IIR filter design
Impulse invariance technique

Dr. Sampath Kumar

Associate Professor

Department of ECE

MIT, Manipal

Design of IIR filters from Analog filters

Analog filter system function can be described

1) Using filter coefficients

$$H_a(s) = \frac{B(s)}{A(s)} = \frac{\sum_{k=0}^{M} \beta_k s^k}{\sum_{k=0}^{N} \alpha_k s^k} = \frac{\beta_0 + \beta_1 s + \beta_2 s^2 + \cdots + \beta_M s^M}{\alpha_0 + \alpha_1 s + \alpha_2 s^2 + \cdots + \alpha_N s^N}$$

$\{\alpha_k\}$ and $\{\beta_k\}$ – analog filter coefficients

2) Using impulse response

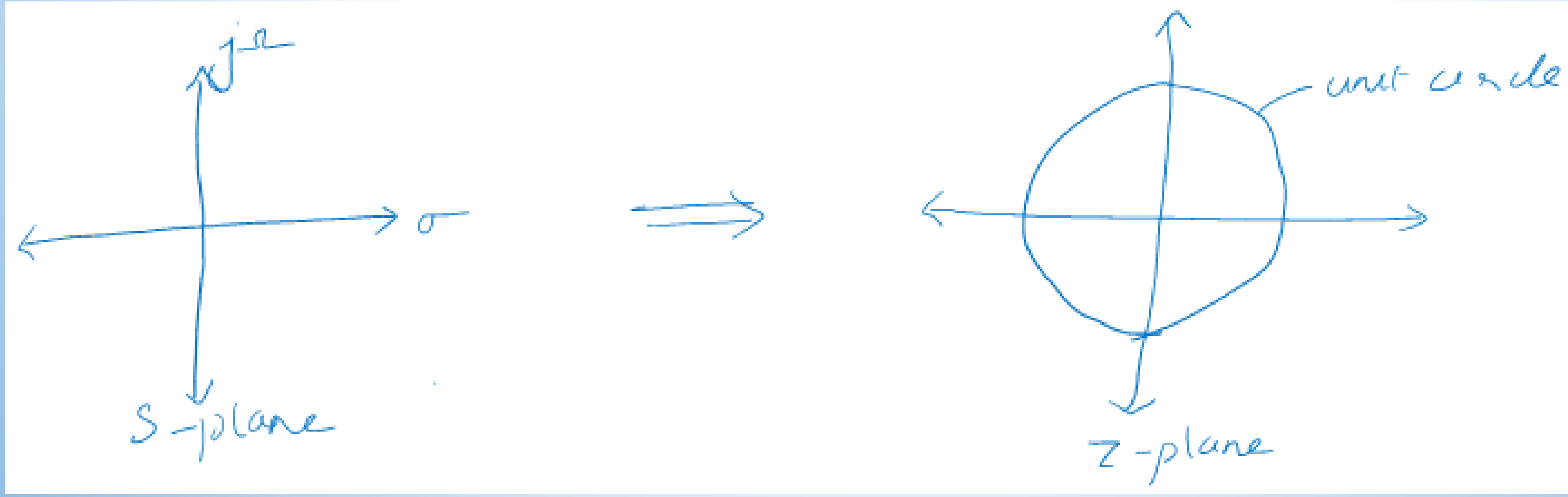
$$H_a(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt$$

3) Using LCCDE

$$\sum_{k=0}^{N} \alpha_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} \beta_k \frac{d^k x(t)}{dt^k}$$

Properties for effective conversion from Analog to Digital (mapping from s-plane to z-plane)

- Direct relationship between the frequency variable in the s-domain and z-domain
  - jΩ axis in s-plane should map into the unit circle in z-plane
- LHP of s-plane should map into the inside of the unit circle in z-plane.
  - A stable analog filter can be converted to a stable digital filter
  - Analog LTI system is stable if all its poles lie in the left half of s-plane



