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Filtering of Nonlinear Chaotic Time-series with Noise [1]

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Abstract

It has been observed that when filtering chaotic time series using a linear Infinite Impulse Response filter, the Lyapunov dimension can become dependent on the contraction rates associated with filter dynamics. In this paper we obtain necessary and sufficient conditions which guarantee that the Lyapunov dimension remains unchanged in the presence of external disturbances that act on the filter. These conditions apply to a certain class of noise sequences, and ensure that the Lyapunov dimension of the attractor in the extended state space, consisting of the chaotic system, filter and noise, is the same as the dimension of the attractor in the chaotic system.

1 Introduction

The classical filtering of time-series in signal processing is carried out by the use of Finite Impulse Response (FIR) or Infinite Impulse Response (IIR) filters. The main difference among the two filters is that an FIR filter, being a simple moving average process, does not affect the ability to reconstruct the state space from the time-series. In contrast, this is not true with IIR filters, since they can be characterized as non-autonomous dynamical systems, whose dynamics necessarily interact with the signal.

An important problem that has recently been studied is the analysis and reconstruction of the state-space of time-series of a non-linear dynamic system that exhibits chaotic behaviour, [1]- [2]. It has been observed that when filtering chaotic time series using a linear Infinite Impulse Response filter, the Lyapunov dimension can become dependent on the contraction rates associated with filter dynamics, [3]-[7]. In [8], necessary and sufficient conditions on the contraction rates of the IIR filter have been obtained which guarantee that the Lyapunov dimension remains unchanged.

The robustness to external noise of the methods for reconstructing the state-space based on measured chaotic time-series is a problem that is widely studied due to its practical importance , e.g. in [9]. However, in the context of the invariance of the Lyapunov dimension, this problem has not yet been addressed [8].

In this work, as a first step along this line, we give necessary and sufficient conditions on a certain class of noise sequences, which will ensure that the Lyapunov dimension of the attractor in the extended state space, consisting of the chaotic system, filter and noise, is the same as the dimension of the attractor in the chaotic system.

2 Notation and Definitions

C^1	:	The space of continuously differentiable functions.
$\mathbb{R}^p$	:	p-dimensional Euclidean space
$\mathbb{R}^{p\times p}$	:	The space of $p \times p$ matrices.
(λ_i^f)	:	The descending sequence of Lyapunov exponents for f with respect to ergodic invariant measure ρ
μ_1^A	:	spectral radius of matrix A
$\mathbb{N}$	:	set of all natural numbers

We denote by, see [5], $c_\rho^f(k) = \sum_{i=1}^k \lambda_i^f$, the sum of the k largest characteristic exponents and extend this definition by linearity between integers by

$$c_\rho^f(s) = \sum_{i=1}^{k} \lambda_i^f + (s-k)\lambda_{k+1}^f \text{ if } k \leq s < k+1.$$

The Lyapunov dimension, d_Λ^f is defined as

$$d_\Lambda^f = max(s : c_\rho^f(s) \geq 0).$$

We define a collection of sequences $\mathcal{T}'$ by

$$\begin{aligned}\mathcal{T}' \;:=\; & \{(w_k)_{k\in\mathbb{N}} \text{ such that there exist } m \in \mathbb{N}, C_w \in \mathbb{R}^{1\times m}, \\ & A_w \in \mathbb{R}^{m\times m} \text{ and sequence } (p_k) \text{ in } \mathbb{R}^m \\ & \text{with } p_{k+1} = A_w p_k \text{ and } w_k = C_w p_k\}.\end{aligned}$$

We say that A_w *is the associated matrix with the sequence* (w_k). Now, we define a collection of sequences

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in $\mathbb{R}^n$ by

$$\mathcal{T} \ := \ \{(W_k)_{k\in\mathbb{N}} \text{ such tha} W_k = (\, w_k \quad 0 \quad \ldots \quad 0\,)^T \text{ where } (w_k)_{k\in\mathbb{N}} \text{ is in} \mathcal{T}'\},$$

and define a collection of matrices $\mathcal{D}_m^f$ by

$$\mathcal{D}_m^f \ := \ \{A \in \mathbb{R}^{m\times m} \text{ such that } \mu_1^A < min(1, e^{\lambda_t^f}) \text{ where, } t-1 < d_\Lambda^f < t\}.$$

