

Meta-dissipation: A framework for quantifying energy dissipation in dissipative discrete periodic metamaterials

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Abstract

Quantifying energy dissipation in non-conservative periodic metamaterial systems lacks a consistent framework, with traditional interpretations of metadamping through damping ratio summations often failing to correspond with actual time-domain energy losses in multi-modal systems. This study addresses this critical gap by introducing a unified approach based on a consistent definition of the unit cell for discrete systems. A single-term exponential decay function derived via least-squares minimization is proposed as an effective descriptor of system-level energy dissipation. This decay coefficient reliably approximates the transient response of both non-conservative phononic crystals and acoustic metamaterials, demonstrating a strong correlation with actual energy decay rates, providing an unambiguous metric previously unavailable to researchers. Parametric investigations reveal that acoustic metamaterials exhibit significantly higher dissipation capabilities than phononic crystals at comparable long-wave sound speeds, a finding with direct implications for vibration control applications. The proposed meta-dissipation framework bridges a fundamental gap between wave dispersion theory and practical energy dissipation quantification, enabling more effective design and optimization of metamaterials for vibration attenuation across diverse application domains.

Keywords: Energy Dissipation; Long-wave sound speed; Decay coefficient; Least-squares minimization; Discrete periodic system

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1. Introduction

The existence of evanescent waves within subwavelength frequency bands has been a central reason behind the growing interest in metamaterials and periodic chains [1]. These wave phenomena give rise to unique dynamic responses, including bandgaps and local resonances, which are critical for vibration attenuation and wave manipulation. However, the influence of damping in metamaterials or periodic chains remains relatively less explored, despite its significant role in practical applications.

Hussein and Frazier [2] were the first to report that damped acoustic metamaterials exhibit enhanced energy dissipation compared to their equivalent damped phononic crystals. This enhanced dissipation was termed metadamping [3], and is typically quantified by summing the integrated values of each damping ratio branch across the Brillouin zone (BZ), as derived from dispersion relationships using Bloch's theorem applied to the unit cell. Since this seminal work, metadamping has been investigated in various lumped mass configurations, including systems with negative stiffness [4], non-local resonances [5], multi-degree-of-freedom units [6, 7], inertial amplifiers [8], active resonators [9, 10], and nonlocal interactions [11]. Further extensions have included viscoelastic relaxation models [12], nonlinear metamaterials [13], and electrically activated local resonators [14]. Metadamping has also been studied in structural configurations such as flexural beams with periodically attached spring-mass-dampers [15–17], and generalized mechanical networks [18]. Existing interpretations of the total damping ratio sum, $\xi_{\text{tot}}^{\text{sum}}$, lack clarity in terms of their physical relation to actual energy dissipation. In higher-degree-of-freedom systems, a simple summation of damping ratios often overpredicts the dissipation, which may not align with physical behavior.

To fill this gap recently, the concept of metadamping has been extended to quantify energy dissipation, understood as the fundamental measure of energy loss or damping in a medium over time. This extension has been effectively demonstrated in continuous structures by evaluating the dispersion characteristics of unit cells under free vibration, such as in metamaterial stepped beams [19], thin-walled beams [20, 21], periodically supported beams [22], and pile-soil interaction systems [23, 24]. However, applying this concept to discrete systems poses challenges due to the ambiguity in defining the unit cell, a problem not encountered in continuous systems. Moreover, a simplified yet comprehensive representation of

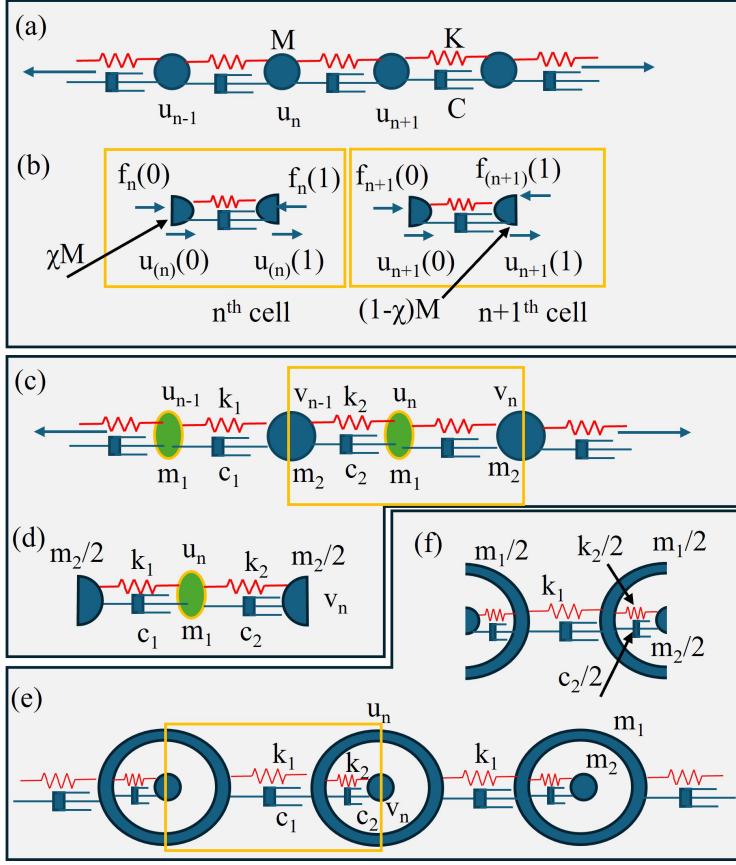


Figure S1: Periodic metamaterial chains and the corresponding unit cells for (a)-(b) monoatomic, (c)-(d) diatomic or phononic crystal, and (e)-(f) mass-in-mass or acoustic metamaterial.

the system's energy dissipation in the time domain, characterized by a single representative parameter, is currently unavailable.

Motivated by this research gap, this study proposes a consistent definition of a unit cell tailored for time-domain response analysis of discrete systems. A single-term exponential decay function, obtained through least-squares approximation, is introduced to effectively characterize system responses. Furthermore, a unified correlation is established between the decay coefficients and system-level energy dissipation.

