# Review of Z transform and DFT

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# Transforms:

- Continuous time – Laplace transform
- Stable and periodic – Fourier Series
- Stable and non-periodic – Fourier Transform

- Discrete time – Z transform
- Stable and periodic – Discrete Time Fourier Series
- Stable and non-periodic – Discrete Time Fourier Transform

# Why z-Transform?

- A generalization of Fourier transform
- Why generalize it?
  - FT does not converge on all sequence
  - Simplifies analysis – convolution becomes multiplication
  - Characterizes the LTI System – Transfer function
  - Its response to various signals – locating poles and zeros

# Definition

- The $z$-transform of sequence $x(n)$ is defined by

$$X(z)=\sum_{n=-\infty}^{\infty}x(n)z^{-n}$$

- Let $z = e^{-j\omega}$.

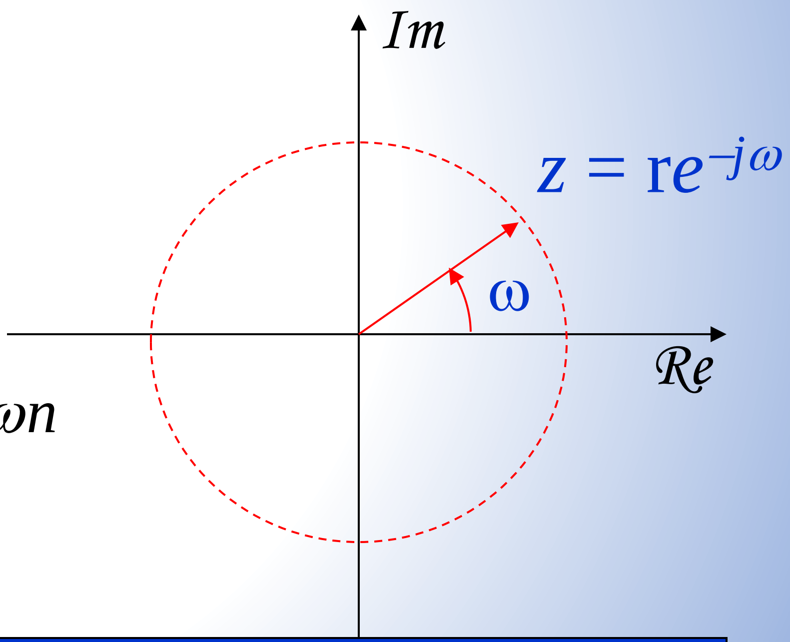
$$X(e^{j\omega})=\sum_{n=-\infty}^{\infty}x(n)e^{-j\omega n}$$

*Fourier Transform*

# z-Plane

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$




Fourier Transform is to *evaluate z-transform on a unit circle* (r = 1)

# Definition

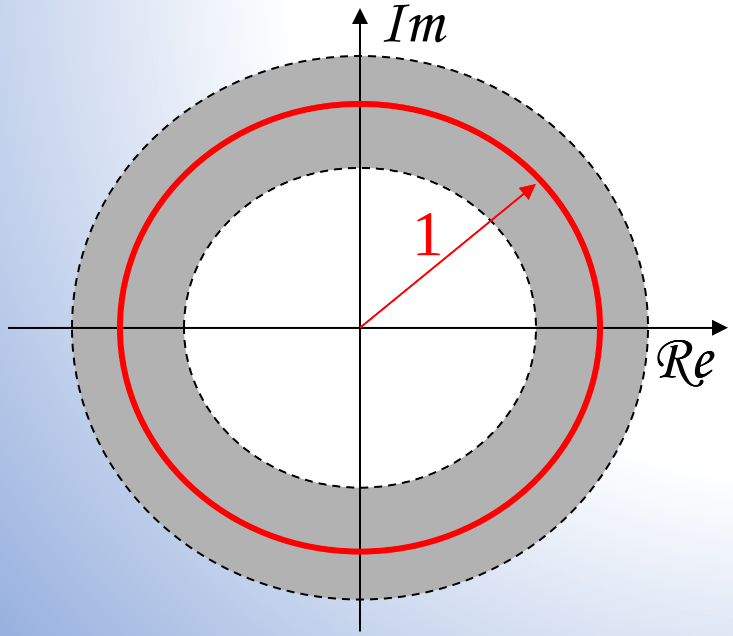
- Give a sequence, the set of values of $z$ for which the $z$-transform converges, i.e., $|X(z)|<\infty$, is called the region of convergence.

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x(n) z^{-n} \right| = \sum_{n=-\infty}^{\infty} |x(n)||z|^{-n} < \infty$$

ROC is centered on origin and consists of a set of rings.

# Stable Systems

- A stable system requires that its Fourier transform is uniformly convergent.




- Fact: Fourier transform is to evaluate $z$-transform on a unit circle.
- A stable system requires the ROC of $z$-transform to include the unit circle.

# Review of Fourier representation of discrete signals

$$x[n] \longleftrightarrow X(e^{j\omega})$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$ – Analysis Equation – DTFT

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$ – Synthesis Equation – Inverse DTFT

- The DTFT and inverse DTFT are not symmetric. One is integration over a finite interval (2π), and the other is summation over infinite terms
- The signal, x[n] is aperiodic, and hence, the transform is a continuous function of frequency
- Not practical for (real-time) computation on a digital computer.

# Examples

1) $x[n] = \delta[n]$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n} = 1$$

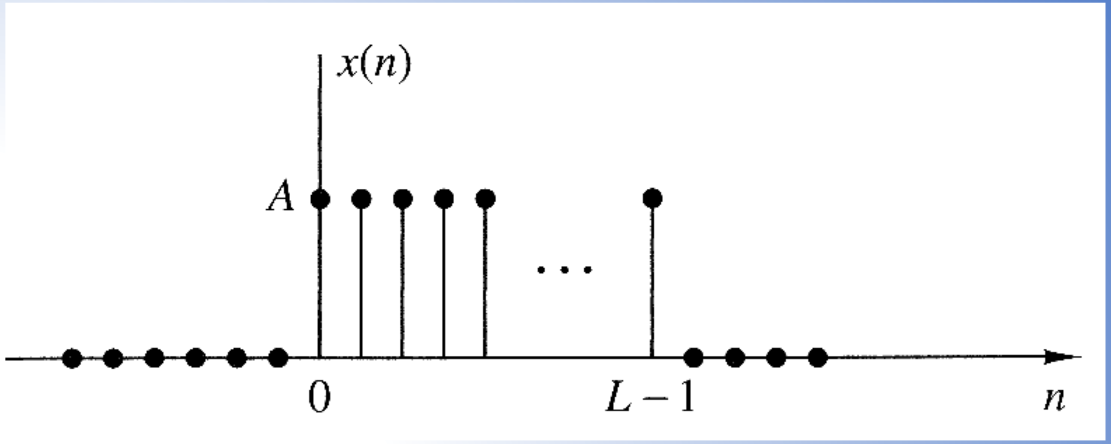
2) $x[n] = \delta[n - n_0]$ - shifted unit sample

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n - n_0] e^{-j\omega n} = e^{-j\omega n_0}$$

Same amplitude (=1) as above,
but with a *linear* phase $-\omega n_0$

Determine the Fourier transform of the sequence

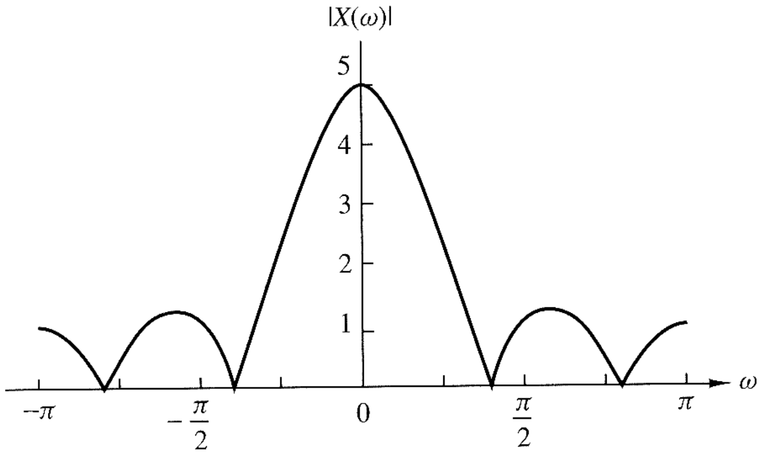
$$x(n) = \begin{cases} A, & 0 \le n \le L-1 \\ 0, & \text{otherwise} \end{cases}$$




$$X(\omega) = \sum_{n=0}^{L-1} A e^{-j\omega n}$$

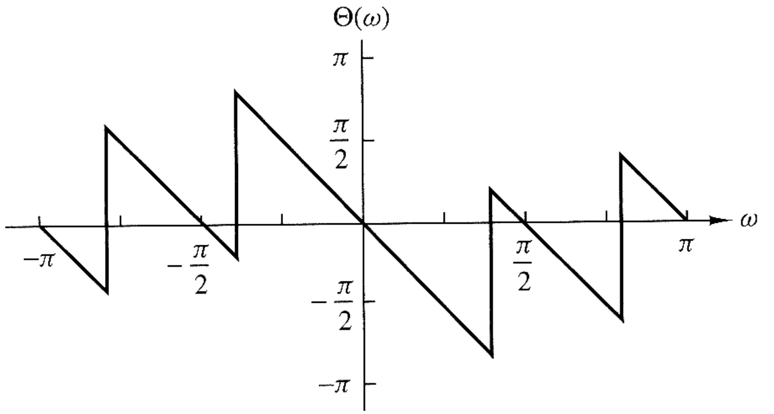
$$= A \frac{1 - e^{-j\omega L}}{1 - e^{-j\omega}}$$

$$= A e^{-j(\omega/2)(L-1)} \frac{\sin(\omega L/2)}{\sin(\omega/2)}$$




$$|X(\omega)| = \begin{cases} |A|L, & \omega = 0 \\ |A| \left| \dfrac{\sin(\omega L/2)}{\sin(\omega/2)} \right|, & \text{otherwise} \end{cases}$$

$$\measuredangle X(\omega) = \measuredangle A - \frac{\omega}{2}(L-1) + \measuredangle \frac{\sin(\omega L/2)}{\sin(\omega/2)}$$



# Convolution Property

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$y[n] = h[n] * x[n]$$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

$$H(e^{j\omega}) = \text{DTFT of } h[n]$$

$$\text{Frequency response} = \text{DTFT of the unit sample response}$$

# Multiplication Property

$$y[n] = x_1[n] \cdot x_2[n]$$

$$\begin{aligned} Y(e^{j\omega}) &= \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta \\ &= \frac{1}{2\pi} X_1(e^{j\omega}) \otimes X_2(e^{j\omega}) \\ &\hookrightarrow \text{ Periodic Convolution} \end{aligned}$$

# Find DTFT of $x[n] = (0.5)^n u[n-4]$

$$a^n u(n) \xleftrightarrow{DTFT} \frac{1}{1-ae^{-j\omega}}, \quad |a|<1$$

Time shifting

$$x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$$

$$x(n-n_0) \xleftrightarrow{DTFT} e^{-j\omega n_0} X(e^{j\omega})$$

$$(0.5)^{n-4} u(n-4) \xleftrightarrow{DTFT} \frac{e^{-j4\omega}}{1-0.5e^{-j\omega}}$$

$$X(e^{j\omega}) = \frac{(0.5)^4 e^{-j4\omega}}{1-0.5e^{-j\omega}}$$

Magnitude Spectrum

$$|X(e^{j\omega})| = \frac{(0.5)^4}{\sqrt{(1-0.5\cos\omega)^2+(0.5\sin\omega)^2}}$$

$|e^{-j4\omega}|=1$

Phase Spectrum

$$\angle X(e^{j\omega}) = -4\omega - \tan^{-1}\left[\frac{0.5\sin\omega}{1-0.5\cos\omega}\right]$$


Thank
you

# Introduction to
# Discrete Fourier Transform
# Frequency domain sampling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# Review of Fourier representation of discrete signals

$$x[n] \longleftrightarrow X(e^{j\omega})$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

