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Indirect Adaptive Control of Spatially Invariant Systems

Azeem Sarwar, Petros G. Voulgaris, and Srinivasa M. Salapaka

Abstract— **We present a general class of indirect adaptive controllers for spatiotemporally invariant systems. The control design is based on certainty-equivalence approach, where at each step system parameters are estimated and the controller is implemented using the estimated parameters. At each estimation stage a modeling error is committed which affects the output of the plant. We show that under suitable assumptions on the rates of variation of the estimated plant, which follow from utilizing a distributed projection algorithm, a globally stable adaptive scheme can be guaranteed.**

I. Introduction

With the advancement in sensing and actuating techniques coupled with the incessant increase in computational power, the idea of developing more and more complex systems by putting together simpler smaller units is turning into a reality. Examples of such systems can now be cited from various areas such as: satellite constellations [1], cross-directional control in paper processing applications [2], airplane formation flight [3],[4], automated highway systems [5] and very recently, microcantilever array control for various nano-robotic applications [9]. Lumped approximations of partial differential equations (PDEs) can also be considered in this regard-examples include the deflection of beams, plates, and membranes, and the temperature distribution of thermally conductive materials [6].

Most of the examples cited above have an inherent distributed structure associated with them. For example, many of these systems have sensing and actuation capabilities at every unit (or subsystem). This can be seen clearly in the case for automated highway systems, airplane formation flight, satellite constellations, and cross-directional control systems. The rapid advancement in micro electromechanical actuators and sensors is now enabling deployment of distributed sensors and actuators for systems governed by partial differential equations only to validate the lumped approximations of such systems.

Although distributed control design is still a daunting task in general, several results on distributed control using recently developed techniques are now available [7], [8] for spatially invariant systems. Spatial invariance is a strong property of a given system, which means that the dynamics of the system do not vary as we translate along some spatial axis.

We note here that control design of any system is as good as the system model. When the system model is not available upfront, system identification and control action have to be implemented in parallel. As the system model gets updated, the control law needs to adapt in order to guarantee stability/performance. Adaptation of spatially invariant systems is an aspect that we look at in this article. Many distributed systems, for example array of identical microcantilevers to be employed in Atomic Force Microscope (AFM) application [9] or large segmented telescope [11], can be approximated as spatially invariant.

The analysis presented in [12] is extended to an indirect adaptive scheme where the plant is estimated recursively via the distributed projection algorithm as presented in [10]. The sequence of estimated plants is viewed as a gradually varying spatiotemporal system. The local control law is designed on the basis of frozen time and frozen space plants. Given an instance in space and time, the plant is thought of as a linear spatiotemporally invariant (LSTI) system, with the defining operators fixed at that time and space instance. The spatiotemporally local controllers are designed for the corresponding frozen LSTI system. Since the frozen plant is LSTI, a frozen LSTI controller can be obtained using various methods, e.g. [7], [8] with different design objectives in mind. Our approach generalizes the one developed for the standard linear time invariant case in [13].

The paper is organized as follows: Section II presents necessary preliminaries. Section III presents characterization of a class of gradually varying spatiotemporal controllers. Section IV presents discussion on convergence of the adaptive scheme. We conclude our discussion in Section V.

II. Preliminaries

A. Notation

The set of reals is denoted by $\mathbb{R}$, and the set of integers is denoted by $\mathbb{Z}$. The set of non-negative integers is denoted by $\mathbb{Z}^+$. We use l_∞^e to denote the set of all real double sequences $f = \{f_i(t)\}_{i=-\infty,\ t=0}^{i=\infty,\ t=\infty}$. These sequences correspond to spatiotemporal signals with a 2-sided spatial support ($-\infty \le i \le \infty$) and one sided temporal ($0 \le t \le \infty$). We use l_∞ to denote the space of such sequences with $\|f\|_\infty := \sup_{i,t} |f_i(t)| < \infty$. Note that for $f \in l_\infty^e$, we can represent it as a one-sided (causal) temporal sequence as $f = \{f(0), f(1), \cdots\}$, where

$$f(t) = \big(\ \cdots,\ f_{-1}(t),\ f_0(t),\ f_{+1}(t),\ \cdots\ \big)',\quad t \in \mathbb{Z}^+$$

Azeem Sarwar is with the Department of Mechanical Science and Engineering, University of Illinois at Urbana Champaign, USA asarwar@illinois.edu

Petros G. Voulgaris is with the Faculty of Aerospace Engineering and with the Coordinated Sciences Laboratory, University of Illinois at Urbana Champaign, USA, and a Visiting ADGAS Chaired Professor, Petroleum Institute, Mechanical Engineering, Abu Dhabi, UAE. voulgari@illinois.edu

Srinivasa M. Salapaka is with the Faculty of Mechanical Science and Engineering, University of Illinois at Urbana Champaign, USA salapaka@illinois.edu

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and each $f_j(t) \in \mathbb{R}$, with $j \in \mathbb{Z}$. The norm ($\|\cdot\|$) throughout this paper is taken to be Euclidian.

