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On l_∞ and l_2 robustness of spatially invariant systems

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SUMMARY

We consider spatiotemporal systems and study their l_∞ and l_2 robustness properties in the presence of spatiotemporal perturbations. In particular, we consider spatially invariant nominal models and provide necessary and sufficient conditions for system robustness for the cases when the underlying perturbations are linear spatiotemporal varying, and nonlinear spatiotemporal invariant, unstructured or structured. It turns out that these conditions are analogous to the scaled small gain condition (which is equivalent to a spectral radius condition and a linear matrix inequality for the l_∞ and l_2 cases, respectively) derived for standard linear time-invariant models subject to time-varying linear and time-invariant nonlinear perturbations. Copyright © 2009 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Considerable work has been carried out recently for the analysis and synthesis of spatiotemporal systems e.g. [1, 2]. In this paper we restrict our focus to a certain class of spatiotemporal systems that are spatially invariant. In particular, we analyze the robustness of these systems in the presence of spatiotemporal perturbations to derive necessary and sufficient conditions for stability.

Spatial invariance is a strong property of a given system, which means that the dynamics of the system do not vary as we translate along some spatial axis. Several complex systems can be modelled as being spatially invariant. Consider, for example, an array of (about 4000) closely packed identical microcantilevers to be employed in an atomic force microscope application [3]. In reality, however, it is impossible to fabricate such a system where all the microcantilevers are identical owing to the imperfections in the fabrication process. Each microcantilever will be having slightly different length, thickness, mass and hence the associated spring constant with regard to the nominal design. Moreover, the actual array is finite and hence the spatial-invariant approximation model with an infinite number of identical elements entails additional errors. It is, therefore, imperative to analyze

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the behavior of these ideal models in the presence of perturbations. These perturbations may not, in general, be spatially invariant or even linear.

Robust l_2 stability analysis for linear spatiotemporal-invariant (LSTI) systems has been carried out for LSTI $\mathscr{H}_\infty$-stable perturbations in [4] and μ-like conditions were established. The focus of this paper is the robust l_∞ and l_2 stability analysis for other types of perturbations. In particular, this paper aims to address the necessary and sufficient conditions for robust stability of LSTI stable systems in the presence of linear spatiotemporal-varying (LSTV) perturbations. We also investigate the robust stability of LSTI stable systems with the underlying perturbations being nonlinear spatiotemporal invariant (NLSTI). This paper capitalizes on the time domain representation of these spatiotemporal systems to show that the robustness conditions are analogous to the scaled small gain condition (which is equivalent to a spectral radius condition and a linear matrix inequality (LMI) for the l_∞ and l_2 cases, respectively) derived for standard linear time-invariant (LTI) models subject to linear time-varying or nonlinear perturbations (see [5–7]). The organization of this paper is as follows: Section 2 presents mathematical preliminaries. Section 3 addresses the conditions for system robustness for l_∞-stable LSTI systems. In Section 4 we present the conditions for system robustness for l_2-stable LSTI systems. Finally, we conclude in Section 5.

2. PRELIMINARIES

2.1. Notation

The set of reals is denoted by $\mathbb{R}$ and the set of integers is denoted by $\mathbb{Z}$. The set of non-negative integers is denoted by $\mathbb{Z}^+$. We use l_∞^e to denote the set of all real double sequences $f=\{f_i(t)\}_{i=-\infty,t=0}^{i=\infty,t=\infty}$. These sequences correspond to spatiotemporal signals with a two-sided spatial support ($-\infty\leqslant i\leqslant\infty$) and a one-sided temporal ($0\leqslant t\leqslant\infty$). We use l_∞ to denote the set of such sequences with $\|f\|_\infty:=\sup_{i,t}|f_i(t)|<\infty$. Similarly, l_2 denotes the set of (double) sequences $f=\{f_i(t)\}_{i=-\infty,t=0}^{i=\infty,t=\infty}$ with $\|f\|_2:=(\sum_{i,t}|f_i(t)|^2)^{1/2}<\infty$. Note that for $f\in l_\infty^e$, we can represent it as a one-sided (causal) temporal sequence as $f=\{f(0),f(1),\ldots\}$, where

$$f(t)=(\ldots,f_{-1}(t),f_0(t),f_{+1}(t),\ldots)',\quad t\in\mathbb{Z}^+$$

and each $f_j(t)\in\mathbb{R}$ with $j\in\mathbb{Z}$.

2.2. Spatially invariant systems

We consider spatiotemporal systems $M:u\rightarrow y$ on l_∞^e given by the convolution

$$y_i(t)=\sum_{\tau=0}^{\tau=t}\sum_{j=-\infty}^{j=\infty}m_{i-j}(t-\tau)u_j(\tau)$$

These systems can be viewed as an infinite array of interconnected LTI systems. These form identical building blocks in the system and the corresponding input–output relationship of the ith block can be given as follows:

$$\begin{pmatrix} y_i(0)\\ y_i(1)\\ y_i(2)\\ \vdots\end{pmatrix}=\begin{pmatrix} m_0(0) & 0 & 0 & \cdots\\ m_0(1) & m_0(0) & 0 & \cdots\\ m_0(2) & m_0(1) & m_0(0) & \cdots\\ \cdots & \ddots & \ddots & \ddots\end{pmatrix}\begin{pmatrix} u_i(0)\\ u_i(1)\\ u_i(2)\\ \vdots\end{pmatrix}$$
$$+\begin{pmatrix} m_{-1}(0) & 0 & 0 & \cdots\\ m_{-1}(1) & m_{-1}(0) & 0 & \cdots\\ m_{-1}(2) & m_{-1}(1) & m_{-1}(0) & \cdots\\ \cdots & \ddots & \ddots & \ddots\end{pmatrix}\begin{pmatrix} u_{i-1}(0)\\ u_{i-1}(1)\\ u_{i-1}(2)\\ \vdots\end{pmatrix}$$
$$+\begin{pmatrix} m_1(0) & 0 & 0 & \cdots\\ m_1(1) & m_1(0) & 0 & \cdots\\ m_1(2) & m_1(1) & m_1(0) & \cdots\\ \cdots & \ddots & \ddots & \ddots\end{pmatrix}\begin{pmatrix} u_{i+1}(0)\\ u_{i+1}(1)\\ u_{i+1}(2)\\ \vdots\end{pmatrix}$$
$$+\cdots$$

where $\{u_i(t)\}$ is the input applied at the ith block with $u_i(t)\in\mathbb{R}$ and $t\in\mathbb{Z}^+$ as the time index and $\{m_i(t)\}$ is the pulse response corresponding to the ith input with $m_i(\cdot)\in\mathbb{R}$. In addition, $\{y_i(t)\}$ is the output sequence of the ith block with $y_i(t)\in\mathbb{R}$. We can write the overall input–output relationship for an LSTI system

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as follows:

$$y(t)=\begin{pmatrix} y(0)\\ y(1)\\ y(2)\\ \vdots\\ \vdots \end{pmatrix} = \begin{pmatrix} M_0 & & & & \\ M_1 & M_0 & & & \\ M_2 & M_1 & M_0 & & \\ M_3 & M_2 & M_1 & M_0 & \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}\begin{pmatrix} u(0)\\ u(1)\\ u(2)\\ u(3)\\ \vdots \end{pmatrix} \quad (1)$$

where $u(t)=(\ldots,u_{-1}(t),u_0(t),u_{+1}(t),\ldots)'$ and

$$M_t=\begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots\\ \cdots & m_{-1}(t) & m_0(t) & m_1(t) & \cdots\\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \quad t\in\mathbb{Z}^+$$

2.2.1. l_∞ Stability. We say that an LSTI SISO system M is l_∞ stable if its l_∞-induced norm is finite. It is a straightforward exercise to show that this condition requires the pulse response of the LSTI system to be absolutely summable. With regard to the system representation of (1), this condition reduces to the requirement that the l_1 norm of M satisfies $\|M\|_1:=\sum_{t=0}^{\infty}\sum_{i=-\infty}^{\infty}|m_i(t)|<\infty$. Note that $\|M\|_1=\sum_{t=0}^{\infty}\|M_t\|_1$.

