Moment Generating Functions (mgf)

Let X be the random variable then mgf of X is denoted by $\boxed{M_X(t)}$ and is defined as

$$M_X(t) = \sum_{i=1}^{\infty} e^{tx_i} P(X = x_i) \rightarrow \text{if } X \text{ is discrete R.V}$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\,dx \longrightarrow \text{if } X \text{ is Continuous R.V.}$$

In general $M_X(t) = E(e^{tx})$

Here 't' is a parameter.

Theorem:

The n^{th} derivative of $M_X(t)$ at $t=0$ gives $E(x^n)$.

Proof: $M_X(t) = E(e^{tx})$

$$e^{tx} = 1 + tx + \frac{t^2x^2}{2!} + \cdots + \frac{t^n x^n}{n!} + \cdots$$

$$E(e^{tx}) = E\left\{1 + tx + \frac{t^2x^2}{2!} + \cdots + \frac{t^n x^n}{n!} + \cdots\right\}$$

$$M_X(t) = \overset{=1}{E(1)} + t\,E(X) + \frac{t^2\,E(X^2)}{2!} + \cdots + \frac{t^n\,E(X^n)}{n!} + \cdots$$

diff w.r.to 't'

$$\frac{d}{dt} M_X(t) = 0 + E(X) + \frac{2t}{2!} E(X^2) + \cdots + \frac{n t^{n-1}}{n!} E(X^n) + \cdots$$

Put $t = 0$

$$M'_X(t)\Big|_{t=0} = E(X)$$

$$\boxed{M'_X(0) = E(X)}$$

Similarly

$$M''_X(t) = 0 + E(X^2) + \cdots +$$

$$M''_X(t)\Big|_{t=0} = E(X^2)$$

$$\Rightarrow \quad M''_X(0) = E(X^2)$$

$\Rightarrow$ Similarly,

$$\boxed{M^n_X(t)\Big|_{t=0} = E(X^n)}$$

$$\Rightarrow \quad V(X) = E(X^2) - (E(X))^2$$

$$V(X) = M''_X(0) - [M'_X(0)]^2$$

Properties of mgf :

1) $M_{ax}(t) = M_x(at)$ ✓

Proof: $M_{ax}(t) = E\left(e^{t(ax)}\right)$ $\quad\because$ Def^n of mgf.

$= E\left(e^{(at)x}\right)$

$= M_x(at)$

2) $M_{ax+b}(t) = e^{bt} M_x(at)$ ✓ $\quad\because$ a & b are Constants

Proof: $M_{ax+b}(t) = E\left(e^{t(ax+b)}\right)$

$= E\left(e^{axt} \cdot e^{bt}\right)$

$= e^{bt} E\left(e^{axt}\right)$

$= e^{bt} M_x(at)$ ✓

3) If x & y are <u>independent random variables</u>

then $M_{x+y}(t) = M_x(t) \cdot M_y(t)$.

Proof: $M_{x+y}(t) = E\left(e^{(x+y)t}\right)$

$= E\left(e^{xt} \cdot e^{yt}\right)$

$= E\left(e^{xt}\right) \cdot E\left(e^{yt}\right)$ $\quad\because E(xy) = E(x) \cdot E(y)$ when x & y are independent

$$\boxed{M_{x+y}(t) = M_x(t) \cdot M_y(t).}$$

Note: $E(x^n)$ is the coefficient of $\frac{t^n}{n!}$ in the expansion
1) of $E(e^{tx})$/mgf

Proof: $E(e^{tx}) = E\left\{1 + tx + \frac{t^2x^2}{2!} + \cdots + \frac{t^nx^n}{n!} + \cdots\right\}$

$$= 1 + tE(x) + \frac{t^2}{2!}E(x^2) + \cdots + \frac{t^n}{n!}E(x^n) + \cdots \quad \text{—} (*)$$

$\Rightarrow$ $\boxed{E(x^n) = \text{coefficient of } \frac{t^n}{n!} \text{ in the above eq}^n.}$

Note 2: whenever a random variable is given to find $E(x)$, using mgf we can directly find,. instead of finding pdf & proceeding.

$\Rightarrow$ This is an alternative way to find Mean & variance.

Examples :

1) Let x be the outcome when a die is rolled. find mgf of x, also find mean & variance of x.