B. Setup

The basic setup is captured in Figure 1, where an infinite string of interconnected subsystems is shown. P_i refers to the i_{th} subsystem, and u_i, y_i refer to the respective input and output. The subsystem P_i is taken as a single input single output linear time invariant. We note here that all subsystems have identical dynamics, however, they may be operating independently. Equivalently, $P_i = P_j = P \ \forall \ i, j$.

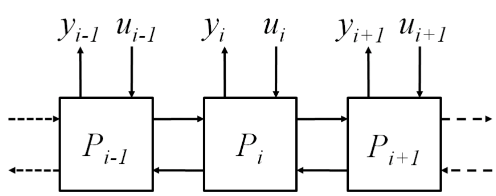

Fig. 1. Basic Setup

C. Spatiotemporal Varying Systems

Linear spatiotemporal varying systems (LSTV) are systems $M : u \to y$ on l_∞^e given by the convolution

$$y_i(t) = \sum_{\tau=0}^{\tau=t} \sum_{j=-\infty}^{j=\infty} m_{i,i-j}(t, t-\tau) u_j(\tau)$$

where $\{m_{i,j}(t,\tau)\}$ is the kernel representation of M. These systems can be viewed as an infinite interconnection of different linear time varying systems. For simplicity, we assume that each of these subsystems is single-input-single-output (SISO). Let $y_i = (y_i(0), y_i(1), y_i(2), \cdots)'$, then the corresponding input-output relationship of the i_{th} block can be given as follows:

$$\begin{pmatrix} y_i(0) \\ y_i(1) \\ y_i(2) \\ \vdots \end{pmatrix} = \begin{pmatrix} m_{i,0}(0,0) & 0 & 0 & \cdots \\ m_{i,0}(1,0) & m_{i,0}(1,1) & 0 & \cdots \\ m_{i,0}(2,0) & m_{i,0}(2,1) & m_{i,0}(2,2) & \cdots \\ \cdots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} u_i(0) \\ u_i(1) \\ u_i(2) \\ \vdots \end{pmatrix}$$
$$+ \sum_{\substack{j=-\infty \\ j \neq i}}^{\infty} \begin{pmatrix} m_{i,j}(0,0) & 0 & 0 & \cdots \\ m_{i,j}(1,0) & m_{i,j}(1,1) & 0 & \cdots \\ m_{i,j}(2,0) & m_{i,j}(2,1) & m_{i,j}(2,2) & \cdots \\ \cdots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} u_{i+j}(0) \\ u_{i+j}(1) \\ u_{i+j}(2) \\ \vdots \end{pmatrix}$$

where $\{u_i(t)\}$ is the input applied at the i_{th} block with $u_i(t) \in \mathbb{R}$ and $t \in \mathbb{Z}^+$ is the time index, and $\{m_{i,j}(t,\tau)\}$ is the kernel representation of M. Also, $\{y_i(t)\}$ is the output sequence of the i_{th} block, with $y_i(\cdot) \in \mathbb{R}$.

We can write the overall input-output relationship for a LSTV system as follows:

$$\begin{pmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} M^{00} & & & & \\ M^{10} & M^{11} & & & \\ M^{20} & M^{21} & M^{22} & & \\ M^{30} & M^{31} & M^{32} & M^{33} & \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} u(0) \\ u(1) \\ u(2) \\ u(3) \\ \vdots \end{pmatrix}$$

Where, $u(t) = (\cdots, u_{-1}(t), u_0(t), u_{+1}(t), \cdots)'$ and

$$M^{t\tau} = \begin{pmatrix} \cdots & \vdots & \vdots & \vdots & \cdots \\ \cdots & m_{i-1,0}(t,\tau) & m_{i-1,1}(t,\tau) & m_{i-1,2}(t,\tau) & \cdots \\ \cdots & m_{i,-1}(t,\tau) & m_{i,0}(t,\tau) & m_{i,1}(t,\tau) & \cdots \\ \cdots & m_{i+1,-2}(t,\tau) & m_{i+1,-1}(t,\tau) & m_{i+1,0}(t,\tau) & \cdots \\ \cdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}$$

where $t, \tau \in \mathbb{Z}^+$. The l_∞ induced operator norm on M in this case is given as

$$\|M\| = \sup_{i,t} \sum_{\tau=0}^{t} \sum_{j=-\infty}^{j=\infty} |m_{i,j}(t,\tau)|$$

