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# Identification of spatiotemporally invariant systems for control adaptation[☆]

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ABSTRACT

We present a distributed projection algorithm for system identification of spatiotemporally invariant systems with the ultimate purpose of utilizing it in an indirect adaptive control scheme. Each subsystem communicates only with its immediate neighbors to share its current estimate along with a cumulative improvement index. On the basis of the cumulative improvement index, the best estimate available is picked in order to carry out the next iteration. For small estimation error, the scheme switches over to a "smart" averaging routine. The proposed algorithm guarantees to bring the local estimates arbitrarily close to one another, developing a "local consensus", which makes it amenable to control by the application of indirect distributed adaptive control schemes. It is also shown through simulations that the proposed algorithm has a clear advantage over the standard projection algorithm. Our proposed algorithm is also suitable for addressing the estimation problem in distributed networks that arise in a variety of applications, such as environment monitoring, target localization and potential sensor network problems.

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## 1. Introduction

With the advances in sensing and actuating techniques coupled with the incessant increase in computational power, the idea of developing more and more complex systems by putting together simpler smaller units is turning into a reality. Examples of such systems can be cited from various areas such as satellite constellations (Shaw, Miller, & Hastings, 1998), cross-directional control in paper processing applications (Stewart, 2000), airplane formation flying (Chichka & Speyer, 1998; Wolfe, Chichka, & Speyer, 1996), automated highway systems (Raza & Ioannou, 1996) and microcantilever array control for various nanorobotic applications (Sarwar, Voulgaris, & Salapaka, 2011a). Lumped approximations of partial differential equations (PDEs) can also be considered in this regard—examples include the deflection of beams, plates, and membranes, and the temperature distribution of thermally conductive materials (Taylor, 1996). Centralized control of such distributed systems becomes increasingly complicated and difficult to implement as the number of underlying subsystems or units increases, hence making distributed control inevitable. The control design of any system, however, is only as good as the system model. When the system model is not available upfront, system identification and control action have to be implemented in parallel. As the system model gets updated, the control law needs to adapt in order to guarantee stability/performance. Hence, adaptation as well as identification need to be carried out in a distributed manner for systems with large numbers of subsystems or units.

Control design of distributed systems is a daunting task in general, and is mostly dominated by the architectural and localization constraints. Such design problems are well known to be difficult, with now nearly three decades of research; see Siljak (1991) and the references therein. Several attempts have been made already to address the problem of distributed adaptive control of interconnected systems employing different approaches while assuming various structures. The most notable early work can be attributed to Ioannou and Kokotovic (1985) in this regard, where weakly interconnected subsystems were studied. Subsequent work includes the $M$-matrix approach in Ossman (1989), and a high-gain approach in Gavel and Siljak (1989) assuming a strict matching condition on the disturbances. A methodology for handling higher-order interconnections in a distributed adaptive control framework was developed in Shi and Singh (1992), whereas neural networks have been used to approximate unknown interconnections in Spooner

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**Notation**

| | |
|---|---|
| $\mathbb{R}$ | Set of reals |
| $\mathbb{Z}$ | Set of integers |
| $\mathbb{Z}^+$ | Set of non-negative integers |
| $l_\infty^e$ | Space of all real spatiotemporal sequences $f = \{f_i(t)\}$ with a two-sided spatial support $(-\infty \le i \le \infty)$ and one-sided temporal support $(0 \le t \le \infty)$ |
| LSTV | Linear spatiotemporally varying system |
| LSTI | Linear spatiotemporally invariant system |
| $\hat{R}(z, \lambda)$ | $z,\ \lambda$ transform of LSTI system $R$ |
| $\mathscr{B}(a, \varepsilon)$ | $\{x \in \mathbb{R} \mid \lvert a - x\rvert \le \varepsilon\}$ |
| $Supp(m)$ | Support of a spatiotemporal sequence $\{m_i(t)\} = \{[i, t] \in \mathbb{Z}^2 : m_i(t) \neq 0\}$ |
| $\lVert \cdot \rVert$ | The Euclidean norm for vectors or the system (induced) norm. |

and Passino (1996) and Spooner and Passino (1999). Prior sharing of information amongst controllers about the reference model has been assumed in Mirkin and Gutman (2003), Narendra and Oleng (2002) and Hovakimyan, Lavretsky, Yang, and Calise (2005) for asymptotic tracking of desired outputs. In each of the above cited works, a very narrow class of systems has been addressed owing to the lack of any unified framework for distributed systems. Systems, therefore, have to be treated on a rather case by case basis for distributed control design.

Systems that are either homogeneous or are made up of similar subsystems or units can be approximated by their infinite abstractions, i.e. they can be considered as spatially invariant (refer to Curtain, Iftime, and Zwart (2010) for instances where these abstractions are valid). Examples of such systems include arrays of identical microcantilevers for atomic force microscope applications (Sarwar et al., 2011a), large segmented telescopes (Jiang, Voulgaris, Holloway, & Thompson, 2006), temperature control of thermally conductive material (Taylor, 1996), and fluid flow control (Bamieh, Paganini, & Dahleh, 2002), to cite but a few. Spatial invariance is a strong property of a given system, which means that the dynamics of the system do not vary as we translate along some spatial axis. While the control design of spatially invariant systems has been worked out in some detail (see e.g. Bamieh et al. (2002) and D'Andrea and Dullerud (2003)), knowledge of the underlying system model is assumed a priori. A distributed identification scheme is, therefore, imperative for control design and implementation where the system model is not available upfront.

This article aims to develop a distributed identification scheme for spatially invariant systems that can be used for the adaptation of control laws that are designed for a known plant model. We focus on systems that are recursively computable within the class of spatially invariant systems. Recursibility is a property of certain difference equations, that allows one to iterate the equation by choosing an indexing scheme such that every output sample can be computed from outputs that have already been found from initial conditions and from samples of the input sequence. Systems that can be represented by a two-dimensional rational transfer functions are recursively computable, i.e. a system $P$ is recursively computable if its transfer function has the form

$$\hat{P}(z, \lambda) = \frac{\hat{B}(z, \lambda)}{\hat{A}(z, \lambda)} \tag{1}$$

where $\hat{B}$ and $\hat{A}$ are polynomials in $z$ (spatial domain) and $\lambda$ (time domain) (Bose, 1982). If $\hat{A}$ and $\hat{B}$ or equivalent state space descriptions are known, current LSTI control design methods can be readily applied. Refer to Bose (1982) (Section 4.5) for a detailed discussion on the conversion of a transfer function formulation into the equivalent state space description. Recursive systems are guaranteed to be well defined and this class encompasses many systems of practical importance, such as discretized partial differential equations (PDEs; deflection of beams, plates, membranes, the temperature distribution of thermally conductive materials (Taylor, 1996)). Systems already discussed above in Sarwar et al. (2011a) and Jiang et al. (2006) are also recursively computable.

Results on parameter estimation for linear lumped systems have been well established (see Goodwin and Sin (1984) and Johnson (1988)). Distributed estimation/identification, on the other hand is still an active area of research. It finds applications in distributed optimization, network consensus, sensor fusion, dynamic systems characterized by PDEs, and wireless networks to name but a few examples. Each of the aforementioned areas brings its own flavor to the quest for distributed estimation/identification. The literature on system identification of distributed systems (assuming a centralized setting) is abundant, with the early attempts geared towards investigations dealing with the 'inverse problem' in heat transfer. For a thorough historic development in this regard see Banks and Kunish (1989), Kubrusly (1977) and Polis and Goodson (1976) and the references therein.

In principle, quite a few recently developed algorithms can be employed for the identification of spatially invariant systems. Diffusion techniques are proposed in Lopes and Syed (2007), where each subsystem combines its current estimate with the estimate of its neighbors, based on some performance criterion, to come up with an aggregate. This aggregate is then used for carrying out the next estimation update. A similar space–time diffusion approach can be found in Xiao, Boyd, and Lall (2006). An iterative optimization algorithm for a networked system is considered in Ram, Nedić, and Veeravalli (2009). Each subsystem (agent) obtains a weighted average of its own iterate with the iterates of its neighbors, and updates the average using the subgradient of its local function to generate the new iterate. Identification of circulant systems is considered in Massioni and Verhaegen (2008) by employing the spatial Fourier transform. The identified data available to each subsystem, however, should be processed centrally in order to construct the global system matrices.

From a control adaptation perspective, however, we are interested in ascertaining whether the following are achieved from a distributed system identification scheme for a spatially invariant system:

(a) the estimation error (the difference between the actual and predicted system output) goes to zero, regardless of the convergence of estimates to the true value;
(b) estimates get close to each other arbitrarily as time grows (at least locally).

While the requirement of (a) is quite clear, the requirement of having (b) is motivated from the fact that such estimated systems can be used for adaptive control using the results recently developed in Sarwar, Voulgaris, and Salapaka (2011b). The literature cited above on system identification, however, does not provide guarantees that we require in (a) and (b). No literature exists, to our knowledge, that addresses the system identification (or adaptive control for that matter) of spatially invariant systems from a control adaptation perspective that provides these guarantees. With this motivation, we develop a distributed projection algorithm for system identification of recursively computable spatiotemporally invariant systems that achieves both of the above mentioned objectives and can, therefore, be employed for adaptive control of spatially invariant systems as demonstrated in Sarwar, Voulgaris, and Salapaka

(2010). The main idea, however, can be extended to other gradient based identification schemes such as least squares. We remark here that although the quest for convergence to true parameters is important in its own right, it is not the focus of our article. Our focus is on having a scheme that is amenable to adaptive control, for which convergence to true parameters is not a requirement.

The paper is organized as follows: Section 2 presents the necessary preliminaries. Section 3 presents the main idea of the distributed projection algorithm for the case where there is no noise present in the system output, along with some simulation results. Section 4 presents the analysis of the distributed projection algorithm in the case where bounded noise is added to the system output. We conclude our discussion in Section 5.

## 2. Preliminaries

### 2.1. Spatiotemporally invariant systems

Linear spatiotemporally invariant systems are spatiotemporal systems $M : u \rightarrow y$ on $l_{\infty}^{e}$ given by the convolution

$$y_i(t) = \sum_{\tau=0}^{\tau=t} \sum_{j=-\infty}^{j=\infty} m_{i-j}(t-\tau)u_j(\tau) \tag{2}$$

where $\{m_i(t)\}$ is the pulse response of $M$. These systems can be viewed as an infinite array of interconnected linear time invariant (LTI) subsystems. The induced $l_\infty$ operator norm on $M$ in this case is given as

$$\|M\| = \sum_{t=0}^{\infty} \sum_{i=-\infty}^{i=\infty} |m_i(t)|. \tag{3}$$

The subspace of stable LSTI systems, i.e. LSTI systems with $\|M\| < \infty$, will be denoted as $\mathscr{L}_{STI}$.

### 2.2. Recursively computable spatiotemporal systems

An LSTI system $P$ is called recursively computable if its input–output relationship is defined by an equation of the form

$$(Ay)_i(t) = (Bu)_i(t) \tag{4}$$

where $y$ is the output and $u$ is the input. With $\{a_i(t)\}$, $\{b_i(t)\}$ being the kernel representations of the operators $A$, $B$ in $\mathscr{L}_{STI}$ respectively, we can write the above equation explicitly as follows;

$$\sum_{\substack{j \\ (j,\tau)\in I_a}} \sum_{\tau} a_j(\tau) y_{i-j}(t-\tau) = \sum_{\substack{j \\ (j,\tau)\in I_b}} \sum_{\tau} b_j(\tau) u_{i-j}(t-\tau) \tag{5}$$

where $I_a$ (output mask) and $I_b$ (input mask) denote, respectively, the finite area region of support for $\{a_i(t)\}$ and $\{b_i(t)\}$. The system in (5) is well defined if $a_0(0) \neq 0$, and $a_j(\tau) \neq 0$ for some $(j, \tau)$, and $Supp(\{a_j(\tau)\})$ is a subset of the lattice sector with vertex $(0, 0)$ of angle less than 180° (Bose, 1982) (a general schematic is shown in Fig. 1).