Linear phase filter must have a system function that satisfies the condition:

$$H(z) = \pm z^{-N} H(z^{-1})$$

Symmetric and antisymmetric response

$$h(n) = \pm h(N-n)$$

$\downarrow ZT$

$$H(z) = \sum_{n=0}^{N} h(n) z^{-n} = h(0) + h(1) z^{-1} + \dots + h_N z^{-N}$$

$$H(z^{-1}) = \sum_{k=0}^{N} h(k) z^{k} = h(0) + h(1) z + \dots + h(N) z^{N}$$

$$z^{-N} H(z^{-1}) = h(0) z^{-N} + h(1) z^{-N+1} + \dots h(N-1) z^{-1} + h(N) z^{0}$$

If condition $H(z) = \pm z^{-N} H(z^{-1})$ is met,

$h(0) = h(N)$, $h(1) = h(N-1)$, ...

This condition implies the roots of polynomial $H(z)$ are identical to the roots of polynomial $H(z^{-1})$

- Roots of $H(z)$ must occur in reciprocal pairs.

If z_1 is a root of $H(z)$, $\frac{1}{z_1}$ is also a root

For every pole inside a unit circle, there will be a mirror image pole outside the unit circle.

Therefore the system is unstable.

A causal and stable IIR system cannot have linear phase.

Design of IIR filters:

- Two steps
- First design an analog filter – Butterworth, Chebyshev or Elliptic
- Convert this analog filter to digital using
  - Impulse invariance technique
  - Bilinear transformation
  - Matched Z – transform

Impulse invariance transformation:

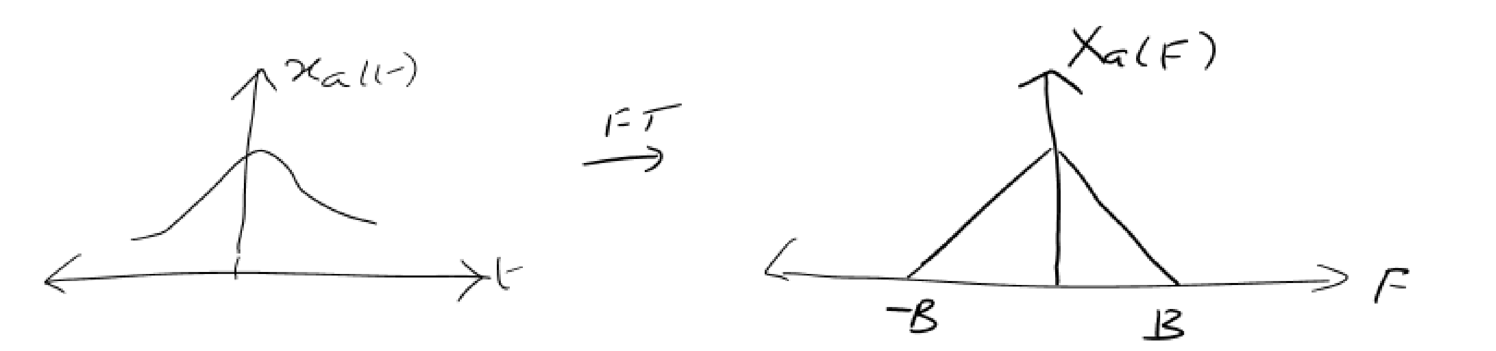
We design an IIR filter having unit sample response

