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On l_∞ stability and performance of spatiotemporally varying systems

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SUMMARY

A characterization of stability for spatiotemporally varying systems based on input–output description of the plant and controller is presented. The overall controller is generated by a collection of controllers indexed in space and time, which are based on frozen spatially and temporally invariant descriptions of the plant. In particular, we consider the case where these frozen controllers are not necessarily adjusted for every instance in space and time, and hence are used for some fixed window in time and space. It is shown that the actual spatiotemporally varying system can be stabilized using frozen in space and time controllers, provided the variations in the spatiotemporal dynamics are small enough. We also show that the l_∞ to l_∞ performance of such systems cannot be much worse than that of the frozen spatially and temporally invariant systems. Copyright © 2010 John Wiley & Sons, Ltd.

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KEY WORDS: spatiotemporally varying systems; distributed systems; spatially invariant systems; l_∞ stability

1. INTRODUCTION

Control of spatially invariant systems has received a lot of attention recently and several techniques have been developed, see for example [1, 2]. Spatial invariance is a strong property of a given system, which means that the dynamics of the system do not vary as we translate along some spatial axis. Many systems, however, may not be spatially invariant in general or even time invariant. Distributed systems with time-dependent spatial domains arise naturally in many physical situations. Consider, for example, the problem of controlled annealing of a solid by dipping it in a fluid medium [3] or shape stabilization of flexible structures [4], where the spatial domain of the underlying distributed system is time dependent.

Spatiotemporally varying systems also arise frequently in the process industry where system dynamics are parameter dependent, and change with, for example, change in temperature, pressure, concentration of chemicals, etc. There are a large number of industrial control problems, which involve transport–reaction processes with moving boundaries, such as crystal growth, metal casting, and gas–solid reaction systems. The motion of boundaries is usually a result of a phase change, such as a chemical reaction, mass and heat transfer, and melting or solidification. A two-dimensional linear PDE model is used to describe the spatiotemporal evolution of the thin film surface coating

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process [5, 6]. The control problem is formulated as regulating the thin film thickness and surface roughness by manipulating the substrate temperature and the adsorption rate.

The most widely used methods for approximating the spatiotemporal dynamical systems are finite difference methods. The goal of all finite difference methods is to transform the distributed equations into a set of difference equations. A variety of implicit and explicit methods are available, which have been discussed in detail in the literature [7]. Although finite difference methods are a powerful tool, they should be used with care and several points may be brought out concerning their use (see [8] for details). Model approximation techniques have been used by several authors [9–11]. Galerkin's methods are also extensively used for PDE's in this regard and it is shown that these methods may be used to yield satisfactory approximations [12]. For applications of these methods, refer to [13, 14]. For an extensive discourse on multidimensional systems see [15, 16] and the references therein.

The advancement in sensing techniques can now enable a very fine capturing of dynamic evolution of spatiotemporal systems (e.g. by having densely populated sensors with a fast sampling) not only in time but also in space, rendering the variations in dynamics rather 'quasi-invariant' from one step to another (in space as well as time). At the same time, dense deployment of actuators can allow very fine control of processes at hand. We also note that spatiotemporally varying systems also relate to combined identification and control of linear spatiotemporally invariant (LSTI) systems.

This motivates us to look at the task of controlling spatiotemporally varying systems in an abstract setting, and as such we seek to answer the following question: under what conditions can a spatiotemporally varying system be stabilized by controllers that are designed based on the local LSTI approximations? In addition, we look into the aspect of not adjusting the controllers for every instance of space and time but use them for some fixed window in time and space before implementing new controllers. The lengths of these windows then enter as additional parameters in the stability analysis.

In this paper, we restrict our focus to a certain class of discrete distributed systems that have varying dynamics in time as well as in space. In particular, we focus on the recursively computable spatiotemporal systems. Recursibility [16] is a property of certain difference equations, which allows one to iterate the equation by choosing an indexing scheme so that every output sample can be computed from outputs that have already been found from initial conditions and from samples of the input sequence. Recursive systems are guaranteed to be well defined, and this class encompasses many systems of practical importance, such as discretized partial differential equations (PDEs) (deflection of beams, plates, membranes, temperature distribution of thermally conductive materials [17]). We emphasize here that the results of this paper are applicable to linear spatiotemporally varying systems as long as they are well defined.

In this paper, we show that these spatiotemporally varying systems can be stabilized using the frozen LSTI controllers provided the rates of the variations in the spatiotemporal dynamics are sufficiently small. Our result is a generalization of the results on slowly time-varying systems presented in [18] and [19]. We also show that the l_∞ to l_∞ performance of global spatiotemporally varying system cannot be much worse than the worst frozen spatially and temporally l_∞ to l_∞ performance given that the rates of variation of the plant and the controller are sufficiently small. This is the generalization of the result presented in [20].

The organization of this paper is as follows: Section 2 presents the mathematical preliminaries. Section 3 elaborates on the frozen space-time control law. The stability analysis is presented in Section 4, while the performance analysis is presented in Section 5. We conclude our discussion in Section 6.

2. PRELIMINARIES

2.1. Notation

$\mathbb{R}$	$:=$ Set of reals
$\mathbb{Z}$	$:=$ Set of integers
$\mathbb{Z}^+$	$:=$ Set of non-negative integers

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l_∞^e	Space of all real spatiotemporal sequences $f=f_i(t)$ with a 2-sided spatial support $(-\infty\leqslant i\leqslant\infty)$ and one sided temporal $(0\leqslant t\leqslant\infty)$
l_∞	Space of l_∞^e sequences with $\|f\|_\infty:=\sup_{i,t}\|f_i(t)\|<\infty$
$M_{i,t}$	Frozen LSTI system at i,t associated to the LSTV system M
$\hat{R}(z,\lambda)$	$:= z,\lambda$ transform of LSTI system R
$\|x\|_\infty$	$:=$ sup norm of a sequence $x\in l_\infty^e$
$\|M\|$	$:= l_\infty$-induced operator norm of operator M
$\|\hat{M}\|_\infty$	$:= \mathscr{H}_\infty$ (or spectral) norm of LSTI operator M
$A_{i,t}\nabla B_{i,t}$	$:= AB-A_{i,t}B_{i,t}$ i.e. the difference between the global and local product of operators given a pair (i,t)
GSTV	:= Gradually spatiotemporally varying system
ITSAE(M)	:= Integral time and space absolute error of M
LSTV	Linear spatiotemporally varying
LSTI	Linear spatiotemporally invariant

Note that for $f\in l_\infty^e$, we can represent it as an one-sided (causal) temporal (infinite) vector sequence as $f=\{f(0), f(1),\ldots\}$, where

$$f(t)=(\ldots, f_{-1}(t), f_0(t), f_{+1}(t),\ldots)', \quad t\in\mathbb{Z}^+ \tag{1}$$

and each $f_j(t)\in\mathbb{R}$, with $j\in\mathbb{Z}$.

2.2. *Spatiotemporal varying systems*

Linear spatiotemporally varying systems (LSTV) are systems $M: u\to y$ on l_∞^e given by the convolution

$$y_i(t)=\sum_{\tau=0}^{\tau=t}\sum_{j=-\infty}^{j=\infty} m_{i,i-j}(t,t-\tau)u_j(\tau) \tag{2}$$

where $\{m_{i,j}(t,\tau)\}$ is the kernel representation of M. These systems can be viewed as an infinite interconnection of different linear time-varying systems. For simplicity, we assume that each of these subsystems is single-input-single-output (SISO). Let $y_i=(y_i(0), y_i(1), y_i(2),\ldots)'$, then their corresponding input–output relationship of the ith block can be given as follows:

$$\begin{aligned}
\begin{pmatrix} y_i(0)\\ y_i(1)\\ y_i(2)\\ \vdots\end{pmatrix} &= \begin{pmatrix} m_{i,0}(0,0) & 0 & 0 & \cdots\\ m_{i,0}(1,0) & m_{i,0}(1,1) & 0 & \cdots\\ m_{i,0}(2,0) & m_{i,0}(2,1) & m_{i,0}(2,2) & \cdots\\ \cdots & \ddots & \ddots & \ddots\end{pmatrix}\begin{pmatrix} u_i(0)\\ u_i(1)\\ u_i(2)\\ \vdots\end{pmatrix}\\
&+\begin{pmatrix} m_{i,-1}(0,0) & 0 & 0 & \cdots\\ m_{i,-1}(1,0) & m_{i,-1}(1,1) & 0 & \cdots\\ m_{i,-1}(2,0) & m_{i,-1}(2,1) & m_{i,-1}(2,2) & \cdots\\ \cdots & \ddots & \ddots & \ddots\end{pmatrix}\begin{pmatrix} u_{i-1}(0)\\ u_{i-1}(1)\\ u_{i-1}(2)\\ \vdots\end{pmatrix}\\
&+\begin{pmatrix} m_{i,1}(0,0) & 0 & 0 & \cdots\\ m_{i,1}(1,0) & m_{i,1}(1,1) & 0 & \cdots\\ m_{i,1}(2,0) & m_{i,1}(2,1) & m_{i,1}(2,2) & \cdots\\ \cdots & \ddots & \ddots & \ddots\end{pmatrix}\begin{pmatrix} u_{i+1}(0)\\ u_{i+1}(1)\\ u_{i+1}(2)\\ \vdots\end{pmatrix}+\cdots
\end{aligned} \tag{3}$$

