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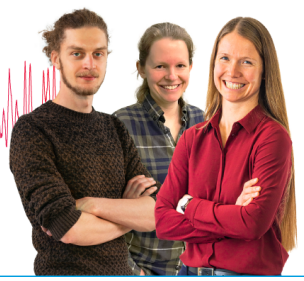

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Meta-dissipation: A framework for quantifying energy dissipation in dissipative discrete periodic metamaterials

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ABSTRACT

Quantifying energy dissipation in dissipative periodic discrete metamaterials remains a challenge, as traditional interpretations of metadamping, based on summing modal damping ratios, do not consistently reflect actual time-domain energy loss in multi-modal systems. The study addresses this critical gap by introducing a framework grounded in a consistent definition of the unit cell for discrete dissipative systems. A single-term exponential energy decay coefficient, obtained through least squares minimization of the transient response, is proposed as a robust descriptor of system-level dissipation. The coefficient approximately captures energy decay behavior in both phononic crystals and acoustic metamaterials, offering a more physically meaningful measure than conventional damping metrics. The proposed meta-dissipation framework establishes a critical link between dispersion-based analysis and transient energy dissipation, enabling more effective design and optimization of metamaterials for vibration attenuation across a wide range of engineering applications.

Evanescent waves within subwavelength frequency bands have been a central reason behind the growing interest in metamaterials and periodic chains.[1] These wave phenomena give rise to unique dynamic responses, including bandgaps and local resonances, critical for vibration attenuation and wave manipulation.[2] However, the influence of energy dissipation in dissipative metamaterials or periodic chains remains relatively less explored, despite its significant role in practical applications.

Hussein and Frazier[3] were the first to report that dissipative acoustic metamaterials exhibit enhanced energy dissipation compared to their statically equivalent dissipative phononic crystals. This enhanced dissipation was termed metadamping,[4] and is typically quantified by summing the integrated values of each damping ratio branch across the Brillouin zone (BZ), as derived from dispersion relationships using Bloch's theorem applied to the unit cell. Since the seminal work, metadamping has been investigated in various lumped mass configurations, including systems with negative stiffness,[5] nonlocal resonances,[6] multi-degree-of-freedom units,[7,8] inertial amplifiers,[9] active resonators,[10,11] and nonlocal interactions.[12] Further extensions have included viscoelastic relaxation models,[13] viscous to viscoelastic transition,[27] nonlinear metamaterials,[14] and electrically activated local resonators.[15] Metadamping has also been studied in structural configurations such as flexural beams with periodically attached spring-mass-dampers,[16–18] generalized mechanical networks,[19] and simultaneous energy harvesting.[28,29] However, existing interpretations of the total damping ratio lack clarity regarding their physical interpretation or direct quantification of actual energy dissipation. In higher-degree-of-freedom systems, a simple summation of modal damping ratios often overpredicts the dissipation, which may not align with physical behavior.

Recently, the concept of metadamping has been extended to quantify energy dissipation, understood as the fundamental measure of energy loss in a dissipative medium over time. This extension has been effectively demonstrated in continuous structures by evaluating the dispersion characteristics of unit cells under free vibration, such as in metamaterial stepped beams,[20] thin-walled beams,[21,22] periodically

supported beams,[23] and pile–soil interaction systems.[24,25] In the context of damping estimation for continuous systems with periodic attachments, the ambiguity in unit cell definition is absent due to homogenization, which transforms localized discontinuities into a uniformly distributed representation over the spatial domain. However, this approach is not directly applicable to a discrete system.

The unit cell of a periodic discrete lattice can be tessellated in many forms. Some of the recent studies on the choice of unit cell elucidate that an asymmetric unit cell often leads to natural frequencies in the attenuation bandgap for a finite chain for undamped systems;[30–34] however, the aspect of energy dissipation has not been discussed. To systematically define the unit cell, let us consider Figs. 1(a) and 1(b), having the mass of the left side of the unit cell as χM and the right side of the unit cell as $(1-\chi)M$, where $0 \leq \chi \leq 1$, leads to the following dispersion relationship:

$$M\lambda^2 + 2(K + \lambda C)(1 - \cos\mu) = 0. \tag{1}$$

Here, M, C, and K denote the mass, viscous damping, and stiffness of the system, respectively, and the associated wave equation can be written as $u_{n+j} = U_n e^{(i\kappa lj + \lambda t)}$, with κ, λ, and l denoting the wavenumber, Laplace variable, and lattice length, respectively. The dimensionless wavenumber is defined as $\mu = \kappa l$. The dispersion relationship is independent of the distribution factor χ, signifying that a unit cell of a periodic spring-mass-damper chain can be conceptualized from any of its portions. Multiple forms of unit cells are possible, and they all will result in an identical dispersion relationship. Although different unit cells represent the same dispersion relationship and thus yield identical wave number-dependent damping ratios, their distinct configurations lead to variations in natural frequencies and time-domain responses. As a result, each unit cell exhibits a different level of energy dissipation despite having analogous dispersion characteristics. This disparity introduces a fundamental inconsistency in evaluating energy dissipation in discrete systems, highlighting the need for a consistent framework. Moreover, a simplified yet comprehensive representation of the system's energy dissipation in the time domain, characterized by a single representative parameter, is currently unavailable.