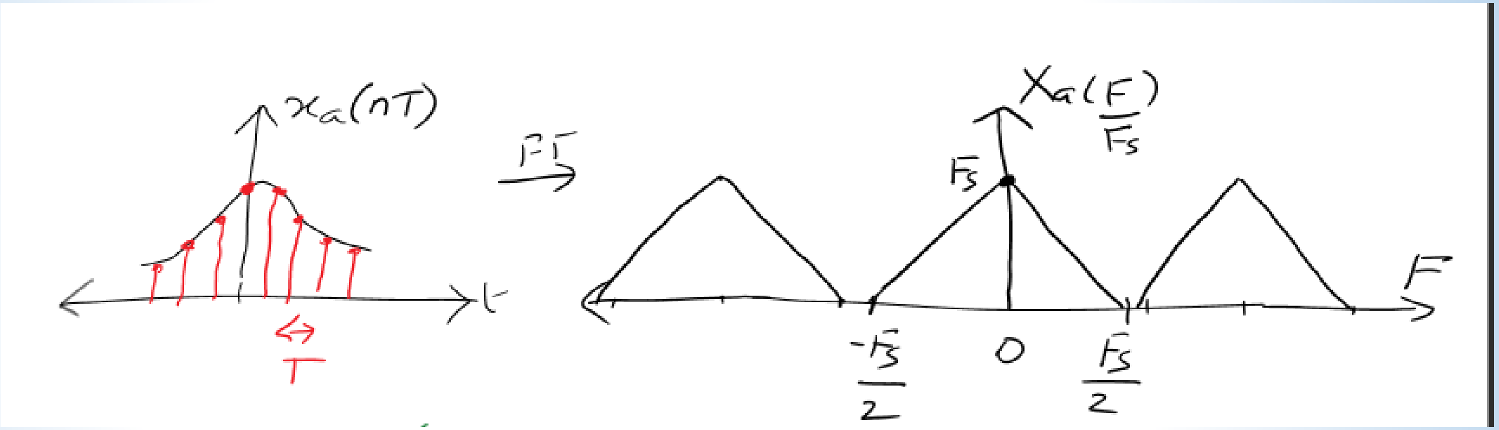
$h(n)$ = sampled version of IR of analog filter

$h(n) \equiv h(nT) \quad , n = 0, 1, 2, \ldots$

T = sampling interval

When a CT signal having a spectrum $X_a(F)$ is sampled at a rate $F_s = \frac{1}{T}$ samples per sec, the spectrum of the sampled signal $x_a(nT)$ is the periodic repetition of the scaled spectrum $F_s X_a(F)$, with period F_s.





$$\therefore X(f) = F_s \sum_{k=-\infty}^{\infty} X_a\left((f-k)F_s\right), \quad f = \frac{F}{F_s}$$

$$= F_s\left[\cdots X_a\left(\left(\frac{F}{F_s}-1\right)F_s\right) + X_a\left(\left(\frac{F}{F_s}\right)F_s\right) + X_a\left(\left(\frac{F}{F_s}+1\right)F_s\right) + \cdots\right]$$

$$= F_s\left[\cdots X_a(F-F_s) + X_a(F) + X_a(F+F_s) + \cdots\right]$$

Aliasing occurs if the sampling rate
$F_s < 2F_{max}$ of $X_a(F)$
$T > 2T_{max}$

As sampling interval T increases,
the copies of the spectrum move closer
spectral overlap → aliasing

With respect to sampling the IR of analog filter
with frequency response $H_a(F)$,
the digital filter with unit-sample response
$h(n) \equiv h_a(nT)$ has the frequency response