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where $\{u_i(t)\}$ is the input applied at the ith block with $u_i(t)\in\mathbb{R}$ and $t\in\mathbb{Z}^+$ is the time index, and $\{m_{i,j}(t,\tau)\}$ is the kernel representation of M. In addition, $\{y_i(t)\}$ is the output sequence of the ith block, with $y_i(t)\in\mathbb{R}$. We can write the overall input–output relationship for an LSTV system as follows:

$$\begin{pmatrix} y(0)\\ y(1)\\ y(2)\\ y(3)\\ \vdots \end{pmatrix}=\begin{pmatrix} M^{0,0} & & & & \\ M^{1,0} & M^{1,1} & & & \\ M^{2,0} & M^{2,1} & M^{2,2} & & \\ M^{3,0} & M^{3,1} & M^{3,2} & M^{3,3} & \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}\begin{pmatrix} u(0)\\ u(1)\\ u(2)\\ u(3)\\ \vdots \end{pmatrix} \tag{4}$$

where, $u(t)=(\ldots,u_{-1}(t),u_0(t),u_{+1}(t),\ldots)'$ and

$$M^{t,\tau}=\begin{pmatrix} \cdots & \vdots & \vdots & \vdots & \cdots\\ \cdots & m_{i-1,0}(t,\tau) & m_{i-1,1}(t,\tau) & m_{i-1,2}(t,\tau) & \cdots\\ \cdots & m_{i,-1}(t,\tau) & m_{i,0}(t,\tau) & m_{i,1}(t,\tau) & \cdots\\ \cdots & m_{i+1,-2}(t,\tau) & m_{i+1,-1}(t,\tau) & m_{i+1,0}(t,\tau) & \cdots\\ \cdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix} \tag{5}$$

where $t,\tau\in\mathbb{Z}^+$. The l_∞-induced operator norm on M in this case is given as

$$\|M\|=\sup_{i,t}\sum_{\tau=0}^{t}\sum_{j=-\infty}^{j=\infty}|m_{i,j}(t,\tau)| \tag{6}$$

The space of l_∞ bounded LSTV systems will be denoted as $\mathscr{L}_{\mathrm{STV}}$.

2.3. *Spatiotemporally invariant systems*

Linear spatially invariant systems are spatiotemporal systems $M:u\to y$ on l_∞^e given by the convolution

$$y_i(t)=\sum_{\tau=0}^{\tau=t}\sum_{j=-\infty}^{j=\infty}m_{i-j}(t-\tau)u_j(\tau) \tag{7}$$

where $\{m_i(t)\}$ is the pulse response of M. These systems can be viewed as an infinite array of interconnected linear time invariant (LTI) systems. The subspace of $\mathscr{L}_{\mathrm{STV}}$ that contains the stable LSTI systems will be denoted as $\mathscr{L}_{\mathrm{STI}}$. The induced l_∞ operator norm on M in this case is given as

$$\|M\|=\sum_{t=0}^{\infty}\sum_{i=-\infty}^{i=\infty}|m_i(t)| \tag{8}$$

2.4. *Frozen spatiotemporal systems*

Given an LSTV system M, consider the lower triangular representation of M as shown in (4). For any given pair (i,t) (where $i\in\mathbb{Z}$ represents a spatial coordinate, and $t\in\mathbb{Z}^+$ represents time), we

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can define an LSTI system $M_{i,t}$ as follows:

$$M_{i,t} := \begin{pmatrix} M_i^{t,t} & & & & & & \\ M_i^{t,t-1} & M_i^{t,t} & & & & & \\ \ddots & \dots & \ddots & & & & \\ M_i^{t,0} & M_i^{t,1} & \cdots & M_i^{t,t} & & & \\ M_i^{t+1,0} & M_i^{t,0} & M_i^{t,1} & \cdots & M_i^{t,t} & & \\ M_i^{t+2,0} & M_i^{t+1,0} & M_i^{t,0} & \cdots & \cdots & M_i^{t,t} & \\ \ddots & \ddots & \dots & \dots & \dots & \ddots & \ddots \end{pmatrix} \tag{9}$$

where each $M_i^{t,\tau}$ is frozen at the ith spatial coordinate, i.e. the ith row is picked in (5) and the matrix representation of $M_i^{t,\tau}$ is assumed to have a Toeplitz structure with respect to this row

$$M_i^{t,\tau} = \begin{pmatrix} \dots & \vdots & \vdots & \vdots & \dots \\ \cdots & m_{i,i}(t,\tau) & m_{i,i+1}(t,\tau) & m_{i,i+2}(t,\tau) & \cdots \\ \cdots & m_{i,i-1}(t,\tau) & m_{i,i}(t,\tau) & m_{i,i+1}(t,\tau) & \cdots \\ \cdots & m_{i,i-2}(t,\tau) & m_{i,i-1}(t,\tau) & m_{i,i}(t,\tau) & \cdots \\ \dots & \vdots & \vdots & \vdots & \dots \end{pmatrix} \tag{10}$$

This ensures that $(M_{i,t}u)_i(t) = (Mu)_i(t)$. We will refer to $M_{i,t}$ as the local or frozen system corresponding to the pair (i,t). Conversely, given a sequence of LSTI operators $\{M_{i,t}\}$, we will say that the LSTV system M is associated with (generated by) the family $M_{i,t}$ if $(M_{i,t}u)_i(t) = (Mu)_i(t)$. For LSTV systems A, B associated with the families $A_{i,t}$, $B_{i,t}$ of frozen LSTI operators, we define a global product AB to mean the usual composition. Given a pair (i,t), the local product of operators A, B corresponds to the product (composition) of the LSTI systems $A_{i,t}$ and $B_{i,t}$, i.e. $A_{i,t}B_{i,t}$.

2.5. *Support of m*

We define the support of a sequence $\{m_i(t)\}$ by Supp(m), i.e.

$$\mathrm{Supp}(m) = \{[i,t] \in \mathbb{Z}^2 : m_i(t) \neq 0\} \tag{11}$$

2.6. *Gradually varying spatiotemporal system*

A LSTV system A is said to be gradually space-time varying if given two pairs (i,t), and $(\mathtt{i},\tau)$, we have

$$\|A_{i,t} - A_{\mathtt{i},\tau}\| \leqslant \gamma(|i-\mathtt{i}| + |t-\tau|) \tag{12}$$

where $\gamma \in \mathbb{Z}^+$ is a constant. Such systems will be denoted by GSTV(γ).

