in the domain (see Figure 2).
3. If $\mathbf{M}$ in $\mathbb{N}^2$ is such that $\sum_{k_0=M_0}^{N_0}\sum_{k_1=-N_1}^{N_1}|a_{\mathbf{k}}|\leqslant\tilde{\varepsilon}$, $\sum_{k_0=M_0}^{N_0}\sum_{k_1=-N_1}^{N_1}|\gamma_{\mathbf{k}}|\leqslant\tilde{\varepsilon}$ and $\sum_{\mathbf{k}=-\mathbf{N}}^{\mathbf{N}}|\gamma_{\mathbf{k}}-\xi_{\mathbf{k}}^{\mathbf{N}}|\leqslant\tilde{\varepsilon}$ for some arbitrarily chosen $\tilde{\varepsilon}>0$ (this is possible by Proposition 1.2), then we impose the condition that $c_i>M_0$ for all $0\leqslant i\leqslant n_c-1$.

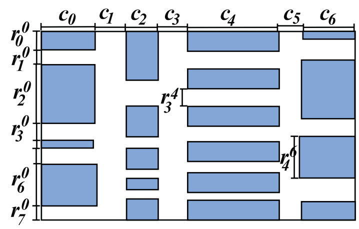


Figure 2. A possible underlying domain in the LRE system.

Thus the extracting matrices are given by

$$L^j = \begin{array}{cc} \begin{pmatrix} I_{m_0^j} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & I_{m_1^j} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} & \begin{array}{l} \}r_0^j \\ \}r_1^j \\ \}r_2^j \\ \vdots \end{array} \end{array}$$

where L^j has n_m^j block-columns and n_r^j block-rows. In this matrix, the kth block-column has only one Identity matrix ($I_{m_k^j}$) (with all other entries in this block-column being 0); and every alternate block-row is a zero block. As we assumed in the previous case, $r_k^j > M_1$ for all $0 \leqslant k \leqslant n_r^j$ and $0 \leqslant j \leqslant n_c - 1$. In the following analysis, we assume that $L^1, L^3, \ldots$ are not in the domain for the sake of simplicity. The analysis of the case in which $L^0, L^2, \ldots$ are not in the domain is identical to this case.

The preconditioner for the LRE system (defined by (4)) with the underlying domain specified by the above constraints is given by $\hat{P} = \mathscr{L}^{\mathrm{T}} P \mathscr{L}$ where

$$\mathscr{L} = \begin{pmatrix} \mathscr{L}_0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & \mathscr{L}_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

is a full rank matrix with $\mathscr{L}_j = I_{c_j} \otimes L^j$. This structure on $\mathscr{L}$ implies that $\hat{P} = \mathscr{L}^{\mathrm{T}} P \mathscr{L} = \mathscr{L}^{\mathrm{T}} \tilde{P} \mathscr{L}$ where

$$\tilde{P} = \begin{pmatrix} \mathscr{P}_{00} & 0 & \mathscr{P}_{02} & \cdots & \cdots & \mathscr{P}_{0n_c-1} \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \mathscr{P}_{20} & 0 & \mathscr{P}_{22} & \cdots & \cdots & \mathscr{P}_{2n_c-3} \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathscr{P}_{n_c-1} & 0 & \cdots & \cdots & \cdots & \mathscr{P}_{n_c-1n_c-1} \end{pmatrix}$$

where the blocks $\mathscr{P}_{kl}$, are $c_k N_1 \times c_l N_1$ subblocks of the preconditioner (having the same structure as $\hat{A}$ in (4)). In particular, for each $0 \leqslant j \leqslant n_c - 1$, $\mathscr{P}_{jj}$ is a $c_j N_1 \times c_j N_1$ two-level Toeplitz matrix which has P_0 in its main diagonal. This matrix can be written as a sum of two two-level Toeplitz matrices as

$$\tilde{P} = \underbrace{\begin{pmatrix} \mathscr{P}_{00} & & & & & \\ & 0 & & & & \\ & & \mathscr{P}_{22} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & \mathscr{P}_{n_c-1 n_c-1} \end{pmatrix}}_{\triangleq \bar{D}_P} + \bar{E}_P$$

where $\bar{E}_P = \tilde{P} - \bar{D}_P$ has a diagonal zero-band (with width greater than M_0) and therefore from Lemmas A.2 and A.3 and Proposition 1.2

$$\|\bar{E}_P\| \leqslant \sum_{k_0 = M_0}^{N_0 - 1} \sum_{k_1 = 0}^{N_1 - 1} |\xi_{\mathbf{k}}^{\mathbf{N}}| \leqslant \sum_{k_0 = M_0}^{N_0 - 1} \sum_{k_1 = 0}^{N_1 - 1} |\gamma_{\mathbf{k}}| + \sum_{\mathbf{k} = 0}^{\mathbf{N} - 1} |\xi_{\mathbf{k}}^{\mathbf{N}} - \gamma_{\mathbf{k}}| \leqslant 2\tilde{\varepsilon}$$

Therefore the preconditioner is given by $\hat{P} = D_P + E_P$ where $D_P = \mathscr{L}^{\mathrm{T}} \bar{D}_P \mathscr{L} = \mathrm{diag}(\mathscr{L}_0^{\mathrm{T}} \mathscr{P}_{00} \mathscr{L}_0\ \mathscr{L}_2^{\mathrm{T}} \mathscr{P}_{22} \mathscr{L}_2 \cdots)$ and $E_P = \mathscr{L}^{\mathrm{T}} \bar{E}_P \mathscr{L}$ and therefore $\|E_P\| \leqslant \|\bar{E}_P\| \leqslant 2\tilde{\varepsilon}$. Similarly, $\hat{A}$ is given by $\hat{A} = D_A + E_A$ where $\|E_A\| \leqslant 2\tilde{\varepsilon}$. Therefore the product, $\hat{P}\hat{A}$ is given by

$$\begin{pmatrix} \mathscr{L}_0^{\mathrm{T}} \mathscr{P}_{00} \mathscr{L}_0 \mathscr{L}_0^{\mathrm{T}} \mathscr{A}_{00} \mathscr{L}_0 & & & \\ & \mathscr{L}_2^{\mathrm{T}} \mathscr{P}_{22} \mathscr{L}_2 \mathscr{L}_2^{\mathrm{T}} \mathscr{A}_{22} \mathscr{L}_2 & & \\ & & \ddots & \\ & & & \mathscr{L}_{n_c-1}^{\mathrm{T}} \mathscr{P}_{n_c-1 n_c-1} \mathscr{L}_{n_c-1} \mathscr{L}_{n_c-1}^{\mathrm{T}} \mathscr{A}_0 \mathscr{L}_{n_c-1} \end{pmatrix}$$

$$+ D_p E_A + E_P \hat{A}$$

Each of the products, $\mathscr{L}_j^{\mathrm{T}} \mathscr{P}_{jj} \mathscr{L}_j \mathscr{L}_j^{\mathrm{T}} \mathscr{A}_{jj} \mathscr{L}_j$, depicts a sub-LRE system with the domain given by a striped domain. Therefore from Lemma 2.1, there exists a D_j such that rank of D_j is at most $v_0^j N_1 + v_1^j c_j$ and $\|I - \mathscr{L}_j^{\mathrm{T}} \mathscr{P}_{jj} \mathscr{L}_j \mathscr{L}_j^{\mathrm{T}} \mathscr{A}_{jj} \mathscr{L}_j - D_j\| \leqslant \tilde{\varepsilon}$. Therefore, if we define $D = \mathrm{diag}(D_0\ D_2 \cdots D_{n_c-1})$, then its rank is at most $\sum_{j=0,2,\ldots,n_c-1} v_0^j N_1 + v_1^j c_j$ and

$$\|I - \hat{P}\hat{A} - D\| \leqslant n_c \tilde{\varepsilon} + 2(c + 2c_0)\tilde{\varepsilon} \triangleq \varepsilon$$

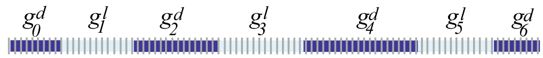


Figure 3. A possible underlying domain in one-level LRE system.

As $\tilde{\varepsilon} > 0$ was chosen arbitrarily, and if we denote the domains which satisfy the conditions described in this section by $\mathscr{G}_{\mathbf{N}}^{\text{od}}(\tilde{\varepsilon})$ and the corresponding set of preconditioned LRE coefficient matrices by $\mathscr{C}_{\mathbf{N}}(\tilde{\varepsilon})$, we have proved the following proposition.

Proposition 2.1
Under Assumptions 1, for every $\varepsilon > 0$ there exists an $\mathbf{M}$ in $\mathbb{N}^2$ such that for each $\hat{P}\hat{A} \in \mathscr{C}_{\mathbf{N}}(\varepsilon)$ and $\mathbf{N} > \mathbf{M}$, there exist $\{\mathbf{v}^j\}$ in $\mathbb{N}^2$ and a matrix D whose rank is at most $\sum_{j=0,2,\dots,n_c-1} v_0^j N_1 + v_1^j c_j$, and which satisfies $\|I - \hat{P}\hat{A} - D\| \leqslant \varepsilon$. The sequence $\{\mathbf{v}^j\}$ depends only on the geometric parameters of the underlying domain and ε.

2.2. Characterization of domains of higher level LRE systems

In the previous section, we studied two-level LRES with a large class of underlying domains. In higher dimensions, the underlying domains are more complex and very difficult to visualize. However, we can describe them as cross-product sets of one-dimensional domains.

Therefore, we develop the notation to describe the one-dimensional geometry which we use to describe geometries in higher dimensions. The underlying domain in a one level LRE system is described by alternating dark (g_i^d) and light (g_i^l) line segments (see Figure 3). Also, the number of grid points in each segment is assumed to be greater than a prespecified number, M, e.g. $g_i^d > M$ and $g_i^l > M$. We represent the number of segments (dark and light together) by $n(G)$, where G denotes the domain given by the union of dark segments; for instance $n(G) = 7$ in Figure 3. We represent the space of all such domains, G, satisfying the above constraints by $\mathscr{G}_1^M$. Here, 1 in the subscript refers to the dimension of the domain and the superscript, M, denotes the lower bound on the number of points in each segment, g_i^x ($x \in \{d, l\}$ and $0 \leqslant i < n(G)$). In this way, to every domain $G \in \mathscr{G}_1^M$, we associate $n(G)$ segments, g_i^x such that

$$G = \bigcup_{\substack{i < n(G),\\ x = d}} g_i^x$$

and the number of grid points in each segment (denoted by $n_i(G)$) is at least M.