2. Methodology

2.1. An anomaly in damping calculation and unit cell definition

The unit cell of a periodic lattice can be tessellating in many forms. For example, as shown in Figure S1 (a) and (b), let us consider the left mass of the unit cell as χM and the right side of the unit cell as $(1 - \chi)M$, where $0 \leq \chi \leq 1$. The wave equation is written as:

$$u_{n+j} = U_n e^{(iklj + \lambda t)} \quad (S1)$$

where, κ is the wave number and l is the lattice length. Bloch's theorem for the displacement and force can be expressed as:

$$u_{n-1} = u_{(n)}(0) = e^{-i\mu} u_{(n)}(l) = e^{-i\mu} u_n, \quad f_{n-1} = f_{(n)}(0) = e^{-i\mu} f_{(n+1)}(0) = -e^{-i\mu} f_n \quad (\text{S2})$$

where $\mu = \kappa l$ is the non-dimensional wavenumber and the subscript within parentheses denotes the unit. The equation of motion of the unit cell ([Figure S1 \(b\)](#)) is written as:

$$\chi M \ddot{u}_{n-1} + (K + \lambda C)(u_{n-1} - u_n) = f_{n-1} \quad (\text{S3})$$

$$(1 - \chi) M \ddot{u}_n + (K + \lambda C)(u_n - u_{n-1}) = f_n \quad (\text{S4})$$

Substitution of [Equation S2](#) in [Equation S3](#) and [Equation S4](#) results in:

$$\chi M \ddot{u}_n e^{-i\mu} + (K + \lambda C)(e^{-i\mu} - 1)u_n = -f_n e^{-i\mu} \quad (\text{S5})$$

$$(1 - \chi) M \ddot{u}_n + (K + \lambda C)(1 - e^{-i\mu})u_n = f_n \quad (\text{S6})$$

Dividing [Equation S5](#) by $e^{-i\mu}$ and adding with the [Equation S6](#), yields

$$M\lambda^2 + 2(K + \lambda C)(1 - \cos \mu) = 0 \quad (\text{S7})$$

It can be noticed that the dispersion relationship is independent of the distribution factor χ . This also signifies that a unit cell of a periodic spring-mass-damper chain can be conceptualized from any of its portions, and multiple forms of unit cells are possible, and they all will result in an identical dispersion relationship.

Although different unit cells represent the same dispersion relationship and thus yield identical wave number-dependent damping ratios, their distinct configurations lead to variations in natural frequencies and time-domain responses. As a result, each unit cell exhibits a different level of energy dissipation despite having analogous dispersion characteristics. This disparity introduces a fundamental inconsistency in evaluating energy dissipation in discrete systems, highlighting the need for a unified and consistent framework.

2.2. Formulation of the consistent unit cell from energy dissipation perspective

A discrete periodic system, as shown in [Figure S1 \(a\)](#), can be expressed as $L(x) = \sum_n \delta(x - nl)$, where n is an integer and l is the distance between two successive units. To convert this discrete model in the wavenumber domain, the Fourier transform can be

performed, which yields:

$$\begin{aligned} R(\kappa) &= \int_{-\infty}^{\infty} e^{i\kappa x} L(x) dx = \sum_n \int_{-\infty}^{\infty} e^{i\kappa x} \delta(x - nl) dx = \sum_n e^{i\kappa nl} \\ &= \sum_n (\cos(\kappa na) + i \sin(\kappa nl)) = \sum_n \cos(\kappa nl) \end{aligned} \quad (\text{S8})$$

Here, κ denotes the wavenumber. Since $\sin(x)$ is an odd function i.e., $\sin(-x) = -\sin(x)$, its contributions cancel out in the summation. As a result, for any arbitrary value of κ , the expression in [Equation S8](#) tends to zero as the number of terms approaches infinity. This cancellation can only be avoided for specific values of κ such that

$$\cos(\kappa na) = 1 \quad \rightarrow \quad \kappa na = 2I\pi \quad \rightarrow \quad \kappa = N \frac{2\pi}{a} \quad \rightarrow \quad \mu = \kappa a = 2N\pi \quad (\text{S9})$$

Here, N is an integer, and $\mu = \kappa a$ denotes the non-dimensional wave number. In the domain of the non-dimensional wave number μ —that is, in the reciprocal lattice—the discrete points are spaced at intervals of 2π . According to the definition, the Wigner-Seitz unit cell in the reciprocal lattice corresponds to the Brillouin zone, which therefore extends from $-\pi$ to π . Owing to the symmetry of the lattice, the irreducible Brillouin zone is taken as the range $0 \leq \mu \leq \pi$.

The dispersion relation for the monoatomic chain, given in [Equation S7](#), involves both the wave number and frequency. To derive the corresponding time-domain equation of motion for a consistent unit cell, we propose integrating this dispersion relation over the entire wave number domain ($\mu \in [0, \pi]$), as follows:

$$\begin{aligned} \frac{1}{\pi} \int_0^\pi [M\lambda^2 + 2(K + \lambda C)(1 - \cos \mu)] d\mu &= 0 \\ \longrightarrow \quad M\lambda^2 + 2(K + \lambda C) &= 0 \\ \longrightarrow \quad \frac{M}{2}\ddot{u}_n + C\dot{u}_n + Ku_n &= 0 \end{aligned} \quad (\text{S10})$$

For a diatomic chain or phononic crystal, the equations of motion for the two masses can be expressed as follows:

$$\begin{aligned} m_1\ddot{u}_n + (c_1 + c_2)\dot{u}_n - c_1\dot{v}_n - c_2\dot{v}_{n-1} + (k_1 + k_2)u_n - k_1v_n - k_2v_{n-1} &= 0 \\ m_2\ddot{v}_n + (c_1 + c_2)\dot{v}_n - c_1\dot{u}_n - c_2\dot{u}_{n+1} + (k_1 + k_2)v_n - k_1u_n - k_2u_{n+1} &= 0 \end{aligned} \quad (\text{S11})$$

Applying Bloch's theorem, the Bloch-transformed equations in the matrix form are obtained

as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_n \\ \ddot{v}_n \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -(k_1 + k_2 e^{-i\mu}) \\ -(k_1 + k_2 e^{i\mu}) & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} u_n \\ v_n \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -(c_1 + c_2 e^{-i\mu}) \\ -(c_1 + c_2 e^{i\mu}) & c_1 + c_2 \end{bmatrix} \begin{Bmatrix} \dot{u}_n \\ \dot{v}_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (\text{S12})$$

[Equation S12](#) yields the following dispersion relation

$$\begin{aligned} D := m_1 m_2 \lambda^4 + (c_1 + c_2)(m_1 + m_2)\lambda^3 + ((k_1 + k_2)(m_1 + m_2) + 2c_1 c_2(1 - \cos \mu))\lambda^2 + \\ 2(c_2 k_1 + c_1 k_2)(1 - \cos \mu)\lambda + 2k_1 k_2(1 - \cos \mu) = 0 \end{aligned} \quad (\text{S13})$$

Integrating the dispersion relation over the wave number domain, following the approach used in [Equation S10](#) for the monoatomic chain, leads to the characteristic equation for the diatomic chain, given as:

$$\begin{aligned} \frac{1}{\pi} \int_0^\pi D d\mu = 0 \\ m_1 m_2 \lambda^4 + (c_1 + c_2)(m_1 + m_2)\lambda^3 + ((k_1 + k_2)(m_1 + m_2) + 2c_1 c_2)\lambda^2 + \\ 2(c_2 k_1 + c_1 k_2)\lambda + 2k_1 k_2 = 0 \end{aligned} \quad (\text{S14})$$

Based on [Equation S14](#), the corresponding time-domain equations of motion for the consistent unit cell of the diatomic chain can be interpreted as:

$$\begin{bmatrix} m_1 & 0 \\ 0 & \frac{m_2}{2} \end{bmatrix} \begin{Bmatrix} \ddot{u}_n \\ \ddot{v}_n \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{Bmatrix} \dot{u}_n \\ \dot{v}_n \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_n \\ v_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (\text{S15})$$