2.2.2. l_2 stability. We first define the z,λ transform for an LSTI system M as

$$\hat{M}(z,\lambda)=\sum_{t=0}^{\infty}\sum_{k=-\infty}^{\infty}(m_k(t)z^k)\lambda^t \quad (2)$$

It is known [1] that the l_2-induced norm of an LSTI system is equal to the $\mathscr{H}_\infty$ norm of $\hat{M}(z,\lambda)$:

$$\|M\|_{l_2\text{-ind}}=\|\hat{M}\|_{\mathscr{H}_\infty}:=\sup_{\theta,\omega}|\hat{M}(\mathrm{e}^{\mathrm{i}\theta},\mathrm{e}^{\mathrm{j}\omega})| \quad (3)$$

We say that an LSTI system M is l_2 stable if $\|\hat{M}\|_{\mathscr{H}_\infty}<\infty$.

2.2.3. Remark. For the case when M is an $n\times p$ LSTI MIMO system, i.e. when $\{m_i(t)\}$ are $n\times p$ (real) matrices for every i and t, and hence M can be represented as a matrix of $n\times p$ LSTI SISO systems $\{M_{kj}\}$, the previous stability and induced norm definitions have the usual generalizations as in the standard LTI systems [7].

2.3. Perturbation models

We will consider various forms of temporally causal perturbations. As usual, by a temporally causal (proper) system T on l_∞^e, we mean that $P_tT=P_tTP_t$ for all $t\in\mathbb{Z}^+$, where P_t is t-step truncation defined as $P_t(x)=P_t(x(0),x(1),\ldots)=(x(0),x(1),\ldots,x(t),0,0,\ldots)$ for any $x\in l_\infty^e$. T is strictly temporally causal (strictly proper) if $P_tT=P_tTP_{t-1}$. In the sequel we will use the terms causal (proper) to mean temporally causal (proper).

2.3.1. LSTV perturbations. The space of LSTV temporally causal and stable perturbations $\Delta: y\to u$ is given by the convolution

$$u_i(t)=\sum_{\tau=0}^{\tau=t}\sum_{j=-\infty}^{j=\infty}\delta_{i,j}(t,\tau)y_j(\tau)$$

These perturbations can also be represented as a temporally causal system

$$\Delta=\begin{pmatrix} \Delta(0,0) & & & & \\ \Delta(1,0) & \Delta(1,1) & & & \\ \Delta(2,0) & \Delta(2,1) & \Delta(2,2) & & \\ \Delta(3,0) & \Delta(3,1) & \Delta(3,2) & \Delta(3,3) & \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \quad (4)$$

The various blocks $\Delta(i,j)$ in the perturbations are infinite matrices, the elements of which are obtained form the spatiotemporal pulse response $\{\delta_{i,j}(t,\tau)\}$. We define the set $\mathbf{B}_{\Delta_{\mathrm{LSTV},p}}$ as

$$\mathbf{B}_{\Delta_{\mathrm{LSTV},p}}=\{\Delta \text{ causal, LSTV, with } \|\Delta\|_{l_p\text{-ind}}<1\} \quad (5)$$

for $p=2,\infty$.

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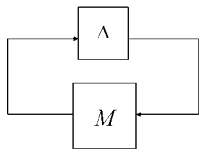

Figure 1. Stability robustness problem.

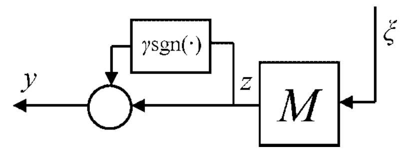

Figure 2. Signal construction for unstructured uncertainty.

2.3.2. NLSTI perturbations. We will also consider NLSTI temporally causal and stable perturbations. For $p=2,\infty$, the set $\mathbf{B}_{\Delta_{\mathrm{NLSTI},p}}$ is defined as

$$\mathbf{B}_{\Delta_{\mathrm{NLSTI},p}}=\{\Delta \text{ causal, NLSTI, with } \|\Delta\|_{l_p\text{-ind}}<1\} \tag{6}$$

2.3.3. System interconnection. Throughout this paper, we will be interested in the stability of the interconnected system shown in Figure 1, with $\Delta\in\mathbf{B}_{\Delta_{\mathrm{LSTV},p}}$ or $\Delta\in\mathbf{B}_{\Delta_{\mathrm{NLSTI},p}}$ strictly causal, and M an LSTI system that is l_p stable. We will investigate the stability in the cases (1) when Δ is unstructured and (2) Δ is structured. By structured, we mean Δ to be of the form $\Delta=\mathrm{diag}(\Delta_1,\Delta_2,\ldots,\Delta_n)$, where Δ_i is LSTV or NLSTI l_p-stable perturbation for all $1\leqslant i\leqslant n$.

2.3.4. Structured norm. Along the lines of Reference [7], which we base our work on, we define the structured norm, SN, as follows. The SN is a map from the space of stable systems to the nonnegative reals defined as

$$\mathrm{SN}_{\Delta,p}(M) = \frac{1}{\inf_\Delta\{\|\Delta\|_{l_p\text{-ind}} \mid (I-\Delta M)^{-1} \text{ is not } l_p \text{ stable}\}}$$

where M is an LSTI l_p-stable system and Δ in a given class i.e. LSTV or NLSTI. It is straightforward to verify directly from the definition that

$$(I-\Delta M)^{-1} \text{ is } l_p \text{ stable for all } \Delta \Leftrightarrow \mathrm{SN}_{\Delta,p}(M)\leqslant 1$$

3. l_∞ STABILITY ROBUSTNESS

In this section we present the robustness analysis of l_∞-stable LSTI systems with regard to unstructured and structured perturbations that are LSTV. We also present an investigation when the underlying perturbations are structured NLSTI.

3.1. LSTV unstructured perturbations

Consider the interconnection of l_∞-stable LSTI system M with a non-structured perturbation $\mathbf{B}_{\Delta_{\mathrm{LSTV},\infty}}$ as shown in Figure 1. The following theorem presents necessary and sufficient conditions for the stability robustness of such a closed-loop system.

Theorem 1
The closed-loop system of Figure 1, with $\Delta\in\mathbf{B}_{\Delta_{\mathrm{LSTV},\infty}}$ and strictly proper, is robustly stable if and only if $\|M\|_1\leqslant 1$.

Proof
The sufficiency follows directly from the small gain theorem and the sub-multiplicative property of the norm, i.e.

$$\|\Delta M\|_{l_\infty\text{-ind}}\leqslant\|\Delta\|_{l_\infty\text{-ind}}\|M\|_1<1 \tag{7}$$

Strict properness guarantees the well posedness of the closed-loop system. For necessity we will show that if $\|M\|_1\geqslant\gamma>1$, then there exists a destabilizing $\Delta\in\mathbf{B}_{\Delta_{\mathrm{LSTV},\infty}}$. For simplicity, we will consider the case when M and Δ are SISO and divide the proof into two steps: (1) we construct an unbounded signal and (2) use this unbounded signal for the construction of a destabilizing perturbation.

3.1.1. Construction of unbounded signal. M is shown in Figure 2 with $\xi\in l_\infty^e$ as its input and $z\in l_\infty^e$ as its output. The signal $y\in l_\infty^e$ is made up of the output z after a bounded signal, the output of a sign function (the operation of which is interpreted componentwise so that the summation $\gamma\,\mathrm{sgn}(z)+z$ makes sense), has been added to it. We interpret this bounded signal as an

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external signal injected for stability analysis. We aim to construct ξ satisfying the following:

1. ξ is unbounded.
2. ξ results in a signal y, such that $\| P_k \xi \|_\infty \leqslant \frac{1}{\gamma} \| P_k y \|_\infty$ where P_k is the truncation operator.

