

Goertzel Algorithm

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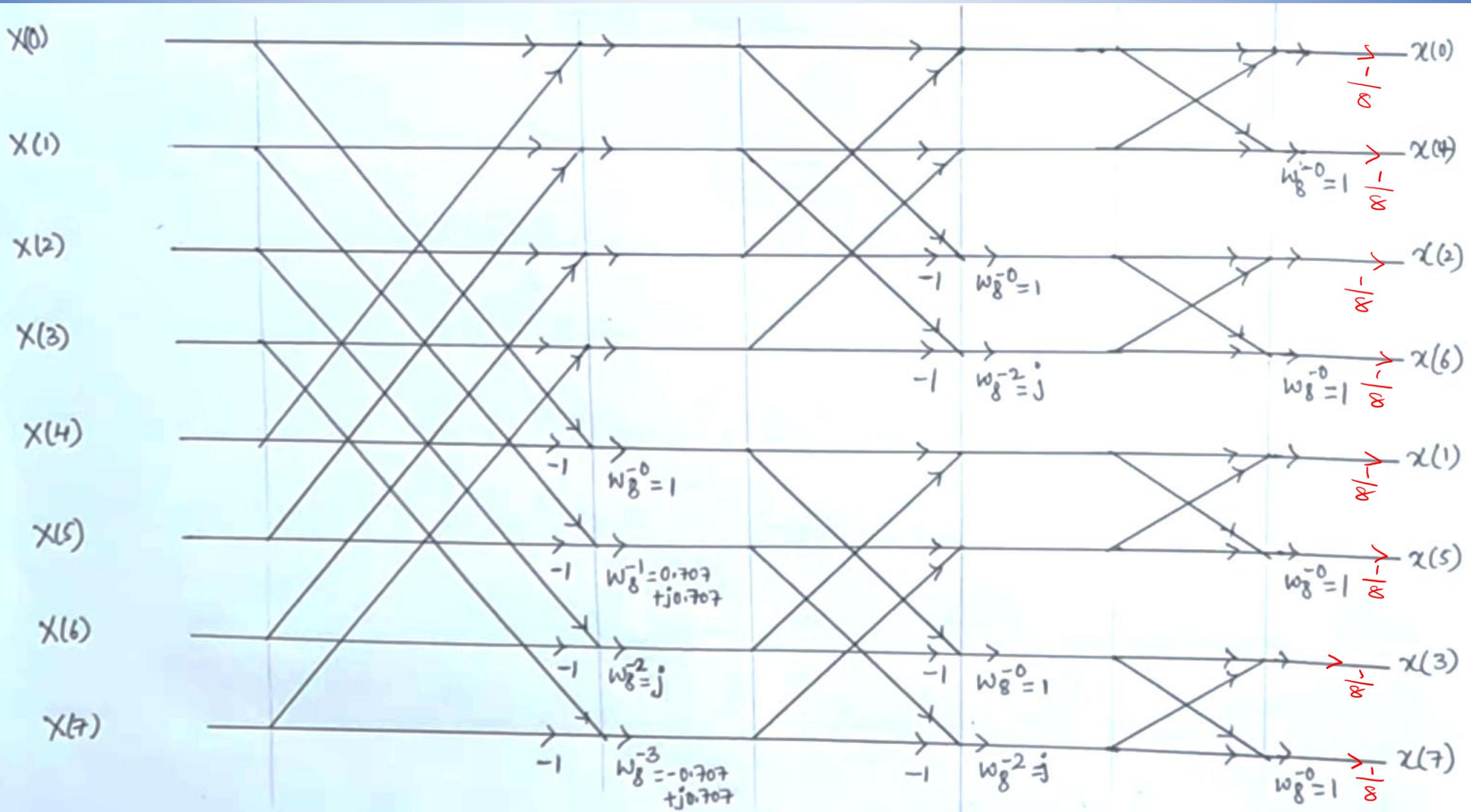
