Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Stability Of Slowly Varying Spatiotemporal Systems

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Abstract— **A characterization of stability for slowly varying spatiotemporal systems based on input-output description of the plant and controller is presented. This approach generalizes the results developed for the standard case for slowly time-varying systems. The controller design is based on frozen spatially and temporally invariant descriptions of the plant. In particular, we consider the case where the controllers are not necessarily adjusted for every instance in space and time, and hence are used for some fixed window in time and space before new controllers are implemented. It is shown that the actual spatiotemporally varying system can be stabilized using frozen in space and time controllers, provided the variations in the spatiotemporal dynamics are sufficiently small.**

I. INTRODUCTION

In this paper we restrict our focus to a certain class of discrete distributed systems that have slowly varying dynamics in time as well as in space. In particular, we focus on the recursively computable spatiotemporal systems. Recursively computable spatiotemporal systems arise naturally in system identification and adaptive control of systems characterized by partial differential equations e.g. [3], [4]. Recursibility is a property of certain difference equations which allows one to iterate the equation by choosing an indexing scheme so that every output sample can be computed from outputs that have already been found from initial conditions and from samples of the input sequence.

The goal of this paper is to analyze the stability of such systems with controllers that are designed based on local linear space and time invariant (LSTI) approximations of these spatiotemporal systems. Moreover, the controllers are not necessarily adjusted for every instance in space and time of these local LSTI approximants, but are used for some fixed window in time and space before new controllers are implemented. We show that the actual spatiotemporally varying system can be stabilized using the frozen LSTI controllers provided the variations in the spatiotemporal dynamics are sufficiently small. Our result is a generalization of the results on slowly time-varying systems presented in [5] and [7]. The organization of this paper is as follows: Section II presents mathematical preliminaries. Section III elaborates on the frozen space-time control law. The stability analysis is presented in Section IV. We present the conclusion of our discussion in Section V.

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II. PRELIMINARIES

A. Notation

The set of reals is denoted by $\mathbb{R}$, and the set of integers is denoted by $\mathbb{Z}$. The set of non-negative integers is denoted by $\mathbb{Z}^+$. We use l_∞^e to denote the set of all real double sequences $f = \{f_i(t)\}_{i=-\infty,\ t=0}^{i=\infty,\ t=\infty}$. These sequences correspond to spatiotemporal signals with a 2-sided spatial support ($-\infty \le i \le \infty$) and one sided temporal ($0 \le t \le \infty$). We use l_∞ to denote the space of such sequences with $\|f\|_\infty := \sup_{i,t} |f_i(t)| < \infty$. Note that for $f \in l_\infty^e$, we can represent it as a one-sided (causal) temporal sequence as $f = \{f(0), f(1), \cdots\}$, where

$$f(t) = \left(\ \cdots,\ f_{-1}(t),\ f_0(t),\ f_{+1}(t),\ \cdots\ \right)', \quad t \in \mathbb{Z}^+$$

and each $f_j(t) \in \mathbb{R}$, with $j \in \mathbb{Z}$.

B. Spatiotemporal Varying Systems

Linear spatiotemporal varying systems (LSTV) are systems $M : u \to y$ on l_∞^e given by the convolution

$$y_i(t) = \sum_{\tau=0}^{\tau=t} \sum_{j=-\infty}^{j=\infty} m_{i,i-j}(t, t-\tau) u_j(\tau)$$

where $\{m_{i,j}(t,\tau)\}$ is the kernel representation of M. These systems can be viewed as an infinite interconnection of different linear time varying systems. For simplicity, we assume that each of these subsystems is single-input-single-output (SISO). Let $y_i = (y_i(0), y_i(1), y_i(2), \cdots)'$, then the corresponding input-output relationship of the i_{th} block can be given as follows:

$$\begin{pmatrix} y_i(0) \\ y_i(1) \\ y_i(2) \\ \\ \vdots \end{pmatrix} = \begin{pmatrix} m_{i,0}(0,0) & 0 & 0 & \cdots \\ m_{i,0}(1,0) & m_{i,0}(1,1) & 0 & \cdots \\ m_{i,0}(2,0) & m_{i,0}(2,1) & m_{i,0}(2,2) & \cdots \\ \\ \cdots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} u_i(0) \\ u_i(1) \\ u_i(2) \\ \\ \vdots \end{pmatrix}$$

$$\sum_{\substack{j=-\infty \\ j\neq i}}^{\infty} + \begin{pmatrix} m_{i,j}(0,0) & 0 & 0 & \cdots \\ m_{i,j}(1,0) & m_{i,j}(1,1) & 0 & \cdots \\ m_{i,j}(2,0) & m_{i,j}(2,1) & m_{i,j}(2,2) & \cdots \\ \\ \cdots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} u_{i+j}(0) \\ u_{i+j}(1) \\ u_{i+j}(2) \\ \\ \vdots \end{pmatrix} + \cdots$$

where $\{u_i(t)\}$ is the input applied at the i_{th} block with $u_i(t) \in \mathbb{R}$ and $t \in \mathbb{Z}^+$ is the time index, and $\{m_{i,j}(t,\tau)\}$ is the kernel representation of M. Also, $\{y_i(t)\}$ is the output sequence of the i_{th} block, with $y_i(\cdot) \in \mathbb{R}$. We can write the overall input-output relationship for a LSTV system as follows:

$$y(t) = \begin{pmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} M^{00} & & & & \\ M^{10} & M^{11} & & & \\ M^{20} & M^{21} & M^{22} & & \\ M^{30} & M^{31} & M^{32} & M^{33} & \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} u(0) \\ u(1) \\ u(2) \\ u(3) \\ \vdots \\ \vdots \end{pmatrix}$$