### 2.3. Setup

We will focus on SISO discrete-time spatiotemporally invariant systems that are recursively computable and assume that all the spatiotemporal systems under consideration are well defined. The basic setup is presented in Fig. 2, where an infinite string of interconnected subsystems is shown. The overall system is assumed to be recursively computable LSTI. The signals $u_i$, $y_i$ refer to the respective scalar input and scalar output of the subsystem $P_i$. All subsystems are assumed to have identical dynamics, i.e., $P_i = P_j \; \forall \; i, j$. The following transfer function representation holds for this system (Bose, 1982):

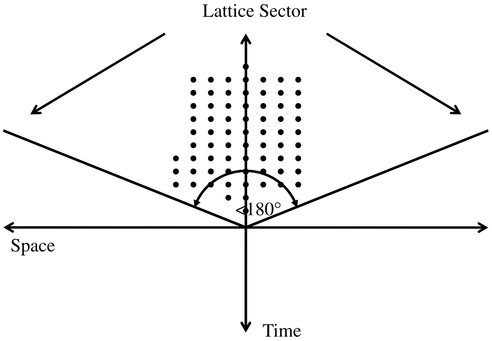

**Fig. 1.** Finite area region of support with the lattice sector having angle less than 180°.

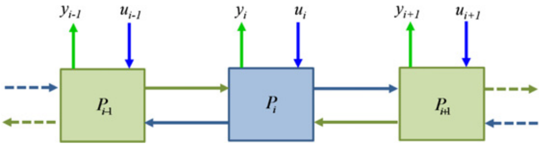

**Fig. 2.** Basic setup.

$$\hat{P}(z, \lambda) = \frac{\hat{B}(z, \lambda)}{\hat{A}(z, \lambda)} \tag{6}$$

where $\hat{B}$ and $\hat{A}$ are polynomials in $z$ and $\lambda$ given by

$$\hat{A}(z, \lambda) = 1 + \sum_{t=0}^{m_1} \sum_{\substack{k=-n_1 \\ k\neq 0}}^{n_1} (a_k(t) z^k) \lambda^t \tag{7}$$

$$\hat{B}(z, \lambda) = \sum_{t=0}^{m_2} \sum_{\substack{k=-n_2 \\ k\neq 0}}^{n_2} (b_k(t) z^k) \lambda^t. \tag{8}$$

The coefficients $\{a_k(t)\}$ and $\{b_k(t)\}$ are not known a priori. However, we will assume knowledge of the bound on the degrees of $\hat{A}$ and $\hat{B}$. The input–output relationship of such systems can also be written as

$$y_i(t) = \phi_i(t-1)^T \theta_0 \tag{9}$$

where $y_i(t)$ denotes the (scalar) system output of subsystem $i$ at time $t$. The regression vector, $\phi_i(t-1)$, is a concatenation of the signals that affect the $i$th subsystem at time $t$ and contribute towards its dynamic evolution, i.e., $\phi_i(t-1) = [Y_i(t-1) \; U_i(t-1)]$ where

$$\begin{aligned} Y_i(t-1) = [&\{y_i(t-1),\; y_i(t-2), \ldots\},\; \{y_{i-1}(t-1), \\ &y_{i-1}(t-2), \ldots\},\; \{y_{i+1}(t-1), y_{i+1}(t-2), \ldots\}, \ldots] \\ U_i(t-1) = [&\{u_i(t-1),\; u_i(t-2), \ldots\},\; \{u_{i-1}(t-1), \\ &u_{i-1}(t-2), \ldots\},\; \{u_{i+1}(t-1),\; u_{i+1}(t-2), \ldots\}, \ldots] \end{aligned}$$

and $\theta_0$ is a vector (unknown) formed from the coefficients $\{a_k(t)\}$ and $\{b_k(t)\}$. We assume that every subsystem has complete knowledge of its regression vector at every time instant. Since each subsystem is assumed to be time invariant, and the overall system is spatially invariant, $\theta_0$ is a constant. Also, since we assume there are finitely many coefficients $\{a_k(t)\}$ and $\{b_k(t)\}$, therefore $\theta_0 \in \mathbb{R}^n$ for some $n$; and consequently the regression vector is also of finite dimension $n$, that is, $\phi_i(t-1) \in \mathbb{R}^n$.

## 3. The distributed projection algorithm

The following notation will be used frequently in the sequel. $\hat{\theta}_i(t-1)$ is the estimate of the true parameter vector $\theta_0$ established by the $i$th subsystem at time $t-1$. $\hat{\theta}(0)$ is the initial estimate/guess which is given. The fused estimate $\hat{\theta}^i(t-1)$ is derived from the estimates available to the $i$th subsystem at time $t$, i.e. from $\{\hat{\theta}_{i-1}(t-1),\ \hat{\theta}_i(t-1),\ \hat{\theta}_{i+1}(t-1)\}$, and is employed by the $i$th subsystem to establish its next estimate $\hat{\theta}_i(t)$. Define

$$\tilde{\theta}^i(t) := \hat{\theta}^i(t) - \theta_0 \tag{10}$$

$$\tilde{\theta}_i(t) := \hat{\theta}_i(t) - \theta_0 \tag{11}$$

$$\begin{aligned} e_i(t) &:= y_i(t) - \phi_i(t-1)^T\hat{\theta}^i(t-1) \\ &= -\phi_i(t-1)^T\tilde{\theta}^i(t-1). \end{aligned} \tag{12}$$

We shall refer to $e_i(t)$ as the estimation error. We summarize our main objectives in the following:

(I) the estimation error (the difference between the actual and predicted system output) should asymptotically approach zero, regardless of the convergence of estimates to the true value;

(II) the estimates should get close to each other arbitrarily as time grows (at least locally) which implies that

$$\lim_{t\to\infty} \|\hat{\theta}_i(t) - \hat{\theta}_k(t)\| = 0 \quad \text{for } k \in \{\, i-1,\ i+1\}.$$

From an adaptive control viewpoint, the necessity of (I) is well known (see Goodwin & Sin, 1984) since a scheme based on certainty equivalence guarantees stability in this case. In certainty equivalence a control law is designed at each time (and space) step to stabilize the estimated system. The requirement of (II) stems from the fact that our overall objective is the stability of the entire system (and not just a subsystem). In an identification scheme that is run independently on each subsystem, the estimates may evolve quite differently from one another as each subsystem is subjected to independent (and hence possibly different) inputs. While several distributed control design methods for LSTI systems exist for a known model, developing an adaptive control scheme for such systems using a certainty equivalence approach is quite challenging since a distributed identification scheme needs to guarantee that the estimated system is also spatially invariant. If this is achieved, recent results for the stability of slowly varying systems (Sarwar et al., 2011b) can show that the slowly varying spatiotemporal systems can be stabilized employing a certainty equivalence approach. With this motivation in mind, we propose below the main idea of a distributed projection algorithm that will result in a slowly varying spatiotemporal system amenable to adaptive control.

### 3.1. The main idea

We assume that all the subsystems start with the same initial guess $\hat{\theta}_i(0) := \hat{\theta}(0)$. Each subsystem runs the same algorithm, described in this section. The main idea of the algorithm is presented in Fig. 3. Step 1: exchange the current estimate (and related information) with immediate neighbors. Step 2: update the current estimate on the basis of the estimates received from immediate neighbors—subsystem $i$ resets its current estimate, $\hat{\theta}_i(t)$, to the best available estimate from the set $\{\hat{\theta}_{i-1}(t), \hat{\theta}_i(t), \hat{\theta}_{i+1}(t)\}$. If there is a tie, i.e., there is more than one estimate that qualifies as the best, an average is taken. The cumulative improvement index ($I_j(t),\ j \in \{i-1, i, i+1\}$) that helps in identifying the best estimate is explained in the next section. The estimate derived at the end of the spatial update is referred to as $\hat{\theta}^i(t)$ (note that the subscript '$i$' is changed to a superscript after the spatial update). Step 3: perform the temporal update by using $\hat{\theta}^i(t)$ to finally derive the estimate $\hat{\theta}_i(t+1)$ based on the current information available as regards the estimation error $e_i(t+1)$ and the regression vector $\phi_i(t)$. The entire process is then repeated. A rigorous mathematical formulation of the algorithm is presented in (13).

```
***Distributed Projection Algorithm***
Start with $\hat{\theta}_i(0) = \theta(0)$. For each subsystem $i$, repeat
  **Share Estimates**:
      Transmit $\{\hat{\theta}_i(t), I_i(t)\}$ to immediate neighbors
  **Spatial Update**:
    $\bar{I}_i(t) = \min\{I_{i-1}(t), I_i(t), I_{i+1}(t)\}$
    If $I_{i-1}(t) = I_{i+1}(t) = \bar{I}_i(t)$
        If $I_i(t) \in \mathscr{B}(\bar{I}_i(t), \varepsilon)$
            $\alpha^i_{i-1}(t) = \frac{1}{3};\ \alpha^i_i(t) = \frac{1}{3};\ \alpha^i_{i+1}(t) = \frac{1}{3}$
        Else
            $\alpha^i_{i-1}(t) = \alpha^i_{i+1}(t) = \frac{1}{2};\ \alpha^i_i(t) = 0$
        End
    ElseIf $I_j(t) = \bar{I}_i(t), j \in \{i-1, i+1\},\ I_{i-1}(t) \neq I_{i+1}(t)$
        If $I_i(t) \in \mathscr{B}(\bar{I}_i(t), \varepsilon)$
            $\alpha^i_j(t) = \alpha^i_i(t) = \frac{1}{2};\ \alpha^i_l(t) = 0;\ l \in \{i-1, i+1\}, l \neq j$
        Else
            $\alpha^i_j(t) = 1;\ \alpha^i_i(t) = 0;\ \alpha^i_l(t) = 0;\ l \in \{i-1, i+1\}, l \neq j$
        end
    Else
        $\alpha^i_i(t) = 1;\ \alpha^i_{i-1}(t) = \alpha^i_{i+1}(t) = 0$
    end
  $\hat{\theta}^i(t) = \sum_{j=i-1}^{i+1} \alpha^i_j(t)\hat{\theta}_j(t)$
  **Temporal Update**:

$\hat{\theta}_i(t+1) = \hat{\theta}^i(t) + \frac{\beta_i(t)\phi_i(t)}{c + \phi_i(t)^T\phi_i(t)}\left[y_i(t+1) - \phi_i(t)^T\hat{\theta}^i(t)\right]$      (13)

  where $c > 0$, and $0 < \beta_i(t) < 2$.
```

### 3.2. The cumulative improvement index $I_i(\cdot)$

The cumulative improvement index $I_i(\cdot)$ is a measure of how much an estimate has improved from the original guess $\hat{\theta}(0)$. Each subsystem shares its current estimate along with the associated cumulative improvement index with its immediate neighbors. In the case of a system with no noise, one can exactly establish $I_i(\cdot)$ for each estimate at every time step $t$. In the case of a system with bounded measurement noise, however, one can establish only a lower bound on the magnitude of $I_i(\cdot)$, and then proceed as discussed here. We define the cumulative improvement index, and its fused counterpart respectively, as follows:

$$I_i(t) := \|\tilde{\theta}_i(t)\|^2 - \|\tilde{\theta}(0)\|^2 \tag{14}$$

$$I^i(t) := \|\tilde{\theta}^i(t)\|^2 - \|\tilde{\theta}(0)\|^2. \tag{15}$$

That is, the cumulative improvement index exactly captures the improvement made by an estimate towards the actual parameter since its initiation at $\theta(0)$. Below we describe how the cumulative improvement index can be established at each iterate. For clarity of exposition, we first assume that for the first $t$ time steps, no averaging of estimates takes place (note that this assumption is at least true for the first time step). This implies that for each time step, the parameter $\hat{\theta}^i(t) \in \{\hat{\theta}_{i-1}(t), \hat{\theta}_i(t), \hat{\theta}_{i+1}(t)\}$. Since the subsequent development (and hence expressions) would be exactly the same but with appropriate adjustment in the spatial indices, we can assume without loss of generality that $\hat{\theta}^i(1) = \hat{\theta}_{i-1}(1)$. Subtracting $\theta_0$ from both sides of (13) at $t = 1$, and using (9) along with (10), we obtain

$$\tilde{\theta}_i(2) = \tilde{\theta}^i(1) - \frac{\beta_i(1)\phi_i(1)}{c + \phi_i(1)^T\phi_i(1)}\phi_i(1)^T\tilde{\theta}^i(1). \tag{16}$$

Using (12) and squaring both sides of (16), we have

$$\begin{aligned} \|\tilde{\theta}_i(2)\|^2 - \|\tilde{\theta}^i(1)\|^2 &= \beta_i(1)\left[-2 + \frac{\beta_i(1)\phi_i(1)^T\phi_i(1)}{c + \phi_i(1)^T\phi_i(1)}\right] \\ &\quad\times \frac{e_i(2)^2}{c + \phi_i(1)^T\phi_i(1)}. \end{aligned} \tag{17}$$

