Chapter4
TEST OF STATISTICAL HYPOTHESES

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Definition 1

A **statistical hypothesis** H is a conjecture about the distribution $f(x;\theta)$ of a population X. This conjecture is usually about the parameter θ if one is dealing with a parametric statistics; otherwise it is about the form of the distribution of X.

Definition 2

A hypothesis H is said to be a **simple hypothesis** if H completely specifies the density $f(x;\theta)$ of the population; otherwise it is called a **composite hypothesis**.

Definition 3

The hypothesis to be tested is called the **null hypothesis**. The negation of the null hypothesis is called the **alternative hypothesis**. The null and alternative hypotheses are denoted by H_o and H_a, respectively.

If θ denotes a population parameter, then the general format of the null hypothesis and alternative hypothesis are

$$H_o : \theta \in \Omega_o \quad \text{and} \quad H_a : \theta \in \Omega_a \qquad (*)$$

where Ω_o and Ω_a are subsets of the parameter space Ω with

$$\Omega_o \cap \Omega_a = \emptyset \quad \text{and} \quad \Omega_o \cup \Omega_a \subseteq \Omega.$$

Remark 1

- If $\Omega_o \cup \Omega_a = \Omega$, then $(*)$ becomes

$$H_o : \theta \in \Omega_o \quad \text{and} \quad H_a : \theta \notin \Omega_o$$

- If Ω_o is a singleton set, then H_o reduces to a simple hypothesis.

Definition 4

A **hypothesis test** is an ordered sequence

$$(X_1, X_2, \ldots, X_n; H_o, H_a; C)$$

where $X_1, X_2, ..., X_n$ is a random sample from a population X with the probability density function $f(x;\theta)$, H_o and H_a are hypotheses concerning the parameter θ in $f(x;\theta)$, and C is called the **critical region or rejection region** in the hypothesis test. The critical region is obtained using a test statistics $W(X_1, X_2, ..., X_n)$. If the outcome of $(X_1, X_2, ..., X_n)$ turns out to be an element of C, then we decide to accept H_a; otherwise we accept H_o. For example, a test might specify that H_o is to be rejected if the sample total $\sum_{i=1}^{n} x_i$ is less than 8. In this case the critical region C is the set

$$\{(x_1, x_2, ..., x_n) | x_1 + x_2 + \ + x_n < 8\}.$$

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There are several methods to find test procedures and they are: (1) Likelihood Ratio Tests, (2) Invariant Tests, (3) Bayesian Tests, and (4) Union Intersection and Intersection-Union Tests. In this section, we only examine likelihood ratio tests.

Definition 5

The **likelihood ratio test statistic** for testing the simple null hypothesis $H_o : \theta \in \Omega_o$ against the composite alternative hypothesis $H_a : \theta \notin \Omega_o$ based on a set of random sample data $x_1, x_2, ..., x_n$ is defined as

$$W\left(x_1, x_2, \ldots, x_n\right) = \frac{\max_{\theta \in \Omega_o} L\left(\theta, x_1, x_2, \ldots, x_n\right)}{\max_{\theta \in \Omega} L\left(\theta, x_1, x_2, \ldots, x_n\right)}$$

where $L(\theta, x_1, x_2, ..., x_n)$ or just $L(\theta)$ denotes the likelihood function of the random sample. A likelihood ratio test (LRT) is any test that has a critical region C of the form

$$C = \{(x_1, x_2, ..., x_n) | W(x_1, x_2, ..., x_n) \leq k\}$$

where k is a number in the unit interval $[0, 1]$.
If $H_o : \theta = \theta_0$ and $H_a : \theta = \theta_a$ are both simple hypotheses, then the likelihood ratio test statistic is defined as

$$W(x_1, x_2, ..., x_n) = \frac{L(\theta_0)}{L(\theta_a)}.$$

Example 1

Let X_1, X_2, X_3 denote three independent observations from a distribution with density

$$f(x; \theta) = \begin{cases} (1+\theta)x^{\theta} & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

What is the form of the LRT critical region for testing $H_o : \theta = 1$ versus $H_a : \theta = 2$?

Example 2

Let $X_1, X_2, ..., X_{12}$ be a random sample from a normal population with mean zero and variance σ^2. What is the form of the LRT critical region for testing the null hypothesis $H_o : \sigma^2 = 10$ versus $H_a : \sigma^2 = 5$?