Where, $u(t) = (\cdots, u_{-1}(t), u_0(t), u_{+1}(t), \cdots)'$ and

$$M^{t\tau} = \begin{pmatrix} \cdots & \vdots & \vdots & \vdots & \cdots \\ \cdots & m_{i-1,0}(t,\tau) & m_{i-1,1}(t,\tau) & m_{i-1,2}(t,\tau) & \cdots \\ \cdots & m_{i,-1}(t,\tau) & m_{i,0}(t,\tau) & m_{i,1}(t,\tau) & \cdots \\ \cdots & m_{i+1,-2}(t,\tau) & m_{i+1,-1}(t,\tau) & m_{i+1,0}(t,\tau) & \cdots \\ \cdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}$$

where $t, \tau \in \mathbb{Z}^+$. The l_∞ induced operator norm on M in this case is given as

$$\|M\| = \sup_{i,t} \sum_{\tau=0}^{t} \sum_{i=-\infty}^{i=\infty} |m_{i,j}(t,\tau)|$$

The space of l_∞ bounded LSTV systems will be denoted as $\mathscr{L}_{STV}$

C. Spatially Invariant Systems

Linear spatially invariant systems are spatiotemporal systems $M : u \rightarrow y$ on l_∞^e given by the convolution

$$y_i(t) = \sum_{\tau=0}^{\tau=t} \sum_{j=-\infty}^{j=\infty} m_{i-j}(t-\tau) u_j(\tau)$$

where $\{m_i(t)\}$ is the pulse response of M. These systems can be viewed as an infinite array of interconnected linear time invariant (LTI) systems. The subspace of $\mathscr{L}_{STV}$ that contains the stable LSTI systems will be denoted as $\mathscr{L}_{STI}$. The induced l_∞ operator norm on M in this case is given as

$$\|M\| = \sum_{t=0}^{\infty} \sum_{i=-\infty}^{i=\infty} |m_i(t)|$$

D. Local and Global Product

For a LSTV system M, we can associate a LSTI system $M_{i,t}$ for any given pair (i,t) (where $i \in \mathbb{Z}$ represents a spatial coordinate, and $t \in \mathbb{Z}^+$ represents time). The input-output time domain description corresponding to the LSTI system $M_{i,t}$ can be given as follows:

$$\begin{pmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} M^0_{i,t} & & & & \\ M^1_{i,t} & M^0_{i,t} & & & \\ M^2_{i,t} & M^1_{i,t} & M^0_{i,t} & & \\ M^3_{i,t} & M^2_{i,t} & M^1_{i,t} & M^0_{i,t} & \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} u(0) \\ u(1) \\ u(2) \\ u(3) \\ \vdots \\ \vdots \end{pmatrix}$$

where, $u(t) = (\cdots, u_{-1}(t), u_0(t), u_{+1}(t), \cdots)'$ and

$$M^\tau_{i,t} = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \cdots & m_{i,-1}(t,\tau) & m_{i,0}(t,\tau) & m_{i,1}(t,\tau) & \cdots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

where $\tau \in \mathbb{Z}^+$. We will refer to $M_{i,t}$ as the local or frozen system corresponding to the pair (i,t). The interpretation is that, $y_i(t) = (Mu)_i(t) = (M_{i,t}u)_i(t)$. For LSTV systems A, B associated with the families $A_{i,t}$, $B_{i,t}$ of frozen LSTI operators, we define a global product $A_{i,t} \cdot B_{i,t}$ to mean an operator associated to the usual composition AB in the sense that, if $u \in l_\infty^e$, then $((A_{i,t} \cdot B_{i,t})u)_i(t) = (ABu)_i(t)$. Given a pair (i,t), the local product of operators A, B corresponds to the product (composition) of the LSTI systems $A_{i,t}$, and $B_{i,t}$, i.e. $A_{i,t}B_{i,t}$.

E. Support of m

We define the support of a sequence $\{m_i(t)\}$ by $Supp(m)$, i.e.

$$Supp(m) = \{[i,t] \in \mathbb{Z}^2 : m_i(t) \neq 0\}$$

F. Slowly Varying Spatiotemporal System

A LSTV system A is said to be slowly space-time varying if given two pairs (i,t), and $(\mathtt{i},\tau)$, we have

$$\|A_{i,t} - A_{\mathtt{i},\tau}\| \leq \gamma(|i-\mathtt{i}| + |t-\tau|)$$

where $\gamma \in \mathbb{Z}^+$ is a constant. Such systems will be denoted by SSTV(γ)

G. Integral Time and Space Absolute Error

Given a LSTI system M, the integral time and space absolute error (ITSAE) is defined as

$$ITSAE(M) = \sum_{t=0}^{\infty} \sum_{i=-\infty}^{i=\infty} (|i| + |t|)|m_i(t)|$$

H. z, λ Transform

We define the z, λ transform for a LSTI SISO system M as

$$\hat{M}(z,\lambda) = \sum_{t=0}^{\infty} \sum_{k=-\infty}^{\infty} (m_k(t) z^k) \lambda^t$$

with the associated spectral or $\mathscr{H}_\infty$ norm

$$\left\|\hat{M}\right\|_\infty := \sup_{\theta,\omega} |\, \hat{M}(e^{i\theta}, e^{j\omega}) \,|$$

