Relativistic Rankine-Hugoniot Equations

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(Received April 13, 1948)

In Part I of this paper the stress energy tensor and the mean velocity vector of a simple gas are expressed in terms of the Maxwell-Boltzman distribution function. The rest density ρ^0, pressure, p, and internal energy per unit rest mass ϵ are defined in terms of invariants formed from these tensor quantities. It is shown that ϵ cannot be an arbitrary function of p and ρ^0 but must satisfy a certain inequality. Thus $\epsilon=(1/\gamma-1)p/\rho^0$ for $\gamma>5/3$ is impossible. It is known that if ϵ is given by this relation and $\gamma>2$, then sound velocity in the medium may be greater than that of light in vacuum. This difficulty is now removed by the inequality mentioned above. In Part II of this paper the relativistic form of the Rankine-Hugoniot equations are derived and it is shown that as a consequence of the inequality mentioned earlier that the shock wave velocity is always less than that of light in vacuum for sufficiently strong shocks.

PART I. SPECIFIC INTERNAL ENERGY

1. Introduction

MACROSCOPIC relativistic theories of fluid dynamics characterize the fluid by giving the internal energy per unit mass, ϵ, measured by an observer at rest with respect to the element of the fluid as a function of the pressure, p and the rest density ρ^0, and also by prescribing the viscosity and the heat conductivity of the fluid. For perfect fluids, for which the latter two quantities vanish, it follows from the work of Lamla[1] and Section 5 below that if

$$\epsilon=(1/\gamma-1)p/\rho^0 \tag{1.1}$$

and γ is a constant greater than 2, then the velocity of sound in the fluid may be greater than the velocity of light in vacuum.

Thus, consistency of hydrodynamics with the special theory of relativity can only be achieved from the macroscopic viewpoint by restricting the allowed relations between specific internal energy, pressure, and density. This restriction applies to fluids with non-vanishing heat conductivity and viscosity as follows from the work of Eckart.[2] For it is evident from the equations Eckart gives for the flow of heat in a gas at rest (Eqs. (43) and (44)) that if ϵ is taken as given by (1.1) then the velocity of propagation is greater than the velocity of light in vacuum.

This functional restriction also enters in the discussion of the relativistic formulation of the Rankine-Hugoniot equations governing the propagation of shock waves, as follows from Part II of this paper.

It is our purpose in Part I of this paper to show how equations of the type of (1.1) are ruled out on the basis of the kinetic theory of gases when this theory is formulated relativistically. In Part II we derive the relativistic Rankine-Hugoniot equations and show that in view of the inequality that must hold on ϵ as a function of p and ρ^0 the shock wave velocity is less than that of light in vacuum.

2. Derivation of the Hydrodynamical Equations

We shall now derive the equations governing the motion of the fluid considered a collection of a number of particles with rest mass m. Let

$$\xi^i=\frac{v^i}{(1-v^2/c^2)^{\frac{1}{2}}},\quad v^i=\frac{\xi^i}{(1+\xi^2/c^2)^{\frac{1}{2}}} \tag{2.1}$$

where

$$v^2=\sum_{i=1}^{3}(v^i)^2,\quad \xi^2=\sum_{i=1}^{3}(\xi^i)^2$$

and v^i are the components of the velocity of a particle. Let $f(x, t, \xi)$ be the number of particles in the region x^i to x^i+dx^i in space at time t and with values of ξ^i between ξ^i and $\xi^i+d\xi^i$. Then the

* The major portion of this work was done while the author was a Guggenheim postservice Fellow in residence at the Institute for Advanced Study, on leave from the University of Washington.

[1] E. Lamla, "Über die Hydrodynamik des Relativitäts prinzips," Dissertation, Berlin (1912).

[2] C. Eckart, "The thermodynamics of irreversible processes. III. Relativistic theory of the simple fluid," Phys. Rev. 58, 919 (1940).