The space of l_∞ bounded LSTV systems will be denoted as $\mathscr{L}_{STV}$

D. Spatially Invariant Systems

Linear spatially invariant systems are spatiotemporal systems $M : u \to y$ on l_∞^e given by the convolution

$$y_i(t) = \sum_{\tau=0}^{\tau=t} \sum_{j=-\infty}^{j=\infty} m_{i-j}(t-\tau) u_j(\tau) \tag{1}$$

where $\{m_i(t)\}$ is the pulse response of M. These systems can be viewed as an infinite array of interconnected linear time invariant (LTI) systems. The space of stable LSTI systems will be denoted as $\mathscr{L}_{STI}$. The induced l_∞ operator norm on M in this case is given as

$$\|M\| = \sum_{t=0}^{\infty} \sum_{i=-\infty}^{i=\infty} |m_i(t)| \tag{2}$$

E. Recursively Computable Spatiotemporal Systems

We will focus on SISO discrete-time spatiotemporally invariant systems that are recursively computable. Such plants can be represented by the following transfer function for the i_{th} subsystem [14].

$$\hat{P}(z,\lambda) = \frac{\hat{B}(z,\lambda)}{\hat{A}(z,\lambda)} \tag{3}$$

Where $\hat{B}$, and $\hat{A}$ are polynomials in z, and λ given by

$$\hat{A}(z,\lambda) = 1 + \sum_{t=0}^{m_1} \sum_{k=-n_1, k\neq 0}^{n_1} (a_k(t) z^k) \lambda^t \tag{4}$$

$$\hat{B}(z,\lambda) = \sum_{t=0}^{m_2} \sum_{k=-n_2, k\neq 0}^{n_2} (b_k(t) z^k) \lambda^t \tag{5}$$

The coefficients $\{a_k(t)\}$, and $\{b_k(t)\}$ are not known a priori. However, we will assume knowledge of the bound on the degrees of $\hat{A}$, and $\hat{B}$. We mark this down as an assumption in the following.

AS-1: The integers $m = \max(m_1, m_2)$, and $n = \max(n_1, n_2)$ are known a priori.

The above model can be written as:

$$y_i(t) = \phi_i(t-1)^T \theta_0 \tag{6}$$

where $y_i(t)$ denotes the (scalar) system output of subsystem i at time 't', $\phi_i(t-1)$ denotes a vector that is a linear function of

$$
\begin{aligned}
Y(t) &= [\{y_i(t-1), y_i(t-2), \cdots\}, \{y_{i-1}(t-1), y_{i-1}(t-2), \cdots\}, \{y_{i+1}(t-1), y_{i+1}(t-2), \cdots\}, \cdots] \\
U(t) &= [\{u_i(t-1), u_i(t-2), \cdots\}, \{u_{i-1}(t-1), u_{i-1}(t-2), \cdots\}, \{u_{i+1}(t-1), u_{i+1}(t-2), \cdots\}, \cdots]
\end{aligned}
$$

θ_0 is a vector that is formed from the coefficients $\{a_k(t)\}$, and $\{b_k(t)\}$.

F. Support of m

We define the support of a sequence $\{m_i(t)\}$ by $Supp(m)$, i.e.

$$Supp(m) = \{[i,t] \in \mathbb{Z}^2 : m_i(t) \neq 0\} \tag{7}$$

G. Gradually Varying Spatiotemporal System

A LSTV system A is said to be gradually space-time varying if given two pairs (i,t), and $(\mathtt{i},\tau)$, we have

$$\|A_{i,t} - A_{\mathtt{i},\tau}\| \leq \gamma(|i-\mathtt{i}| + |t-\tau|) \tag{8}$$

where $\gamma \in \mathbb{Z}^+$ is a constant. Such systems will be denoted by GSTV(γ)

H. Local and Global Product

For a LSTV system M, we can associate a LSTI system $M_{i,t}$ for any given pair (i,t) (where $i \in \mathbb{Z}$ represents a spatial coordinate, and $t \in \mathbb{Z}^+$ represents time). The LSTI system $M_{i,t}$ is obtained by freezing M at i spatial coordinate and at t time instance: $(Mu)_i(t) = (M_{i,t}u)_i(t)$. We will refer to $M_{i,t}$ as the local or frozen system corresponding to the pair (i,t). As an example, let $i = 0$, and $t = 3$, then the input output relationship of $M_{1,3}$ can be given as:

$$y(t) = \begin{pmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} M^{33} & & & & \\ M^{32} & M^{33} & & & \\ M^{31} & M^{32} & M^{33} & & \\ M^{30} & M^{31} & M^{32} & M^{33} & \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} u(0) \\ u(1) \\ u(2) \\ u(3) \\ \vdots \end{pmatrix} \tag{9}$$