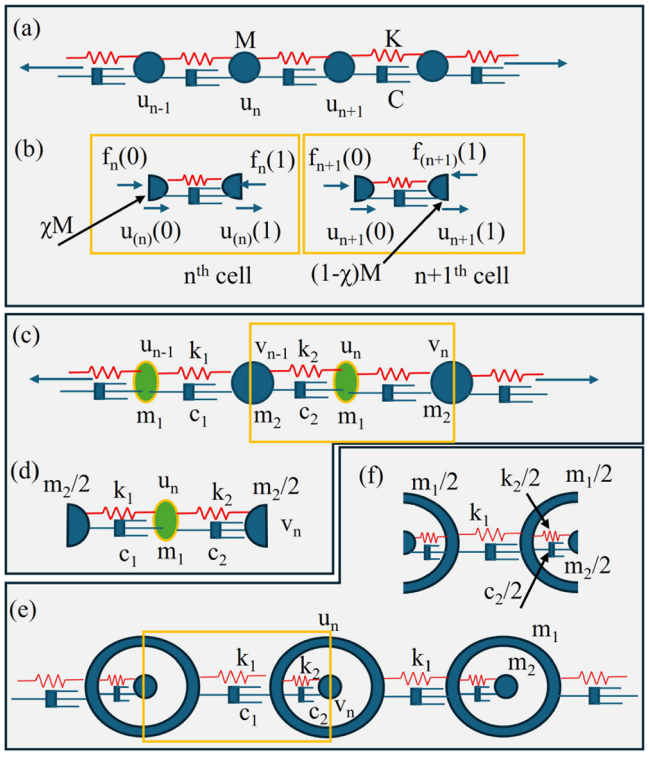

FIG. 1. Periodic metamaterial chains and the corresponding unit cells for (a) and (b) monatomic, (c) and (d) diatomic or phononic crystal, and (e) and (f) mass-in-mass or acoustic metamaterial.

Motivated by the research gap, this study proposes a consistent definition of a unit cell tailored for time-domain response analysis of discrete systems. A single-term exponential displacement decay function, obtained through a least squares approximation, is introduced to characterize the responses of the masses effectively. Furthermore, a single-term energy decay coefficient is proposed to correlate with system-level energy dissipation.

A discrete periodic system, as shown in Fig. 1(a), can be expressed as $L(x) = \sum_n \delta(x - nl)$, where n is an integer and l is the distance between two successive units (i.e., the lattice length). To convert the discrete model in the wavenumber domain, the Fourier transform can be performed, which yields

$$R(\kappa) = \int_{-\infty}^{\infty} e^{i\kappa x} L(x)dx = \sum_n \cos(\kappa nl). \tag{2}$$

Here, κ denotes the wavenumber. For any arbitrary value of κ, Eq. (2) tends to zero as the number of terms approaches infinity, and can be avoided for specific values of κ such that $\cos(\kappa na) = 1 \rightarrow \mu = \kappa a = 2N\pi$, with N being an integer. In the non-dimensional wave number domain (μ), that is, in the non-dimensional reciprocal lattice, the discrete points are spaced at intervals of 2π. According to the definition, the Wigner–Seitz unit cell in the reciprocal lattice corresponds to the Brillouin zone, which therefore extends from $-\pi$ to π. Owing to the symmetry of the lattice, the irreducible Brillouin zone is taken as the range $0 \leq \mu \leq \pi$.

The dispersion relation for the monatomic chain, given in Eq. (1), involves both the wave number and frequency. To derive the corresponding time-domain equation of motion for a consistent unit cell, we propose integrating the dispersion relation over the entire wave number domain ($\mu \in [0, \pi]$), as follows:

$$\frac{1}{\pi}\int_0^{\pi} \left[M\lambda^2 + 2(K + \lambda C)(1 - \cos\mu)\right] d\mu = 0$$
$$\rightarrow M\lambda^2 + 2(K + \lambda C) = 0. \tag{3}$$