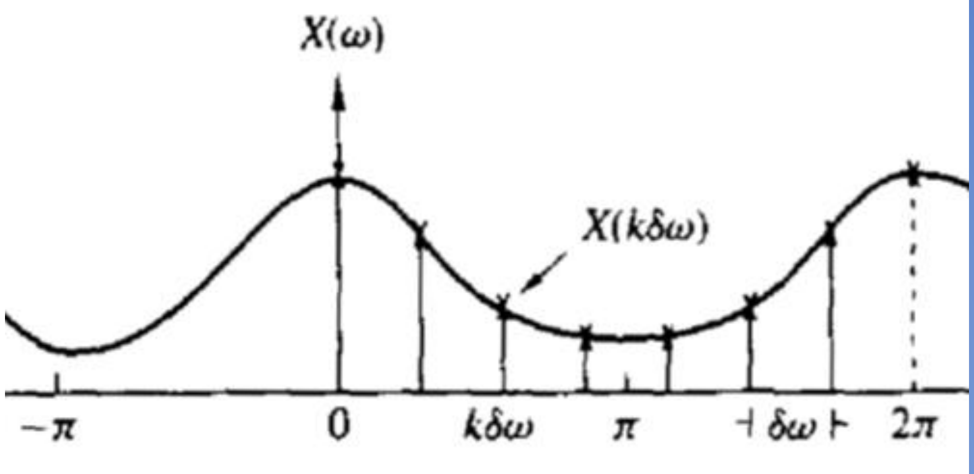
– Analysis Equation
– DTFT

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$$

– Synthesis Equation
– Inverse DTFT

- The DTFT and inverse DTFT are not symmetric. One is integration over a finite interval (2π), and the other is summation over infinite terms
- The signal, x[n] is aperiodic, and hence, the transform is a continuous function of frequency
- Not practical for (real-time) computation on a digital computer
- Go for Discrete Fourier Transform

# Frequency Domain Sampling




- Consider an aperiodic signal $x(n)$ finite duration signal with FT

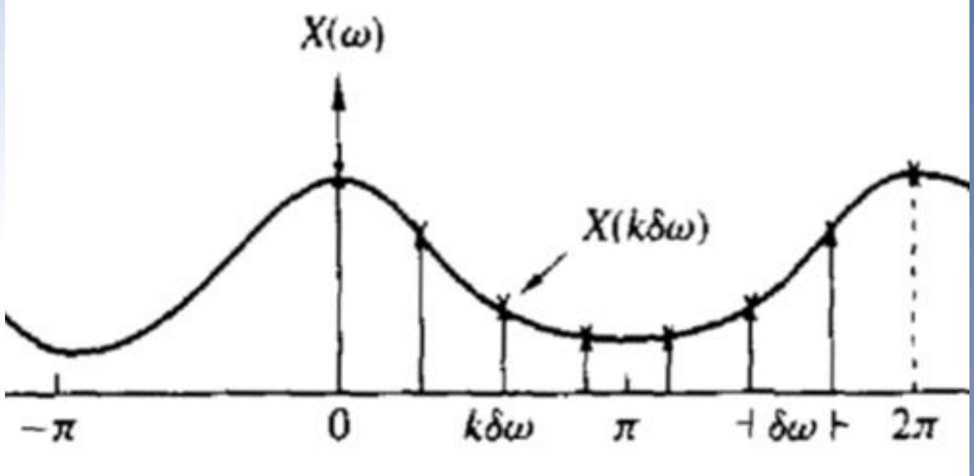
$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \qquad (1)$$

- Suppose we sample $X(\omega)$ periodically in frequency at spacing $\delta\omega$ radians between successive samples.

- Since $X(\omega)$ is periodic with period $2\pi$, therefore only samples in the fundamental frequency range are required.
- Let we take *N* equidistant samples ➔ $\delta\omega = \dfrac{2\pi}{N}$

**This derivation is not there for the exam**

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \qquad (1)$$




- Now calculate Eq. (1) at $\omega_k = \dfrac{2\pi k}{N}$, i.e.,

$$X\left(\frac{2\pi k}{N}\right) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\frac{2\pi kn}{N}} \qquad (2)$$

- Summation in Eq. (2) can be written as

$$X\left(\frac{2\pi k}{N}\right) = \cdots + \sum_{n=-N}^{-1} x(n) e^{-j\frac{2\pi kn}{N}} + \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi kn}{N}}$$

$$+ \sum_{n=N}^{2N-1} x(n) e^{-j\frac{2\pi kn}{N}} + \cdots$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n) e^{-j\frac{2\pi kn}{N}} \qquad (3)$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n) e^{-j\frac{2\pi kn}{N}} \qquad (3)$$

- If we change the index in the inner summation from $n \rightarrow n - lN$ & interchanging the order of summation, we have,

$$X\left(\frac{2\pi k}{N}\right) = \sum_{n=0}^{N-1} \left[\sum_{l=-\infty}^{\infty} x(n - lN)\right] e^{-j\frac{2\pi kn}{N}} \qquad (4)$$

For $k = 0, 1, \ldots, N-1$

- Here, signal

$$x_p(n) = x(n - lN)$$

is a periodic sequence with fundamental period $N$

$$X\left(\frac{2\pi k}{N}\right) = \sum_{n=0}^{N-1} x_p(n) e^{-j\frac{2\pi kn}{N}}$$

For $k = 0, 1, \ldots, N-1$

Discrete Fourier Transform (DFT)

$$X\left(\frac{2\pi k}{N}\right)=\sum_{n=0}^{N-1}x_p(n)\,e^{-j\frac{2\pi kn}{N}} \qquad (4)$$

- The $x_p(n)$ can be expresses using FS as,

$$x_p(n)=\sum_{k=0}^{N-1}c_k\,e^{j\frac{2\pi kn}{N}} \quad n=0,1,\cdots,N-1 \qquad (5)$$

where Fourier coefficient $c_k$ is given as,

$$c_k=\frac{1}{N}\sum_{n=0}^{N-1}x_p(n)\,e^{-j\frac{2\pi kn}{N}} \quad k=0,1,\cdots,N-1 \qquad (6)$$

By comparing Eq. (4) & (6) we observe,

$$c_k=\frac{1}{N}X\left(\frac{2\pi k}{N}\right) \qquad (7)$$

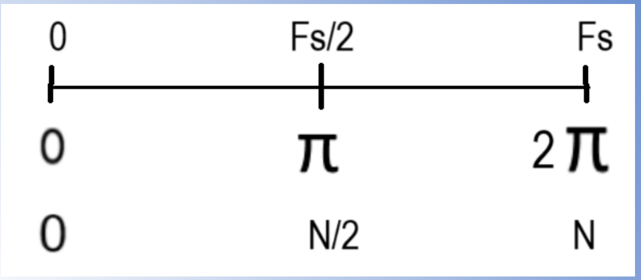
**Substitute Eq. (7) in (5)**

$$x_p(n)=\frac{1}{N}\sum_{k=0}^{N-1}X\left(\frac{2\pi k}{N}\right)e^{j\frac{2\pi kn}{N}} \quad n=0,1,\cdots,N-1 \qquad (8)$$

Inverse Discrete Fourier Transform (IDFT)

## Discrete Fourier Transform (DFT)

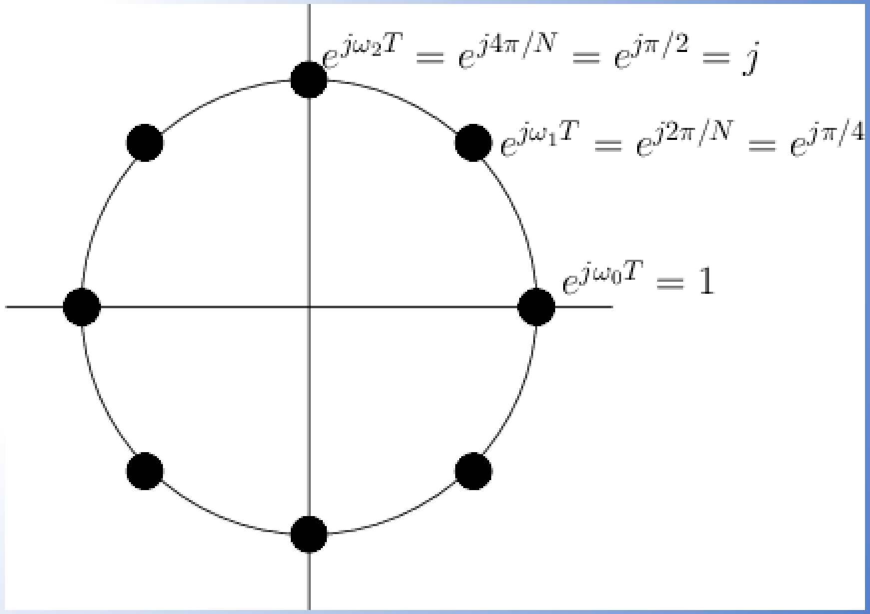
$$X\left(\frac{2\pi k}{N}\right)=\sum_{n=0}^{N-1} x_p(n)\, e^{-j\frac{2\pi kn}{N}} \quad \text{For } k = 0,\ 1,\ \ldots,\ N-1$$




## Inverse Discrete Fourier Transform (IDFT)

Fs = sampling frequency

$$x_p(n)=\frac{1}{N}\sum_{k=0}^{N-1} X\left(\frac{2\pi k}{N}\right) e^{j\frac{2\pi kn}{N}} \quad n=0,1,\cdots,N-1$$




$$\text{DFT:} \quad X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, \qquad k = 0, 1, 2, \ldots, N-1$$

$$\text{IDFT:} \quad x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N}, \qquad n = 0, 1, 2, \ldots, N-1$$

# Properties of the DFT

$N$-point DFT pair $x(n)$ and $X(k)$ is $x(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X(k)$

**Periodicity.** If $x(n)$ and X(k) are an $N$-point DFT pair, then

$$x(n+N) = x(n) \qquad \text{for all } n$$

$$X(k+N) = X(k) \qquad \text{for all } k$$

**Linearity.** If

$$x_1(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_1(k)$$

and

$$x_2(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_2(k)$$

then for any real-valued or complex-valued constants $a_1$ and $a_2$,

$$a_1x_1(n) + a_2x_2(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} a_1X_1(k) + a_2X_2(k)$$

**TABLE 7.1** Symmetry Properties of the DFT

| $N$-Point Sequence $x(n)$, $0 \le n \le N-1$ | $N$-Point DFT |
|---|---|
| $x(n)$ | $X(k)$ |
| $x^*(n)$ | $X^*(N-k)$ |
| $x^*(N-n)$ | $X^*(k)$ |
| $x_R(n)$ | $X_{ce}(k) = \frac{1}{2}[X(k) + X^*(N-k)]$ |
| $jX_I(n)$ | $X_{co}(k) = \frac{1}{2}[X(k) - X^*(N-k)]$ |
| $x_{ce}(n) = \frac{1}{2}[x(n) + x^*(N-n)]$ | $X_R(k)$ |
| $x_{co}(n) = \frac{1}{2}[x(n) - x^*(N-n)]$ | $jX_I(k)$ |
| **Real Signals** | |
| Any real signal | $X(k) = X^*(N-k)$ |
| $x(n)$ | $X_R(k) = X_R(N-k)$ |
| | $X_I(k) = -X_I(N-k)$ |
| | $\lvert X(k)\rvert = \lvert X(N-k)\rvert$ |
| | $\angle X(k) = -\angle X(N-k)$ |
| $x_{ce}(n) = \frac{1}{2}[x(n) + x(N-n)]$ | $X_R(k)$ |
| $x_{co}(n) = \frac{1}{2}[x(n) - x(N-n)]$ | $jX_I(k)$ |

Next slide contains the proof

# Example: Symmetry property

Any real signal $X(k) = X^*(N-k)$

$$X^*(k) = \left[\sum_{n=0}^{N-1} x(n)\ e^{-j2\pi kn/N}\right]^*$$

$$= \left[\sum_{n=0}^{N-1} x(n)\ e^{j2\pi kn/N} . 1\right] \qquad \{x^*(n) = x(n) \text{ for real sequence}\}$$