Where, $u(t) = (\cdots, u_{-1}(t), u_0(t), u_{+1}(t), \cdots)'$ and

$$M^{t\tau} = \begin{pmatrix} \cdots & \vdots & \vdots & \vdots & \cdots \\ \cdots & m_{0,0}(t,\tau) & m_{0,1}(t,\tau) & m_{0,2}(t,\tau) & \cdots \\ \cdots & m_{0,-1}(t,\tau) & m_{0,0}(t,\tau) & m_{0,1}(t,\tau) & \cdots \\ \cdots & m_{0,-2}(t,\tau) & m_{0,-1}(t,\tau) & m_{0,0}(t,\tau) & \cdots \\ \cdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix} \tag{10}$$

For LSTV systems A, B associated with the families $A_{i,t}$, $B_{i,t}$ of frozen LSTI operators, we define a global product AB to mean the usual composition. Given a pair (i,t), the local product of operators A, B corresponds to the product (composition) of the LSTI systems $A_{i,t}$ and $B_{i,t}$, i.e. $A_{i,t}B_{i,t}$.

III. Characterization of a Class of Gradually Varying Spatiotemporal Controllers

We shall employ the distributed projection algorithm (DPA) as presented in [10] for recursive estimation part of the adaptive scheme. At each instant of time the estimation algorithm supplies an estimate $\theta_i(t)$ at the i_{th} subsystem from which we obtain the estimates $\hat{A}_{i,t}$, and $\hat{B}_{i,t}$. For the sake of completion, we list below the properties of the distributed projection algorithm in the absence of additive noise from [10].

1) $\left\|\hat{\theta}_i(t) - \theta_0\right\| \leq \left\|\hat{\theta}^i(t-1) - \theta_0\right\| \leq \left\|\hat{\theta}(0) - \theta_0\right\| \;\; t \geq 1$
2) $\lim_{t\to\infty} \frac{e_i(t)}{[c+\phi_i(t-1)^T\phi_i(t-1)]^{1/2}} = 0$
3) $\lim_{t\to\infty} \left\|\hat{\theta}_i(t) - \hat{\theta}_k(t)\right\| = 0$ for $k \in \{i,\ i-1,\ i+1\}$
4) $\lim_{t\to\infty} \left\|\hat{\theta}_i(t) - \hat{\theta}_i(t-1)\right\| = 0 \; \forall \; i$. This together with 3) implies

$$\lim_{t\to\infty} \left\|\hat{\theta}_i(t) - \hat{\theta}_k(t+l)\right\| = 0 \text{ for } k \in \{i,\ i-1,\ i+1\}$$

and for finite l.

The estimation algorithm implies that the estimates remain bounded, their variation slows down locally, and the normalized estimation error gets small as time progresses.
In the case when there is bounded additive noise present in the system output, the properties of the distributed projection algorithm (with dead-zone) are given as:

1) $\left\|\hat{\theta}_i(t) - \theta_0\right\| \leq \left\|\hat{\theta}^i(t-1) - \theta_0\right\| \leq \left\|\hat{\theta}(0) - \theta_0\right\| \quad t \geq 1$
2) $\lim_{t\to\infty} a(t-1)\frac{e_i(t)^2 - 4\Delta^2}{[c+\phi_i(t-1)^T\phi_i(t-1)]} = 0$
3) $\lim_{t\to\infty} \left\|\hat{\theta}_i(t) - \hat{\theta}_k(t)\right\| = 0$ with $k \in \{i-1, i+1\}$.
4) $\lim_{t\to\infty} \left\|\hat{\theta}_i(t) - \hat{\theta}_i(t-1)\right\| \leq \frac{2\Delta}{\sqrt{c}}$

The sequence of estimated plant, at large enough time, is viewed as a gradually varying spatiotemporal system. The local control law $\hat{K}_{i,t} = \frac{\hat{M}_{i,t}}{\hat{L}_{i,t}}$ is designed on the basis of frozen time and frozen space plants. Given an instance in space and time, the plant is thought of as a LSTI system, with the defining operators fixed at that time and space. The controllers are designed for the corresponding frozen LSTI system. The overall controller, thus, forms a sequence that can be regarded as a spatiotemporally varying controller. The analysis presented in [12] is extended to an indirect adaptive scheme, where the plant is estimated recursively via the DPA, to prove stability. Since the frozen plant is LSTI, a frozen LSTI controller can be obtained using various methods, e.g. [7], [8] with different design objectives. Our approach covers these cases with the advantage of being applicable to more elaborate control techniques.
The frozen space and time operator that defines the control law satisfies the following Bezout identity

$$L_{i,t}A_{i,t} + M_{i,t}B_{i,t} = G_{i,t} \tag{11}$$

where $G_{i,t}^{-1} \in \mathscr{L}_{STI}$ for each fixed pair (i,t), is the closed loop polynomial. The following result gives sufficient conditions for the l_∞ stability of a class of adaptive controllers.