Using a similar approach, the time-domain equation of motion for a consistent unit cell of a mass-in-mass system or acoustic metamaterial chain is derived as:

$$\begin{bmatrix} \frac{m_1}{2} & 0 \\ 0 & \frac{m_2}{2} \end{bmatrix} \begin{Bmatrix} \ddot{u}_n \\ \ddot{v}_n \end{Bmatrix} + \begin{bmatrix} c_1 + \frac{c_2}{2} & -\frac{c_2}{2} \\ -\frac{c_2}{2} & \frac{c_2}{2} \end{bmatrix} \begin{Bmatrix} \dot{u}_n \\ \dot{v}_n \end{Bmatrix} + \begin{bmatrix} k_1 + \frac{k_2}{2} & -\frac{k_2}{2} \\ -\frac{k_2}{2} & \frac{k_2}{2} \end{bmatrix} \begin{Bmatrix} u_n \\ v_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (\text{S16})$$

Here, u_n and v_n denote the displacements associated with the n^{th} unit cell as shown in Fig. [S1](#). The detailed calculation can be found in the supplementary material.

These equations suggest that consistent energy dissipation can be achieved using a symmetric unit cell with one side fixed. The rate of decay in the free vibration response of a symmetric unit cell fixed at one end serves as a measure of energy dissipation, reflecting how

efficiently the unit cell dissipates vibrational energy. To quantify this dissipation, a single exponential function of time is introduced, and its exponent is defined as the decay coefficient. This coefficient provides a consistent and comparable metric for evaluating different unit cell configurations.

2.3. Proposition of a decay coefficient to approximate energy dissipation

In a periodic metamaterial chain, a single unit cell serves as a representative of the entire structure and characterizes the system's dispersion and energy dissipation behavior. Therefore, to study the decay of the time-domain response under an impulse-type loading, we analyze a single unit cell and use it to estimate how rapidly the entire chain dissipates energy.

Consider [Equation S10](#), which describes the consistent unit cell of a monoatomic chain. The time-domain response is given by:

$$u_n(t) = Ce^{-\frac{C}{M}t} \cos(\omega_d t - \phi) \quad (\text{S17})$$

The peak values of the displacement response, u_{np} , which decay exponentially over time, are expressed as:

$$u_{np} = X_{pu} e^{-\theta_0 t} \quad \text{where} \quad \theta_0 = \frac{C}{M} \quad (\text{S18})$$

Here, θ_0 is the decay coefficient that quantifies how rapidly the vibrations attenuate. Thus, this single parameter θ_0 effectively captures both the decay of the displacement response and the energy dissipation of the system over time. This direct interpretation is possible for the monoatomic system because its governing equation of motion corresponds to a single-degree-of-freedom (SDOF) system.

However, for the unit cell of a diatomic or acoustic metamaterial chain, the governing equations represent a two-degree-of-freedom (2DOF) system, making it non-trivial to describe the decay using a single parameter. To address this, we propose a least-squares-based minimization approach to extract an equivalent decay coefficient for each degree of freedom. The first step in this process involves identifying the peak values of the time-domain responses, as outlined below.

Consider the modal transformation, $\mathbf{u} = \Phi \mathbf{q}$, where $\mathbf{u} = [u_n, v_n]^T$ is the displacement vector, $\Phi = [\phi_{11}, \phi_{12}; \phi_{21}, \phi_{22}]$ is the mass normalized modal matrix, and \mathbf{q} is the modal coordinates. Invoking the modal transformation, [Equation S15](#) or [Equation S16](#) yields the

two uncoupled equations of the following form

$$\begin{aligned}\ddot{q}_1 + 2\xi_1\omega_1\dot{q}_1 + \omega_1^2 q_1 &= 0 \\ \ddot{q}_2 + 2\xi_2\omega_2\dot{q}_2 + \omega_2^2 q_2 &= 0\end{aligned}\tag{S19}$$

Here, ξ_1 and ξ_2 are the modal damping ratios associated with the natural frequencies ω_1 and ω_2 , corresponding to the first and second modes, respectively. When a unit impulse load is applied to mass-1 of the unit cell, the initial conditions are defined as: $u_{n0} = v_{n0} = \dot{v}_{n0} = 0$ and $\dot{u}_{n0} = \frac{1}{m_1}$. These conditions result in zero initial displacements in the modal coordinates, while the initial modal velocities are given by $\dot{\mathbf{q}}(t = 0) = [\phi_{11}, \phi_{12}]^T$. Consequently, the time-domain response of the two-degree-of-freedom system is obtained as:

$$\begin{Bmatrix} u_n \\ v_n \end{Bmatrix} = \sum_{p=1}^2 \begin{Bmatrix} \phi_{1p} \\ \phi_{2p} \end{Bmatrix} \frac{\phi_{1p}}{\omega_{dp}} e^{-\xi_p \omega_p t} \sin \omega_{dp} t \tag{S20}$$

Here, $\omega_{dp} = \omega_{np} \sqrt{1 - \xi_p^2}$, $p = 1, 2$ are the damped natural frequencies. The response u_n can also be expressed as [25]

$$u_n = (A_u e^{-\xi_1 \omega_1 t} + B_u e^{-\xi_2 \omega_2 t}) \sin \omega_m t \cos \Delta \omega t - (A_u e^{-\xi_1 \omega_1 t} - B_u e^{-\xi_2 \omega_2 t}) \cos \omega_m t \sin \Delta \omega t \tag{S21}$$

where, $A_u = \phi_{11}^2 / \omega_{d1}$, $B_u = \phi_{12}^2 / \omega_{d2}$, $\omega_m = (\omega_{d1} + \omega_{d2})/2$, and $\Delta \omega = (\omega_{d2} - \omega_{d1})/2$. The expression for the curve representing the corresponding peak values of the response u_n can be written as [25]:

$$y_{pu}(t) = A_u e^{-\xi_1 \omega_1 t} + B_u e^{-\xi_2 \omega_2 t} \tag{S22}$$

It is evident that two exponential terms, corresponding to the two modal frequencies and damping ratios, are required to accurately represent the exponential decay of the response. A similar expression can be derived for v_n , given by $y_{pv}(t) = A_v e^{-\xi_1 \omega_1 t} - B_v e^{-\xi_2 \omega_2 t}$, where $A_v = \phi_{11} \phi_{21} / \omega_{d1}$ and $B_v = \phi_{12} \phi_{22} / \omega_{d2}$. Following the same reasoning, one can conclude that for an n -degree-of-freedom system, n exponential decay terms are needed to describe the decay of the response. However, in the present study - specifically for the two-degree-of-freedom unit cells typical of diatomic or acoustic metamaterial systems - we demonstrate that a single-term approximation can still yield reasonably accurate results. This allows for a simplified quantification of the exponential decay behavior. To this end, we approximate Equation S22 as $y_{pu}(t) \approx X_{pu} e^{-\theta_u t}$ and define the associated residual as:

$$R_u = X_{pu} e^{-\theta_u t} - A_u e^{-\xi_1 \omega_1 t} - B_u e^{-\xi_2 \omega_2 t} \tag{S23}$$