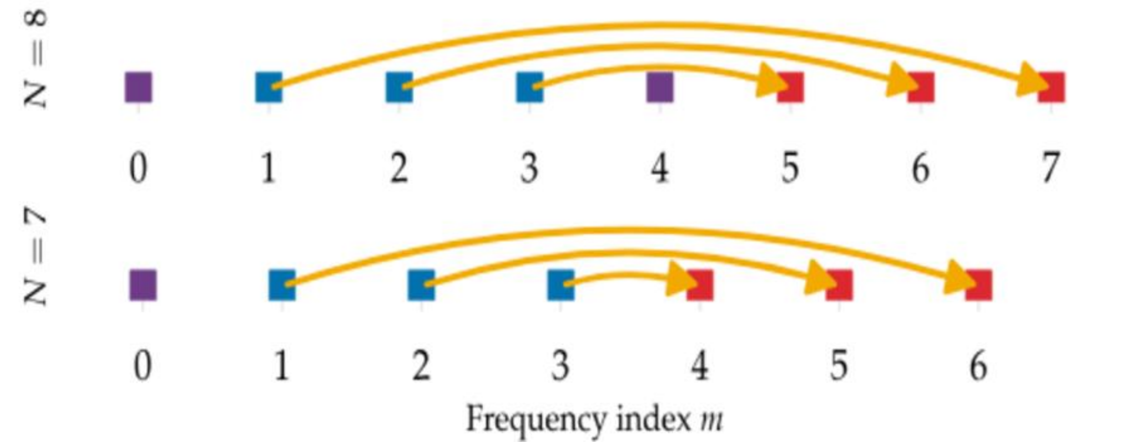
$$= \left[\sum_{n=0}^{N-1} x(n)\ e^{j2\pi kn/N} . e^{-j2\pi n}\right] \qquad \{\because e^{-j2\pi n} = 1\}$$

$$= \left[\sum_{n=0}^{N-1} x(n)\ e^{j2\pi kn/N} . e^{-j2\pi nN/N}\right]$$

$$= \left[\sum_{n=0}^{N-1} x(n)\ e^{-j2\pi (N-k)n/N}\right]$$

$$X^*(k) = X(N-k)$$

Take conjugate on both sides to prove $X(k) = X^*(N-k)$



# Circular shift of a sequence:

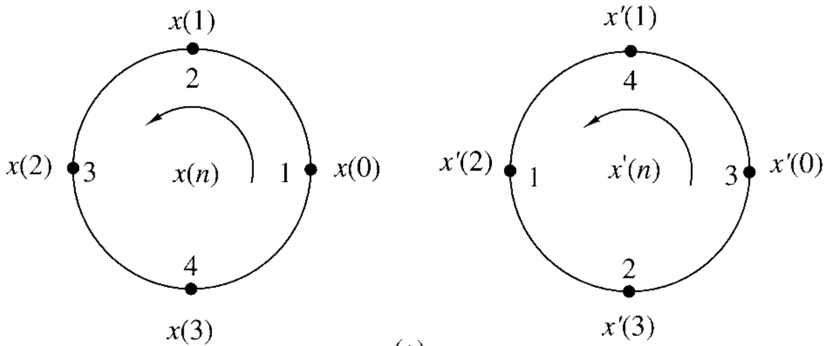
In general, the circular shift of the sequence can be represented as the index modulo $N$. Thus we can write

$$x'(n) = x(n - k, \text{modulo } N)$$

$$\equiv x((n - k))_N$$

For example, if $k = 2$ and $N = 4$, we have

$$x'(n) = x((n - 2))_4$$



(e)

which implies that

$$x'(0) = x((-2))_4 = x(2)$$

$$x'(1) = x((-1))_4 = x(3)$$

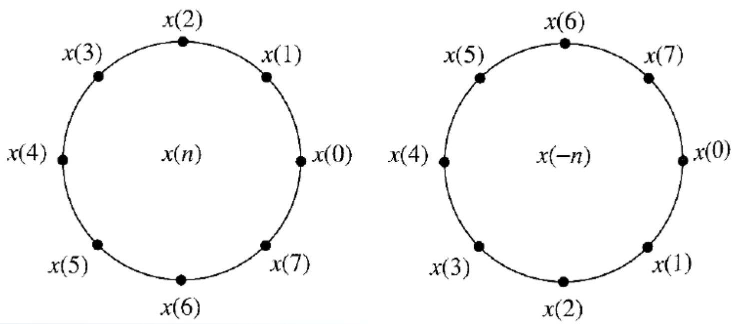
$$x'(2) = x((0))_4 = x(0)$$

$$x'(3) = x((1))_4 = x(1)$$

Hence $x'(n)$ is simply $x(n)$ shifted circularly by two units in time, where the counter-clockwise direction has been arbitrarily selected as the positive direction.

# Next property: **Time reversal of a sequence.**

$$x((-n))_N = x(N-n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X((-k))_N = X(N-k)$$




$$\text{DFT}\{x(N-n)\} = \sum_{n=0}^{N-1} x(N-n)e^{-j2\pi kn/N}$$

change the index from $n$ to $m = N - n$, then

$$\text{DFT}\{x(N-n)\} = \sum_{m=0}^{N-1} x(m)e^{-j2\pi k(N-m)/N}$$

$$= \sum_{m=0}^{N-1} x(m)e^{j2\pi km/N}$$

$$= \sum_{m=0}^{N-1} x(m)e^{-j2\pi m(N-k)/N} = X(N-k)$$


Thank
you

# Properties of DFT contd…

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# Circular time shift of a sequence

$$x((n-l))_N \xleftrightarrow[N]{\text{DFT}} X(k)e^{-j2\pi kl/N}$$

$$\text{DFT}\{x((n-l))_N\} = \sum_{n=0}^{N-1} x((n-l))_N e^{-j2\pi kn/N}$$

$$= \sum_{n=0}^{l-1} x((n-l))_N e^{-j2\pi kn/N}$$

$$+ \boxed{\sum_{n=l}^{N-1} x(n-l)e^{-j\pi kn/N}}$$

But $x((n-l))_N = x(N-l+n)$.

$$\sum_{n=0}^{l-1} x((n-l))_N e^{-j2\pi kn/N} = \sum_{n=0}^{l-1} x(N-l+n)e^{-j2\pi kn/N}$$

$$= \sum_{m=N-l}^{N-1} x(m)e^{-j2\pi k(m+l)/N}$$

Put N – *l* + n = m

$$\sum_{n=l}^{N-1} x(n-l)e^{-j2\pi kn/N} = \sum_{m=0}^{N-1-l} x(m)e^{-j2\pi k(m+l)/N}$$

Put n - *l* = m

$$\text{Therefore, } \text{DFT}\{x((n-l))\} = \sum_{m=0}^{N-1} x(m)e^{-j2\pi k(m+l)/N}$$

$$= X(k)e^{-j2\pi kl/N}$$

## Circular frequency shift.

$$x(n)e^{j2\pi ln/N} \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X((k-l))_N$$

This is the dual to the circular time-shifting property and its proof is similar

## Complex-conjugate properties.

$$x^*(n) \xleftrightarrow[N]{\text{DFT}} X^*((-k))_N = X^*(N-k)$$

Proof is an exercise

The IDFT of $X^*(k)$ is

$$\frac{1}{N}\sum_{k=0}^{N-1} X^*(k)e^{j2\pi kn/N} = \left[\frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi k(N-n)/N}\right]$$

Therefore,

$$x^*((-n))_N = x^*(N-n) \xleftrightarrow[N]{\text{DFT}} X^*(k)$$

## Circular correlation.

$$\tilde{r}_{xy}(l) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} \tilde{R}_{xy}(k) = X(k)Y^*(k)$$

where $\tilde{r}_{xy}(l)$ is the (unnormalized) circular crosscorrelation sequence, defined as

$$\tilde{r}xy(l) = \sum_{n=0}^{N-1} x(n)y^*((n-l))N$$

circular autocorrelation of $x(n)$,

$$\tilde{r}_{xx}(l) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} \tilde{R}_{xx}(k) = |X(k)|^2$$

## Multiplication of two sequences.

$$x_1(n)x_2(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} \frac{1}{N}X_1(k) \, Ⓝ \, X_2(k)$$

Its proof follows simply by interchanging the roles of time and frequency in the expression for the circular convolution of two sequences.

# Parseval's Theorem.

$$\sum_{n=0}^{N-1} x(n)y^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$$

*Proof* The property follows immediately from the circular correlation property

$$\sum_{n=0}^{N-1} x(n)y^*(n) = \tilde{r}_{xy}(0)$$

and

$$\tilde{r}_{xy}(l) = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{R}_{xy}(k)e^{j2\pi kl/N}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)e^{j2\pi kl/N}$$

Evaluate IDFT at $l = 0$

special case where $y(n) = x(n)$

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X(k)|^2$$

**TABLE 7.2** Properties of the DFT

| Property | Time Domain | Frequency Domain |
|---|---|---|
| Notation | $x(n), y(n)$ | $X(k), Y(k)$ |
| Periodicity | $x(n) = x(n+N)$ | $X(k) = X(k+N)$ |
| Linearity | $a_1x_1(n) + a_2x_2(n)$ | $a_1X_1(k) + a_2X_2(k)$ |
| Time reversal | $x(N-n)$ | $X(N-k)$ |
| Circular time shift | $x((n-l))_N$ | $X(k)e^{-j2\pi kl/N}$ |
| Circular frequency shift | $x(n)e^{j2\pi ln/N}$ | $X((k-l))_N$ |
| Complex conjugate | $x^*(n)$ | $X^*(N-k)$ |
| Circular convolution | $x_1(n)$ Ⓝ $x_2(n)$ | $X_1(k)X_2(k)$ |
| Circular correlation | $x(n)$ Ⓝ $y^*(-n)$ | $X(k)Y^*(k)$ |
| Multiplication of two sequences | $x_1(n)x_2(n)$ | $\frac{1}{N}X_1(k)$ Ⓝ $X_2(k)$ |
| Parseval's theorem | $\sum_{n=0}^{N-1} x(n)y^*(n)$ | $\frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$ |

# The DFT as a Linear Transformation

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \qquad k = 0, 1, \ldots, N-1$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}, \qquad n = 0, 1, \ldots, N-1$$

$$W_N = e^{-j2\pi/N}$$ which is an $N$th root of unity.

We note that the computation of each point of the DFT can be accomplished by $N$ complex multiplications and $(N-1)$ complex additions. Hence the $N$-point DFT values can be computed in a total of $N^2$ complex multiplications and $N(N-1)$ complex additions.

# DFT matrix

$$\mathbf{x}_N = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}, \qquad \mathbf{X}_N = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}$$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \qquad k = 0, 1, \ldots, N-1$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}, \qquad n = 0, 1, \ldots, N-1$$

$$W_N = e^{-j2\pi/N}$$

$$\mathbf{W}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots\vdots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

$$\mathbf{x}_N = \mathbf{W}_N^{-1} \mathbf{X}_N$$

$$\mathbf{X}_N = \mathbf{W}_N \mathbf{x}_N$$

$$\begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots\vdots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

$$\mathbf{x}_N = \frac{1}{N} \mathbf{W}_N^* \mathbf{X}_N$$

where $\mathbf{W}_N^*$ denotes the complex conjugate of the matrix $\mathbf{W}_N$

$$\mathbf{W}_N^{-1} = \frac{1}{N} \mathbf{W}_N^*$$

$$\mathbf{W}_N \mathbf{W}_N^* = N \mathbf{I}_N$$

orthogonal (unitary) matrix.

Compute the DFT of the four-point sequence

$$x(n) = (0 \quad 1 \quad 2 \quad 3)$$

$W_N^{kn} \Rightarrow$ Twiddle factor

e.g. $W_4^1 = \left(e^{-j\frac{2\pi}{4}}\right)^1 = \underline{-j}$

$$W_8^1 = \left(e^{-j\frac{2\pi}{8}}\right)^1 = \underline{\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}}$$

**Solution.** The first step is to determine the matrix $\mathbf{W}_4$. By exploiting the periodicity property of $\mathbf{W}_4$ and the symmetry property $W_N^{k+N/2} = -W_N^k$

$$\mathbf{W}_4 = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^0 & W_4^2 \\ 1 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$\mathbf{X}_4 = \mathbf{W}_4\mathbf{x}_4 = \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

The IDFT of $\mathbf{X}_4$ may be determined by conjugating the elements in $\mathbf{W}_4$ to obtain $\mathbf{W}_4^*$


Thank you

# DFT - convolution

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1) Find the 4 Point DFT of $x(n)$

$$x(n) = \{2, 1, 2, -1\}$$

$$X(k) = \sum_{n=0}^{4-1} x(n) e^{\frac{-j2\pi kn}{4}}, \quad 0 \le k \le 4-1$$

$$= 2 + e^{-j\pi k/2} + 2e^{-j\pi k} - e^{-j3\pi k/2}$$

Alternate method (using Twiddle factor)

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

- Ans: 4, -2j, 4 , 2j

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

Det the 4 point IDFT of $X(K) = [\quad]$

$$x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{\frac{j2\pi nk}{N}} , \quad n = 0, 1, \dots N-1$$

Alternate method (Twiddle factor)

$$x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) W_N^{*nK} , \quad n = 0, 1 \dots N-1$$

(e9) If we have $x(n) = [\boxed{8}, 1, 2, 2]$

In this case find 4 point DFT $X(k)$

then $x(-n) = x(-n+N)$

N point DFT

So in this example

$x(-1) = 8$ So $x(-1+4) = x(3) = 8$

Now $x(n) = [1, 2, 2, 8]$

Then proceed to find 4-point DFT $X(k)$.