2.7. *Integral time and space absolute error*

Given an LSTI system M, the integral time and space absolute error (ITSAE) is defined as

$$\mathrm{ITSAE}(M) = \sum_{t=0}^{\infty} \sum_{i=-\infty}^{i=\infty} (|i|+|t|)|m_i(t)| \tag{13}$$

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2.8. z, λ transform

We define z, λ transform for an LSTI SISO system M as

$$\hat{M}(z,\lambda)=\sum_{t=0}^{\infty}\sum_{k=-\infty}^{\infty}(m_k(t)z^k)\lambda^t \tag{14}$$

with the associated spectral or $\mathscr{H}_\infty$ norm

$$\|\hat{M}\|_\infty := \sup_{\theta,\omega}|\hat{M}(e^{i\theta}, e^{j\omega})| \tag{15}$$

It is well known (see e.g. [16]) that for an M in $\mathscr{L}_{\mathrm{STI}}$, M^{-1} is in $\mathscr{L}_{\mathrm{STI}}$ if and only if

$$\inf_{|z|=1,|\lambda|\leqslant 1}|\hat{M}(z,\lambda)|>0 \tag{16}$$

3. FROZEN SPACE-TIME CONTROL

Consider the general form of a closed-loop system given in Figure 1. The plant P is an LSTV recursively computable spatiotemporal system with the input–output relationship defined by an equation of the form

$$(A_{i,t}y_1)_i(t)=(B_{i,t}y_4)_i(t) \tag{17}$$

with $\{a_{i,j}(t,\tau)\}, \{b_{i,j}(t,\tau)\}$, being the kernel representations of the operators $A_{i,t}$, $B_{i,t}$ in $\mathscr{L}_{\mathrm{STI}}$ respectively. We can write the above equation explicitly as follows:

$$\sum_{\substack{j \quad \tau \\ (j,\tau)\in I_{a(i,t)}}}\sum a_{i,j}(t,\tau)y_{1,i-j}(t-\tau)=\sum_{\substack{j \quad \tau \\ (j,\tau)\in I_{b(i,t)}}}\sum b_{i,j}(t,\tau)y_{4,i-j}(t-\tau) \tag{18}$$

where $I_{a(i,t)}$ (output mask) and $I_{b(i,t)}$ (input mask) denote, respectively, the finite area region of support for $\{a_{i,j}(t,\tau)\}$ and $\{b_{i,j}(t,\tau)\}$. The system in (18) is well defined if $\{a_{i,0}(t,0)\}\neq 0$, and $\{a_{i,j}(t,\tau)\}\neq 0$ for some (j,τ), and $\mathrm{Supp}(\{a_{i,j}(t,\tau)\})$ is a subset of the lattice sector with vertex (0,0) of angle less than 180°, for every pair (i,t) [16]. We will assume that all the spatiotemporal systems under consideration are well defined. We note here that the recursively computable system formulation is not an essential assumption for the developments that follow. Indeed, as long as the LSTV operator A associated with $A_{i,t}$ through the relationship $(Ay_1)_i(t)=(A_{i,t}y_1)_i(t)$ is invertible, then (17) will represent a well-defined system. Given this property for the systems involved, the results that follow still hold. It should also be noted though that a general LSTV system may not posses this 'coprime factor' from of (17). In this sense, the paper is limited to LSTV systems described by (17), which represents a rich class on its own right.

The control law is designed on the basis of frozen time and frozen space plants. Given an instance in space and time, the plant is thought of as an LSTI system, with the defining operators fixed at that time and space. The controllers are designed for the corresponding frozen LSTI system. Allowing ourselves the flexibility of using a designed controller for several instances in time and

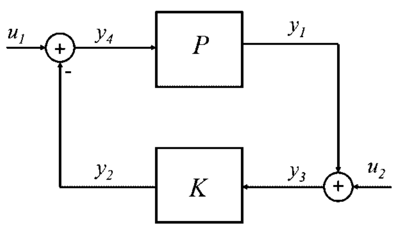

Figure 1. General form of a closed loop.

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space, we will consider the controller design every T steps in time and every S steps in space. Define $n_t = nT$ and $k_i = kS$, where n and k are smallest integers such that t and i lie in the interval $[nT,\ (n+1)T]$ and $[kS,\ (k+1)S]$, respectively. The controller is designed at intervals of nT, and kS in time and space, respectively. We note here that this abstract setting also addresses the control problem of distributed systems where each subsystem has knowledge of its own dynamics, but other wise only knows that the rest of the subsystems are 'similar'. The closed loop is stable if the map from u_1, u_2 to y_1, y_2 is bounded. The dynamics of the control law is given by

$$(L_{k_i,n_t} y_2)_i(t) = (M_{k_i,n_t} y_3)_i(t) \tag{19}$$

where $L_{k_i,n_t},\ M_{k_i,n_t} \in \mathscr{L}_{\text{STI}}$ for each pair of indices (k_i, n_t). The evolution of these operators is given by

$$(L_{k_i,n_t} y_2)_i(t) = \sum_{\substack{j \quad \tau \\ (j,\tau)\in I_{l(k_i,n_t)}}}\sum l_{k_i,i-j}(n_t, t-\tau) y_{2,j}(\tau) \tag{20}$$

$$(M_{k_i,n_t} y_3)_i(t) = \sum_{\substack{j \quad \tau \\ (j,\tau)\in I_{m(k_i,n_t)}}}\sum m_{k_i,i-j}(n_t, t-\tau) y_{3,j}(\tau) \tag{21}$$

where $I_{l(k_i,n_t)}$ and $I_{m(k_i,n_t)}$ denote, respectively, the finite area region of support for $\{l_{k_i,j}(n_t,\tau)\}$ and $\{m_{k_i,j}(n_t,\tau)\}$. The frozen space and time operator that defines the above control law satisfies the following Bezout identity:

$$L_{k_i,n_t} A_{k_i,n_t} + M_{k_i,n_t} B_{k_i,n_t} = G_{k_i,n_t} \tag{22}$$

where $G^{-1}_{k_i,n_t} \in \mathscr{L}_{\text{STI}}$ for each fixed pair (k_i, n_t). That is, for every frozen plant given by A_{k_i,n_t}, B_{k_i,n_t}, the control generated by L_{k_i,n_t}, M_{k_i,n_t} is such that the 'frozen' closed-loop map $G^{-1}_{k_i,n_t}$ is stable. Note that the frozen plant is an LSTI, and hence a frozen LSTI controller that satisfies the frozen closed loop can be obtained using various methods, e.g. [1, 2]. Here, we are not interested in any specific method. We only require that K operates as described above and provides frozen stability.

We investigate the interaction of the gradual variations of the plant and the controller in time as well as in space. In particular, we show how these variations play their role with regard to the stability of the closed loop. The fact that the controller is updated only every T steps in time and after every S number of plants in space introduces a new parameter in the stability analysis. In the sequel, we show as to how large T and S can be without endangering the stability of the closed-loop system. Intuitively, the larger the T and S are, the slower the plant variations should be in the respective time and spatial domains. On the other hand, for the extreme case such that $T \to \infty$, the system should be time invariant, and for $S \to \infty$, the system should be spatially invariant. Equivalently, if we require $S, T \to \infty$ simultaneously, the system should be an LSTI. From Figure 1, we can write down the closed-loop equations for the controlled system as follows:

$$(A_{i,t} y_1)_i(t) = (B_{i,t}(u_1 - y_2))_i(t) \tag{23}$$

$$(L_{k_i,n_t} y_2)_i(t) = (M_{k_i,n_t}(u_2 + y_1))_i(t) \tag{24}$$

$$L_{k_i,n_t} A_{k_i,n_t} + M_{k_i,n_t} B_{k_i,n_t} = G_{k_i,n_t} \tag{25}$$

In the following, we obtain a relation that connects the input sequences $\{u_{1,i}(t)\}$, $\{u_{2,i}(t)\}$ to the outputs $\{y_{1,i}(t)\}$ and $\{y_{2,i}(t)\}$. Operating on Equation (23) by L_{k_i,n_t}, we get

$$(LAy_1)_i(t) = (LBu_1)_i(t) - (LBy_2)_i(t) \tag{26}$$

Similarly, operating on Equation (24) by $B_{i,t}$, we get

$$(BLy_2)_i(t) = (BM(u_2 + y_1))_i(t) \tag{27}$$

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Adding Equations (26) and (27) together and adding and subtracting $(L_{k_i,n_t}B_{i,t}y_2)_i(t)$, $(L_{k_i,n_t}A_{i,t}y_1)_i(t)$, $(B_{i,t}M_{k_i,n_t}y_1)_i(t)$, $(L_{k_i,n_t}A_{k_i,n_t}y_1)_i(t)$, and $(B_{k_i,n_t}M_{k_i,n_t}y_1)_i(t)$ and grouping certain terms we get