Now, the least squares method is employed to minimize this residual with respect to the unknowns X_{pu} and θ_u , leading to the following equations:

$$\frac{\partial}{\partial X_{pu}} \int_0^\infty R_u^2 dt = 0 \rightarrow \frac{X_{pu}}{2\theta_u} - \frac{A_u}{\theta_u + \xi_1\omega_1} - \frac{B_u}{\theta_u + \xi_2\omega_2} = 0 \quad (\text{S24})$$

$$\frac{\partial}{\partial \theta_u} \int_0^\infty R_u^2 dt = 0 \rightarrow \frac{X_{pu}}{4\theta_u^2} - \frac{A_u}{(\theta_u + \xi_1\omega_1)^2} - \frac{B_u}{(\theta_u + \xi_2\omega_2)^2} = 0 \quad (\text{S25})$$

The simultaneous solution of [Equation S24](#) and [Equation S25](#) yields the values of X_{pu} and θ_u . An analytical method for solving these equations is provided in the Appendix. Using a similar approach, the decay coefficient θ_v corresponding to the response of the second mass can also be determined.

The total energy of the unit cell consists of both kinetic and potential energy components. Accordingly, the total energy expressions for the diatomic (phononic crystal, PC) and acoustic metamaterial (AM) chains are given as follows:

$$E_{\text{PC}}(t) = \mathcal{T}_{\text{PC}}(t) + \mathcal{V}_{\text{PC}}(t) = \frac{1}{2} \left(m_1 \dot{u}_n^2 + \frac{m_2}{2} \dot{v}_n^2 \right) + \frac{1}{2} \left(k_1 u_n^2 + k_2 (u_n - v_n)^2 \right) \quad (\text{S26})$$

$$E_{\text{AM}}(t) = \mathcal{T}_{\text{AM}}(t) + \mathcal{V}_{\text{AM}}(t) = \frac{1}{2} \left(\frac{m_1}{2} \dot{u}_n^2 + \frac{m_2}{2} \dot{v}_n^2 \right) + \frac{1}{2} \left(k_1 u_n^2 + \frac{k_2}{2} (u_n - v_n)^2 \right) \quad (\text{S27})$$

Now, the peak of the energy envelope can be obtained approximately in the case of the phononic crystal as:

$$E_{\text{PC,p}}(t) \approx \frac{1}{2} \left(m_1 \dot{y}_{pu}^2 + \frac{m_2}{2} \dot{y}_{pv}^2 \right) + \frac{1}{2} \left(k_1 y_{pu}^2 + k_2 (y_{pu} - y_{pv})^2 \right) \quad (\text{S28})$$

Substituting $y_{pu}(t) \approx X_{pu} e^{-\theta_u t}$ and $y_{pv}(t) \approx X_{pv} e^{-\theta_v t}$ in Eq. [\(S28\)](#), we obtain

$$E_{\text{PC,p}} \approx \underbrace{\frac{1}{2} (k_1 + k_2 + m_1 \theta_u^2) X_{pu}^2 e^{-2\theta_u t}}_{\alpha_u} + \underbrace{\frac{1}{2} \left(k_2 + \frac{m_2}{2} \theta_v^2 \right) X_{pv}^2 e^{-2\theta_v t}}_{\alpha_v} - \underbrace{k_2 X_{pu} X_{pv} e^{-(\theta_u + \theta_v)t}}_{\alpha_c} \quad (\text{S29})$$

Similarly, for the acoustic metamaterial, one obtains

$$\begin{aligned} E_{\text{AM,p}}(t) &\approx \frac{1}{2} \left(\frac{m_1}{2} \dot{y}_{pu}^2 + \frac{m_2}{2} \dot{y}_{pv}^2 \right) + \frac{1}{2} \left(k_1 y_{pu}^2 + \frac{k_2}{2} (y_{pu} - y_{pv})^2 \right) \\ &\approx \underbrace{\frac{1}{4} (2k_1 + k_2 + m_1 \theta_u^2) X_{pu}^2 e^{-2\theta_u t}}_{\alpha_u} + \underbrace{\frac{1}{4} (k_2 + m_2 \theta_v^2) X_{pv}^2 e^{-2\theta_v t}}_{\alpha_v} - \underbrace{\frac{k_2}{2} X_{pu} X_{pv} e^{-(\theta_u + \theta_v)t}}_{\alpha_c} \end{aligned} \quad (\text{S30})$$

Since the total energy of the unit cell also exhibits exponential decay and receives contributions from both degrees of freedom in a two-degree-of-freedom system, such as those found in

diatomic or acoustic metamaterial chains. To this end, considering a single-term exponential approximation, the energy decay for PC or AM is represented in terms of the displacement decay coefficients θ_u and θ_v as $E_0 e^{-2\Theta t}$. Here, $E_0 = E_{PC,p}(t = 0)$ or $E_{AM,p}(t = 0)$ for PC and AM, respectively. Once the rate of the displacement decay coefficients of the two masses (θ_u and θ_v) is comparable, then the energy decay rate is the average of θ_u and θ_v . However, when the displacement decay coefficients are not comparable, at the initial time, both modes contribute, but at a later stage slowly decaying mode dominates the energy decay rate. Therefore, the single term energy decay coefficient (Θ) must be a convex combination of both θ_u and θ_v . Performing the convex combination, using weighted harmonic mean and ignoring the cross-coupling terms, we can obtain an approximate simplified energy decay coefficient as:

$$\frac{\alpha_u}{\theta_u} + \frac{\alpha_v}{\theta_v} = \frac{\alpha_u + \alpha_v}{\Theta} \rightarrow \Theta = \frac{\alpha_u + \alpha_v}{\frac{\alpha_u}{\theta_u} + \frac{\alpha_v}{\theta_v}} \quad (\text{S31})$$

This approximation demonstrates excellent agreement with the numerical response, particularly in the context of PC and AM with varying mass and stiffness matrices. The corresponding Normalized Root Mean Squared Error (NRMSE) is obtained as

$$\text{NRMSE} = \frac{\sqrt{\frac{1}{n} \sum_{i=1}^n (E_{PC/AM}|_i - E_0 e^{-2\Theta t}|_i)^2}}{\max(E_{PC/AM}) - \min(E_{PC/AM})} \quad (\text{S32})$$

To evaluate the energy dissipation profile under varying stiffness characteristics of the system, the long-wave speed of sound (C_{stat}) is considered. This quantity represents the slope of the lowest dispersion branch as the wave number tends to zero. The expressions for C_{stat} in phononic crystals and acoustic metamaterials are given by [2]:

$$C_{\text{stat}}|_{\text{PC}} = l \sqrt{\frac{k_1 k_2}{(m_1 + m_2)(k_1 + k_2)}}, \quad C_{\text{stat}}|_{\text{AM}} = l \sqrt{\frac{k_1}{m_1 + m_2}} \quad (\text{S33})$$

To examine the variation of the energy decay coefficient (Θ) with the long-wavelength speed of sound (C_{stat}) for both PC and AM, having constant c_1 and c_2 , k_1 is varied and k_2 is calculated as $k_2 = k_1(c_2/c_1)$ so that damping matrix always remains proportional to the stiffness matrix. This ensures the validity of modal decomposition as shown in Eq. (S19).