We express the class of two-dimensional domains that we considered in the previous section as

$$\mathscr{G}_2^{\mathbf{M}} = \left\{ G \subset \mathbb{R}^2 \mid G = \bigcup_{\substack{j < n(H),\\ x = d}} \tilde{G}_j \times h_j^x,\ \tilde{G}_j \in \mathscr{G}_1^{M_0} \text{ and } H \in \mathscr{G}_1^{M_1} \right\}$$

where '$\times$' is the set cross product. Figure 4 depicts how the two-dimensional domain G (that we considered in previous section) is written in terms of one-dimensional domains.

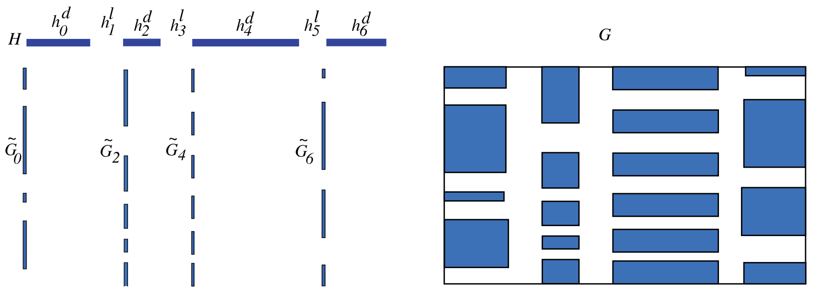


Figure 4. The two-dimensional domain, G on the right is given in terms of the one-dimensional domains $\tilde{G}_0,\ \tilde{G}_2,\ \tilde{G}_4,\ \tilde{G}_6$ and H. In fact $G=(\tilde{G}_0\times h_0^d)\cup(\tilde{G}_2\times h_2^d)\cup(\tilde{G}_4\times h_4^d)\cup(\tilde{G}_6\times h_6^d)$.

Similarly, we generalize the space of underlying domains to k dimensions and specify it by the following recursion relation:

$$\mathscr{G}_{k+1}^{\mathbf{M}}=\left\{G\subset\mathbb{R}^{k+1}\mid G=\bigcup_{\substack{j<n(H),\\ x=d}}\tilde{G}_j\times h_j^x,\ \tilde{G}_j\in\mathscr{G}_k^{\mathbf{M}/M_k}\text{ and }H\in\mathscr{G}_1^{M_k}\right\}$$

These constraints on the underlying domain of an LRE system translate to a specific structure on the corresponding extracting matrix, $\mathscr{L}$, that picks the LRE coefficient matrix from the Toeplitz matrix (i.e. $\hat{A}=\mathscr{L}^{\mathrm{T}}A\mathscr{L}$). Here we develop the notation which describes the structure of this extracting matrix.

For an underlying domain in $\mathscr{G}_p^{\mathbf{M}}$, the corresponding extracting matrix, $\mathscr{L}$ has n_c^0 block-columns and n_r^0 block-rows and is given by

$$\mathscr{L}\triangleq\mathscr{L}^0=\begin{array}{c}\begin{pmatrix}\mathscr{L}_0^0 & 0 & 0 & \cdots\\ 0 & 0 & 0 & \cdots\\ 0 & \mathscr{L}_2^0 & 0 & \cdots\\ \vdots & \vdots & \vdots & \vdots\end{pmatrix}\end{array}\begin{array}{l}\}c_0\\ \}c_1\\ \}c_2\\ \vdots\end{array}\quad\text{where }\mathscr{L}_{j_0}^0=I_{c_{j_0}}\otimes\mathscr{L}_{j_0}^1$$

for $j_0=0,2,\ldots,2(n_r^0-1)$; $L_{j_0}^1$ is itself an extracting matrix with $n_c^{j_0}$ block-columns and $n_r^{j_0}$ block-rows and is given by

$$\mathscr{L}_{j_0}^1=\begin{pmatrix}\mathscr{L}_{j_00}^1 & 0 & 0 & \cdots\\ 0 & 0 & 0 & \cdots\\ 0 & \mathscr{L}_{j_02}^1 & 0 & \cdots\\ \vdots & \vdots & \vdots & \vdots\end{pmatrix}\begin{array}{l}\}c_{j_00}\\ \}c_{j_01}\\ \}c_{j_02}\\ \vdots\end{array}\quad\text{where }\mathscr{L}_{j_0j_1}^1=I_{c_{j_0j_1}}\otimes\mathscr{L}_{j_0j_1}^2$$

for $j_1 = 0, 2, \ldots, 2(n_r^{j_0} - 1)$; and again $L^1_{j_0 j_1}$ is itself an extracting matrix with $n_c^{j_0 j_1}$ block-columns and $n_r^{j_0 j_1}$ block-rows and so on. These matrices satisfy the following recursion relation:

$$\mathscr{L}^{k-1}_{\mathbf{j}_{k-2}} = \mathscr{L}^{k-1}_{j_0 j_1 \cdots j_{k-2}} = \begin{array}{c} \left(\begin{array}{cccc} \mathscr{L}^{k-1}_{\mathbf{j}_{k-2}0} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & \mathscr{L}^{k-1}_{\mathbf{j}_{k-2}2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{array} \right) \end{array} \begin{array}{l} \}c_{\mathbf{j}_{k-2}0} \\ \}c_{\mathbf{j}_{k-2}1} \\ \}c_{\mathbf{j}_{k-2}2} \\ \quad\vdots \end{array} \quad \text{where } \mathscr{L}^{k-1}_{\mathbf{j}_{k-1}} = I_{c_{\mathbf{j}_{k-1}}} \otimes \mathscr{L}^{k}_{\mathbf{j}_{k-1}}$$

for all $1 \leqslant k \leqslant p-1$. (Note that $\mathscr{L}_{\mathbf{j}_{k-2}}$ denotes $\mathscr{L}_{j_0 \cdots j_{k-2}}$ and therefore $\mathscr{L}_{\mathbf{j}_{k-1}} = \mathscr{L}_{\mathbf{j}_{k-2} j_{k-1}}$. See the Notation section for details). Here, $\mathscr{L}^{p-1}_{\mathbf{j}_{p-2}}$ has the form given by

$$\mathscr{L}^{p-1}_{\mathbf{j}_{p-2}} = \left(\begin{array}{cccc} I & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & I & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{array} \right) \begin{array}{l} \}c_{\mathbf{j}_{p-2}0} \\ \}c_{\mathbf{j}_{p-2}1} \\ \}c_{\mathbf{j}_{p-2}2} \\ \quad\vdots \end{array}$$

where $\mathscr{L}^{p-1}_{\mathbf{j}_{p-2}}$ has $n_c^{\mathbf{j}_{p-2}}$ block-columns and $n_r^{\mathbf{j}_{p-2}}$ block-rows. In this matrix, the kth block-column has only one identity matrix (with all other entries in this block-column being 0); and every alternate block-row is a zero block. Note that the matrix, $\tilde{\mathscr{L}}^{p-1}_{\mathbf{j}_{p-2}}$, defined by

$$\mathscr{L}^{p-1}_{\mathbf{j}_{p-2}} \triangleq \left(\begin{array}{cccc} 0 & 0 & 0 & \cdots \\ I & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & I & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{array} \right) \begin{array}{l} \}c_{\mathbf{j}_{p-2}0} \\ \}c_{\mathbf{j}_{p-2}1} \\ \}c_{\mathbf{j}_{p-2}2} \\ \quad\vdots \end{array}$$

satisfies

$$\mathscr{L}^{p-1}_{\mathbf{j}_{p-2}} \mathscr{L}^{(p-1)\mathrm{T}}_{\mathbf{j}_{p-2}} + \tilde{\mathscr{L}}^{p-1}_{\mathbf{j}_{p-2}} \tilde{\mathscr{L}}^{(p-1)\mathrm{T}}_{\mathbf{j}_{p-2}} = I$$

This in turn implies that $\tilde{\mathscr{L}}^{p-2}_{\mathbf{j}_{p-2}} \triangleq I_{c_{\mathbf{j}_{p-2}}} \otimes \tilde{\mathscr{L}}^{p-1}_{\mathbf{j}_{p-2}}$ satisfies

$$\mathscr{L}^{p-2}_{\mathbf{j}_{p-2}} \mathscr{L}^{(p-2)\mathrm{T}}_{\mathbf{j}_{p-2}} + \tilde{\mathscr{L}}^{p-2}_{\mathbf{j}_{p-2}} \tilde{\mathscr{L}}^{(p-2)\mathrm{T}}_{\mathbf{j}_{p-2}} = I$$

In the same way, we can show that

$$\mathscr{L}^{k-1}_{\mathbf{j}_{k-2}} \mathscr{L}^{(k-1)\mathrm{T}}_{\mathbf{j}_{k-2}} + \tilde{\mathscr{L}}^{k-1}_{\mathbf{j}_{k-2}} \tilde{\mathscr{L}}^{(k-1)\mathrm{T}}_{\mathbf{j}_{k-2}} = I \quad \text{and} \quad \mathscr{L}^{k}_{\mathbf{j}_{k-1}} \mathscr{L}^{k\mathrm{T}}_{\mathbf{j}_{k-1}} + \tilde{\mathscr{L}}^{k}_{\mathbf{j}_{k-1}} \tilde{\mathscr{L}}^{k\mathrm{T}}_{\mathbf{j}_{k-1}} = I$$

where

$$\tilde{\mathscr{L}}^{k-1}_{\mathbf{j}_{k-2}} \triangleq \begin{array}{c} \begin{pmatrix} \tilde{\mathscr{L}}^{k-1}_{\mathbf{j}_{k-2}0} & 0 & 0 & 0 & \cdots \\ 0 & I & 0 & 0 & \cdots \\ 0 & 0 & \tilde{\mathscr{L}}^{k-1}_{\mathbf{j}_{k-2}2} & 0 & \cdots \\ 0 & 0 & 0 & I & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \end{array} \begin{array}{l} \}c_{\mathbf{j}_{k-2}0} \\ \}c_{\mathbf{j}_{k-2}1} \\ \}c_{\mathbf{j}_{k-2}2} \\ \vdots \end{array} \quad \text{and} \quad \tilde{\mathscr{L}}^{k-1}_{\mathbf{j}_{k-1}} \triangleq I_{c_{\mathbf{j}_{k-1}}} \otimes \tilde{\mathscr{L}}^{k}_{\mathbf{j}_{k-1}}$$

for all $1 \leqslant k \leqslant p-1$.