Equation (3) can be realized for the system with mass, stiffness, and damping as $M/2$, K, and C, respectively, in the Laplace domain, which leads to the corresponding equation of motion in the time domain as

$$\frac{M}{2}\ddot{u}_n + C\dot{u}_n + Ku_n = 0. \tag{4}$$

Using a similar argument, the equations of motion in the time domain of the consistent unit cell for the diatomic and acoustic metamaterial chain are obtained as

$$\begin{bmatrix} m_1 & 0 \\ 0 & \frac{m_2}{2} \end{bmatrix} \begin{Bmatrix} \ddot{u}_n \\ \ddot{v}_n \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{Bmatrix} \dot{u}_n \\ \dot{v}_n \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_n \\ v_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{5}$$

and

$$\begin{bmatrix} \frac{m_1}{2} & 0 \\ 0 & \frac{m_2}{2} \end{bmatrix} \begin{Bmatrix} \ddot{u}_n \\ \ddot{v}_n \end{Bmatrix} + \begin{bmatrix} c_1 + \frac{c_2}{2} & -\frac{c_2}{2} \\ -\frac{c_2}{2} & \frac{c_2}{2} \end{bmatrix} \begin{Bmatrix} \dot{u}_n \\ \dot{v}_n \end{Bmatrix} + \begin{bmatrix} k_1 + \frac{k_2}{2} & -\frac{k_2}{2} \\ -\frac{k_2}{2} & \frac{k_2}{2} \end{bmatrix} \begin{Bmatrix} u_n \\ v_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{6}$$

respectively. Here, u_n and v_n denote the displacements associated with the nth unit cell as shown in Fig. 1. The detailed calculation is provided in the supplementary material.

Equations (4)–(6) indicate that a consistent measure of energy dissipation is achieved by employing a symmetric unit cell with a fixed–free configuration, which arises naturally from the equations of motion. In this configuration, the rate of decay of the free vibration response serves as a robust metric for energy dissipation, reflecting the efficiency with which the unit cell attenuates vibrational energy. To quantify the dissipation, a single exponential function of time is introduced, and its exponent is defined as the decay coefficient. The proposed framework yields a scalar dissipation coefficient that is physically interpretable and provides a consistent and comparable metric for evaluating different unit cell configurations, limited to 2DOF unit cells.

In a periodic metamaterial chain, a single unit cell sufficiently represents the entire structure and governs its dispersion and energy dissipation behavior. To analyze the decay of the time-domain response under impulse-type loading, we focus on the symmetric unit cell. For a monatomic chain described by Eq. (3), the displacement response is given by $u_n(t) = Ae^{-\frac{C}{M}t}\cos(\omega_d t - \phi)$, where A is amplitude, ω_d is damped natural frequency, and ϕ is the phase angle. The peak displacement decays exponentially as $u_{np} = Ae^{-\theta_0 t}$, where $\theta_0 = \frac{C}{M}$. The displacement decay coefficient, θ_0, effectively characterizes how quickly vibrations attenuate and energy dissipates in the system. Since the governing dynamics correspond to a single-degree-of-freedom system, θ_0 offers a direct measure of the system's dissipation rate. The rate of energy dissipation is proportional to $2\theta_0$ in which the factor 2 can be justified through the Lyapunov energy function $V(t) = \frac{1}{2}Ku_n^2 + \frac{1}{2}M\dot{u}_n^2$, where for a proportionally damped system, $\frac{dV}{dt} = -C\dot{u}_n^2 \approx -2\theta_0 V(t)$.

In contrast to monatomic systems, the unit cell of a diatomic or acoustic metamaterial chain leads to a coupled two-degree-of-freedom (2DOF) system, whose response is governed by two distinct natural frequencies and damping ratios. The time-domain solution involves a superposition of two decaying exponentials with different decay rates and oscillatory components, making it non-trivial to describe the decay using a single parameter. To this end, we propose an effective approach that fits the envelope of the multi-component response using a least squares method, thereby obtaining a single energy decay coefficient (Θ) that consistently reflects the net energy dissipation of the unit cell. Note that Θ is fundamentally different from the metadamping metric $\xi_{\text{tot}}^{\text{sum}}$ originally introduced[3] to quantify the relative enhancement of damping in acoustic metamaterials compared to their statically equivalent phononic crystals. In contrast, the meta-dissipation framework introduces a scalar energy decay coefficient (θ) that directly quantifies the energy dissipation in the time domain. Although $\xi_{\text{tot}}\omega$ corresponds to the decay rate in single degree of freedom (SDOF) systems, no such direct relation exists for the 2DOF systems, and no prior literature establishes such a link.