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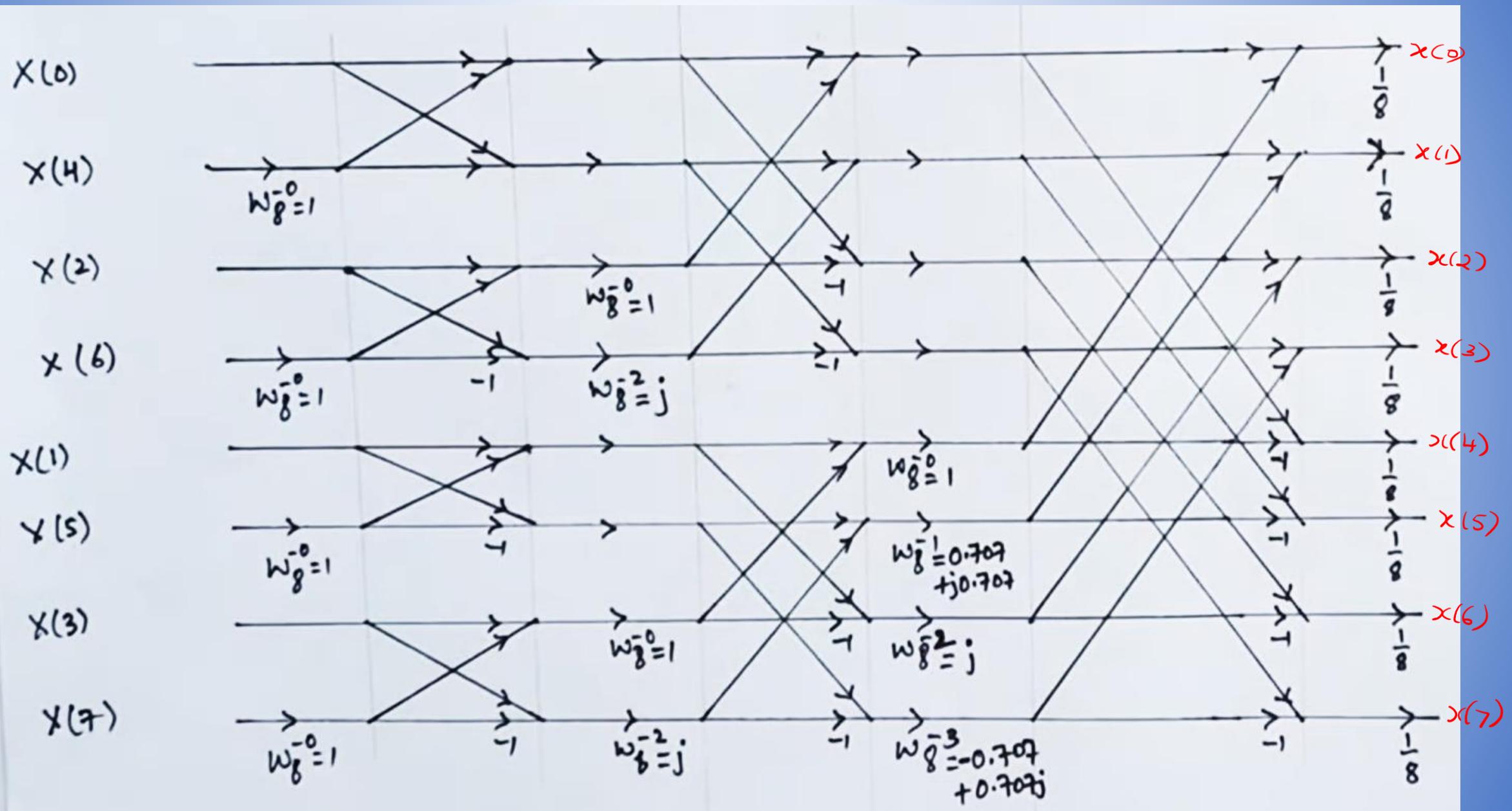
Inverse FFT