If y and ξ satisfy the second condition, then it is always possible to find Δ so that Δ is causal, has induced norm less than one and satisfies $\Delta y = \xi$. If the first requirement is also met then this Δ is also a destabilizing perturbation.

For simplicity of exposition, we assume that M has finite temporal impulse response of length N. While keeping $|\xi(k)| \leqslant 1$ for $k = 0, \ldots, N-1$, the first N components can be constructed so that $\| M \|_1$ is achieved, where here, with some abuse of notation, we use $|\xi(k)|$ to indicate $\sup_i |\xi_i(k)|$. This implies that $\| P_{N-1} z \|_\infty \geqslant \gamma$ since $\| M \|_1 \geqslant \gamma$. This in turn implies that $\| P_{N-1} y \|_\infty \geqslant 2\gamma$. Note that from the way ξ has been constructed, we have

$$\| P_{N-1} y \|_\infty \geqslant \gamma \| P_{N-1} \xi \|_\infty + \gamma \tag{8}$$

This relationship allows us to increase the magnitude of $|\xi(k)|$ for $k > N-1$ without violating the second requirement on ξ. In particular we let $|\xi(k)|$ to be as large as 2 for $k = N, \ldots, 2N-1$. Again, we can pick $|\xi(k)|$ for this range of k as proceeded before so as to satisfy

$$\| P_{2N-1} y \|_\infty \geqslant \gamma \| P_{2N-1} \xi \|_\infty + \gamma \tag{9}$$

which allows us to further increase $|\xi(k)|$ by 1 for the next N components of ξ and follow the entire procedure again. It is evident from this construction that when ξ is completely specified it will be unbounded, hence, meeting the first requirement. The second requirement is also met since we kept choosing $\xi(k)$ in a way such that it does not grow too fast.

3.1.2. Construction of destabilizing perturbation. Given $\xi = \{\xi_i(t)\}_{i=-\infty, t=0}^{i=\infty, t=\infty}$ and $y = \{y_i(t)\}_{i=-\infty, t=0}^{i=\infty, t=\infty}$ from the previous section, we construct a destabilizing perturbation as follows. The construction of $\Delta(0,0)$ is trivial if $y(0) = 0$; therefore, we assume that there is at least one $i \in \mathbb{Z}$ such that $y_i(0) \neq 0$. Without loss of generality, we assume that $i = 0$. We can now specify $\Delta(0,0)$ as follows:

$$\Delta(0,0) = \begin{pmatrix} & & \vdots & & \\ \cdots & 0 & \vdots & 0 & \cdots \\ \cdots & 0 & \dfrac{\xi_{-1}(0)}{y_0(0)} & 0 & \cdots \\ \cdots & 0 & \dfrac{\xi_0(0)}{y_0(0)} & 0 & \cdots \\ \cdots & 0 & \dfrac{\xi_{+1}(0)}{y_0(0)} & 0 & \cdots \\ \cdots & 0 & \vdots & 0 & \cdots \\ & & \vdots & & \end{pmatrix} \tag{10}$$

Clearly, $\xi(0) = \Delta(0,0) y(0)$. In addition, note that $\Delta(0,0)$ has a spatially varying structure, and $\|\Delta(0,0)\|_{l_\infty\text{-ind}} \leqslant 1/\gamma < 1$ by construction, since each row has exactly one element and the magnitude of each element is strictly less than 1. Next we pick $\Delta(1,0) = 0$ and specify $\Delta(1,1)$ as follows. If $y_i(1) = 0$ for all $i \in \mathbb{Z}$, we simply pick $\Delta(1,1) = 0$. We, hence, assume that there is at least one i such that $y_i(1) \neq 0$. Again, without loss of generality, we assume $i = 0$. The construction of $\Delta(1,1)$ is given as follows:

$$\Delta(1,1) = \begin{pmatrix} & & \vdots & & \\ \cdots & 0 & \vdots & 0 & \cdots \\ \cdots & 0 & \dfrac{\xi_{-1}(1)}{y_0(1)} & 0 & \cdots \\ \cdots & 0 & \dfrac{\xi_0(1)}{y_0(1)} & 0 & \cdots \\ \cdots & 0 & \dfrac{\xi_{+1}(1)}{y_0(1)} & 0 & \cdots \\ \cdots & 0 & \vdots & 0 & \cdots \\ & & \vdots & & \end{pmatrix} \tag{11}$$

Clearly, $\xi(1) = \Delta(1,1) y(1)$. In addition, note that $\Delta(1,1)$ has a spatially varying structure and $\| \Delta(1,1) \|_{l_\infty\text{-ind}} \leqslant 1/\gamma < 1$ by construction. Next we pick

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$\Delta(2,0)=\Delta(2,1)=0$ and specify $\Delta(2,2)$ in the same way as above. It is clear that when Δ is completely specified, it will have a diagonal structure as shown in (12) with only one element in any given row guaranteeing that $\|\Delta\|_{l_\infty\text{-ind}}\leqslant 1/\gamma<1$ and satisfying $\Delta y=\xi$. Moreover, Δ is causal and the construction above can be repeated by introducing delay in the construction of ξ so that Δ is strictly causal. This will guarantee the well posedness of the closed-loop system implying that $(I-\Delta M)^{-1}$ exists and is unstable by construction.

$$\Delta=\begin{pmatrix} \Delta(0,0) & & & & \\ 0 & \Delta(1,1) & & & \\ 0 & 0 & \Delta(2,2) & & \\ 0 & 0 & 0 & \Delta(3,3) & \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \tag{12}$$

3.2. LSTV structured perturbations

For simplicity we will show our result only for two SISO perturbation blocks. The entire result can be generalized for n perturbation blocks in a straightforward manner. It is easy to show that, since $\|D\Delta D^{-1}\|_{l_\infty\text{-ind}}=\|\Delta\|_{l_\infty\text{-ind}}$, the SN satisfies [7]

$$\mathrm{SN}_{\Delta_{\mathrm{LSTV},\infty}}(M)\leqslant \inf_{D\in\mathbf{D}}\|D^{-1}MD\|_1 \tag{13}$$

where $\mathbf{D}=\{\mathrm{diag}(d_1,d_2),\ d_i\in\mathbb{R},\ d_i>0\}$.[§] Corresponding to the two perturbation blocks, i.e $\Delta=\mathrm{diag}(\Delta_1,\Delta_2)$, we can partition M as follows:

$$M=\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \tag{14}$$

where each M_{ij} is a temporally causal LSTI stable system. We now introduce the following nonnegative matrix associated with (14):

$$\widetilde{M}=\begin{pmatrix} \|M_{11}\|_1 & \|M_{12}\|_1 \\ \|M_{21}\|_1 & \|M_{22}\|_1 \end{pmatrix} \tag{15}$$

[§]An element $D\in\mathbf{D}$ is a spatially and temporally constant operator and its z,λ transform is the matrix $\{\mathrm{diag}(d_1,d_2)\}$.

Proposition 1

The following conditions are equivalent:

1. $\rho(\widetilde{M})\leqslant 1$, where $\rho(\cdot)$ denotes the spectral radius.
2. The system of inequalities $x<\widetilde{M}x$ and $x\geqslant 0$ has no solutions, where the vector inequalities are to be interpreted componentwise.
3. $\inf_{D\in\mathbf{D}}\|D^{-1}MD\|_1\leqslant 1$.