$$\begin{aligned}
&\{(L_{k_i,n_t}A_{k_i,n_t}+B_{k_i,n_t}M_{k_i,n_t})y_1+(L_{k_i,n_t}\nabla A_{i,t}+(L_{k_i,n_t}A_{i,t}-L_{k_i,n_t}A_{k_i,n_t})+B_{i,t}\nabla M_{k_i,n_t}\\
&\quad+(B_{i,t}M_{k_i,n_t}-B_{k_i,n_t}M_{k_i,n_t}))y_1+(L_{k_i,n_t}\nabla B_{i,t}-B_{i,t}\nabla L_{k_i,n_t})y_2\}_i(t)\\
&=(LBu_1)_i(t)-(BMu_2)_i(t)
\end{aligned}\tag{28}$$

where we have used the notation; $A_{i,t}\nabla B_{i,t}=AB-A_{i,t}B_{i,t}$, i.e. $A_{i,t}\nabla B_{i,t}$ is the difference between the global and local product of operators given a pair (i,t). To obtain the second closed-loop equation, operate on Equation (23) by M_{k_i,n_t}

$$(MAy_1)_i(t)=(MBu_1)_i(t)-(MBy_2)_i(t)\tag{29}$$

Similarly, operate on Equation (24) by $A_{i,t}$, we get

$$(ALy_2)_i(t)=(AM(u_2+y_1))_i(t)\tag{30}$$

Again adding Equations (29) and (30) together and adding and subtracting $(M_{k_i,n_t}B_{k_i,n_t}y_2)_i(t)$, $(A_{k_i,n_t}L_{k_i,n_t}y_2)_i(t)$, $(M_{k_i,n_t}B_{i,t}y_2)_i(t)$, $(A_{i,t}L_{k_i,n_t}y_2)_i(t)$, and $(A_{i,t}M_{k_i,n_t}y_1)_i(t)$ and grouping certain terms we get

$$\begin{aligned}
&\{(M_{k_i,n_t}B_{k_i,n_t}+A_{k_i,n_t}L_{k_i,n_t})y_2+(M_{k_i,n_t}\nabla B_{i,t}+(M_{k_i,n_t}B_{i,t}-M_{k_i,n_t}B_{k_i,n_t})+A_{i,t}\nabla L_{k_i,n_t}\\
&\quad+(A_{i,t}L_{k_i,n_t}-A_{k_i,n_t}L_{k_i,n_t}))y_2+(A_{i,t}\nabla M_{k_i,n_t}-M_{k_i,n_t}\nabla A_{i,t})y_1\}_i(t)\\
&=(MBu_1)_i(t)+(AMu_2)_i(t)
\end{aligned}\tag{31}$$

For $t\in\mathbb{Z}^+$, $i\in\mathbb{Z}$, define the following:

$$X_{i,t}=L_{k_i,n_t}\nabla A_{i,t}+(L_{k_i,n_t}A_{i,t}-L_{k_i,n_t}A_{k_i,n_t})+B_{i,t}\nabla M_{k_i,n_t}+(B_{i,t}M_{k_i,n_t}-B_{k_i,n_t}M_{k_i,n_t})\tag{32}$$

$$Y_{i,t}=L_{k_i,n_t}\nabla B_{i,t}-B_{i,t}\nabla L_{k_i,n_t}\tag{33}$$

$$Z_{i,t}=M_{k_i,n_t}\nabla A_{i,t}-A_{i,t}\nabla M_{k_i,n_t}\tag{34}$$

$$W_{i,t}=M_{k_i,n_t}\nabla B_{i,t}+(M_{k_i,n_t}B_{i,t}-M_{k_i,n_t}B_{k_i,n_t})+A_{i,t}\nabla L_{k_i,n_t}+(A_{i,t}L_{k_i,n_t}-A_{k_i,n_t}L_{k_i,n_t})\tag{35}$$

Denote by X,Y,Z,W,G the spatiotemporally varying operators associated with the families $X_{i,t}$, $Y_{i,t}$, $Z_{i,t}$, $W_{i,t}$, G_{k_i,n_t}, respectively. Using (25) we can write the closed-loop equation as follows:

$$\left(\begin{pmatrix}G+X & Y\\ -Z & G+W\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}\right)_i(t)=\left(\begin{pmatrix}LB & -BM\\ MB & AM\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix}\right)_i(t)\tag{36}$$

The idea is to analyze the above system by considering the operators X, Y, Z, W as perturbations. We state below the main result of this paper regarding stability of the system given in (36). We prove this result in the following section.

Theorem 1

Assume the following for system (36):

(A1) The operators defining the plant are gradually time and space varying with rates γ_A and γ_B, i.e. $A_{i,t}\in$ GSTV(γ_A), and $B_{i,t}\in$ GSTV(γ_B).

(A2) The sequence of controllers is gradually time and space varying, i.e. $M_{k_i,n_t}\in$ GSTV(γ_M) and $L_{k_i,n_t}\in$ GSTV(γ_L).

(A3) The l_∞-induced norms and the ITSAE of the operators $A_{i,t}$, $B_{i,t}$, L_{k_i,n_t}, M_{k_i,n_t} are uniformly bounded in i, and t. From this and A1, A2, and the Bezout identity it follows that the operator G_{k_i,n_t} will also be gradually varying in space and time and, hence, we can write $G_{k_i,n_t}\in$ GSTV(γ_G).

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(A4) The l_∞ to l_∞ norms and the ITSAE of the LSTI operators $G^{-1}_{k_i,n_t}$ are bounded uniformly in i, and t.

Then there exists a non-zero constant γ such that if $\gamma_A, \gamma_B, \gamma_M, \gamma_L, \gamma_G \leqslant \gamma$, the closed-loop system is internally stable.

4. STABILITY ANALYSIS

In this section, we study the stability of the closed-loop system arising from the frozen time and space control design. From Equation (36), we see that the map G_{k_i,n_t} is perturbed by a few operators, each of which falls in one of the two categories:

(1) $A_{i,t}\nabla L_{k_i,n_t}$.
(2) $L_{k_i,n_t}(A_{i,t}-A_{k_i,n_t})$.

In the following lemmas, we show how the l_∞-induced norms of these operators can be made small by controlling the rates of spatiotemporal variations involved in the problem at hand.

Lemma 1
Let $L_{k_i,n_t} \in \mathrm{GSTV}(\gamma_L)$, and $A_{i,t} \in \mathrm{GSTV}(\gamma_A)$ and R denote the varying spatiotemporal operator associated with $A_{i,t}\nabla L_{k_i,n_t}$. Then, $R \in \mathscr{L}_{\mathrm{STV}}$ and its induced norm satisfies

$$\begin{aligned}\|R\| &= \sup_{i,t}\|A_{i,t}\nabla L_{k_i,n_t}\|\\ &\leqslant \gamma_L\left(2(S+T)\sup_{i,t}\sum_{\tau=0}^{t}\sum_{j=-\infty}^{\infty}|a_{i,j}(t,\tau)| + \sup_{i,t}\sum_{\tau=0}^{t}\sum_{j=-\infty}^{\infty}|j|\,|a_{i,j}(t,\tau)|\right.\\ &\qquad\left. + \sup_{i,t}\sum_{\tau=0}^{t}\sum_{j=-\infty}^{\infty}\tau|a_{i,j}(t,\tau)|\right)\end{aligned} \tag{37}$$

Proof
Let $u \in l_\infty$, then the operator $A_{i,t}\nabla L_{k_i,n_t}$ acts on u as follows:

$$\begin{aligned}A_{i,t}\nabla L_{k_i,n_t}u_i(t) &= \sum_{\tau=0}^{t}\sum_{j=-\infty}^{\infty}a_{i,i-j}(t,t-\tau)\\ &\quad\times\sum_{r=0}^{\tau}\sum_{s=-\infty}^{\infty}(l_{k_j,j-s}(n_\tau,\tau-r)-l_{k_i,i-s}(n_t,\tau-r))u_s(r)\end{aligned} \tag{38}$$