Sol^n :

X :	1,	2,	3,	4,	5,	6
$p(x)$:	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

$$M_X(t) = E(e^{tX})$$

$$= \sum_{x=1}^{6} e^{tx} p(x) \qquad \because \text{Def}^n\ E(X)$$

$$= \frac{1}{6} \cdot e^{t} + \frac{1}{6} e^{2t} + \frac{1}{6} e^{3t} + \frac{1}{6} e^{4t} + \frac{1}{6} e^{5t} + \frac{1}{6} e^{6t}$$

$$M_X(t) = \frac{1}{6}\left\{e^{t} + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t}\right\} \quad - \quad (*)$$

We know that $E(X) = M_X'(0)$

$$= \frac{1}{6}\left\{ e^{(0)} + 2e^{2(0)} + 3e^{3(0)} + 4e^{4(0)} + 5e^{5(0)} + 6e^{6(0)} \right\}$$

$$\Rightarrow \quad E(X) = \frac{1+2+3+4+5+6}{6} = \boxed{\frac{21}{6}}$$

Next $V(X) = E(X^2) - (E(X))^2$

$$\Rightarrow \quad E(X^2) = M_X''(0)$$

$$= \frac{1}{6}\left\{ e^{(0)} + 4e^{2(0)} + 9e^{3(0)} + 16e^{4(0)} + 25e^{5(0)} + 36e^{6(0)} \right\}$$

$$= \frac{91}{6}$$

$$\Rightarrow \quad V(X) = \frac{91}{6} - \left(\frac{21}{6}\right)^2 = \boxed{\frac{35}{12}} = 2.9166$$

2) If x is a random variable taking the values $0, 1, 2, \dots \infty$ and $P(x) = ab^x$ where a, b are +ve numbers such that $a + b = 1$

i) find the mgf of x

ii) If $E(x) = m_1$, $E(x^2) = m_2$ Show that

$$\boxed{m_2 = m_1(2m_1 + 1)} \longrightarrow \text{we need to prove}$$

Sol^n: We know that $M_x(t) = E(e^{tx})$

Given : $\quad p(x) = ab^x \quad \forall x : 0, 1, 2, \dots \infty$.

$$M_x(t) = E(e^{tx})$$
$$= \sum_{x=0}^{\infty} e^{tx} p(x)$$
$$= \sum_{x=0}^{\infty} e^{tx} ab^x$$
$$= a \sum_{x=0}^{\infty} (be^t)^x$$
$$= a\{1 + \underbrace{be^t + b^2e^{2t} + b^3e^{3t} + \dots}\}$$

$$\boxed{M_x(t) = a\left(\frac{1}{1-be^t}\right)}$$

Geometric progression where $|be^t| < 1$

$a = 1$

$1 + r + r^2 + r^3 + \dots \infty$

$S_\infty = \frac{1}{1-r} \quad |r|<1$

$r < 1$

G.P.

Now $\quad E(x) = M'_x(0)$

$$= a\left\{\frac{(1-be^t)(0) - (-be^t)}{(1-be^t)^2}\right\}_{t=0} \quad \text{—} \; (*)$$

$$\boxed{\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}}$$

$$= \left.\frac{abe^t}{(1-be^t)^2}\right|_{t=0} = \frac{ab}{(1-b)^2} = \boxed{\frac{b}{a}} \;\because a+b=1$$

$$M'_x(t) = \frac{abe^t}{(1-be^t)^2} \quad \text{—— } ** $$

$$E(x^2) = M''_x(t)\Big|_{t=0} = a\left\{\frac{(1-be^t)^2(be^t) - (be^t)\,2(1-be^t)(-be^t)}{(1-be^t)^4}\right\}_{t=0}$$

$$= a\left\{\frac{(1-b)^2 b + b^2\,2(1-b)}{(1-b)^4}\right\}$$

$$= a\left\{\frac{\cancel{(1-b)}\left[(1-b)b + 2b^2\right]}{(1-b)^{\cancel{4}\,3}}\right\}$$

$$= a\left\{\frac{b - b^2 + 2b^2}{(1-b)^3}\right\}$$

$$= \frac{a(1+b)b}{(1-b)^3}$$

$$E(x^2) = \frac{\cancel{(1-b)}(1+b)b}{(1-b)^{\cancel{3}\,2}} = \frac{b(1+b)}{(1-b)^2} = \frac{b(1+b)}{a^2}$$