2.3. *p-level LRES*

In this section, we present the analysis of preconditioners for p-level LRES. As in Section 2.1, to every p-level Toeplitz matrix, $A_{\mathbf{N}}$ with its underlying rectangular domain G_A and $\varepsilon > 0$, we associate a set of domains $\mathscr{G}_{\mathbf{N}}(\varepsilon)$ in the following way:

$$\mathscr{G}_{\mathbf{N}}(\varepsilon) = \{G \subset G_A \mid G \in \mathscr{G}^{\mathbf{M}}_p\}$$

where $\mathbf{M}$ is such that $\sum_{k_j=N_j}^{\infty} \sum_{\mathbf{k}\backslash k_j=\mathbf{1}}^{\infty} |a_{\mathbf{k}}| \leqslant \varepsilon$, $\sum_{k_j=N_j}^{\infty} \sum_{\mathbf{k}\backslash k_j=\mathbf{1}}^{\infty} |\gamma_{\mathbf{k}}| \leqslant \varepsilon$ and $\sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{N}} |\gamma_{\mathbf{k}} - \xi^{\mathbf{N}}_{\mathbf{k}}| \leqslant \varepsilon$ for all $\mathbf{N} > \mathbf{M}$ (this can be done using Proposition 1.2). To each domain $G \in \mathscr{G}_{\mathbf{N}}(\varepsilon)$, we have an LRE system with coefficient matrix, $\hat{A}_G$ and its preconditioner given by $\hat{P}_G$. We denote the set of all preconditioned matrices $(\hat{P}_G \hat{A}_G)$ by $\mathscr{C}_{\mathbf{N}}(\varepsilon)$, i.e

$$\mathscr{C}_{\mathbf{N}}(\varepsilon) = \{\hat{P}_G \hat{A}_G | G \in \mathscr{G}_{\mathbf{N}}(\varepsilon)\}$$

In the following analysis, we assume an LRE system with its domain in $\mathscr{G}_{\mathbf{N}}(\tilde{\varepsilon})$ for some arbitrarily chosen $\tilde{\varepsilon}$ and $\mathbf{N} > M(\tilde{\varepsilon})$. Therefore the preconditioned coefficient matrix, $\hat{P}\hat{A} \in \mathscr{C}_{\mathbf{N}}(\tilde{\varepsilon})$.

The assumption on the geometry of the underlying domain of the p-level LRE system implies that the extracting matrix, $\mathscr{L}$ (in $\hat{A} = \mathscr{L}^{\mathrm{T}} A \mathscr{L}$) inherits the structure described in the Section 2.2. The analysis of p-level LRE matrices is a natural generalization of the concepts that were used in Section 2.1 to deal with two-level LRE matrices. However, the notation to describe the higher level structure is more complex than the two-level case. A generalization of the Proposition 2.1 to higher dimensions is given by the following proposition (which for the sake of brevity is presented without proof).

Proposition 2.2
Under Assumptions 1, for every $\varepsilon > 0$, there exists an $\mathbf{M}$ in $\mathbb{N}^p$ such that for each $\hat{P}\hat{A} \in \mathscr{C}_{\mathbf{N}}(\varepsilon)$ and $\mathbf{N} > \mathbf{M}$, there exists a $\mathbf{v}$ in $\mathbb{N}^p$ and a matrix D whose rank is at most $2\sum_{k=0}^{p-1} v_k(\mathbf{N}/N_k)$ and which satisfies $\|I - \hat{P}\hat{A} - D\| \leqslant \varepsilon$. The constant $\mathbf{v}$ depends only on the geometrical parameters of the underlying domain and ε.

In Proposition 2.2, we showed that $I - \hat{P}\hat{A}$ can be approximated by a matrix which has all but $2\sum_{k=0}^{p-1} v_k(\mathbf{N}/N_k)$ eigenvalues that are zero. In the following proposition, we use the positive definiteness of $\hat{A}$ and Theorem 2 in Reference [22] (see also the Theorem A.3 in Appendix A) to prove that the spectra of these two matrices are also close.

Proposition 2.3
Under Assumptions 1, for every $\varepsilon > 0$ there exists an $\mathbf{M}$ in $\mathbb{N}^p$ such that for each $\hat{P}\hat{A} \in \mathscr{C}_{\mathbf{N}}(\varepsilon)$ and $\mathbf{N} > \mathbf{M}$, there are at least $\mathbf{N} - 2\sum_{k=0}^{p-1} v_k(\mathbf{N}/N_k)$ eigenvalues λ_j of $\hat{P}\hat{A}$ such that $|\lambda_j - 1| \leqslant \varepsilon \log \mathbf{N}$. The constant $\mathbf{v}$ in $\mathbb{N}^p$ depends only on the geometrical parameters of the underlying domain and ε.

Proof
From Proposition 1.1-(2), we have that $C_{2\mathbf{N}} > 0$ for large enough $\mathbf{N}$. Now $\hat{A}$ is a principal sub-matrix of $C_{2\mathbf{N}}$ (i.e. $\hat{A} = \bar{\mathscr{L}}^{\mathrm{T}} C_N \bar{\mathscr{L}}$ for some $\bar{\mathscr{L}}$) and therefore it is also positive. Therefore $\hat{A}^{1/2}$ and $\hat{A}^{-1/2}$ are well defined and we can construct the symmetric matrix, $T \triangleq \hat{A}^{1/2}(\hat{P}\hat{A})\hat{A}^{-1/2} = \hat{A}^{1/2}\hat{P}\hat{A}^{1/2}$. Since $\hat{P}\hat{A}$ and T are similar, their spectra are the same. Also, since $C_{2\mathbf{N}}$ and $C_{2\mathbf{N}}^{-1}$ are uniformly bounded, the matrices $\hat{A}$ and $\hat{A}^{-1}$ are also uniformly bounded. This, in turn implies that $\hat{A}^{1/2}$ and $\hat{A}^{-1/2}$ are uniformly bounded.

Let $\tilde{\varepsilon} > 0$. From Proposition 2.2, there exists an $\mathbf{M} \in \mathbb{N}^p$ and matrix $\hat{D}$ whose rank is at most $2\sum_{k=0}^{p-1} v_k(\mathbf{N}/N_k)$ such that

$$\|\hat{P}\hat{A} - (I - \hat{D})\| \leqslant \tilde{\varepsilon} \quad \text{for all } \mathbf{N} > \mathbf{M}$$

If we define $\bar{E} \triangleq \hat{A}^{1/2}(I - \hat{D})\hat{A}^{-1/2}$, then $\|T - \bar{E}\| \leqslant \|\hat{A}^{1/2}\|\|\hat{A}^{-1/2}\|\tilde{\varepsilon}$.

In Theorem A.3, if we set $X = T$ and $E = \bar{E} - T$, then we have $\|C\| \leqslant \|E\|$ and $\|D\| \leqslant \|E\|$. Also, note that $\eta_j = 1$ and $\mu_j = 0$ for all but $2\sum_{k=0}^{p-1} v_k(\mathbf{N}/N_k)$ values of j. Therefore,

$$\begin{aligned} |\lambda_j - 1| &\leqslant \|E\|_2 + \|E\|_2(\log_2 \mathbf{N} + 0.038) \leqslant \left(1 + \sum_{k=0}^{p-1} \log_2(N_k) + 0.038\right) \|\hat{A}^{1/2}\|\|\hat{A}^{-(1/2)}\|\tilde{\varepsilon} \\ &\leqslant \underbrace{\tilde{c}\tilde{\varepsilon}}_{\triangleq\, \varepsilon} \log \mathbf{N} \end{aligned}$$

for these values of j and $\tilde{c}$ is a sufficiently large constant. Since, the spectrum of $\hat{P}\hat{A}$ and T are the same and $\varepsilon > 0$ can be chosen arbitrarily, we have proved the proposition. □

Remark
A bound on the error at the kth iteration of a conjugate gradient method (and therefore the proposed algorithm) for a system that has N_{out} eigenvalues outside the interval $(1-\varepsilon, 1+\varepsilon)$ is $\varepsilon^{2k} e_0$ where e_0 is the error at the first step [11, pp. 246–251]. This further implies that the number of iterations to achieve a desired accuracy of δ is $N_{\text{out}} + (\log \delta / \log \varepsilon)$. For e.g. a system with $\varepsilon = 10^{-8}$ and desired accuracy of 10^{-16} would require just two iterations in excess of N_{out}. We note from this proposition that there are at most $2\sum_{k=0}^{p-1} v_k\,(\mathbf{N}/N_k) = O(\mathbf{N}^{1-(1/p)})$ eigenvalues which are not clustered around 1 and since each iteration takes $O(\mathbf{N} \log \mathbf{N})$ operations, the PCG with the proposed preconditioner takes at most $O(\mathbf{N}^{1-(1/p)}\mathbf{N} \log \mathbf{N}) = O(\mathbf{N}^{2-(1/p)} \log \mathbf{N})$ operations. It can be shown that the clustering properties of the spectra of the preconditioned matrices (shown in the previous section) guarantee a substantial reduction in the number of iterations of the PCG. It should also be emphasized that unlike the one-level systems in Reference [9], here the number of iterations is not independent of the matrix size $A_{\mathbf{N}}$.

Remark

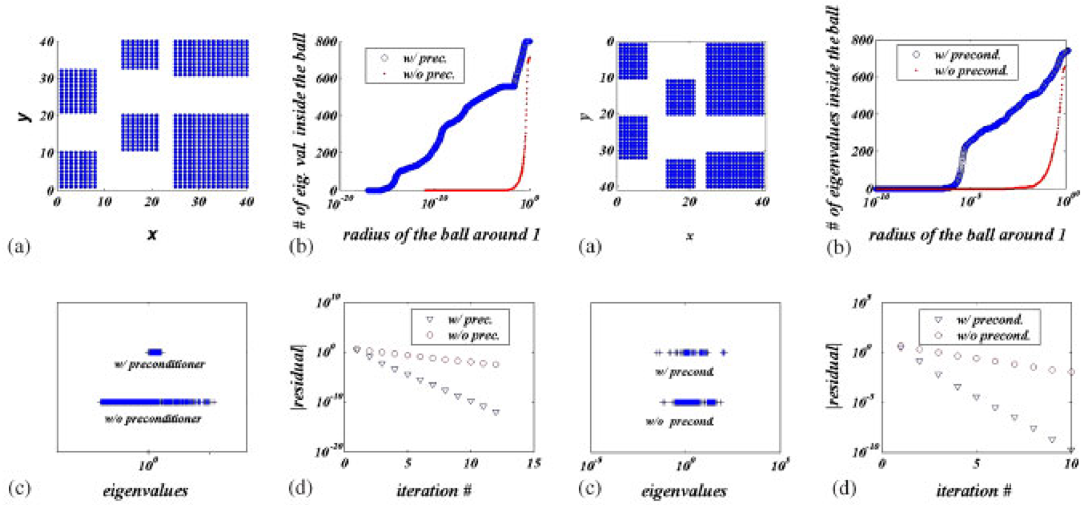
It should be observed that we assumed 'smoothness' condition on the generating function $\mathbf{f}$ $(\sum \mathbf{k}^{2+2\delta}|a_{\mathbf{k}}|^2)$ to prove the above clustering result. If we relax this condition, i.e. if we assume only absolute summability, then we can show that the sequence of eigenvalues $\{\lambda_j\}$ of $\hat{P}\hat{A}$ and the constant sequence $\{1\}$ are *equally distributed* (see Reference [23] for the definition). This result follows from Proposition 1.2 and Theorems 2.1 from Reference [24] and 3.1 from Reference [23]. However, we cannot guarantee that the sequences are *strongly equally distributed* and therefore cannot guarantee the clustering results proven here.