Taking the absolute value of the above equation we get:

$$|A_{i,t}\nabla L_{k_i,n_t}u_i(t)| \leqslant \sum_{\tau=0}^{t}\sum_{j=-\infty}^{\infty}|a_{i,i-j}(t,t-\tau)| \times \sum_{r=0}^{\tau}\sum_{s=-\infty}^{\infty}|l_{k_j,j-s}(n_\tau,\tau-r)-l_{k_i,i-s}(n_t,\tau-r)|\|u\|_\infty \tag{39}$$

$$= \sum_{\tau=0}^{t}\sum_{j=-\infty}^{\infty}|a_{i,i-j}(t,t-\tau)| \times \sum_{r=0}^{\tau}\sum_{s=-\infty}^{\infty}|l_{k_j,j-s}(n_\tau,r)-l_{k_i,i-s}(n_t,r)|\|u\|_\infty \tag{40}$$

$$\leqslant \sum_{\tau=0}^{t}\sum_{j=-\infty}^{\infty}|a_{i,i-j}(t,t-\tau)|\|L_{k_j,n_\tau}-L_{k_i,n_t}\|\|u\|_\infty \tag{41}$$

$$\leqslant \sum_{\tau=0}^{t}\sum_{j=-\infty}^{\infty}|a_{i,i-j}(t,t-\tau)|\gamma_L(|k_j-k_i|+|n_\tau-n_t|)\|u\|_\infty \tag{42}$$

Now

$$|k_j - k_i| + |n_\tau - n_t| = |k_j - j + j - i + i - k_i| + |n_\tau - \tau + \tau - t + t - n_t| \tag{43}$$

$$\leqslant 2S + 2T + |j - i| + |\tau - t| \tag{44}$$

since, $|k_j - j| \leqslant S$, and $|n_\tau - \tau| \leqslant T$. The above inequality can now be written as:

$$|A_{i,t} \nabla L_{k_i,n_t} u_i(t)| \leqslant \sum_{\tau=0}^{t} \sum_{j=-\infty}^{\infty} |a_{i,j}(t,\tau)| \gamma_L (2S + 2T + |j-i| + |\tau - t|) \|u\|_\infty \tag{45}$$

$$\leqslant \gamma_L \left(2(T+S) \sum_{\tau=0}^{t} \sum_{j=-\infty}^{\infty} |a_{i,i-j}(t, t-\tau)| + \sum_{\tau=0}^{t} \sum_{j=-\infty}^{\infty} |j|\,|a_{i,j}(t,\tau)| + \sum_{\tau=0}^{t} \sum_{j=-\infty}^{\infty} \tau |a_{i,j}(t,\tau)| \right) \|u\|_\infty \tag{46}$$

□

Lemma 2

Let the assumptions in Lemma 1 hold. Let R now denote the varying spatiotemporal operator associated with the family of $L_{k_i,n_t}(A_{i,t} - A_{k_i,n_t})$, then $R \in$ LSTV and its induced norm satisfies

$$\|R\| = \sup_{i,t} \|L_{k_i,n_t}(A_{i,t} - A_{k_i,n_t})\| \leqslant \gamma_A (T+S) \sup_{i,t} \sum_{\tau=0}^{t} \sum_{j=-\infty}^{\infty} |l_{k_i,j}(n_t,\tau)| \tag{47}$$

Proof

The proof follows in a similar fashion as above and is hence omitted. □

We now proceed to present the proof of Theorem 1.

Proof of Theorem 1

Consider the first equation in (36), expressed in operator form

$$G y_1 + X y_1 + Y y_2 = v \tag{48}$$

where $v_i(t) = (LBu_1)_i(t) - (BMu_2)_i(t)$. Let $(\mathtt{i}, \tau)$ be a fixed instance in space and time, we can write

$$G_{k_\mathtt{i},n_\tau} y_1 + (G - G_{k_\mathtt{i},n_\tau}) y_1 + X y_1 + Y y_2 = v \tag{49}$$

where $G_{k_\mathtt{i},\tau} \in \mathscr{L}_{STI}$. Denote by $H_{k_\mathtt{i},\tau}$ the inverse of $G_{k_\mathtt{i},\tau}$. By assumption (A4), $H_{k_\mathtt{i},\tau} \in \mathscr{L}_{STI}$. The above equation can, therefore, be written as

$$y_1 + H_{k_\mathtt{i},n_\tau}(G - G_{k_\mathtt{i},n_\tau}) y_1 + H_{k_\mathtt{i},n_\tau} X y_1 + H_{k_\mathtt{i},n_\tau} Y y_2 = H_{k_\mathtt{i},n_\tau} v \tag{50}$$

Evaluating the above operator equation at $(\mathtt{i}, \tau)$ we obtain

$$y_{1,\mathtt{i}}(\tau) + (H_{k_\mathtt{i},n_\tau}(G - G_{k_\mathtt{i},n_\tau}) y_1)_\mathtt{i}(\tau) + (H_{k_\mathtt{i},n_\tau} X y_1)_\mathtt{i}(\tau) + (H_{k_\mathtt{i},n_\tau} Y y_2)_\mathtt{i}(\tau) = (H_{k_\mathtt{i},n_\tau} v)_\mathtt{i}(\tau) \tag{51}$$

Similarly, we can write

$$-(H_{k_\mathtt{i},n_\tau} Z y_1)_\mathtt{i}(\tau) + y_{2,\mathtt{i}}(\tau) + (H_{k_\mathtt{i},n_\tau}(G - G_{k_\mathtt{i},n_\tau}) y_2)_\mathtt{i}(\tau) + (H_{k_\mathtt{i},n_\tau} W y_2)_\mathtt{i}(\tau) = (H_{k_\mathtt{i},n_\tau} w)_\mathtt{i}(\tau) \tag{52}$$

where $w_i(t) = (MBu_1)_i(t) + (AMu_2)_i(t)$. Combining the above equations, we get the following closed-loop system:

$$\left((I+F) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right)_\mathtt{i} (\tau) = \begin{pmatrix} H_{k_\mathtt{i},n_\tau} v \\ H_{k_\mathtt{i},n_\tau} w \end{pmatrix}_\mathtt{i} (\tau) \tag{53}$$

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where

$$F=\begin{pmatrix} H_{k_{\mathrm{i}},n_\tau}(G-G_{k_{\mathrm{i}},n_\tau})+H_{k_{\mathrm{i}},n_\tau}X & H_{k_{\mathrm{i}},n_\tau}Y \\ -H_{k_{\mathrm{i}},n_\tau}Z & H_{k_{\mathrm{i}},n_\tau}(G-G_{k_{\mathrm{i}},n_\tau})+H_{k_{\mathrm{i}},n_\tau}W \end{pmatrix} \tag{54}$$

The idea is to show that the induced norm of the spatiotemporally varying perturbing operator F can be made less than one by choosing the rates of variations sufficiently small. From the previous lemmas, and the fact that $H_{k_{\mathrm{i}},n_\tau}$ is uniformly bounded, it is clear that each of the spatiotemporally varying operators generated from each family of operators $H_{k_{\mathrm{i}},n_\tau}X$, $H_{k_{\mathrm{i}},n_\tau}Y$, $H_{k_{\mathrm{i}},n_\tau}Z$, $H_{k_{\mathrm{i}},n_\tau}W$, have induced norms that are controlled by the rates of variation γ_A, γ_B, γ_L, γ_M, γ_G. The internal stability will follow from the small gain theorem, if we show that the induced norm of the operator $H_{k_{\mathrm{i}},n_\tau}(G-G_{k_{\mathrm{i}},n_\tau})$ can be analogously controlled. We present in the following a calculation of an upper bound of the norm of the operator $H_{k_{\mathrm{i}},n_\tau}(G-G_{k_{\mathrm{i}},n_\tau})$. Let $y\in l_\infty$ and the output of the system be x, then

$$\begin{aligned} x_{\mathrm{i}}(\tau)=\sum_{m=0}^{\tau}\sum_{j=-\infty}^{\infty} h_{k_{\mathrm{i}},\mathrm{i}-j}(n_\tau,\tau-m)\times\sum_{r=0}^{m}\sum_{s=-\infty}^{\infty}(g_{k_j,j-s}(n_m,m-r) \\ -g_{k_{\mathrm{i}},\mathrm{i}-s}(n_\tau,m-r))y_s(r) \end{aligned} \tag{55}$$