It is well known (see e.g. [8]) that for a M in $\mathscr{L}_{STI}$, M^{-1} is in $\mathscr{L}_{STI}$ if and only if

$$\inf_{|z|=1, |\lambda| \leq 1} |\hat{M}(z,\lambda)| > 0$$

III. Frozen Space-Time Control

Consider the general form of closed loop system given in Figure 1. The plant P is a LSTV recursively computable spatiotemporal system with the input-output relationship defined by an equation of the form

$$(A_{i,t} y_1)_i(t) = (B_{i,t} y_4)_i(t)$$

with $\{a_{i,j}(t,\tau)\}, \{b_{i,j}(t,\tau)\}$, being the kernel representations of the operators $A_{i,t}$, $B_{i,t}$ in $\mathscr{L}_{STI}$ respectively. We can write the above equation explicitly as follows;

$$\sum_{\substack{j \\ (j,\tau) \in I_{a(i,t)}}} \sum_{\tau} a_{i,j}(t,\tau) y_{1,i-j}(t-\tau) = \sum_{\substack{j \\ (j,\tau) \in I_{b(i,t)}}} \sum_{\tau} b_{i,j}(t,\tau) y_{4,i-j}(t-\tau) \quad (1)$$

where $I_{a(i,t)}$ (output mask) and $I_{b(i,t)}$ (input mask) denote, respectively, the area region of support for $\{a_{i,j}(t,\tau)\}$ and $\{b_{i,j}(t,\tau)\}$. The system in (1) is well defined if $\{a_{i,0}(t,0)\} \neq 0$, and $\{a_{i,j}(t,\tau)\} \neq 0$ for some (j,τ), and $Supp(\{a_{i,j}(t,\tau)\})$ is a subset of the lattice sector with vertex $(0,0)$ of angle less than 180°, for every pair (i,t) [8]. We will assume that all the spatiotemporal systems under consideration are well defined.

Given an instance in space and time, the plant is thought of as a LSTI system, with the defining operators fixed at that time and space. The controllers are designed for the corresponding frozen LSTI system. Allowing ourselves the flexibility of using a designed controller for several instances in time and space, we will consider the controller design every T steps in time and every S steps in space. Define $n_t = nT$ and $k_i = kS$, where n and k are smallest integers such that t and i lie in the interval $[nT,\ (n+1)T]$ and $[kS,\ (k+1)S]$ respectively. The controller is designed at intervals of nT, and kS in time and space respectively. The closed loop is stable if the map from

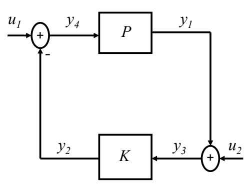

Fig. 1. General form of closed loop.

$u_1,\ u_2$ to $y_1,\ y_2$ is bounded. The dynamics of the control law are given by

$$(L_{k_i,n_t} y_2)_i(t) = (M_{k_i,n_t} y_3)_i(t)$$

where $L_{k_i,n_t},\ M_{k_i,n_t} \in \mathscr{L}_{STI}$ for each pair of indices (k_i,n_t). The evolution of these operators is given by

$$\begin{aligned}
(L_{k_i,n_t} y_2)_i(t) &= \sum_{\tau=0}^{\tau=t} \sum_{j=-\infty}^{j=\infty} l_{k_i,i-j}(n_t, t-\tau) y_{2,j}(\tau) \\
(M_{k_i,n_t} y_3)_i(t) &= \sum_{\tau=0}^{\tau=t} \sum_{j=-\infty}^{j=\infty} m_{k_i,i-j}(n_t, t-\tau) y_{3,j}(\tau)
\end{aligned}$$

The frozen space and time operator that defines the above control law satisfies the following Bezout identity

$$L_{k_i,n_t} A_{k_i,n_t} + M_{k_i,n_t} B_{k_i,n_t} = G_{k_i,n_t}$$

where $G^{-1}_{k_i,n_t} \in \mathscr{L}_{STI}$ for each fixed pair (k_i,n_t). That is, for every frozen plant given by $A_{k_i,n_t},\ B_{k_i,n_t}$, the control generated by $L_{k_i,n_t},\ M_{k_i,n_t}$ is such that the "frozen" closed loop map $G^{-1}_{k_i,n_t}$ is stable. Note that the frozen plant is LSTI, and hence a frozen LSTI controller that satisfies the frozen closed loop can be obtained using various methods, e.g. [1], [2]. Here, we are not interested on any specific method. We only require that K operates as described above and provides frozen stability.

The fact that the controller is updated only every T steps in time and after every S number of plants in space introduces a new parameter in the stability analysis. In the sequel, we show as to how large T and S can be without endangering the stability of the closed loop system. From Figure 1, we can write down the closed loop equations for the controlled system as follows:

$$(A_{i,t} y_1)_i(t) = (B_{i,t}(u_1 - y_2))_i(t) \tag{2}$$

$$(L_{k_i,n_t} y_2)_i(t) = (M_{k_i,n_t}(u_2 + y_1))_i(t) \tag{3}$$

$$L_{k_i,n_t} A_{k_i,n_t} + M_{k_i,n_t} B_{k_i,n_t} = G_{k_i,n_t} \tag{4}$$

In the following we obtain a relation that connects the input sequences $\{u_{1,i}(t)\}$, $\{u_{2,i}(t)\}$ to the outputs $\{y_{1,i}(t)\}$ and $\{y_{2,i}(t)\}$. Operating on equation (2) by L_{k_i,n_t}, we get

$$(L_{k_i,n_t} \cdot A_{i,t} y_1)_i(t) = (L_{k_i,n_t} \cdot B_{i,t} u_1)_i(t) - (L_{k_i,n_t} \cdot B_{i,t} y_2)_i(t)$$