Theorem 3.1: Given $\hat{P} = \frac{\hat{B}}{\hat{A}}$ a LSTI plant, and N an integer such that the degrees of $\hat{A}$, and $\hat{B}$ are bounded by N. Let $A_{i,t}$, and $B_{i,t}$ be the estimates of A, and B at the i_{th} subsystem at time t. Assume that a spatiotemporal varying controller K is implemented as follows.

$$L_{i,t}u_i(t) = M_{i,t}(r_i(t) - y_i(t)) \tag{12}$$

$$L_{i,t}A_{i,t} + M_{i,t}B_{i,t} = G_{i,t} \tag{13}$$

where $L_{i,t}$, $M_{i,t}$, $G_{i,t} \in \mathscr{L}_{STI}$, and $\{r_i(t)\}$ is a bounded reference input. Let the following conditions hold:

1) The operators defining estimates of the plant are gradually time and space varying after time $T_p < \infty$ with rates γ_A and γ_B, i.e. $A_{i,t} \in$ GSTV(γ_A), and $B_{i,t} \in$ GSTV(γ_B).
2) The sequence of controllers are gradually time and space varying after time $T_k < \infty$, i.e. $M_{i,t} \in$ GSTV(γ_M),$L_{i,t} \in$ GSTV(γ_L), and $G_{i,t} \in$ GSTV(γ_G).
3) There exists an integer N_2 such that the degrees of $L_{i,t}$, $M_{i,t}$ are bounded by N_2 for all (i,t)
4) The zeros in λ of $\hat{G}_{i,t}(e^{j\theta}, \lambda)$, lie outside a disc of radius $1+\varepsilon$, for some $\varepsilon > 0$ and for all θ.
5) The l_∞ to l_∞ norms of the LSTI operators $G_{i,t}^{-1}\, L_{i,t}$, $M_{i,t}$ are bounded uniformly in i, and t.

Then there exists a non-zero constant γ such that if $\gamma_A,\ \gamma_B,\ \gamma_M,\ \gamma_L,\ \gamma_G \le \gamma$, the spatiotemporally varying controller will result in stable adaptive scheme.

Proof: Spatiotemporally varying polynomials $A_{i,t}$, and $B_{i,t}$ are obtained from the DPA, driven by the error term $e_i(t) = y_i(t) - \phi_i(t)\hat{\theta}^i(t-1)$. The following equations are the basic components of the adaptive scheme:

$$e_i(t) = A_{i,t-1}y_i(t) - B_{i,t-1}u_i(t) \tag{14}$$

$$L_{i,t}u_i(t) = M_{i,t}(-y_i(t) + r_i(t)) \tag{15}$$

$$G_{i,t} = L_{i,t}A_{i,t} + M_{i,t}B_{i,t} \tag{16}$$

The basic idea is to relate the sequences $\{u_i(t)\}$ and $\{y_i(t)\}$ to the sequence $\{e_i(t)\}$ and $\{r_i(t)\}$, and show that the resulting operator is l_∞ stable. Using the Equations (14)-(16), this can be easily done and the resulting equations can be written as :

$$\begin{bmatrix} G_{i,t}+X_{i,t} & Y_{i,t} \\ -Z_{i,t} & G_{i,t}+W_{i,t} \end{bmatrix} \begin{bmatrix} u_i(t) \\ y_i(t) \end{bmatrix} = \begin{bmatrix} w_i(t) - M_{i,t}e_i(t) \\ z_i(t) + L_{i,t}e_i(t) \end{bmatrix} \tag{17}$$

where

$$X_{i,t} = A_{i,t}\nabla L_{i,t} + M_{i,t}\nabla B_{i,t-1} + M_{i,t}(B_{i,t} - B_{i,t-1}) \tag{18}$$

$$Y_{i,t} = A_{i,t}\nabla M_{i,t} - M_{i,t}\nabla A_{i,t-1} + M_{i,t}(A_{i,t} - A_{i,t-1}) \tag{19}$$