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-nk}$$

IDFT using DIT-FFT algorithm



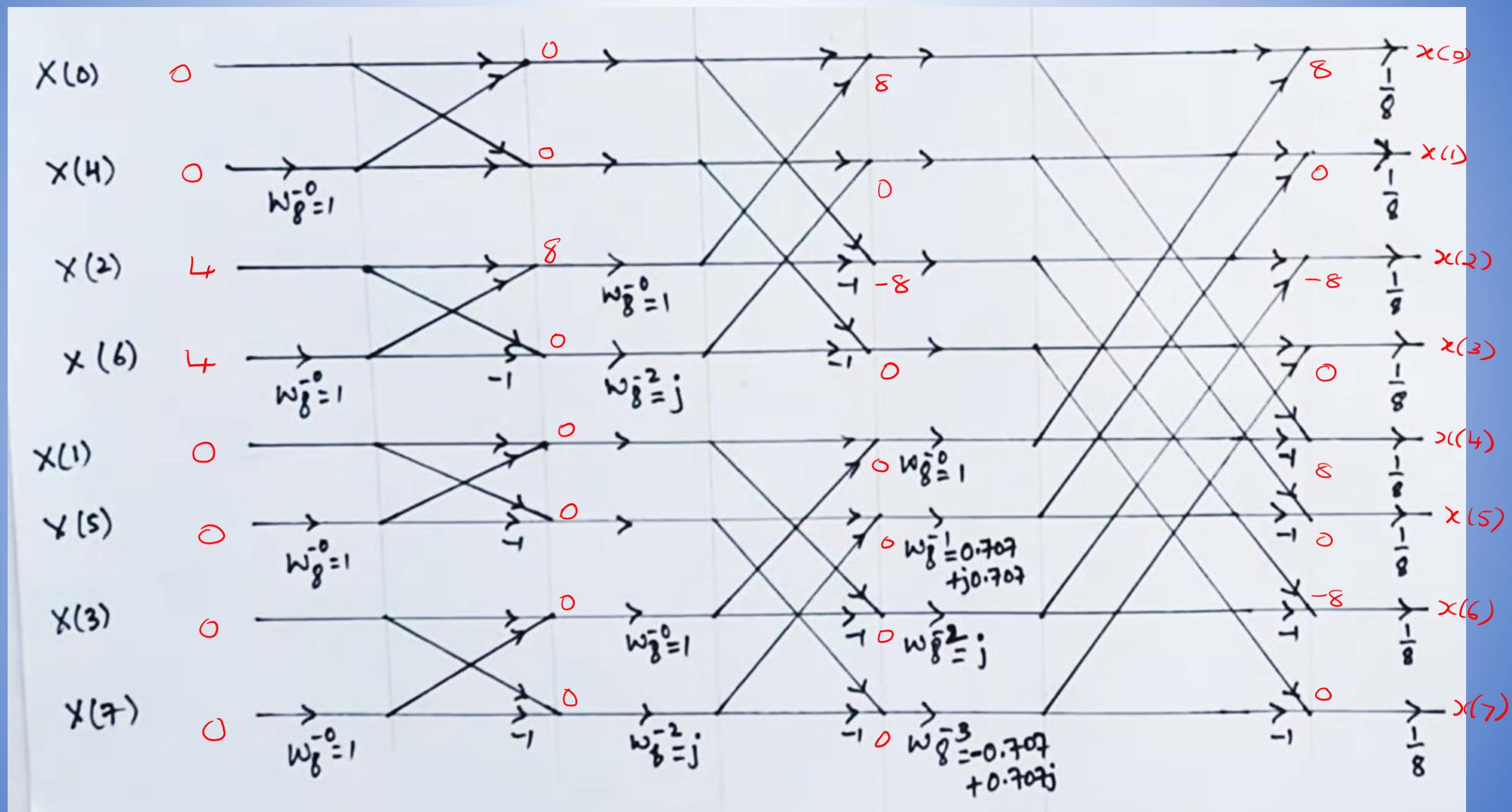
IDFT using DIF-FFT algorithm



$$X(k) = \{0, 0, 4, 0, 0, 0, 4, 0\}$$

IDFT using DIF-FFT algorithm

$$x(n) = \{1, 0, -1, 0, 1, 0, -1, 0\}$$



Linear filtering approach to computation of DFT

- Geortzel Algorithm

- Exploits the periodicity of the phase factors $\{W_N^k\}$
- DFT is computed as a linear convolution operation

We know that $W_N^{-Nk} = 1$. We multiply this to DFT expression and we get

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{nk} \cdot W_N^{-Nk} = \sum_{m=0}^{N-1} x(m) W_N^{-(N-m)k} \quad \text{--- ①}$$

Let us define a sequence $y_k(n)$ as follows:

$$y_k(n) = \sum_{m=-\infty}^{\infty} x(m) W_N^{-(N-m)k} \quad \text{--- ②}$$

The above equation looks as if $y_k(n)$ is a convolution of two sequences $x(n)$ and $W_N^{-kn} u(n)$.