Boltzman equation for f is

$$Df \equiv \frac{\partial f}{\partial t} + \frac{\xi^i}{(1+\xi^2/c^2)^{\frac{1}{2}}} \frac{\partial f}{\partial x^i} + F^i \frac{\partial f}{\partial \xi^i} = \Delta_e f, \quad (2.2)$$

where F^i is the external force per unit mass and $\Delta_e f$ is the time rate of change in f due to encounters between the particles.

An integration by parts shows that

$$\int \phi D f d_3\xi = \frac{\partial}{\partial t}(n\langle\varphi\rangle) + \frac{\partial}{\partial x^i}\left[n\left\langle\frac{\varphi\xi^i}{(1+\xi^2/c^2)^{\frac{1}{2}}}\right\rangle\right] - n\left\{\left\langle\frac{\partial\varphi}{\partial t}\right\rangle + \left\langle\frac{\xi^i}{(1+\xi^2/c^2)^{\frac{1}{2}}}\frac{\partial\varphi}{\partial x^i}\right\rangle + F^i\left\langle\frac{\partial\varphi}{\partial x^i}\right\rangle\right\} \quad (2.3)$$

where

$$n\langle h\rangle = \int h(\xi)f(\xi)d_3\xi, \quad n = \int f(\xi)d_3\xi,$$

and the integrations are carried out over the entire volume of the ξ^1, ξ^2, ξ^3 space. The notation $\int \cdots d_3\xi$ indicates such a volume integral.

The laws of conservation of mass, energy, and momentum follow from Eqs. (2.2) and (2.3) by observing that

$$\int \varphi^q \Delta_e f d_3\xi = 0, \quad q = 0, 1, \cdots, 4$$

where

$$\varphi^0 = m, \quad \varphi^i = m\xi^i \qquad (i = 1, 2, 3)$$

and

$$\varphi^4 = m(1+\xi^2/c^2)^{\frac{1}{2}}.$$

Multiplying Eq. (2.2) by the various φ^q in turn and integrating over all ξ's we obtain five equations which may be written as

$$mU^\alpha{}_{,\alpha} = 0, \quad (2.4)$$

$$T^{\alpha\beta}{}_{,\beta} = \rho^0 F^\alpha, \quad \alpha, \beta = 1, 2, 3, 4, \quad (2.5)$$

respectively, where the comma denotes differentiation the summation convention is used, and the quantities involved are defined as follows: U^α is the mass current four vector given by

$$U^\alpha = \int V^\alpha(\xi)f(\xi)\frac{d_3\xi}{(1+\xi^2/c^2)^{\frac{1}{2}}} \equiv \int V^\alpha(\xi)d\mu(\xi), \quad (2.6)$$

where

$$V^i = \xi^i/c, \quad V^4 = (1+\xi^2/c^2)^{\frac{1}{2}};$$

hence

$$V^\alpha V^\beta g_{\alpha\beta} = V^\alpha V_\alpha = -1, \quad (2.7)$$

where now

$$g_{\alpha\beta} = 0, \quad \alpha \neq \beta \quad \text{and} \quad g_{11} = g_{22} = g_{33} = -g_{44} = 1.$$

That is,

$$U^4 = \int f(\xi)d_3\xi \equiv \int (1+\xi^2/c^2)^{\frac{1}{2}} d\mu(\xi) = n,$$

$$U_i = \int \frac{\xi^i/c}{(1+\xi^2/c^2)^{\frac{1}{2}}} f(\xi)d_3\xi \equiv \int \xi^i/c\, d\mu(\xi) = n\left\langle\frac{\xi^i}{(1+\xi^2/c^2)^{\frac{1}{2}}}\right\rangle, \quad i = 1, 2, 3.$$

$T^{\alpha\beta}$ is the stress energy tensor and it is defined as

$$T^{\alpha\beta} = mc^2 \int V^\alpha(\xi)V^\beta(\xi)\frac{f(\xi)}{(1+\xi^2/c^2)^{\frac{1}{2}}}d_3\xi \equiv mc^2\int V^\alpha(\xi)V^\beta(\xi)d\mu(\xi). \quad (2.8)$$