ii) Given $E(x) = m_1$, $E(x^2) = m_2$

$$m_2 = \frac{b(1+b)}{a^2} = \frac{b}{a^2}(1+b)$$

$$= \frac{b}{a}\left(\frac{1+b}{a}\right)$$

$$= m_1\left(\frac{1}{a} + \frac{b}{a}\right)$$

$$\boxed{\frac{b}{a} = m_1}$$

$$\boxed{a+b=1}$$

$$m_2 = m_1\left(\frac{a+b}{a} + \frac{b}{a}\right) = m_1\left(1 + \frac{2b}{a}\right)$$

$$\boxed{m_2 = m_1(1+2m_1)}$$

3) Find the mgf of a random variable which is uniformly distributed from $-a < x < a$, Hence evaluate $E(x^{2n})$

Soln: Let $X \sim U(-a, a)$ $\because$ given

$\Rightarrow \boxed{f(x) = \frac{1}{2a}}$ $\boxed{f(x) = \frac{1}{b-a}}$

$$M_X(t) = E(e^{tx})$$

$$= \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\, dx$$

$$= \int_{-a}^{a} e^{tx} \cdot \frac{1}{2a}\, dx = \frac{1}{2a} \cdot \left[\frac{e^{tx}}{t}\right]_{-a}^{a}$$

$$\boxed{M_X(t) = \frac{1}{at}\left(\frac{e^{at} - e^{-at}}{2}\right)} \text{ — } *$$

$$\boxed{M_X(t) = \frac{1}{at} \sinh at}$$ $\boxed{\because \sinh x = \frac{e^{x} - e^{-x}}{2}}$

To find $E(x^{2n})$ we know that

$$E(x^n) = \text{Coefficient of } \frac{t^n}{n!} \text{ of } M_X(t)$$

Similarly, $E(x^{2n}) = \text{coefficient of } \frac{t^{2n}}{2n!} \text{ of } M_X(t)$

$$\Rightarrow M_X(t) = \frac{\sinh at}{at}$$

$$M_x(t) = \frac{\sinh at}{at}$$

check Power Series for Sinhx

$$= \frac{1}{at}\left\{ at + \frac{(at)^3}{3!} + \frac{(at)^5}{5!} + \cdots + \frac{(at)^{2n+1}}{(2n+1)!} + \cdots \right\}$$

$$= \frac{at}{at}\left\{ 1 + \frac{(at)^2}{3!} + \frac{(at)^4}{5!} + \cdots + \frac{(at)^{2n}}{(2n+1)!} + \cdots \right\}$$

$$M_x(t) = \left\{ 1 + \frac{(at)^2}{3!} + \frac{(at)^4}{5!} + \cdots + \frac{a^{2n}}{(2n+1)} \cdot \frac{t^{2n}}{2n!} + \cdots \right\} \quad (*)$$

$3! = 3 \times 2!$

$$\Rightarrow \boxed{E(x^{2n}) = \text{coeff}^{nt} \text{ of } \frac{t^{2n}}{2n!} \text{ in } M_x(t)}$$

From $(*)$ $$\boxed{E(x^{2n}) = \frac{a^{2n}}{2n+1}} \qquad \boxed{E(x^2) = \frac{a^2}{3}}$$

$$\boxed{E(x) = \frac{a}{2}} \qquad E(x) = M'_x(t)\Big|_{t=0} = \frac{a}{2}$$

4) Find the mgf of Binomial distribution. Hence find the mean & variance of Binomial distribution.

Soln: If $X \sim B(n, p)$

$$P(x) = {}^nC_x \, p^x q^{n-x} \quad , \ \forall x = 0, 1, 2, \dots n$$

$$p + q = 1$$

$$M_X(t) = E(e^{tx}) \quad \text{— (A)}$$

$$= \sum_{x=0}^{n} e^{tx} \cdot \underbrace{{}^nC_x \, p^x q^{n-x}}_{\to P(x)}$$

$$= \sum_{x=0}^{n} (e^t p)^x \cdot {}^nC_x \, q^{n-x} \quad \text{— (*)}$$

from the Binomial expansion we know that

$$\sum_{x=0}^{\infty} {}^nC_x \, p^x q^{n-x} = (p+q)^n \quad \text{— (**)}$$

$\Rightarrow$ from eqn (*)