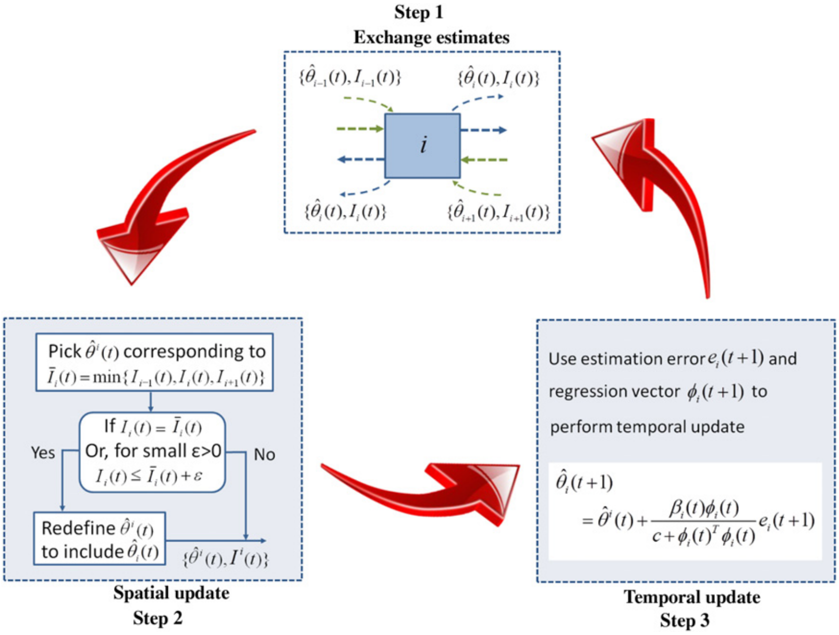
Step 1
Exchange estimates
$\{\hat{\theta}_{i-1}(t), I_{i-1}(t)\}$
$\{\hat{\theta}_{i}(t), I_{i}(t)\}$
$i$
$\{\hat{\theta}_{i}(t), I_{i}(t)\}$
$\{\hat{\theta}_{i+1}(t), I_{i+1}(t)\}$
Pick $\hat{\theta}^i(t)$ corresponding to $\bar{I}_i(t) = \min\{I_{i-1}(t), I_i(t), I_{i+1}(t)\}$
If $I_i(t) = \bar{I}_i(t)$ Or, for small ε>0 $I_i(t) \leq \bar{I}_i(t) + \varepsilon$
Yes
No
Redefine $\hat{\theta}^i(t)$ to include $\hat{\theta}_i(t)$
$\{\hat{\theta}^i(t), I^i(t)\}$
Spatial update
Step 2
Use estimation error $e_i(t+1)$ and regression vector $\phi_i(t+1)$ to perform temporal update
$\hat{\theta}_i(t+1) = \hat{\theta}^i(t) + \frac{\beta_i(t)\phi_i(t)}{c+\phi_i(t)^T\phi_i(t)} e_i(t+1)$
Temporal update
Step 3

**Fig. 3.** Main idea of the distributed projection algorithm.

Note that

$$\beta_i(1)\left[-2 + \frac{\beta_i(1)\phi_i(1)^T\phi_i(1)}{1 + \phi_i(1)^T\phi_i(1)}\right] < 0. \tag{18}$$

The expansion of $\|\tilde{\theta}^i(1)\|^2 = \|\tilde{\theta}_{i-1}(1)\|^2$ can be derived by subtracting $\theta_0$ from both sides of (13) at $t = 0$ and using (9) along with (10). Squaring the resulting expression we get

$$\begin{aligned}
&\|\tilde{\theta}_{i-1}(1)\|^2 - \|\tilde{\theta}(0)\|^2 \\
&\quad = \beta_{i-1}(0)\left[-2 + \frac{\beta_{i-1}(0)\phi_{i-1}(0)^T\phi_{i-1}(0)}{c + \phi_{i-1}(0)^T\phi_{i-1}(0)}\right] \\
&\qquad \times \frac{e_{i-1}(1)^2}{c + \phi_{i-1}(0)^T\phi_{i-1}(0)} \\
&\quad = I_{i-1}(1) = I^i(1).
\end{aligned} \tag{19}$$

Using this expression for $\|\tilde{\theta}_{i-1}(1)\|^2$ in (17) we can write

$$\begin{aligned}
\|\tilde{\theta}_i(2)\|^2 - \|\tilde{\theta}(0)\|^2 &= \beta_{i-1}(0)\left[-2 + \frac{\beta_{i-1}(0)\phi_{i-1}(0)^T\phi_{i-1}(0)}{c + \phi_{i-1}(0)^T\phi_{i-1}(0)}\right] \\
&\quad \times \frac{e_{i-1}(1)^2}{c + \phi_{i-1}(0)^T\phi_{i-1}(0)} \\
&\quad + \beta_i(1)\left[-2 + \frac{\beta_i(1)\phi_i(1)^T\phi_i(1)}{c + \phi_i(1)^T\phi_i(1)}\right] \\
&\quad \times \frac{e_i(2)^2}{c + \phi_i(1)^T\phi_i(1)}
\end{aligned} \tag{20}$$

$$\begin{aligned}
&= I^i(1) + \beta_i(1)\left[-2 + \frac{\beta_i(1)\phi_i(1)^T\phi_i(1)}{c + \phi_i(1)^T\phi_i(1)}\right] \\
&\quad \times \frac{e_i(2)^2}{c + \phi_i(1)^T\phi_i(1)} = I_i(2).
\end{aligned} \tag{21}$$

Note that similar expressions hold for other subsystems as well. Again, since the development (and hence the expressions) would be exactly the same but with appropriate adjustment in the spatial indices, we now assume without loss of generality that for next iterate, $\hat{\theta}^i(2) = \theta_{i+1}(2)$. This implies that we have the following expression for $\hat{\theta}_i(3)$:

$$\hat{\theta}_i(3) = \hat{\theta}^i(2) + \frac{\beta_i(2)\phi_i(2)}{c + \phi_i(2)^T\phi_i(2)} e_i(3). \tag{22}$$

Again, using (12), subtracting $\theta_0$ from both sides, and squaring we have

$$\begin{aligned}
\|\tilde{\theta}_i(3)\|^2 &= \|\tilde{\theta}^i(2)\|^2 + \beta_i(2)\left[-2 + \frac{\beta_i(2)\phi_i(2)^T\phi_i(2)}{c + \phi_i(2)^T\phi_i(2)}\right] \\
&\quad \times \frac{e_i(3)^2}{c + \phi_i(2)^T\phi_i(2)}.
\end{aligned} \tag{23}$$

Plugging in the expansion of $\|\tilde{\theta}^i(2)\|^2$, we can rewrite (23) as

$$\begin{aligned}
\|\tilde{\theta}_i(3)\|^2 - \|\tilde{\theta}(0)\|^2 &= \sum_{t=0}^{1} \beta_k(t)\left[-2 + \frac{\beta_k(t)\phi_k(t)^T\phi_k(t)}{c + \phi_k(t)^T\phi_k(t)}\right] \\
&\quad \times \frac{e_k(t+1)^2}{c + \phi_k(t)^T\phi_k(t)} + \beta_i(2) \\
&\quad \times \left[-2 + \frac{\beta_i(2)\phi_i(2)^T\phi_i(2)}{c + \phi_i(2)^T\phi_i(2)}\right] \\
&\quad \times \frac{e_i(3)^2}{c + \phi_i(2)^T\phi_i(2)}
\end{aligned} \tag{24}$$

$$\begin{aligned}
&= I^i(2) + \beta_i(2)\left[-2 + \frac{\beta_i(2)\phi_i(2)^T\phi_i(2)}{c + \phi_i(2)^T\phi_i(2)}\right] \\
&\quad \times \frac{e_i(3)^2}{c + \phi_i(2)^T\phi_i(2)} = I_i(3)
\end{aligned} \tag{25}$$

where the subscript $k$ captures the evolution path of $\hat{\theta}^i(2)$, i.e. the sequence of the subsystems that were involved in establishing $\hat{\theta}^i(2)$. Clearly, this expression can be generalized up to time step $t$ and the cumulative improvement index can be calculated iteratively up to $t$. We note here that the cumulative improvement is a sum of non-positive terms as is clear from (24). Also note that $I_i(t) \leq I_i(t-1)$, i.e. it is a non-increasing sequence up to time step $t$. We now look at time step $t+1$ when the averaging of estimates takes place. Again, for clarity of exposition let us assume

$\hat{\theta}^i(t) = \frac{\hat{\theta}_{i-1}(t)+\hat{\theta}_i(t)}{2}$. Define

$$E_i(t) := \hat{\theta}_i(t) - \hat{\theta}(0). \tag{26}$$

Note that we can equivalently write

$$E_i(t) = \sum_{\tau=1}^{t}\left[\frac{\beta_k(\tau-1)\phi_k(\tau-1)}{c+\phi_k(\tau-1)^T\phi_k(\tau-1)}\right]e_k(\tau). \tag{27}$$

The calculation of $I^i(t)$ can now be given as follows:

$$\begin{aligned}\hat{\theta}^i(t) &= \frac{\hat{\theta}_i(t)+\hat{\theta}_{i-1}(t)}{2} = \frac{\hat{\theta}(0)+E_i(t)+\hat{\theta}(0)+E_{i-1}(t)}{2}\\ &= \hat{\theta}(0) + \frac{E_i(t)+E_{i-1}(t)}{2}.\end{aligned} \tag{28}$$

Subtracting $\theta_0$ from both sides and taking the inner product of $\tilde{\theta}^i(t)$ with itself, we get

$$\begin{aligned}\|\tilde{\theta}^i(t)\|^2 = {}&\|\tilde{\theta}(0)\|^2 + \tilde{\theta}(0)^T[E_i(t)+E_{i-1}(t)]\\ &+ \frac{1}{4}[E_i(t)+E_{i-1}(t)]^T[E_i(t)+E_{i-1}(t)].\end{aligned} \tag{29}$$

From the above expression, we can write $I^i(t)$ as follows:

$$\begin{aligned}I^i(t) = {}&\tilde{\theta}(0)^T[E_i(t)+E_{i-1}(t)]\\ &+ \frac{1}{4}[E_i(t)+E_{i-1}(t)]^T[E_i(t)+E_{i-1}(t)].\end{aligned} \tag{30}$$

Now the expression $\tilde{\theta}(0)^TE_i(t)$ can be iteratively computed as follows:

$$\begin{aligned}\tilde{\theta}(0)^TE_i(t) &= \tilde{\theta}(0)^T\left(\sum_{\tau=1}^{t}\beta_k(\tau-1)\right.\\ &\quad\left.\times\left[\frac{\phi_k(\tau-1)}{c+\phi_k(\tau-1)^T\phi_k(\tau-1)}\right]e_k(\tau)\right)\end{aligned} \tag{31}$$

$$= \sum_{\tau=1}^{t}\beta_k(\tau-1)\left[\frac{\hat{\theta}(0)^T\phi_k(\tau-1)-y_k(\tau)}{c+\phi_k(\tau-1)^T\phi_k(\tau-1)}\right]e_k(\tau) \tag{32}$$

$$\begin{aligned}&= \tilde{\theta}(0)^TE_i(t-1) + \beta_i(t-1)\\ &\quad\times\left[\frac{\hat{\theta}(0)^T\phi_i(t-1)-y_i(t)}{c+\phi_i(t-1)^T\phi_i(t-1)}\right]e_i(t).\end{aligned} \tag{33}$$

Similarly, if for the time step $t+2$, we have $\hat{\theta}^i(t+1) = \frac{\hat{\theta}_{i-1}(t+1)+\hat{\theta}_i(t+1)+\hat{\theta}_{i+1}(t+1)}{3}$, the cumulative improvement index $I^i(t+1)$ can be established as follows:

$$\hat{\theta}^i(t+1) = \hat{\theta}(0) + \frac{E_{i-1}(t+1)+E_i(t+1)+E_{i+1}(t+1)}{3}. \tag{34}$$

Subtracting $\theta_0$ from both sides and taking the inner product of $\tilde{\theta}^i(t+1)$ with itself, we get

$$\begin{aligned}\|\tilde{\theta}^i(t+1)\|^2 = {}&\|\tilde{\theta}(0)\|^2 + \frac{2}{3}\tilde{\theta}(0)^T[E_{i-1}(t+1)\\ &+E_i(t+1)+E_{i+1}(t+1)] + \frac{1}{9}[E_{i-1}(t+1)\\ &+E_i(t+1)+E_{i+1}(t+1)]^T[E_{i-1}(t+1)\\ &+E_i(t+1)+E_{i+1}(t+1)].\end{aligned} \tag{35}$$

The expression for $I^i(t+1)$ is, therefore, given as

$$\begin{aligned}I^i(t+1) = {}&\frac{2}{3}\tilde{\theta}(0)^T[E_{i-1}(t+1)+E_i(t+1)+E_{i+1}(t+1)]\\ &+\frac{1}{9}[E_{i-1}(t+1)+E_i(t+1)+E_{i+1}(t+1)]^T\\ &\times[E_{i-1}(t+1)+E_i(t+1)+E_{i+1}(t+1)].\end{aligned} \tag{36}$$

Now, the expression for $E_{i-1}(t+1)$ is given as

$$\begin{aligned}E_{i-1}(t+1) = {}&\hat{\theta}_{i-1}(t+1) - \theta(0) = \sum_{k=-2}^{0}\alpha_{i+k}^{i-1}\hat{\theta}_{i+k}(t)\\ &+\frac{\beta_{i-1}(t)\phi_{i-1}(t)}{c+\phi_{i-1}^T\phi_{i-1}}e_{i-1}(t+1) - \theta(0)\end{aligned} \tag{37}$$

where $\alpha_{i+k}^{i-1} \in \{1, 1/2, 1/3\}$. Note that (37) above can be equivalently written as

$$\begin{aligned}E_{i-1}(t+1) = {}&\sum_{k=-2}^{0}\alpha_{i+k}^{i-1}E_{i+k}(t)\\ &+\frac{\beta_{i-1}(t)\phi_{i-1}(t)}{c+\phi_{i-1}(t)^T\phi_{i-1}(t)}e_{i-1}(t+1).\end{aligned} \tag{38}$$

The expression $\tilde{\theta}(0)^TE_{i-1}(t+1)$ can, therefore, be formulated as

$$\begin{aligned}\tilde{\theta}(0)^TE_{i-1}(t+1) = {}&\sum_{k=-2}^{0}\alpha_{i+k}^{i-1}\tilde{\theta}(0)^TE_{i+k}(t) + \beta_{i-1}(t)\\ &\times\left[\frac{\hat{\theta}(0)^T\phi_{i-1}(t)-y_{i-1}(t+1)}{c+\phi_{i-1}(t)^T\phi_{i-1}(t)}\right]\\ &\times e_{i-1}(t+1)\end{aligned} \tag{39}$$

where the information from the previous time step (i.e. $\tilde{\theta}(0)^TE_{i+k}(t)$) is already available. Similarly, expressions for $\tilde{\theta}(0)^TE_i(t+1)$ and $\tilde{\theta}(0)^TE_{i+1}(t+1)$ can be calculated using information available from the previous time step. We now show that $I^i(t) \le I_i(t)$. It suffices to show that $\|\tilde{\theta}^i(t)\| \le \|\tilde{\theta}_i(t)\|$. Note that

$$\begin{aligned}\|\tilde{\theta}^i(t)\| = \|\hat{\theta}^i(t) - \theta_0\| &= \left\|\frac{\hat{\theta}_i(t)+\hat{\theta}_{i-1}(t)}{2} - \theta_0\right\|\\ &\le \frac{1}{2}\|\hat{\theta}_i(t)-\theta_0\| + \frac{1}{2}\|\hat{\theta}_{i-1}(t)-\theta_0\|.\end{aligned}$$

Since, by assumption, $\|\tilde{\theta}_{i-1}(t)\| \le \|\tilde{\theta}_i(t)\|$, we have

$$\le \frac{1}{2}\|\hat{\theta}_i(t)-\theta_0\| + \frac{1}{2}\|\hat{\theta}_i(t)-\theta_0\| = \|\tilde{\theta}_i(t)\|. \tag{40}$$

Note that since $I_i(t+1) \le I^i(t)$, we can conclude that $I_i(\cdot)$ is a non-increasing sequence of non-positive numbers for all time steps.