F^α is the four-dimensional force vector:

$$F^4 = F^i u^i/c,$$

$$u^i = \int \frac{\xi^i}{(1+\xi^2/c^2)^{\frac{1}{2}}} f(\xi)d_3\xi \Big/ \int f(\xi)d_3\xi, \quad (2.9)$$

$$F^i = \frac{F^i}{(1-u^2/c^2)^{\frac{1}{2}}},$$

and ρ^0 is the rest density of the gas defined as

$$(\rho^0)^2 = -m^2 U^\alpha U_\alpha. \quad (2.10)$$

It follows from Eq. (2.7) that

$$T_\alpha{}^\alpha = -mc^2\int d\mu(\xi). \quad (2.11)$$

It is evident from Eq. (2.5) and the fact that $\rho^0 F^\alpha$ is a four-dimensional vector that $T^{\alpha\beta}$ is a tensor. From Eq. (2.8) it then follows that $f(x, \xi)$ is a scalar function under Lorentz transformations since the Lorentz invariant volume measure in ξ space is

$$\frac{d_3\xi}{(1+\xi^2/c^2)^{\frac{1}{2}}}.$$

3. Specific Internal Energy

We now define the function ϵ in terms of the components of $T^{\alpha\beta}$ and U^α as is done by Eckart.[3] We write

$$T^{\alpha\beta}=\frac{m^2w}{(\rho^0)^2}U^\alpha U^\beta+\frac{m}{\rho^0}U^\alpha W^\beta+\frac{m}{\rho^0}U^\beta W^\alpha+W^{\alpha\beta}, \quad (3.1)$$

where W^α is the heat flow vector and $W^{\alpha\beta}$ is the stress tensor. We require that

$$W^\alpha U_\alpha=0, \quad W^{\alpha\beta}U_\beta=0. \quad (3.2)$$

The scalar w is the energy density as measured by some one instantaneously at rest with respect to an element of the fluid.

From Eq. (3.1) we have

$$T_\alpha{}^\alpha=3p-w, \quad (3.3)$$

where

$$W_\alpha{}^\alpha=3p$$

and p is the hydrostatic pressure. It follows from (3.1), (3.2), and (2.10) that

$$w=m^2T_{\alpha\beta}\frac{U^\alpha U^\beta}{(\rho^0)^2}\equiv\rho^0(c^2+\epsilon). \quad (3.4)$$

The last of these equations will be regarded as our definition of ϵ, the internal energy per unit rest mass of the fluid.

4. The Fundamental Inequality

We now propose to show that ϵ defined by Eq. (3.4) when considered as a function of p and ρ^0 must satisfy an inequality which rules out functions of the type given by (1.1). Let us define

$$g(\xi')=\int V^\alpha(\xi')V_\alpha(\xi)d\mu(\xi)=V^\alpha(\xi')\int V_\alpha(\xi)d\mu(\xi). \quad (4.1)$$

[3] See reference 2, p. 921.

It follows from Eq. (3.4) that

$$w=\frac{m^3c^2}{(\rho^0)^2}\int V_\alpha(\xi)V_\beta(\xi)d\mu(\xi)\times\int V^\alpha(\xi')d\mu(\xi')\int V^\beta(\xi'')d\mu(\xi'')$$
$$=\frac{m^3c^2}{(\rho^0)^2}\int\left(V_\beta(\xi)\int g(\xi)V^\beta(\xi')d\mu(\xi')\right)d\mu(\xi) \quad (4.2)$$
$$=\frac{m^3c^2}{(\rho^0)^2}\int g^2(\xi)d\mu(\xi). \quad (4.2)$$

From Eq. (2.10) we have

$$-\left(\frac{\rho^0}{m}\right)^2=U^\alpha U_\alpha=\int V^\alpha(\xi)d\mu(\xi)\times\int V_\alpha(\xi')d\mu(\xi')=\int g(\xi)d\mu(\xi). \quad (4.3)$$