$$\boxed{M_X(t) = (pe^t + q)^n} \quad \text{— (1)}$$

$\Rightarrow$ $E(X) = M'_X(0)$

$$M'_X(t) = n(pe^t + q)^{n-1} \cdot pe^t$$

from ① $\quad M'_x(t) = n(pe^t + q)^{n-1} \cdot pe^t \quad$ — ②

when $t = 0$

$$M'_x(0) = n(p+q)^{n-1} \cdot p \quad ✓$$

$\because p + q = 1$

$$= n(1)p$$

$$\boxed{M'_x(0) = np = E(x)} \quad \text{— ③}$$

$\Rightarrow \quad V(x) = E(x^2) - (E(x))^2$

$E(x^2) = M''_x(0)$

from ② $\quad M''_x(t) = np\left[e^t (n-1)(pe^t + q)^{n-2} \cdot pe^t + (pe^t + q)^{n-1} \cdot e^t\right]$

when $t = 0$

$$M''_x(0) = np\left[(n-1)(1)p + (1)\right]$$

$$= np[np - p + 1]$$

$$M''_x(0) = n^2p^2 - np^2 + np = E(x^2) \quad \text{— ④}$$

$\Rightarrow \quad V(x) = n^2p^2 - np^2 + np - n^2p^2$

$$= np(1-p)$$

$\because p + q = 1$

$1 - p = q$

$$\boxed{V(x) = npq}$$

$$V(x) = npq$$

$\Rightarrow$ **mgf of Poisson Distribution**

Let $X \sim P(\alpha)$

$$P(X = x) = \frac{e^{-\alpha} \cdot \alpha^x}{x!} \quad \forall \; x = 0, 1, 2, \ldots \infty$$

$$M_x(t) = E(e^{tx}) \quad \text{——— Def}^n.$$

$$= \sum_{x=0}^{\infty} e^{tx} \cdot \frac{e^{-\alpha} \cdot \alpha^x}{x!} \quad \text{— Ⓐ}$$

$$= e^{-\alpha}\left\{1 + \frac{e^t\alpha}{1!} + \frac{(e^t\alpha)^2}{2!} + \cdots + \frac{(e^t\alpha)^n}{n!} + \cdots\right\}$$

$$= e^{-\alpha} \cdot e^{e^t\alpha}$$

$$= e^{e^t\alpha - \alpha}$$

$$\boxed{M_x(t) = e^{\alpha(e^t - 1)}} \quad \text{—} \quad (1) \Rightarrow \text{mgf of P.D. . .}$$

To find mean

$$E(x) = M'_x(0)$$

from (1),

$$M'_x(t) = e^{\alpha(e^t-1)} \cdot \alpha e^t \quad \text{—} \quad (2)$$

when $t = 0$

$$\boxed{M'_x(0) = \alpha = E(x)} \quad \text{—} \quad (3)$$

$$\Rightarrow V(x) = E(x^2) - (E(x))^2$$

$$E(x^2) = M''_x(0)$$

from (2),

$$M''_x(t) = \alpha\left[e^t \cdot e^{\alpha(e^t-1)} + e^t e^{\alpha(e^t-1)} \cdot \alpha e^t\right]$$

when $t = 0$

$$M''_x(0) = \alpha(1+\alpha)$$

$$V(x) = \alpha + \alpha^2 - \alpha^2$$

$$\Rightarrow \boxed{V(x) = \alpha} \quad \text{—} \quad (4)$$

$\Rightarrow$ Moment generating function of Gamma Distribution

Let $X \sim G(r, \alpha)$

pdf of X is given by

$$f(x) = \frac{\alpha e^{-\alpha x} (x\alpha)^{r-1}}{\Gamma(r)}, \quad x > 0 \longrightarrow (*)$$

$$M_X(t) = E(e^{tx})$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx \longrightarrow \because \text{ by Def}^n \text{ of } E(X)$$

$$= \int_0^{\infty} e^{tx} \frac{\alpha e^{-\alpha x} (x\alpha)^{r-1}}{\Gamma(r)}\, dx$$