$$Z_{i,t} = B_{i,t}\nabla L_{i,t} - L_{i,t}\nabla B_{i,t-1} + L_{i,t}(B_{i,t} - B_{i,t-1}) \tag{20}$$

$$W_{i,t} = B_{i,t}\nabla M_{i,t} + L_{i,t}\nabla A_{i,t-1} - L_{i,t}(A_{i,t} - A_{i,t-1}) \tag{21}$$

$$w_i(t) = (AMr)_i(t) \tag{22}$$

$$z_i(t) = (BMr)_i(t) \tag{23}$$

where the notation; $A_{i,t}\nabla B_{i,t} = AB - A_{i,t}B_{i,t}$ has been used, i.e. $A_{i,t}\nabla B_{i,t}$ is the difference between the global and local product of operators given a pair (i,t).

For any pair $(\mathtt{i},\tau)$, we factor $G_{\mathtt{i},\tau}$, evaluate the equations at $i = \mathtt{i}, t = \tau$, and consider the evolution of the system as a function of $\mathtt{i},\tau$. The equations can be written as

$$\left(\left(\ I+F \ \right) \begin{pmatrix} u \\ y \end{pmatrix} \right)_{\mathtt{i}} (\tau) = \begin{pmatrix} H_{\mathtt{i},\tau}w_{\mathtt{i}}(\tau) - H_{\mathtt{i},\tau}M_{i,t}e_{\mathtt{i}}(\tau) \\ H_{\mathtt{i},\tau}z_{\mathtt{i}}(\tau) + H_{\mathtt{i},\tau}L_{i,t}e_{\mathtt{i}}(\tau) \end{pmatrix} \tag{24}$$

where

$$F = \begin{pmatrix} H_{\mathtt{i},\tau}(G_{i,t} - G_{\mathtt{i},\tau}) + H_{\mathtt{i},\tau}X_{i,t} & H_{\mathtt{i},\tau}Y_{i,t} \\ H_{\mathtt{i},\tau}Z_{i,t} & H_{\mathtt{i},\tau}(G_{i,t} - G_{\mathtt{i},\tau}) + H_{\mathtt{i},\tau}W_{i,t} \end{pmatrix} \tag{25}$$

$H_{\mathtt{i},\tau}$ is the inverse of $G_{\mathtt{i},\tau}$. By assumption, $H_{\mathtt{i},\tau} \in \mathscr{L}_{STI}$. The operator F can be seen as a perturbation. The key idea is that for large enough time, when the spatiotemporal variations become sufficiently small as guaranteed by DPA, $\|F\|$ shall be small. The invertibility of $I+F$ can then be argued from small gain theorem. From the proof of Theorem I ([12]), we know that there exists an integer T, such that $\|(I-P_T)F\| < 1$ where P_T is a temporal truncation operator. Note that the invertibility of $I+F$ is in essence concerned with the solvability of the equation

$$(\tilde{y} + F\tilde{y})_i(t) = (\tilde{e})_i(t) \tag{26}$$

for $(\tilde{e})_i(t) \in l_\infty$. Let $f_{i,\mathtt{i}}(t,\tau)$ be the kernel representing the operator F. As argued above, there exists an integer T such that

$$C_1 = \sup_{i,t>T} \sum_{k=-\infty}^{\infty} \sum_{j=0}^{t} \left\| f_{i,k}(t,j) \right\| < 1 \tag{27}$$

Let us investigate the operator $I+F$ on the time segment $[0,T]$. On this time segment the operator $I+F$ is finite dimensional (temporally) and is given by.

$$\begin{pmatrix} I+F^{00} & 0 & \cdots & 0 \\ F^{10} & I+F^{11} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ F^{T0} & F^{T1} & \cdots & I+F^{TT} \end{pmatrix} \begin{pmatrix} \tilde{y}(0) \\ \tilde{y}(1) \\ \vdots \\ \tilde{y}(T) \end{pmatrix} = \begin{pmatrix} \tilde{e}(0) \\ \tilde{e}(1) \\ \vdots \\ \tilde{e}(T) \end{pmatrix} \tag{28}$$

Where, $\tilde{y}(t) = (\cdots, \tilde{y}_{i-1}(t), \tilde{y}_i(t), \tilde{y}_{i+1}(t), \cdots)'$ and

$$F^{t\tau} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \cdots \\ \cdots & f_{i-1,i-1}(t,\tau) & f_{i-1,i}(t,\tau) & f_{i-1,i+1}(t,\tau) & \cdots \\ \cdots & f_{i,i-1}(t,\tau) & f_{i,i}(t,\tau) & f_{i,i+1}(t,\tau) & \cdots \\ \cdots & f_{i+1,i-1}(t,\tau) & f_{i+1,i}(t,\tau) & f_{i+1,i+1}(t,\tau) & \cdots \\ \cdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{29}$$