$$H(f) = F_s \sum_{k=-\infty}^{\infty} H_a(f-k)F_s$$

$$H(f) = F_s \sum_{k=-\infty}^{\infty} H_a[(f-k)F_s]$$

$$H(\omega) = F_s \sum_{k=-\infty}^{\infty} H_a(\omega - 2\pi k)F_s]$$

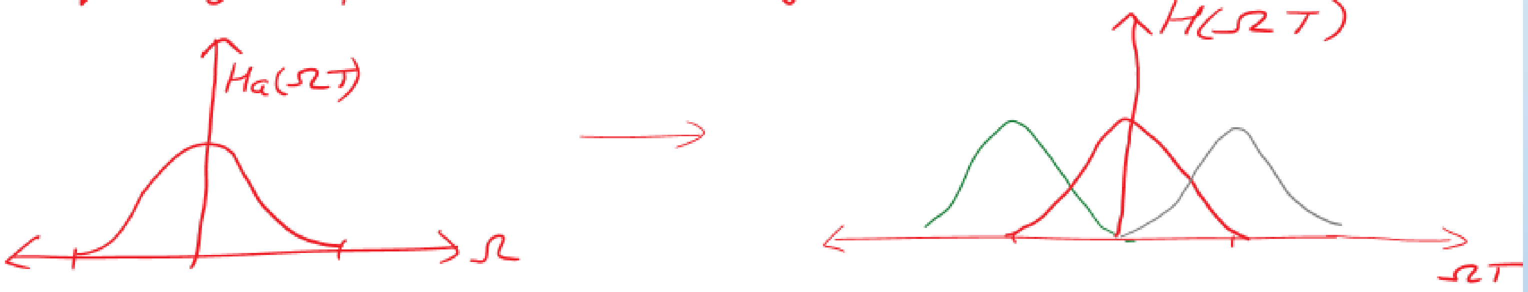
ω is periodic with 2π

$\downarrow \Omega T$

$$H(\Omega T) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_a(\Omega T - 2\pi k)\frac{1}{T}]$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} H_a\left(\Omega - \frac{2\pi k}{T}\right)$$

Frequency response of analog filter:



Effects of aliasing can be minimized if the sampling interval T is selected sufficiently small.

Mapping of points between z-plane & s-plane implied by the sampling process:

Relation between z-transform of $h(n)$ and Laplace transform of $h_a(t)$:

$$H_a(s) = \int_0^{\infty} h_a(t)\, e^{-st}\, dt$$

$$H_a(s)\Big|_{\text{sampled at } nT} = \int_0^{\infty} \sum_{n=0}^{\infty} h(n)\, \delta(t-nT)\, e^{-st}\, dt$$

$$= \sum_{n=0}^{\infty} h(n) \int_0^{\infty} \delta(t-nT)\, e^{-st}\, dt$$

$$= \sum_{n=0}^{\infty} h(n)\, e^{-snT} \qquad \left[\because \mathcal{L}(\delta(t-a)) = e^{-sa}\right]$$

$$= H(z)\Big|_{z=e^{sT}} \qquad \left[H(z) = \sum_{n=0}^{\infty} h(n)\, z^{-n}\right]$$

Frequency response of digital filter with unit sample resp. $h_a(nT)$:

$$H(\Omega T) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_a\left(\Omega - \frac{2\pi k}{T}\right)$$

$$\Rightarrow H(z)\Big|_{z=e^{sT}} = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_a\left(s - j\frac{2\pi k}{T}\right) \Leftarrow \text{Relation}$$

Mapping points from s-plane to z-plane:

$$z = e^{sT}$$

$$re^{j\omega} = e^{(\sigma + j\Omega)T} = e^{\sigma T} \cdot e^{j\Omega T}$$

$$\therefore \; r = e^{\sigma T}$$

$$\omega = \Omega T$$

$\sigma < 0, \; 0 < r < 1$ ⇒ LHP in s-plane is mapped inside unit ○ in z-plane

$\sigma > 0, \; r > 1$ ⇒ RHP in s-plane is mapped outside the unit ○ in z-plane

$\sigma = 0, \; r = 0$ ⇒ $j\Omega$ axis is mapped onto the unit ○ in z-plane.