[Figure S2](#) illustrates the steps of the proposed methodology.

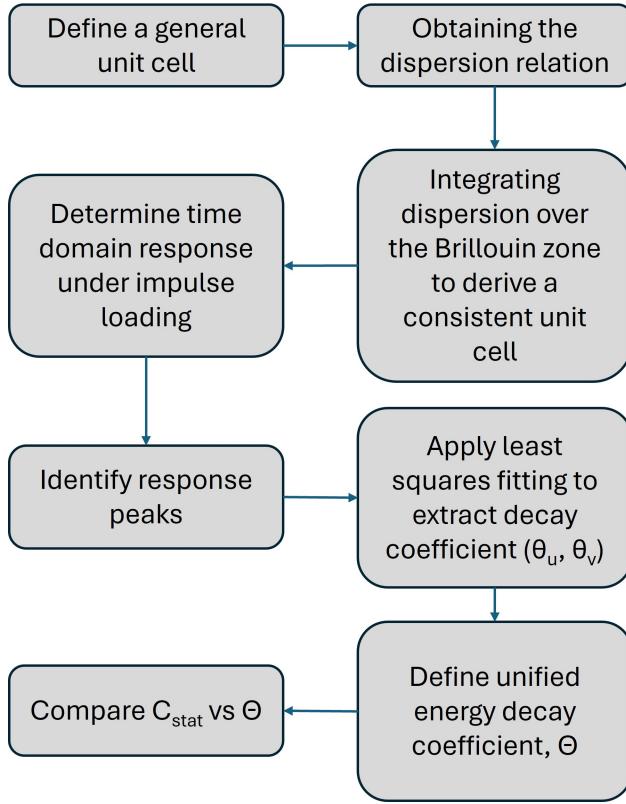


Figure S2: Flow chart for the complete solution procedure.

3. Results and Discussion

3.1. Validation of the proposed concept

To elucidate the accuracy of the developed approximate formulas, a phononic crystal (PC) characterized by $m_1 = 1$, $m_2 = 0.8$, $k_1 = 40906$, $k_2 = 18000$, and $c_1 = 20$, $c_2 = 8.8$ and its statically equivalent acoustic metamaterial (AM) characterized by $m_1 = 1$, $m_2 = 0.8$, $k_1 = 12500$, $k_2 = 5500$, and $c_1 = 20$, $c_2 = 8.8$ have been considered. Both the system has $C_{\text{stat}} = 83.33$. All the parameters in this work may be read in any consistent system of physical units. The time domain responses of PC and AM are illustrated in Fig. S3 (a-c) and (d-f), respectively. Subfigures of Fig. S3 (a), (b), (d), and (e) illustrate the free vibration responses of a unit cell for both PC and AM. These figures also include a comparison between the classical two-term approximation [25] and the proposed single-term displacement decay for both the masses. It is evident that (θ_u) or (θ_v) closely captures the amplitude peaks and overall response. The deviation from the two-term approximation is insignificant, indicating that the proposed method effectively captures the decay rate of the displacements. This simplification not only retains accuracy but also reduces computational complexity, making

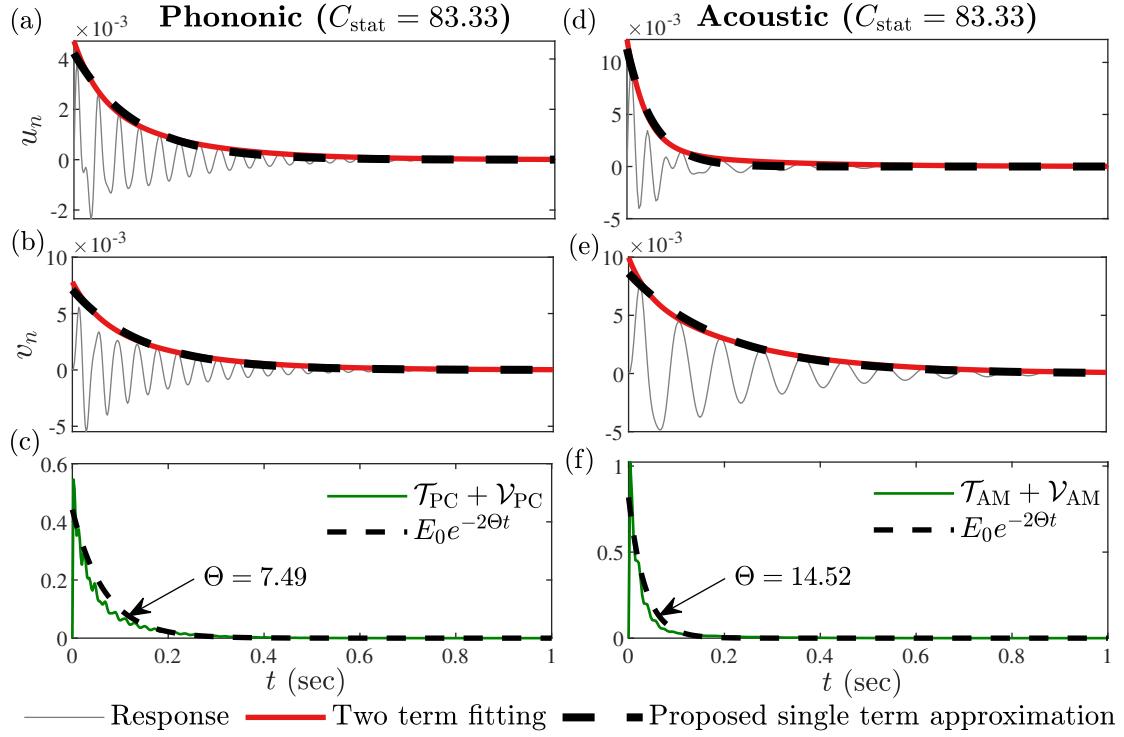


Figure S3: Time-domain responses (u_n and v_n) to a unit impulse load, along with peak fitting using both the two-term and proposed single-term expressions for (a)–(b) a phononic crystal, and (d)–(e) a statically equivalent acoustic metamaterial unit cell. Here, $C_{\text{stat}}|_{\text{PC}} = C_{\text{stat}}|_{\text{AM}} = 83.33$. The corresponding variation in total system energy and its approximation using the energy decay coefficient are shown in (c) and (f) for PC and AM, respectively.

