Sr. No.	Title	Page No.	Sign./Remarks
	$(AB)^{-1} = B^{-1}A^{-1}$ symetric matrix	$A = A^T$	
	$(AB)^T = B^T A^T$		
	$(A+B)^T = A^T + B^T$		
	$(A^{-1})^T = (A^T)^{-1} = A^{-T}$		

que A company produces product $N_1, N_2 \dots N_n$ for which resources $R_1, R_2, \dots R_m$ are required. To produce a unit of product N_j, a_{ij} unit of resorce R_i are needed, where $i = 1, 2 \dots, m$ and
$j = 1, 2 \dots n$

The object is to find an optimal production plan. i.e., a plan of how many units x_j of product N_j should be produced if a total of b_i unit of resource R_i are available and (ideally) no resources are left over.

If we produce $x_i, \dots x_n$ units of the corresponding products. we need

$a_{i1}x_1 + \dots + a_{in}x_n$ many units of resource R_i

An optimal production plan $(x_1, \dots x_n) \in \mathbb{R}^n$ therefore, has to satisfy the following system of linear eqns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

where $a_{ij} \in \mathbb{R}$ and $b_i \in \mathbb{R}$

$$\begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} x_2 + \dots \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} x_n = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$range(f) = \{ f(x) : x \in R \}$

$range(A) = \{ A\vec{x} : \vec{x} \in R^n \}$

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⇒ Sum of symmetric matrixs is symmetric. However Product is generally not symmetric

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* Particular solution and general solution

let consider system of linear eqⁿs

$$\underset{2\times4}{\begin{bmatrix} 1 & 0 & 8 & -4 \\ 0 & 1 & 2 & 12 \end{bmatrix}} \underset{4\times1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}} = \begin{bmatrix} 42 \\ 8 \end{bmatrix}$$

Here two eqⁿ fore variable so ∞ solutions

we want to find $x_1, \dots, x_4$ such that $\sum_{i=1}^{4} x_i c_i = b$

where c_i to be ith column of matrix and b is $\begin{bmatrix} 42 \\ 8 \end{bmatrix}$

Here we can say $b = \begin{bmatrix} 42 \\ 8 \end{bmatrix} = 42\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 8\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Therefor solution is $[42, 8, 0, 0]^T = \begin{bmatrix} 42 \\ 8 \\ 0 \\ 0 \end{bmatrix}$ This solution called particular solution of system of linear eqⁿs

Here $c_3 = 8c_1 + 2c_2$

so $8c_1 + 2c_2 - c_3 + 0c_3 = 0$

Also $Au = x_1 c_1 + x_2 c_2 + x_3 c_3 + x_4 c_4 = 0$

so we can clearly say

$$u = \begin{bmatrix} 8 \\ 2 \\ -1 \\ 0 \end{bmatrix} \Rightarrow Ax = 0$$

So same $c_4 = -4c_1 + 12c_2$

$4c_1 + c_4 - 12c_2 + 0c_3 = 0$

$AV = x_1 c_1 + x_2 c_2 + x_3 c_3 + x_4 c_4 = 0$

$$V = \begin{bmatrix} -4 \\ 12 \\ 0 \\ -1 \end{bmatrix} \Rightarrow AV = 0$$

$Ax = b$ $\quad Au = 0$

Then

$A(x + \lambda u)$ also b

$Ax + \lambda uA$

$= Ax + 0 = Ax = b$

Here $Ax = b$

so $A(x + \lambda_1 u + \lambda_2 v) = 0$ $\quad (\because Au = AV = 0)$

general solution, particular solution, $Ax=0$, $Ax=0$

$$x = \begin{bmatrix}42\\8\\0\\0\end{bmatrix} + \lambda_1\begin{bmatrix}8\\2\\-1\\0\end{bmatrix} + \lambda_2\begin{bmatrix}-4\\12\\0\\-1\end{bmatrix} = \begin{bmatrix}42\\8\end{bmatrix} \quad \lambda_1, \lambda_2 \in R$$

step

1st Find particular solution to $Ax=b$

2nd find all solution to $Ax=0$

3rd combine solution from step 1 and 2 to the general solution

Neither the general solution or particular solution is unique

* Elementary Transformations

que Find all solution of following system of linear eq^ns.

$$-2x_1 + 4x_2 - 2x_3 - x_4 + 4x_5 = -3$$
$$4x_1 - 8x_2 + 3x_3 - 3x_4 + x_5 = 2$$
$$x_1 - 2x_2 + x_3 - x_4 + x_5 = 0$$
$$x_1 - 2x_2 \quad - 3x_4 + 4x_5 = a$$

$$\left[\begin{array}{ccccc|c} -2 & 4 & -2 & -1 & 4 & -3 \\ 4 & -8 & 3 & -3 & 1 & 2 \\ 1 & -2 & 1 & -1 & 1 & 0 \\ 1 & -2 & 0 & -3 & 4 & a \end{array}\right]$$