# Applications of DFT – IDFT tools

- Frequency (spectrum) analysis
- Power spectrum estimations
- Linear filtering
- Presence of computationally efficient algorithms

# Multiplication of two DFT and circular convolution

- Let

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n)e^{-j2\pi nk/N}, \qquad k = 0, 1, \ldots, N-1$$

$$X_2(k) = \sum_{n=0}^{N-1} x_2(n)e^{-j2\pi nk/N}, \qquad k = 0, 1, \ldots, N-1$$

We have

$$X_3(k) = X_1(k)X_2(k), \qquad k = 0, 1, \ldots, N-1$$

Let us determine the relationship between $x_3(n)$ and the sequences $x_1(n)$ and $x_2(n)$.

The IDFT of $\{X_3(k)\}$ is

$$x_3(m) = \frac{1}{N}\sum_{k=0}^{N-1} X_3(k)e^{j2\pi km/N}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X_1(k)X_2(k)e^{j2\pi km/N}$$

$$x_3(m) = \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{n=0}^{N-1} x_1(n)e^{-j2\pi kn/N}\right]\left[\sum_{l=0}^{N-1} x_2(l)e^{-j2\pi kl/N}\right]e^{j2\pi km/N}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x_1(n)\sum_{l=0}^{N-1} x_2(l)\left[\sum_{k=0}^{N-1} e^{j2\pi k(m-n-l)/N}\right]$$

$$x_3(m) = \frac{1}{N} \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \left[ \sum_{k=0}^{N-1} e^{j2\pi k(m-n-l)/N} \right]$$

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N, & a = 1 \\ \dfrac{1-a^N}{1-a}, & a \neq 1 \end{cases}$$

where $a$ is defined as

$$a = e^{j2\pi(m-n-l)/N}$$

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N, & l = ((m-n))_N \\ 0, & \text{otherwise} \end{cases}$$

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n) x_2((m-n))_N, \qquad m = 0, 1, \ldots, N-1$$ *circular convolution.*

$$x_1(n) \; \textcircled{N} \; x_2(n) \xleftrightarrow[N]{\text{DFT}} X_1(k) X_2(k)$$


Thank you

# DFT - convolution

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Perform the circular convolution of the following two sequences:

$$x_1(n) = \{\underset{\uparrow}{2}, 1, 2, 1\}$$

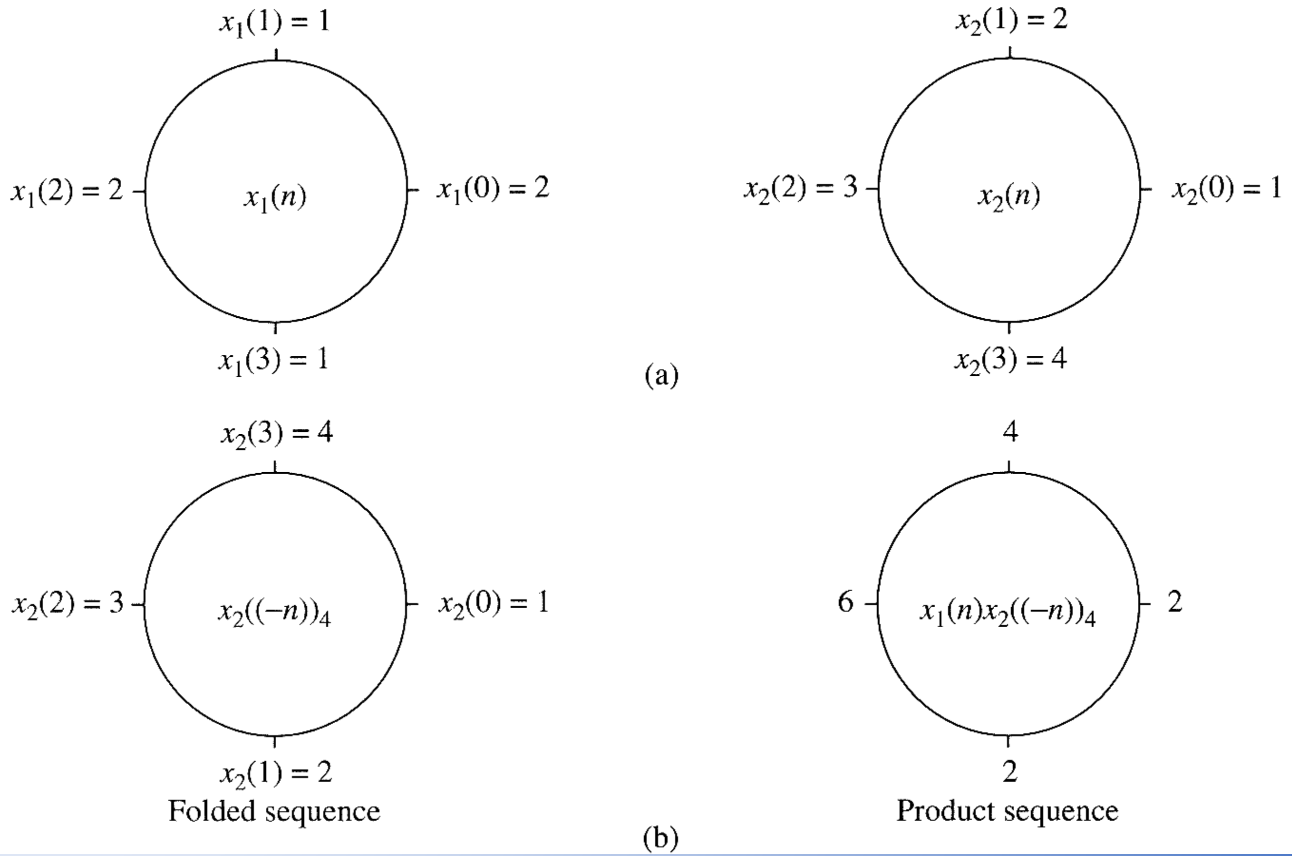
$$x_2(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$$

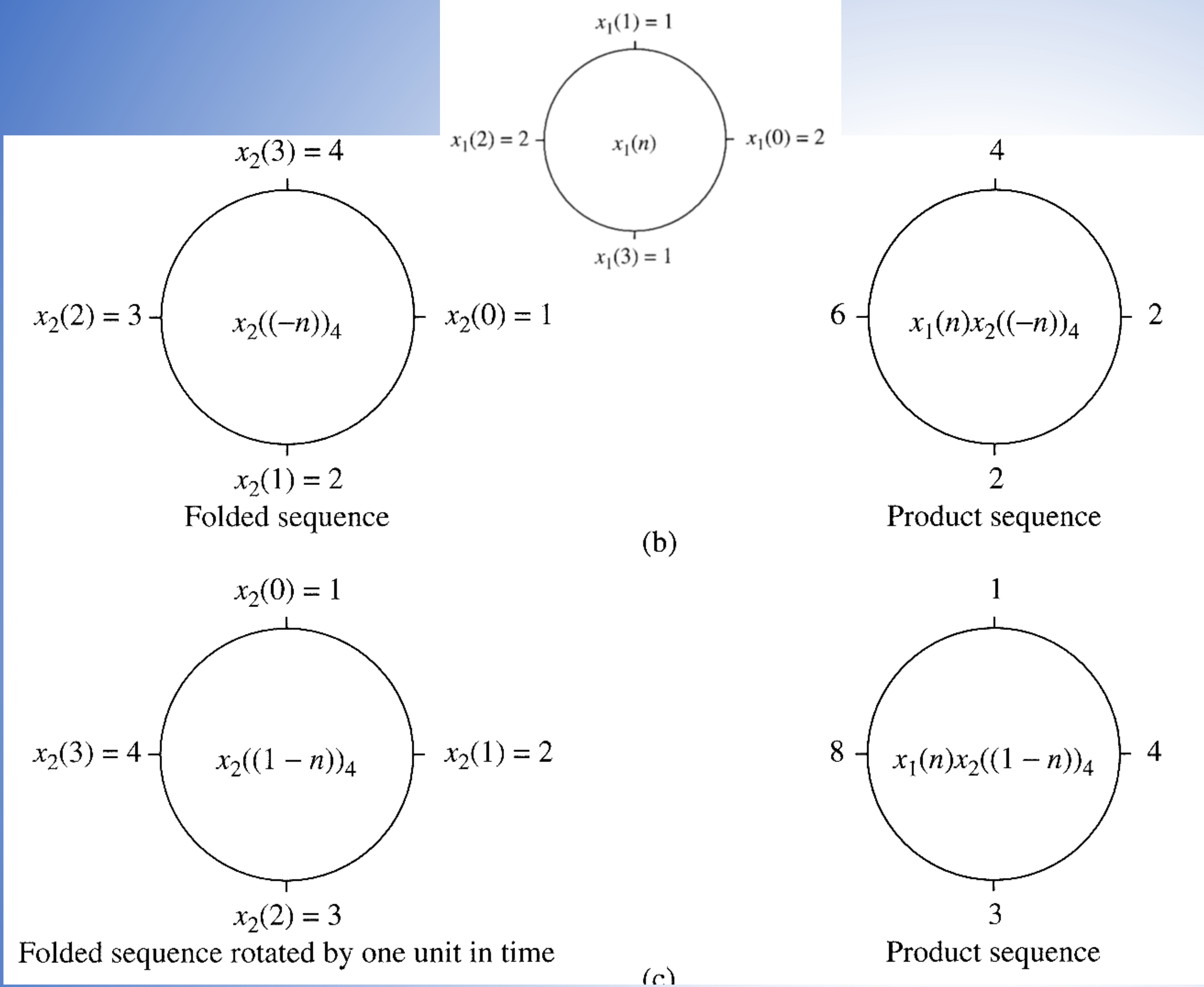
- Method 1: Circular convolution – Time domain approach

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n)x_2((m-n))_N, \qquad m = 0, 1, \ldots, N-1$$

$$x_3(0) = \sum_{n=0}^{3} x_1(n)x_2((-n))N$$

$$x_3(0) = 14$$



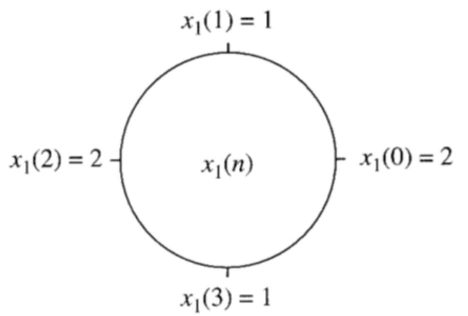


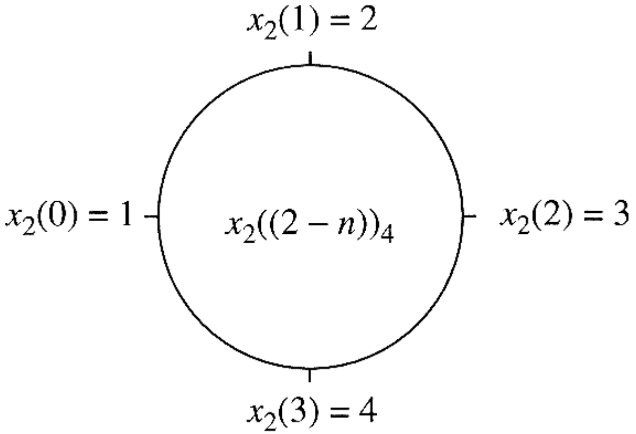
$x_1(1) = 1$
$x_1(2) = 2$
$x_1(n)$
$x_1(0) = 2$
$x_1(3) = 1$
$x_2(3) = 4$
$x_2(2) = 3$
$x_2((-n))_4$
$x_2(0) = 1$
$x_2(1) = 2$
Folded sequence
4
6
$x_1(n)x_2((-n))_4$
2
2
Product sequence
$x_2(0) = 1$
$x_2(3) = 4$
$x_2((1-n))_4$
$x_2(1) = 2$
$x_2(2) = 3$
Folded sequence rotated by one unit in time
1
8
$x_1(n)x_2((1-n))_4$
4
3
Product sequence