it a practical and efficient metric for quantifying amplitude decay in such systems. Furthermore, the energy decay within each unit cell is effectively characterized using the proposed energy decay coefficient, Θ . The dissipation profiles for both the phononic crystal (PC) and acoustic metamaterial (AM), as shown in Fig. S3(c) and (f), respectively, demonstrate that this coefficient accurately captures the temporal decay of vibrational energy. The successful validation of both the dynamic response and the corresponding energy dissipation using the proposed framework reinforces its applicability and robustness. It is noteworthy that the computed values of Θ are 7.49 for the phononic crystal and 14.52 for the acoustic metamaterial. This indicates that the energy decay coefficient for the AM is nearly twice that of the statically equivalent PC, despite both having identical viscous damping. This observation further substantiates the concept of metadamping [2] and highlights the superior damping capabilities of acoustic metamaterials, which can dissipate vibrational energy more efficiently than their phononic counterparts. While previous methods for characterizing damping in

metamaterial unit cells often lacked a direct correlation with energy dissipation, the proposed decay coefficient provides a quantitative and physically meaningful measure of energy loss within a unit cell.

To examine the variation of the energy decay coefficient (Θ) with the long-wavelength speed of sound (C_{stat}) for both PC and AM, having constant c_1 and c_2 , k_1 is varied and k_2 is calculated as $k_2 = k_1(c_2/c_1)$ so that damping matrix always remains proportional to the stiffness matrix. This ensures the validity of modal decomposition as shown in Eq. (S19). As shown in Fig. S4(a), increasing C_{stat} results in a decrease in the energy decay coefficient Θ . This reduction is more pronounced at lower values of C_{stat} , while the variation becomes marginal at higher values. This trend aligns with physical intuition, as stiffer materials (i.e., higher C_{stat}) inherently dissipate less energy, which is reflected in the lower values of Θ . To further validate the robustness of the proposed approach, the dynamic responses and energy dissipation characteristics of the two masses in the phononic crystal are examined for representative cases of $C_{\text{stat}} = 20$ and 50 . The results confirm that the proposed framework effectively captures both the mechanical behavior and the dissipation characteristics across varying stiffness conditions. Similar observation can be perceived for acoustic metamaterial as illustrated in Fig. S5.

Numerical validation across a wide range of parameter sets confirms the robustness of the proposed approximation, as the normalized root mean squared error (NRMSE), defined in Eq. (S32), remains close to zero for the entire range of C_{stat} , except at very low values. Furthermore, the correlation coefficient consistently exceeds 0.9 for both phononic crystals and acoustic metamaterials, reinforcing the accuracy of the approach. These trends are illustrated in Fig. S6 (a) and (b), respectively.

4. Conclusion

This study established a consistent framework for quantifying energy dissipation in non-conservative periodic discrete systems, including phononic crystals and acoustic metamaterials. By proposing a single-term exponential decay function derived through least-squares fitting, the research successfully captured both transient response characteristics and energy dissipation behavior with a simple yet effective metric. The proposed decay coefficient demonstrated a strong correlation with actual energy decay patterns, providing a more accurate and meaningful measure of dissipation than conventional damping ratio summations.

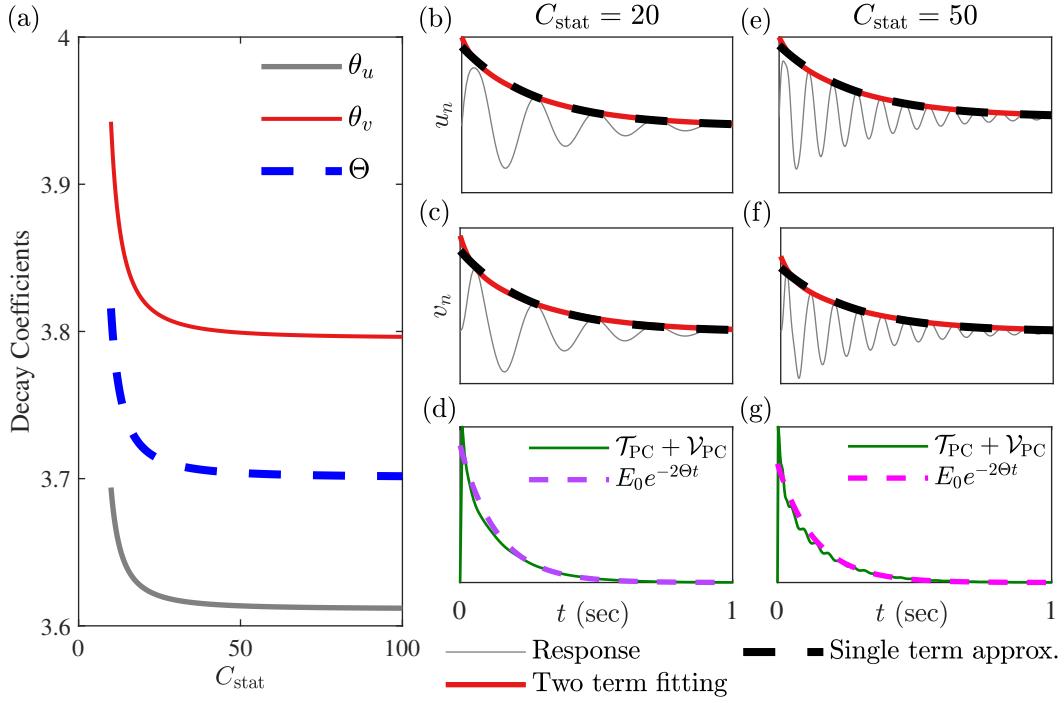


Figure S4: (a) Variation of θ_u , θ_v , and Θ with the long-wave speed of sound (C_{stat}) in a phononic crystal. Sub-figures (b)–(c) and (e)–(f) show the time-domain responses (u_n and v_n) to a unit impulse load, along with peak fitting using both the two-term and proposed single-term expressions. Sub-figures (d) and (g) present the corresponding variation in total system energy and its approximation using the energy decay coefficient.

The key innovations include: 1) the consistent definition of unit cells for discrete systems that resolves ambiguities in existing approaches, and 2) the introduction of a unified energy decay coefficient that directly quantifies dissipation across different metamaterial configurations. These advances offer significant implications for metamaterial design, enabling precise optimization of energy dissipation properties for applications ranging from vibration control to impact mitigation. The finding that acoustic metamaterials exhibit substantially higher dissipation than phononic crystals at comparable sound speeds particularly highlights their potential for transient vibration attenuation. Future research should extend this framework to more complex metamaterial geometries, nonlinear systems, and incorporate a more realistic damping model to further enhance energy dissipation capabilities in practical engineering applications.