The displacement decay coefficients are extracted for each degree of freedom (θ_u and θ_v). The first step involves identifying the peak values of the time-domain responses, as outlined below. Consider the modal transformation, $\boldsymbol{u} = \boldsymbol{\Phi q}$, where $\boldsymbol{u} = [u_n, v_n]^T$ is the displacement vector, $\boldsymbol{\Phi} = [\phi_{11}, \phi_{12}; \phi_{21}, \phi_{22}]$ is the mass normalized modal matrix, and $\boldsymbol{q}$ is the modal coordinates. Invoking the modal transformation, Eq. (5) or Eq. (6) yields the two uncoupled equations of the following form:

$$\ddot{q}_j + 2\xi_j\omega_j\dot{q}_j + \omega_j^2 q_j = 0, \quad j = 1, 2. \tag{7}$$

Here, ξ_1 and ξ_2 are the modal damping ratios associated with the natural frequencies ω_1 and ω_2, corresponding to the first and second modes, respectively. Note that the present study considers linear systems with proportional viscous damping, which leads to a Hermitian system. When a unit impulse load is applied to mass-1 of the unit cell, the initial conditions are defined as $u_{n0} = v_{n0} = \dot{v}_{n0} = 0$ and $\dot{u}_{n0} = \frac{1}{m_1}$. These conditions result in zero initial displacements in the modal coordinates, while the initial modal velocities are given by $\dot{\boldsymbol{q}}(t=0) = [\phi_{11}, \phi_{12}]^T$. Consequently, the time-domain response of the two-degree-of-freedom system is obtained as

$$\begin{Bmatrix} u_n \\ v_n \end{Bmatrix} = \sum_{j=1}^{2} \begin{Bmatrix} \phi_{1j} \\ \phi_{2j} \end{Bmatrix} \frac{\phi_{1j}}{\omega_{dj}} e^{-\xi_j\omega_j t} \sin\omega_{dj} t. \tag{8}$$

Here, $\omega_{dj} = \omega_{nj}\sqrt{1-\xi_j^2}$, $j = 1, 2$ are the damped natural frequencies. Now, the curve representing the peak values of response, u_n, can be written as[26]

$$y_{pu}(t) = A_u e^{-\xi_1\omega_1 t} + B_u e^{-\xi_2\omega_2 t}, \tag{9}$$

where $A_u = \phi_{11}^2/\omega_{d1}$ and $B_u = \phi_{12}^2/\omega_{d2}$. It is evident that two exponential terms, corresponding to the two modal frequencies and damping ratios, are required to represent the exponential decay of the response accurately. A similar expression can be derived for v_n, given by $y_{pv}(t) = A_v e^{-\xi_1\omega_1 t} + B_v e^{-\xi_2\omega_2 t}$, where $A_v = \phi_{11}\phi_{21}/\omega_{d1}$ and $B_v = \phi_{12}\phi_{22}/\omega_{d2}$. Following the same reasoning, one can conclude that for an n-degree-of-freedom system, n exponential decay terms are needed to describe the decay of the response. Additionally, for a more than 2DOF system, an equation for peaks in the form of Eq. (9) is not analytically available.

However, in the present study, specifically for the two-degree-of-freedom unit cells typical of diatomic or acoustic metamaterial systems, we demonstrate that a single-term approximation can still yield reasonably accurate results and thus allows for a simplified quantification of the exponential decay behavior.

To this end, we approximate Eq. (9) as $y_{pu}(t) \approx X_{pu}e^{-\theta_u t}$ and define the associated residual as

$$R_u = X_{pu}e^{-\theta_u t} - A_u e^{-\xi_1\omega_1 t} - B_u e^{-\xi_2\omega_2 t}. \tag{10}$$

The least squares method is employed to minimize the residual (R_u) with respect to the unknowns X_{pu} and θ_u, leading to the following equations:

$$\frac{\partial}{\partial X_{pu}}\int_0^\infty R_u^2 dt = 0, \tag{11}$$

$$\frac{\partial}{\partial \theta_u}\int_0^\infty R_u^2 dt = 0. \tag{12}$$

The solution of the simultaneous equations, Eqs. (11) and (12), yields the values of X_{pu} and θ_u. The detailed derivation is not included in the manuscript for brevity; however, readers may refer to the supplementary material for the complete derivation and the solution procedure. Using a similar approach, the decay coefficient θ_v corresponding to the response of the second mass can also be determined.