### 3.3. The $\varepsilon$-rule

The key idea of the algorithm that we propose is that we can write any estimate as the sum of the initial guess and some improvement terms. At each time step, the cumulative improvement index is a sum of non-positive real numbers. At the same time, $\|\tilde{\theta}_i(t)\|$ is a bounded function (bounded below by 0, and above by $\|\tilde{\theta}(0)\|$). As $t \to \infty$, the cumulative improvement index $I_i(t)$ will come to contain infinitely many non-positive terms. Since, however, $\|\tilde{\theta}_i(t)\|$ is bounded, $\lim_{t\to\infty} I_i(t)$ should exist. This implies that the improvement in estimates will slow down as time increases. We note here that it is possible to have, for example, $\|\tilde{\theta}_i(t)\| \approx \|\tilde{\theta}_{i-1}(t)\|$, but for $\hat{\theta}_i(t)$ and $\hat{\theta}_{i-1}(t)$ to still be far apart. This is depicted in Fig. 4. For such a situation, when the neighboring estimates (in space as well as time) are at almost 'equal distance' from the true value but still far from each other, we introduce the $\varepsilon$-rule, which is as follows:

- There is a small number $\varepsilon > 0$ specified a priori and known for each subsystem.
- At a given time $t$, if $I_i(t) \le I^i(t) + \varepsilon$, redefine $\hat{\theta}^i(t)$ to include $\hat{\theta}_i(t)$. For example:

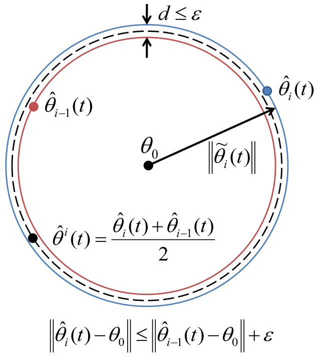
$d \le \varepsilon$
$\hat{\theta}_{i-1}(t)$
$\hat{\theta}_i(t)$
$\theta_0$
$\|\tilde{\theta}_i(t)\|$
$\hat{\theta}^i(t) = \frac{\hat{\theta}_i(t)+\hat{\theta}_{i-1}(t)}{2}$
$\|\hat{\theta}_i(t)-\theta_0\| \le \|\hat{\theta}_{i-1}(t)-\theta_0\| + \varepsilon$

**Fig. 4.** Figure showing the $\varepsilon$-rule.

(I) If $\hat{\theta}^i(t) = \hat{\theta}_j(t)$, with $j \in \{i-1, i+1\}$ redefine $\hat{\theta}^i(t)$ as $\hat{\theta}^i(t) = \frac{\hat{\theta}_j(t)+\hat{\theta}_i(t)}{2}$.

(II) If $\hat{\theta}^i(t) = \frac{\hat{\theta}_{i-1}(t)+\hat{\theta}_{i+1}(t)}{2}$, redefine $\hat{\theta}^i(t)$ as $\hat{\theta}^i(t) = \frac{\hat{\theta}_{i-1}(t)+\hat{\theta}_i(t)+\hat{\theta}_{i+1}(t)}{3}$.

The purpose of the $\varepsilon$-rule is to bring the estimates close to each other locally when their respective rate of improvement slows down. At the same time, this rule ensures that the next estimate will be at least as good as the current one, if not better, in terms of its distance from the true value. The algorithm, therefore, guarantees improvement throughout its execution, making sure that the estimation error at each subsystem will eventually become small. The proposed algorithm also serves to bring the estimates 'close' to one another asymptotically, as shown below.

### 3.4. Properties of the distributed projection algorithm

The properties of the distributed projection algorithm are summarized below.

**Lemma 1.** *For the algorithm (13) and subject to (9) it follows that:*

(1) $\|\hat{\theta}_i(t) - \theta_0\| \le \|\hat{\theta}^i(t-1) - \theta_0\| \le \|\hat{\theta}(0) - \theta_0\|$ *for* $t \ge 1$;
(2) $\lim_{t\to\infty} \frac{e_i(t)}{[c+\phi_i(t-1)^T\phi_i(t-1)]^{1/2}} = 0$;
(3) $\lim_{t\to\infty} \|\hat{\theta}_i(t) - \hat{\theta}_k(t)\| = 0$ *for* $k \in \{\, i-1,\ i+1\}$;
(4) $\lim_{t\to\infty} \|\hat{\theta}_i(t) - \hat{\theta}_i(t-1)\| = 0 \forall\, i$; *this together with* (3) *implies*

$$\lim_{t\to\infty} \|\hat{\theta}_i(t) - \hat{\theta}_k(t+l)\| = 0 \quad \text{for } k \in \{i,\ i-1,\ i+1\}$$

*and for finite l.*

**Proof.** The proof of (1) is evident from the discussion provided above.

For (2), we observe that $\|\tilde{\theta}_i(t)\|^2$ is a bounded non-increasing function, and its expression is given by

$$\begin{aligned}\|\tilde{\theta}_i(t)\|^2 &= \|\tilde{\theta}^i(t-1)\|^2 + \beta_i(t-1) \\ &\quad \times \left[-2 + \frac{\beta_i(t-1)\phi_i(t-1)^T\phi_i(t-1)}{c+\phi_i(t-1)^T\phi_i(t-1)}\right] \\ &\quad \times \frac{e_i(t)^2}{c+\phi_i(t-1)^T\phi_i(t-1)}.\end{aligned} \tag{41}$$

Note that

$$\beta_i(t-1)\left[-2 + \frac{\beta_i(t-1)\phi_i(t-1)^T\phi_i(t-1)}{c+\phi_i(t-1)^T\phi_i(t-1)}\right] < 0. \tag{42}$$

Now since $\|\tilde{\theta}_i(t)\|^2$ is non-negative and (42) holds, we conclude that (2) holds.

In order to prove (3), first note the following:

(1) Given $\varepsilon_0 > 0$, $\exists\, T_{\varepsilon_0} \mid \forall\, t \ge T_{\varepsilon_0}$, and $\forall\, i$ we have

$$\left|\frac{e_i(t)}{[c+\phi_i(t-1)^T\phi_i(t-1)]^{1/2}}\right| \le \varepsilon_0, \tag{43}$$

i.e. the (normalized) estimation error will be small (and will remain small) for all subsystems after time $T_{\varepsilon_0}$.

(2) Given $\varepsilon_1 > 0$, $\exists\, T_{\varepsilon_1} \mid \forall\, t \ge T_{\varepsilon_1}$ $\forall i$, we have

$$\|\tilde{\theta}_i(t-1)\| - \|\tilde{\theta}_i(t)\| \le \varepsilon_1, \tag{44}$$

i.e. the improvement in the parameter estimates will be small (and will remain small) for all subsystems after time $T_{\varepsilon_1}$. Also note that if $\varepsilon_1 \le \varepsilon$, where $\varepsilon$ is the constant used for the $\varepsilon$-rule, then the $\varepsilon$-rule will stay in place $\forall\, t \ge T_{\varepsilon_1}$.

(3) If $\|\tilde{\theta}_i(t)\| = \|\tilde{\theta}_{i-1}(t)\|$ and $\hat{\theta}_i(t) \ne \hat{\theta}_{i-1}(t)$, then

$$\begin{aligned}\left\|\theta_0 - \frac{\hat{\theta}_i(t)+\hat{\theta}_{i-1}(t)}{2}\right\| &< \frac{1}{2}\|\theta_0 - \hat{\theta}_i(t)\| + \frac{1}{2}\|\theta_0 - \hat{\theta}_{i-1}(t)\| \\ &\quad \times \text{(triangle inequality is strict since the} \\ &\quad \text{estimates are not aligned)} \\ &= \|\theta_0 - \hat{\theta}_i(t)\| \\ &= \|\theta_0 - \hat{\theta}_{i-1}(t)\|.\end{aligned} \tag{45}$$

To see the main idea, let $\hat{\theta}^i(t) = \frac{\hat{\theta}_i(t)+\hat{\theta}_{i-1}(t)}{2}$ and $\hat{\theta}_i(t) \ne \hat{\theta}_{i-1}(t)$; then

$$\begin{aligned}\|\tilde{\theta}_i(t+1)\|^2 &= \left\|\frac{\tilde{\theta}_i(t)+\tilde{\theta}_{i-1}(t)}{2}\right\|^2 \\ &\quad + \beta_i(t)\left[-2 + \frac{\beta_i(t)\phi_i(t)^T\phi_i(t)}{c+\phi_i(t)^T\phi_i(t)}\right] \\ &\quad \times \underbrace{\frac{e_i(t+1)^2}{c+\phi_i(t)^T\phi_i(t)}}_{\le \varepsilon_0^2}\end{aligned} \tag{46}$$

$$\|\tilde{\theta}_i(t+1)\|^2 \le \left\|\frac{\tilde{\theta}_i(t)+\tilde{\theta}_{i-1}(t)}{2}\right\|^2 < \|\tilde{\theta}_i(t)\|^2. \tag{47}$$

Since $\|\tilde{\theta}_i(t)\| - \|\tilde{\theta}_i(t+1)\| \le \varepsilon_1$, the improvement resulting from averaging should be less than $\varepsilon_1$. We show in the following that if the estimates are far apart then the improvement caused by averaging will exceed $\varepsilon_1$. For $t \ge \max\{T_{\varepsilon_1}, T_{\varepsilon_0}\}$ and $\varepsilon_1 \le \varepsilon$, we demonstrate below that:

- either the estimate(s) from the immediate neighbor(s) ($\hat{\theta}_{i+1}(t)$ or $\hat{\theta}_{i-1}(t)$, or both) will be used to establish the estimate at the $i$th subsystem ($\hat{\theta}_i(t+1)$); or
- the estimate of the $i$th subsystem ($\hat{\theta}_i(t)$) is used in establishing the estimate(s) of the immediate neighbor(s) ($\hat{\theta}_{i+1}(t+1)$ or $\hat{\theta}_{i-1}(t+1)$, or both); or
- both of the above (in the case where the estimates are equally good).

In other words, the estimation algorithm has strong interdependences in a given local neighborhood. At any given time instance $t$, we have one of the following possibilities for the subsystem $i$:

(a) $\hat{\theta}^i(t) = \hat{\theta}_i(t)$ in which case we will have $\hat{\theta}^{i+1}(t) \in \{(1/2)\sum_{j=i}^{i+1}\hat{\theta}_j(t), (1/3)\sum_{j=i}^{i+2}\hat{\theta}_j(t)\}$ and $\hat{\theta}^{i-1}(t) \in \{(1/2)\sum_{j=i-1}^{i}\hat{\theta}_j(t), (1/3)\sum_{j=i-1}^{i+1}\hat{\theta}_j(t)\}$;

(b) $\hat{\theta}^i(t) = (1/2)\sum_{j=i-1}^{i}\hat{\theta}_j(t)$ in which case we will have $\hat{\theta}^{i+1}(t) \in \{(1/2)\sum_{j=i}^{i+1}\hat{\theta}_j(t),\ (1/3)\sum_{j=i}^{i+2}\hat{\theta}_j(t)\}$;

(c) $\hat{\theta}^i(t) = (1/2)\sum_{j=i}^{i+1}\hat{\theta}_j(t)$ in which case we will have $\hat{\theta}^{i-1}(t) \in \{(1/2)\sum_{j=i-1}^{i}\hat{\theta}_j(t),\ (1/3)\sum_{j=i-2}^{i}\hat{\theta}_j(t)\}$;

(d) $\hat{\theta}^i(t) = (1/3)\sum_{j=i-1}^{i+1}\hat{\theta}_j(t)$.