Adding, subtracting, and grouping certain terms we get:

$$\begin{aligned}
&\{(L_{k_i,n_t} A_{k_i,n_t} + B_{k_i,n_t} M_{k_i,n_t}) y_1 + (L_{k_i,n_t} \nabla A_{i,t} + (L_{k_i,n_t} A_{i,t} \\
&- L_{k_i,n_t} A_{k_i,n_t}) + B_{i,t} \nabla M_{k_i,n_t} + (B_{i,t} M_{k_i,n_t} - B_{k_i,n_t} M_{k_i,n_t})) y_1 \\
&+ (L_{k_i,n_t} \nabla B_{i,t} - B_{i,t} \nabla L_{k_i,n_t}) y_2\}(i,t) \\
&= (L_{k_i,n_t} \cdot B_{i,t} u_1)(i,t) - (B_{i,t} \cdot M_{k_i,n_t} u_2)(i,t)
\end{aligned}$$

where we have used the notation; $A_{i,t} \nabla B_{i,t} = A_{i,t} \cdot B_{i,t} - A_{i,t} B_{i,t}$, i.e. $A_{i,t} \nabla B_{i,t}$ is the difference between the global and local product of operators given a pair (i,t). To obtain a second closed loop equation, operate on equation (2) by M_{k_i,n_t}:

$$(M_{k_i,n_t} \cdot A_{i,t} y_1)_i(t) = (M_{k_i,n_t} \cdot B_{i,t} u_1)_i(t) - (M_{k_i,n_t} \cdot B_{i,t} y_2)_i(t)$$

Again adding, subtracting, and grouping certain terms we get:

$$\begin{aligned}
&\{(M_{k_i,n_t} B_{k_i,n_t} + A_{k_i,n_t} L_{k_i,n_t}) y_2 + (M_{k_i,n_t} \nabla B_{i,t} + (M_{k_i,n_t} B_{i,t} \\
&- M_{k_i,n_t} B_{k_i,n_t}) + A_{i,t} \nabla L_{k_i,n_t} + (A_{i,t} L_{k_i,n_t} - A_{k_i,n_t} L_{k_i,n_t})) y_2 \\
&+ (A_{i,t} \nabla M_{k_i,n_t} - M_{k_i,n_t} \nabla A_{i,t}) y_1\}(i,t) \\
&= (M_{k_i,n_t} \cdot B_{i,t} u_1)(i,t) + (A_{i,t} \cdot M_{k_i,n_t} u_2)(i,t)
\end{aligned}$$

For $t \in \mathbb{Z}^+,\ i \in \mathbb{Z}$, define the following

$$\begin{aligned}
X_{i,t} &= L_{k_i,n_t} \nabla A_{i,t} + (L_{k_i,n_t} A_{i,t} - L_{k_i,n_t} A_{k_i,n_t}) \\
&\quad + B_{i,t} \nabla M_{k_i,n_t} + (B_{i,t} M_{k_i,n_t} - B_{k_i,n_t} M_{k_i,n_t}) \\
Y_{i,t} &= L_{k_i,n_t} \nabla B_{i,t} - B_{i,t} \nabla L_{k_i,n_t} \\
Z_{i,t} &= M_{k_i,n_t} \nabla A_{i,t} - A_{i,t} \nabla M_{k_i,n_t} \\
W_{i,t} &= M_{k_i,n_t} \nabla B_{i,t} + (M_{k_i,n_t} B_{i,t} - M_{k_i,n_t} B_{k_i,n_t}) \\
&\quad + A_{i,t} \nabla L_{k_i,n_t} + (A_{i,t} L_{k_i,n_t} - A_{k_i,n_t} L_{k_i,n_t})
\end{aligned}$$

Using (4) we can write the closed loop equation as follows:

$$\begin{aligned}
&\begin{bmatrix} G_{k_i,n_t} + X_{i,t} & Y_{i,t} \\ -Z_{i,t} & G_{k_i,n_t} + W_{i,t} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} (i,t) \\
&\qquad = \begin{bmatrix} L_{k_i,n_t} \cdot B_{i,t} & -B_{i,t} M_{k_i,n_t} \\ -M_{k_i,n_t} \cdot B_{i,t} & A_{i,t} M_{k_i,n_t} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} (i,t)
\end{aligned} \tag{5}$$

Denote by X, Y, Z, W, G the spatiotemporal varying operators associated with the families $X_{i,t}$, $Y_{i,t}$, $Z_{i,t}$, $W_{i,t}$, G_{k_i,n_t} respectively. The idea is to analyze the above system by considering the operators X, Y, Z, W as perturbations. We state below the main result of this paper regarding stability of the system given in (5). We prove this result in the next section.

Theorem 1: Assume the following for system (5):

A1. The operators defining the plant are slowly time and space varying with rates γ_A and γ_B, i.e. $A_{i,t} \in$ SSTV(γ_A), and $B_{i,t} \in$ SSTV(γ_B).
A2. The sequence of controllers are slowly time and space varying, i.e. $M_{k_i,n_t} \in$ SSTV(γ_M) and $L_{k_i,n_t} \in$ SSTV(γ_L).
A3. The l_∞ induced norms and the ITSAE of the operators $A_{i,t}$, $B_{i,t}$, L_{k_i,n_t}, M_{k_i,n_t} are uniformly bounded in i, and t. From this and A1, A2, and the Bezout identity it follows that the operator G_{k_i,n_t} will also be slowly varying in space and time and, hence, we can write $G_{k_i,n_t} \in$ SSTV(γ_G)
A4. The l_∞ to l_∞ norms and the ITSAE of the LSTI operators $G^{-1}_{k_i,n_t}$ are bounded uniformly in i, and t.

Then there exists a non-zero constant γ such that if γ_A, γ_B, γ_M, γ_L, $\gamma_G \le \gamma$, the closed loop system is internally stable.