Example 3

Suppose that X is a random variable about which the hypothesis $H_o : X \sim UNIF(0, 1)$ against $H_a : X \sim N(0, 1)$ is to be tested. What is the form of the LRT critical region based on one observation of X?

Example 4

Let X be a single observation from a population with probability density

$$f(x; \theta) = \begin{cases} \frac{\theta^x e^{-\theta}}{x!} & \text{for } x = 0, 1, 2, \dots \\ 0 & \text{otherwise.} \end{cases}$$

where $\theta \geq 0$. Find the likelihood ratio critical region for testing the null hypothesis $H_o : \theta = 2$ against the composite alternative $H_a : \theta \neq 2$.

Contents

There are several criteria to evaluate the goodness of a test procedure. Some well known criteria are: (1) Powerfulness, (2) Unbiasedness and Invariancy, and (3) Local Powerfulness. In order to examine some of these criteria, we need some terminologies such as error probabilities, power functions, type I error, and type II error. First, we develop these terminologies.

In hypothesis test, the basic problem is to decide, based on the sample information, whether the null hypothesis is true. There are four possible situations that determines our decision is correct or in error. These four situations are summarized below:

	H_o is true	H_o is false
Accept H_o	Correct Decision	Type II Error
Reject H_o	Type I Error	Correct Decision

Definition 6

Let $H_o : \theta \in \Omega_o$ and $H_a : \theta \notin \Omega_o$ be the null and alternative hypothesis to be tested based on a random sample $X_1, X_2, ..., X_n$ from a population X with density $f(x; \theta)$, where θ is a parameter.

- The **significance level** of the hypothesis test $H_o : \theta \in \Omega_o$ and $H_a : \theta \notin \Omega_o$ denoted by α, is defined as

$$\alpha = P(\text{Type I Error}) = P(\text{Reject } H_0 | H_0 \text{ is true}).$$

- The **probability of type II error** of the hypothesis test $H_o : \theta \in \Omega_o$ and $H_a : \theta \notin \Omega_o$ denoted by β, is defined as

$$\beta = P(\text{Type II Error}) = P(\text{Accept } H_0 | H_0 \text{ is false}).$$

Remark 2

Note that α can be numerically evaluated if the null hypothesis is a simple hypothesis and rejection rule is given. Similarly, β can be evaluated if the alternative hypothesis is simple and rejection rule is known. If null and the alternatives are composite hypotheses, then α and β become functions of θ.

Example 5

Let $X_1, X_2, ..., X_{20}$ be a random sample from a Bernoulli distribution $Ber(p)$, where $0 < p \leq \frac{1}{2}$ is a parameter. The hypothesis $H_o : p = \frac{1}{2}$ to be tested against $H_a : p < \frac{1}{2}$. If H_o is rejected when $\sum_{i=1}^{20} x_i \leq 6$, then what is the probability of type I error?

Example 6

Let p represent the proportion of defectives in a manufacturing process. To test $H_o : p \leq \frac{1}{4}$ versus $H_a : p > \frac{1}{4}$, a random sample of size 5 is taken from the process. If the number of defectives is 4 or more, the null hypothesis is rejected. What is the probability of rejecting H_o if $p = \frac{1}{5}$?

Example 7

A random sample of size 4 is taken from a normal distribution with unknown mean μ and variance $\sigma^2 > 0$. To test $H_o : \mu = 0$ against $H_a : \mu < 0$ the following test is used: "Reject H_o if and only if $X_1 + X_2 + X_3 + X_4 < -20$." Find the value of σ so that the significance level of this test will be closed to 0.14.

Example 8

Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal population $N(0, \sigma^2)$, where $\sigma = 16$. The critical region for testing $H_o : \mu = 5$ versus the alternative $H_a : \mu = k$ is $\bar{x} > k - 2$. What would be the value of the constant k and the sample size n which would allow the probability of Type I error to be 0.0228 and the probability of Type II error to be 0.1587.