The operator $P_T(I+F)P_T$ maps $P_T(l_\infty)$ into $P_T(l_\infty)$. Note that we have $F^{tt} = 0$, and hence $P_T(I+F)P_T$ is invertible and the inverse is algebraic (does not require inversion of operators). As an example, for $T = 3$, the inverse of

$$\begin{pmatrix} I & 0 & 0 & 0 \\ F^{10} & I & 0 & 0 \\ F^{20} & F^{21} & I & 0 \\ F^{30} & F^{31} & F^{32} & I \end{pmatrix} \tag{30}$$

is given as:

$$\begin{pmatrix} I & 0 & 0 & 0 \\ -F^{10} & I & 0 & 0 \\ F^{21}F^{10} - F^{20} & -F^{21} & I & 0 \\ F^{32}F^{20} - F^{32}F^{21}F^{10} + F^{31}F^{10} - F^{30} & -F^{31} + F^{32}F^{21} & -F^{32} & I \end{pmatrix} \tag{31}$$

and can be verified by direct multiplication. Therefore, there exists a constant C, such that

$$\|P_T\tilde{y}\|_\infty = C\,\|\tilde{e}\|_\infty \tag{32}$$

Since the 'tail' of the operator is small, we should be able to bound the term $(I-P_T)\tilde{y}_i(t)$ in terms of $\tilde{e}(t)$, arising from the solution of $((I+F)\tilde{y})_i(t) = (\tilde{e})_i(t)$. We have

$$(I-P_T)\tilde{y}_i(t) + (I-P_T)F\tilde{y}_i(t) = (I-P_T)\tilde{e}_i(t) \tag{33}$$

which implies that

$$\|(I-P_T)\tilde{y}\|_\infty - \|(I-P_T)F\tilde{y}\|_\infty \leq \|(I-P_T)\tilde{e}\|_\infty \leq \|\tilde{e}\|_\infty \quad (34)$$

Investigating the term

$$(I-P_T)F\tilde{y}_i(t) = \begin{cases} \sum_{k=-\infty}^{\infty}\sum_{j=0}^{t} f_{i,k}(t,j) & \text{if } t > T \\ 0 & t \leq T \end{cases}$$

This implies that

$$\begin{aligned}
\|(I-P_T)F\tilde{y}\| &\leq \sum_{k=-\infty}^{\infty}\sum_{j=0}^{t}\left\|f_{i,k}(t,j)\right\|\|\tilde{y}_i(t)\| \text{ for } t > T & (35)\\
&= \sum_{k=-\infty}^{\infty}\sum_{j=0}^{T}\left\|f_{i,k}(t,j)\right\|\|\tilde{y}_i(t)\| \\
&+ \sum_{k=-\infty}^{\infty}\sum_{j=T+1}^{t}\left\|f_{i,k}(t,j)\right\|\|\tilde{y}_i(t)\| & (36)\\
&\leq \|P_T\tilde{y}\|_\infty \sup_{i,t>T}\sum_{k=-\infty}^{\infty}\sum_{j=0}^{T}\left\|f_{i,k}(t,j)\right\| \\
&+ \|(I-P_T)\tilde{y}\|_\infty \sup_{i,t>T}\sum_{k=-\infty}^{\infty}\sum_{j=T+1}^{t}\left\|f_{i,k}(t,j)\right\| & (37)
\end{aligned}$$

From Equation (34) we have

$$\|(I-P_T)\tilde{y}\|_\infty \leq M_1\|P_T\tilde{y}\|_\infty + M_2\|(I-P_T)\tilde{y}\|_\infty + \|\tilde{e}\|_\infty \quad (38)$$

which implies that

$$\|(I-P_T)\tilde{y}\|_\infty \leq \frac{M_2C}{1-M_1}\|\tilde{e}\|_\infty + \frac{1}{1-M_1}\|\tilde{e}\|_\infty \quad (39)$$

Combining the results from (32), (34), and (39) we get the following bound,

$$\|\tilde{y}\|_\infty \leq \|P_T\tilde{y}\|_\infty + \|(I-P_T)\tilde{y}\|_\infty \leq k_1\|\tilde{e}\|_\infty \quad (40)$$

for some positive constant k_1. We have, therefore, established that in 17, the sequences $\{u_i(t), y_i(t)\}$ are bounded by the sequences $\{e_i(t), w_i(t), z_i(t)\}$. Equivalently,

$$\|\phi_i(t)\| \leq K_1 + K_2 \max_{i,\tau\leq t}\|e_i(\tau)\| \quad (41)$$

In order to complete the proof, we now introduce the following technical lemma [15].