IV. Stability Analysis

In this section we study the stability of the closed loop system arising from the frozen time and space control design. From equation (5) we see that the map G_{k_i,n_t} is perturbed by a few operators, each of which falls in one of the two categories: 1) $A_{i,t}\nabla L_{k_i,n_t}$ 2) $L_{k_i,n_t}(A_{i,t} - A_{k_i,n_t})$. In the following lemmas we show how the l_∞ induced norms of these operators can be made small by controlling the rates of spatiotemporal variations involved in the problem at hand.

Lemma 1: Let $L_{k_i,n_t} \in$ SSTV(γ_L), and $A_{i,t} \in$ SSTV(γ_A) and R denote the varying spatiotemporal operator associated with $A_{i,t}\nabla L_{k_i,n_t}$. Then $R \in \mathscr{L}_{STV}$ and its induced norm satisfies

$$\begin{aligned}
&\|R\| = \sup_{i,t} \left\|A_{i,t}\nabla L_{k_i,n_t}\right\| \\
&\le \gamma_L \left(2(S+T) \sup_{i,t} \sum_{\tau=0}^{t} \sum_{j=-\infty}^{\infty} |a_{i,j}(t,\tau)| \right. \\
&\left. + \sup_{i,t} \sum_{\tau=0}^{t} \sum_{j=-\infty}^{\infty} |j||a_{i,j}(t,\tau)| + \sup_{i,t} \sum_{\tau=0}^{t} \sum_{j=-\infty}^{\infty} \tau |a_{i,j}(t,\tau)| \right)
\end{aligned}$$

Proof: Let $u \in l_\infty$, then the operator $A_{i,t}\nabla L_{k_i,n_t}$ acts on u as follows

$$\begin{aligned}
A_{i,t}\nabla L_{k_i,n_t} u_i(t) &= \sum_{\tau=0}^{t} \sum_{j=-\infty}^{\infty} a_{i,j}(t,t-\tau) \times \\
&\sum_{r=0}^{\tau} \sum_{s=-\infty}^{\infty} \left(l_{k_j,j-s}(n_\tau, \tau - r) - l_{k_i,i-s}(n_t, \tau - r) \right) u_s(r)
\end{aligned}$$

Taking absolute value of the above equation we get:

$$\begin{aligned}
|A_{i,t}\nabla L_{k_i,n_t} u_i(t)| \quad &\le \\
&\le \sum_{\tau=0}^{t} \sum_{j=-\infty}^{\infty} |a_{i,j}(t,t-\tau)| \\
&\times \sum_{r=0}^{\tau} \sum_{s=-\infty}^{\infty} |l_{k_j,j-s}(n_\tau, r) - l_{k_i,i-s}(n_t, r)| \, \|u\|_\infty \\
&\le \sum_{\tau=0}^{t} \sum_{j=-\infty}^{\infty} |a_{i,j}(t,\tau)| \left\| L_{k_j,n_\tau} - L_{k_i,n_t} \right\| \|u\|_\infty \\
&\le \sum_{\tau=0}^{t} \sum_{j=-\infty}^{\infty} |a_{i,j}(t,\tau)| \gamma_L(|k_j - k_i| + |n_\tau - n_t|) \, \|u\|_\infty
\end{aligned}$$

Now,

$$\begin{aligned}
|k_j - k_i| + |n_\tau - n_t| \quad &= \quad |k_j - j + j - i + i - k_i| \\
&\quad + |n_\tau - \tau + \tau - t + t - n_t| \\
&\le \quad 2S + 2T + |j - i| + |\tau - t|
\end{aligned}$$

since, $|k_j - j| \le S$, and $|n_\tau - \tau| \le T$. The above inequality can now be written as:

$$\begin{aligned}
&|A_{i,t}\nabla L_{k_i,n_t} u_i(t)| \quad \le \\
&\le \sum_{\tau=0}^{t} \sum_{j=-\infty}^{\infty} |a_{i,j}(t,\tau)| \gamma_L(2S + 2T + |j-i| + |\tau - t|) \, \|u\|_\infty \\
&\le \gamma_L \left(2(T+S) \sum_{\tau=0}^{t} \sum_{j=-\infty}^{\infty} |a_{i,j}(t,\tau)| + \sum_{\tau=0}^{t} \sum_{j=-\infty}^{\infty} |j||a_{i,j}(t,\tau)| \right. \\
&\left. + \sum_{\tau=0}^{t} \sum_{j=-\infty}^{\infty} \tau |a_{i,j}(t,\tau)| \right) \|u\|_\infty
\end{aligned}$$

■

Lemma 2: Let the assumptions in Lemma 1 hold. Let R now denote the varying spatiotemporal operator associated with the family of $L_{k_i,n_t}(A_{i,t} - A_{k_i,n_t})$, then $R \in$ LSTV and its induced norm satisfies

$$\|R\| = \sup_{i,t} \left\| L_{k_i,n_t}(A_{i,t} - A_{k_i,n_t}) \right\| \le \gamma_A (T+S) \sup_{i,t} \sum_{\tau=0}^{t} \sum_{j=-\infty}^{\infty} |l_{k_i,j}(n_t,\tau)|$$

Proof: The proof follows in a similar fashion as above and is hence omitted ■

We now proceed to present the proof of Theorem 1.