$$= \frac{\alpha^r}{\Gamma(r)} \int_0^{\infty} e^{(t-\alpha)x} x^{r-1}\, dx$$

$$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1}\, dx$$

$$= \frac{\alpha^r}{\Gamma(r)} \int_0^{\infty} e^{-(\alpha-t)x} x^{r-1}\, dx$$

Put $(\alpha - t)x = u \Rightarrow x = \frac{u}{\alpha - t}$

$$dx = \frac{du}{\alpha - t}$$

$$\Rightarrow M_X(t) = \frac{\alpha^r}{\Gamma(r)} \int_0^{\infty} e^{-u} \left(\frac{u}{\alpha - t}\right)^{r-1} \frac{du}{\alpha - t}$$

$$= \frac{\alpha^r}{\Gamma(r)(\alpha - t)^r} \int_0^{\infty} e^{-u} u^{r-1}\, du \Rightarrow \Gamma(r)$$

$$= \frac{\alpha^r \cancel{\gamma(r)}}{\cancel{\gamma(r)}\,(\alpha - t)^r}$$

$$\boxed{M_x(t) = \frac{\alpha^r}{(\alpha - t)^r}} \quad \text{— (1)} \longrightarrow \text{mgf of G.D.}$$

To find $E(X)$

from (1), $$M_x'(t) = \alpha^r \left\{ -\frac{(\alpha - t)^{r-1} \cdot r(-1)}{(\alpha - t)^{2r}} \right\}$$

$$= \frac{r\alpha^r (\alpha - t)^{r-1}}{(\alpha - t)^{2r}} = \frac{r\alpha^r}{(\alpha - t)^{r+1}} \quad \text{— (2)}$$

$$M_x'(0) = \frac{r\alpha^r}{\alpha^{r+1}} = \boxed{\frac{r}{\alpha}}.$$

$$\Rightarrow \boxed{M_x'(0) = \frac{r}{\alpha} = E(X)} \quad \text{— (3)}$$

from (2) $$M_x''(t) = \frac{r\alpha^r (r+1)(\alpha - t)^r}{(\alpha - t)^{2r+2}}$$

when $t = 0$

$$M_x''(0) = \frac{r(r+1)}{\alpha^2} = E(x^2).$$

$$V(X) = E(x^2) - (E(x))^2$$

$$V(X) = \frac{r^2 + r}{\alpha^2} - \frac{r^2}{\alpha^2} = \boxed{\frac{r}{\alpha^2}} \quad \text{— (4)}$$

$\Rightarrow$ mgf of Exponential Distribution

Let $X \sim \exp(\alpha)$ then,

pdf of X is given by

$$f(x) = \alpha e^{-\alpha x} \quad \forall\ x > 0$$

$$M_X(t) = \int_0^\infty \alpha e^{-\alpha x} e^{tx}\, dx$$

$$= \alpha \int_0^\infty e^{-(\alpha - t)x}\, dx$$

$$= \alpha \left[\frac{e^{-(\alpha-t)x}}{-(\alpha-t)} \right]_0^\infty$$

$$= \frac{\alpha}{-(\alpha-t)} \left[e^{-\infty} - e^{0} \right]$$

$$\boxed{M_X(t) = \frac{\alpha}{\alpha - t}} \quad \text{— (1)}$$

$$E(X) = M'_X(0)$$

$$E(X^2) = M''_X(0)$$

$$= \frac{\alpha}{t-\alpha}\left(e^{-\infty} - e^{0}\right)$$

$$\boxed{M_x(t) = \frac{\alpha}{\alpha - t}} \longleftarrow (1)$$

H.W find $E(x)$ & $V(x)$ when $x \sim \exp(\alpha)$

$\Rightarrow$ mgf of χ^2 - distribution

Let $x \sim \chi^2(n)$

$\Rightarrow$ pdf of x is given by

$$f(x) = \frac{e^{-x/2}\, x^{n/2 - 1}}{2^{n/2}\, \Gamma(n/2)} \quad \forall\ x > 0 \longleftarrow (A)$$

$$M_x(t) = \int_0^\infty e^{tx} \frac{e^{-x/2}\, x^{n/2-1}}{2^{n/2}\, \Gamma(n/2)}\, dx$$