Lemma 1: If

$$\lim_{t\to\infty}\frac{e_i(t)}{[c+\phi_i(t-1)^T\phi_i(t-1)]^{1/2}} = 0 \quad (42)$$

and (41) holds, then it follows that

$$\lim_{t\to\infty} e_i(t) = 0 \quad (43)$$

and $\{\|\phi_i(t)\|\}$ is bounded.
It now follows that e, u, and y are bounded.

■

A. Discussion

We consider two cases here.

1) System With No Noise : Condition 1 in Theorem 3.1 is immediately satisfied from property 2 of the estimation scheme. In fact γ_A, γ_B are smaller than any positive γ for T large enough. If one designs a compensator uniformly continuous with respect to the coefficients in $A_{i,t}$ and $B_{i,t}$, with the stability region in the complement of the disc of radius $1+\varepsilon$ for all θ, conditions 2, 4 will be satisfied. The boundedness conditions 3, 5 are generally satisfied when $\|G_{i,t}\|$ does not approach zero. Hence, any frozen space-time control design methodology that stabilizes the estimates and at the same time is continuous with respect to these estimates will result in stabilizing the unknown system.

2) System With Noise: While the estimates in this case may be slowly varying in space, they may not be slowly varying in time, hence not guaranteeing condition 1 of Theorem 3.1. This means that the speed of estimation has to be controlled after some finite time T. This can be done by choosing the step size for each iterate to be small enough (instead of being constantly 1 outside the dead-zone). Also, it is worth noting that the speed of the estimation scheme need not be controlled for all time but it has to be controlled for large enough time. The question of how small the step size have to be is difficult to answer a priori. The estimates derived in Theorem 3.1 give a very clear idea about the tradeoffs involved, but the issue remains dependent on the control scheme employed. We note that the results hold without any assumptions of persistence of excitation to force the parameters to converge, and their value is obvious in showing the limitation of the adaptive control in the presence of noise. Also, this characterization has the advantage of providing us with large class of stabilizing adaptive controllers, which makes it possible to satisfy performance specifications by choosing an appropriate one.

IV. Convergence of Adaptive Scheme

We have already established the following,

1) The estimates $A_{i,t}$ and $B_{i,t}$ remain bounded, and are gradually varying in space and time
2) $\{u_i(t)\}$ is a bounded sequence
3) $\{y_i(t)\}$ is a bounded sequence

Theorem 4.1: Subject to assumption Given in Theorem 3.1 (AS 1-5), the following holds for the closed loop polynomial

$$\lim_{t\to\infty}[G_{i,t}y_i(t) - B_{i,t}M_{i,t}r(t+1)] = 0 \quad (44)$$

Proof: We have already concluded in the proof of Theorem 3.1 that $\lim_{t\to\infty} e_i(t) = 0$. Since $e_i(t) = A_{i,t-1}y_i(t) - B_{i,t-1}u_i(t)$, we can write

$$\begin{aligned}
L_{i,t}e_i(t+1) &= (LAy)_i(t+1) - (LBu)_i(t+1)\\
&= ([LA - L_{i,t}A_{i,t}]y)_i(t+1) - ([LB - L_{i,t}B_{i,t}]u)_i(t+1)\\
&+ L_{i,t}A_{i,t}y_i(t+1) - (B_{i,t}M_{i,t}r - B_{i,t}M_{i,t}y)_i(t)\\
&= ([LA - L_{i,t}A_{i,t}]y)_i(t+1) - ([LB - L_{i,t}B_{i,t}]u)_i(t+1)\\
&+ G_{i,t}y_i(t+1) - B_{i,t}M_{i,t}r_i(t+1) \qquad (45)
\end{aligned}$$

where

$$G_{i,t} = M_{i,t}B_{i,t} + A_{i,t}L_{i,t} \quad (46)$$

Taking limit as $t \to \infty$ of both sides of the above expression, and using the boundedness of $A_{i,t}$, $B_{i,t}$, $L_{i,t}$, $\{y_i(t)\}$, $\{u_i(t)\}$, $\{r_i(t)\}$ we get

$$\lim_{t\to\infty}[G_{i,t}y_i(t) - B_{i,t}M_{i,t}r(t+1)] = 0 \quad (47)$$

■

V. CONCLUSION

We have presented an indirect adaptive control scheme for LSTI systems that is independent of the underlying control design methodology. We employ certainty-equivalence approach, where at each step system parameters are estimated and the controller is implemented using the estimated parameters. We showed that under suitable assumptions, a globally stable adaptive scheme can be guaranteed.

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