Proof

$1\Leftrightarrow 2$ follows exactly along the lines of Reference [7]. We will only show $1\Leftrightarrow 3$ by showing that $\rho(\widetilde{M})=\inf_{D\in\mathbf{D}}\|D^{-1}MD\|_1$. By definition,

$$\begin{aligned}\|D^{-1}MD\|_1 &= \max_i \sum_{j=1}^{2}\|(d_i^{-1}M_{ij}d_j)\|_1\\ &= \max_i \sum_{j=1}^{2}\left(\sum_{t=0}^{\infty}\sum_{k=-\infty}^{\infty}\left|\frac{d_j}{d_i}M_{ij,k}(t)\right|\right)\\ &= \max_i \sum_{j=1}^{2}\frac{d_j}{d_i}\left(\sum_{t=0}^{\infty}\sum_{k=-\infty}^{\infty}|M_{ij,k}(t)|\right)\\ &= \max_i \sum_{j=1}^{2}\frac{d_j}{d_i}\|M_{ij}\|_1 \end{aligned}\tag{16}$$

The expression on the right is also equal to the standard 1-norm of the matrix $D^{-1}\widetilde{M}D$. Denoting this norm by $|\cdot|_1$, we have $\|D^{-1}MD\|_1=|D^{-1}\widetilde{M}D|_1$. Since the spectral radius of a matrix is bounded from above by any matrix norm of that matrix, we have

$$\begin{aligned}\inf_{D\in\mathbf{D}}\|D^{-1}MD\|_1 &= \inf_{D\in\mathbf{D}}|D^{-1}\widetilde{M}D|_1\\ &\geqslant \inf_{D\in\mathbf{D}}\rho(D^{-1}\widetilde{M}D)=\rho(\widetilde{M})\end{aligned}$$

Choosing $D=\mathrm{diag}(d_1,d_2)$ where $(d_1,d_2)'$ is the positive eigenvector corresponding to the eigenvalue $\rho(\widetilde{M})$, the inequality becomes an equality hence establishing an equivalence between 1 and 3. □

The fact that for the optimum scaling $D=\mathrm{diag}(d_1,d_2)$ all the rows $D^{-1}MD$ have the same norm will be exploited next in showing the necessity of condition 3 in the above proposition.

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Theorem 2
The system in Figure 1 achieves robust stability for all structured $\Delta \in \mathbf{B}_{\Delta_{\text{LSTV},\infty}}$ if and only if

$$\min_{D \in \mathbf{D}} \|D^{-1}MD\|_1 \leqslant 1$$

Equivalently, the SN can be computed exactly and is given by

$$\mathrm{SN}_{\Delta_{\text{LSTV},\infty}}(M) = \min_{D \in \mathbf{D}} \|D^{-1}MD\|_1 = \rho(\widetilde{M})$$

Proof
The sufficiency of this condition follows from (13). We now demonstrate that $\inf_{D\in\mathbf{D}} \|D^{-1}MD\|_1 \leqslant 1$ is necessary for robust stability. The approach is to show how to construct a destabilizing perturbation whenever $\inf_{D\in\mathbf{D}} \|D^{-1}MD\|_1 > 1$. Suppose that $\inf_{D\in\mathbf{D}} \|D^{-1}MD\|_1 \geqslant \gamma > 1$. Given that this infimum is in fact a minimum, and the fact that the rows of $D^{-1}MD$ will have equal norms, we have the following relationship for $n=2$:

$$\|(D^{-1}MD)_1\|_1 = \|(D^{-1}MD)_2\|_1 \tag{17}$$

where $(D^{-1}MD)_i$ denotes the ith row of $D^{-1}MD$. The proof follows along the footsteps of the previous section, with the first step being the construction of an unbounded signal that gets amplified componentwise by $\|D^{-1}MD\|_1$ at the optimum D and the second being the construction of a destabilizing perturbation using this signal.

3.2.1. Construction of unbounded signals. $D^{-1}MD$ shown in Figure 3 has $\xi = (\xi_1,\ \xi_2)$ as its input, where each $\xi_i \in l^e_\infty$, and $z = (z_1, z_2)$ as the output, with each $z_i \in l^e_\infty . y = (y_1, y_2)$ is made up of the output $z = (z_1, z_2)$ after a bounded signal, the output of a sign function, has been added to it, where the sign function operates on z_i componentwise. Again, we interpret this bounded signal as an external signal injected for stability analysis. We aim to construct ξ satisfying the following:

1. ξ is unbounded.
2. ξ results in a signal y, satisfying $\| P_k\xi_i \|_\infty \leqslant 1/\gamma\ \| P_k y_i \|_\infty$ for $i = 1, 2$, with P_k a temporal truncation operator.

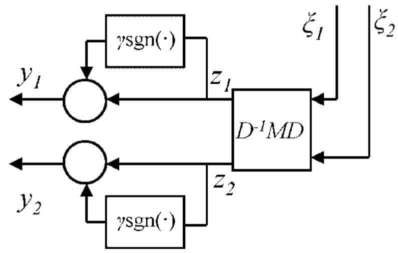

Figure 3. Signal construction.

Simplifying the exposition, we assume that all M_{ij}'s have finite temporal pulse response of length N. While keeping $|\xi(k)| \leqslant 1$ for $k = 0, \ldots, N-1$, the first N temporal components of ξ can be constructed so as to achieve $\| (D^{-1}MD)_1 \|_1$. Since $\| (D^{-1}MD)_1 \|_1 \geqslant \gamma$, this implies that $\| P_{N-1}z_1 \|_\infty \geqslant \gamma$, which in turn implies that $\| P_{N-1}y_1 \|_\infty \geqslant 2\gamma$. Next, while still keeping $|\xi(k)| \leqslant 1$, we pick the next N temporal components of ξ so as to achieve the second row norm $\| (D^{-1}MD)_2 \|_1$. As a result we have $\| P_{2N-1}z_1 \|_\infty \geqslant \gamma$, which implies that $\| P_{2N-1}y_1 \|_\infty \geqslant 2\gamma$. Note that the second requirement mentioned above is met for $k = 0, \ldots, 2N-1$. In addition, from the way the first $2N$ terms of ξ have been constructed, we have

$$\| P_{2N-1}y_i \|_\infty \geqslant \gamma \| P_{2N-1}\xi_i \|_\infty + \gamma, \quad i = 1, 2 \tag{18}$$

This relationship allows us to increase the magnitude of $|\xi(k)|$ to 2 for $k = 2N, \ldots, 4N-1$ while satisfying the following relationship:

$$\| P_{4N-1}y_i \|_\infty \geqslant \gamma \| P_{4N-1}\xi_{i,j} \|_\infty + \gamma, \quad i = 1, 2 \tag{19}$$

This allows us to increase $|\xi(\cdot)|$ by 1 for the next $2N$ temporal components and repeat the whole procedure again. Hence, by the time when ξ is completely specified it will be unbounded, meeting the first requirement. The second requirement is also met since ξ was not allowed to grow too fast.

3.2.2. Construction of destabilizing perturbation. We shall only show the construction of Δ_1 as the construction of Δ_2 follows in its footsteps. Given $\xi_1 = \{\xi_{1,i}(t)\}_{i=-\infty,t=0}^{i=\infty,t=\infty}$ and $y_1 = \{y_{1,i}(t)\}_{i=-\infty,t=0}^{i=\infty,t=\infty}$ from the previous section, we proceed as follows for the construction of destabilizing perturbation. The

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construction of $\Delta_1(0,0)$ is trivial if $y_1(0)=0$; therefore, we assume that there is at least one i such that $y_{1,i}(0)\neq 0$. Without loss of generality, we assume that $i=0$. We can specify $\Delta_1(0,0)$ as follows:

$$\Delta_1(0,0)=\begin{pmatrix} & & \vdots & & \\ \cdots & 0 & \vdots & 0 & \cdots \\ \cdots & 0 & \dfrac{\xi_{1,-1}(0)}{y_{1,0}(0)} & 0 & \cdots \\ \cdots & 0 & \dfrac{\xi_{1,0}(0)}{y_{1,0}(0)} & 0 & \cdots \\ \cdots & 0 & \dfrac{\xi_{1,+1}(0)}{y_{1,0}(0)} & 0 & \cdots \\ \cdots & 0 & \vdots & 0 & \cdots \\ & & \vdots & & \end{pmatrix} \tag{20}$$