Taking absolute values we get,

$$\begin{aligned} |x_{\mathrm{i}}(\tau)|\leqslant\sum_{m=0}^{\tau}\sum_{j=-\infty}^{\infty}|h_{k_{\mathrm{i}},\mathrm{i}-j}(n_\tau,\tau-m)|\times\sum_{r=0}^{m}\sum_{s=-\infty}^{\infty}|g_{k_j,j-s}(n_m,m-r) \\ -g_{k_{\mathrm{i}},\mathrm{i}-s}(n_\tau,m-r)|\|y\|_\infty \end{aligned} \tag{56}$$

By an argument similar to the one given in the proof of Lemma 1, it follows that

$$\begin{aligned} \|x\|_\infty\leqslant\gamma_G\Bigg(2(T+S)\sup_{\mathrm{i},\tau}\sum_{m=0}^{\tau}\sum_{j=-\infty}^{\infty}|h_{k_{\mathrm{i}},j}(n_\tau,m)|+\sup_{\mathrm{i},\tau}\sum_{m=0}^{\tau}\sum_{j=-\infty}^{\infty}m|h_{k_{\mathrm{i}},j}(n_\tau,m)| \\ +\sup_{\mathrm{i},\tau}\sum_{m=0}^{\tau}\sum_{j=-\infty}^{\infty}|j||h_{k_{\mathrm{i}},j}(n_\tau,m)|\Bigg)\|y\|_\infty \end{aligned} \tag{57}$$

It follows by assumption (A4) that there exist constants C_1, $C_2\geqslant 0$ such that

$$\|x\|_\infty\leqslant\gamma_G(2(S+T)C_1+C_2+C_3)\|y\|_\infty \tag{58}$$

We have, hence, shown that the induced norms of all the perturbing operators that comprise F can be made small by choosing the rates of variations small enough. This, in turn, implies that the induced norm of F can be made arbitrarily small. Now, from the small gain theorem we know that $(I+F)^{-1}$ must exist (and is stable) if $\|F\|<1$. Hence, by choosing γ sufficiently small, we can ensure that $\|F\|<1$. The application of small gain theorem then implies that $(I+F)^{-1}$ exists (and is stable) and that $y_1(t)$, $y_2(t)$ are bounded, for bounded input, hence concluding the proof of Theorem 1. □

Theorem 1 shows that if the assumptions (A1–A4) are satisfied and if the variations are small enough, then the closed-loop system will be l_∞ stable. The assumptions (A1 and A2) are quite reasonable and are typically satisfied for the recursively computable spatiotemporal system that we focus on. The first part of assumption (A3), requiring uniform bounds on the operators, is also quite reasonable. Intuitively, the second part of assumption (A3), that requires uniform bounds on the ITSAE of operators, implies that the LTV building blocks of the LSTV system have a decaying memory (temporal), and decaying spatial dependence on the neighbors (as one goes away from the reference in space). Assumption (A4), however, is harder to satisfy. This assumption implies

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that the zeros in λ of $\hat{G}_{i,t}(e^{j\theta}, \lambda)$, lie outside a disc of radius $1+\varepsilon$, for some $\varepsilon>0$ and for all θ. Precisely, this assumption implies:

$$\inf_{|z|=1,|\lambda|\leqslant 1} |\hat{G}_{i,t}(z,\lambda)| \geqslant \delta > 0 \quad \forall i \quad \forall t \tag{59}$$

Satisfying this condition only, however, does not imply that the l_∞-induced norms and the ITSAE of $G_{i,t}^{-1}$ are uniformly bounded, i.e. assumption (A4) is not satisfied in general. The above spectral condition does imply that the $\mathscr{H}_\infty$ norm of $G_{i,t}^{-1}$ is uniformly bounded in i, and t, since:

$$\|\hat{G}_{i,t}^{-1}\|_\infty = \sup_{|z|=1,|\lambda|\leqslant 1} \frac{1}{|\hat{G}_{i,t}(z,\lambda)|} = \frac{1}{\inf_{|z|=1,|\lambda|\leqslant 1} |\hat{G}_{i,t}(z,\lambda)|} \leqslant \frac{1}{\delta} \quad \forall i \quad \forall t \tag{60}$$

In the following, we show that with some additional mild assumptions on the $\mathscr{H}_\infty$ norm of $\hat{G}_{i,t}(z,\lambda)$ and its partial derivatives, the spectral condition is enough to verify the uniform bounds on the l_∞-induced norms and the ITSAE of $G_{i,t}^{-1}$. For partial derivatives, we define the notation $\hat{G}_{i,t,(\xi)} = \partial \hat{G}_{i,t}/\partial\xi$, $\hat{G}_{i,t,(\xi\zeta)} = \partial^2 \hat{G}_{i,t}/\partial\xi\partial\zeta$, $\hat{G}_{i,t,(\xi\xi\zeta)} = \partial^3 \hat{G}_{i,t}/\partial\xi\partial\xi\partial\zeta$, where ξ, ζ can be z or λ.

Theorem 2

Given the following conditions:

(1) $\|\hat{G}_{i,t}\|_\infty \leqslant M_1, \quad \forall i,t.$
(2) $\|\hat{G}_{i,t,(z)}\|_\infty \leqslant M_2, \quad \forall i,t.$
(3) $\|\hat{G}_{i,t,(\lambda)}\|_\infty \leqslant M_3, \quad \forall i,t.$
(4) $\|\hat{G}_{i,t,(zz)}\|_\infty \leqslant M_4, \quad \forall i,t.$
(5) $\|\hat{G}_{i,t,(\lambda\lambda)}\|_\infty \leqslant M_5, \ \forall i,t.$
(6) $\|\hat{G}_{i,t,(z\lambda)}\|_\infty \leqslant M_6, \quad \forall i,t.$
(7) $\|\hat{G}_{i,t,(zz\lambda)}\|_\infty \leqslant M_7, \quad \forall i,t.$
(8) $\|\hat{G}_{i,t,(\lambda zz)}\|_\infty \leqslant M_8, \quad \forall i,t.$
(9) $\inf_{|z|=1,\lambda\leqslant 1} |\hat{G}_{i,t}(z,\lambda)| \geqslant \delta > 0, \quad \forall i,t$ (spectral condition).

Then, the l_∞-induced norm and ITSAE of $G_{i,t}^{-1}$ are uniformly bounded in i, and t.

Proof

The proof is based on Hardy's theorem [21], and its extension for two variables. We present Hardy's theorem in one variable along with its extension form in two variables without proof in the following.

Hardy's Theorem: Given a function $\hat{R} \in \mathscr{H}_\infty$ with

$$\hat{R}(\lambda) = \sum_{t=0}^{\infty} m(t)\lambda^t \tag{61}$$

there exists a constant $0 < C < +\infty$ such that the coefficients $m(t)$ satisfy:

$$\sum_{t=1}^{\infty} \frac{1}{t} |m(t)| \leqslant C \|\hat{R}\|_\infty \tag{62}$$

Similarly, given a function $\hat{R} \in \mathscr{H}_\infty$ with

$$\hat{R}(z,\lambda) = \sum_{t=0}^{\infty} \sum_{k=-\infty}^{\infty} (m_k(t) z^k) \lambda^t \tag{63}$$

then there exists a constant $0 < C < +\infty$ such that the coefficients $m_k(t)$ satisfy:

$$\sum_{t=1}^{\infty} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{1}{t|k|} |m_k(t)| \leqslant C \|\hat{R}\|_\infty \tag{64}$$

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To show that $\|G_{i,t}^{-1}\|$ is uniformly bounded, we apply Hardy's theorem on $\hat{R}(z,\lambda) = \hat{G}^{-1}_{i,t,(z\lambda)}$. Note that

$$\frac{\partial^2 \hat{G}_{i,t}^{-1}}{\partial z \partial \lambda} = \frac{-\hat{G}_{i,t,(z\lambda)}\hat{G}_{i,t} + 2\hat{G}_{i,t,(z)}\hat{G}_{i,t,(\lambda)}}{\hat{G}_{i,t}^3} \tag{65}$$

Hence,

$$\|\hat{R}\|_\infty \leqslant \frac{\|\hat{G}_{i,t,(z\lambda)}\|_\infty \|\hat{G}_{i,t}\|_\infty + 2\|\hat{G}_{i,t,(z)}\|_\infty \|\hat{G}_{i,t,(\lambda)}\|_\infty}{\delta^3} \leqslant \frac{M_9}{\delta^3} \tag{66}$$

where $M_9 := M_6 M_1 + 2M_2 M_3$. Let $\hat{G}_{i,t}^{-1}$ be given by:

$$\hat{G}_{i,t}^{-1} = \sum_{\tau=0}^{\infty} \sum_{k=-\infty}^{\infty} (h_{i,k}(t,\tau) z^k) \lambda^\tau \tag{67}$$