Now from Schwartz's inequality we have

$$\left(\int g(\xi)d\mu(\xi)\right)^2\leqslant\left(\int g^2(\xi)d\mu(\xi)\right)\left(\int d\mu(\xi)\right). \quad (4.4)$$

Substituting in (4.4) from (2.11), (4.3), and (4.2) we have

$$\frac{(\rho^0)^2}{m^4c^4}w(-T_\alpha{}^\alpha)\geqslant\frac{(\rho^0)^4}{m^2}.$$

Using (3.3) we obtain

$$w(w-3p)\geqslant(\rho^0)^2c^4;$$

hence

$$w\geqslant\tfrac{3}{2}p+\rho^0c^2\left(1+\frac{9}{4}\left(\frac{p}{\rho^0c^2}\right)^2\right)^{\frac{1}{2}}$$

and

$$\epsilon\geqslant\tfrac{3}{2}p/\rho^0+c^2\left(\left(1+\frac{9}{4}\left(\frac{p}{\rho^0c^2}\right)^2\right)^{\frac{1}{2}}-1\right). \quad (4.5)$$

The existence of this inequality shows that the kinetic formulation of the theory of gases and the formalism of the special theory of relativity together are in contradiction with the macroscopic viewpoint which allows ϵ to be any function of p

and ρ^0. In particular the functions of the type (1.1) with $\gamma \geqslant 5/3$ are not permitted. Thus the restrictions on the types of functions $\epsilon(p, \rho^0)$ have been shown to be furnished by kinetic theory and are not *ad hoc* ones.

PART II. RANKINE-HUGONIOT EQUATIONS

5. One-Dimensional Motion of a Perfect Gas

The equations governing the motion of a perfect gas subject to no external forces are (2.4) and (2.5); these may be written as

$$(\rho^0 u^\alpha)_{,\alpha} = 0, \tag{5.1}$$

$$T_{,\beta}{}^{\alpha\beta} = 0, \tag{5.2}$$

where

$$u^\alpha = (m/\rho^0) U^\alpha \tag{5.3}$$

and

$$T^{\alpha\beta} = \rho^0 (c^2 + \epsilon + p/\rho^0) u^\alpha u^\beta + p g^{\alpha\beta}, \tag{5.4}$$

since, for a perfect gas,

$$W^\alpha = 0 \tag{5.5}$$

and

$$W^{\alpha\beta} = p(g^{\alpha\beta} + u^\alpha u^\beta). \tag{5.6}$$

Substituting (5.4) into (5.2) and taking account of (5.1) we obtain

$$\rho^0 u^\beta (c^2 \mu u^\alpha)_{,\beta} + p_{,\beta} g^{\alpha\beta} = 0 \tag{5.7}$$

where

$$\mu = 1 + (1/c^2)(\epsilon + p/\rho^0). \tag{5.8}$$

Multiplying (5.7) by $-u_\alpha$ and summing we obtain the equation of conservation of energy

$$\rho^0(\epsilon + p/\rho^0)_{,\beta} u^\beta - p_{,\beta} u^\beta = \rho^0 (\epsilon_{,\beta} u^\beta + p(1/\rho^0)_{,\beta} u^\beta) = 0. \tag{5.9}$$

Defining absolute temperature θ and specific entropy S as measured by an observer at rest with respect to the gas from the equations

$$d\epsilon + p d(1/\rho^0) = \theta dS, \tag{5.9'}$$

where ϵ is a function of p and ρ^0, Eqs. (5.9) may be written as

$$\rho^0 \theta S_{,\beta} u^\beta = 0. \tag{5.10}$$

Hence, the conservation of energy along the stream lines is equivalent to the statement that entropy is constant along a stream line. If all the gas is initially at the same entropy and the flow is continuous, then we shall have

$$S = \text{constant}. \tag{5.11}$$

This equation determines p as a function of ρ^0. It is then sufficient to consider only three of the four Eqs. (5.7).