$R_3 \leftrightarrow R_1$ & $-4R_1$ $+2R_1$ $-R_1$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -3 & 2 \\ 0 & 0 & 0 & -3 & 6 & -3 \\ 0 & 0 & -1 & -2 & 3 & a \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 3 & -2 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & a+1 \end{array}\right]$$

$$x_1 - 2x_2 + x_3 - x_4 + x_5 = 0$$
$$x_3 - x_4 + 3x_5 = -2$$
$$x_4 - 2x_5 = 1$$
$$0 = a+1$$

only for $a=-1$ this system can be solved

column 3 & 5 don't have pivot so x_2 and x_5 are free variable they can assum any real values

so let $x_2 = s$
$x_5 = t$

$$-3x_4 + 6t = -3$$
$$\boxed{x_4 - 2t = 1}$$
$$x_3 - (1+2t) + 3t = -2$$
$$x_3 = 1 + 2t - 2 - 3t = -1 - t$$
$$\boxed{x_3 = -1 - t}$$

$$x_1 - 2s + (-1-t) - (1+2t) + t = 0$$

so $x_1 = 2 + 2s - 2t$
$x_2 = s$
$x_3 = -1 - t$
$x_4 = 1 + 2t$
$x_5 = t$

$$x = \begin{bmatrix}2+2s-2t\\s\\-1-t\\1+2t\\t\end{bmatrix} = \begin{bmatrix}2\\0\\-1\\1\\0\end{bmatrix} + s\begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix} + t\begin{bmatrix}-2\\0\\-1\\2\\1\end{bmatrix}$$

$$\text{so } x = \begin{bmatrix}2\\0\\-1\\1\\0\end{bmatrix} + \lambda_1\begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix} + \lambda_2\begin{bmatrix}-2\\0\\-1\\2\\1\end{bmatrix} \quad \lambda_1, \lambda_2 \in R$$

particular solution, $Au=0$, $Av=0$

or for Particular solution

$$b = \sum_{i=1}^{P} \lambda_i P_i \qquad i = 0, \dots P$$

P_i is pivot columns

$$\lambda_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_3 = 1, \ \lambda_2 = -1, \ \lambda_1 = 2$$

for non-pivot columns coefficient = 0

so $x = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

* Row-Reduced echelon form

$$A = \begin{bmatrix} 1 & 3 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 9 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix}$$

Key idea for find $Ax = 0$ is to look at the non-Pivot columns. which will need to express as a (linear) combination of pivot columns. The reduced row echelon form make this relatively straightforward, and we can express the non-pivot column in term of sums and ~~multiplication~~ multiples of the pivot columns that are on their left.

Ex 2nd column (1st non-Pivot column) is 3 times of 1st column (1st pivot-column)

we ignor pivot column here

for 5th column (2nd non-Pivot column) we can express by $(3 \times 1^{st}$ pivot$) + (9 \times 2^{nd}$ pivot$) + (-4 \times 3^{rd}$ pivot$)$

Then we need to substract for obtain 0

all solution for $Ax = 0$, $x \in \mathbb{R}^5$ are given by

$$x \in \mathbb{R}^5 \quad x = \lambda_1 \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 3 \\ 0 \\ 9 \\ -4 \\ -1 \end{bmatrix} \qquad \lambda_1, \lambda_2 \in \mathbb{R}$$

Short cut

minus-1 $A = \begin{bmatrix} 1 & 3 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 9 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix}$

now we augment this matrix 5×5 by adding rows of the form at place where the pivot on the diagonal are missing.

$$A = \begin{bmatrix} 1 & 3 & 0 & 0 & 3 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 9 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$x = \lambda_1 \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 3 \\ 0 \\ 9 \\ -4 \\ -1 \end{bmatrix}$$

* Groups

Consider a set G and an operation $\otimes : G \times G \to G$ defined on G. Then $G := (G, \otimes)$ is called a group if following hold:

1) Closure of G under $\otimes$: $\forall x, y \in G : x \otimes y \in G$
2) Associativity: $\forall x, y, z \in G : (x \otimes y) \otimes z = x \otimes (y \otimes z)$
3) Neutral element: $\exists e \in G \ \forall x \in G : x \otimes e = x$ and $e \otimes x = x$
4) Inverse element: $\forall x \in G \ \exists y \in G : x \otimes y = e$ and $y \otimes x = e$

where e is neutral element we often write x^{-1} to denote the inverse element of x.

but inverse element is defined with respect to the operation $\otimes$ and does not necessarily mean $\frac{1}{x}$

⇒ If $\forall x, y \in G : x \otimes y = y \otimes x$, then $G = (G, \otimes)$ is an Abelian group (commutative)

* Vector space

⇒ linear combination: Let $x_1, x_2, \dots, x_n$ be any set of vector in a vector space V over R, and $\alpha_i \in R$, $i = 1, 2 \dots n$. Then the vector $\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n \in V$ is called linear combination of vector $x_1, x_2 \dots x_n$.