Clearly, $\xi_1(0)=\Delta_1(0,0)y_1(0)$. In addition, note that $\Delta_1(0,0)$ has a spatially varying structure and $\|\Delta_1(0,0)\|_{l_\infty\text{-ind}}\leqslant 1/\gamma<1$ by construction. Next we pick $\Delta_1(1,0)=0$ and specify $\Delta_1(1,1)$ as follows. If $y_{1,i}(1)=0$ for all i, we simply pick $\Delta_1(1,1)=0$. We, hence, assume that there is at least one i such that $y_{1,i}(1)\neq 0$. Again, without loss of generality, we assume that $i=0$. The construction of $\Delta_1(1,1)$ is given as follows:

$$\Delta_1(1,1)=\begin{pmatrix} & & \vdots & & \\ \cdots & 0 & \vdots & 0 & \cdots \\ \cdots & 0 & \dfrac{\xi_{1,-1}(1)}{y_{1,0}(1)} & 0 & \cdots \\ \cdots & 0 & \dfrac{\xi_{1,0}(1)}{y_{1,0}(1)} & 0 & \cdots \\ \cdots & 0 & \dfrac{\xi_{1,+1}(1)}{y_{1,0}(1)} & 0 & \cdots \\ \cdots & 0 & \vdots & 0 & \cdots \\ & & \vdots & & \end{pmatrix} \tag{21}$$

Clearly, $\xi_1(1)=\Delta_1(1,1)y_1(1)$. In addition, note that $\Delta_1(1,1)$ has a spatially varying structure and $\|\Delta_1(1,1)\|_{l_\infty\text{-ind}}\leqslant 1/\gamma<1$ by construction. Next we pick $\Delta_1(2,0)=\Delta_1(2,1)=0$ and specify $\Delta_1(2,2)$ in the same way as above. It is clear that when Δ is completely specified, it will have a diagonal structure as shown in (22) with only one element in any given row guaranteeing that $\|\Delta_1\|_{l_\infty\text{-ind}}\leqslant 1/\gamma<1$ and satisfying $\Delta_1 y_1=\xi_1$. Moreover, Δ_1 is causal and the construction above can be repeated so that Δ_1 is strictly causal. This will guarantee the well posedness of the closed-loop system implying that $(I-\Delta M)^{-1}$ exists and is unstable by construction.

$$\Delta_1=\begin{pmatrix} \Delta_1(0,0) & & & & \\ 0 & \Delta_1(1,1) & & & \\ 0 & 0 & \Delta_1(2,2) & & \\ 0 & 0 & 0 & \Delta_1(3,3) & \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \tag{22}$$

3.3. NLSTI structured perturbations

We will only discuss the case of structured NLSTI perturbations as the case of unstructured NLSTI perturbations will become obvious from it. From the definition of SN, it follows that

$$\mathrm{SN}_{\Delta_{\mathrm{NLSTI},\infty}}(M)\leqslant \inf_{D\in\mathbf{D}}\|D^{-1}MD\|_1 \tag{23}$$

In the following, we show that the equality also holds in (23).

Theorem 3
The SN for NLSTI structured perturbations satisfies

$$\mathrm{SN}_{\Delta_{\mathrm{NLSTI},\infty}}(M)=\mathrm{SN}_{\Delta_{\mathrm{LSTV},\infty}}(M)=\inf_{D\in\mathbf{D}}\|D^{-1}MD\|_1$$

Proof
The proof of this fact follows exactly as the proof of Theorem 2, except for the construction of the destabilizing perturbation. Given the signals y and ξ, we show that an NLSTI structured perturbation can be constructed to destabilize the closed loop. Let the signals y_i and ξ_i be given as before $(i=1,\ 2)$. Δ_i must

be such that $\|\Delta_i\|_{l_\infty\text{-ind}}<1$ and $\Delta_i y_i = \xi_i$. We define Δ_i as follows:

$$(\Delta_i f)(k) = \begin{cases} \xi_{i,l-m}(k-j) & \text{if for some integers } m \\ & \quad \text{and } j \text{ with } j \geqslant 0 \\ & \quad P_k f = P_k S_{m,j} y_i \\ 0 & \text{otherwise} \end{cases}$$

where $S_{m,j}$ is the shift operator by m spatial and j temporal steps.[¶] It can be seen that Δ_i is a causal, NLSTI system. It has a norm less than one and maps y_i to ξ_i. □

We note here that if, instead of NLSTI, the underlying perturbations are causal l_∞-stable nonlinear spatiotemporal varying (NLSTV), the SN is obviously the same as in the above theorem since LSTV perturbations are a subset of NLSTV.

3.4. Numerical example

As an example to calculate the sub-optimal scaling D, we consider an LSTI approximation that models the microcantilever array presented in [3]. The system consists of infinitely many microcantilevers connected to a base, each forming a micro-capacitor, with the second rigid plate located underneath the microcantilever. The vertical displacement of each microcantilever can be controlled by applying a voltage across the plates. Although each microcantilever is independently actuated, its dynamics are influenced by the presence of other microcantilevers. As elaborated in Figure 4, this coupling has two sources of origin: (1) mechanical, since the microcantilevers are attached to the same base, and (2) electrical, due to the fringing fields generated by the micro-capacitors nearby. The dynamics for the ith microcantilever can be given as follows:

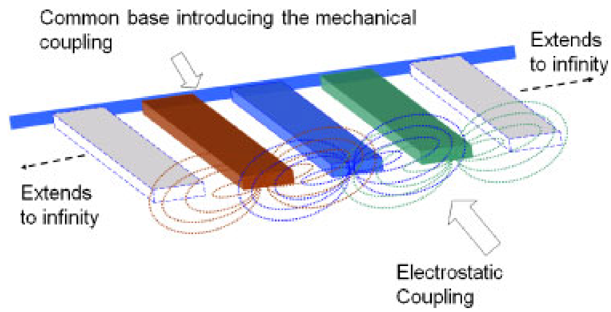

Figure 4. Schematic showing the layout of the infinite-dimensional microcantilever array with mechanical and electrostatic coupling.

$$\dot{x}_i = Ax_i + Bu_i + \sum_{j=-\infty, j\neq i}^{\infty} G_{i,j} x_j$$
$$y_i = Cx_i + Du_i$$

where $G_{i,j}$ captures the mechanical and electrostatic coupling effects from the neighboring microcantilevers. A distributed controller was designed to decouple the dynamics of this system allowing independent actuation of each microcantilever. We are interested in assessing the robust stability of the (nominal) closed-loop system M (system formed by closing the loop of the plant and the designed controller). The (nominal) closed loop M is a 2×2 LSTI system. The (nominal) closed-loop system satisfies

$$\|M\|_1 = \gamma \approx 2.14$$

Considering the perturbation block of $\Delta = \text{diag}\{\Delta_1, \Delta_2\}$ to assess the robust stability of the system, we calculate the corresponding matrix $\widetilde{M}$ (necessary details are given in the Appendix). The spectral radius $\rho(\widetilde{M})$ comes out to be ≈ 0.0011. This implies that the system can tolerate any structured spatiotemporal-varying perturbation $\Delta = \text{diag}\{\Delta_1, \Delta_2\}$ with $\|\Delta_1\|_1 < 1/\rho(\widetilde{M}) \approx 874$

[¶]For example, if $x \in l_\infty^e$, then

$$\begin{aligned} &S_{1,1}((\ldots, x_{-1}(0), x_0(0), x_1(0), \ldots)', \\ &\quad (\ldots, x_{-1}(1), x_0(1), x_1(1), \ldots)', \\ &\quad (\ldots, x_{-1}(2), x_0(2), x_1(2), \ldots)' \ldots) \\ &= (0, (\ldots, x_{-2}(0), x_{-1}(0), x_0(0), \ldots)', \\ &\quad (\ldots, x_{-2}(1), x_{-1}(1), x_0(1), \ldots)', \ldots) \end{aligned}$$

and $\|\Delta_2\|_1 < 1/\rho(\widetilde{M}) \approx 874$. Note that if the diagonal structure of Δ is ignored and we use the l_1 norm criterion for robustness, the size of allowable perturbations reduces dramatically to $1/\gamma = 0.467$.

4. l_2 STABILITY ROBUSTNESS

In this section we present the robustness analysis of l_2-stable LSTI systems with regard to unstructured and structured perturbations that are LSTV. We also present an investigation when the underlying perturbations are structured NLSTI.