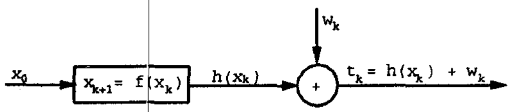

Figure 1: Discrete nonlinear systems affected by noise

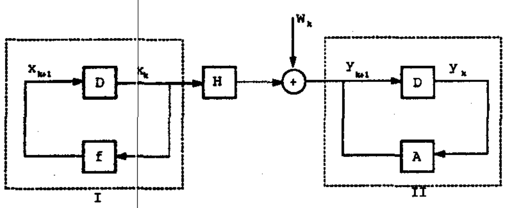

Figure 2: I: The *base system* and II: The *filter dynamics*

3 Problem Formulation

We have $f : M \to M$ a C^1 diffeomorphism, [see Fig. 1] defined on a compact manifold M, and a C^1 function, $h : M \to \mathbb{R}$, which define the discrete nonlinear dynamics given by $x_{k+1} = f(x_k)$ and the output sequence $h(x_k)$. We consider filtering this output using a linear IIR filter which we can identify by the vector $(\, a_1 \quad \ldots \quad a_p\,)^T$ such that

$$l_{k+1} = \sum_{i=1}^{p} a_i l_{k+1-i} + h(x_k) + w_k,$$

where (w_k) in $\mathcal{T}'$. Now, if we define a matrix A in $\mathbb{R}^{p\times p}$ by

$$A := \begin{pmatrix} a_1 & a_2 & \ldots & a_p \\ 1 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 1 & 0 \end{pmatrix},$$

$H : M \to \mathbb{R}^p$ by

$$H(x) = (\, h(x) \quad 0 \quad \ldots \quad 0\,)^T, \text{ for all } x \text{ in } M,$$

the vector sequence, $(y_k)_{k\in\mathbb{N}}$ in $\mathbb{R}^p$ by

$$(\, l_k \quad l_{k-1} \quad \ldots \quad l_{k-p+1}\,)^T,$$

and the vector $W_k \in \mathbb{R}^p$ by $(\, w_k \quad 0 \quad \ldots \quad 0\,)^T$, then we can rewrite the system described above in the following form:

$$\begin{aligned} y_{k+1} &= Ay_k + H(x_k) + W_k \\ x_{k+1} &= f(x_k). \end{aligned}$$

This system has been represented in Fig. 2. Here, system I represents the discrete nonlinear dynamics. In this paper we will call this the *base system dynamics.* System II represents the *filter dynamics.* In this work, we assume that the Lyapunov dimensions of the attractors in the base system and the system consisting of the base system and the filter (without noise), are the same. Then, the results in [8] imply that A is in $\mathcal{D}_p^f$.

Now, (W_k) is a noise sequence in $\mathcal{T}$ that acts on the filter. Therefore, there exists an integer m in $\mathbb{N}$, A_W in $\mathbb{R}^{q\times q}$ and C_W in $R^{p\times q}$, so that $(W_k), k \in \mathbb{N}$ is given by

$$\begin{aligned} p_{k+1} &= A_W p_k \text{ and} \\ W_k &= C_W p_k, \end{aligned}$$

where $A_W = A_w$ and $C_W = (\, C_w^T \quad 0_q \quad \ldots \quad 0_q\,)^T, 0_q$ is a zero vector in $\mathbb{R}^q$. Thus, the whole system represented by Fig. 2 is given by

$$x_{k+1} = f(x_k), \tag{1}$$
$$y_{k+1} = Ay_k + H(x_k) + W_k, \tag{2}$$
$$p_{k+1} = A_W p_k\ , \tag{3}$$
$$W_k = C_W p_k. \tag{4}$$

The dynamics given by Equation (1) is the *base dynamics* and the dynamics given by equation (2) is the *filter dynamics.* Now, we define the extended system $F : M \times \mathbb{R}^{p+q} \to M \times \mathbb{R}^{p+q}$ by

$$F(x, y, p) = \begin{pmatrix} f(x) \\ Ay + H(x) + C_W p \\ A_W p \end{pmatrix}.$$

We also define, collections of sequences $\mathcal{F}$ and $\mathcal{B}$ by

$$\mathcal{F} \ := \ \{(W_k)_{k\in\mathbb{N}} \in \mathcal{T} \text{ such that attractor of the extended system, F exists and } d_\Lambda^F = d_\Lambda^f\}.$$

$$\mathcal{B} \ := \ \{(W_k)_{k\in\mathbb{N}} \in \mathcal{T} \text{ such that } \mu_1^{Aw} < min(1, e^{\lambda_t^f}) \text{ where } A_W \text{ is the matrix associated with the sequence } (W_k) \text{ and } t-1 < d_\Lambda^f < t\}.$$

Theorem
$\mathcal{B} = \mathcal{F}$

Proof: Let $(W_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{B}$. Then, there exists an integer $m \in \mathbb{N}$, $C_W \in \mathbb{R}^{p\times m}$ and $A_W \in \mathcal{D}_m^f$ such that

$$p_{k+1} = A_W p_k \text{ and} \tag{5}$$
$$W_k = C_W p_k \tag{6}$$

Therefore, the system dynamics given by equations (1), (2), (3) and (4) are the same as

$$x_{k+1} = f(x_k), \quad (7)$$
$$y_{k+1} = Ay_k + H(x_k) + C_W p_k \text{ and} \quad (8)$$
$$p_{k+1} = A_W p_k. \quad (9)$$