We will provide an upper bound on $\|\hat{\theta}_i(t) - \hat{\theta}_k(t)\|$, where $k \in \{i-1, i+1\}$. Note that the farther apart the estimates are, the closer the resulting averaged estimate will be to the true parameter. We shall assume that we have situation (d) as this covers the rest, (a)–(c). This becomes obvious if we further assume that $\hat{\theta}_i(t) = \hat{\theta}_{i+1}(t)$ and let $e_i(t) = 0$. This ensures that the improvement can be attributed to the process of averaging alone. We also assume that $\|\tilde{\theta}_i(t)\| = \|\tilde{\theta}_{i-1}(t)\|$. Our goal is to see how far $\hat{\theta}_{i-1}(t)$ can be from $\hat{\theta}_i(t)$ such that the resulting improvement does not exceed $\varepsilon_1$. The calculation is presented in Fig. 5. As shown in Fig. 5(a), we are interested in establishing the distance $c$. It is, however, easier to calculate the distance $\bar{c}$ where $\bar{c} > c$ (see Fig. 5(b)). Solving the right triangle $AB\theta_0$ we see that $\bar{c} = 2\sqrt{3a\varepsilon_1 - 9/4\varepsilon_1^2}$. Clearly, this distance is controlled by $\varepsilon_1$. We can, therefore, conclude that $\|\hat{\theta}_i(t) - \hat{\theta}_k(t)\| \le 2\sqrt{3a\varepsilon_1 - 9/4\varepsilon_1^2}$, where $k \in \{i-1, i+1\}$.

To prove (d) we can bound the successive iterates at subsystem $i$ as follows:

$$\|\hat{\theta}_i(t+1) - \hat{\theta}_i(t)\| \le \left\| \frac{\hat{\theta}_i(t) + \hat{\theta}_{i-1}(t) + \hat{\theta}_{i+1}(t)}{3} + \frac{\beta_i(t+1)\phi_i(t)}{c + \phi_i(t)^T\phi_i(t)} e_i(t+1) - \hat{\theta}_i(t) \right\| \tag{48}$$

$$\le \frac{1}{3}\|\hat{\theta}_{i-1}(t) - \hat{\theta}_i(t)\| + \frac{1}{3}\|\hat{\theta}_{i+1}(t) - \hat{\theta}_i(t)\| + \varepsilon_0 \tag{49}$$

$$\le \frac{4}{3}\sqrt{3a\varepsilon_1 - 9/4\varepsilon_1^2} + \varepsilon_0. \tag{50}$$

Similarly we can calculate an upper bound on

$\|\hat{\theta}_i(t+1) - \hat{\theta}_k(t)\|$ for $k \in \{i-1, i+1\}$

controlled by $\varepsilon_1$ and $\varepsilon_0$. Clearly as $\varepsilon_0$ and $\varepsilon_1$ approach zero with $t \to \infty$, the difference between the estimates also approaches zero. This completes the proof. □

### 3.5. Information exchange

In this section we provide details about the information exchange that has to take place between the subsystems to execute the proposed algorithm.

(1) $\{\hat{\theta}_i(t), I_i(t)\}$.

After each iteration or update, each subsystem must provide its estimate and the associated cumulative improvement index to its immediate neighbors. The cumulative improvement index $I_i(t)$ can be computed iteratively; e.g. at each step, all one needs to do is calculate the term

$$\beta_i(t-1)\left[-2 + \frac{\beta_i(t-1)\phi_i(t-1)^T\phi_i(t-1)}{c + \phi_i(t-1)^T\phi_i(t-1)}\right] \times \frac{e_i(t)^2}{c + \phi_i(t-1)^T\phi_i(t-1)}$$

and add it to $I^i(t-1)$ to obtain $I_i(t)$.

(2) $[\{y_i(t-1), y_i(t-2), \ldots\}, \{y_{i-1}(t-1), y_{i-1}(t-2), \ldots\}, \{y_{i+1}(t-1), y_{i+1}(t-2), \ldots\}, \ldots]$, $[\{u_i(t-1), u_i(t-2), \ldots\}, \{u_{i-1}(t-1), u_{i-1}(t-2), \ldots\}, \{u_{i+1}(t-1), u_{i+1}(t-2), \ldots\}, \ldots]$.

Depending on the structure of the model of the system, each system must receive information about the outputs and inputs from other subsystems that affect it in order to construct the regression vector $\phi_i(t-1)$.

(3) $\{E_i(t), \tilde{\theta}(0)^T E_i(t)\}$.

More information exchange is required when the $\varepsilon$-rule comes into play or averaging takes place in order to keep track of the cumulative improvement index. We summarize this information exchange below.

(a) $E_i(t)$.

The vector $E_i(t)$ should be shared amongst the immediate neighbors in order to evaluate the expression $E_i(t)^T E_{i-1}(t)$. The record of this vector can be kept iteratively.

(b) $\tilde{\theta}(0)^T E_i(t)$.

Each subsystem should also share the scalar $\tilde{\theta}(0)^T E_i(t)$ with its immediate neighbors. The record of this scalar can be kept iteratively.

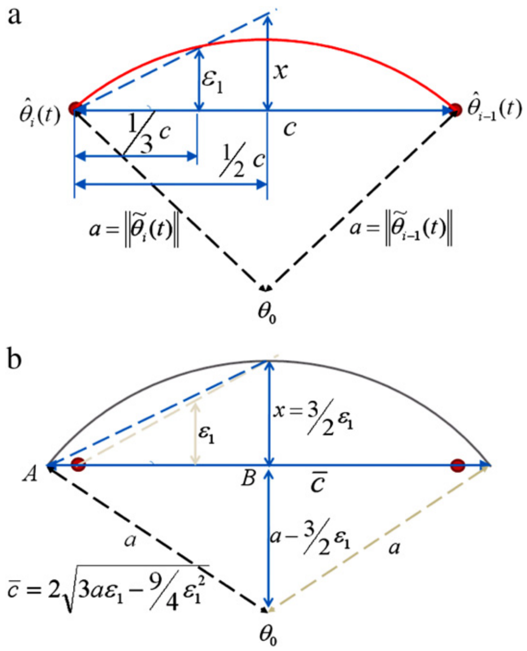
a
$\hat{\theta}_i(t)$
$\hat{\theta}_{i-1}(t)$
$\varepsilon_1$
$x$
$^1/_3\,c$
$c$
$^1/_2\,c$
$a = \|\tilde{\theta}_i(t)\|$
$a = \|\tilde{\theta}_{i-1}(t)\|$
$\theta_0$
b
$x = ^3/_2\,\varepsilon_1$
$\varepsilon_1$
$A$
$B$
$\bar{c}$
$a$
$a - ^3/_2\,\varepsilon_1$
$a$
$\bar{c} = 2\sqrt{3a\varepsilon_1 - ^9/_4\,\varepsilon_1^2}$
$\theta_0$

**Fig. 5.** Calculation of the upper bound on the distance between local estimates.

### 3.6. Simulation

We remark here that the results developed in this article are valid and applicable for a general class of linear spatiotemporally invariant systems that have rational transfer functions with finite spatial interactions. Within this class also fall the circulant systems that 'wrap around' themselves and have rational transfer functions. These systems are finite analogues of infinite spatiotemporally invariant systems. Systems that can be modeled as circulant include deformable mirrors used in adaptive optics (Denis, 1998), and cross-directional control in paper machines (Laughlin, Morari, & Braatz, 1993). In this section we present an implementation of our algorithm on a simple circulant system to illustrate its performance. The basic structure of a circulant system is presented in Fig. 6. Each subsystem has the same dynamics. For $i = N$, $i+1 = 1$. Likewise, for $i = 1$, $i-1 = N$. The dynamics of the

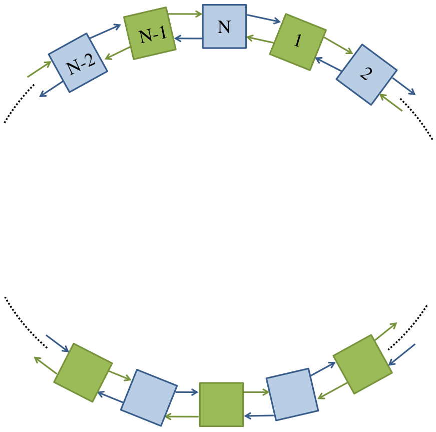

**Fig. 6.** Circulant system.

$i$th subsystem is given as follows:

$$y_i(t+1) = -0.6y_i(t) + 0.1y_{i-1}(t) + 0.1y_{i+1}(t) + u_i(t). \tag{51}$$

The initial guess for all the subsystems was taken as $\hat{\theta}(0) = [1, 1, 1, 1]^T$. The input for the subsystems was chosen as given below:

$$u_i(t) = \cos(t+i) + \sin(t - i/2). \tag{52}$$

A system comprising $N = 100$ subsystems was simulated. The standard projection algorithm as presented in Goodwin and Sin (1984) was simulated along with the proposed distributed projection algorithm. The update equation for the standard projection algorithm, using the notation used in this paper, is presented below:

$$\hat{\theta}_i(t+1) = \hat{\theta}_i(t) + \frac{\beta_i(t)\phi_i(t)}{c + \phi_i(t)^T\phi_i(t)}[y_i(t+1) - \phi_i(t)^T\hat{\theta}_i(t)] \tag{53}$$

with $0 < \beta_i(t) < 2$. Each subsystem starts with the same initial guess $\hat{\theta}_i(0) = \theta(0)$. There is no spatial update for the standard projection algorithm, as estimates are not exchanged amongst the immediate neighbors—each estimate, hence, evolves independently.

For the implementation of the distributed projection algorithm, a constant value of $\varepsilon = 10^{-5}$ was chosen. The results are captured in Fig. 7. The top plots in Fig. 7 present the distances of the estimates of subsystem 1 from the neighbors, four to its right and four to its left. The bottom plots, on the other hand, present the distances of the subsystems $(N-3, \dots, N, 1, \dots 5)$ from the true parameter $\theta_0$. A clear difference can be seen between the two plots. For the standard projection algorithm, the parameter estimates cease to change after about 20 ($\times 10$) time steps. Also, the parameter estimates of the neighbors for subsystem 1 are quite far apart from it. The performance of the distributed projection algorithm, on the other hand, is quite outstanding. Not only do the parameters for the subsystems converge to the true value, but also their distance from subsystem 1 can be seen to go to zero. It is quite interesting to note that the input signal was not 'exciting' enough for the standard projection algorithm, whereas convergence to the true parameter was seen for the distributed projection algorithm. Establishing excitation conditions necessary for true system identification will be part of future research work.

### 3.6.1. Choice of $\varepsilon$

Simulations were also carried out to demonstrate the choice of $\varepsilon$ for the $\varepsilon$-rule. A system of 10 subsystems, having the same dynamics as given in (51), was simulated with the initial guess for all the subsystems taken as $\hat{\theta}(0) = [1, 1, 1, 1]^T$. The input for the subsystems was chosen as follows:

$$u_i(t) = cos(t+i). \tag{54}$$

Fig. 8 presents a comparison of choosing $\varepsilon = 10^{-5}$ against choosing $\varepsilon = 10^{-1}$. It is interesting to note that a smaller value of $\varepsilon$ results in bringing the estimates closer at a faster rate than a bigger value that enforces averaging. This is due to the fact that a smaller value of $\varepsilon$ emphasizes improving the estimates rather than bringing the estimates closer. In doing this, the estimates get closer to the true parameter and hence to each other as a result.

## 4. Parameter estimation with bounded noise

We now consider the following system model:

$$y_i(t) = \phi_i(t-1)^T\theta_0 + w_i(t) \tag{55}$$

where $w_i(t)$ denotes a bounded 'noise' term that can account for measurement noise, inaccurate modeling, computer round-off error etc. We know, however, that $w_i(t)$ is bounded, i.e. $\sup_{i,t}|w_i(t)| \leq \Delta$. We modify the distributed projection algorithm presented above as follows to account for the measurement noise:

$$\hat{\theta}_i(t+1) = \hat{\theta}^i(t) + \frac{\beta_i(t)\phi_i(t)}{c + \phi_i(t)^T\phi_i(t)}[y_i(t+1) - \phi_i(t)^T\hat{\theta}^i(t)] \tag{56}$$

where $\theta(0)$ is given, $c > 0$, and

$$\beta_i(t) = \begin{cases} 1 & \text{if } |y_i(t+1) - \phi_i(t)^T\hat{\theta}^i(t)| > 2\Delta \\ 0 & \text{otherwise.} \end{cases} \tag{57}$$

The motivation above is to turn off the algorithm when the prediction error is small compared to the size of the noise.