Mapping of $j\Omega$ axis onto the unit circle is not one-to-one

Mapping $\omega = \Omega T$,

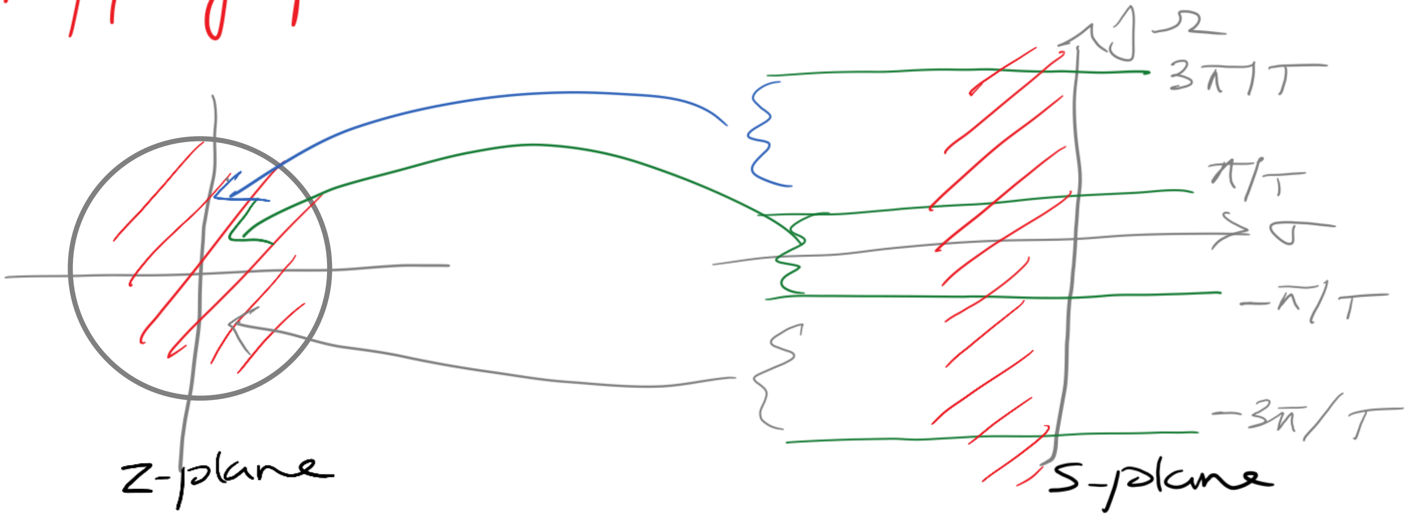
Analog freq interval maps into freq interval in digital

$$-\frac{\pi}{T} \le \Omega \le \frac{\pi}{T} \qquad\qquad -\pi \le \omega \le \pi$$

$$\frac{\pi}{T} \le \Omega \le \frac{3\pi}{T} \qquad\qquad -\pi \le \omega \le \pi$$

$$(2k-1)\frac{\pi}{T} \le \Omega \le (2k+1)\frac{\pi}{T} \qquad\qquad -\pi \le \omega \le \pi, \quad k = \text{integer}$$

$\therefore$ Mapping from Ω to ω is many-to-one



Let us express the system function of analog filter in partial fraction form.

Assuming distinct poles of analog filter:

$$H_a(s) = \sum_{k=1}^{N} \frac{C_k}{s-p_k}$$

coefficients in partial fraction

distinct poles

$$h_a(t) = \sum_{k=1}^{N} C_k e^{p_k t} \quad , t \geq 0$$

$$\left[\mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}\right]$$

Sampling $h_a(t)$ periodically at $t = nT$,

$$h(n) = h_a(nT)$$

$$= \sum_{k=1}^{N} C_k e^{p_k nT}$$

System function of resulting digital IIR filter,

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n}$$, substitute $h(n)$

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} \sum_{k=1}^{N} C_k e^{p_k n T} \cdot z^{-n}$$

$$\boxed{= \sum_{k=1}^{N} C_k \sum_{n=0}^{\infty} \left(e^{p_k T} z^{-1}\right)^{n}}$$

$$\frac{1}{1 - e^{p_k T} \cdot z^{-1}}$$

Geometric Series exp

$$\sum_{i=0}^{\infty} a_i r^{i} = \frac{a_1}{1-r}$$

$$H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}}$$

We observe that the digital filter has poles at

$$z_k = e^{p_k T}, \qquad k = 1, 2, \ldots, N$$

Impulse invariance transformation

$$H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}}$$

We observe that the digital filter has poles at

$$z_k = e^{p_k T}, \qquad k = 1, 2, \ldots, N$$

This equation holds for IIR filter having distinct poles.
Due to presence of aliasing, impulse invariance is suitable for LPF and BPF only.

Convert the analog filter with system function

$$H_a(s) = \frac{s + 0.1}{(s + 0.1)^2 + 9}$$

into a digital IIR filter by means of the impulse invariance method.

$$H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}}$$

Solution. We note that the analog filter has a zero at $s = -0.1$ and a pair of complex-conjugate poles at