3. SIMULATION RESULTS

Typically p-level LRES, $\hat{A}x = b$ are large. Since the unknown vector, x represents a field on a p-level domain, its length is in the order of N^p where N is the number of points used to discretize each dimension. For instance, in a two level LRE system defined on a domain embedded in a rectangle represented by a 256×256 grid, has the length of x in $O(256^2)$, and therefore the corresponding coefficient matrix is in $O(256^2 \times 256^2)$. The application of the PCG with the proposed preconditioner has two advantages: first, it reduces the computational expense by exploiting the structure of $\hat{A}$; and, secondly it computes the solution without explicitly forming the large coefficient matrices.

In the following simulations, we present some two-level LRES of different sizes, and with different kernels and underlying domains. B and X represent the given and the unknown fields corresponding to the vectors b and x, respectively. In Figure 5(A), the kernel, A is generated by the two-sequence $\{(i^2 + j^2 + 1)^{-3/2}\}$. Here the $\hat{A}$ is a 800×800 matrix and the associated two-level Toeplitz matrix is 1600×1600. The underlying domain is shown in (a). In (b) and (c), we compare the clustering of the spectra of the preconditioned matrix ($\hat{P}\hat{A}$) and the non-preconditioned matrix $\hat{A}$. In (b), we plot the number of eigenvalues of the preconditioned matrix, $\hat{P}\hat{A}$, in a ball of radius r centred at 1 vs the radius, r. We observe that a majority ($\approx$ 600 out of 800) of the eigenvalues are clustered around 1 (within a radius of 10^{-4}). This clustering of the eigenvalues is exploited by the PCG and the rapid convergence of the PCG can be observed in (d). In fact, the solution within a tolerance of 10^{-14} is achieved in just 12 iterations while the error is much larger (10^{-3}) in 12 iterations when the proposed preconditioner is not used. These trends persist as the size of the LRES is increased (by making finer grids to describe the domain) and the corresponding results are shown in Table I. In Figure 6, the LRES with the same kernel as in Figure 5 but with different geometries have been presented. In plot (A), the underlying domain which is represented by 7128 points (the size of the corresponding coefficient matrix is 7128×7128) is shown in (a), the given (B) and the unknown (X) fields in (b) and (c), respectively, and the comparison of convergence rates of the preconditioned and non-preconditioned cases in (d). Here we see that the preconditioner reduces the number of iterations by a large amount for a given residual error. Similar results are observed in plot (B), in which the underlying domain is represented by 26 532 points.

In Figure 7(A), we simulate a image restoration example where the problem is to determine the original image from a blurred image. The two images are related through an integral equation where the kernel is called the *blurring* function. The domain of reconstruction, the blurring function and the distorted image are shown in (a)–(c) In (d), the reconstructed



(A) $\left\{(i^2+j^2+1)^{-\frac{3}{2}}\right\}$ $(\boldsymbol{N} = 40 \times 40,\ s = 800)$ (B)$\{(ij+1)^{-1}\}$ $(\boldsymbol{N} = 40 \times 40,\ s = 800)$

Figure 5. (a) The underlying domain of the LRE system; (b) plot of the number of eigenvalues (of $\hat{P}\hat{A}$) within a ball around 1 versus the radius of the ball; (c) clustering of eigenvalues of $\hat{P}\hat{A}$ compared to that of $\hat{A}$; and (d) comparison of the convergence rates of PCG between the preconditioned and non-preconditioned cases. s represents the size of the coefficient matrix.

image is presented. We see that there are only two peaks in the distorted image while the reconstruction of the original image has eight peaks. In (e), we compare the convergence rates of the PCG with and without the proposed preconditioner. Once again, we see that the preconditioner greatly enhances the performance of the algorithm.

In Figure 7(B), we use our preconditioner to solve an LRE system with a circular underlying domain which is not covered in our analysis because our analysis is restricted to rectangular subdomains described in Section 2.2. This example represents the crack opening displacement of a penny shaped crack subjected to a constant pressure. The simulations show that the performance is greatly improved by using the proposed preconditioner. The underlying geometry, the given and solution fields are shown in (a)–(c). In (d) we compare the diagonal preconditioner (it is formed by taking the reciprocals of the diagonal elements of $\hat{A}$) with ours and find that the error goes below 10^{-3} in about 30 iterations of the 'diagonal' algorithm while it takes seven steps with ours.

In Figure 5(B), we simulate an LRE system using the two-sequence $\{(ij+1)^{-1}\}$. Note that the corresponding series is not even convergent, but still the algorithm works satisfactorily. The fact that these algorithms continue to perform well even beyond the limitations of our analysis demonstrates their robustness. In Table I, we present the computation times in solving the LRES presented in this section. For each geometry, the computation times have been recorded for different grid sizes (and therefore different matrix sizes). The clustering, convergence, and robustness properties were found to persist in many other simulations (which we do not present here) that we did by changing the kernels, domain geometries and matrix sizes. In this paper, the analytical results concerning the clustering of the preconditioned matrices are

Table I. This table compares the CPU time and the no. of iterations to solve the LRES presented in Section 3 for different grid sizes **N** and coefficient matrices of size s.

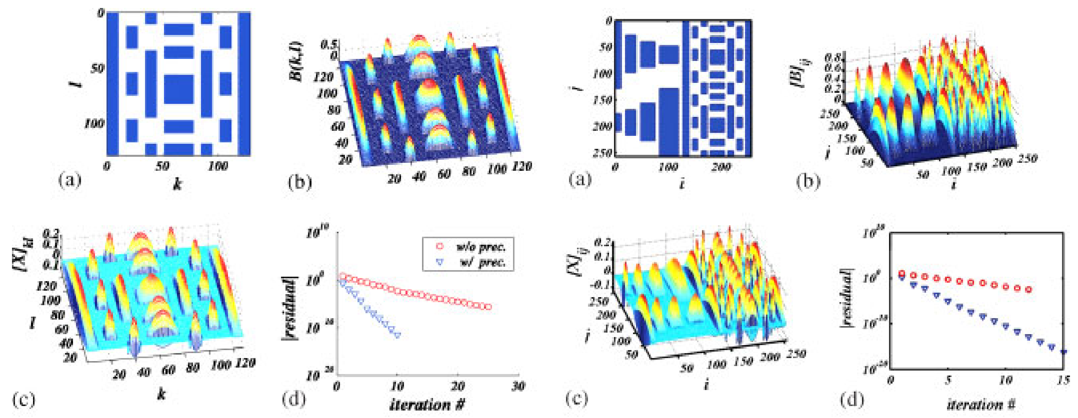
$a_{ij} = (i^2 + j^2 + 1)^{-3/2}$ geom. (Figure 5(A)) tol. $= 2.2 \times 10^{-16}$	$\mathbf{N} = 40 \times 40$, s = 800 w/prec.-w/o prec.	$\mathbf{N} = 80 \times 80$, s = 3200 w/prec.-w/o prec.	$\mathbf{N} = 160 \times 160$, s = 12 800 w/prec.-w/o prec.
No. of iterations	16 66	16 75	15 80
CPU time	2.9 s 8.0 s	19.8 s 62.3 s	211.3 s 753.2 s
$a_{ij} = (i^2 + j^2 + 1)^{-3/2}$ geom. (Figure 6(A)) tol. $= 2.2 \times 10^{-16}$	$\mathbf{N} = 64 \times 64$, s = 1782 w/prec.-w/o prec.	$\mathbf{N} = 128 \times 128$, s = 7128 w/prec.-w/o prec.	$\mathbf{N} = 256 \times 256$, s = 28512 w/prec.-w/o prec.
No. of iterations	16 68	16 75	16 80
CPU time	9 s 25.6 s	100.5 s 303.3 s	25.4 min 119.4 min
$a_{ij} = (i^2 + j^2 + 1)^{-3/2}$ geom. (Figure 6(B)) tol. $= 2.2 \times 10^{-16}$	$\mathbf{N} = 64 \times 64$, s = 1656 w/prec.-w/o prec.	$\mathbf{N} = 128 \times 128$, s = 6622 w/prec.-w/o prec.	$\mathbf{N} = 256 \times 256$, s = 26488 w/prec.-w/o prec.
No. of iterations	15 70	16 73	15 76
CPU time	10.1 s 31.3 s	95.7 s 288.9 s	24.12 min 81.8 min
$a_{ij} = e^{-(i^2+j^2)/10^4)}$ geom. (Figure 7(A)) tol. $= 10^{-6}$	$\mathbf{N} = 32 \times 32$, s = 896 w/prec.-w/o prec.	$\mathbf{N} = 64 \times 64$, s = 3456 w/prec.-w/o prec.	$\mathbf{N} = 128 \times 128$, s = 13 824 w/prec.-w/o prec.
No. of iterations	25 890	17 3024	8 2105
CPU time	4.1 s 105.5 s	14.6 s 30.3 min	86.8 s 4.1 h
$a_{ij} = (ij + 1)^{-1}$ geom. (Figure 5(B)) tol. $= 2.2 \times 10^{-16}$	$\mathbf{N} = 40 \times 40$, s = 800 w/prec.-w/o prec.	$\mathbf{N} = 80 \times 80$, s = 3200 w/prec.-w/o prec.	$\mathbf{N} = 160 \times 160$, s = 12 800 w/prec.-w/o prec.
No. of iterations	45 436	44 643	43 876
CPU time	9.7 s 62.3 s	54.3 s 8.7 min	10.0 min 2.3 h

The simulations were done in MATLAB on a 512 Mb RAM/1 GHz PC.

presented for a specific class of generating functions; however, the simulations show that the algorithm converges quickly for various generating functions that are not limited to the assumptions made in our analysis.