The total energy of the phononic crystal is expressed as

$$E_{\mathrm{PC}}(t) = \frac{1}{2}\left(m_1\dot{u}_n^2 + \frac{m_2}{2}\dot{v}_n^2\right) + \frac{1}{2}\left(k_1u_n^2 + k_2(u_n - v_n)^2\right). \tag{13}$$

Substituting: $u_n(t) \approx X_{pu}e^{-\theta_u t}$ and $v_n(t) \approx X_{pv}e^{-\theta_v t}$ in Eq. (13), the peak of the energy envelope is approximately obtained as

$$E_{\mathrm{PC,p}} \approx \underbrace{\frac{1}{2}(k_1 + k_2 + m_1\theta_u^2)X_{pu}^2}_{\alpha_u} e^{-2\theta_u t} + \underbrace{\frac{1}{2}\left(k_2 + \frac{m_2}{2}\theta_v^2\right)X_{pv}^2}_{\alpha_v} e^{-2\theta_v t} - \underbrace{k_2X_{pu}X_{pv}}_{\alpha_c} e^{-(\theta_u+\theta_v)t}. \tag{14}$$

A similar expression for AM can also be derived as

$$E_{\mathrm{AM,p}}(t) \approx \underbrace{\frac{1}{4}(2k_1 + k_2 + m_1\theta_u^2)X_{pu}^2}_{\alpha_u} e^{-2\theta_u t} + \underbrace{\frac{1}{4}(k_2 + m_2\theta_v^2)X_{pv}^2}_{\alpha_v} e^{-2\theta_v t} - \underbrace{\frac{k_2}{2}X_{pu}X_{pv}}_{\alpha_c} e^{-(\theta_u+\theta_v)t}. \tag{15}$$

Since the total energy of the unit cell decays exponentially, with contributions from both degrees of freedom in a two-degree-of-freedom system such as diatomic or acoustic metamaterial chains. To this end, considering a single-term exponential approximation, the energy decay for PC or AM is represented in terms of the displacement decay coefficients θ_u and θ_v as $E_0e^{-2\Theta t}$. Here, $E_0 = E_{\mathrm{PC,p}}(t=0)$ or $E_{\mathrm{AM,p}}(t=0)$ for PC and AM, respectively. When the displacement decay coefficients of the two masses, θ_u and θ_v, are comparable, the energy decay rate can be approximated by their arithmetic average. In contrast, if the decay rates differ significantly, both modes contribute to the energy dissipation during the initial period; however, over time, the mode with the slower decay rate becomes dominant. Consequently, the overall energy decay coefficient, denoted by Θ, should be expressed as a convex combination of θ_u and θ_v. By employing a weighted harmonic mean and neglecting cross coupling effects, an approximate expression for Θ can be obtained as

$$\frac{\alpha_u}{\theta_u} + \frac{\alpha_v}{\theta_v} = \frac{\alpha_u + \alpha_v}{\Theta} \rightarrow \Theta = \frac{\alpha_u + \alpha_v}{\frac{\alpha_u}{\theta_u} + \frac{\alpha_v}{\theta_v}}. \tag{16}$$

Equation (16) demonstrates excellent agreement with the numerical response, particularly in the context of PC and AM with varying mass and stiffness matrices. The corresponding normalized root mean squared error (NRMSE) is obtained as

$$\mathrm{NRMSE} = \frac{\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(E_{\mathrm{PC/AM}}\big|_i - E_0e^{-2\Theta t}\big|_i\right)^2}}{\max(E_{\mathrm{PC/AM}}) - \min(E_{\mathrm{PC/AM}})}. \tag{17}$$

To evaluate the energy dissipation profile under varying stiffness characteristics of the system, the long-wave speed of sound (C_{stat}) is considered. C_{stat} represents the slope of the lowest dispersion branch as the wave number tends to zero. The expressions for C_{stat} in phononic crystals and acoustic metamaterials are given by[6]

$$C_{\mathrm{stat}}|_{\mathrm{PC}} = l\sqrt{\frac{k_1k_2}{(m_1+m_2)(k_1+k_2)}}, \qquad C_{\mathrm{stat}}|_{\mathrm{AM}} = l\sqrt{\frac{k_1}{m_1+m_2}}. \tag{18}$$

Here, l denotes the unit cell length, which, without loss of generality, can be taken as unity.

To elucidate the accuracy of the developed approximate formulas, a phononic crystal (PC) characterized by $m_1 = 1$, $m_2 = 0.8$, $k_1 = 40\,906$, $k_2 = 18\,000$, and $c_1 = 20$, $c_2 = 8.8$ and its statically equivalent acoustic metamaterial (AM) characterized by $m_1 = 1$, $m_2 = 0.8$, $k_1 = 12\,500$, $k_2 = 5500$, and $c_1 = 20$, $c_2 = 8.8$ have been considered. Both the systems have $C_{\mathrm{stat}} = 83.33$. All the parameters may be read in any consistent system of physical units. The time-domain responses of PC and AM are illustrated in Figs. 2(a)–2(c) and Figs. 2(d)–2(f), respectively. Subfigures of Figs. 2(a), 2(b), 2(d), and 2(e) illustrate the free vibration responses of a unit cell for both PC and AM, including a comparison between the classical two-term approximation[26] and the proposed single-term displacement decay for both the masses. It is evident that (θ_u) or (θ_v) closely captures the amplitude peaks and overall response. The deviation from the two-term approximation is insignificant, indicating that the proposed method effectively captures the decay rate of the displacements.