Definition 7

Let $H_o : \theta \in \Omega_o$ and $H_a : \theta \notin \Omega_o$ be the null and alternative hypothesis to be tested based on a random sample $X_1, X_2, ..., X_n$ from a population X with density $f(x; \theta)$, where θ is a parameter. The **power function** of a hypothesis test

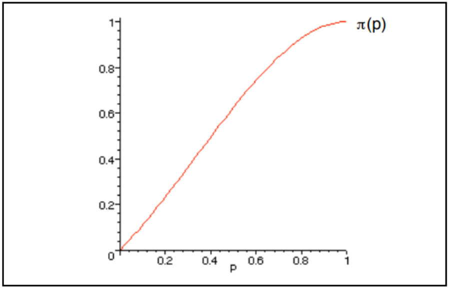
$$H_0 : \theta \in \Omega_o \qquad \text{versus} \qquad H_a : \theta \notin \Omega_o$$

is a function $\pi : \Omega \to [0, 1]$ defined by

$$\pi(\theta) = \begin{cases} P \text{ (Type I Error)} & \text{if } \mathrm{H_o} \text{ is true} \\ 1 - P \text{ (Type II Error)} & \text{if } \mathrm{H_a} \text{ is true} \end{cases}$$

Example 9

A manufacturing firm needs to test the null hypothesis H_o that the probability p of a defective item is 0.1 or less, against the alternative hypothesis $H_a : p > 0.1$. The procedure is to select two items at random. If both are defective, H_o is rejected; otherwise, a third is selected. If the third item is defective H_o is rejected. If all other cases, H_o is accepted, what is the power of the test in terms of p (if H_o is true)?



Example 10

Let X be the number of independent trails required to obtain a success where p is the probability of success on each trial. The hypothesis $H_o : p = 0.1$ is to be tested against the alternative $H_a : p = 0.3$. The null hypothesis H_o is rejected if $X \leq 4$. What is the power of the test if H_a is true?

Example 11

Let $X_1, X_2, ..., X_{25}$ be a random sample of size 25 drawn from a normal distribution with unknown mean μ and variance $\sigma^2 = 100$. It is desired to test the null hypothesis $\mu = 4$ against the alternative $\mu = 6$. What is the power at $\mu = 6$ of the test with rejection rule: reject $\mu = 4$ if $\sum_{i=1}^{25} x_i \geq 125$?

Definition 8

Given $0 \leq \delta \leq 1$, a test (or test procedure) T for testing the null hypothesis $H_o : \theta \in \Omega_o$ against the alternative $H_a : \theta \in \Omega_a$ is said to be a **test of level** δ if

$$\max_{\theta \in \Omega_o} \pi(\theta) \leq \delta,$$

where $\pi(\theta)$ denotes the power function of the test T.

Definition 9

Given $0 \leq \delta \leq 1$, a test (or test procedure) T for testing the null hypothesis $H_o : \theta \in \Omega_o$ against the alternative $H_a : \theta \in \Omega_a$ is said to be a **test of size** δ if

$$\max_{\theta \in \Omega_o} \pi(\theta) = \delta,$$

where $\pi(\theta)$ denotes the power function of the test T.

Definition 10

Let T be a test procedure for testing the null hypothesis $H_o : \theta \in \Omega_o$ against the alternative $H_a : \theta \in \Omega_a$. The test (or test procedure) T is said to be the **uniformly most powerful (UMP)** test of level δ if T is of level δ and for any other test W of level δ,

$$\pi_T(\theta) \geq \pi_W(\theta)$$

for all $\theta \in \Omega_a$. Here $\pi_T(\theta)$ and $\pi_W(\theta)$ denote the power functions of tests T and W, respectively.

Remark 3

If T is a test procedure for testing $H_o : \theta = \theta_0$ against $H_a : \theta = \theta_a$ based on a sample data $x_1, ..., x_n$ from a population X with a continuous probability density function $f(x; \theta)$, then there is a critical region C associated with the test procedure T, and power function of T can be computed as

$$\pi_T = \int_C L(\theta_a, x_1, \ldots, x_n)\, dx_1 \cdots dx_n.$$

Similarly, the size of a critical region C, say α, can be given by

$$\pi_T = \int_C L(\theta_0, x_1, \ldots, x_n)\, dx_1 \cdots dx_n.$$

Theorem 1 (Neyman-Pearson Lemma)

Let $X_1, X_2, ..., X_n$ be a random sample from a population with probability density function $f(x;\theta)$. Let $L(\theta)$ be the likelihood function of the sample. Then any critical region C of the form

$$C = \{(x_1, x_2, \dots, x_n) | \frac{L(\theta_0)}{L(\theta_a)} \leq k\}$$

for some constant $0 \leq k < \infty$ is best (or uniformly most powerful) critical region of its size for testing $H_o : \theta = \theta_0$ against $H_a : \theta = \theta_a$.