→ Another interesting fact about the standard vector is that $\alpha_1 \varepsilon_1 + \alpha_2 \varepsilon_2 + \dots + \alpha_n \varepsilon_n = 0$ implies that all coefficients $\alpha_1, \alpha_2, \alpha_3 \dots \alpha_n = 0$

→ geometrically it means. no ~~two~~ more then two vector lie in a plane!

⇒ linearly independed

$x_1, x_2, \dots, x_n$ in a vector space V over R are linearly independed if for $\alpha_i \in R$, $i = 1, 2 \dots n$

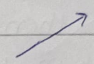
$$\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n = 0$$

that means $\alpha_i = 0$, $i = 1, 2 \dots n$

→ The standard vectors have another interesting property that any vector in R^n can be written as a linear combinations of it, i.e for $x := (x_1, x_2, \dots x_n)$ in R^n,

$$x = x_1(1,0,0\dots,0) + x_2(0,1,0,\dots,0) + \dots + x_n(0,0,\dots 0,1)$$

* vector

for physics 

for CS

vector is orderd list of number

for Ex $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 5 \\ 0 \\ 0 \\ -3 \end{bmatrix}$ $\begin{bmatrix} 2.3 \\ -7.1 \\ 0.1 \end{bmatrix}$

square footage $= 2600 \text{ ft}^3$

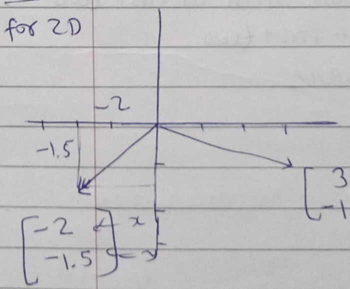
Price - \$300,000

$\begin{bmatrix} 2600 \text{ ft}^2 \\ \$300,000 \end{bmatrix}$

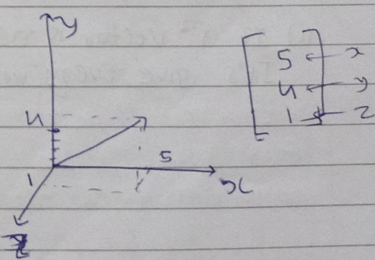
for mathematician

both

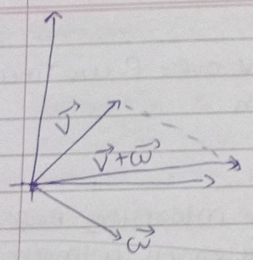
for 2D



for 3D



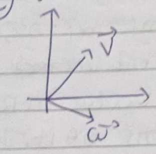
basis : The basis of vector space is a set of linearly independent vector that span the full space



*basis $\hat{i}$ and $\hat{j}$ are the "basis vectors" of xy coordinate system

→ we can chose diff basis vector also

ex any vector $\vec{v}$ & $\vec{w}$



* span of vector

The span of $\vec{v}$ and $\vec{w}$ is the set of all their linear combinations

$a\vec{v} + b\vec{w}$

$a, b \in R$

$a\vec{v} \neq b\vec{w}$ then we can reach every point in two dimention space

Here $\vec{v}$ and $\vec{w}$ are linearly dependent

⇒ span of 2D vector is All vector in 2D space

→ for $a\vec{v} = b\vec{w}$ span of $a\vec{v}$ & $\vec{w}$ is line on 2D space

⇒ In 3D space span of 2 vector give us 2D plan

⇒ Linear combination of 3 vector

$\vec{v}, \vec{w}$ and $\vec{u}$

$a\vec{v} + b\vec{w} + c\vec{u}$ span of this $\vec{v}, \vec{w}$ and $\vec{u}$ give us

i) If 3rd vector is on span of first two then It's give every 2D vector on span of first two.