4. CONCLUSIONS

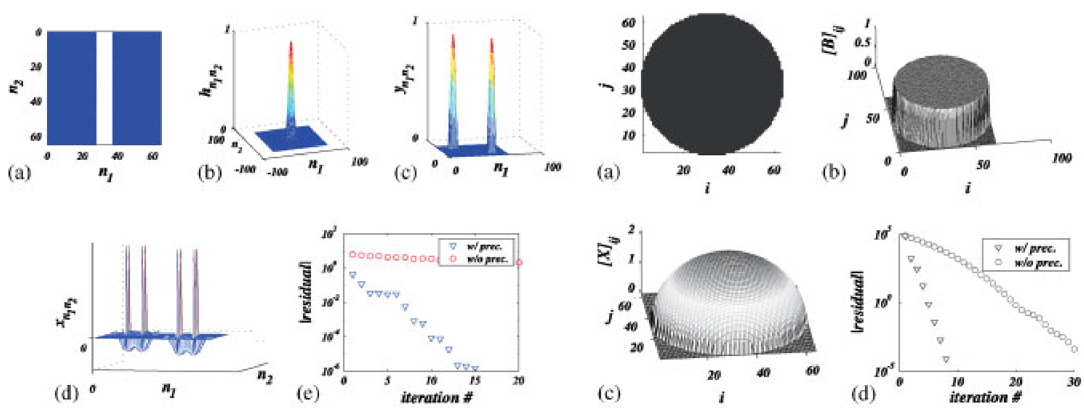
In this paper we have introduced and analysed preconditioners ($\hat{P}$) in PCG for the efficient solution of p-level LRES, $\hat{A}x = b$. The elements of the preconditioners are shown to approximate the Fourier coefficients of the reciprocal of the generating function associated with the p-level LRE system. Under fairly mild assumptions on the generating function, $f(\boldsymbol{\theta})$, or equivalently, on the generating sequence $\{a_{\mathbf{N}}\}$, these properties are exploited in order to prove clustering of the eigenvalues of the matrices $\hat{P}\hat{A}$. Also, these systems are shown to be subsystems of p-level Toeplitz systems, $A_{\mathbf{N}}x = b$. For p-level LRES, the PCG converges



(A) $\left\{(i^2+j^2+1)^{-\frac{3}{2}}\right\}$ $(\boldsymbol{N} = 128 \times 128,\ p = 7128)$

(B) $\left\{(i^2+j^2+1)^{-\frac{3}{2}}\right\}$ $(\boldsymbol{N} = 256 \times 256,\ s = 26532)$

Figure 6. (a) The underlying domain of the LRE system; (b) the given field in the integral equation; (c) the solution field; and (d) comparison of the convergence rates of PCG between the preconditioned (triangles) and non-preconditioned (circles) cases.



(A) the image restoration problem

(B) the penny-crack problem

Figure 7. (A)(a) the underlying domain; (b) the blurring function ($h_{n_1n_2} = \mathrm{e}^{-(n_1^2+n_2^2)/10^4}$); (c) the distorted two-sequence; (d) the reconstructed two-sequence; and (e) comparison of the convergence rates of PCG between the preconditioned and non-preconditioned cases. (B) (a) the underlying domain of the LRE system; (b) the given field in the integral equation; (c) the solution field; and (d) comparison of the convergence rates of PCG between the preconditioned (triangles) and diagonal preconditioner (circles) cases. s represents the size of the coefficient matrix.

to a specified tolerance in $O(\mathbf{N}^{2-1/p} \log \mathbf{N})$ operations where $\mathbf{N}$ is the size of $A_{\mathbf{N}}$. To study the preconditioner, $\hat{P}$, many simulations of two-level LRES with different kernels, sizes, and domains have been presented. Simulation results corroborate the theoretical findings regarding clustering of the spectra of the preconditioned matrices and the associated convergence rates.

In particular, the majority of the eigenvalues of $\hat{P}\hat{A}$ fall in the vicinity of 1. In addition, the simulations demonstrate that the algorithm is robust in that it still yields significant clustering even for two-level LRE matrices derived from sequences or domains which did not satisfy the restrictions imposed by the hypotheses of the propositions presented. This indicates that theoretical results established in this paper might be proved under more relaxed conditions.

APPENDIX A

A.1. *Proof of* $\sum \mathbf{k}^{2+2\delta} a_{\mathbf{k}}^2 < \infty \Rightarrow \sum \mathbf{k}^{2+2\sigma} \gamma_{\mathbf{k}}^2 < \infty$ *for some* $\sigma > 0$

Lemma A.1
If $f(\boldsymbol{\theta}) = \sum a_{\mathbf{k}} \mathrm{e}^{i\mathbf{k}\boldsymbol{\theta}}$ is real and such that $0 < \eta \leqslant f(\boldsymbol{\theta}) \leqslant c$ for all $\boldsymbol{\theta} \in \mathbb{T}^p$ and $\sum \mathbf{k}^{2+2\delta} a_{\mathbf{k}}^2 < \infty$ (with $\delta > \max\{0, \frac{p}{4} - \frac{1}{2}\}$) then $g \triangleq \frac{1}{f} = \sum \gamma_{\mathbf{k}} \mathrm{e}^{i\mathbf{k}\boldsymbol{\theta}}$ is such that $\sum \mathbf{k}^{2+2\sigma} \gamma_{\mathbf{k}}^2 < \infty$ for some $\sigma > 0$.

To prove this, we first state two lemmas from Reference [25, Lemmas A1 and A4, pp. 301–304]:

Theorem A.1
Let $F \in \mathscr{C}^1(\mathbb{R}; \mathbb{R})$ and $G \in \mathscr{C}^1(\mathbb{R}; \mathbb{R}^+)$ such that

$$|F'(\theta x) + (1-\theta)y| \leqslant \mu(\theta)(G(x) + G(y)), \quad x, y \in \mathbb{R}, \quad 0 \leqslant \theta \leqslant 1$$

where $\mu(\cdot) \in L^1(0,1)$. Then we have for $0 < s < 1$:

$$\|D^s F(\phi)\|_r \leqslant c_1(\|G(\phi)\|_{\bar{q}} \|D^s \phi\|_p), \quad 1/r = 1/p + 1/\bar{q} \ (D = (-\Delta)^{1/2})$$

$p, r \in (1, \infty)$, $\bar{q} \in (1, \infty]$, where c depends on $\mu, s, p, \bar{q}, r$. (ϕ denotes a generic function on $\mathbb{R}^m$; in this paper we apply this lemma only to those ϕ for which the existence of $D^s\phi$ is obvious.)

Theorem A.2
Let $s \geqslant 0$. Then

$$\|D^s(\phi_1\phi_2)\|_r \leqslant c_2(\|D^s\phi_1\|_{p_1} \|\phi_2\|_{q_1} + \|\phi_1\|_{q_2} \|D^s\phi_2\|_{p_2})$$

where

$$\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}, \quad p_1, p_2 \in (1, \infty) \quad \text{and} \quad q_1, q_2 \in (1, \infty]$$

Now we prove Lemma A.1 using these lemmas. Let

$$H(x) = \begin{cases} \dfrac{1}{x}, & \eta \leqslant x \leqslant c \\ \dfrac{1}{2c}, & x \geqslant 2c \\ \dfrac{2}{\eta}, & x \leqslant \dfrac{\eta}{2} \end{cases}$$

and H be defined on the intervals $(\eta/2, \eta)$ and $(c, 2c)$ such that $H''(x)$ is monotonic in $(\eta/2, \eta)$ and $(c, 2c)$. Let $F(x) \triangleq H'(x)$, $\mu(\theta) \triangleq 4c^3(\max_{x \in (\eta/2, \eta) \cup (c, 2c)} |H'(x)| + 1)$ and $G(x) \triangleq 2/x^3$ on the interval (η, c) and smoothly extended over the whole real line to satisfy the conditions of Lemma A.1. Then we have $\mu \in L^1(0,1)$ and $|F'(\theta x + (1-\theta)y)| \leqslant \mu(G(x)+G(y))$, $x, y \in \mathbb{R}$ since

$$\left|\frac{2}{(\theta x + (1-\theta)y)^3}\right| \leqslant \frac{2}{x^3} + \frac{2}{y^3}$$

is true for all $x, y \in (\eta, c)$.

Let σ be such that $2\delta - \sigma > (p/2) - 1$. Now $\|D^{1+\sigma}H(f)\|_2 = \|D^{\sigma}(H'(f)f')\|_2 = \|D^{\sigma}(F(f)f')\|_2$ and using Theorem A.2 we have

$$\|D^{1+\sigma}H(f)\|_2 \leqslant c_2(\|D^{\sigma}F(f)\|_{p_1}\|f'\|_{q_1} + \|F(f)\|_{\infty}\|D^{\delta}f'\|_2) \tag{A1}$$

where $(1/p_1) + (1/q_1) = \frac{1}{2}$. The term, $\|D^{\sigma}F(f)\|_{p_1}$, on the right-hand side can be estimated from Theorem A.1 as $\|D^{\sigma}F(f)\|_{p_1} \leqslant c_1\|G(f)\|_{\infty}\|D^{\sigma}f\|_{p_1}$ and substituting this in (A1) we have

$$\|D^{1+\sigma}H(f)\|_2 \leqslant c_2(c_1\|G(f)\|_{\infty}\|D^{\sigma}f\|_{p_1}\|f'\|_{q_1} + \|F(f)\|_{\infty}\|D^{1+\delta}f\|_2) \tag{A2}$$

From theory of Sobolev spaces we have $\|D^{\mu_1}\phi\|_{r_1} \leqslant c_3\|D^{\mu_2}\phi\|_{r_2}$ for some $c_3 > 0$ if $(1/r_1) - (\mu_1/p) \geqslant (1/r_2) - (\mu_2/p)$ where domain of ϕ is $\mathbb{T}^p$. Since $2\delta - \sigma > (p/2) - 1$, we can choose p_1 and q_1 such that

$$\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{2}, \quad \frac{1}{p_1} - \frac{\sigma}{p} \geqslant \frac{1}{2} - \frac{1+\delta}{p} \quad \text{and} \quad \frac{1}{q_1} - \frac{1}{p} \geqslant \frac{1}{2} - \frac{1+\delta}{p}$$

which implies (A2) can be rewritten as

$$\begin{aligned}\|D^{1+\sigma}H(f)\|_2 &\leqslant c_2(c_1c_3^2\|G(f)\|_{\infty}\|D^{1+\delta}f\|_2\|D^{1+\delta}f\|_2 + \|F(f)\|_{\infty}\|D^{1+\delta}f\|_2)\\ &\leqslant c_2(c_1c_3^2\|G(f)\|_{\infty}\|D^{1+\delta}f\|_2^2 + \|F(f)\|_{\infty}\|D^{1+\delta}f\|_2)\end{aligned} \tag{A3}$$

Therefore $\|D^{1+\delta}f\|_2 < \infty \Rightarrow \|D^{1+\sigma}H(f)\|_2 < \infty$. Therefore $\|D^{1+\delta}f\|_2 < \infty \Rightarrow \|D^{1+\sigma}g\|_2 < \infty$ which is equivalent to $\sum \mathbf{k}^{2+2\delta}a_{\mathbf{k}}^2 < \infty \Rightarrow \sum \mathbf{k}^{2+2\sigma}\gamma_{\mathbf{k}}^2 < \infty$. □

A.2. Lemmas on norms of matrices

Lemma A.2

Let T be a finite or infinite dimensional matrix and $T_k, 1 \leqslant k \leqslant 4$ be its subblocks such that

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$$

then

1. $\|T_k\| \leqslant \|T\|$, $1 \leqslant k \leqslant 4$.
2. $\|T\| \leqslant \|T_1\| + \|T_2\| + \|T_3\| + \|T_4\|$.