Proof of Theorem 1: Consider the first equation in (5), expressed in operator form,

$$G y_1 + X y_1 + Y y_2 = v$$

where $v_i(t) = (L_{k_i,n_t} \cdot B_{i,t} u_1)_i(t) - (B_{i,t} \cdot M_{k_i,n_t} u_2)_i(t)$. Let $(\mathtt{i}, \tau)$ be a fixed instance in space and time, we can write

$$G_{k_\mathtt{i},n_\tau} y_1 + (G - G_{k_\mathtt{i},n_\tau}) y_1 + X y_1 + Y y_2 = v$$

where $G_{k_\mathtt{i},\tau} \in \mathscr{L}_{STI}$. Denote by $H_{k_\mathtt{i},\tau}$ the inverse of $G_{k_\mathtt{i},\tau}$. By assumption (A4), $H_{k_\mathtt{i},\tau} \in \mathscr{L}_{STI}$. The above equation can, therefore, be written as

$$y_1 + H_{k_\mathtt{i},n_\tau}(G - G_{k_\mathtt{i},n_\tau}) y_1 + H_{k_\mathtt{i},n_\tau} X y_1 + H_{k_\mathtt{i},n_\tau} Y y_2 = H_{k_\mathtt{i},n_\tau} v$$

Evaluating the above operator equation at $(\mathtt{i},\tau)$ we obtain

$$y_{1,\mathtt{i}}(\tau)+(H_{k_\mathtt{i},n_\tau}(G-G_{k_\mathtt{i},n_\tau})y_1)_\mathtt{i}(\tau)+(H_{k_\mathtt{i},n_\tau}Xy_1)_\mathtt{i}(\tau) +(H_{k_\mathtt{i},n_\tau}Yy_2)_\mathtt{i}(\tau)=(H_{k_\mathtt{i},n_\tau}v)_\mathtt{i}(\tau)$$

Similarly we can write

$$(H_{k_\mathtt{i},n_\tau}Zy_1)_\mathtt{i}(\tau)+y_{2,\mathtt{i}}(\tau)+(H_{k_\mathtt{i},n_\tau}(G-G_{k_\mathtt{i},n_\tau})y_2)_\mathtt{i}(\tau) +(H_{k_\mathtt{i},n_\tau}Wy_2)_\mathtt{i}(\tau)=(H_{k_\mathtt{i},n_\tau}w)_\mathtt{i}(\tau)$$

where $w_i(t)=(M_{k_i,n_t}\cdot B_{i,t}u_1)_i(t)+(A_{i,t}\cdot M_{k_i,n_t}u_2)_i(t)$. Combining the above equations, we get the following closed loop system:

$$\left(\left(\ I+F\ \right)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right)_\mathtt{i}(\tau)=\begin{pmatrix} H_{k_\mathtt{i},n_\tau}v \\ H_{k_\mathtt{i},n_\tau}w \end{pmatrix}_\mathtt{i}(\tau)$$

where

$$F=\begin{pmatrix} H_{k_\mathtt{i},n_\tau}(G-G_{k_\mathtt{i},n_\tau})+H_{k_\mathtt{i},n_\tau}X & H_{k_\mathtt{i},n_\tau}Y \\ H_{k_\mathtt{i},n_\tau}Z & H_{k_\mathtt{i},n_\tau}(G-G_{k_\mathtt{i},n_\tau})+H_{k_\mathtt{i},n_\tau}W \end{pmatrix}$$

The idea is to show that the induced norm of the spatiotemporal varying perturbing operator F can be made less than one by choosing the rates of variations sufficiently small. From the previous lemmas, and the fact that $H_{k_\mathtt{i},n_\tau}$ is uniformly bounded, it is clear that each of the spatiotemporal varying operators generated from each family of operators $H_{k_\mathtt{i},n_\tau}X,\ H_{k_\mathtt{i},n_\tau}Y,\ H_{k_\mathtt{i},n_\tau}Z,\ H_{k_\mathtt{i},n_\tau}W$, have induced norms that are controlled by the rates of variation $\gamma_A,\ \gamma_B,\ \gamma_L,\ \gamma_M,\ \gamma_G$. The internal stability will follow from the small gain theorem if we show that the induced norm of the operator $H_{k_\mathtt{i},n_\tau}(G-G_{k_\mathtt{i},n_\tau})$ can be analogously controlled. We present in the following a calculation of an upper bound of the norm of the operator $H_{k_\mathtt{i},n_\tau}(G-G_{k_\mathtt{i},n_\tau})$. Let $y\in l_\infty$ and the output of the system be x, then

$$x_\mathtt{i}(\tau)=\sum_{m=0}^{\tau}\sum_{j=-\infty}^{\infty}h_{k_\mathtt{i},j}(n_\tau,\tau-m)\times \sum_{r=0}^{m}\sum_{s=-\infty}^{\infty}\Big(g_{k_j,j-s}(n_m,m-r)-g_{k_\mathtt{i},i-s}(n_\tau,m-r)\Big)y_s(r)$$

Taking absolute values we get,

$$|x_\mathtt{i}(\tau)|\le\sum_{m=0}^{\tau}\sum_{j=-\infty}^{\infty}|h_{k_\mathtt{i},j}(n_\tau,\tau-m)|\times \sum_{r=0}^{m}\sum_{s=-\infty}^{\infty}|g_{k_j,j-s}(n_m,m-r)-g_{k_\mathtt{i},i-s}(n_\tau,m-r)|\,\|y\|_\infty$$

By an argument similar to one given in the proof of Lemma 1, it follows that:

$$\|x\|_\infty\le\gamma_G\Bigg(2(T+S)\sup_{\mathtt{i},\tau}\sum_{m=0}^{\tau}\sum_{j=-\infty}^{\infty}|h_{k_\mathtt{i},j}(\tau,m)| +\sup_{\mathtt{i},\tau}\sum_{m=0}^{\tau}\sum_{j=-\infty}^{\infty}m|h_{k_\mathtt{i},j}(\tau,m)|+\sup_{\mathtt{i},\tau}\sum_{m=0}^{\tau}\sum_{j=-\infty}^{\infty}|j||h_{k_\mathtt{i},j}(\tau,m)|\Bigg)\|y\|_\infty$$

It follows by assumption (A4) that there exist constants $C_1,\ C_2\ge 0$ such that

$$\|x\|_\infty\le\gamma_G\left(2(S+T)C_1+C_2+C_3\right)\|y\|_\infty$$

We have, hence, shown that the induced norms of all the perturbing operators that comprise F can be made small by choosing the rates of variations small enough. Internal stability now follows by an application of the small gain theorem. This concludes the proof of Theorem 1.