linearly dependent

$\vec{u} = a\vec{v} + b\vec{w}$

for some value of a & b

ii) If 3rd vector is not on span of first two It's give every vector on 3D space

* Linear transformation

function

5 → 25

2 → $f(x)$ → 4

−3 → 9

input output

$\begin{bmatrix} 5 \\ 7 \end{bmatrix}$ $L(\vec{v})$ $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$

vector input vector output

transformation transfer input vector into output vector

Linear transformation

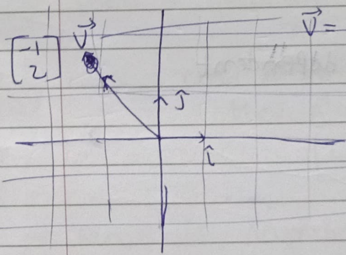
A transformation is linear If It has two Properies

i) All Lines remain lines without keeping curves

ii) origin must remain fixed. grid lines remain parallel and evenly spaced

$\vec{v} = -\hat{i} + 2\hat{j}$

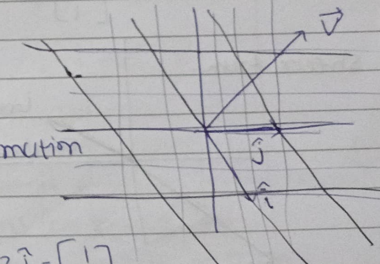
Tranformed $\vec{v}$ = −(Tranformed $\hat{i}$) + 2(Transformed $\hat{j}$)



$\vec{v} = -1\hat{i} + 2\hat{j}$

$L(\vec{v})$

some transformation



$\hat{i}_{new}$ ⇒ $\hat{i} - 2\hat{j} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\hat{j}_{new}$ ⇒ $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$

tranformed

$\vec{v} = -1\,\hat{i}_{new} + 2\,\hat{j}_{new}$

same linear combination

$\vec{v}_{new} = -1\begin{bmatrix} 1 \\ -2 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix}}_{\text{Tranformation}} \underbrace{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}_{\vec{v}\text{ input vector}}$

$\hat{i}_{new}$ $\hat{j}_{new}$

for any 2x2 matrix let basis $\hat{i}$ convert into (a, c)

$\hat{i} \Rightarrow \begin{bmatrix} a \\ c \end{bmatrix}$

and $\hat{j} \Rightarrow \begin{bmatrix} b \\ d \end{bmatrix}$

and we apply this tranformation on $\vec{v} = x\hat{i} + y\hat{j}$

$\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$

$$x\begin{bmatrix} a \\ c \end{bmatrix} + y\begin{bmatrix} b \\ d \end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\text{Tranformation}}\begin{bmatrix} x \\ y \end{bmatrix} \leftarrow \text{input vector}$$

$\hat{i}$ lands ↑ $\hat{j}$ lands

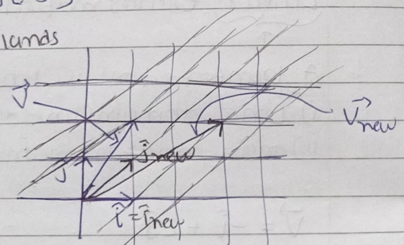
ex

$\hat{i} \Rightarrow \hat{i}$

$\hat{j} \Rightarrow \hat{i} + \hat{j}$

tranfered

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

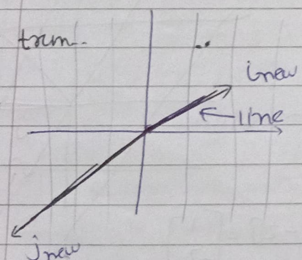


If $\hat{i}$ and $\hat{j}$ linearly dependent

let $\hat{i} \Rightarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\hat{j} \Rightarrow \begin{bmatrix} -4 \\ -2 \end{bmatrix}$

linearly dependent

shear tram.



* matrix multiplication as composition

First rotation then shear

composition of a rotation and shear Record where $\hat{i}$ & $\hat{j}$ land:

$\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$

$$\underset{\text{Shear}}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}\left(\underset{\text{Rotation}}{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}\begin{bmatrix} x \\ y \end{bmatrix}\right) = \underset{\text{Composition}}{\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

* Determinant

The factor by which linear tranformation chang any area called determinant

for 2D ↑

for 3D Volume

* Inverse matrices, column space and null space

let A tranform $\hat{i} \Rightarrow \hat{i}_{new}$, $\hat{j} \Rightarrow \hat{j}_{new}$

then A^{-1} tranform $\hat{i}_{new} \Rightarrow \hat{i}$ and $\hat{j}_{new} \Rightarrow \hat{j}$

$A\vec{x} = \vec{v}$

and det(A) ≠ 0

then $A^{-1}A\vec{x} = A^{-1}\vec{v}$

$\boxed{\vec{x} = A^{-1}\vec{v}}$

number of dimention in the column space.

⇒ Rank is Number of dimensions in the output

→ when the output of tranformation is line meaning it's one-dimensional we say tranformation has a rank = 1

→ for output of tranformation plane, rank = 2

→ for 3D, rank = 3

set of all posible outputs $A\vec{v}$ is column space of A

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ span of columns ⇒ Column space

$\hat{i}$ lands ↑ $\hat{j}$ lands

ex $\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$ rank=2 (2D plane)

$\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$ c_1 and c_2 are linearly dependent so rank=1 (line)