The proof follows directly from the definitions.

Lemma A.3
If $\{\alpha_k\}$ with $\alpha_k = \alpha_{-k}$ is a sequence of real numbers such that $\sum_{k=-\infty}^{\infty} |\alpha_k| < \infty$, and H and T are, respectively, infinite dimensional Hankel and Toeplitz matrices given by

$$H = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots \\ \alpha_2 & \iddots & \iddots \\ \vdots & \iddots & \iddots \end{pmatrix}, \quad T = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots \\ \alpha_2 & \alpha_1 & \cdots \\ \vdots & \ddots & \ddots \end{pmatrix}$$

then,

1. $\|H\| \leqslant \sum_1^{\infty} |\alpha_k|$.
2. $\|T\| \leqslant \sum_{-\infty}^{\infty} |\alpha_k|$.
3. If H_N and T_N are, respectively, $N \times N$ symmetric Hankel ($[H_N]_{ij} = \alpha_{i+j}$) and Toeplitz ($[T_N]_{ij} = \alpha_{|i-j|}$) matrices, then $\|H_N\| \leqslant 2\sum_1^N |\alpha_k|$ and $\|T_N\| \leqslant \sum_{-N}^{N} |\alpha_k|$.

Lemma A.4
Let T be a $p \times q$ block matrix given by

$$T = \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & \cdots & \cdots & T_{2n} \\ \vdots & \cdots & \cdots & \vdots \\ T_{m1} & \cdots & \cdots & T_{mn} \end{pmatrix}, \quad \text{then } \|T\| \leqslant \left\| \begin{pmatrix} \|T_{11}\| & \|T_{12}\| & \cdots & \|T_{1n}\| \\ \|T_{21}\| & \cdots & \cdots & \|T_{2n}\| \\ \vdots & \cdots & \cdots & \vdots \\ \|T_{m1}\| & \cdots & \cdots & \|T_{mn}\| \end{pmatrix} \right\| \triangleq \|\tilde{T}\|$$

Note that by using Lemma A.3 and repeatedly applying this lemma, we can show that the norm of a p-block Toeplitz matrix T with the generating sequence, $\{t_{\mathbf{n}}\}$, is given by $\sum_{\mathbf{n}=-\infty}^{\infty} |t_{\mathbf{n}}|$.

A.3. Theorem 2 in Reference [22]

Theorem A.3
Let X be a Hermitian matrix in $\mathbf{c}^{n \times n}$ with eigenvalues arranged in ascending order $\alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_n$. Let E be an arbitrary $n \times n$ matrix. Let $\{\eta_j + i\mu_j\}$, $1 \leqslant j \leqslant n$ be the eigenvalues of $X + E$ such that $\eta_1 \leqslant \eta_2 \leqslant \cdots \leqslant \eta_n$. Let $E = C + iD$ where $C = (E + E^*)/2$ and $iD = (E - E^*)/2$. Then

1. $|\eta_j - \alpha_j| \leqslant \|C\|_2 + \|D\|_2(\log_2 n + 0.038), |\mu_j| \leqslant \|D\|$.
2. $\sum \mu_j^2 \leqslant \|D\|_F^2$ and $\sqrt{\sum(\eta_j - \alpha_j)^2} \leqslant \|C\|_F + \sqrt{\|D\|_F^2 - \sum \mu_j^2}$.

APPENDIX B: p-LEVEL TOEPLITZ SYSTEMS

Here, we show clustering results (with our preconditioner) for a special class of LRE matrices, the case of Toeplitz matrices. The one-level Toeplitz systems has been analysed and clustering results have also been shown in Reference [17]. Similar preconditioners have been proposed to solve band Toeplitz matrices [15] and BTTB systems (two-block Toeplitz matrices) [14]. We still present this case, because our analysis of these systems is different from that given in Reference [17], and the results used in this section are used in dealing with LRES. It should be noted that in this case, the sets $\mathscr{A}_{\mathbf{N}}$ are singletons and the coefficient matrix is the Toeplitz matrix itself ($\hat{A}=A$), and therefore the corresponding preconditioner, $\hat{P}=P$.

B.1. The structural properties of PA

In this section, we study some properties of the matrix product, PA, which will later be used to establish that its spectrum clusters around 1. First, we introduce some notation to describe this structure.

As mentioned earlier, we exploit the fact that the p-block Toeplitz matrices can be 'embedded' in p-block circulant matrices (the structure of p-block matrix has been described in Section 1 and notation). A block in the $(k-1)$th level of a p-block circulant matrix has the structure,

$$\left(\begin{array}{ccc|ccc} T_0 & \cdots & T_{N_k-1} & T_{N_k} & \cdots & T_1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ T_{N_k-1} & \ddots & T_0 & \ddots & \ddots & T_{N_k} \\ \hline T_{N_k} & \ddots & \ddots & T_0 & \ddots & T_{N_k-1} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ T_1 & \cdots & \cdots & T_{N_k-1} & \cdots & T_0 \end{array}\right)$$

This matrix can be partitioned into four submatrices as shown by the solid lines. Each of these submatrices is a Toeplitz matrix. The two diagonal submatrices are identical and in these submatrices, the index of the subblocks (T_j) increases as one moves away from the main diagonal. We call Toeplitz matrices having this structure to be of *type-*$\mathscr{R}$. On the other hand, in the counter-diagonal submatrices the index of the subblocks (T_j) decreases as one moves away from the main diagonal. We call Toeplitz matrices having this structure to be of *type-*$\mathscr{L}$; The matrices having the same structure as the second or the fourth quadrant of a symmetric circulant matrix are of type-$\mathscr{R}$ and those having the either the structure of the first or the third quadrants of a symmetric circulant matrix is of type-$\mathscr{L}$.

Let C^{ξ} and C^{a} be p-block circulant matrices constructed from the sequences $\{\xi_{\mathbf{k}}\}_{\mathbf{k}=-(\mathbf{N}-\mathbf{1})}^{\mathbf{N}}$ and $\{a_{\mathbf{k}}\}_{\mathbf{k}=-(\mathbf{N}-\mathbf{1})}^{\mathbf{N}}$ respectively. We construct the matrix product PA from these circulant matrices which is shown in the following steps:

1. We choose the extracting matrices, $L_1{}^{\mathrm{T}}=[0\ \ I]$ and $\tilde{L}_1{}^{\mathrm{T}}=[I\ \ 0]$, where 0 and I are $(\mathbf{2N}/2)\times(\mathbf{2N}/2)$ matrices. Note that $L_1^{\mathrm{T}}C^{\xi}L_1$ is of type-$\mathscr{R}$ and $L_1^{\mathrm{T}}C^{\xi}\tilde{L}_1$ is of type-$\mathscr{L}$. Also, $C^{\xi}C^{a}=I$ (from Proposition 1.1-(3)) implies that $L_1^{\mathrm{T}}C^{\xi}C^{a}L_1=I$. Since $L_1L_1^{\mathrm{T}}+\tilde{L}_1\tilde{L}_1{}^{\mathrm{T}}=I$,

we have

$$\underbrace{L_1^{\mathrm{T}} C^{\xi} \tilde{L}_1 \tilde{L}_1{}^{\mathrm{T}} C^{a} L_1}_{\triangleq E^1} + \underbrace{L_1^{\mathrm{T}} C^{\xi} L_1}_{\triangleq P^1} \underbrace{L_1^{\mathrm{T}} C^{a} L_1}_{\triangleq A^1} = I \tag{B1}$$

2. In the same manner for $2 \leqslant k \leqslant p$, if we choose $(\mathbf{2N}/2^k) \times (\mathbf{2N}/2^{k-1})$ extracting matrices,

$$L_k^{\mathrm{T}} = \begin{pmatrix} (0 \;\; I) & 0 & 0 & 0 & 0 & & \\ 0 & 0 & (0 \;\; I) & 0 & 0 & \cdots & \\ 0 & 0 & 0 & 0 & (0 \;\; I) & & \\ & & \vdots & & & & \end{pmatrix} \quad \text{and} \quad \tilde{L}_k^{\mathrm{T}} = \begin{pmatrix} (I \;\; 0) & 0 & 0 & 0 & 0 & & \\ 0 & 0 & (I \;\; 0) & 0 & 0 & \cdots & \\ 0 & 0 & 0 & 0 & (I \;\; 0) & & \\ & & \vdots & & & & \end{pmatrix}$$

and starting from Equation (B1) we have the following recursion relation:

$$E^k + P^k A^k = I$$

where

$$E^k = L_k^{\mathrm{T}} E^{k-1} L_k + L_k^{\mathrm{T}} P^{k-1} \tilde{L}_k \tilde{L}_k^{\mathrm{T}} A^{k-1} L_k$$

$$P^k = L_k^{\mathrm{T}} P^{k-1} L_k \quad \text{and} \quad A^k = L_k^{\mathrm{T}} A^{k-1} L_k$$

for $2 \leqslant k \leqslant p$. Note that from our construction of circulant matrices in relation to the corresponding Toeplitz matrices, the matrix $P^p = P$ and $A^p = A$ and therefore the product $P^p A^p = PA$. Therefore, we have that

$$PA = I - E^p \tag{B2}$$

where E^p is an $\mathbf{N} \times \mathbf{N}$ p-block matrix and using the above recursion relationships, we have

$$E^p = \sum_{j=1}^{p} L_p^{\mathrm{T}} \cdots \underbrace{L_j^{\mathrm{T}} \cdots L_1^{\mathrm{T}} C^{\xi} L_1 \cdots L_{j-1} \tilde{L}_j}_{\triangleq \Gamma_j^{\xi}} \underbrace{\tilde{L}_j^{\mathrm{T}} L_{j-1}^{\mathrm{T}} \cdots L_1^{\mathrm{T}} C^{a} L_1 \cdots L_j}_{\triangleq \Gamma_j^{a}} \cdots L_p \tag{B3}$$

where Γ_j^{ξ} and Γ_j^{a} are p-block matrices whose $(j-1)$th level blocks are of type-$\mathscr{L}$ and all the blocks of other levels are of type-$\mathscr{R}$.

It is easier to analyse the p-block matrices which have zeroth level of type-$\mathscr{L}$ than the matrices that have zeroth level of type-$\mathscr{R}$. Now, we show that there exist permutation matrices, R_j, $1 \leqslant j \leqslant p$ such that $R_j^{\mathrm{T}} \Gamma_j^{a} R_j$ and $R_j^{\mathrm{T}} \Gamma_j^{\xi} R_j$ are matrices whose zero-level blocks are of type-$\mathscr{L}$.