In the case of one-dimensional motion all quantities are assumed to be functions of $x^1 = x$ and $x^4 = ct$ and $u^2 = u^3 = 0$. Then we may write for u^α

$$u^1 = \frac{u}{(1-u^2)^{\frac{1}{2}}}, \quad u^4 = \frac{1}{(1-u^2)^{\frac{1}{2}}}, \tag{5.12}$$

where u is the velocity of the gas in units in which the velocity of light is one. Equations (5.1) and (5.7) with $\alpha = 1$ become

$$\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\rho^0}{(1-u^2)^{\frac{1}{2}}}\right) + \frac{\partial}{\partial x}\left(\frac{\rho^0 u}{(1-u^2)^{\frac{1}{2}}}\right) = 0, \tag{5.13}$$

$$\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{c^2 \mu \rho^0 u}{1-u^2}\right) + \frac{\partial}{\partial x}\left(\frac{c^2 \mu \rho^0 u^2}{1-u^2} + p\right) = 0. \tag{5.14}$$

If we now introduce the two auxilliary quantities

$$\alpha = \frac{a}{c} = \left(\frac{\rho^0}{\mu}\frac{d\mu}{d\rho^0}\right)^{\frac{1}{2}}, \quad \varphi = \frac{1}{c}\int \frac{a}{\rho^0} d\rho^0, \tag{5.15}$$

Eqs. (5.13) and (5.14) may be written as

$$(1-u^2)\left(\frac{1}{c}\frac{\partial \varphi}{\partial t} + u\frac{\partial \varphi}{\partial x}\right) + \alpha\left(u\frac{1}{c}\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\right) = 0,$$

$$\alpha(1-u^2)\left(u\frac{1}{c}\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial x}\right) + \frac{1}{c}\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0.$$

These in turn may be written as

$$D_+ u + (1-u^2) D_+ \varphi = 0,$$
$$D_- u - (1-u^2) D_- \varphi = 0,$$

where

$$D_+ f = (1 + \alpha u)\frac{1}{c}\frac{\partial}{\partial t} f + (\alpha + u)\frac{\partial f}{\partial x},$$

$$D_- f = (1 - \alpha u)\frac{1}{c}\frac{\partial}{\partial t} f - (\alpha - u)\frac{\partial f}{\partial x},$$

and hence

$$\frac{D_\pm u}{1-u^2} = D_\pm \log\left(\frac{1+u}{1-u}\right)^{\frac{1}{2}}.$$

Thus Eqs. (5.13) and (5.14) become

$$(1-u^2)D_+\left[\log\left(\frac{1+u}{1-u}\right)^{\frac{1}{2}}+\varphi\right]=0,$$

$$(1-u^2)D_-\left[\log\left(\frac{1+u}{1-u}\right)^{\frac{1}{2}}-\varphi\right]=0.$$

If we now define

$$\begin{aligned} r&=\varphi+\log\left(\frac{1+u}{1-u}\right)^{\frac{1}{2}}, \\ s&=\varphi-\log\left(\frac{1+u}{1-u}\right)^{\frac{1}{2}}, \end{aligned} \tag{5.16}$$

Eqs. (5.13) and (5.14) become equivalent to

$$\begin{aligned} (1+\alpha u)(1/c)(\partial r/\partial t)+(\alpha+u)(\partial r/\partial x)=0, \\ (1-\alpha u)(1/c)(\partial s/\partial t)-(\alpha-u)(\partial s/\partial x)=0. \end{aligned} \tag{5.17}$$

From these equations it is evident that r and s are the relativistic analogs of the Riemann functions which occur in the classical theory of propagation of one-dimensional waves of finite amplitude. In particular we have that

$r=$ constant along the curve (dx/dt) $=(\alpha+u/1+\alpha u)$,

$s=$ constant along the curve (dx/dt) $=-(\alpha-u/1-\alpha u)$.