Let us define

$$A_3 := \begin{pmatrix} A & C_W \\ 0 & A_W \end{pmatrix} \text{ in } \mathbb{R}^{(p+m)\times(p+m)}, \quad (10)$$
$$z_k := \begin{pmatrix} y_k \\ p_k \end{pmatrix} \text{ in } \mathbb{R}^{(p+m)} \text{ and} \quad (11)$$
$$H_3(x_k) := \begin{pmatrix} H \\ 0 \end{pmatrix} \text{ in } \mathbb{R}^{(p+m)}. \quad (12)$$

Thus, the system dynamics given by equations above can be rewritten as

$$x_{k+1} = f(x_k) \text{ and} \quad (13)$$
$$z_{k+1} = A_3 z_k + H_3(x_k). \quad (14)$$

Also, note that from the definition of A_3, we have $\mu_1^{A_3} = max(\mu_1^A, \mu_1^{A_W})$. Therefore, $\mu_1^{A_3}$ is less than $min(1, e^{\lambda_t^f})$ where $t - 1 < d_\Lambda^f < t, t \in \mathbb{N}$. Hence A_3 is an element of $\mathcal{D}_{m+p}^f$.

Here, the extended system, F is given by

$$F(x,z) = \begin{pmatrix} f(x) \\ A_3 z + H_3(x) \end{pmatrix}.$$

Having defined F as above and under the assumptions that the eigenvalues of A_3 lie inside the unit disc, f is a C^1 diffeomorphism and that h is a C^1 function, it has been shown in [8] that the attractor in this extended system exists and the Lyapunov dimension of the attractor of both this system and *the base system* are the same. Therefore, the sequence $(W_k)_{k\in\mathbb{N}}$ belongs to $\mathcal{F}$. As $(W_k)_{k\in\mathbb{N}}$ was chosen arbitrarily from $\mathcal{B}$ and we have shown that $(W_k)_{k\in\mathbb{N}}$ belongs to $\mathcal{F}$, therefore, $\mathcal{B} \subset \mathcal{F}$. Conversely, let $(W_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{F}$. Therefore, there exists an integer $m \in \mathbb{N}, C_W \in \mathbb{R}^{p\times m}$ and a stable matrix A_W such that

$$p_{k+1} = A_W p_k, \quad (15)$$
$$W_k = C_W p_k, \quad (16)$$
$$d_\Lambda^F = d_\Lambda^f, \quad (17)$$

where F is the extended system obtained in the same way as above. In [8] it has been shown that the Lyapunov exponents associated with the extended system F, are given by $\{log(\mu_i^{A_3})\} \cup \{\lambda_j^f\}$,where $\lambda_j^f, 1 \le j \le r$ are the Lyapunov exponents associated with the base system f, and $\mu_i^{A_3}, 1 \le i \le m+p$ are the moduli of eigenvalues of A_3.

Let $t - 1 < d_\Lambda^f < t$ and suppose $log(\mu_1^{A_3}) \ge \lambda_t^f$, then it can be seen from the definition of the Lyapunov dimension that the Lyapunov dimension of the attractor in the extended system will be greater than the Lyapunov dimension for the attractor in the base system. This contradicts (17) and so, our supposition is incorrect. Therefore $log(\mu_1^{A_3}) < \lambda_t^f$. Hence A_3 belongs to $\mathcal{D}_{m+p}^f$. This, in turn implies that A_W belongs to $\mathcal{D}_m^f$. So we have shown that, the sequence $(W_k)_{k\in\mathbb{N}}$ belongs to $\mathcal{B}$. As $(W_k)_{k\in\mathbb{N}}$ was chosen arbitrarily from $\mathcal{F}$ and we have shown that$(W_k)_{k\in\mathbb{N}}$ belongs to $\mathcal{B}$, therefore, $\mathcal{F} \subset \mathcal{B}$. Hence, $\mathcal{B} = \mathcal{F}$

■

4 Conclusion

It has been observed that when filtering chaotic time series using a linear Infinite Impulse Response filter, the Lyapunov dimension can become dependent on the contraction rates associated with filter dynamics. In this paper necessary and sufficient conditions have been obtained which guarantee that the Lyapunov dimension remains unchanged when external disturbances act on the filter are considered. These conditions on invariance of Lyapunov dimensions are important guarantees in the evaluation of Lyapunov dimensions in the embedding space, reconstructed from noisy time-series.

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