If $x(n)$ is a finite length sequence taking values from 0 to $N-1$, then we can write

$$y_k(n) = \sum_{m=0}^{N-1} x(m) w_N^{-(n-m)k} \quad \text{--- (3)}$$

Rewriting eqn (1)

$$x(k) = \sum_{m=0}^{N-1} x(m) w_N^{-(N-m)k}$$

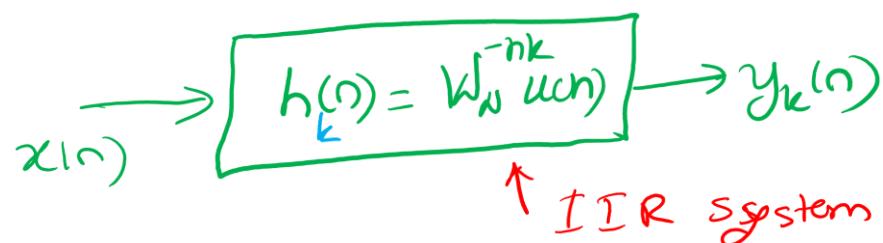
From equations (1) & (3) we can say that

$$x(k) = y_k(n) \Big|_{n=N} \quad \text{--- (4)}$$

We can think of $x(n)$ as an input to a linear time invariant system (filter)

whose impulse response is $h(n) = w_N^{-nk} u(n)$

Then the output observed at time $n=N$ yields the value of the DFT at the frequency $\omega_k = \frac{2\pi k}{N}$



The system function for this filter can be defined as follows:

$$h_k(n) = W_N^{-nk} u(n) \Leftrightarrow H_k(z) = \frac{1}{1 - W_N^{-k} z^{-1}} \quad \text{--- (5)}$$

This filter has a pole on the unit circle at the frequency $\omega_k = \frac{2\pi k}{N}$

The entire DFT can be computed by passing the block of input data into a parallel bank of N single pole filters.

We can use the difference equation corresponding to the filter given by equation(5)

we have $H_k(z) = \frac{Y_k(z)}{X(z)} = \frac{1}{1 - W_N^{-k} z^{-1}}$

$$Y_k(z)(1 - W_N^{-k} z^{-1}) = X(z)$$

or $y_k(n) - W_N^{-k} y_k(n-1) = x(n)$

$$\boxed{y_k(n) = W_N^{-k} y_k(n-1) + x(n)}$$

Now we can realize the structure for equation(5) as follows:

$$H_k(z) = \frac{1}{1 - w_N^{-k} z^{-1}} \frac{(1 - w_N^k z^{-1})}{(1 - w_N^k z^{-1})} = \frac{1 - w_N^k z^{-1}}{1 - (w_N^k + w_N^{-k})z^{-1} + z^{-2}}$$

$$H_k(z) = \frac{1 - w_N^k z^{-1}}{1 - 2\cos\left(\frac{2\pi k}{N}\right)z^{-1} + z^{-2}} \quad \text{--- (6)}$$

$$\text{Let } H_k(z) = \frac{Y_k(z)}{X(z)} = \frac{1 - w_N^k z^{-1}}{1 - 2\cos\left(\frac{2\pi k}{N}\right)z^{-1} + z^{-2}}$$

$$\text{Let } H_k(z) = \frac{Y_k(z)}{V_k(z)} \cdot \frac{V_k(z)}{X(z)} = (1 - w_N^k z^{-1}) \cdot \frac{1}{1 - 2 \cos\left(\frac{2\pi k}{N}\right) z^{-1} + z^{-2}}$$

$$\text{Let } \frac{V_k(z)}{X(z)} = \frac{1}{1 - 2 \cos\left(\frac{2\pi k}{N}\right) z^{-1} + z^{-2}} \quad \text{--- } \textcircled{7}$$

$$V_k(z) \left[1 - 2 \cos\left(\frac{2\pi k}{N}\right) z^{-1} + z^{-2} \right] = X(z)$$

$$V_k(z) - 2 \cos\left(\frac{2\pi k}{N}\right) z^{-1} V_k(z) + V_k z^{-2} = X(z)$$