4.1. LSTV unstructured perturbations

In [8], l_2 stability analysis of multidimensional systems subject to specific types of structured perturbations was carried under an LMI framework to obtain robustness conditions equivalent to a scaled small gain condition. Our approach here produces the same outcome and, hence, in a sense is equivalent to [8], although the overall development is different as it relies on the same ideas presented in the previous section for l_∞ robustness and generalizes the one in [7]. In the sequel we elaborate in some detail on these developments. Let M be an LSTI l_2-stable system. It follows from the small gain theorem argument that

$$\mathrm{SN}_{\Delta_{\mathrm{LSTV},2}}(M) \leqslant \|\hat{M}\|_{\mathscr{H}_\infty} \tag{24}$$

As in the l_∞ case, the upper bound is equal to the SN as stated in the following theorem.

Theorem 4
The system in Figure 1 achieves robust stability for all $\Delta \in \mathbf{B}_{\Delta_{\mathrm{LSTV},2}}$ if and only if $\|\hat{M}\|_{\mathscr{H}_\infty} \leqslant 1$. Equivalently, the SN is given as

$$\mathrm{SN}_{\Delta_{\mathrm{LSTV},2}}(M) = \|\hat{M}\|_{\mathscr{H}_\infty} \tag{25}$$

Proof
We will show the result for the case of SISO block only. To establish this result, we first show the following lemmas. For simplicity, we assume that M is temporally Finite Impulse Response (FIR) of length N_1. The result generalizes in a straightforward way. Define:

$$k(f) := \|Mf\|_2^2 - \|f\|_2^2 \quad \text{where } f \in l_2 \tag{26}$$

Lemma 1
If $\|\hat{M}\|_{\mathscr{H}_\infty} > 1$, then there exists $f \in l_2$ such that $k(f) > 0$.

Proof
Suppose on the contrary, for every $f \in l_2$, we have $k(f) \leqslant 0$, then

$$k(f) = \|Mf\|_2^2 - \|f\|_2^2 \leqslant 0 \tag{27}$$

$$\Longrightarrow \sup \frac{\|Mf\|_2}{\|f\|_2} = \|\hat{M}\|_{\mathscr{H}_\infty} \leqslant 1 \tag{28}$$

which is a contradiction. □

Lemma 2
There exists a destabilizing perturbation $\Delta \in \mathbf{B}_{\Delta_{\mathrm{LSTV},2}}$ of the system in Figure 1, if there exists an $f \in l_2$ such that $k(f) > 0$.

Proof
Since $k(f) > 0$, then there exists an $N_2 \geqslant N_1$ and a $\gamma^2 \geqslant 1$ such that

$$\|P_{N_2-1} Mf\|_2^2 \geqslant \gamma^2 \|P_{N_1-1} f\|_2^2 \tag{29}$$

where P_k is the temporal truncation operator. Without loss of generality, we assume that f has a finite temporal length N_2, i.e. $f(k) = 0$ for all $k \geqslant N_2$. The proof is divided into two steps. The first step is the construction of a signal $\xi \in l_\infty^e \setminus l_2$ such that the output is amplified by γ^2. The next step is to use this signal to construct a destabilizing perturbation.

4.1.1. Construction of unbounded signal. Define the signal ξ as follows:

$$\xi = \sum_{k=0}^{\infty} S_{k(N_1+N_2)} f \tag{30}$$

where $S_{k(N_1+N_2)}$ is the temporal shift operator by $k(N_1+N_2)$ temporal steps. We remark here that this operator is same as the spatiotemporal shift operator $S_{m,j}$ presented in Section III (C), with $m=0$. For simplicity, we omit the subscript identifying the spatial shift. The signal ξ can be visualized as a signal made up from the nonzero components of f (which we denote by f) by shifting it, and adding zeros in between, i.e.

$$\xi=\left\{\underbrace{f}_{N_2},\overbrace{0}^{N_1},f,0,\ldots\right\} \tag{31}$$

The action of M on ξ can be decomposed as follows:

$$\begin{aligned}y=M\xi&=\sum_{k=0}^{\infty}S_{k(N_1+N_2)}Mf\\&=\sum_{k=0}^{\infty}S_{k(N_1+N_2)}(P_{N_2-1}Mf\\&\quad+(P_{N_1+N_2-1}-P_{N_2-1})Mf)\end{aligned}$$

Define M_0 and M_1 as follows:

$$\begin{aligned}M_0&:=P_{N_2-1}MP_{N_2-1}\\M_1&:=S_{-N_2}(P_{N_1+N_2-1}-P_{N_2-1})MP_{N_2-1}\end{aligned}$$

Then y can be written as

$$y=\{M_0f,M_1f,M_0f,M_1f\ldots\} \tag{32}$$

Defining $\hat{y}=M_0f$, and $\tilde{y}=M_1f$, we can write y as follows:

$$y=\{\hat{y},\tilde{y},\hat{y},\tilde{y}\ldots\} \tag{33}$$

It follows from (29) and (32) that for any $k\geqslant 0$

$$\|P_{k(N_1+N_2-1)}y\|_2^2\geqslant\gamma^2\|P_{k(N_1+N_2-1)}\xi\|_2^2 \tag{34}$$

4.1.2. Construction of destabilizing perturbation. We construct $\hat{\Delta}$ satisfying the following:

$$f=\hat{\Delta}\hat{y},\quad\|\hat{\Delta}\|_{l_2-\mathrm{ind}}\leqslant\frac{1}{\gamma} \tag{35}$$

The construction of such a $\hat{\Delta}$ can be given as:

$$\hat{\Delta}=\begin{pmatrix}\hat{\Delta}_{0,0}&\cdots&\hat{\Delta}_{0,N_2-1}\\\vdots&&\vdots\\\hat{\Delta}_{N_2-1,0}&\cdots&\hat{\Delta}_{N_2-1,N_2-1}\end{pmatrix}$$

where

$$\begin{aligned}\hat{\Delta}_{i,j}&=\frac{1}{\|\hat{y}\|_2^2}\cdot\begin{pmatrix}\vdots\\f_{-1}(i)\\f_0(i)\\f_{+1}(i)\\\vdots\end{pmatrix}(\cdots y_{-1}(j)y_0(j)y_{+1}(j)\cdots)\\&=\frac{1}{\|\hat{y}\|_2^2}\cdot f(i)y(j)'\end{aligned}$$

Equivalently, we can write

$$\hat{\Delta}=\frac{1}{\|\hat{y}\|_2^2}\cdot\begin{pmatrix}f(0)y'(0)&\cdots&f(0)y'(N_2-1)\\\vdots&&\vdots\\f(N_2-1)y'(0)&\cdots&f(N_2-1)y'(N_2-1)\end{pmatrix}$$

or succinctly

$$\hat{\Delta}=\frac{f\hat{y}'}{\|\hat{y}\|_2^2}$$

and $\|\hat{\Delta}\|_{l_2\text{-ind}}$ can be evaluated as follows:

$$\begin{aligned}\|\hat{\Delta}\|_{l_2\text{-ind}}&:=\sup\frac{\|\hat{\Delta}g\|_2}{\|g\|_2}=\frac{\sqrt{g'\hat{\Delta}'\hat{\Delta}g}}{\|g\|_2}=\sqrt{\frac{g'\hat{y}f'f\hat{y}'g}{\|\hat{y}\|_2^4\|g\|_2^2}}\\&=\sqrt{\frac{\|f\|_2^2g'\hat{y}\hat{y}'g}{\|\hat{y}\|_2^4\|g\|_2^2}}\\&\leqslant\sqrt{\frac{\|f\|_2^2(\|g\|_2\|y\|_2)(\|y\|_2\|g\|_2)}{\|\hat{y}\|_2^4\|g\|_2^2}}\leqslant\frac{1}{\gamma}\end{aligned}$$

Now, define the perturbation:

$$\Delta=\begin{pmatrix} 0 & \cdots & & & \\ 0 & 0 & \cdots & & \\ \hat{\Delta} & 0 & 0 & \cdots & \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \hat{\Delta} & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

With this perturbation, the output in each channel for the input ξ is given by

$$(f,0,f-\hat{\Delta}\hat{y},0,f-\hat{\Delta}\hat{y},\ldots)=(f,0,0,\ldots)\in l_2$$

This immediately implies that $(I-\Delta M)^{-1}$ is not l_2-stable since it maps a signal in l_2 to a signal $l_\infty^e\setminus l_2$. Notice that $(I-\Delta M)^{-1}$ is well defined since Δ is strictly proper.