### 4.1. Computing bounds on $I_i(\cdot)$

We note here that it no longer remains possible to exactly establish the cumulative improvement index. We can, however, calculate a lower bound on its magnitude. This lower bound can be used to identify the better estimate as we explain in the sequel. To simplify the exposition, as in the noiseless case, we first consider the initial $t$ time steps when no averaging takes place and $\hat{\theta}^i(t) \in \{\hat{\theta}_{i-1}(t), \hat{\theta}_i(t), \hat{\theta}_{i+1}(t)\}$. Let us examine the update of the system $i$ from time $t = 0$ to time $t = 1$:

$$\hat{\theta}_i(1) = \hat{\theta}(0) + \frac{\beta_i(0)\phi_i(0)}{c + \phi_i(0)^T\phi_i(0)}[y_i(1) - \phi_i(0)^T\hat{\theta}(0)]. \tag{58}$$

Subtracting $\theta_0$ from both sides and taking the inner product with itself, we get

$$\begin{aligned}\|\tilde{\theta}_i(1)\|^2 &= \|\tilde{\theta}(0)\|^2 + \frac{\beta_i(0)^2\phi_i(0)^T\phi_i(0)}{[c + \phi_i(0)^T\phi_i(0)]^2}e_i(1)^2 \\ &\quad + \frac{2\beta_i(0)\tilde{\theta}(0)^T\phi_i(0)}{c + \phi_i(0)^T\phi_i(0)}e_i(1).\end{aligned} \tag{59}$$

Using the fact that

$$\tilde{\theta}(0)^T\phi_i(0) = w_i(1) - e_i(1) \tag{60}$$

we can rewrite (59) as follows:

$$\begin{aligned}\|\tilde{\theta}_i(1)\|^2 &= \|\tilde{\theta}(0)\|^2 + \beta_i(0)\left[-2 + \frac{\beta_i(0)\phi_i(0)^T\phi_i(0)}{c + \phi_i(0)^T\phi_i(0)}\right] \\ &\quad \times \frac{e_i(1)^2}{c + \phi_i(0)^T\phi_i(0)} + \frac{\beta_i(0)}{c + \phi_i(0)^T\phi_i(0)} \\ &\quad \times [2e_i(1)w_i(1)]\end{aligned} \tag{61}$$

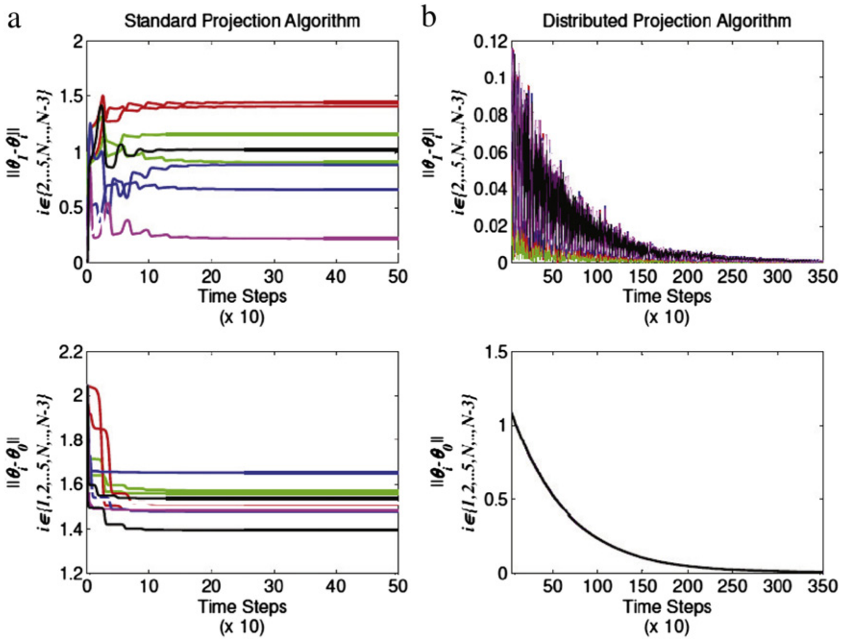

**Fig. 7.** Comparison of the distributed projection algorithm (a) with the standard projection algorithm (b).

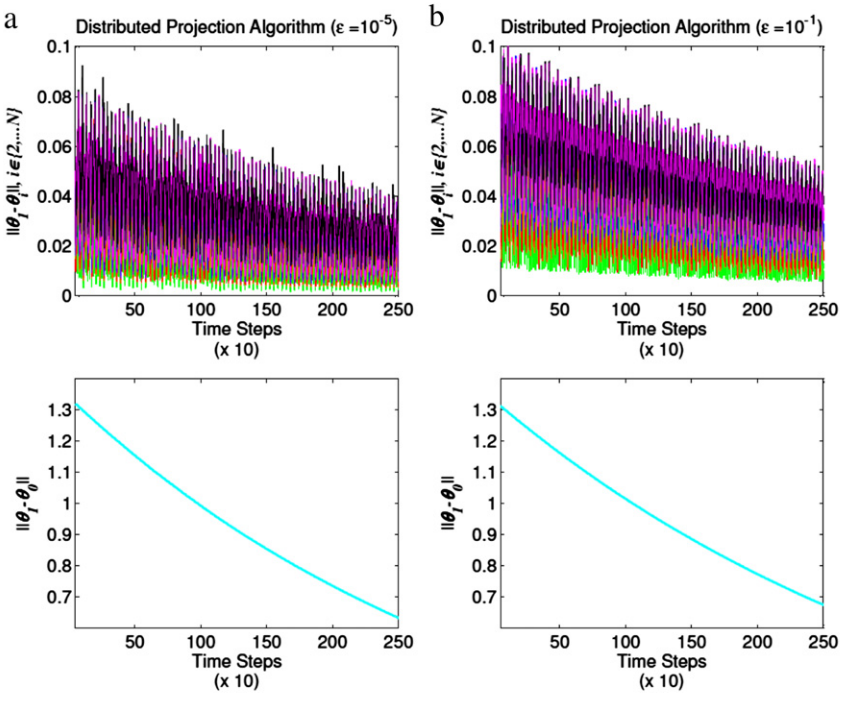

**Fig. 8.** Figure showing the effect of choice of $\varepsilon$ for the $\varepsilon$-rule in the distributed projection algorithm.

$$\|\tilde{\theta}_i(1)\|^2 \leq \|\tilde{\theta}(0)\|^2 + \beta_i(0)\left[-2 + \frac{\beta_i(0)\phi_i(0)^T\phi_i(0)}{c+\phi_i(0)^T\phi_i(0)}\right] \times \frac{e_i(1)^2}{c+\phi_i(0)^T\phi_i(0)} + \frac{\beta_i(0)}{c+\phi_i(0)^T\phi_i(0)} \times [2|e_i(1)|\Delta]. \tag{62}$$

We define $\hat{I}_i(1)$ as follows:

$$\hat{I}_i(1) := \beta_i(0)\left[-2 + \frac{\beta_i(0)\phi_i(0)^T\phi_i(0)}{c+\phi_i(0)^T\phi_i(0)}\right]\frac{e_i(1)^2}{c+\phi_i(0)^T\phi_i(0)} + \frac{\beta_i(0)}{c+\phi_i(0)^T\phi_i(0)}[2|e_i(1)|\Delta]. \tag{63}$$

$\hat{I}_i(1)$ is an overestimate (or its magnitude serves as a lower bound) for the actual cumulative improvement index $I_i(1)$. Note that the use of a dead-zone in the algorithm above makes sure that $\hat{I}_i(1)$ remains non-positive. To see this note that

$$\beta_i(0)\left[-2 + \frac{\beta_i(0)\phi_i(0)^T\phi_i(0)}{c+\phi_i(0)^T\phi_i(0)}\right] \leq -1 \quad \text{if } |y_i(t+1) - \phi_i(t)^T\hat{\theta}_i(t)| > 2\Delta \tag{64}$$

and

$$\frac{e_i(1)^2}{c+\phi_i(0)^T\phi_i(0)} \geq \frac{2|e_i(1)|\Delta}{c+\phi_i(0)^T\phi_i(0)}. \tag{65}$$

One can write any update in a similar fashion as described above and the upper bound on the cumulative improvement index can be established iteratively. For example, from the update relationship of $\hat{\theta}_i(t)$, we can establish $\hat{I}_i(t)$ as follows:

$$\begin{aligned}\|\tilde{\theta}_i(t)\|^2 = {} & \|\tilde{\theta}^i(t-1)\|^2 + \beta_i(t-1) \\ & \times \left[-2 + \frac{\beta_i(t-1)\phi_i(t-1)^T\phi_i(t-1)}{c+\phi_i(t-1)^T\phi_i(t-1)}\right] \\ & \times \frac{e_i(t)^2}{c+\phi_i(t-1)^T\phi_i(t-1)} \\ & + \frac{\beta_i(t-1)}{c+\phi_i(t-1)^T\phi_i(t-1)} \\ & \times [2e_i(t)w_i(t)]\end{aligned} \tag{66}$$

$$\begin{aligned}\|\tilde{\theta}_i(t)\|^2 \leq {} & \|\tilde{\theta}(0)\|^2 + \hat{I}^i(t-1) + \beta_i(t-1) \\ & \times \left[-2 + \frac{\beta_i(t-1)\phi_i(t-1)^T\phi_i(t-1)}{c+\phi_i(t-1)^T\phi_i(t-1)}\right] \\ & \times \frac{e_i(t)^2}{c+\phi_i(t-1)^T\phi_i(t-1)} \\ & + \frac{\beta_i(t-1)}{c+\phi_i(t-1)^T\phi_i(t-1)} \\ & \times [2|e_i(t)|\Delta]\end{aligned} \tag{67}$$

and $\hat{I}_i(t+1)$ is given as

$$\begin{aligned}\hat{I}_i(t+1) = {} & \hat{I}^i(t) + \beta_i(t)\left[-2 + \frac{\beta_i(t)\phi_i(t)^T\phi_i(t)}{c+\phi_i(t)^T\phi_i(t)}\right] \\ & \times \frac{e_i(t+1)^2}{c+\phi_i(t)^T\phi_i(t)} + \frac{\beta_i(t)}{c+\phi_i(t)^T\phi_i(t)} \\ & \times [2|e_i(t+1)|\Delta].\end{aligned} \tag{68}$$

The upper bound on the cumulative improvement index, whenever averaging needs to take place, can be calculated as explained in the following. For clarity of exposition, assume that for the $t+1$ iterate we have $\hat{\theta}^i(t) = \frac{\hat{\theta}_i(t)+\hat{\theta}_{i-1}(t)}{2}$ for the subsystem *i*. Note that any other case for the choice of $\hat{\theta}^i(t)$ will follow a similar development. The calculation of $I^i(t)$ can now be given as

$$\begin{aligned}\hat{\theta}^i(t) &= \frac{\hat{\theta}_i(t)+\hat{\theta}_{i-1}(t)}{2} = \frac{\hat{\theta}(0)+E_i(t)+\hat{\theta}(0)+E_{i-1}(t)}{2} \\ &= \hat{\theta}(0) + \frac{E_i(t)+E_{i-1}(t)}{2}\end{aligned} \tag{69}$$

and subtracting $\theta_0$ from both sides and taking the inner product with itself, we get

$$\begin{aligned}\|\tilde{\theta}^i(t)\|^2 = {} & \|\tilde{\theta}(0)\|^2 + \tilde{\theta}(0)^T[E_i(t)+E_{i-1}(t)] \\ & + \frac{1}{4}[E_i(t)+E_{i-1}(t)]^T[E_i(t)+E_{i-1}(t)].\end{aligned} \tag{70}$$

From the above expression, we can write $I^i(t)$ as follows:

$$\begin{aligned}I^i(t) = {} & \tilde{\theta}(0)^T[E_i(t)+E_{i-1}(t)] \\ & + \frac{1}{4}[E_i(t)+E_{i-1}(t)]^T[E_i(t)+E_{i-1}(t)].\end{aligned} \tag{71}$$

The upper bound on the expression $\tilde{\theta}(0)^TE_i(t)$ can be iteratively computed as follows:

$$\begin{aligned}\tilde{\theta}(0)^TE_i(t) &= \tilde{\theta}(0)^T\left(\sum_{\tau=1}^{t}\left[\frac{\beta_k(\tau-1)\phi_k(\tau-1)}{c+\phi_k(\tau-1)^T\phi_k(\tau-1)}\right]e_k(\tau)\right) \\ &= \sum_{\tau=1}^{t}\beta_k(\tau-1)\left[\frac{\hat{\theta}(0)^T\phi_k(\tau-1)-y_k(\tau)+w_k(\tau)}{c+\phi_k(\tau-1)^T\phi_k(\tau-1)}\right]e_k(\tau) \\ &\leq \sum_{\tau=1}^{t}\beta_k(\tau-1)\left[\frac{\hat{\theta}(0)^T\phi_k(\tau-1)-y_k(\tau)}{c+\phi_k(\tau-1)^T\phi_k(\tau-1)}\right]e_k(\tau) \\ &\quad + \sum_{\tau=1}^{t}\beta_k(\tau-1)\left[\frac{\Delta}{c+\phi_k(\tau-1)^T\phi_k(\tau-1)}\right]|e_k(\tau)| \\ &= \underbrace{\sum_{\tau=1}^{t}\beta_k(\tau-1)\left(\left[\frac{\hat{\theta}(0)^T\phi_k(\tau-1)-y_k(\tau)}{c+\phi_k(\tau-1)^T\phi_k(\tau-1)}\right]e_k(\tau) + \left[\frac{\Delta}{c+\phi_k(\tau-1)^T\phi_k(\tau-1)}\right]|e_k(\tau)|\right)}_{=:\tilde{\theta}(0)^T\hat{E}_i(t)} \\ &= \tilde{\theta}(0)^T\hat{E}_i(t-1) + \beta_i(t-1)\left[\frac{\hat{\theta}(0)^T\phi_i(t-1)-y_i(t)}{c+\phi_i(t-1)^T\phi_i(t-1)}\right] \\ &\quad \times e_i(t) + \left[\frac{\beta_i(t-1)\Delta}{c+\phi_i(t-1)^T\phi_i(t-1)}\right]|e_i(t)|.\end{aligned} \tag{72}$$