H.W find $E(x)$ ✓ & $V(x)$ ✓

$$= \frac{1}{2^{n/2}\, \Gamma(n/2)} \int_0^\infty e^{-(1/2 - t)x}\, x^{n/2-1}\, dx$$

$$= \frac{1}{2^{n/2}\, \Gamma(n/2)} \underbrace{\int_0^\infty e^{(t-1/2)x}\, x^{n/2-1}\, dx}$$

using property of gamma function

$$= \frac{1}{2^{n/2}\, \cancel{\Gamma(n/2)}} \cdot \frac{\cancel{\Gamma(n/2)}}{-(t-1/2)^{n/2}}$$

$$\boxed{M_x(t) = (1-2t)^{-n/2}} \text{ — } (1)$$

$\Rightarrow$ mgf of <u>Normal Distribution</u>

Let $X \sim N(\mu, \sigma^2)$

pdf of X is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$-\infty < x < \infty$

$$M_X(t) = E(e^{tx}) \quad \text{— By Def}^n.$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{tx} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Put $\frac{x-\mu}{\sigma} = u \Rightarrow \boxed{x = \sigma u + \mu}$ $\boxed{dx = \sigma\, du}$

$$= \frac{\sigma}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{t(\mu+\sigma u)}\, e^{-u^2/2}\, du$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t\sigma u}\, e^{-u^2/2}\, du$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\boxed{(u^2+2tu\sigma)}}\, du$$

Completing the square by adding & substracting the term "$(t\sigma)^2$".

i,e $\quad \underbrace{u^2+2tu\sigma+(t\sigma)^2}-(t\sigma)^2$

$$\Rightarrow \quad (u+t\sigma)^2-(t\sigma)^2$$

$$\Rightarrow \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[(u+t\sigma)^2-(t\sigma)^2\right]} du$$

$$\Rightarrow \frac{e^{\mu t+\frac{\sigma^2 t^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(u+t\sigma)^2}{2}} du$$

put $u+t\sigma = s$

$du = ds$

$$\Rightarrow \frac{e^{\mu t+\frac{\sigma^2 t^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{e^{-s^2/2}}_{\hookrightarrow \text{ even fun}^n.} ds$$

$$\Rightarrow \frac{e^{ut+\frac{\sigma^2 t^2}{2}}}{\sqrt{2\pi}} (2) \int_0^\infty e^{-s^2/2} \, ds$$

Put $\frac{s^2}{2} = y \Rightarrow s = \sqrt{2y}$

$$\frac{2s\,ds}{2} = dy$$

$$s\,ds = dy$$

$$\Rightarrow \frac{\sqrt{2}}{\sqrt{\pi}} \cdot e^{ut+\frac{\sigma^2 t^2}{2}} \int_0^\infty e^{-y} \frac{dy}{\sqrt{2y}}$$

$$\Rightarrow \sqrt{\frac{2}{\pi}}\, e^{\mu t + \frac{\sigma^2 t^2}{2}} \cdot \frac{1}{\cancel{\sqrt{2}}} \int_0^\infty e^{-y} y^{-1/2} dy$$

$$= \frac{1}{\sqrt{\pi}}\, e^{\mu t + \frac{\sigma^2 t^2}{2}} \cdot \Gamma(1/2)$$

$$= \frac{1}{\cancel{\sqrt{\pi}}}\, e^{\mu t + \frac{\sigma^2 t^2}{2}} \cdot \cancel{\sqrt{\pi}}$$

$$\boxed{M_x(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}} \quad — (1)$$

$$E(x) = M'_x(t)\Big|_{t=0}$$

$$M'_x(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}} \left(\mu + \frac{1}{2} 2t\sigma^2\right)$$

when $t = 0$

$$\Rightarrow M'_x(0) = \mu = E(x)$$

$E(x^2) = \mu^2 + \sigma^2$ (verify)

$V(x) = \mu^2 + \sigma^2 - \mu^2$

$\boxed{V(x) = \sigma^2}$ — check later.

H.W.

⇒ Let X be the random variable & its mgf is given by $M_x(t) = \left(\frac{1}{3} + \frac{2}{3}e^t\right)^5$ then find probability at $x=2$, & $x=3$

Soln: $M_x(t) = \left(\frac{1}{3} + \frac{2}{3}e^t\right)^5$

it is similar to $(pe^t + q)^n$

where $n = 5$, $p = 2/3$ $q = \frac{1}{3}$

$\Rightarrow$ $X \sim B(n, p)$

$$P(X = x) = {}^{n}C_{x}\, p^{x}\, q^{n-x}$$

$$P(X = 2) = {}^{5}C_{2}\left(2/3\right)^{2}\left(\frac{1}{3}\right)^{3}$$

$$= 0.1646$$

$$P(X = 3) = 0.3292.$$