Furthermore, the energy decay within each unit cell is effectively characterized using the proposed energy decay coefficient, Θ. The dissipation profiles for both the phononic crystal (PC) and acoustic metamaterial (AM), as shown in Figs. 2(c) and 2(f), respectively, demonstrate that Θ accurately captures the temporal decay of vibrational energy. The validation of both the dynamic response and the corresponding energy dissipation using the proposed framework reinforces its applicability and robustness. It is noteworthy that the computed values of Θ are 7.49 for the phononic crystal and 14.52 for the acoustic metamaterial, indicating that the Θ for the AM is nearly twice that of the statically equivalent PC, despite both having identical viscous damping. This observation further substantiates the concept of

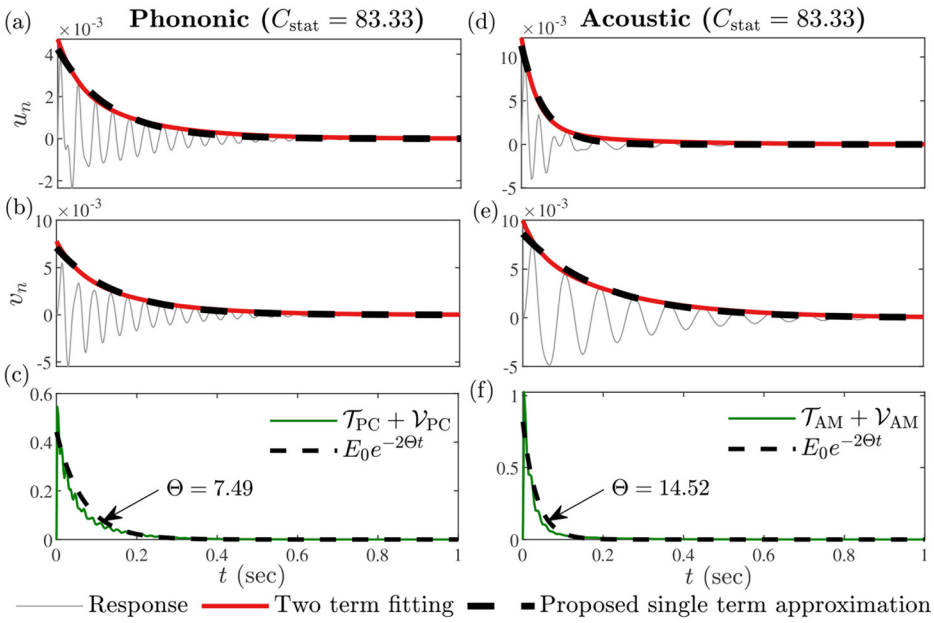

FIG. 2. Time-domain responses (u_n and v_n) to a unit impulse load, along with peak fitting using both the two-term and proposed single-term expressions for (a) and (b) a phononic crystal, and (d) and (e) a statically equivalent acoustic metamaterial unit cell. Here, $C_{stat}|_{PC} = C_{stat}|_{AM} = 83.33$. The corresponding variation in total system energy and its approximation using the energy decay coefficient are shown in (c) and (f) for PC and AM, respectively.

metadamping[3] and highlights the superior damping capabilities of acoustic metamaterials, which can dissipate vibrational energy more efficiently than their phononic counterparts. While previous methods for characterizing damping in metamaterial unit cells often lacked a direct correlation with energy dissipation, the proposed decay coefficient provides a quantitative and physically meaningful measure of energy loss within a unit cell.

To examine the variation of the energy decay coefficient (Θ) with the long-wavelength speed of sound (C_{stat}) for both PC and AM, having constant c_1 and c_2, k_1 is varied and k_2 is calculated as $k_2 = k_1(c_2/c_1)$ so that the damping matrix always remains proportional to the stiffness matrix. This ensures the validity of modal decomposition as shown in Eq. (7).