Then,

$$\hat{R}_{i,t} = \sum_{\tau=0}^{\infty} \sum_{k=-\infty}^{\infty} \tau k (h_{i,k}(t,\tau) z^{k-1}) \lambda^{\tau-1} \tag{68}$$

Note that $z\lambda\hat{R}_{i,t} \in H_\infty$, since it is a bounded composition of analytical function. We apply Hardy's theorem on $z\lambda\hat{R}_{i,t}$ as follows:

$$\sum_{\tau=1}^{\infty} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} |h_{i,k}(t,\tau)| \leqslant C \frac{M_9}{\delta} \tag{69}$$

Note that in the last expression, we are missing the terms $\sum_{k=-\infty}^{\infty} |h_{i,k}(t,0)|$, and $\sum_{\tau=0}^{\infty} |h_{i,0}(t,\tau)|$ to establish a bound on $\|G_{i,t}^{-1}\|$. Let

$$\hat{h}1 = \sum_{k=-\infty}^{\infty} h_{i,k}(t,0) z^k, \quad \hat{h}2 = \sum_{\tau=1}^{\infty} h_{i,0}(t,\tau) \lambda^\tau \tag{70}$$

Since we have

$$\hat{h}1 + \hat{h}2 + \sum_{\tau=1}^{\infty} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} (h_{i,k}(t,\tau) z^k) \lambda^\tau = \hat{G}_{i,t}^{-1} \tag{71}$$

Also

$$\|\hat{h}1\|_\infty = \sup_\theta |\hat{G}_{i,t}^{-1}(z = e^{i\theta}, \lambda = 0)| \leqslant \|\hat{G}_{i,t}^{-1}\|_\infty < +\infty \tag{72}$$

and from (69) we have

$$\sum_{\tau=1}^{\infty} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} |h_{i,k}(t,\tau)| < +\infty \tag{73}$$

we must have $\|\hat{h}2\|_\infty < +\infty$. Also the $\mathscr{H}_\infty$ norms of $\hat{h}1_{(z)}$, and $\hat{h}2_{(\lambda)}$ are finite since $\|\hat{R}_{i,t}\|_\infty$ is finite. Using Hardy's theorem again and reasoning in a similar fashion as above, we can establish the bounds on, $\sum_{k=-\infty k\neq 0}^{\infty} |h_{i,k}(t,0)|$, and $\sum_{\tau=1}^{\infty} |h_{i,0}(t,\tau)|$. Let the sum of their bounds be denoted by C_h. We now have the following bound for $\|G_{i,t}^{-1}\|$:

$$\|G_{i,t}^{-1}\| \leqslant C \frac{M_9}{\delta} + C_h + |h_{i,0}(t,0)| \leqslant C \frac{M_9}{\delta} + C_h + \delta^{-1} \tag{74}$$

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A similar argument works for ITSAE($G_{i,t}^{-1}$) by considering $G_{i,t,(zz\lambda)}^{-1}$, and $G_{i,t,(\lambda\lambda z)}^{-1}$. We omit the details as they follow in the footsteps of the above argument. □

The benefit of the above theorem is that one can check if assumption (A4) is satisfied by checking easily computable $\mathscr{H}_\infty$ norms.

Theorem 3
Let the assumptions (A1–A3), and the spectral condition (9, Theorem 2) hold along with the uniform boundedness of the following quantities:

- $\|\hat{A}_{i,t,(zz)}\|_\infty$, $\|\hat{A}_{i,t,(zz\lambda)}\|_\infty$, $\|\hat{A}_{i,t,(\lambda\lambda z)}\|_\infty$
- $\|\hat{B}_{i,t,(zz)}\|_\infty$, $\|\hat{B}_{i,t,(zz\lambda)}\|_\infty$, $\|\hat{B}_{i,t,(\lambda\lambda z)}\|_\infty$
- $\|\hat{L}_{i,t,(zz)}\|_\infty$, $\|\hat{L}_{i,t,(zz\lambda)}\|_\infty$, $\|\hat{L}_{i,t,(\lambda\lambda z)}\|_\infty$
- $\|\hat{M}_{i,t,(zz)}\|_\infty$, $\|\hat{M}_{i,t,(zz\lambda)}\|_\infty$, $\|\hat{M}_{i,t,(\lambda\lambda z)}\|_\infty$

Then, the closed-loop system (36) is stable.

Proof
The proof is straightforward since the conditions in Theorem 2 will be satisfied from the Bezout identity relating $\hat{G}_{i,t}$ to the above quantities. □

5. PERFORMANCE ANALYSIS

In this section, we seek a relationship between the performance of the frozen time pair $(P_{i,t}, K_{k_i,n_t})$ and the actual time-varying feedback pair (P, K). This is addressed in the following theorem.

Theorem 4
Let S^{lk} $(k=1,2,\ l=1,2,3,4)$ represent the map from u_k to y_l in the system of Figure 1 and $S_{i,t}^{lk}$ the LSTI map from u_k to y_l for the frozen system $(P_{i,t}, K_{k_i,n_t})$. Now, let the assumptions of Theorem 1 hold. Given $0<\varepsilon<1$, there exists a non-zero constant γ_p with $\gamma\leqslant\gamma_p$ such that

$$(1-\varepsilon)\|S^{lk}\|\leqslant\sup_{i,t}\|S_{i,t}^{lk}\|+\varepsilon$$

Proof
Let $u_1=0$ and $\|u_2\|\leqslant 1$. From the system equations, we get

$$y_{1,i}(t)=-(H_{k_i,n_t}(G-G_{k_i,n_t})y_1)_i(t)-(H_{k_i,n_t}Xy_1)_i(t)-(H_{k_i,n_t}Yy_2)_i(t)-(H_{k_i,n_t}(BMu_2))_i(t) \tag{75}$$

Consider now the frozen LSTI feedback system given a pair (i,t) and subjected to the same input u_2. Let $\hat{y}_1$ denote the output that corresponds to y_1 in the time-varying loop. Evaluating $\hat{y}_1$ at (i,t) we have

$$\hat{y}_{1,i}(t)=-(H_{k_i,n_t}B_{i,t}M_{k_i,n_t}u_2)_i(t) \tag{76}$$

Subtracting (75) from (76), we obtain

$$\begin{aligned}\hat{y}_{1,i}(t)-y_{1,i}(t)=&(H_{k_i,n_t}(G-G_{k_i,n_t})y_1)_i(t)+(H_{k_i,n_t}Xy_1)_i(t)+(H_{k_i,n_t}Yy_2)_i(t)\\&+(H_{k_i,n_t}(BM-B_{i,t}M_{k_i,n_t})u_2)_i(t)\end{aligned} \tag{77}$$

The idea is to bound $|(H_{k_i,n_t}(BM-B_{i,t}M_{k_i,n_t})u_2)_i(t)|$ by some constant. For this purpose, define the operator $Q\in\mathscr{L}_{\mathrm{STV}}$ as

$$(Qz)_{\mathtt{i}}(\tau)=(B_{\mathtt{i},\tau}M_{k_{\mathtt{i}},n_\tau}z)_{\mathtt{i}}(\tau),\quad \mathtt{i}\in\mathbb{Z},\ \ \tau\in\mathbb{Z}^+ \tag{78}$$

then

$$(H_{k_i,n_t}(BM - B_{i,t}M_{k_i,n_t})u_2)_i(t) = (H_{k_i,n_t}(BM - Q)u_2)_i(t) + (H_{k_i,n_t}(Q - B_{i,t}M_{k_i,n_t})u_2)_i(t) \quad (79)$$