The quantity α will be shown to be equal to the velocity of sound in units where the velocity of light is one. The expressions for (dx/dt) are then the relativistic sum and difference of particle velocity and sound velocity.

6. Progressive Waves

A disturbance will be said to propagate as a progressive wave if either r or s is constant. Let us suppose

$$s=\varphi_0=\text{constant}.$$

Then from (5.16) we have

$$u=\tanh(\varphi-\varphi_0). \tag{6.1}$$

and the first of (5.17) becomes

$$(1/c)(\partial/\partial t)\varphi+\Gamma(\varphi)(\partial\varphi/\partial x)=0 \tag{6.2}$$

where

$$\Gamma(\varphi)=(\alpha+u/1+\alpha u) \tag{6.3}$$

and α is considered as a function of φ determined by Eqs. (5.15).

The general solution of Eq. (6.2) is

$$f(\varphi)=x-\Gamma(\varphi)ct, \tag{6.4}$$

where $f(\varphi)$ is an arbitrary function. Equation (6.4) states that φ is constant along the straight lines in the x, ct plane of slope $\Gamma(\varphi)$, and hence $\Gamma(\varphi)$ is the velocity of propagation of φ. From Eq. (6.3) it is evident that for weak disturbances for which $u\rightarrow 0$, $\Gamma(\varphi)\rightarrow\alpha$, and hence α is the velocity of propagation of sound in units where the velocity of light is one.

If we define

$$\epsilon=(1/\gamma-1)p/\rho^0,$$

then Eqs. (5.9′) and (5.11) lead to

$$p/p_1=(\rho^0/\rho_1{}^0)^\gamma$$

as the equation for the adiabatics. It may then be verified that

$$\alpha=\left(\frac{\gamma p/c^2\rho^0}{1+\dfrac{\gamma}{\gamma-1}\dfrac{1}{c^2}p/\rho^0}\right)^{\frac{1}{2}}.$$

From this it follows that if a sound wave is progressing into a medium of high temperature, that is, if $p/c^2\rho^0$ is large, then

$$\alpha\rightarrow(\gamma-1)^{\frac{1}{2}}.$$

Hence for $\gamma>2$ sound waves propagate with velocity greater than that of light in vacuum. For gases the equation used for ϵ is not a possible one as follows from the argument given in the introduction.

7. Derivation of the Rankine-Hugoniot Equations

It follows from Eq. (6.4) that for certain functions $f(\varphi)$ which are determined by the boundary conditions the curves $\varphi=$constant in the x, ct plane intersect. This is impossible physically and hence continuous one-dimensional motions are impossible under these conditions. From classical theory it is to be expected that shock waves form. We now derive the equations which must hold across these waves.

We first write the equations governing the motion in integral form. Thus Eqs. (5.1) and

(5.2) may be written as

$$\int (\rho^0 u^\alpha)_{,\alpha} d_4x = \int \rho^0 u^\alpha \lambda_\alpha d_3x = 0 \tag{7.1}$$

and

$$\int T_{,\beta}{}^{\alpha\beta} d_4x = \int T^{\alpha\beta}\lambda_\beta d_3x = 0, \tag{7.2}$$

where the integrals on the left of (7.1) and (7.2) are taken over a volume in space-time and the integrals on the right are taken over the three-dimensional hypersurface bounding this volume. λ_α are the covariant components of the outward drawn normal to the hypersurface.

Next we suppose that the three-dimensional volume is a shell of thickness ϵ enclosing a surface of discontinuity $\sum$ whose three-dimensional normal vector is Λ_i. If we choose our coordinate system so that the discontinuity is at rest, then since

$$\lambda_\alpha \lambda^\alpha = 1, \quad \sum_{i=1}^{3} \Lambda_i{}^2 = 1,$$

we have

$$\lambda_i = \Lambda_i \quad \text{and} \quad \lambda_4 = 0.$$

Hence Eqs. (7.1) and (7.2) become, as ϵ goes to zero,

$$[\rho^0 u^i \Lambda_i] = 0, \tag{7.3}$$

$$[T^{\alpha i}\Lambda_i] = 0, \tag{7.4}$$

where

$$[f] = f_+ - f_-$$

and represents the discontinuity in the function involved. That is f_+ represents the value of the function f on one side of the surface $\sum$ and f_- represents that on the other side. We shall take f_+ and f_- as the values of the function f on the side of the surface opposite to and in the direction of the outwardly drawn normal, respectively.