Taking Inverse Z.T.

$$v_k(n) - 2 \cos\left(\frac{2\pi k}{N}\right) v_k(n-1) + v_k(n-2) = x(n) \quad \text{--- } \textcircled{8}$$

$$v_k(n) = x(n) + 2 \cos\left(\frac{2\pi k}{N}\right) v_k(n-1) - v_k(n-2) \quad \text{--- } \textcircled{9}$$

Now Let

$$\frac{Y_k(z)}{V_k(z)} = (1 - W_N^k z^{-1}) \quad \text{--- (10)}$$

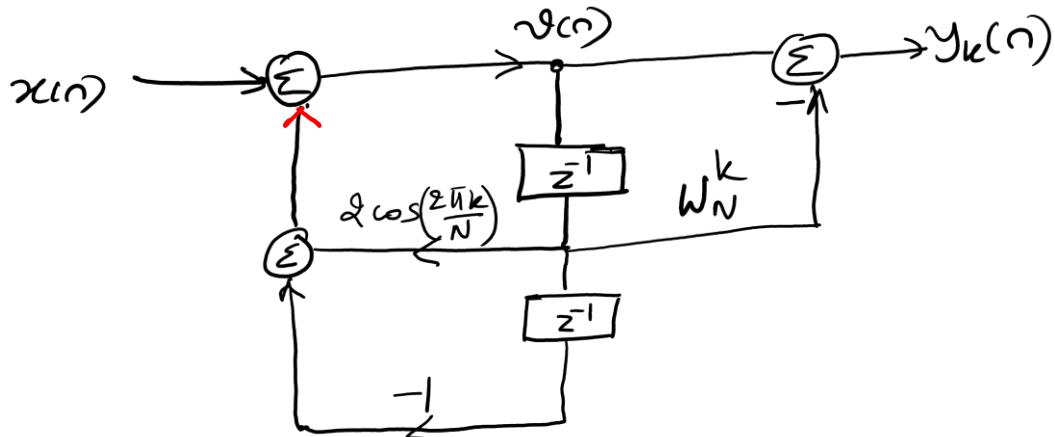
$$Y_k(z) = V_k(z) - W_N^k z^{-1} V_k(z) \quad \text{--- (11)}$$

Taking inverse Z.T. we get

$$y_k(n) = v_k(n) - W_N^k v_k(n-1) \quad \text{--- (12)}$$

$$v_k(n) = x(n) + 2\cos\left(\frac{2\pi k}{N}\right)v_k(n-1) - v_k(n-2) \quad \text{--- (9)}$$

Let us now realize a system using eqns (9) and (12) as follows:



computed at time $n=N$.

Recursive relation is iterated for $n=0, 1, \dots, N$

Groertzel algorithm is attractive when the DFT to be computed at a relatively smaller number M of values where $M \leq \log_2 N$

Direct form II realization for computing k^{th} DFT point

If $x(n) = \{2, 0, 2, 0\}$

find $X(2)$

Soln $X(k) = y_k(n) \Big|_{n=n}$

$$k=2, n=4$$

$$X(2) = y_2(n) \Big|_{n=4}$$

and $y_k(n) - h_k^{-1}y_k(n-1) = x(n)$

$$y_2(n) - h_2^{-1}y_2(n-1) = x(n)$$

where $h_2^{-1} = e^{-j\frac{2\pi}{4}} = e^{-j\frac{\pi}{2}} = -i$

$$\boxed{y_2(n) + y_2(n-1) = x(n)}$$

$$X(2) = y_2(4)$$

$$y_2(n) + y_2(n-1) = x(n)$$

$$\therefore n = 0, 1, 2, 3$$

$$y_2(0) = x(0) - y_2(-1)$$
$$= 2 - 0 = 2$$

$$y_2(1) = x(1) - y_2(0)$$
$$= 0 - 2 = -2$$

$$y_2(2) = x(2) - y_2(1)$$
$$= 2 - (-2) = 4$$

$$y_2(3) = x(3) - y_2(2)$$
$$= 0 - 4 = -4$$

$$y_2(4) = x(4) - y_2(3) = 0 - (-4)$$
$$= 4$$

$$x[2] = y_2[4] = 4$$

*Thank
you*