(b)

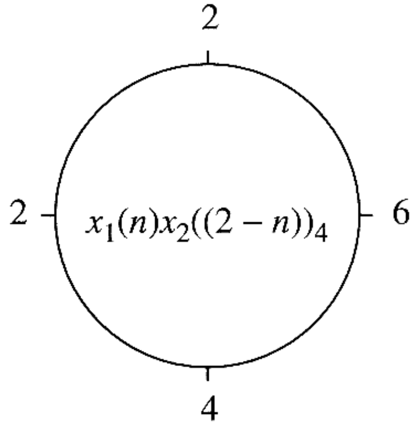
(c)

$$x_3(1) = 16$$



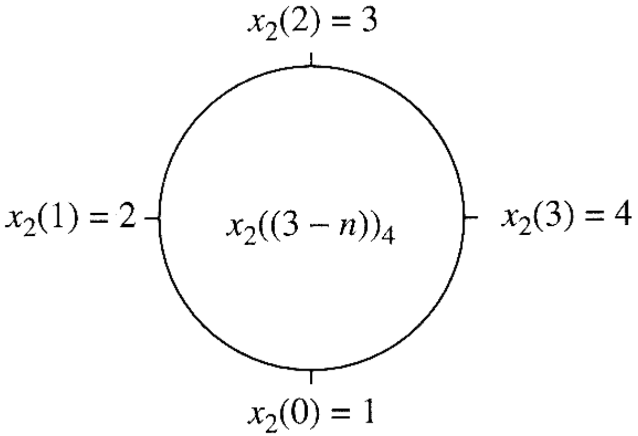

Folded sequence rotated by two units in time



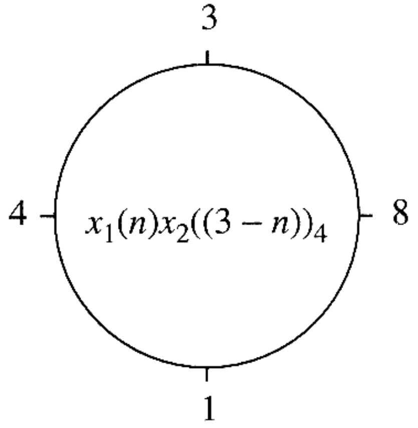

Product sequence

(d)

$$x_3(2) = 14$$




Folded sequence rotated by three units in time




Product sequence

(e)

$$x_3(3) = 16$$

$$x_3(n) = \{\underset{\uparrow}{14}, 16, 14, 16\}$$

- Same 4 operations of convolution – Fold, shift, multiply and add
- Folding and shifting are circular in nature.

- Method 2: Circular convolution: DFT – IDFT approach

Perform the circular convolution of the following two sequences:

$$x_1(n) = \{\underset{\uparrow}{2}, 1, 2, 1\}$$

$$x_2(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$$

$$X_1(k) = \sum_{n=0}^{3} x_1(n) e^{-j2\pi nk/4}, \qquad k = 0, 1, 2, 3$$

$$= 2 + e^{-j\pi k/2} + 2e^{-j\pi k} + e^{-j3\pi k/2}$$

$$X_1(0) = 6, \qquad X_1(1) = 0, \qquad X_1(2) = 2, \qquad X_1(3) = 0$$

$$X_2(k) = \sum_{n=0}^{3} x_2(n) e^{-j2\pi nk/4}, \qquad k = 0, 1, 2, 3$$

$$= 1 + 2e^{-j\pi k/2} + 3e^{-j\pi k} + 4e^{-j3\pi k/2}$$

$$X_2(0) = 10, \qquad X_2(1) = -2 + j2, \qquad X_2(2) = -2, \qquad X_2(3) = -2 - j2$$

$$X_3(k) = X_1(k) X_2(k)$$

$$X_3(0) = 60, \qquad X_3(1) = 0, \qquad X_3(2) = -4, \qquad X_3(3) = 0$$

Now, the IDFT of $X_3(k)$ is

$$x_3(n) = \frac{1}{N} \sum_{k=0}^{3} X_3(k) e^{j2\pi nk/4}, \qquad n = 0, 1, 2, 3$$

$$= \frac{1}{4}(60 - 4e^{j\pi n})$$

$$x_3(n) = \{\underset{\uparrow}{14}, 16, 14, 16\}$$

# Circular convolution using matrix method

$$x_1(n) = \{1, -1, -2, 3, -1\}$$

$$x_2(n) = \{1, 2, 3, 0, 0\}$$

$$\overset{x_2(n)}{\begin{bmatrix} 1 & 0 & 0 & 3 & 2 \\ 2 & 1 & 0 & 0 & 3 \\ 3 & 2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 3 & 2 & 1 \end{bmatrix}} \overset{x_1(n)}{\begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \\ -1 \end{bmatrix}} = \overset{y(n)}{\begin{bmatrix} 8 \\ -2 \\ -1 \\ -4 \\ -1 \end{bmatrix}}$$

$$y(n) = \{8, -2, -1, -4, -1\}$$


Thank
you

# Linear Filtering based on DFT

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# Use of the DFT in Linear Filtering

- Product of 2 DFTs is equivalent to circular convolution
- Not a linear convolution
- But output of an LTI system is y[n] = x(n) * h[n] (Linear convolution of input with impulse response)

Suppose that we have a finite-duration sequence $x(n)$ of length $L$ which excites an FIR filter of length $M$.

$$x(n) = 0, \qquad n < 0 \text{ and } n \geq L$$

$$h(n) = 0, \qquad n < 0 \text{ and } n \geq M$$

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k)$$

where $h(n)$ is the impulse response of the FIR filter.

Since $h(n)$ and $x(n)$ are finite-duration sequences, their convolution is also finite in duration. In fact, the duration of $y(n)$ is $L + M - 1$.

- In frequency domain $Y(\omega) = X(\omega)H(\omega)$
- If y(n) is represented by spectrum Y(ω), the number of samples must be equal or exceed L+M-1
- Therefore, DFT of size N ≥ L+M-1 is required
- DFT of length N is increased to L+M-1 by zero padding
- Thus, DFT can be used for linear filtering

By means of the DFT and IDFT, determine the response of the FIR filter with impulse response

$h(n) = \{\underset{\uparrow}{1}, 2, 3\}$ to the input sequence $x(n) = \{\underset{\uparrow}{1}, 2, 2, 1\}$

- L = 3, M = 4, therefore N = 3+4-1 = 6 . We can take 8-point DFT for the convenance (Fast algorithm exists – Next chapter)

$$X(k) = \sum_{n=0}^{7} x(n)e^{-j2\pi kn/8}$$

$$= 1 + 2e^{-j\pi k/4} + 2e^{-j\pi k/2} + e^{-j3\pi k/4}, \qquad k = 0, 1, \ldots, 7$$

$$X(0) = 6, \qquad X(1) = \frac{2+\sqrt{2}}{2} - j\left(\frac{4+3\sqrt{2}}{2}\right)$$

$$X(2) = -1 - j, \qquad X(3) = \frac{2-\sqrt{2}}{2} + j\left(\frac{4-3\sqrt{2}}{2}\right)$$

$$X(4) = 0, \qquad X(5) = \frac{2-\sqrt{2}}{2} - j\left(\frac{4-3\sqrt{2}}{2}\right)$$

$$X(6) = -1 + j, \qquad X(7) = \frac{2+\sqrt{2}}{2} + j\left(\frac{4+3\sqrt{2}}{2}\right)$$

Homework: Try this using matrix method

# Matrix method:

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^0 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W^6 & W_8^7 \\ W_8^0 & W_8^2 & W_8^4 & W_8^6 & W_8^8 & W_8^{10} & W_8^{12} & W_8^{14} \\ W_8^0 & W_8^3 & W_8^6 & W_8^9 & W_8^{12} & W_8^{15} & W_8^{18} & W_8^{21} \\ W_8^0 & W_8^4 & W_8^8 & W_8^{12} & W_8^{16} & W_8^{20} & W_8^{24} & W_8^{28} \\ W_8^0 & W_8^5 & W_8^{10} & W_8^{15} & W_8^{20} & W_8^{25} & W_8^{30} & W_8^{35} \\ W_8^0 & W_8^6 & W_8^{12} & W_8^{18} & W_8^{24} & W_8^{30} & W_8^{36} & W_8^{42} \\ W_8^0 & W_8^7 & W_8^{14} & W_8^{21} & W_8^{28} & W_8^{35} & W_8^{42} & W_8^{49} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix}$$

$$W_8^1 = \left(e^{-j\frac{2\pi}{8}}\right)^1 = \underline{\frac{1}{\sqrt{2}}} - \frac{j}{\sqrt{2}}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1-j}{\sqrt{2}} & -j & \frac{-(1+j)}{\sqrt{2}} & -1 & \frac{-(1-j)}{\sqrt{2}} & j & \frac{1+j}{\sqrt{2}} \\ 1 & -j & -1 & j & 1 & -j & -1 & j \\ 1 & \frac{-(1+j)}{\sqrt{2}} & j & \frac{1-j}{\sqrt{2}} & -1 & \frac{1+j}{\sqrt{2}} & -j & \frac{-(1-j)}{\sqrt{2}} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & \frac{-(1-j)}{\sqrt{2}} & -j & \frac{1+j}{\sqrt{2}} & -1 & \frac{1-j}{\sqrt{2}} & j & \frac{-(1+j)}{\sqrt{2}} \\ 1 & j & -1 & -j & 1 & j & -1 & -j \\ 1 & \frac{1+j}{\sqrt{2}} & j & \frac{-(1-j)}{\sqrt{2}} & -1 & \frac{-(1+j)}{\sqrt{2}} & -j & \frac{(1-j)}{\sqrt{2}} \end{bmatrix}$$

Problem contd from slide 4: h(n) = {1,2,3}

$$H(k) = \sum_{n=0}^{7} h(n)e^{-j2\pi kn/8}$$

$$= 1 + 2e^{-j\pi k/4} + 3e^{-j\pi k/2}$$

$$H(0) = 6, \qquad H(1) = 1 + \sqrt{2} - j\left(3 + \sqrt{2}\right), \qquad H(2) = -2 - j2$$

$$H(3) = 1 - \sqrt{2} + j\left(3 - \sqrt{2}\right), \qquad H(4) = 2$$

$$H(5) = 1 - \sqrt{2} - j\left(3 - \sqrt{2}\right), \qquad H(6) = -2 + j2$$

$$H(7) = 1 + \sqrt{2} + j\left(3 + \sqrt{2}\right)$$

The product of these two DFTs yields $Y(k)$, which is

$$Y(0) = 36, \qquad Y(1) = -14.07 - j17.48, \qquad Y(2) = j4, \qquad Y(3) = 0.07 + j0.515$$

$$Y(4) = 0, \qquad Y(5) = 0.07 - j0.515, \qquad Y(6) = -j4, \qquad Y(7) = -14.07 + j17.48$$

Finally, the eight-point IDFT is

$$y(n) = \sum_{k=0}^{7} Y(k)e^{j2\pi kn/8}, \qquad n = 0, 1, \ldots, 7$$

This computation yields the result $y(n) = \{\underset{\uparrow}{1}, 4, 9, 11, 8, 3, 0, 0\}$

Observe – last 2 digits are zero

Also, note: Same result can be obtained using circular and linear convolutions


Thank you

# Linear Filtering based on DFT

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An LTI system has impulse response $h(n) = \cos\left(\frac{n\pi}{2}\right)$ & i/p $x(n) = 2^n$.
using DFT-IDFT approach compute the response of LTI system. choose $0 \le n \le 3$.