We first consider an $N \times N$ p-block matrix, T whose (k)th-level blocks are of type-$\mathscr{L}$ while all the other level blocks are of type-$\mathscr{R}$. Note that this matrix has the same structure as Γ_k^{a}. We are analysing T just for the sake of convenience in notation. Let V be a $(k-1)$-level block. Then all the entries in V have the form $t_{j_0 j_1 \cdots j_{k-2} xx \cdots x}$ where the fixed sequence $j_0, j_1, \ldots, j_{k-2}$ specifies the fixed block, V. In the same way any entry in the kth-level subblock of V is of the form $t_{j_0j_1\cdots j_{k-1}xx\cdots x}$. Now, we show how a permutation of columns of V gives a matrix, $\tilde{V}$, whose kth-level subblocks have entries that are of the form $t_{j_0j_1\cdots j_{k-2}xj_kx\cdots x}$. If we define the permutation matrix, $\tilde{R}$ by

$$\tilde{R} = \left(I_{N_{k-1}} \otimes e_0^{k+1} \;\; I_{N_{k-1}} \otimes e_1^{k+1} \;\; \cdots \;\; I_{N_{k-1}} \otimes e_{N_k-1}^{k+1}\right)$$

where e_j^{k+1}, $0 \leqslant j \leqslant N_k - 1$ are defined in the Notation section; then $\tilde{V} = \tilde{R}^{\mathrm{T}} V \tilde{R}$ is a matrix whose kth-level subblocks have entries of the form $t_{j_0j_1\cdots j_{k-2}xj_kx\cdots x}$. Also $\tilde{V}$ has the dimensional structure of $N_k \times N_{k-1} \times \cdots \times N_{p-1}$ as opposed to the structure of $N_{k-1} \times N_k \times \cdots \times N_{p-1}$ of V. If we apply such permutations to all $(k-1)$th-level blocks of T to obtain a new matrix, $\tilde{T}$, then $\tilde{T}$ is a p-block matrix with dimensional structure $N_0 \times N_1 \times \cdots \times N_{k-2} \times N_k \times N_{k-1} \times \cdots \times N_{p-1}$ whose $(k-1)$th-level blocks are of type-$\mathscr{L}$. Thus by doing permutations on rows and columns of T, we have moved the type-$\mathscr{L}$ structure from kth-level to $(k-1)$th-level. Thus by recursively applying this procedure, we can obtain a permutation matrix, R, such that $R^{\mathrm{T}}TR$ is a matrix which is of type-$\mathscr{L}$. Now we present an example of a three-block matrix, in which the second-level blocks are of type-$\mathscr{L}$ and we construct the permutation matrix that convert it into a three-block matrix which is of type-$\mathscr{L}$.

Example

T is an $\mathbf{N}\times\mathbf{N}$ three-block matrix with second-level blocks of type-$\mathscr{L}$ and $\mathbf{N} \triangleq N_0 \times N_1 \times N_2 = 3 \times 4 \times 3$; i.e.

$$T = \begin{pmatrix} T_{0x_0x_1} & T_{1x_0x_1} & T_{2x_0x_1} \\ T_{1x_0x_1} & T_{0x_0x_1} & T_{1x_0x_1} \\ T_{2x_0x_1} & T_{1x_0x_1} & T_{0x_0x_1} \end{pmatrix}$$

where

$$T_{j_0x_0x_1} = \begin{pmatrix} T_{j_00x} & T_{j_01x} & T_{j_02x} & T_{j_03x} \\ T_{j_01x} & T_{j_00x} & T_{j_01x} & T_{j_02x} \\ T_{j_02x} & T_{j_01x} & T_{j_00x} & T_{j_01x} \\ T_{j_03x} & T_{j_02x} & T_{j_01x} & T_{j_00x} \end{pmatrix} \quad \text{and} \quad T_{j_0j_1x} = \begin{pmatrix} t_{j_0j_12} & t_{j_0j_11} & t_{j_0j_10} \\ t_{j_0j_11} & t_{j_0j_12} & t_{j_0j_11} \\ t_{j_0j_10} & t_{j_0j_11} & t_{j_0j_12} \end{pmatrix}$$

Now, if $R_0 = (I_{N_1} \otimes e_0^3 \;\; I_{N_1} \otimes e_1^3 \;\; I_{N_1} \otimes \;\; e_2^3)$, then

$$R_0^{\mathrm{T}} T_{j_0x_0x_1} R_0 = \underbrace{\begin{pmatrix} T_{j_0x2} & T_{j_0x1} & T_{j_0x0} \\ T_{j_0x1} & T_{j_0x2} & T_{j_0x1} \\ T_{j_0x0} & T_{j_0x1} & T_{j_0x2} \end{pmatrix}}_{\triangleq T_{j_0x_1x_0}}, \quad \text{where} \quad T_{j_0xj_2} = \begin{pmatrix} t_{j_00j_2} & t_{j_01j_2} & t_{j_02j_2} & t_{j_03j_2} \\ t_{j_01j_2} & t_{j_00j_2} & t_{j_01j_2} & t_{j_02j_2} \\ t_{j_02j_2} & t_{j_01j_2} & t_{j_00j_2} & t_{j_01j_2} \\ t_{j_02j_2} & t_{j_02j_2} & t_{j_01j_2} & t_{j_00j_2} \end{pmatrix}$$

for $\mathbf{0} \leqslant \mathbf{j} \leqslant \mathbf{N} - 1$. Therefore if $R_1 = I_{N_0} \otimes R_0$, then

$$\tilde{T} \triangleq R_1^{\mathrm{T}} T R_1 = \begin{pmatrix} T_{0x_1x_0} & T_{1x_1x_0} & T_{2x_1x_0} \\ T_{1x_1x_0} & T_{0x_1x_0} & T_{1x_1x_0} \\ T_{2x_1x_0} & T_{1x_1x_0} & T_{0x_1x_0} \end{pmatrix}$$

Note that the one-level blocks in $\tilde{T}$ are of type-$\mathscr{L}$ and other blocks are of type-$\mathscr{R}$. Also its dimension, $\tilde{\mathbf{N}} \triangleq \tilde{N}_0 \times \tilde{N}_1 \times \tilde{N}_2 = N_0 \times N_2 \times N_1$. If we choose $\tilde{R}_0 = \left(I_{\tilde{N}_0} \otimes \tilde{e}_0^2 \; I_{\tilde{N}_0} \otimes \tilde{e}_1^2 \; I_{\tilde{N}_0} \otimes \tilde{e}_2^2 \right)$, then

$$\hat{T} \triangleq \tilde{R}_0^{\mathrm{T}} \tilde{T} \tilde{R}_0 = \begin{pmatrix} T_{x_0x_1 2} & T_{x_0x_1 1} & T_{x_0x_1 0} \\ T_{x_0x_1 1} & T_{x_0x_1 2} & T_{x_0x_1 1} \\ T_{x_0x_1 0} & T_{x_0x_1 1} & T_{x_0x_1 2} \end{pmatrix}$$

Note that $\hat{T}$ is of type-$\mathscr{L}$ and its dimension $\hat{\mathbf{N}} = N_2 \times N_0 \times N_1$. Also, since $\hat{T} = \tilde{R}_0^{\mathrm{T}} R_1^{\mathrm{T}} T R_1 \tilde{R}_0$ and the product of permutation matrices is also a permutation matrix, we have a permutation matrix, $R \triangleq R_1 \tilde{R}_0$ such that $\hat{T} = R^{\mathrm{T}} T R$.

So we have shown that there exist permutation matrices, R_j, such that $H_j^a \triangleq R_j^{\mathrm{T}} \Gamma_j^a R_j$ and $H_j^\xi \triangleq R_j^{\mathrm{T}} \Gamma_j^\xi R_j$ are matrices of type-$\mathscr{L}$. Since, the transpose of every permutation matrix is its inverse [26, pp. 25–26], we also have the relations, $\Gamma_j^a = R_j H_j^a R_j^{\mathrm{T}}$ and $\Gamma_j^\xi = R_j H_j^\xi R_j^{\mathrm{T}}$. We rewrite Equations (B2) and (B3) in terms of the type-$\mathscr{L}$ matrices, H_j^a and H_j^ξ, as

$$I - PA = \sum_{j=1}^{p} L_p^{\mathrm{T}} \cdots L_{j+1} (R_j H_j^\xi H_j^a R_j^{\mathrm{T}}) L_{j+1} \cdots L_p \tag{B4}$$

As the product, PA, has been written in terms of type-$\mathscr{L}$ matrices, we study their structure. We decompose a type-$\mathscr{L}$ matrix, H given by

$$H \triangleq \begin{pmatrix} H_N & \cdots & H_{M+1} & H_M & \cdots & H_1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ H_{M+1} & \ddots & H_N & \ddots & \ddots & H_M \\ H_M & \ddots & \ddots & \ddots & \ddots & H_{M+1} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ H_1 & \cdots & H_M & H_{M+1} & \cdots & H_N \end{pmatrix}$$

into the following sum:

$$
H = \underbrace{\begin{pmatrix}
 & & & H_M & \cdots & H_1 \\
 & 0 & 0 & & \ddots & \vdots \\
 & & & & & H_M \\
 & 0 & 0 & 0 & & \\
H_M & & & & & \\
\vdots & \ddots & 0 & 0 & & \\
H_1 & \cdots & H_M & & &
\end{pmatrix}}_{\triangleq \Delta_H^M}
$$

$$
+ \underbrace{\begin{pmatrix}
H_N & \cdots & H_{M+1} & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
H_{M+1} & \ddots & H_N & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & H_{M+1} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & H_{M+1} & \cdots & H_N
\end{pmatrix}}_{\triangleq E_H^M}
$$

Note that Δ_H^M is a good approximation of H if $\|E_H^M\|$ is small. To compute $\|E_H^M\|$, we observe that

$$
E_H^M J = \underbrace{\begin{pmatrix}
0 & \cdots & 0 & H_{M+1} & \cdots & H_N \\
\vdots & ⋰ & ⋰ & ⋰ & ⋰ & \vdots \\
0 & ⋰ & ⋰ & ⋰ & ⋰ & H_{M+1} \\
H_{M+1} & ⋰ & ⋰ & ⋰ & ⋰ & 0 \\
\vdots & ⋰ & ⋰ & ⋰ & ⋰ & \vdots \\
H_N & \cdots & H_{M+1} & 0 & \cdots & 0
\end{pmatrix}}_{\triangleq E}
$$

where J is a counter diagonal identity matrix. Since $J^2 = I$, we have $\|E_H^M\| \leqslant 2\|E\|$ and from Lemma B.3-(3), we have the following lemma.