As shown in Fig. 3(a), increasing C_{stat} results in a decrease in the energy decay coefficient Θ which is more pronounced at lower values of C_{stat}, while the variation becomes marginal at higher values. The trend aligns with physical intuition, as stiffer materials (i.e., higher C_{stat}) inherently dissipate less energy, which is reflected in the lower values of Θ. To further validate the robustness of the proposed approach, the dynamic responses and energy dissipation

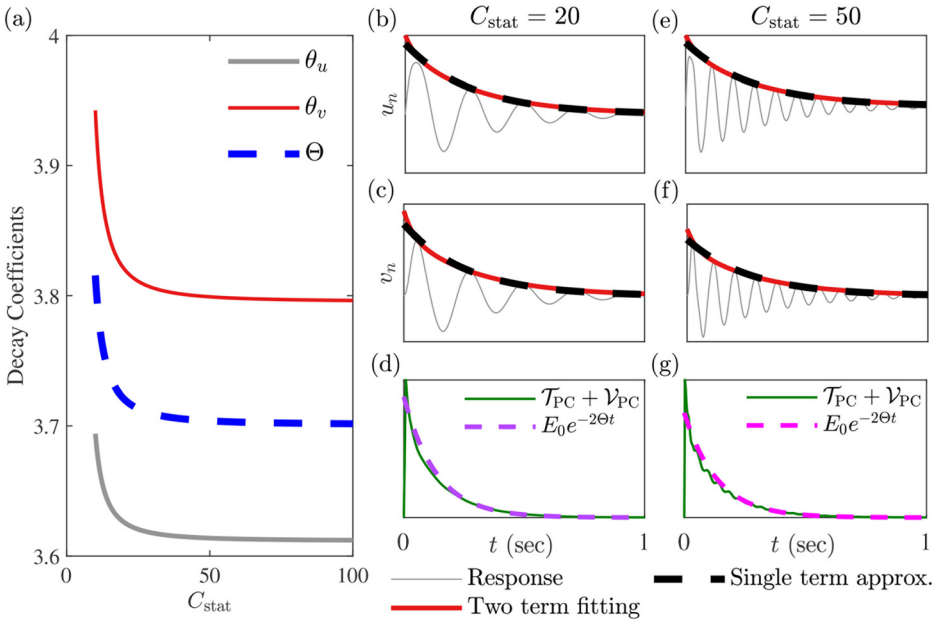

FIG. 3. (a) Variation of θ_u, θ_v, and Θ with the long-wave speed of sound (C_{stat}) in a phononic crystal. Subfigures (b) and (c) and (e) and (f) show the time-domain responses (u_n and v_n) to a unit impulse load, along with peak fitting using both the two-term and proposed single-term expressions. Subfigures (d) and (g) present the corresponding variation in total system energy and its approximation using the energy decay coefficient.

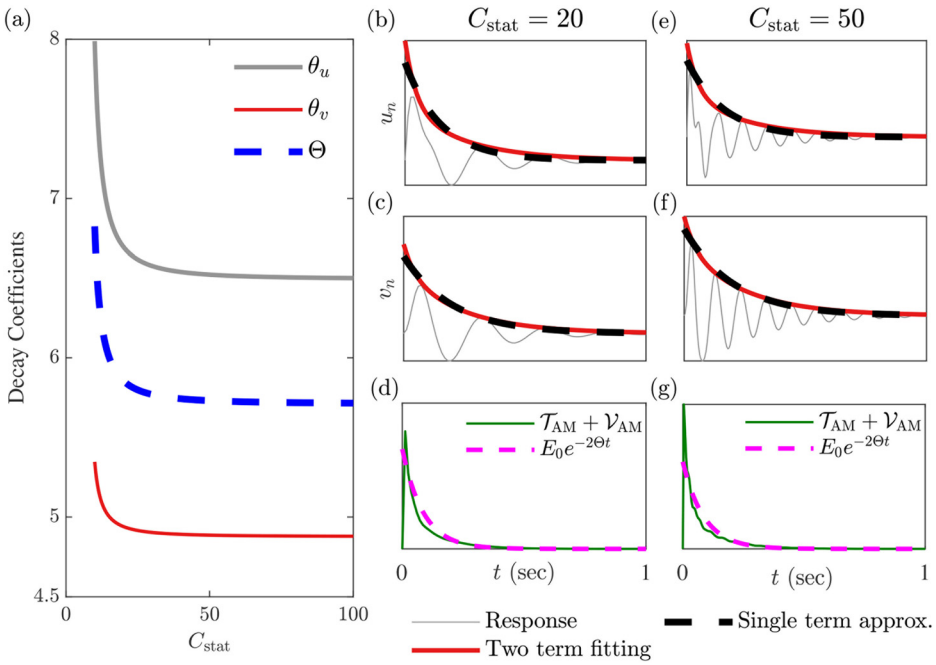

FIG. 4. (a) Variation of θ_u, θ_v, and Θ with C_{stat} in an acoustic metamaterial. Subfigures (b) and (c) and (e) and (f) show the time-domain responses (u_n and v_n) to a unit impulse load, along with peak fitting using both the two-term and proposed single-term expressions. Subfigures (d) and (g) present the corresponding variation in total system energy and its approximation using the energy decay coefficient.

characteristics of the two masses in the phononic crystal are examined for representative cases of $C_{\text{stat}} = 20$ and 50. The results confirm that the proposed framework effectively captures both the mechanical behavior and the dissipation characteristics across varying stiffness conditions. A similar observation can be perceived for acoustic metamaterial, as illustrated in Fig. 4.