Example 12

Let $X_1, X_2, ..., X_{12}$ be a random sample from a normal population with mean zero and variance σ^2. What is the most powerful test of size 0.025 for testing the null hypothesis $H_o : \sigma^2 = 10$ versus $H_a : \sigma^2 = 5$?

Remark 4

Suppose that our goal is to find the UMP level α test of $H_0 : \theta = \theta_0$ versus $H_a : \theta > \theta_0$. Instead of considering this simple-versus-composite test, we first "pretend" like we have the level α simple-versus-simple test $H_0 : \theta = \theta_0$ versus $H_a : \theta = \theta_a$, where $\theta_a > \theta_0$ is arbitrary. If we can then show that the critical region C for the most powerful level α simple-versus-simple test **does not depends on** θ_a, then the test with the same rejection region will be UMP level α for the simple-versus-composite test $H_0 : \theta = \theta_0$ versus $H_a : \theta > \theta_0$.

Example 13

Suppose that $X_1, X_2, ..., X_n$ is a random sample from a normal distribution with mean μ and variance 16. Find the UMP test with the sample size of $n = 16$ and a significance level $\alpha = 0.05$ to the test

$$H_0 : \mu = 10 \text{ versus } H_a : \mu > 10.$$

Remark 5

If either H_o or H_a is not simple, then it is not always possible to find the most powerful test and corresponding critical region. In this situation, hypothesis test is found by using the likelihood ratio. A test obtained by using likelihood ratio is called the likelihood ratio test and the corresponding critical region is called the likelihood ratio critical region.

Example 14

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the normal distribution $N(\mu, \sigma_0^2)$, where σ_0^2 is known but μ is unknown.

(a) Find the likelihood ratio test for $H_0 : \mu = \mu_0$ against $H_a : \mu \neq \mu_0$. Show that the critical region for a test with significance level α is given by $|\bar{X} - \mu_0| > z_{\alpha/2} \frac{\sigma_0}{\sqrt{n}}$.

(b) Test $H_0 : \mu = 59$ against $H_a : \mu \neq 59$ when $\sigma^2 = 225$ and a sample of size $n = 100$ yielded $\bar{x} = 56.13$. Let $\alpha = 0.05$.

Remark 6

The hypothesis $H_a : \mu \neq \mu_0$ is called a **two-sided alternative hypothesis**. An alternative hypothesis of the form $H_a : \mu > \mu_0$ is called a **right-sided alternative**. Similarly, $H_a : \mu < \mu_0$ is called the a **left-sided alternative**. In the above example, if we had a right-sided alternative, that is $H_a : \mu > \mu_0$, then the critical region would have been

$$C = \{(x_1, x_2, \ldots, x_n) | z \geq z_\alpha\}$$

Similarly, if the alternative would have been left-sided, that is $H_a : \mu < \mu_0$, then the critical region would have been

$$C = \{(x_1, x_2, \ldots, x_n) | z \leq -z_\alpha\}$$

We summarize the three cases of hypotheses test of the mean (of the normal population with known variance) in the following table.

H_o	H_a	Critical Region (or Test)
$\mu = \mu_o$	$\mu > \mu_o$	$z = \frac{\overline{x}-\mu_o}{\frac{\sigma}{\sqrt{n}}} \geq z_\alpha$
$\mu = \mu_o$	$\mu < \mu_o$	$z = \frac{\overline{x}-\mu_o}{\frac{\sigma}{\sqrt{n}}} \leq -z_\alpha$
$\mu = \mu_o$	$\mu \neq \mu_o$	$\lvert z \rvert = \left\lvert \frac{\overline{x}-\mu_o}{\frac{\sigma}{\sqrt{n}}} \right\rvert \geq z_{\frac{\alpha}{2}}$

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P-values

Definition 11

The P-value is the probability, calculated assuming that the null hypothesis is true, of obtaining a value of the test statistic at least as contradictory to H_0 as the value calculated from the available sample.

DECISION RULE BASED ON THE P-VALUE

Select a significance level α. Then

- If P-value $\leq \alpha$, reject the null hypothesis H_o.
- If P-value $> \alpha$, do not reject the null hypothesis H_o.