We can, therefore, write the expression for $\hat{I}^i(t)$ as follows:

$$\begin{aligned}\hat{I}^i(t) = {} & \tilde{\theta}(0)^T[\hat{E}_i(t)+\hat{E}_{i-1}(t)] \\ & + \frac{1}{4}[E_i(t)+E_{i-1}(t)]^T[E_i(t)+E_{i-1}(t)].\end{aligned} \tag{73}$$

It is, however, not clear at this time that the upper bound $\hat{I}^i(t)$ calculated in this way is at least as good as $\hat{I}_i(t)$. This is required for correct execution of the algorithm and to make sure that the algorithm guarantees continuous improvement, at least in terms of establishing bounds on the successive terms. To see that $\hat{I}^i(t)$ is indeed as good as $\hat{I}_i(t)$, note that we can write $\hat{I}^i(t)$ as follows:

$$\begin{aligned}\hat{I}^i(t) & \\ = {} & \underbrace{\tilde{\theta}(0)^T[\hat{E}_i(t)+\hat{E}_{i-1}(t)] + \frac{1}{2}E_i(t)^TE_i(t) + \frac{1}{2}E_{i-1}(t)^TE_{i-1}(t)}_{=\frac{\hat{I}_i(t)+\hat{I}_{i-1}(t)}{2}} \\ & + \frac{1}{2}E_i(t)^TE_{i-1}(t) - \frac{1}{4}(E_i(t)^TE_i(t)+E_{i-1}(t)^TE_{i-1}(t))\end{aligned} \tag{74}$$

$$\hat{I}^i(t) = \frac{\hat{I}_i(t)+\hat{I}_{i-1}(t)}{2} - \frac{1}{4}[E_i(t)-E_{i-1}(t)]^T[E_i(t)-E_{i-1}(t)] \tag{75}$$

$$\hat{I}^i(t) \leq \frac{\hat{I}_i(t)+\hat{I}_i(t)}{2} - \frac{1}{4}[E_i(t)-E_{i-1}(t)]^T[E_i(t)-E_{i-1}(t)] \tag{76}$$

since $\hat{I}_{i-1}(t) \leq \hat{I}_i(t)$

$$\hat{I}^i(t) \leq \hat{I}_i(t). \tag{77}$$

Note that for $\Delta = 0$, the above modified algorithm coincides with the one described for the noiseless system. We now sum up the properties of this algorithm in the following lemma.

**Lemma 2.** *For the algorithm (56) and (57) and subject to (55) with the upper bound $\hat{I}_i(t)$ as discussed above, it follows that:*

(1) $\|\hat{\theta}_i(t)-\theta_0\| \leq \|\hat{\theta}^i(t-1)-\theta_0\| \leq \|\hat{\theta}(0)-\theta_0\|,\ t \geq 1$;
(2) $\lim_{t\to\infty}\beta_i(t-1)\frac{e_i(t)^2-4\Delta^2}{[c+\phi_i(t-1)^T\phi_i(t-1)]} \leq 0$;
(3) $\lim_{t\to\infty}\|\hat{\theta}_i(t)-\hat{\theta}_k(t)\| = 0$ *with* $k \in \{i-1, i+1\}$;
(4) $\lim_{t\to\infty}\|\hat{\theta}_i(t)-\hat{\theta}_i(t-1)\| \leq \frac{2\Delta}{\sqrt{c}}$.

**Proof.** (1) The proof is obvious from the discussion above.

(2) From (66), we have

$$\|\tilde{\theta}_i(t+1)\|^2 \leq \|\tilde{\theta}(0)\|^2 + \hat{I}^i(t) - \beta_i(t)\frac{e_i(t+1)^2}{c+\phi_i(t)^T\phi_i(t)} + \frac{\beta_i(t)2e_i(t+1)w_i(t+1)}{c+\phi_i(t)^T\phi_i(t)} \tag{78}$$

$$\leq \|\tilde{\theta}(0)\|^2 + \hat{I}^i(t) - \beta_i(t)\frac{e_i(t+1)^2}{c+\phi_i(t)^T\phi_i(t)} + \frac{\beta_i(t)}{c+\phi_i(t)^T\phi_i(t)} \times \left[\frac{e_i(t+1)^2}{2} + 2w_i(t+1)^2\right] \tag{79}$$

since $2ab \leq ka^2 + \frac{b^2}{k}$ for any $k$, and

$$\leq \|\tilde{\theta}(0)\|^2 + \hat{I}^i(t) - \beta_i(t)\frac{e_i(t+1)^2}{c+\phi_i(t)^T\phi_i(t)} + \frac{\beta_i(t)}{c+\phi_i(t)^T\phi_i(t)}\left[\frac{e_i(t+1)^2}{2} + 2\Delta^2\right] \tag{80}$$

$$\leq \|\tilde{\theta}(0)\|^2 + \hat{I}^i(t) - \frac{\beta_i(t)}{2}\left[\frac{e_i(t+1)^2 - 4\Delta^2}{c+\phi_i(t)^T\phi_i(t)}\right]. \tag{81}$$

Since the left hand side above is bounded below by zero, the result follows.

The proof of (3) follows exactly as presented in Section 2.

To prove (4) note that given $\varepsilon_0 > 0$, $\exists\, T_{\varepsilon_0} \mid \forall\, t \geq T_{\varepsilon_0}$, and $\forall\, i$ we have

$$\left|\beta_i(t-1)\frac{e_i(t)}{[c+\phi_i(t-1)^T\phi_i(t-1)]^{1/2}}\right| \leq \beta_i(t-1)\frac{2\Delta}{\sqrt{c}} + \varepsilon_0, \tag{82}$$

i.e. the (normalized) estimation error will be close to $2\Delta$ (and will remain so) for all subsystems after time $T_{\varepsilon_0}$. We have

$$\|\hat{\theta}_i(t+1) - \hat{\theta}_i(t)\| \leq \left\|\frac{\hat{\theta}_i(t) + \hat{\theta}_{i-1}(t) + \hat{\theta}_{i+1}(t)}{3} + \frac{\beta_i(t)\phi_i(t)}{c+\phi_i(t)^T\phi_i(t)}e_i(t+1) - \hat{\theta}_i(t)\right\| \tag{83}$$

$$\leq \frac{1}{3}\|\hat{\theta}_{i-1}(t) - \hat{\theta}_i(t)\| + \frac{1}{3}\|\hat{\theta}_{i+1}(t) - \hat{\theta}_i(t)\| + \beta_i(t)\frac{2\Delta}{\sqrt{c}} + \varepsilon_0 \tag{84}$$

$$\leq \frac{4}{3}\sqrt{3a\varepsilon_1 - 9/4\varepsilon_1^2} + \beta_i(t)\frac{2\Delta}{\sqrt{c}} + \varepsilon_0 \tag{85}$$

$$\leq \frac{4}{3}\sqrt{3a\varepsilon_1 - 9/4\varepsilon_1^2} + \frac{2\Delta}{\sqrt{c}} + \varepsilon_0. \tag{86}$$

The result follows as $t \to \infty$. □

### 4.2. Simulation

In this section we present an implementation of our algorithm on the same circulant system as was presented in Section 2 with noise added to the output. The dynamics of the $i$th subsystem is given as follows:

$$y_i(t+1) = -0.6y_i(t) + 0.1y_{i-1}(t) + 0.1y_{i+1}(t) + u_i(t) + w_i(t) \tag{87}$$

where $|w_i(t)| \leq \Delta = 0.4$. The initial guess for all the subsystems was taken as $\theta(0) = [1, 1, 1, 1]^T$. The input for the subsystems was chosen as follows:

$$u_i(t) = \cos(t+i) + \sin(t - i/2). \tag{88}$$

A system comprising $N = 100$ subsystems was simulated. The standard projection algorithm as described in Section 3.6 was simulated along with the proposed distributed projection algorithm. For the implementation of the distributed projection algorithm, $\varepsilon = 10^{-5}$ was chosen. The results are captured in Fig. 9. The top plots for Fig. 9 present the distances of the estimates of the subsystem 1 from the neighbors, four to its right and four to its left. The bottom plots, on the other hand, present the distances of the subsystems $(N-3, \ldots, N, 1, \ldots, 5)$ from the true parameter $\theta_0$. A clear difference can be seen between the standard projection and distributed projection algorithm. For the standard projection algorithm, the parameter estimates cease to change at about 250 $(\times 10)$ time steps. Also, the parameter estimates of the neighbors for subsystem 1 are quite far apart from it. While the distributed projection algorithm shows no significant improvement in terms of getting closer to the true estimate, it settles to the final estimate a lot faster. The parameter estimation in this case ceases to improve after about 30 $(\times 10)$ time steps. It should be noted that the convergence of the parameter estimates is in both of the algorithms enforced by the defined dead-zone, which remains the same in the implementations of both algorithms. For the projection algorithm, it can be seen that the distance between subsystem 1 and its neighbors vanishes completely at around 30 $(\times 10)$ time steps.

## 5. Conclusions

We have presented a distributed projection algorithm for system identification of spatiotemporally invariant systems, with a perspective of adaptive control. Each subsystem receives information from all of its neighbors affecting it, in order to construct the regression vector. Each subsystem, however, communicates only with its immediate neighbor to share its current estimate along with the related information (cumulative improvement index etc). The best available estimate is picked in order to carry out the next iteration. For small estimation error, the scheme switches over to a "smart" averaging routine. The scheme ensures continuous decay of the estimation error and serves to bring the local estimates arbitrarily close to one another. For a system with bounded noise added to the output, it was shown that for a given time step, the dead-zone algorithm serves to bring the local estimates arbitrarily close to one another. The scheme was seen to operate in a significantly better fashion than the standard projection algorithm, even in the presence of bounded noise. It was also seen that the parameter estimates converge to the true value even when the standard projection algorithm fails to do so. This calls for an investigation into the excitation conditions necessary for true parameter identification for the distributed projection algorithm. This work will be part of our future research work. We remark here, however, that such conditions should be a function of the underlying network structure, and should change with changes in the subsystem interconnections.

We also comment here that although our focus is on system identification of spatiotemporally invariant systems, the proposed algorithm is very well equipped to address the estimation problem in distributed networks. Distributed networks embedded with cooperative algorithms have been proposed for addressing estimation problems that arise in a variety of applications, such as environment monitoring, target localization and potential sensor network problems (Estrin, Pottie, & Srivastava, 2001). Hence, a

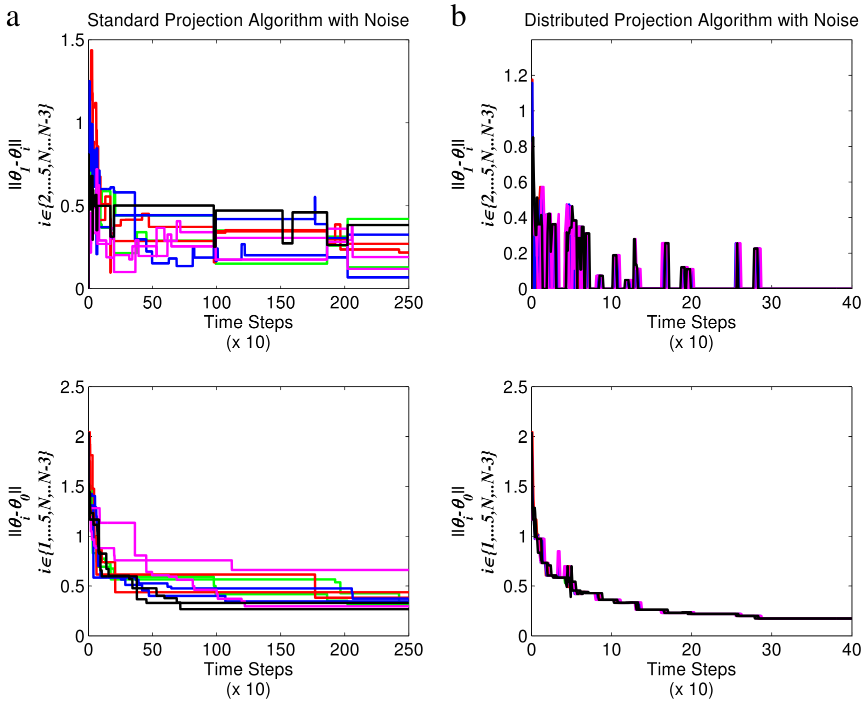

**Fig. 9.** Parameter estimation with bounded noise: comparison of the distributed projection algorithm (a) with the standard projection algorithm (b).

brief discussion on the mean stability of the proposed algorithm is provided in the Appendix for the interested reader.