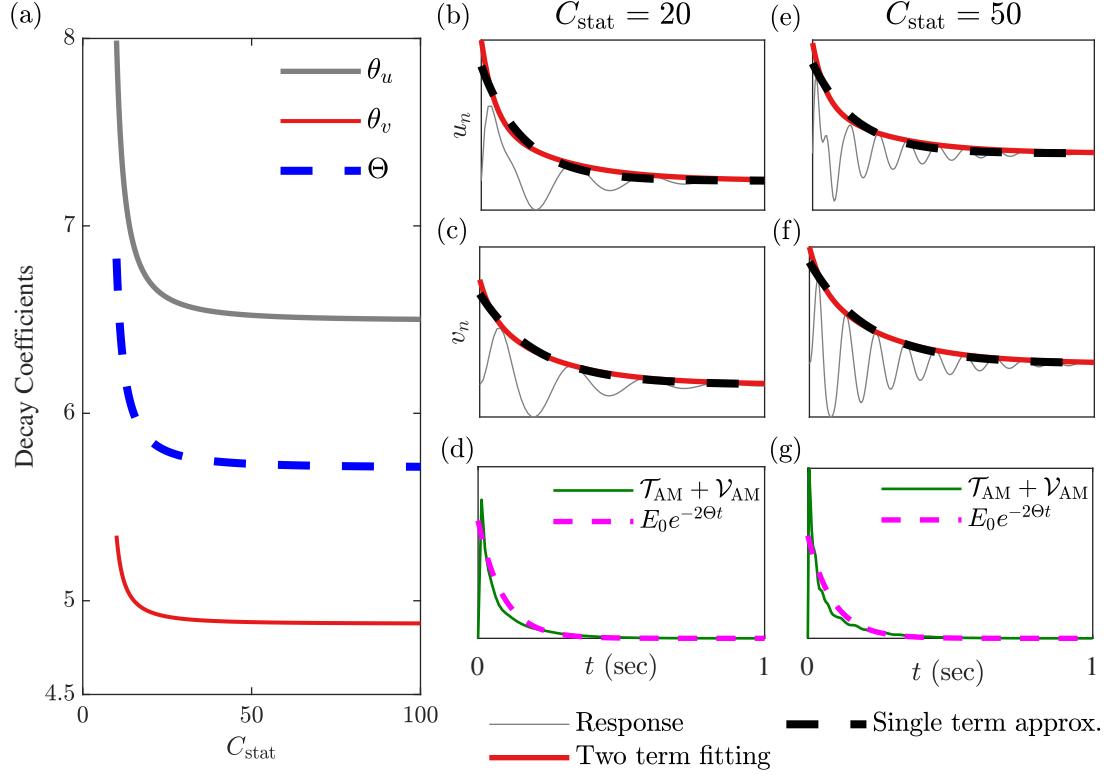


Figure S5: (a) Variation of θ_u , θ_v , and Θ with C_{stat} in an acoustic metamaterial. Sub-figures (b)–(c) and (e)–(f) show the time-domain responses (u_n and v_n) to a unit impulse load, along with peak fitting using both the two-term and proposed single-term expressions. Sub-figures (d) and (g) present the corresponding variation in total system energy and its approximation using the energy decay coefficient.

Appendix

Substituting X_{pu} from Equation S24 into Equation S25, and realizing that $X_{pu} \neq 0$, the following cubic equation in θ_u is obtained.

$$a\theta_u^3 + b\theta_u^2 + c\theta_u + d = 0 \quad (\text{A1})$$

where,

$$\begin{aligned} a &= A_u + B_u; \quad c = A_u\omega_2^2\xi_2^2 + B_u\omega_1^2\xi_1^2 - 2A_u\omega_1\omega_2\xi_1\xi_2 - 2B_u\omega_1\omega_2\xi_1\xi_2 \\ b &= 2A_u\omega_2\xi_2 - A_u\omega_1\xi_1 + 2B_u\omega_1\xi_1 - B_u\omega_2\xi_2; \quad d = -B_u\omega_1^2\omega_2\xi_1^2\xi_2 - A_u\omega_1\omega_2^2\xi_1\xi_2^2 \end{aligned} \quad (\text{A2})$$

Introducing the change of variable, $\theta_u = \tau - \frac{b}{3a}$, Equation A1 can be written in the following depressed cubic polynomial form

$$\tau^3 + r\tau + s = 0 \quad (\text{A3})$$

where

$$r = \frac{3ac - b^2}{3a^2}; \quad s = \frac{2b^3 - 9abc + 27a^2d}{27a^3} \quad (\text{A4})$$

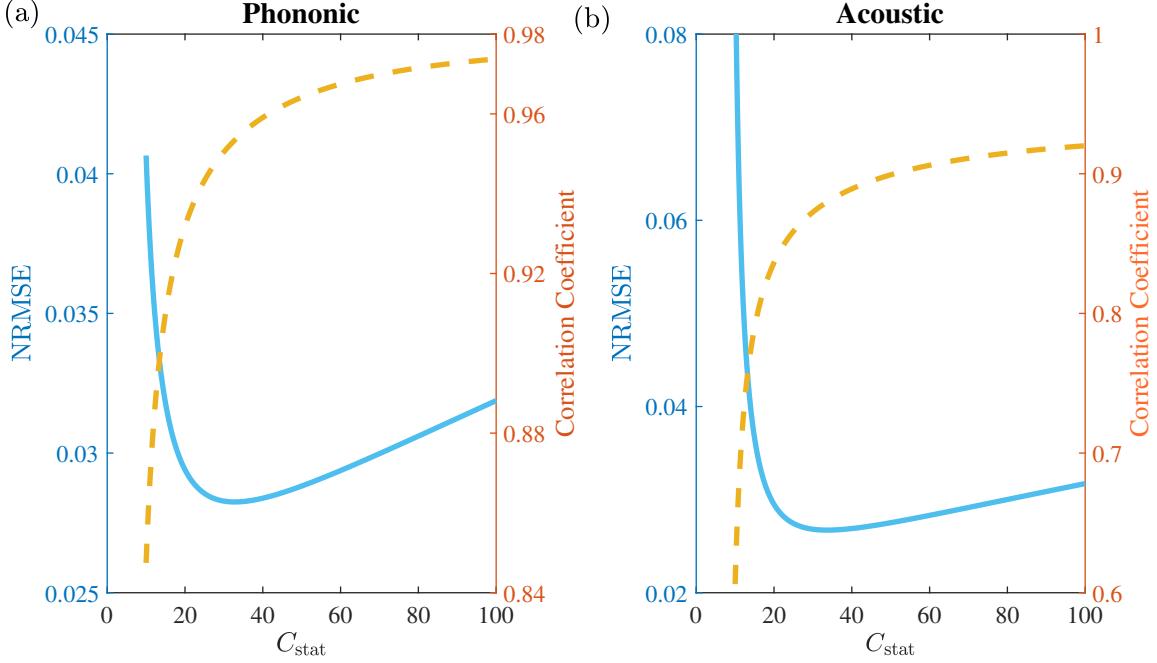


Figure S6: Variation of the Normalized Root Mean Squared Error (NRMSE) and correlation coefficient with long wave sound speed (C_{stat}) for (a) phononic crystal, and (b) acoustic metamaterial. For each case, the y-axis for the correlation coefficient is shown on the right side of the respective plot.