Numerical validation across a wide range of parameter sets confirms the robustness of the proposed approximation, as the normalized root mean squared error (NRMSE), defined in Eq. (17), remains close to zero for the entire range of C_{stat}, except at very low values. Furthermore, the correlation coefficient consistently exceeds 0.9 for both phononic crystals and acoustic metamaterials, reinforcing the accuracy of the approach, as illustrated in Figs. 5(a) and 5(b), respectively.

In conclusion, this study established a consistent framework for quantifying energy dissipation in dissipative periodic discrete systems, including phononic crystals and acoustic metamaterials. The proposed single-term exponential decay function, derived through least squares fitting, captured both transient response characteristics and energy dissipation behavior with a simple yet effective metric. The proposed decay coefficient demonstrated a strong correlation with actual energy decay patterns, providing a meaningful measure of dissipation than

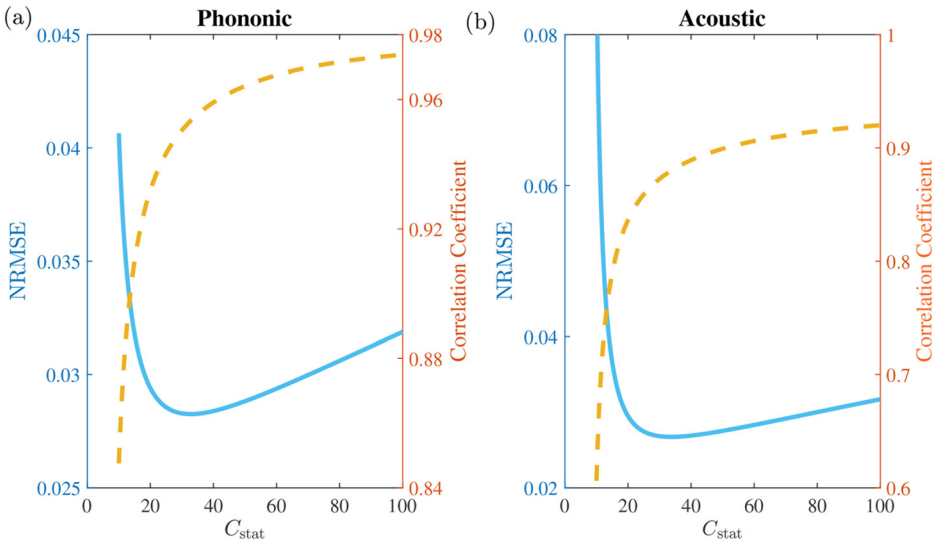

FIG. 5. Variation of the normalized root mean squared error (NRMSE) and correlation coefficient with a long-wave sound speed (C_{stat}) for (a) phononic crystal and (b) acoustic metamaterial. For each case, the y axis for the correlation coefficient is shown on the right side of the respective plot.

conventional damping ratio summations. It is worth mentioning that the proposed energy decay coefficient Θ is fundamentally different from the metadamping metric $\zeta_{\text{tot}}^{\text{sum}}$, both in purpose and formulation.

The key innovations include (1) a consistent definition of unit cells for discrete metamaterials that resolves ambiguities in existing approaches for estimation of energy dissipation, and (2) the introduction of a single energy decay coefficient that directly quantifies dissipation across different metamaterial configurations having a two-degree-of-freedom unit cell. These advances offer significant implications for metamaterial design, enabling precise optimization of energy dissipation properties for applications ranging from vibration control to impact mitigation. Future research should aim to generalize the framework to accommodate more complex metamaterial topologies involving higher degrees of freedom, as well as to incorporate nonlinear dynamics and more realistic damping models along with experimental validation. Such extensions will be essential for accurately predicting energy dissipation in practical engineering scenarios.

See the supplementary material for the detailed mathematical derivation.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Arnab Banerjee: Conceptualization (equal); Data curation (equal); Formal analysis (equal); Funding acquisition (equal); Investigation (equal); Methodology (equal); Project administration (equal); Resources (equal); Software (equal); Validation (equal); Visualization (equal); Writing – original draft (equal); Writing – review & editing (equal). **Kamal Krishna Bera:** Conceptualization (equal); Data curation (equal); Formal analysis (equal); Investigation (equal); Methodology (equal); Software (equal); Validation (equal); Visualization (equal); Writing – original draft (equal); Writing – review & editing (equal). **Sondipon Adhikari:** Conceptualization (equal); Funding acquisition (equal); Writing – review & editing (equal).

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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