By Lemma 1, and the fact that H_{k_i,n_t} has uniformly bounded norm, it follows that

$$|(H_{k_i,n_t}(BM - Q)u_2)_i(t)| \leqslant \gamma c_1 \quad (80)$$

where c_1 is a positive constant. We have the following for the term $(H_{k_i,n_t}(Q - B_{i,t}M_{k_i,n_t})u_2)_i(t)$:

$$\|B_{\mathtt{i},\tau}M_{k_\mathtt{i},n_\tau} - B_{i,t}M_{k_i,n_t}\| \leqslant \|B_{\mathtt{i},\tau}\|\|M_{k_\mathtt{i},n_\tau} - M_{k_i,n_t}\| + \|M_{k_i,n_t}\|\|B_{i,t} - B_{\mathtt{i},\tau}\| \quad (81)$$

$$\leqslant \|B_{\mathtt{i},\tau}\|\gamma_M(|i-\mathtt{i}| + |t-\tau|) + \|M_{k_i,n_t}\|\gamma_B(|i-\mathtt{i}| + |t-\tau|) \quad (82)$$

$$\leqslant \gamma c_2(|i-\mathtt{i}| + |t-\tau|) \quad (83)$$

Hence, if $z_\mathtt{i}(\tau) = ((Q - B_{i,t}M_{k_i,n_t})u_2)_\mathtt{i}(\tau)$, then $|z_\mathtt{i}(\tau)| \leqslant \gamma c_2(|i-\mathtt{i}| + |t-\tau|)$, $\mathtt{i} \in \mathbb{Z}$, $\tau \in \mathbb{Z}^+$, with $c_2 > 0$. However, from the fact that H_{k_i,n_t} has bounded (uniformly in t, and i) ITSAE, it follows that

$$|(H_{k_i,n_t}(Q - B_{i,t}M_{k_i,n_t})u_2)_i(t)| = \left|\sum_{\tau=0}^{t}\sum_{\mathtt{i}=-\infty}^{\infty}(h_{k_i,i-\mathtt{i}}(n_t, t-\tau))z_\mathtt{i}(\tau)\right| \quad (84)$$

$$\leqslant \gamma c_2 \sum_{\tau=0}^{t}\sum_{\mathtt{i}=-\infty}^{\infty}(h_{k_i,\mathtt{i}}(n_t,\tau))(|\mathtt{i}| + |\tau|) \quad (85)$$

$$\leqslant \gamma c_3, \quad c_3 > 0 \quad (86)$$

Looking at the rest of the terms, and since $\|u_2\| \leqslant 1$, we have $|(H_{k_i,n_t}Xy_1)_i(t)| \leqslant \gamma c_4\|S^{12}\|$, $|(H_{k_i,n_t}Yy_2)_i(t)| \leqslant \gamma c_5\|S^{22}\|$ and $|(H_{k_i,n_t}(G - G_{k_i,n_t})y_1)_i(t)| \leqslant \gamma c_6\|S^{12}\|$. Putting everything together, it follows that there are constants c, c_{12}, $c_{22} > 0$ such that

$$|\hat{y}_{1,i}(t) - y_{1,i}(t)| \leqslant \gamma c + \gamma c_{12}\|S^{12}\| + \gamma c_{22}\|S^{22}\| \quad (87)$$

Since $\|u_2\| \leqslant 1$, we have $|\hat{y}_{1,i}(t)| \leqslant \|S^{12}_{i,t}\|$, and therefore

$$\sup_{i,t}|y_{1,i}(t)| \leqslant \sup_{i,t}\|S^{12}_{i,t}\| + \gamma c + \gamma c_{12}\|S^{12}\| + \gamma c_{22}\|S^{22}\| \quad (88)$$

Since u_2 is arbitrary

$$\|S^{12}\| \leqslant \sup_{i,t}\|S^{12}_{i,t}\| + \gamma c + \gamma c_{12}\|S^{12}\| + \gamma c_{22}\|S^{22}\| \quad (89)$$

$$(1 - \gamma c_{12})\|S^{12}\| \leqslant \sup_{i,t}\|S^{12}_{i,t}\| + \gamma(c + c_{22}\|S^{22}\|) \quad (90)$$

Similarly working for $\|S^{22}\|$, we get

$$(1 - \gamma k_{22})\|S^{22}\| \leqslant \sup_{i,t}\|S^{22}_{i,t}\| + \gamma(k + k_{12}\|S^{12}\|) \quad (91)$$

Noting that $\|H_{k_i,n_t}\|$ is uniformly bounded, we have $\sup_{i,t}\|S^{12}_{i,t}\|, \sup_{i,t}\|S^{22}_{i,t}\| < \infty$. Let $D := \max\{\sup_{i,t}\|S^{12}_{i,t}\|, \sup_{i,t}\|S^{22}_{i,t}\|\}$. Using (91), we can write (90) as:

$$(1 - \gamma c_{12})\|S^{12}\| \leqslant \sup_{i,t}\|S^{12}_{i,t}\| + \gamma\left(c + \frac{c_{22}}{1 - \gamma k_{22}}(D + \gamma(k + k_{12}\|S^{12}\|))\right) \quad (92)$$

Rearranging we get:

$$\left(1 - \gamma c_{12} - \gamma^2\frac{c_{22}k_{12}}{1 - \gamma k_{22}}\right)\|S^{12}\| \leqslant \sup_{i,t}\|S^{12}_{i,t}\| + \gamma\left(c + \frac{c_{22}}{1 - \gamma k_{22}}(D + \gamma k)\right) \quad (93)$$

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Similarly using (90), we can write (91) as follows:

$$\left(1-\gamma k_{22}-\gamma^2\frac{k_{12}c_{22}}{1-\gamma c_{12}}\right)\|S^{22}\|\leqslant \sup_{i,t}\|S^{22}_{i,t}\|+\gamma\left(k+\frac{k_{12}}{1-\gamma c_{12}}(D+\gamma c)\right) \tag{94}$$

Let

$$\varepsilon_1 := \gamma c_{12}+\gamma^2\frac{c_{22}k_{12}}{1-\gamma k_{22}}$$

$$\varepsilon_2 := \gamma\left(c+\frac{c_{22}}{1-\gamma k_{22}}(D+\gamma k)\right)$$

$$\varepsilon_3 := \gamma k_{22}+\gamma^2\frac{k_{12}c_{22}}{1-\gamma c_{12}}$$

$$\varepsilon_4 := \gamma\left(k+\frac{k_{12}}{1-\gamma c_{12}}(D+\gamma c)\right)$$

Picking γ_p such that ε_1, ε_2, ε_3, $\varepsilon_4<\varepsilon$ concludes the proof for S^{12} and S^{22} maps. Similar proof holds for S^{21}, S^{11} maps. □

Before closing, some final remarks are in order. First, we would like to note that although the previous stability and performance analysis rely on a sufficiently small rate of change in time and space of local approximants, this does not mean that all the local approximants are close to each other. If the spatiotemporal distance is large, the local systems may be very different. Second, the specific overall controller structure can be viewed as a collection of local controllers distributed in space and time. Each local station has to know the measurements and controls of the others, in fact only a limited number that depends on the specific (polynomial) order of L_{k_i,n_t}, M_{k_i,n_t} that we choose for the local, frozen design. Each local station though, does not have to know the dynamics of the others as it assumes they are the same as its own. Therefore, only their own, local models need to be available to the local stations.

6. CONCLUSION

We have considered the stability analysis of systems that have gradually varying dynamics in time as well as in space. In particular, we have looked at the case where the controllers were not necessarily adjusted for every instance in space and time, and hence were used for some fixed spatiotemporal window before new controllers were implemented. We showed how the length of these windows entered in the stability analysis. It was shown that the actual time-varying system can be stabilized using the frozen space-time controllers provided the variations in the spatiotemporal dynamics are sufficiently small. We have also shown that the l_∞ to l_∞ performance of global spatiotemporally varying system cannot be much worse than the worst frozen spatially and temporally l_∞ to l_∞ performance given that the rates of variation of the plant and the controller are sufficiently small.

We remark here that while we have looked at systems that have infinite spatial extent, this is never the case in reality. The infinite spatial dimensionality lends itself to more elegant, unified, and easier mathematical treatment. For the real systems, boundary effects are always present and do need special considerations. Currently, the boundary effects are treated in an *ad hoc* fashion, see e.g. [22], which lack rigor and may even lead to instability. More elaborated approaches are required to incorporate the boundary effects in a unified manner to guarantee the system stability and performance. In this direction Dullerud and D'Andrea [23] have worked on the control design for distributed systems, where they do not require the underlying system dynamics to be spatiotemporally invariant and when quadratic criteria are of interest.

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