We may further restrict our coordinate system so that $\Lambda_i = \delta_i{}^1$, that is, the discontinuity is perpendicular to the x axis. Then in view of Eqs. (5.4) and (5.12), Eqs. (7.3) and (7.4) may be written as

$$\frac{\rho_+{}^0 u_+}{(1-u_+{}^2)^{\frac{1}{2}}} = \frac{\rho_-{}^0 u_-}{(1-u_-{}^2)^{\frac{1}{2}}} = m \tag{7.5}$$

and

$$\frac{mc^2\mu_+}{(1-u_+{}^2)^{\frac{1}{2}}} = \frac{mc^2\mu_-}{(1-u_-{}^2)^{\frac{1}{2}}},$$
$$\frac{mc^2\mu_+ u_+}{(1-u_+{}^2)^{\frac{1}{2}}} + p_+ = \frac{mc^2\mu_- u_-}{(1-u_-{}^2)^{\frac{1}{2}}} + p_-. \tag{7.6}$$

The second of Eqs. (7.6) may be written as

$$m^2c^2\left(\frac{\mu_+}{\rho_+{}^0} - \frac{\mu_-}{\rho_-{}^0}\right) = p_- - p_+$$

or

$$m = \frac{1}{c}\left(\frac{p_+ - p_-}{\mu_-/\rho_-{}^0 - \mu_+/\rho_+{}^0}\right)^{\frac{1}{2}}. \tag{7.7}$$

The second of Eqs. (7.6) may also be written as

$$m^2c^4\left(\frac{\mu_+{}^2 u_+{}^2}{1-u_+{}^2} - \frac{\mu_-{}^2 u_-{}^2}{1-u_-{}^2}\right) = (p_- - p_+) m^2 c^2 \left(\frac{\mu_+}{\rho_+{}^0} + \frac{\mu_-}{\rho_-{}^0}\right).$$

Subtracting this from the square of the first of (7.6) we obtain

$$m^2c^2(\mu_+{}^2 - \mu_-{}^2) = m^2(p_+ - p_-)\left(\frac{\mu_+}{\rho_+{}^0} + \frac{\mu_-}{\rho_-{}^0}\right). \tag{7.8}$$

Equations (7.5), (7.7), and (7.8) are the relativistic Rankine-Hugoniot equations. It may readily be verified that if we neglect terms of order $p/c^2\rho^0$ compared to one, then these equations reduce to the classical ones. In the next section we investigate the effect of these terms when they are not negligible.

8. The Shock Velocity

Let us write

$$\xi = p_+/p_-, \quad \eta = \rho_+{}^0/\rho_-{}^0,$$
$$\epsilon_+ = \frac{1}{\gamma_+ - 1} p_+/\rho_+{}^0 = \frac{1}{\gamma_+ - 1}\xi/\eta \frac{p_-}{\rho_-{}^0} = \frac{\beta}{\gamma_+} c^2 \xi/\eta,$$
$$\beta = \frac{\gamma_+}{\gamma_+ - 1} \frac{1}{c^2} \frac{p_-}{\rho_-{}^0}, \tag{8.1}$$
$$\mu_+ = 1 + \beta\xi/\eta,$$

where γ_+ and hence β may be functions of $p_+/\rho_+{}^0$. We shall assume that they are slowly varying functions and for the purposes of the discussion to follow γ_+ will be treated as a constant. From the inequality (4.5) and the requirement that $\epsilon \geqslant 0$, we have

$$1 < \gamma_+ \leqslant 5/3. \tag{8.2}$$

Equation (7.8) may now be written as

$$(1+\beta\xi/\eta)^2-\mu_-{}^2=\frac{\gamma_+-1}{\gamma_+}\beta(\xi-1)\left(\frac{1}{\eta}+\frac{\beta\xi}{\eta^2}+\mu_-\right).$$