P-values

STEPS TO FIND THE P-VALUE

1. Let T be the test statistic that its distribution is known.
2. Compute the value of T using the sample $X_1, \ldots, X_n$. Say it is t.
3. The P-value is given by

$$P\text{-Value} = \begin{cases} P(T \leq t|H_0), & \text{for left-tail test,} \\ P(T \geq a|H_0), & \text{for right-tail test} \\ 2\min\{P(T \geq t|H_0), P(T \leq t|H_0)\}, & \text{for two-sided test.} \end{cases}$$

Remark 7

If the distribution of T is symmetric about zero, then

$$P\text{-value} = P(|T| \geq |t||H_0)$$

Tests about a Population Mean

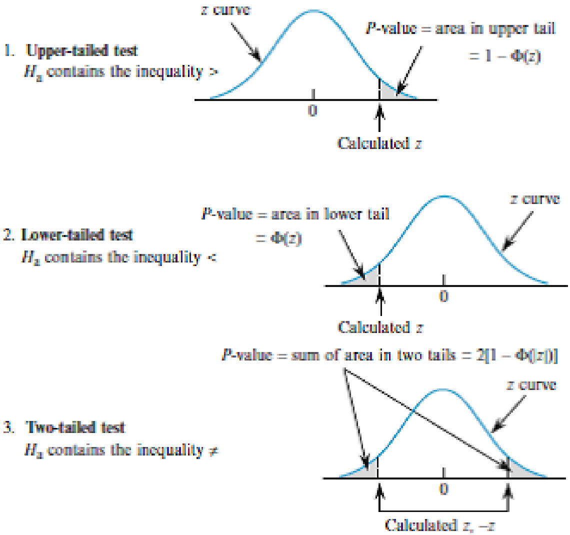
Case I: A Normal Population with Known σ

Assumption: Let $X_1, X_2, \ldots, X_n$ represent a random sample of size n from the normal population $N(\mu, \sigma^2)$, where σ is known value.

- Null hypothesis: $H_0 : \mu = \mu_0$
- Test statistic value: $z = \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}$

Alternative Hypothesis	Rejection Region for Level α Test
$H_a : \mu > \mu_0$	$R = \{z : z \geq z_\alpha\}$
$H_a : \mu < \mu_0$	$R = \{z : z \leq -z_\alpha\}$
$H_a : \mu \neq \mu_0$	$R = \{z : z \leq -z_{\alpha/2} \text{ or } z \geq z_{\alpha/2}\}$

P-value for z Tests



Test about a Population Mean

P-value for z test

$$P - \text{Value} : P = \begin{cases} 1 - \Phi(z) & \text{for an upper-tailed test} \\ \Phi(z) & \text{for an lower-tailed test} \\ 2[1 - \Phi(|z|)] & \text{for a two-tailed } z \text{ test} \end{cases}$$

Example 15

A researcher wishes to test the claim that the average cost of tuition and fees at a four-year public college is greater than \$5700. She selects a random sample of 36 four-year public colleges and finds the mean to be \$5950. The population standard deviation is \$659. Is there evidence to support the claim at $\alpha = 0.05$? Use the P-value method.

Test about a Population Mean

β and Sample Size Determination

Alternative Hypothesis	$\beta(\mu')$ for a Level α Test
$H_a : \mu > \mu_0$	$\Phi\left(z_\alpha + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}\right)$
$H_a : \mu < \mu_0$	$1 - \Phi\left(-z_\alpha + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}\right)$
$H_a : \mu \neq \mu_0$	$\Phi\left(z_{\alpha/2} + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}\right) - \Phi\left(-z_{\alpha/2} + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}\right)$

where $\Phi(z)$=the standard normal cdf.

Test about a Population Mean

β and Sample Size Determination

The sample size n for which a level α test also has at the $\beta(\mu') = \beta$ at the alternative value μ' is

$$n = \begin{cases} \left[\dfrac{\sigma(z_\alpha + z_\beta)}{\mu_0 - \mu'}\right]^2 & \text{for a one-tailed (lower or upper) test} \\ \left[\dfrac{\sigma(z_{\alpha/2} + z_\beta)}{\mu_0 - \mu'}\right]^2 & \text{for a two-tailed test (an proximation solution)} \end{cases}$$

β and Sample Size Determination

Example 16

Let μ denote the true average tread life of a certain type of tire. Consider testing $H_0 : \mu = 30,000$ versus $H_a : \mu > 30,000$ based on a sample of size $n = 16$ from a normal population distribution with $\sigma = 1500$.

