Linear Filtering based on DFT

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Use of the DFT in Linear Filtering

- Product of 2 DFTs is equivalent to circular convolution
- Not a linear convolution
- But output of an LTI system is y[n] = x(n) * h[n] (Linear convolution of input with impulse response)

Suppose that we have a finite-duration sequence $x(n)$ of length L which excites an FIR filter of length M.

$$x(n) = 0, \qquad n < 0 \text{ and } n \geq L$$

$$h(n) = 0, \qquad n < 0 \text{ and } n \geq M$$

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k)$$

where $h(n)$ is the impulse response of the FIR filter.

Since $h(n)$ and $x(n)$ are finite-duration sequences, their convolution is also finite in duration. In fact, the duration of $y(n)$ is $L + M - 1$.

- In frequency domain $Y(\omega) = X(\omega)H(\omega)$
- If y(n) is represented by spectrum Y(ω), the number of samples must be equal or exceed L+M-1
- Therefore, DFT of size N ≥ L+M-1 is required
- DFT of length N is increased to L+M-1 by zero padding
- Thus, DFT can be used for linear filtering

By means of the DFT and IDFT, determine the response of the FIR filter with impulse response

$h(n) = \{\underset{\uparrow}{1}, 2, 3\}$ to the input sequence $x(n) = \{\underset{\uparrow}{1}, 2, 2, 1\}$

- L = 3, M = 4, therefore N = 3+4-1 = 6 . We can take 8-point DFT for the convenance (Fast algorithm exists – Next chapter)

$$X(k) = \sum_{n=0}^{7} x(n)e^{-j2\pi kn/8}$$

$$= 1 + 2e^{-j\pi k/4} + 2e^{-j\pi k/2} + e^{-j3\pi k/4}, \qquad k = 0, 1, \ldots, 7$$

$$X(0) = 6, \qquad X(1) = \frac{2+\sqrt{2}}{2} - j\left(\frac{4+3\sqrt{2}}{2}\right)$$

$$X(2) = -1 - j, \qquad X(3) = \frac{2-\sqrt{2}}{2} + j\left(\frac{4-3\sqrt{2}}{2}\right)$$

$$X(4) = 0, \qquad X(5) = \frac{2-\sqrt{2}}{2} - j\left(\frac{4-3\sqrt{2}}{2}\right)$$

$$X(6) = -1 + j, \qquad X(7) = \frac{2+\sqrt{2}}{2} + j\left(\frac{4+3\sqrt{2}}{2}\right)$$

Homework: Try this using matrix method

Matrix method:

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^0 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W^6 & W_8^7 \\ W_8^0 & W_8^2 & W_8^4 & W_8^6 & W_8^8 & W_8^{10} & W_8^{12} & W_8^{14} \\ W_8^0 & W_8^3 & W_8^6 & W_8^9 & W_8^{12} & W_8^{15} & W_8^{18} & W_8^{21} \\ W_8^0 & W_8^4 & W_8^8 & W_8^{12} & W_8^{16} & W_8^{20} & W_8^{24} & W_8^{28} \\ W_8^0 & W_8^5 & W_8^{10} & W_8^{15} & W_8^{20} & W_8^{25} & W_8^{30} & W_8^{35} \\ W_8^0 & W_8^6 & W_8^{12} & W_8^{18} & W_8^{24} & W_8^{30} & W_8^{36} & W_8^{42} \\ W_8^0 & W_8^7 & W_8^{14} & W_8^{21} & W_8^{28} & W_8^{35} & W_8^{42} & W_8^{49} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix}$$

$$W_8^1 = \left(e^{-j\frac{2\pi}{8}}\right)^1 = \underline{\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1-j}{\sqrt{2}} & -j & \frac{-(1+j)}{\sqrt{2}} & -1 & \frac{-(1-j)}{\sqrt{2}} & j & \frac{1+j}{\sqrt{2}} \\ 1 & -j & -1 & j & 1 & -j & -1 & j \\ 1 & \frac{-(1+j)}{\sqrt{2}} & j & \frac{1-j}{\sqrt{2}} & -1 & \frac{1+j}{\sqrt{2}} & -j & \frac{-(1-j)}{\sqrt{2}} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & \frac{-(1-j)}{\sqrt{2}} & -j & \frac{1+j}{\sqrt{2}} & -1 & \frac{1-j}{\sqrt{2}} & j & \frac{-(1+j)}{\sqrt{2}} \\ 1 & j & -1 & -j & 1 & j & -1 & -j \\ 1 & \frac{1+j}{\sqrt{2}} & j & \frac{-(1-j)}{\sqrt{2}} & -1 & \frac{-(1+j)}{\sqrt{2}} & -j & \frac{(1-j)}{\sqrt{2}} \end{bmatrix}$$

Problem contd from slide 4: h(n) = {1,2,3}

$$H(k) = \sum_{n=0}^{7} h(n) e^{-j2\pi kn/8}$$

$$= 1 + 2e^{-j\pi k/4} + 3e^{-j\pi k/2}$$

$$H(0) = 6, \qquad H(1) = 1 + \sqrt{2} - j\left(3 + \sqrt{2}\right), \qquad H(2) = -2 - j2$$

$$H(3) = 1 - \sqrt{2} + j\left(3 - \sqrt{2}\right), \qquad H(4) = 2$$

$$H(5) = 1 - \sqrt{2} - j\left(3 - \sqrt{2}\right), \qquad H(6) = -2 + j2$$

$$H(7) = 1 + \sqrt{2} + j\left(3 + \sqrt{2}\right)$$

The product of these two DFTs yields $Y(k)$, which is

$$Y(0) = 36, \qquad Y(1) = -14.07 - j17.48, \qquad Y(2) = j4, \qquad Y(3) = 0.07 + j0.515$$

$$Y(4) = 0, \qquad Y(5) = 0.07 - j0.515, \qquad Y(6) = -j4, \qquad Y(7) = -14.07 + j17.48$$

Finally, the eight-point IDFT is

$$y(n) = \sum_{k=0}^{7} Y(k) e^{j2\pi kn/8}, \qquad n = 0, 1, \ldots, 7$$

This computation yields the result $y(n) = \{\underset{\uparrow}{1}, 4, 9, 11, 8, 3, 0, 0\}$

Observe – last 2 digits are zero

Also, note: Same result can be obtained using circular and linear convolutions


Thank you