Treating γ_+ and β as constants we obtain a quadratic equation for $1/\eta$ with one positive and one negative root. Since η must be positive we have

$$\beta/\eta=\frac{R-[(\gamma_++1)\xi+\gamma_+-1]}{2\xi(\xi+\gamma_+-1)},\tag{8.3}$$

where

$$R=[(\gamma_+-1)^2(\xi-1)^2+4\xi(\xi+\gamma_+-1)\times(\gamma_+\mu_-{}^2+\beta\mu_-(\gamma_+-1)(\xi-1))]^{\frac{1}{2}}.\tag{8.4}$$

Hence

$$\mu_+=\frac{R-(\gamma_+-1)(\xi-1)}{2(\xi+\gamma_+-1)}.\tag{8.5}$$

It follows from these equations that

$$\mu_--\frac{\mu_+}{\eta}=\mu_--\frac{\gamma_+\mu_-{}^2+\beta\mu_-(\gamma_+-1)(\xi-1)-\gamma_+\mu_+}{\beta(\xi+\gamma_+-1)}.$$

However, it is a consequence of (7.8) and the fact that μ_+ and μ_- are positive that

$$\mu_+\geqslant\mu_-.$$

Hence we have

$$\mu_--\mu_+/\eta\geqslant\frac{\mu_-}{\beta(\xi+\gamma_+-1)}\times((\xi-1)\beta(2-\gamma_+)+\gamma_+(1+\beta-\mu_-)).\tag{8.6}$$

Now we may write

$$\mu_-=1+(\gamma_-/\gamma_--1)p/c^2\rho_-{}^0.$$

Then

$$1+\beta-\mu_-=(\gamma_--\gamma_+/\gamma_+(\gamma_--1))\beta,$$

and

$$\mu_--\mu_+/\eta\geqslant\frac{\mu_-}{\xi+\gamma_+-1}\times\left((\xi-1)(2-\gamma_+)+\frac{\gamma_--\gamma_+}{\gamma_--1}\right).\tag{8.7}$$

Since Eq. (7.7) may be written as

$$\frac{u_-}{(1-u_-{}^2)^{\frac{1}{2}}}=\left(\frac{(\gamma_+-1)\beta(\xi-1)}{\gamma_+(\mu_--\mu_+/\eta)}\right)^{\frac{1}{2}},\tag{8.8}$$

we will have u_- less than one whenever the right-hand side of the inequality (8.7) is positive. That is, in this case the velocity of the shock wave relative to the gas into which it is traveling will be less than that of light in vacuum. Now in view of the inequality (8.5) which must hold for both γ_+ and γ_- it follows that for $\xi>3$ the right hand side of (8.7) is positive. Thus, for sufficiently strong shocks the shock velocity must be less than that of light in vacuum.

It is evident that if

$$\gamma_-\geqslant\gamma_+,\tag{8.9}$$

then this result holds for all values of ξ. The inequality (8.9) is satisfied for a monotonic gas as follows from the expression derived by Jüttner[4] for the internal energy of such a gas.

In case $\gamma_-=\gamma_+$ or in the general case for weak shocks where we may assume this equality because of the slowly varying nature of γ, it may be shown that as β becomes large Eq. (8.8) becomes

$$u_-=\left(\frac{(\gamma-1)(\xi+\gamma-1)}{(\gamma-1)\xi+1}\right)^{\frac{1}{2}}.$$

It is evident from this that we must have $\gamma<2$ in order for u_- to be less than one. The inequality (4.5) insures that $\gamma<2$.

[4] F. Jüttner, "Das Maxwellsche Gesetz der Geschwindigkeitsverteilung in der Relativtheorie," Ann. der Phys. **34**, 856–882 (1911).