solution

$h(n) = [1, 0, -1, 0]$

$x(n) = [1, 2, 4, 8]$

$N \ge L+M-1$ $\quad N \ge 7$, $\quad N = 8$

$X(k) \Rightarrow$ 8 Point DFT of $x(n)$

$H(k) \Rightarrow$ 8 Point DFT of $h(n)$

$Y(k) = X(k)\, H(k)$

$Y(k) \xrightarrow[\text{IDFT}]{\text{8 point}} y(n)$

$\downarrow$

Response of LTI system

# Linear convolution using matrix method (Time domain) (For cross verification of the result)

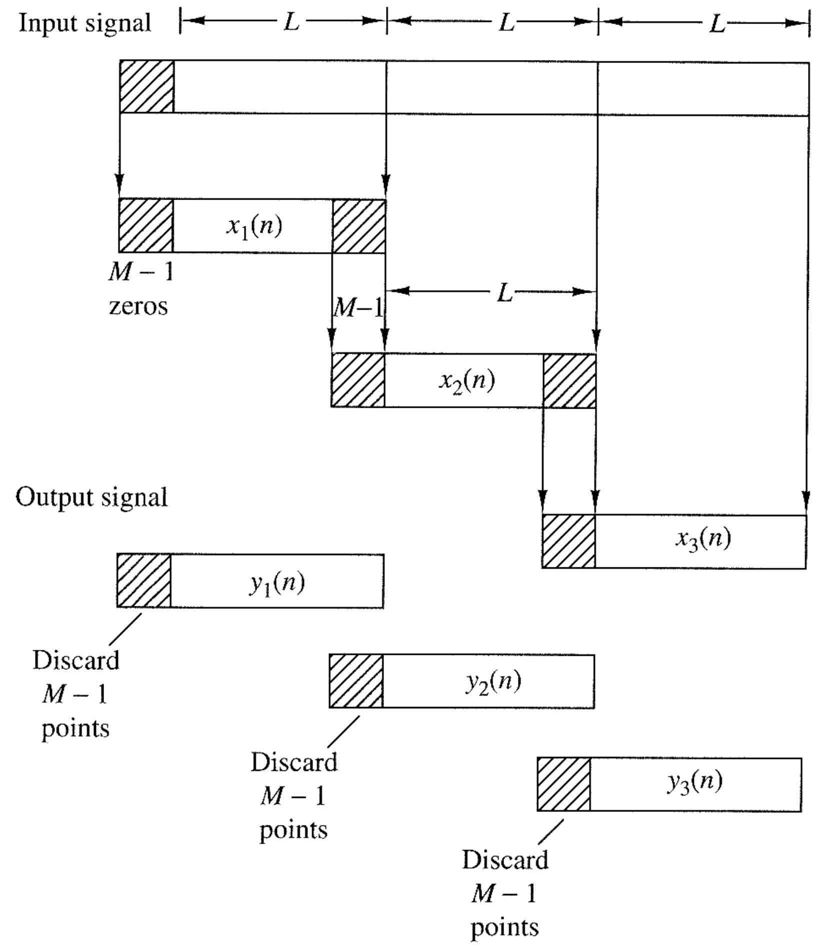
$$\boldsymbol{h} = [1,\ 2, -1,\ 1]$$
$$\boldsymbol{x} = [1,\ 1,\ 2,\ 1,\ 2,\ 2,\ 1,\ 1]$$

$$\boldsymbol{y} = H\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \\ 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \\ 5 \\ 3 \\ 7 \\ 4 \\ 3 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

# Filtering of Long Data Sequences

## overlap save method

1) The i/p sequence x(n) is subdivided into block of length L (L samples in a block)
2) Let the FIR h(n) has a length M.
For N-point circular convolution $N = L+M-1$
3) For the first block, the first N samples of x(n) are taken & left of it is padded with M-1 zeros. This sequence is circularly convolved with h(n) after padding as many zeros to its right, (N-M) zero to make it of length N.

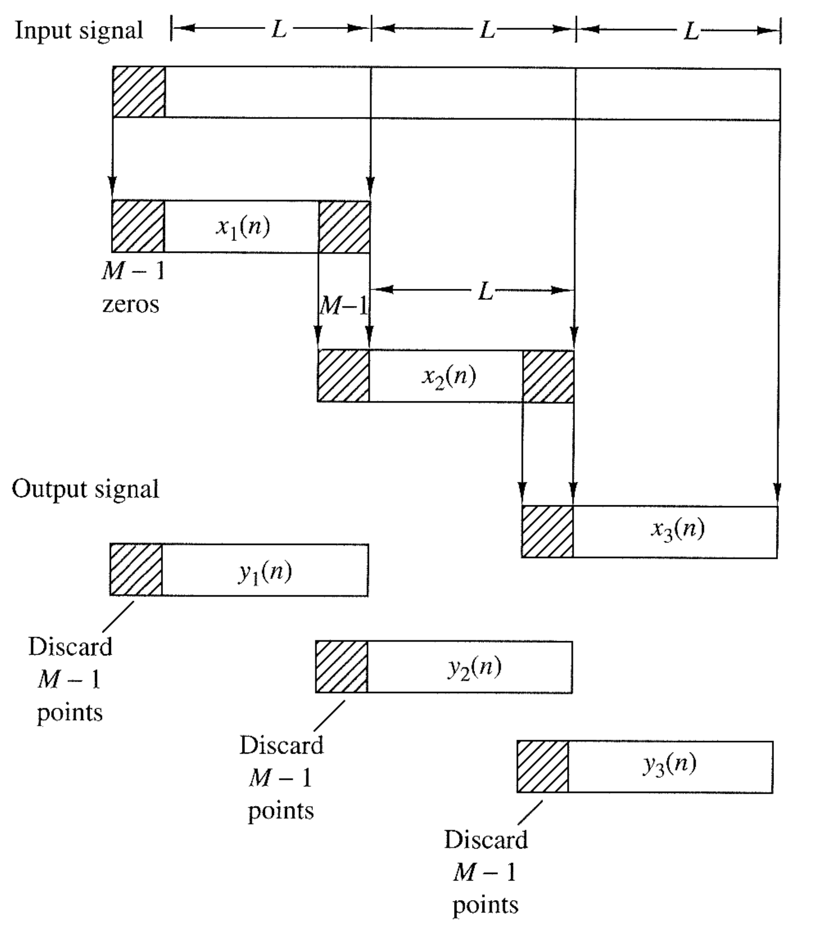


4) From the result of convolution the first (M-1) elements are discarded & the remaining is saved.

5) For the second block, next L samples of $x(n)$ are selected. To its left the (M-1) elements of the previous block are added. This is overlapping of data of $x(n)$

6) The convolution is performed. From this first (M-1) elements corresponding to the overlapped data are rejected and the remaining L elements are saved (To overcome aliasing)

This is continued till all blocks are convolved. Finally the saved result are cascaded to get the final result $y(n)$.



Using **overlap save method** find the output sequence y(n). The input sequence is x(n)=[3, 2, 1, 1, 2, 2, 0, 1, 2, 0, 1, 3] and h(n)= [1, 1, 1].

Let $L=4$, $M=3$ $\therefore N = L+M-1 = 6$

$x_1(n) = \{0, 0, 3, 2, 1, 1\}$

$x_2(n) = \{1, 1, 2, 2, 0, 1\}$

$x_3(n) = \{0, 1, 2, 0, 1, 3\}$

$x_4(n) = \{1, 3, 0, 0, 0, 0\}$

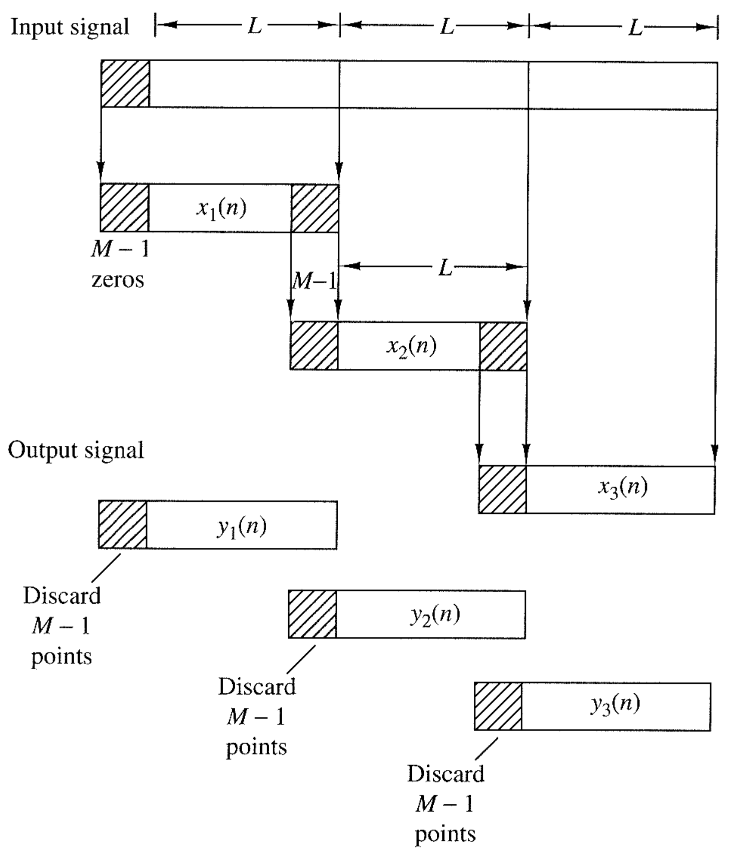
$h(n) = \{1, 1, 1, 0, 0, 0\}$

$x_1(n) \, (N) \, h(n) = \{\cancel{2}, \cancel{1}, 3, 5, 6, 4\}$

$x_2(n) \, (N) \, h(n) = \{\cancel{2}, \cancel{3}, 4, 5, 4, 3\}$

$x_3(n) \, (N) \, h(n) = \{\cancel{4}, \cancel{4}, 3, 3, 3, 4\}$

$x_4(n) \, (N) \, h(n) = \{\cancel{1}, \cancel{4}, 4, 3, 0, 0\}$



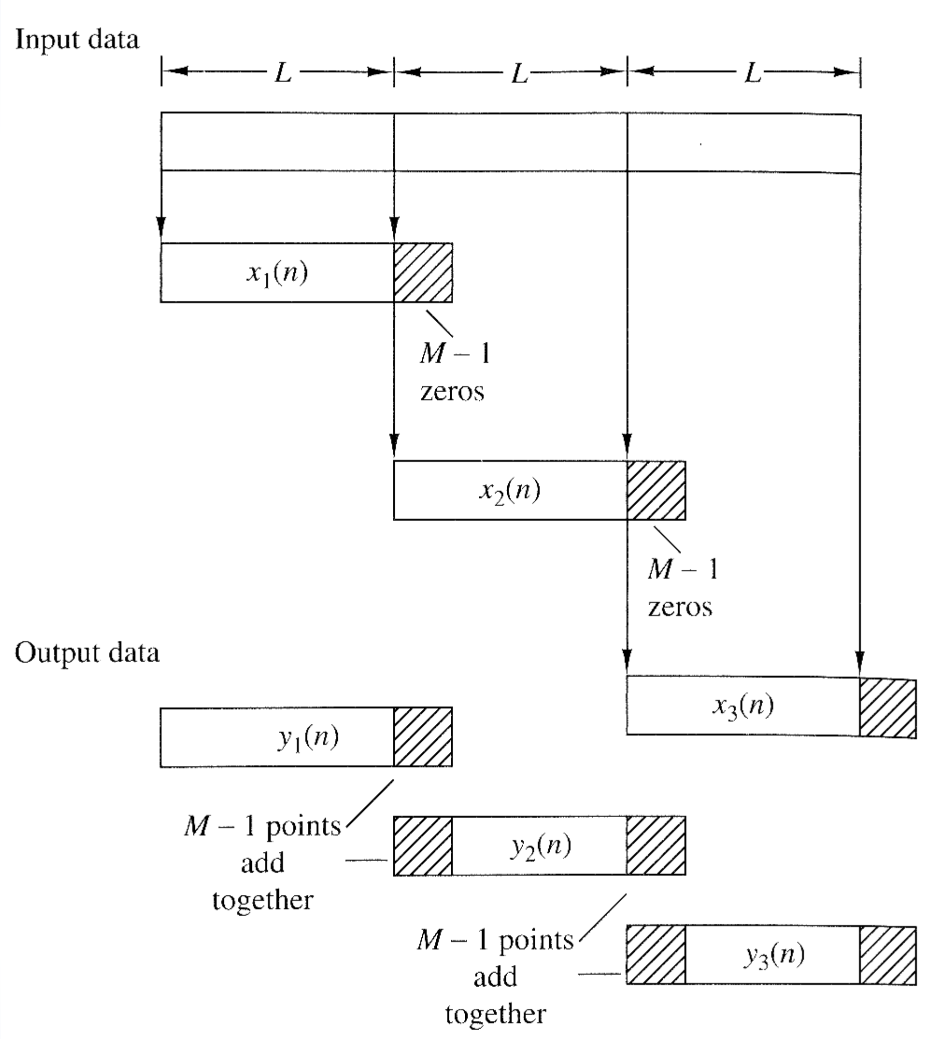

Use DFT – IDFT method for finding circular convolution

Try with L = 8

Ans: [3,5,6,4,4,5,4,3,3,3,3,4,4,3]

## Overlap add method

1) The input sequence $x(n)$ is subdivided into blocks of L samples.
2) Let the FIR $h(n)$ has M-samples. For N point circular convolution $N = L + M - 1$
3) For the first block $x(n)$ the first L samples are taken & M-1 zeros are padded to its right.
4) The sequence $h(n)$ is also order of length N by padding to its right (N - M) zeros.
5) Circular convolution is performed to first block.