If $\|\hat{M}\|_{\mathscr{H}_\infty}\leqslant 1$ then the system is stable. Suppose that $\|\hat{M}\|_{\mathscr{H}_\infty}>1$. By Lemma 1, there exists a function $f\in l_2$ such that $k(f)>0$. It follows by Lemma 2 that there exists a destabilizing perturbation of the system in Figure 1. □

4.2. LSTV Structured perturbations

For simplicity, we will show our result only for two SISO perturbation blocks. The entire result can be generalized in a straightforward manner. Let M be an LSTI l_2-stable system. It follows from the small gain theorem argument that

$$\mathrm{SN}_{\Delta_{\mathrm{LSTV},2}}(M)\leqslant \inf_{D\in\mathbf{D}}\|D^{-1}\hat{M}D\|_{\mathscr{H}_\infty} \tag{36}$$

Theorem 5
The system in Figure 1 achieves robust stability for all structured $\Delta\in\mathbf{B}_{\Delta_{\mathrm{LSTV},2}}$ if and only if $\inf_{D\in\mathbf{D}}\|D^{-1}\hat{M}D\|_{\mathscr{H}_\infty}\leqslant 1$. Equivalently, the SN is given as

$$\mathrm{SN}_{\Delta_{\mathrm{LSTV},2}}(M)=\inf_{D\in\mathbf{D}}\|D^{-1}\hat{M}D\|_{\mathscr{H}_\infty} \tag{37}$$

Proof
For simplicity, we assume that M is temporally FIR of length N_1. The generalization is straightforward. Define:

$$k_i(f):=\|(Mf)_i\|_2^2-\|f_i\|_2^2 \quad \text{for } i=1,2 \text{ and } f\in l_2 \tag{38}$$

$(Mf)_i$ denotes the ith row of Mf. In order to establish the result of Theorem 5, we first invoke Lemma 7.3.2 in [7], that states the following.

Lemma 3
Suppose that $\hat{M}$ is such that $\inf_{D\in\mathbf{D}}\|D^{-1}\hat{M}D\|_{\mathscr{H}_\infty}$ has a finite nonzero minimizer. If

$$\inf_{D\in\mathbf{D}}\|D^{-1}\hat{M}D\|_{\mathscr{H}_\infty}>1 \tag{39}$$

then there exists a function $f\in l_2$ such that

$$k_i(f)>0 \quad \text{for } i=1,2 \tag{40}$$

We now show the following lemma.

Lemma 4
There exists a destabilizing perturbation $\Delta\in\mathbf{B}_{\Delta_{\mathrm{LSTV},2}}$ of the system in Figure 1, if there exists an $f\in l_2$ such that $k_i(f)>0$ for $i=1,2$.

Proof
The proof follows in the footsteps of the proof of Lemma 2. Since $k_i(f)>0$, then there exists an $N_2\geqslant N_1$ and a $\gamma^2\geqslant 1$ such that

$$\|P_{N_{2-1}}(Mf)_i\|_2^2\geqslant\gamma^2\|P_{N_{1-1}}f_i\|_2^2 \quad \text{for } i=1,2 \tag{41}$$

where P_k is the temporal truncation operator. Without loss of generality, we assume that f has a finite temporal length N_2, i.e. $f(k)=0$ for all $k\geqslant N_2$. □

4.2.1. Construction of unbounded signal. With the definitions of ξ, M_0, and M_1 presented in the previous section, we can directly write

$$y=\{M_0f,M_1f,M_0f,M_1f\ldots\}=\{\hat{y},\tilde{y},\hat{y},\tilde{y}\ldots\} \tag{42}$$

It follows from (41), and (42) that for any $k\geqslant 0$

$$\|P_{k(N_1+N_{2-1})}y_i\|_2^2\geqslant\gamma\|P_{k(N_1+N_{2-1})}\xi_i\|_2^2 \quad \text{for } i=1,2 \tag{43}$$

4.2.2. Construction of destabilizing perturbation. We construct $\hat{\Delta}_i$ as follows:

$$\hat{\Delta}_i = \frac{f_i \hat{y}_i'}{\|\hat{y}_i\|_2^2}, \quad \|\hat{\Delta}_i\|_{l_2-\text{ind}} \leqslant \frac{1}{\gamma} \text{ for } i=1,2$$

Now, define the perturbation:

$$\Delta = \begin{pmatrix} 0 & \cdots & & & \\ 0 & 0 & \cdots & & \\ \hat{\Delta} & 0 & 0 & \cdots & \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \hat{\Delta} & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{with } \hat{\Delta} = \begin{pmatrix} \hat{\Delta}_1 & 0 \\ 0 & \hat{\Delta}_2 \end{pmatrix}$$

With this perturbation, the output in each channel for the input ξ is given by

$$(f_i, 0, f_i - \hat{\Delta}_i \hat{y}_i, 0, f_i - \hat{\Delta}_i \hat{y}_i, \cdots) = (f_i, 0, 0, \cdots) \in l_2$$

This immediately implies that $(I - \Delta M)^{-1}$ is not l_2-stable since it maps a signal in l_2 to a signal $l_\infty^e \setminus l_2$. Notice that $(I - \Delta M)^{-1}$ is well defined since Δ is strictly proper.

If $\|D^{-1}\hat{M}D\|_{\mathscr{H}_\infty} \leqslant 1$ then the system is stable. Suppose that $\|D^{-1}\hat{M}D\|_{\mathscr{H}_\infty} > 1$. By Lemma 3, there exists a function $f \in l_2$ such that $k(f) > 0$. It follows by Lemma 4 that there exists a destabilizing perturbation of the system in Figure 1. □

4.3. NLSTI structured perturbations

Here we present the case of structured NLSTI perturbations only, since the case of unstructured NLSTI perturbations will become obvious from it. From the definition of SN, it follows that

$$\text{SN}_{\Delta_{\text{NLSTI},2}}(M) \leqslant \inf_{D \in \mathbf{D}} \|D^{-1}\hat{M}D\|_{\mathscr{H}_\infty} \tag{44}$$

We remark here that the above inequality also holds if the underlying perturbations are causal l_2-stable NLSTV. However, since this set contains LSTV perturbations, it follows that equality holds. Similar to the l_∞ case, the equality in (44) also holds as shown in the following theorem.