Lemma B.1
If H is a type-$\mathscr{L}$ matrix, then H can be written as $H=\Delta_H^M+E_H^M$, where Δ_H^M and E_H^M have the structure as defined above and $\|E_H^M\|=\|H-\Delta_H^M\|\leqslant 2\sum_{k=M+1}^{N}\|H_k\|$.

After determining this structure of type-$\mathscr{L}$, we present the following proposition which states that the matrix product, PA can be approximated by a block diagonal matrix with a large zero-block.

Proposition B.1
Under Assumptions 1, there exists an $\mathbf{M}$ in $\mathbb{N}^p$ and a matrix D whose rank is at most $2\sum_{j=0}^{p-1}M_j(\mathbf{N}/N_j)$ and for which $\|I-P_{\mathbf{N}}A_{\mathbf{N}}-D_{\mathbf{N}}\|\leqslant\varepsilon$ for all $\mathbf{N}>\mathbf{M}$.

Proof
Let $\tilde{\varepsilon}>0$. From Proposition 1.2, there exists an $\mathbf{M}$ such that $\sum_{\mathbf{k}=-(\mathbf{N}-\mathbf{1})}^{\mathbf{N}}|\gamma_{\mathbf{k}}-\xi_{\mathbf{k}}^{\mathbf{N}}|\leqslant\tilde{\varepsilon}$ for all $\mathbf{N}>\mathbf{M}$; and, $\sum_{k_j=M_j}^{\infty}\sum_{\mathbf{k}\backslash k_j=-\infty}^{\infty}|a_{\mathbf{k}}|\leqslant\tilde{\varepsilon}$ and $\sum_{k_j=M_j}^{\infty}\sum_{\mathbf{k}\backslash k_j=-\infty}^{\infty}|\gamma_{\mathbf{k}}|\leqslant\tilde{\varepsilon}$ for all $0\leqslant j\leqslant p-1$. Now H_j^a is a type-$\mathscr{L}$ matrix and therefore can be written as $H_j^a=\Delta_j^a+E_j^a$ (using Lemma B.1), where $\Delta_j^a\triangleq\Delta_{H_j^a}^{M_0}$ and $E_j^a=E_{H_j^a}^{M_0}$. Similarly, $H_j^\gamma=\Delta_j^\gamma+E_j^\gamma$, where H_j^γ is of the same structure as H_j^a but is formed from the sequence, $\{\gamma_{\mathbf{k}}\}$. For each j, we have

$$\|H_j^\xi H_j^a-\underbrace{\Delta_j^\gamma\Delta_j^a}_{\triangleq D_j}\|\leqslant\|H_j^\xi-H_j^\gamma\|\|H_j^a\|+\|(\Delta_j^\gamma+E_j^\gamma)(\Delta_j^a+E_j^a)-\Delta_j^\gamma\Delta_j^a\|$$

$$\leqslant\|H_j^\xi-H_j^\gamma\|\|H_j^a\|+\|\Delta_j^a\|\|E_j^\gamma\|+\|E_j^a\|\|H_j^\gamma\|$$

Note that D_j is a block diagonal matrix with only two non-zero one-level blocks. Now $(H_j^\xi-H_j^\gamma)J$ (J is counter diagonal block identity matrix) is a block Hankel matrix and using Lemmas A.2 and A.3, we have $\|H_j^\xi-H_j^\gamma\|\leqslant\sum_{-(\mathbf{N}-\mathbf{1})}^{\mathbf{N}}|\gamma_{\mathbf{k}}-\xi_{\mathbf{k}}^{\mathbf{N}}|$; and using Lemmas A.2 and A.4, we have $\|H_j^a\|,\|\Delta_j^a\|\leqslant\sum_k|a_k|=c$ and $\|H_j^\gamma\|\leqslant\sum_k|\gamma_k|\leqslant c_0$. With these estimates and by using Proposition 1.2 and Lemma B.1, we can rewrite the above equation as

$$\|H_j^\xi H_j^a-D_j\|\leqslant c\sum_{-(\mathbf{N}-\mathbf{1})}^{\mathbf{N}}|\gamma_{\mathbf{k}}-\xi_{\mathbf{k}}^{\mathbf{N}}|+2c\sum_{k_j=M_j+1}^{\infty}\sum_{\mathbf{k}\backslash k_j=-\infty}^{\infty}|\gamma_{\mathbf{k}}|$$

$$+2c_0\sum_{k_j=M_j+1}^{\infty}\sum_{\mathbf{k}\backslash k_j=-\infty}^{\infty}|a_{\mathbf{k}}|\leqslant(3c+2c_0)\tilde{\varepsilon}$$

Therefore,

$$\|I-P_{\mathbf{N}}A_{\mathbf{N}}-\underbrace{\sum_{j=1}^{p}L_p^{\mathrm{T}}\cdots L_{j+1}R_j^{\mathrm{T}}D_jR_jL_{j+1}\cdots L_p}_{\triangleq D}\|$$

$$\leqslant\left\|\sum_{j=1}^{p}L_p^{\mathrm{T}}\cdots L_{j+1}R_j^{\mathrm{T}}\left(H_j^\xi H_j^a-D_j\right)R_jL_{j+1}\cdots L_p\right\|$$

$$\leqslant(3c+2c_0)p\tilde{\varepsilon}\ \triangleq\ \varepsilon$$

Therefore, we have $\|I - P_{\mathbf{N}}A_{\mathbf{N}} - D\| \leqslant \varepsilon$ for all $\mathbf{N} > \mathbf{M}$, where D is a matrix with at most $2\sum_{j=0}^{p-1} M_j(\mathbf{N}/N_j)$ non-zero eigenvalues. □

In Proposition B.1, we have shown that the product $I - P_{\mathbf{N}}A_{\mathbf{N}}$ can be approximated by a rank deficient matrix $D_{\mathbf{N}}$. The following proposition establishes the clustering of the spectra of preconditioned Toeplitz matrices and can be proved in the same way as Proposition 2.3.

Proposition B.2

Under Assumptions 1, there exists an $\mathbf{M}$ in $\mathbb{N}^p$ such that there are $\mathbf{N} - 2\sum_{k=0}^{p-1} M_k(\mathbf{N}/N_k)$ eigenvalues λ_j of $P_{\mathbf{N}}A_{\mathbf{N}}$ such that $|\lambda_j - 1| \leqslant \varepsilon \log \mathbf{N}$ for all $\mathbf{N} > \mathbf{M}$.

REFERENCES

1. Chan R, Wong C. Sine transform based preconditioners for elliptic problems. *Numerical Linear Algebra with Applications* 1997; **4**(5):351–368.
2. Lirkov I, Margenov S. MPI parallel implementation of CBF preconditioning for 3d elasticity problems. *Mathematics and Computers in Simulation* 1999; **50**(1999):247–254.
3. Chan R, Chan T. Circulant preconditioners for elliptic problems. *Numerical Linear Algebra with Applications* 1992; **1**:77–101.
4. Vogel CR. *Computational Methods for Inverse Problems* (1st edn). Frontiers in Applied Mathematics. SIAM: Philadelphia, PA, 2002.
5. Dudgeon DE, Russel M, Mersereau M. *Multidimensional Digital Signal Processing* (1st edn). Prentice-Hall Inc.: Englewood Cliffs, NJ 07632, 1984.
6. Chan RH, Ng MK. Conjugate gradient methods for Toeplitz systems. *SIAM Review* 1996; **38**(3): 427–482.
7. Peirce AP, Spottiswoode S, Napier JAL. The spectral boundary element method: a new window on boundary elements in rock mechanics. *International Journal of Rock Mechanics and Mining Sciences and Geomechanics Abstracts* 1992; **29**(4):379–400.
8. Zhuang Y. On the use of hybrid full-wave cg-fft and spectrally preconditioned cg-fft methods for analyzing large microstrip antenna arrays. *Ph.D. Thesis*, Mc Master University, 7 August 1995.
9. Salapaka S, Peirce A, Dahleh M. Analysis of a circulant based preconditioner for a class of lower rank extracted systems. *Numerical Linear Algebra with Applications* 2005; **12**:9–32.
10. Quarteroni A, Sacco R, Saleri F. *Numerical Mathematics* (1st edn). Springer: Berlin, 2000.
11. Luenberger DG. *Linear and Nonlinear Programming* (2nd edn). Addison-Wesley Publishing Company: Reading, MA, 1984.
12. Chan T, Olkin J. Circulant preconditioners for Toeplitz-block matrices. *Numerical Algorithms* 1994; **6**: 89–101.
13. Chan R, Jin X. A family of block preconditioners for block systems. *SIAM Journal on Statistical and Scientific Computing* 1992; **13**:1218–1235.
14. Nagy JG, Plemmons RJ, Torgersen TC. Iterative image restoration using approximate inverse preconditioning. *IEEE Transactions on Image Processing* 1996; **5**(7):1151–1161.
15. Hanke M, Nagy J. Toeplitz-approximate inverse preconditioner for banded Toeplitz matrices. *Numerical Algorithms* 1994; **7**:183–199.
16. Capizzano SS, Tyrtyshnikov EE. Any circulant-like preconditioner for multilevel matrices is not superlinear. *SIAM Journal on Matrix Analysis and Applications* 2000; **21**(2):431–439.
17. Chan RH, Ng KP. Toeplitz preconditioners for Hermitian Toeplitz systems. *Linear Algebra and its Applications* 1993; **190**:181–208.
18. Golub G, Van Loan C. *Matrix Computations* (3rd edn). The Johns Hopkins University Press: London, Baltimore, 1989.
19. Rudin W. *Real and Complex Analysis* (3rd edn). Mathematics Series. McGraw-Hill: New York, 1987.
20. Oppenheim AV, Schafer RW. *Discrete-Time Signal Processing* (1st edn). Prentice-Hall: Englewood Cliffs, NJ, 1989.
21. Bamieh B, Paganini F, Dahleh M. Distributed control of spatially-invariant systems. *IEEE Transactions on Automatic Control*, in press. Preprint downloadable at http://www.engineering.ucsb.edu/~bamieh/research.html
22. Kahan W. Spectra of nearly Hermitian matrices. *Proceedings of American Mathematical Society* 1975; **48**(1):11–17.

23. Tyrtyshnikov E. A unifying approach to some old and new theorems on distribution and clustering. *Linear Algebra and its Applications* 1996; **232**:1–43.
24. Tyrtyshnikov E. Influence of matrix operations on the distribution of eigenvalues and singular values of Toeplitz matrices. *Linear Algebra and its Applications* 1994; **207**:225–249.
25. Kato T. On nonlinear Schrödinger equations, II h^s-solutions and unconditional well-posedness. *Journal D'Analyse Mathématique* 1995; **67**:281–306.
26. Horn RA, Johnson CR. *Matrix Analysis* (1st edn). Cambridge University Press: Cambridge, MA, 1990.