Finally the idea presented in this paper can be extended easily to other identification schemes such as least squares. This is obvious since the key point is the ability to write the given estimate as the sum of the original guess and some improvement terms.

## Appendix. Mean stability

This section presents discussion about the stability of the proposed algorithm in the mean, for the interested reader who may want to look at it from the perspective of adaptive filtering, although the intent of the developed algorithm is for control adaptation. Define the following global variables:

$$\Theta_t := \{\cdots, \hat{\theta}_{i-1}(t), \hat{\theta}_i(t), \hat{\theta}_{i+1}(t), \ldots\}^T, \tag{A.1}$$

$$\Theta^t := \{\cdots, \hat{\theta}^{i-1}(t), \hat{\theta}^i(t), \hat{\theta}^{i+1}(t), \ldots\}^T, \tag{A.2}$$

$$\Theta := \{\cdots, \theta_0, \theta_0, \theta_0, \ldots\}^T, \tag{A.3}$$

$$\tilde{\Theta}_t := \Theta_t - \Theta, \tag{A.4}$$

$$\hat{\beta}_i(t) := \frac{\beta_i(t)}{c + \phi_i(t)^T \phi_i(t)}, \tag{A.5}$$

$$\mathbb{B}_t := \text{diag}\{\cdots, \hat{\beta}_{i-1}(t), \hat{\beta}_i(t), \hat{\beta}_{i+1}, \ldots\}, \tag{A.6}$$

$$\Phi_t := \text{diag}\{\cdots, \phi_{i-1}(t), \phi_i(t), \phi_{i+1}(t), \ldots\}, \tag{A.7}$$

$$Y_t := \{\cdots, y_{i-1}(t), y_i(t), y_{i+1}(t), \ldots\}, \tag{A.8}$$

$$\alpha^i(t) := [\alpha^i_{i-1}(t) \;\; \alpha^i_i(t) \;\; \alpha^i_{i+1}(t)], \tag{A.9}$$

$$\mathbb{G}_t := \text{diag}\{\cdots, \alpha^{i-1}(t), \alpha^i(t), \alpha^{i+1}(t), \ldots\}, \tag{A.10}$$

$$W_t := \{\cdots, w_{i-1}(t), w_i(t), w_{i+1}(t), \ldots\}^T. \tag{A.11}$$

Before proceeding with the analysis, certain customary assumptions are made on the data for such algorithms (see, for example, Cattivelli and Sayed (2011) and the references therein) in order to make the analysis tractable and to gain insights into the behavior. The noise, $w_i(t)$, is assumed to be independent over time and space with zero mean, i.e., independent of $w_j(\tau)$ for $i \neq j$ and $t \neq \tau$, and is also independent of $\phi_i(t)$ for all $i$ and $t$. The regression vector, $\phi_i(t)$, is also independent over space and time. While these assumptions can be restrictive, it has been shown that (Sayed, 2003) performance results of similar algorithms that are obtained under these assumptions tend to match well with practice for sufficiently small step sizes, and can be achieved by choosing $\beta_i(t) \ll 1$ for all $i$ and $t$ in this case. Also, we shall assume that the $\varepsilon$-rule is in place across the entire system such that $\mathbb{G}_t = \mathbb{G} = \text{diag}\{\cdots, \alpha^{i-1}, \alpha^i, \alpha^{i+1}, \ldots\}$, where $\alpha^j = [\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]\ \forall j \in \mathbb{Z}$. For the entire system, the update equation, Eq. (13), can now be written as

$$\Theta_{t+1} = \Theta^t + \mathbb{B}_t \Phi_t (Y_{t+1} - \Phi_t^T \Theta^t). \tag{A.12}$$

Note that since $\Theta^t = \mathbb{G}\Theta_t$, we can write (A.12) as

$$\Theta_{t+1} = \mathbb{G}\Theta_t + \mathbb{B}_t \Phi_t (\Phi_t^T \Theta + W_t - \Phi_t^T \Theta^t) \tag{A.13}$$

$$\Theta_{t+1} = \mathbb{G}\Theta_t + \mathbb{B}_t \Phi_t (\Phi_t^T \Theta - \Phi_t^T \mathbb{G}\Theta_t) + \mathbb{B}_t \Phi_t W_t. \tag{A.14}$$

Subtracting $\Theta$ from both sides of the last expression above and noting that $\mathbb{G}\Theta = \Theta$, we get

$$\tilde{\Theta}_{t+1} = \mathbb{G}\tilde{\Theta}_t - \mathbb{B}_t \Phi_t \Phi_t^T \mathbb{G}\tilde{\Theta}_t + \mathbb{B}_t \Phi_t W_t \tag{A.15}$$

$$\tilde{\Theta}_{t+1} = (\mathbb{I} - \mathbb{B}_t \Phi_t \Phi_t^T)\mathbb{G}\tilde{\Theta}_t + \mathbb{B}_t \Phi_t W_t \tag{A.16}$$

where $\mathbb{I}$ is the identity. Taking the expectation of both sides of (A.16) we get

$$\mathscr{E}[\tilde{\Theta}_{t+1}] = (\mathbb{I} - \mathscr{E}[\mathbb{B}_t \Phi_t \Phi_t^T])\mathbb{G}\mathscr{E}[\tilde{\Theta}_t]. \tag{A.17}$$

Define $\mathbb{K} := \mathscr{E}[\mathbb{B}_t \Phi_t \Phi_t^T]$; then we require $\|(\mathbb{I} - \mathbb{K})\mathbb{G}\| < 1$ in order to achieve mean stability. Note that we have

$$\|(\mathbb{I} - \mathbb{K})\mathbb{G}\| \le \|(\mathbb{I} - \mathbb{K})\| \|\mathbb{G}\| \tag{A.18}$$

and given that $\mathbb{G}$ is a banded diagonal matrix, with the sum of elements in each row being equal to 1, we have $\|\mathbb{G}\| = 1$. Hence, requiring $\|(\mathbb{I} - \mathbb{K})\| < 1$ guarantees mean stability. Note that this condition depends heavily on the structure of $\Phi_t$, and in turn depends on the system inputs, and thus is an excitation condition. For the standard non-cooperative case, it is well know that excitation conditions are required in order to identify the true parameter (Goodwin & Sin, 1984). As discussed above, we, however, do not require mean stability or identification of the true parameter for control adaptation.

## References

Bamieh, B., Paganini, B. F., & Dahleh, M. A. (2002). Distributed control of spatially invariant systems. *IEEE Transactions on Automatic Control*, *47*(7), 1091–1107.

Banks, H. T., & Kunish, K. (1989). Estimation techniques for distributed parameter systems. *Systems & Control: Foundation & Applications*, *1*, Note(s); XIII-315.

Bose, N. K. (1982). *Applied multidimensional systems theory*. Van Nostrand Reinhold Company.

Cattivelli, F. S., & Sayed, A. H. (2011). Analysis of spatial and incremental LMS processing for distributed estimation. *IEEE Transactions on Signal Processing*, *59*(4), 1465–1480.

Chichka, D. F., & Speyer, J. L. (1998). Solar-powered, formation-enhanced aerial vehicle system for sustained endurance. In *Proc. Amer. control conf.* (pp. 684–688).

Curtain, R., Iftime, O., & Zwart, H. (2010). A comparison between LQR control for a long string of SISO systems and LQR control of the infinite spatially invariant version. *Automatica*, *46*(10), 1604–1615.

D'Andrea, R., & Dullerud, G. E. (2003). Distributed control design for spatially interconnected systems. *IEEE Transaction on Automatic Control*, *48*(9).

Denis, N. (1998). Solution of optimization problems with spatial symmetry and applications to adaptive optics. *Ph.D. thesis*. University of Massachusetts, August 1998.

Estrin, D., Pottie, G., & Srivastava, M. (2001). Instrumenting the world with wireless sensor networks. In *Proc. IEEE ICASSP. Salt Lake City, UT. May* (pp. 2033–2036).

Gavel, D. T., & Siljak, D. D. (1989). Decentralized adaptive control: structural conditions for stability. *IEEE Transactions on Automatic Control*, *34*(4), 413–426.

Goodwin, G. C., & Sin, K. S. (1984). *Adaptive filtering prediction and control*. Prentice Hall.

Ioannou, P., & Kokotovic, P. (1985). Decentralized adaptive control of interconnected systems with reduced-order models. *Automatica*, *21*(4), 401–412.

Jiang, S., Voulgaris, P. G., Holloway, L. E., & Thompson, L. A. (2006). Distributed control of large segmented telescopes. In *Proc. ACC*.

Kubrusly, C. S. (1977). Distributed parameter system identification: a survey. *International Journal of Control*, *26*(4), 509–535.

Laughlin, D. L., Morari, M., & Braatz, R. D. (1993). Robust performance of cross-directional basis-weight control in paper machines. *Automatica*, *29*(6), 1395–1410.

Lopes, C. G., & Syed, A. H. (2007). Diffusion least-mean squares over adaptive networks. In *Proc. IEEE ICASSP. Hawaii*.

Massioni, P., & Verhaegen, M. (2008). Subspace identification of circulant systems. *Automatica*, *44*, 2825–2833.

Mirkin, B. M., & Gutman, P. O. (2003). Decentralized output-feedback MRAC of linear state delay systems. *IEEE Transactions on Automatic Control*, *48*(9), 1613.

Narendra, K. S., & Oleng, N. O. (2002). Exact output tracking in decentralized adaptive control systems. *IEEE Transactions on Automatic Control*, *47*(2), 390–395.

Ossman, K. A. (1989). Indirect adaptive control for interconnected systems. *IEEE Transactions on Automatic Control*, *34*(8), 908–911.

Polis, M. P., & Goodson, R. E. (1976). Parameter identification in distributed systems: a synthesizing overview. *Proceedings of the IEEE*, *64*(1), 45–61.

Ram, S. S., Nedić, A., & Veeravalli, V. V. (2009). Distributed subgradient projection algorithm for convex optimization. In *Proc. ICASSP*.

Raza, H., & Ioannou, P. (1996). Vehicle following control design for automated highway systems. *IEEE Control System Magazine*, *16*(June), 43–60.

Johnson, C. R. (1988). *Lectures on adaptive parameter estimation*. Englewood Cliffs, NJ: Prentice Hall.

Sarwar, A., Voulgaris, P. G., & Salapaka, S. M. (2011). On the control design and robustness analysis for high-density microcantilever arrays. *Journal of Vibration and Control*, *17*, 1195–1210. July.

Sarwar, A., Voulgaris, P. G., & Salapaka, S. M. (2011). On $l_\infty$ stability and performance of spatiotemporally varying systems. *International Journal of Robust and Nonlinear Control*, *21*(9), 957–973.

Sarwar, A., Voulgaris, P. G., & Salapaka, S. M. (2010). Indirect adaptive control of spatially invariant systems. In *American Control Conference (ACC) June 30 2010–July 2 2010* (pp. 3235– 3240).

Sayed, A. H. (2003). *Fundamentals of adaptive filtering*. New Jersey: Wiley.

Shaw, G. B., Miller, D. W., & Hastings, D. E. (1998). Generalized information network analysis methodology for distributed satellite systems, *Ph.D. dissertation*, Mass. Inst. Tech. Cambridge, MA.

Shi, L., & Singh, S. K. (1992). Decentralized adaptive controller design for large-scale systems with higher order interconnections. *IEEE Transactions on Automatic Control*, *37*(8), 1106–1118.

Siljak, D. D. (1991). *Decentralized control of complex systems*. Academic Press.

Spooner, J. T., & Passino, K. M. (1996). Adaptive control of a class of decentralized nonlinear systems. *IEEE Transactions on Automatic Control*, *41*(2), 280–284.

Spooner, J. T., & Passino, K. M. (1999). Decentralized adaptive control of nonlinear systems using radial basis neural networks. *IEEE Transactions on Automatic Control*, *44*(11), 2050–2057.

Stewart, G. E. (2000). Two-dimensional loop shaping controller design for paper machine cross-directional processes, *Ph.D. dissertation*, Univ. British Columbia. Vancouver, BC, Canada.

Taylor, M. E. (1996). *Partial differential equations (I): basic theory*. New York: Springer-Verlag.

Wolfe, J. D., Chichka, D. F., & Speyer, J. L. (1996). *Decentralized controllers for unmanned aerial vehicle formation flight*. Amer. Inst. Aeronautics Astronautics.

Xiao, L., Boyd, S., & Lall, S. (2006). A space–time diffusion scheme for peer-to-peer least squares estimation. In *Proc. the fifth international conference on information processing in sensor networks*.

Hovakimyan, N., Lavretsky, E., Yang, B.-J., & Calise, A. J. (2005). Coordinated decentralized adaptive output feedback control of interconnected systems. *IEEE Transactions on Neural Networks*, *16*(1), 185–194.

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