- **a.** If $\bar{x} = 30,960$ and a level $\alpha = 0.01$ test is used, what is the decision?
- **b.** If a level 0.01 test is used, what is $\beta(30,500)$?
- **c.** If a level 0.01 test is used and it is also required that $\beta(30,500) = 0.05$, what sample size n is necessary?
- **d.** If $\bar{x} = 30,960$, what is the smallest α at which H_0 can be rejected (based on $n = 16$)?

Test about a Population Mean

Case II: Large-Sample Test

Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with common mean μ and finite variance σ^2. A large n implies that the standardized variable,

$$z = \frac{\bar{X} - \mu}{S/\sqrt{n}}$$

has approximately a standard normal distribution. Substitution of the null value μ_0 in place of μ yields the test statistic

$$z = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}$$

which has approximately a standard normal distribution when H_0 is true. The rule of thumb $n \geq 30$ will again be used to characterize a large sample size.

Test about a Population Mean

Example 17

A random sample of 78 observations produced the following sums:

$$\sum_{i=1}^{78} x_i = 22.8 \qquad \sum_{i=1}^{78} (x_i - \bar{x})^2 = 2.05.$$

(a) Test the null hypothesis that $\mu = 0.45$ against the alternative hypothesis that $\mu < 0.45$ using $\alpha = 0.01$. Also find the P-value.

(b) Test the null hypothesis that $\mu = 0.45$ against the alternative hypothesis that $\mu \neq 0.45$ using $\alpha = 0.01$. Also find the P-value.

(c) What assumptions did you make for solving (a) and (b)?

Test about a Population Mean

Case III: small sample size (The One-Sample t Test)

Assumption: Let $X_1, X_2, \ldots, X_n$ represent a random sample of size n from the normal population $N(\mu, \sigma^2)$, where both parameters μ and σ^2 are unknown and the sample size n is small $(n < 30)$.

- Null hypothesis: $H_0 : \mu = \mu_0$
- Test statistic value: $t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$

Alternative Hypothesis	Rejection Region for Level α Test
$H_a : \mu > \mu_0$	$R = \{t : t \geq t_{\alpha, n-1}\}$
$H_a : \mu < \mu_0$	$R = \{t : t \leq -t_{\alpha, n-1}\}$
$H_a : \mu \neq \mu_0$	$R = \{t : t \geq t_{\alpha/2, n-1} \text{ or } t \leq -t_{\alpha/2, n-1}\}$

P-value for t Tests

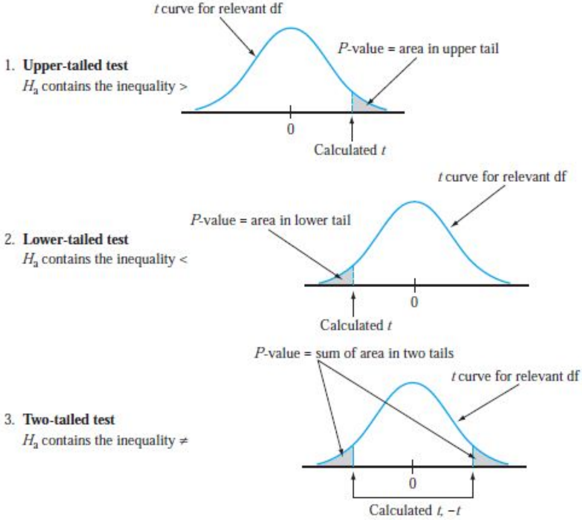


Figure 8.10 P-values for t tests

Test about a Population Mean

Example 18

Consider the test $H_0 : \mu = 35$ versus $H_a : \mu > 35$ for a population that is normally distributed.

(a) A random sample of 18 observations taken from this population produced a sample mean of 40 and a sample standard deviation of 5. Using $\alpha = 0.025$, would you reject the null hypothesis?

(b) Another random sample of 18 observations produced a sample mean of 36.8 and a sample standard deviation of 6.9. Using $\alpha = 0.025$, would you reject the null hypothesis?

(c) Compare and discuss the decisions of parts (a) and (b).

Test for a Variance or Standard Deviation

χ^2 Test for a Variance or Standard Deviation

Assumption: Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the normal distribution $N(\mu, \sigma^2)$, where μ is known value.