■

Theorem 1 shows that if the assumptions (A1-A4) are satisfied and if the variations are small enough, then the closed loop system will be l_∞ stable. The assumptions (A1-A2) are quite reasonable and are typically satisfied for the recursively computable spatiotemporal system that we focus on. The first part of assumption (A3), requiring uniform bounds on the operators, is also quite reasonable. Intuitively the second part of assumption (A3), that requires uniform bounds on the ITSAE of operators, implies that the LTV building blocks of the LSTV system have decaying memory (temporal), and decaying spatial dependence on the neighbors (as one goes away from the reference in space). Assumption (A4), however, is harder to satisfy. This assumption implies that the zeros in λ of $\hat{G}_{i,t}(e^{j\theta},\lambda)$, lie outside a disc of radius $1+\varepsilon$, for some $\varepsilon>0$ and for all θ. Precisely, this assumption implies:

$$\inf_{|z|=1,|\lambda|\le 1}|\hat{G}_{i,t}(z,\lambda)|\ge\delta>0,\qquad\forall i,\ \forall t$$

Satisfying this condition only, however, does not imply that the l_∞ induced norms and the ITSAE of $G_{i,t}^{-1}$ are uniformly bounded, i.e., assumption (A4) is not satisfied in general. The above spectral condition does imply that the $\mathscr{H}_\infty$ norm of $G_{i,t}^{-1}$ is uniformly bounded in i, and t, since:

$$\left\|\hat{G}_{i,t}^{-1}\right\|_\infty=\sup_{|z|=1,|\lambda|\le 1}=\frac{1}{|\hat{G}_{i,t}(z,\lambda)|} =\frac{1}{\inf_{|z|=1,|\lambda|\le 1}|\hat{G}_{i,t}(z,\lambda)|}\le\frac{1}{\delta},\qquad\forall i,\ \forall t$$

In the following we show that with some additional mild assumptions on the $\mathscr{H}_\infty$ norm of $\hat{G}_{i,t}(z,\lambda)$ and its partial derivatives, the spectral condition is enough to verify the uniform bounds on the l_∞ induced norms and the ITSAE of $G_{i,t}^{-1}$. For partial derivatives, we define the notation $\hat{G}_{i,t,(\xi)}=\frac{\partial\hat{G}_{i,t}}{\partial\xi}$, $\hat{G}_{i,t,(\xi\zeta)}=\frac{\partial^2\hat{G}_{i,t}}{\partial\xi\partial\zeta}$, $\hat{G}_{i,t,(\xi\xi\zeta)}=\frac{\partial^3\hat{G}_{i,t}}{\partial\xi\partial\xi\partial\zeta}$, where $\xi,\ \zeta$ can be z or λ.

Theorem 2: Given the following conditions:

- $\left\|\hat{G}_{i,t}\right\|_\infty\le M_1,\ \left\|\hat{G}_{i,t,(z)}\right\|_\infty\le M_2,\ \forall i,\ t$
- $\left\|\hat{G}_{i,t,(\lambda)}\right\|_\infty\le M_3,\ \left\|\hat{G}_{i,t,(zz)}\right\|_\infty\le M_4,\ \forall i,\ t$
- $\left\|\hat{G}_{i,t,(\lambda\lambda)}\right\|_\infty\le M_5,\ \left\|\hat{G}_{i,t,(z\lambda)}\right\|_\infty\le M_6,\ \forall i,\ t$
- $\left\|\hat{G}_{i,t,(zz\lambda)}\right\|_\infty\le M_7,\ \left\|\hat{G}_{i,t,(\lambda zz)}\right\|_\infty\le M_8,\ \forall i,\ t$
- $\inf_{|z|=1,\lambda\le 1}|\hat{G}_{i,t}(z,\lambda)|\ge\delta>0\ \forall i,\ t$ (spectral condition)

Then the l_∞ induced norm and ITSAE of $G_{i,t}^{-1}$ are uniformly bounded in i, and t.

Proof: The proof is based on Hardy's theorem [9], and its extension for two variables. We present Hardy's theorem in two variables without proof in the following. Given a function $\hat{R} \in \mathscr{H}_\infty$ with $\hat{R}(z,\lambda) = \sum_{t=0}^{\infty}\sum_{k=-\infty}^{\infty}(m_k(t)z^k)\lambda^t$ then there exists a constant $0 < C < +\infty$ such that the coefficients $m_k(t)$ satisfy:

$$\sum_{t=1}^{\infty}\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{1}{t|k|}|m_k(t)| \leq C\left\|\hat{R}\right\|_\infty$$

To show that $\left\|G_{i,t}^{-1}\right\|$ is uniformly bounded, we apply Hardy's theorem on $\hat{R}(z,\lambda) = \hat{G}_{i,t,(z\lambda)}^{-1}$. Note that

$$\frac{\partial^2 \hat{G}_{i,t}^{-1}}{\partial z \partial \lambda} = \frac{-\hat{G}_{i,t,(z\lambda)}\hat{G}_{i,t} + \hat{G}_{i,t,(z)}G_{i,t,(\lambda)}}{\hat{G}_{i,t}^3}$$