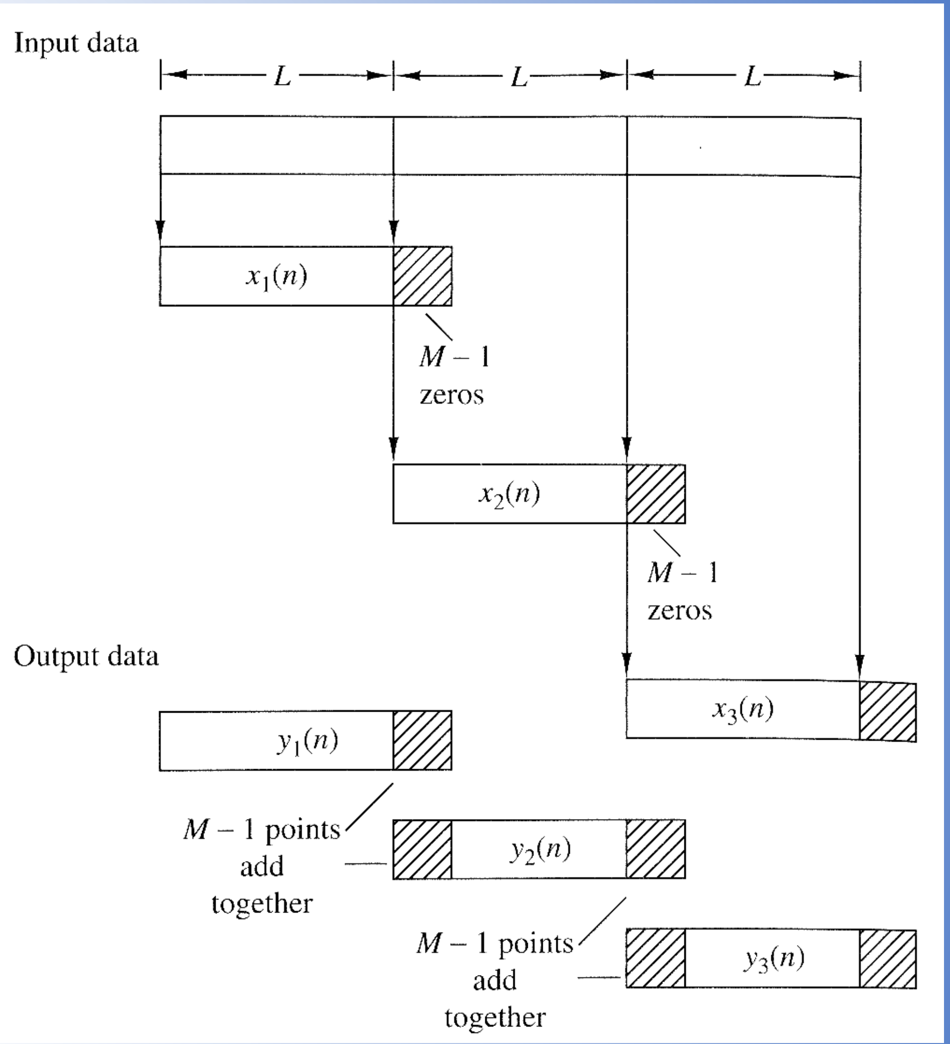


6) For the second block the rest L samples are taken and to its right (M-1) zeros are added

7) The circular convolution of the samples of second block and h(n) is performed.

8) From this result of second convolution, first (M-1) elements are overlapped and added to the last (M-1) samples of the result of first block convolution. This generates the first 2L samples of final result.

9) This procedure is continued untill all blocks are convolved & final result is obtained.



Using **overlap add method** find the output sequence y(n). The input sequence is x(n)=[3,2,1,1,2,2,0,1,2,0,1,3] and h(n)=[1,1,1].

## Homework

Take L = 4

M = 3

N = 6

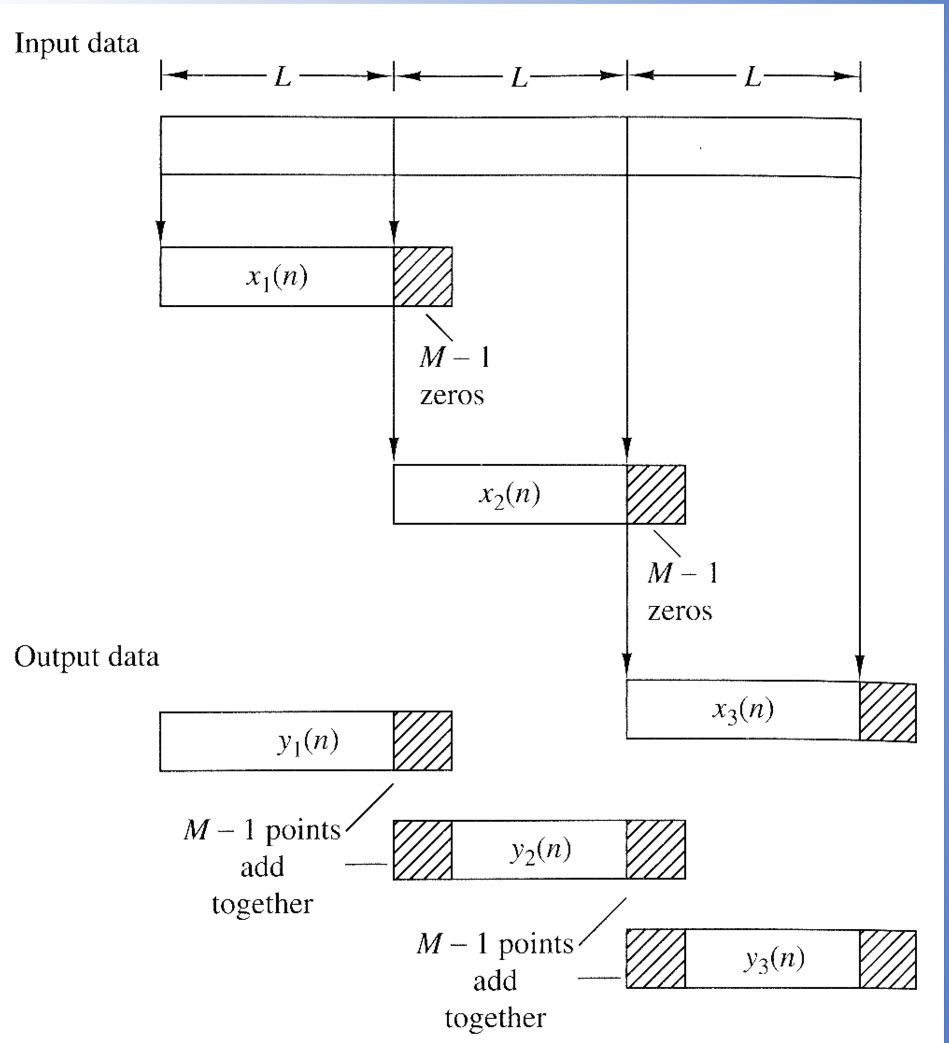
$x_1(n)$, $x_2(n)$, $x_3(n)$ - Add 2 zeros

h(n) – Add 3 zeros

Compute 6 point circular convolutions

Overlap and add to get y(n)

Ans: [3,5,6,4,4,5,4,3,3,3,3,4,4,3]




Thank
you

# Efficient Computation of the DFT: Fast Fourier Transform Algorithms

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DFT plays an important role in many applications of Digital Signal Processing

- Linear Filtering
- Correlation Analysis
- Spectrum analysis

Efficient algorithms are available for DFT computation

**Complexity in Direct computation of DFT:**

Each DFT point direct computation of X(K) involves

- N complex multiplications (**4N real multiplications**)
- N-1 complex additions (**4N-2 real additions**)
- To compute All N points of the DFT we require $N^2$ **complex multiplications and** $N^2 - N$ **complex additions**

Two approaches for efficient computation:

1. Divide and Conquer approach (for composite N)
    - Decomposition of an N-point DFT into successively smaller DFTs
        - Decimation In Time FFT (DITFFT) Algorithm
        - Decimation In Frequency (DIFFFT) Algorithm

2. Formulation of the DFT as a linear filtering operation on the data
    - Goertzel Algorithm
    - Chirp-z Transform

# Radix-2 FFT Algorithms

## Decimation In Time FFT (DITFFT) Algorithm

- Decompose the computation into successively smaller DFT computations.
- We exploit both the symmetry and periodicity property of the complex exponential $W_N^{nk} = e^{-j\left(\frac{2\pi}{N}\right)nk}$
- In this algorithm the decomposition is based on the sequence $x(n)$ into successively smaller subsequences

Consider $N$ an integer power of 2, $N = 2^{\nu}$

We know that $X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk}, \quad k = 0, 1, \ldots N-1$ —— ①

Let us now separate $x(n)$ into even- and odd-numbered points

$$X(k) = \sum_{n \text{ even}} x(n) W_N^{nk} + \sum_{n \text{ odd}} x(n) W_N^{nk}$$

Let us switch to new variables, $n = 2r$ (even) and $n = (2r+1)$ (odd)

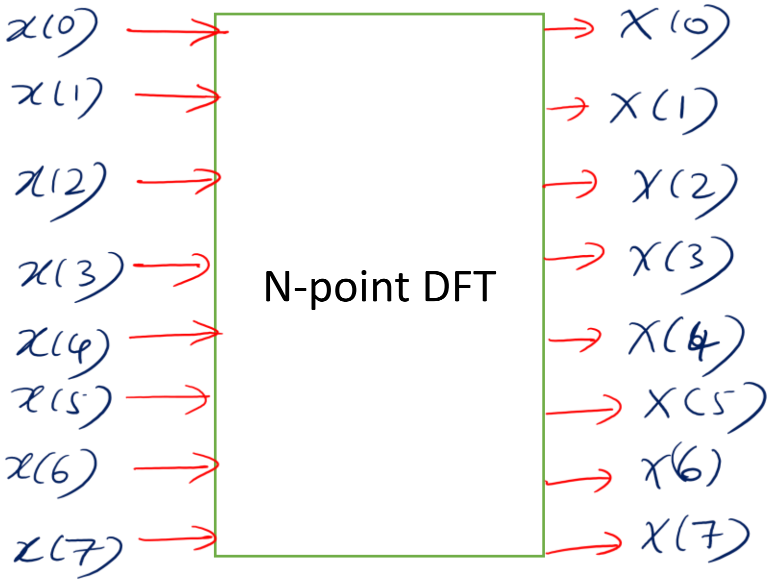
$$X(k) = \sum_{r=0}^{\frac{N}{2}-1} x(2r) W_N^{2rk} + \sum_{r=0}^{\frac{N}{2}-1} x(2r+1) W_N^{(2r+1)k} \quad \text{—— ②}$$