- Null hypothesis: $H_0 : \sigma^2 = \sigma_0^2$
- Test statistic value: $\chi^2 = \frac{(n-1)S^2}{\sigma_0^2}$

Alternative Hypothesis	Rejection Region for Level α Test
$H_a : \sigma^2 > \sigma_0^2$	$R = \{\chi^2 : \chi^2 \geq \chi^2_{\alpha, n-1}\}$
$H_a : \sigma^2 < \sigma_0^2$	$R = \{\chi^2 : \chi^2 \leq \chi^2_{1-\alpha, n-1}\}$
$H_a : \sigma^2 \neq \sigma_0^2$	$R = \{\chi^2 : \chi^2 \leq \chi^2_{1-\alpha/2, n-1}$ or $\chi^2 \geq \chi^2_{\alpha/2}\}$

Test for a Variance or Standard Deviation

Example 19

The standard deviation for the Math SAT test is 100. The variance is 10,000. An instructor wishes to see if the variance of the 23 randomly selected students in her school is less than 10,000. The variance for the 23 test scores is 7225. Is there enough evidence to support the claim that the variance of the students in her school is less than 10,000 at $\alpha = 0.05$? Assume that the scores are normally distributed

Example 20

A researcher knows from past studies that the standard deviation of the time it takes to inspect a car is 16.8 minutes. A sample of 24 cars is selected and inspected. The standard deviation is 12.5 minutes. At $\alpha = 0.05$, can it be concluded that the standard deviation has changed? Use the P-value method.

Test about a Population Proportion

Assumption: $X_1, X_2, ..., X_n$ is an iid Bernoulli(p) sample.

- Null hypothesis: $H_0 : p = p_0$
- Test statistic value: $z = \dfrac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)/n}}$

Alternative Hypothesis	Rejection Region for Level α Test
$H_a : p > p_0$	$R = \{z : z \geq z_\alpha\}$
$H_a : p < p_0$	$R = \{z : z \leq -z_\alpha\}$
$H_a : p \neq p_0$	$R = \{z : z \leq -z_{\alpha/2} \text{ or } z \geq z_{\alpha/2}\}$

These test procedures are valid provided that $np_0 \geq 5$ and $n(1-p_0) \geq 5$

Test about a Population Proportion

Example 21

A machine in a certain factory must be repaired if it produces more than 12% defectives among the large lot of items it produces in a week. A random sample of 175 items from a week's production contains 45 defectives, and it is decided that the machine must be repaired.

(a) Does the sample evidence support this decision? Use $\alpha = 0.02$.

(b) Compute the P-value.

β and Sample Size Determination

Alternative Hypothesis	$\beta(p')$
$H_a : p > p_0$	$\Phi\left[\dfrac{p_0 - p' + z_\alpha\sqrt{p_0(1-p_0)/n}}{\sqrt{p'(1-p')/n}}\right]$
$H_a : p < p_0$	$1 - \Phi\left[\dfrac{p_0 - p' - z_\alpha\sqrt{p_0(1-p_0)/n}}{\sqrt{p'(1-p')/n}}\right]$
$H_a : p \neq p_0$	$\Phi\left[\dfrac{p_0 - p' + z_{\alpha/2}\sqrt{p_0(1-p_0)/n}}{\sqrt{p'(1-p')/n}}\right] - \Phi\left[\dfrac{p_0 - p' - z_{\alpha/2}\sqrt{p_0(1-p_0)/n}}{\sqrt{p'(1-p')/n}}\right]$

β and Sample Size Determination

The sample size n for which the level α test also satisfies $\beta(p') = \beta$ is

$$n = \begin{cases} \left[\dfrac{z_\alpha \sqrt{p_0(1-p_0)} + z_\beta \sqrt{p'(1-p')}}{p' - p_0} \right]^2 & \text{one-tailed test} \\ \left[\dfrac{z_{\alpha/2} \sqrt{p_0(1-p_0)} + z_\beta \sqrt{p'(1-p')}}{p' - p_0} \right]^2 & \text{Two-tailed test(approximat} \end{cases}$$

β and Sample Size Determination

Example 22

A package-delivery service advertises that at least 90% of all packages brought to its office by 9 A.M. for delivery in the same city are delivered by noon that day. Let p denote the true proportion of such packages that are delivered as advertised and consider the hypotheses $H_0 : p = 0.9$ versus $H_a : p < 0.9$. If only 80% of the packages are delivered as advertised, how likely is it that a level 0.01 test based on $n = 225$ packages will detect such a departure from H_0 ? What should the sample size be to ensure that $\beta(0.8) = 0.01$?