Hence,

$$\left\|\hat{R}\right\|_\infty \leq \frac{\left\|\hat{G}_{i,t,(z\lambda)}\right\|_\infty\left\|\hat{G}_{i,t}\right\|_\infty + \left\|\hat{G}_{i,t,(z)}\right\|_\infty\left\|G_{i,t,(\lambda)}\right\|_\infty}{\delta^3} \leq \frac{M_9}{\delta^3}$$

Where $M_9 := M_6M_1 + M_2M_3$. Let $\hat{G}_{i,t}^{-1}$ be given by:

$$\hat{G}_{i,t}^{-1} = \sum_{\tau=0}^{\infty}\sum_{k=-\infty}^{\infty}(h_{i,k}(t,\tau)z^k)\lambda^\tau$$

$$\text{Then;} \quad \hat{R}_{i,t} = \sum_{\tau=0}^{\infty}\sum_{k=-\infty}^{\infty}\tau k(h_{i,k}(t,\tau)z^{k-1})\lambda^{\tau-1}$$

Applying Hardy's theorem on $z\lambda\hat{R}_{i,t}$, we get

$$\sum_{\tau=1}^{\infty}\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty}|h_{i,k}(t,\tau)| \leq C\frac{M_9}{\delta}$$

Note that in the last expression above, we are missing the terms $\sum_{k=-\infty}^{\infty}|h_{i,k}(t,0)|$, and $\sum_{\tau=0}^{\infty}|h_{i,0}(t,\tau)|$ in order to establish a bound on $\left\|G_{i,t}^{-1}\right\|$. Let

$$\hat{h}1 = \sum_{k=0}^{\infty}h_{i,k}(t,0)z^k; \quad \hat{h}2 = \sum_{k=-\infty}^{-1}h_{i,k}(t,0)z^k; \quad \hat{h}3 = \sum_{\tau=0}^{\infty}h_{i,0}(t,\tau)\lambda^\tau$$

The $\mathscr{H}_\infty$ norms of the operators $\hat{h}1$, $\hat{h}2$, $\hat{h}3$ are bounded, since $\left\|\hat{G}_{i,t}^{-1}\right\|_\infty$ is bounded. Also the $\mathscr{H}_\infty$ norms of $\hat{h}1_{(z)}$, $\hat{h}2_{(z)}$, and $\hat{h}3_{(\lambda)}$ are bounded since $\left\|\hat{R}_{i,t}\right\|_\infty$ is bounded. Using the Hardy's theorem for one variable and reasoning in a similar fashion as above, we can establish bounds on $\sum_{k=1}^{\infty}|h_{i,k}(t,0)|$, $\sum_{k=-\infty}^{-1}|h_{i,k}(t,0)|$, and $\sum_{\tau=1}^{\infty}|h_{i,0}(t,\tau)|$. Let the sum of their bounds be denoted by C_h. We now have the following bound for $\left\|G_{i,t}^{-1}\right\|$:

$$\left\|G_{i,t}^{-1}\right\| \leq C\frac{M_9}{\delta} + C_h + |h_{i,0}(t,0)| \leq C\frac{M_9}{\delta} + C_h + \delta^{-1}$$

A similar argument works for ITSAE($G_{i,t}^{-1}$) by considering $G_{i,t,(zz\lambda)}^{-1}$, and $G_{i,t,(\lambda\lambda z)}^{-1}$. We omit the details as they follow in the footsteps of the above argument. ■

The benefit of the above theorem is that one can check if assumption (A4) is satisfied by checking easily computable $\mathscr{H}_\infty$ norms.

Theorem 3: Let the assumptions (A1-A3), and the spectral condition (last condition, Theorem 2) hold along with the uniform boundedness of the following quantities

- $\left\|\hat{A}_{i,t,(zz)}\right\|_\infty$, $\left\|\hat{A}_{i,t,(zz\lambda)}\right\|_\infty$, $\left\|\hat{A}_{i,t,(\lambda\lambda z)}\right\|_\infty$, $\left\|\hat{B}_{i,t,(zz)}\right\|_\infty$
- $\left\|\hat{B}_{i,t,(zz\lambda)}\right\|_\infty$, $\left\|\hat{B}_{i,t,(\lambda\lambda z)}\right\|_\infty$, $\left\|\hat{L}_{i,t,(zz)}\right\|_\infty$, $\left\|\hat{L}_{i,t,(zz\lambda)}\right\|_\infty$
- $\left\|\hat{L}_{i,t,(\lambda\lambda z)}\right\|_\infty$, $\left\|\hat{M}_{i,t,(zz)}\right\|_\infty$, $\left\|\hat{M}_{i,t,(zz\lambda)}\right\|_\infty$, $\left\|\hat{M}_{i,t,(\lambda\lambda z)}\right\|_\infty$

Then the closed loop system (5) is stable.

Proof: The proof is straight forward since the conditions in Theorem 2 will be satisfied from the Bezout identity relating $\hat{G}_{i,t}$ to the above quantities. ■

V. Conclusion

We have considered the stability analysis of distributed systems that have slowly varying dynamics in time as well as in space. In particular we have looked at the case where the controllers were not necessarily adjusted for every instance in space and time, and hence were used for some fixed spatiotemporal window before new controllers were implemented. We showed how the length of these windows entered in the stability analysis. It was shown that the actual time varying system can be stabilized using the frozen space-time controllers provided the variations in the spatiotemporal dynamics are sufficiently small. Current research is directed on the characterization of l_∞ to l_∞ performance of the slowly varying spatiotemporal systems and adaptive control applications.

VI. ACKNOWLEDGMENTS

This material is based upon work supported in part by the National Science Foundation under NSF Awards No CCR 03-25716 ITR, and by AFOSR grant FA9950-06-1-0252.

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