From [Equation A3](#), one real solution of τ can be written using the hyperbolic functions as follows

$$\begin{aligned} \tau &= -2 \frac{|s|}{s} \sqrt{-\frac{r}{3}} \cosh \left[\frac{1}{3} \operatorname{arccosh} \left(\frac{-3|s|}{2r} \sqrt{\frac{-3}{r}} \right) \right] && \text{if } 4r^3 + 27s^2 > 0 \text{ and } r < 0 \\ \tau &= -2 \sqrt{\frac{r}{3}} \sinh \left[\frac{1}{3} \operatorname{arcsinh} \left(\frac{3s}{2r} \sqrt{\frac{3}{r}} \right) \right] && \text{if } r > 0 \end{aligned} \quad (\text{A5})$$

Once τ is obtained, the real root for the original cubic equation ([Equation A1](#)) can be obtained as

$$\theta_u = \tau - \frac{b}{3a} \quad (\text{A6})$$

and the corresponding X_{pu} can be calculated by substituting θ_u in [Equation S24](#) i.e.,

$$X_{pu} = 2\theta_u \left(\frac{A_u}{\theta_u + \xi_1 \omega_1} + \frac{B_u}{\theta_u + \xi_2 \omega_2} \right) \quad (\text{A7})$$

Data availability statement

Data sharing does not apply to this article as no new data were created or analyzed in this study.

References

- [1] A. Banerjee, R. Das, E. P. Calius, Waves in Structured Mediums or Metamaterials: A Review, *Archives of Computational Methods in Engineering* 26 (2019) 1029–1058.
- [2] M. I. Hussein, M. J. Frazier, Metadamping: An emergent phenomenon in dissipative metamaterials, *Journal of Sound and Vibration* 332 (20) (2013) 4767–4774.
- [3] C. L. Bacquet, H. Al Ba’ba’a, M. J. Frazier, M. Nouh, M. I. Hussein, Metadamping: dissipation emergence in elastic metamaterials, *Advances in Applied Mechanics* 51 (2018) 115–164.
- [4] I. Antoniadis, D. Chronopoulos, V. Spitas, D. Koulocheris, Hyper-damping properties of a stiff and stable linear oscillator with a negative stiffness element, *Journal of Sound and Vibration* 346 (2015) 37–52.
- [5] D. DePauw, H. Al Ba’ba’a, M. Nouh, Metadamping and energy dissipation enhancement via hybrid phononic resonators, *Extreme Mechanics Letters* 18 (2018) 36–44.
- [6] Y. Chen, M. V. Barnhart, J. Chen, G. Hu, C. Sun, G. Huang, Dissipative elastic metamaterials for broadband wave mitigation at subwavelength scale, *Composite Structures* 136 (2016) 358–371.
- [7] Q. Li, Z. He, E. Li, Dissipative multi-resonator acoustic metamaterials for impact force mitigation and collision energy absorption, *Acta Mechanica* 230 (8) (2019) 2905–2935.
- [8] M. I. Hussein, I. Patrick, A. Banerjee, S. Adhikari, Metadamping in inertially amplified metamaterials: Trade-off between spatial attenuation and temporal attenuation, *Journal of Sound and Vibration* 531 (2022) 116977.
- [9] K. K. Bera, A. Banerjee, Active metadamping: A phenomenon of damping enhancement in metamaterial via feedback control, *Journal of Sound and Vibration* 546 (2023) 117452.
- [10] T. Chatterjee, K. K. Bera, A. Banerjee, Machine learning enabled quantification of stochastic active metadamping in acoustic metamaterials, *Journal of Sound and Vibration* 567 (2023) 117938.
- [11] A. Banerjee, K. K. Bera, S. Adhikari, Enhanced damping and bandwidth in roton-like dispersion of a beyond nearest neighbor periodic chain, *Physics Letters A* 524 (2024) 129821.
- [12] T. U. R. Abbasi, H. Zheng, Wave dispersion and dissipation performance of locally resonant acoustic metamaterials using an internal variable model, *Wave Motion* 93 (2020) 102483.
- [13] Y.-L. Chen, L. Ma, A minimalist elastic metamaterial with meta-damping mechanism, *International Journal of Solids and Structures* 302 (2024) 112977.
- [14] H. Al Ba’ba’a, Z. Lin, S. Tol, Metadamping enhancement and tunability via scissor-like electromechanical metamaterials, *Journal of Applied Physics* 130 (18) (2021) 184901.
- [15] C. Du, D. Yu, J. Liu, J. Wen, Flexural vibration band gaps for a phononic crystal beam with x-shaped local resonance metadamping structure, *Acta Phys Sin* 66 (14) (2017) 140701.
- [16] A. Aladwani, M. Nouh, Strategic damping placement in viscoelastic bandgap structures: dissecting the metadamping phenomenon in multiresonator metamaterials, *Journal of Applied Mechanics* 88 (2) (2021).
- [17] A. Aladwani, M. Nouh, Mechanics of metadamping in flexural dissipative metamaterials: analysis and design in frequency and time domains, *International Journal of Mechanical Sciences* 173 (2020) 105459.

- [18] A. Aladwani, A. Mohammed, M. Nouh, Tunable dissipation in elastic metamaterials via methodic reconfiguration of inertant mechanical networks, *Meccanica* 57 (6) (2022) 1337–1352.
- [19] S. Biswas, A. Banerjee, K. K. Bera, An analytical and experimental investigation into overall dissipation of flexural mode in a periodically damped beam, *Mechanics of Materials* 198 (2024) 105113.
- [20] A. Das, K. K. Bera, A. Banerjee, Emergence of metadamping in thin-walled mono-symmetric metabeam: Homogenization approach, *International Journal of Mechanical Sciences* 283 (2024) 109621.
- [21] A. Das, K. K. Bera, A. Banerjee, Investigation on metadamping and energy dissipation in thin-walled mono-symmetric metabeam, *Engineering Structures* 318 (2024) 118700.
- [22] A. Bhatt, K. K. Bera, A. Banerjee, Wave dispersion in a damped beam supported by cubic nonlinear springs: A multiscale freewave approach, *Physical Review E* 110 (4) (2024) 044213.
- [23] R. Das, A. Banerjee, B. Manna, Estimation of the dissipation due to radiation damping for the pile embedded in soil: A closed-form solution, *Computers and Geotechnics* 163 (2023) 105716.
- [24] H. Liu, Y. Gao, J. Li, X. Yang, X. Liu, M. H. El Naggar, G. Mei, W. Wu, A meta-damping model for soil-pile interaction in large-diameter pipe piles under horizontal dynamic loading, *Computers and Geotechnics* 184 (2025) 107233.
- [25] J. Cooper, Extending the logarithmic decrement method to analyse two degree of freedom transient responses, *Mechanical systems and signal processing* 10 (4) (1996) 497–500.