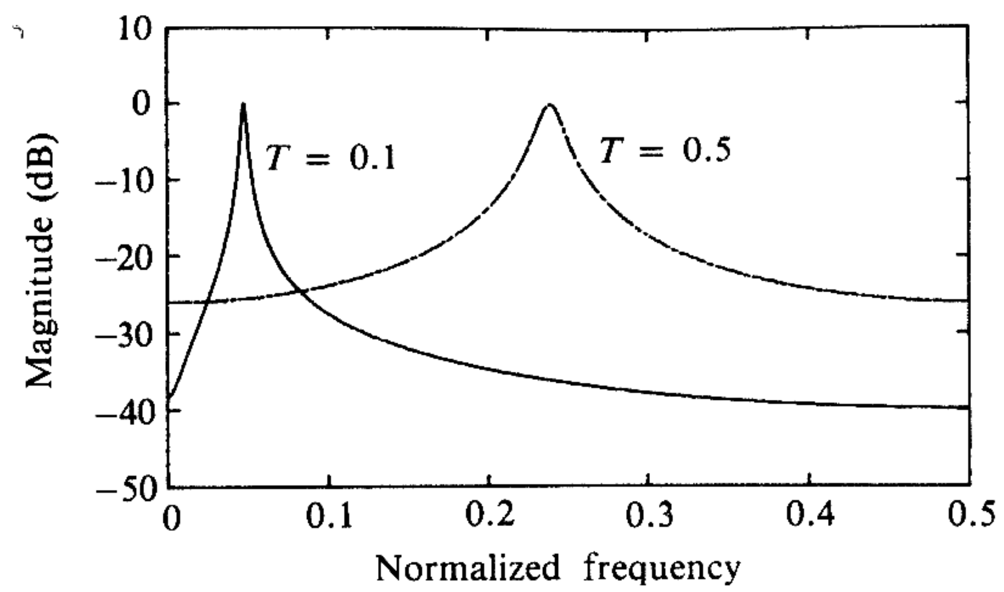
$$p_k = -0.1 \pm j3$$

from the partial-fraction expansion of $H_a(s)$

$$H(s) = \frac{\frac{1}{2}}{s + 0.1 - j3} + \frac{\frac{1}{2}}{s + 0.1 + j3}$$

$$H(z) = \frac{\frac{1}{2}}{1 - e^{-0.1T} e^{j3T} z^{-1}} + \frac{\frac{1}{2}}{1 - e^{-0.1T} e^{-j3T} z^{-1}}$$

$$H(z) = \frac{1 - (e^{-0.1T}\cos 3T)z^{-1}}{1 - (2e^{-0.1T}\cos 3T)z^{-1} + e^{-0.2T}z^{-2}}$$



We note that aliasing is significantly more prevalent when $T = 0.5$ than when $T = 0.1$. Also, note the shift of the resonant frequency as T changes.

Remember this

$$\frac{s+a}{(s+a)^2+b^2} \Rightarrow \frac{1-e^{-aT}\cos bT\, z^{-1}}{1-2e^{-aT}\cos bT\, z^{-1}+e^{-2aT}z^{-2}}$$

$$\frac{b}{(s+a)^2+b^2} \Rightarrow \frac{e^{-aT}\sin bT\, z^{-1}}{1-2e^{-aT}\cos bT\, z^{-1}+e^{-2aT}z^{-2}}$$

Convert the analog filter in to its equivalent digital filter whose system function is given by $H(s) = \frac{s+0.4}{s^2+0.8s+25.16}$ using impulse invariance technique. Assume sampling frequency of 10Hz.

4B)

$$H(s) = \frac{s+0.4}{(s+0.4)^2 + 5^2}$$

$$T = \frac{1}{f_s} = \frac{1}{10} = 0.1$$

$$\frac{s+a}{(s+a)^2+b^2} \Rightarrow \frac{1 - e^{-aT}\cos bT\, z^{-1}}{1 - 2e^{-aT}\cos bT\, z^{-1} + e^{-2aT} z^{-2}}$$

(or)
They can solve by Partial fraction method (1M)

$$H(z) = \frac{1 - e^{-0.4\times0.1}\cos(0.5)\, z^{-1}}{1 - 2e^{-0.4\times0.1}\cos(0.5) z^{-1} + e^{-2\times0.4\times0.1} z^{-2}}$$

$$H(z) = \frac{1 - 0.843\, z^{-1}}{1 - 1.686 z^{-1} + 0.923 z^{-2}}$$ (2M)


Thank
you