Theorem 6
The SN for NLSTI structured perturbations satisfies

$$\text{SN}_{\Delta_{\text{NLSTI},2}}(M) = \text{SN}_{\Delta_{\text{LSTV},2}}(M) = \inf_{D \in \mathbf{D}} \|D^{-1}\hat{M}D\|_{\mathscr{H}_\infty}$$

Proof
The proof follows exactly as in the LSTV case if we can show that a NLSTI perturbation can be constructed such that

$$\Delta = \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix}, \quad \Delta_i y_i = (0, 0, f_i, 0, f_i, \ldots)$$

$$= S_{N_1+N_2}\xi_i, \quad i=1,2$$

with $\|\Delta\|_{l_2-\text{ind}} < 1$. Consider the following perturbations:

$$(\Delta_i g)(k) = \begin{cases} 0 & \text{if } k < N_1 + N_2 \\ \xi_{i,l-m}(k-j-N_1-N_2) & \text{if for some integers } m \text{ and } j \text{ with } j \geqslant 0 \\ & P_k g = P_k S_{m,j} y_i \\ 0 & \text{otherwise} \end{cases}$$

It can be verified that Δ is a causal NLSTI perturbation. It satisfies $\|\Delta\|_{l_2-\text{ind}} < 1$ with $\Delta_i y_i = S_{N_1+N_2}\xi_i$. □

4.4. Remark

The condition requiring the existence of a scaling matrix D can be readily cast into a family of LMIs over the spatial Fourier frequency parameter θ. For $\{\hat{A}(z), \hat{B}(z), \hat{C}(z), \hat{D}(z)\}$ a state space realization of

M parameterized by spatial Fourier transform (see [1] for details), it is a straightforward exercise to show (using the KYP lemma for discrete systems) that the condition $\|D^{-1}\hat{M}D\|_{\mathscr{H}_\infty}<1$ is equivalent to the feasibility condition of the following LMI over the Fourier frequency parameter θ:

$$\begin{bmatrix} \hat{A}(e^{i\theta}) & \hat{B}(e^{i\theta}) \\ \hat{C}(e^{i\theta}) & \hat{D}(e^{i\theta}) \end{bmatrix}^* \begin{bmatrix} X(e^{i\theta}) & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} \hat{A}(e^{i\theta}) & \hat{B}(e^{i\theta}) \\ \hat{C}(e^{i\theta}) & \hat{D}(e^{i\theta}) \end{bmatrix} - \begin{bmatrix} X(e^{i\theta}) & 0 \\ 0 & D \end{bmatrix} < 0$$

$$\begin{aligned} X(e^{i\theta}) &> 0 \\ D = \mathrm{diag}(d_1, \ldots, d_n) &> 0 \end{aligned} \tag{45}$$

4.5. Numerical example

As an example to calculate the SN sub-optimal scaling D, we consider the same model as in the previous example. The system satisfies

$$\|M(e^{i\theta})\|_{\mathscr{H}_\infty} = \left\| \left[\begin{array}{c|c} \hat{A}(e^{i\theta}) & \hat{B}(e^{i\theta}) \\ \hline \hat{C}(e^{i\theta}) & \hat{D}(e^{i\theta}) \end{array} \right] \right\|_{\mathscr{H}_\infty} = \gamma \approx 0.58 \quad \text{for all } \theta \in [0\ 2\pi]$$

With a structured perturbation block of size two i.e. $\Delta = \mathrm{diag}\{\Delta_1, \Delta_2\}$ we grid θ over the interval $[0\ 2\pi]$, stack the resulting LMI's in (45) together with constraints for $X(e^{i\theta})$ and D to form a single LMI. We then check the feasibility of the resulting LMI over various values of $\hat{\gamma}$, where $\hat{\gamma}$ is the upper bound on $\|D^{-1}\hat{M}D\|_{\mathscr{H}_\infty}$. An optimal (almost) value of $\hat{\gamma}=0.0015$ was found. This implies that the system can tolerate any structured spatiotemporal-varying perturbation $\Delta = \mathrm{diag}\{\Delta_1, \Delta_2\}$ with $\|\Delta_1\|_{l_2-\mathrm{ind}}<1/\hat{\gamma}=666$ and $\|\Delta_2\|_{l_2-\mathrm{ind}}<1/\hat{\gamma}=666$. Note that if the diagonal structure of Δ is ignored and we use the $\mathscr{H}_\infty$ norm criterion for robustness, the size of allowable perturbations reduces dramatically to $1/\gamma = 1.72$.

5. CONCLUSION

We have presented necessary and sufficient conditions for robust stability for LSTI stable systems with respect to l_∞-induced norm and l_2 induced norm when the underlying perturbations are LSTV and NLSTI stable (in the sense of corresponding induced norms). We have shown that the SN has the same value for two classes of perturbations (1) NLSTI perturbations (2) LSTV perturbations. These conditions turn out to be analogous to the robustness conditions of standard LTI stable (l_∞, and l_2) systems.

Future research directions include application of the developed tools to distributively controlled real systems such as presented in [3]. We are also interested in the problem of remote mean square stabilization of LSTI systems when independent fading channels are dedicated to every actuator and sensor. For standard MIMO systems, the stochastic variables responsible for the fading can be seen as a source of model uncertainty, leading to robust control analysis and synthesis problems with a deterministic nominal system with stochastic, structured, model uncertainty [9].

APPENDIX

This section contains necessary details for the numerical examples presented in Sections 3.4 and 4.5. For clarity of exposition, we have only considered parametric uncertainty on the lumped parameters of the plant model given in [3]. A time step of 0.01 ms was used to obtain a discrete time approximation of the plant model. The closed loop $M=\{A^M, B^M, C^M, D^M\}$ is a stable LSTI system. To ease the calculations for the example at hand, we considered only an immediate neighbor interaction model for M. Although limited for the purpose of the example, this is in fact a very good approximation of M as rest of the interactions are very small. The corresponding $\{A^M, B^M, C^M, D^M\}$

matrices are given below:

$$A_0^M = \begin{pmatrix} 0.99 & 1\times10^{-5} & 0 & 0 & 0 & 0 & 0 & 0 \\ -1.14 & 0.99 & 3.04\times10^{-5} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.99 & -6.40 & -0.01 & -1.41\times10^{-5} & 3235.89 & 0 \\ 0 & 0 & 0 & 0.99 & 1.00\times10^{-5} & 0 & 0 & 0 \\ 0 & 0 & 0.82 & -1.14 & 0.99 & -3.20\times10^{-4} & 0 & 0 \\ 0 & 0 & 0.02 & -15005.07 & -17.02 & 0.96 & 7.58\times10^{6} & 0 \\ 0 & 0 & 0 & -1\times10^{-5} & 0 & 0 & 0.99 & 0 \\ 0 & 1.20\times10^{-7} & 0 & -6.65\times10^{-18} & -5.13\times10^{-11} & -4.71\times10^{-22} & 1.59\times10^{-14} & 0.99 \end{pmatrix}$$

$$A_1^M = \begin{pmatrix} -7.22\times10^{-16} & -1.10\times10^{-20} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.11 & -1.44\times10^{-15} & -4.41\times10^{-20} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -7.22\times10^{-16} & -0.06 & -2.91\times10^{-5} & -2.66\times10^{-8} & 0.02 & 0 \\ 0 & 0 & 0 & -7.22\times10^{-16} & -1.37\times10^{-20} & 0 & 0 & 0 \\ 0 & 0 & -2.34\times10^{-15} & 0.11 & -1.44\times10^{-15} & 7.90\times10^{-19} & 0 & 0 \\ 0 & 0 & -5.06\times10^{-17} & -133.72 & -0.07 & -6.24\times10^{-5} & 48.97 & 0 \\ 0 & 0 & 0 & 2.20\times10^{-20} & 0 & 0 & -1.44\times10^{-15} & 0 \\ 0 & -5.36\times10^{-22} & 0 & 1.10\times10^{-17} & 6.64\times10^{-20} & 2.64\times10^{-22} & -8.99\times10^{-15} & -7.22\times10^{-16} \end{pmatrix}$$

$$B_0^M = \begin{pmatrix} 0 & 0 \\ -0.01 & -4.10\times10^{-8} \\ 0_{8\times1} & 0_{8\times1} \end{pmatrix}$$

$$B_1^M = \begin{pmatrix} 0 & 0 \\ 0.00124 & 4.30\times10^{-23} \\ 0_{8\times1} & 0_{8\times1} \end{pmatrix}$$

$$C_0^M = \begin{pmatrix} 1.00 & -3.33\times10^{-18} & 0_{1\times8} \\ 0 & 1.00 & 0_{1\times8} \end{pmatrix}$$

$$C_1^M = \begin{pmatrix} -7.22\times10^{-16} & 1.70\times10^{-18} & 0_{1\times8} \\ 0 & 1.00 & 0_{1\times8} \end{pmatrix}$$

Since there is spatial symmetry in the system, we have $A_1^M = A_{-1}^M$. Also $B_1^M = B_{-1}^M$, and $C_1^M = C_{-1}^M$. The D term is simply zero.

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