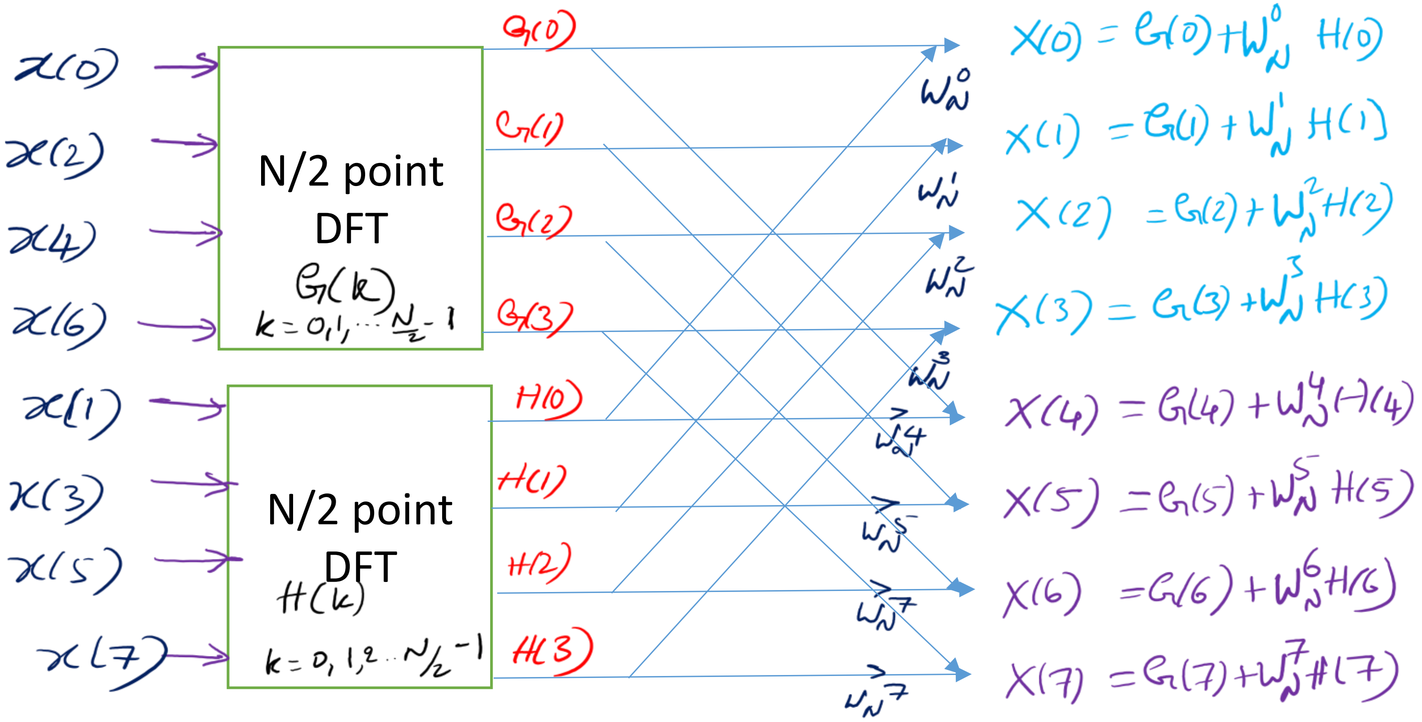
$$X(k) = \sum_{r=0}^{\frac{N}{2}-1} x(2r)\, W_N^{2rk} + \sum_{r=0}^{\frac{N}{2}-1} x(2r+1)\, W_N^{(2r+1)k} \quad \text{—— (2)}$$

$$= \underbrace{\sum_{r=0}^{\frac{N}{2}-1} x(2r)\cdot W_{\frac{N}{2}}^{rk}}_{G(k) \Rightarrow N/2 \text{ pt DFT}} + \underbrace{\sum_{r=0}^{\frac{N}{2}-1} x(2r+1)\, W_{\frac{N}{2}}^{rk}}_{H(k) \Rightarrow \frac{N}{2} \text{ pt DFT}} \cdot \underline{W_N^k} \quad \text{—— (3)}$$

$$W_N^2 = W_{N/2} = e^{-j2(\frac{2\pi}{N})} = e^{-j\frac{2\pi}{N/2}}$$

$$X(k) = G(k) + W_N^k\, H(k) \quad \text{—— (4)}$$



$$X(k) = \sum_{r=0}^{\frac{N}{2}-1} x(2r) \cdot W_{\frac{N}{2}}^{rk} + \sum_{r=0}^{\frac{N}{2}-1} x(2r+1)\, W_{\frac{N}{2}}^{rk} \cdot W_N^{k} \quad \text{—— (3)}$$

Now letus consider first term in the eqn(3)

$$G(k) = \sum_{r=0}^{\frac{N}{2}-1} x(2r)\, W_{\frac{N}{2}}^{rk} \quad \text{—————} \quad \boxed{5}$$

We can rewrite it as

$$G(k) = \sum_{r=0}^{\frac{N}{2}-1} g(r)\, W_{N/2}^{rk} \qquad k = 0, 1, \ldots N/2 - 1$$

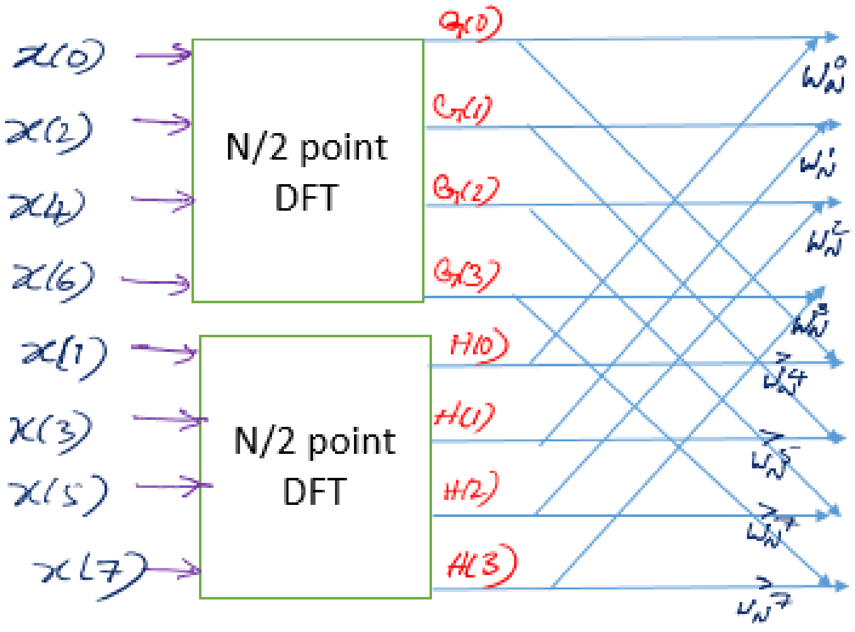
We can again decompose g(r) into even and odd indexed sequence

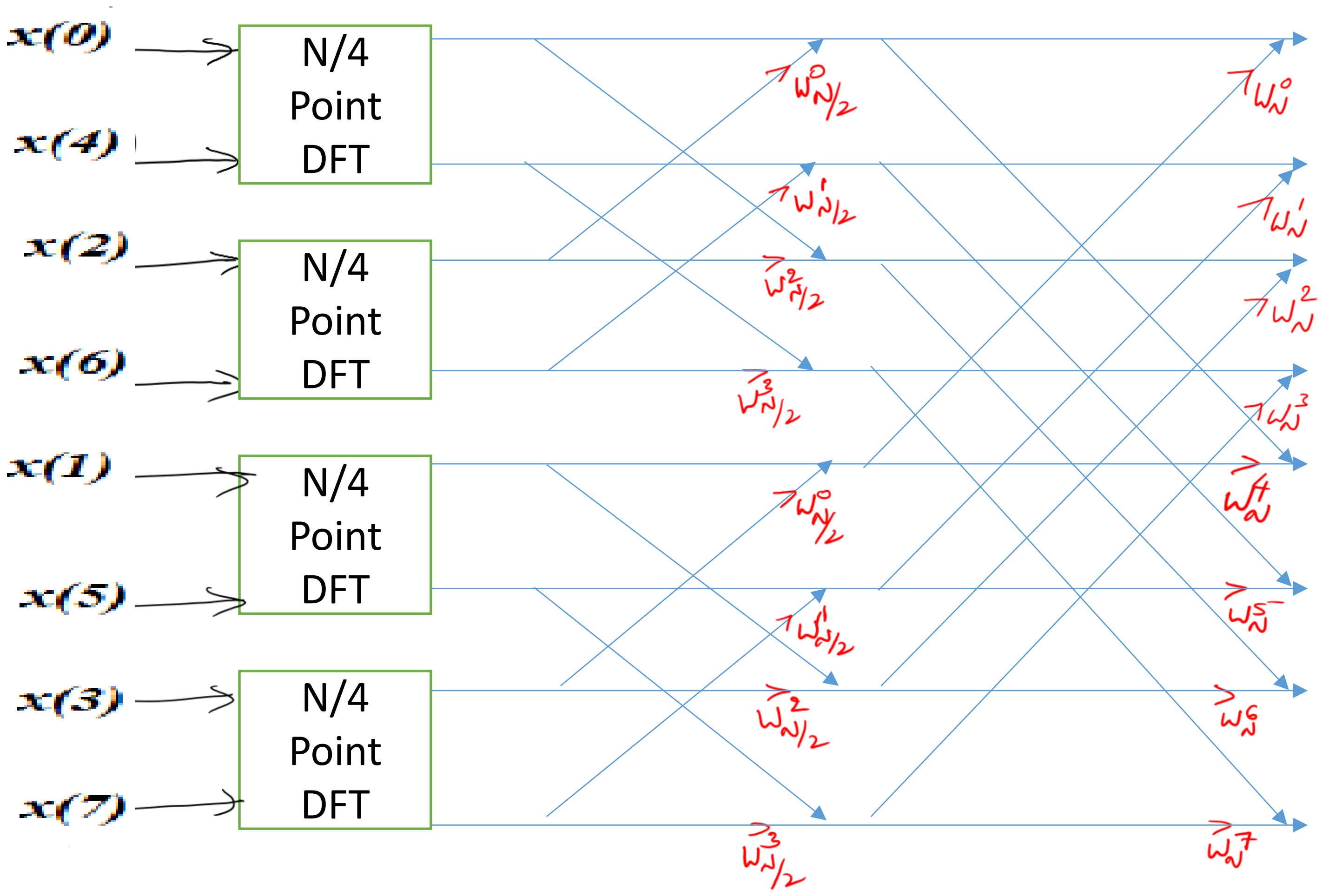
$$G(k) = \sum_{l=0}^{N/4-1} g(2l)\, W_{N/2}^{2lk} + \sum_{l=0}^{\frac{N}{4}-1} g(2l+1)\, W_{N/2}^{(2l+1)k}$$

$$= \sum_{l=0}^{N/4-1} g(2l)\, W_{N/4}^{lk} + W_{N/2}^{k} \sum_{l=0}^{\frac{N}{4}-1} g(2l+1)\, W_{\frac{N}{4}}^{lk}$$

Similarly

$$H(k) = \sum_{l=0}^{\frac{N}{2}-1} h(l)\, W_{\frac{N}{2}}^{lk} = \sum_{l=0}^{N/4-1} h(2l)\, W_{\frac{N}{4}}^{lk} + W_{\frac{N}{2}}^{k} \sum_{l=0}^{\frac{N}{4}-1} h(2l+1)\, W_{\frac{N}{4}}^{lk}$$



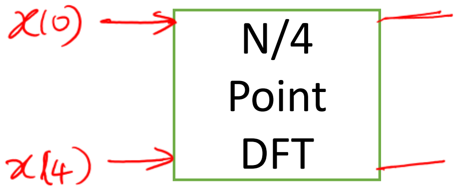



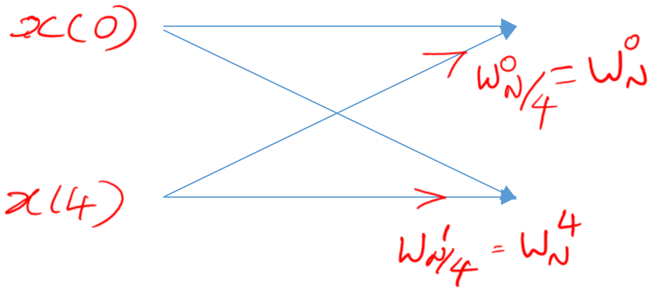
We can keep on decomposing $x(n)$ in this fashion

One final stage we reach is a two point DFT computation.

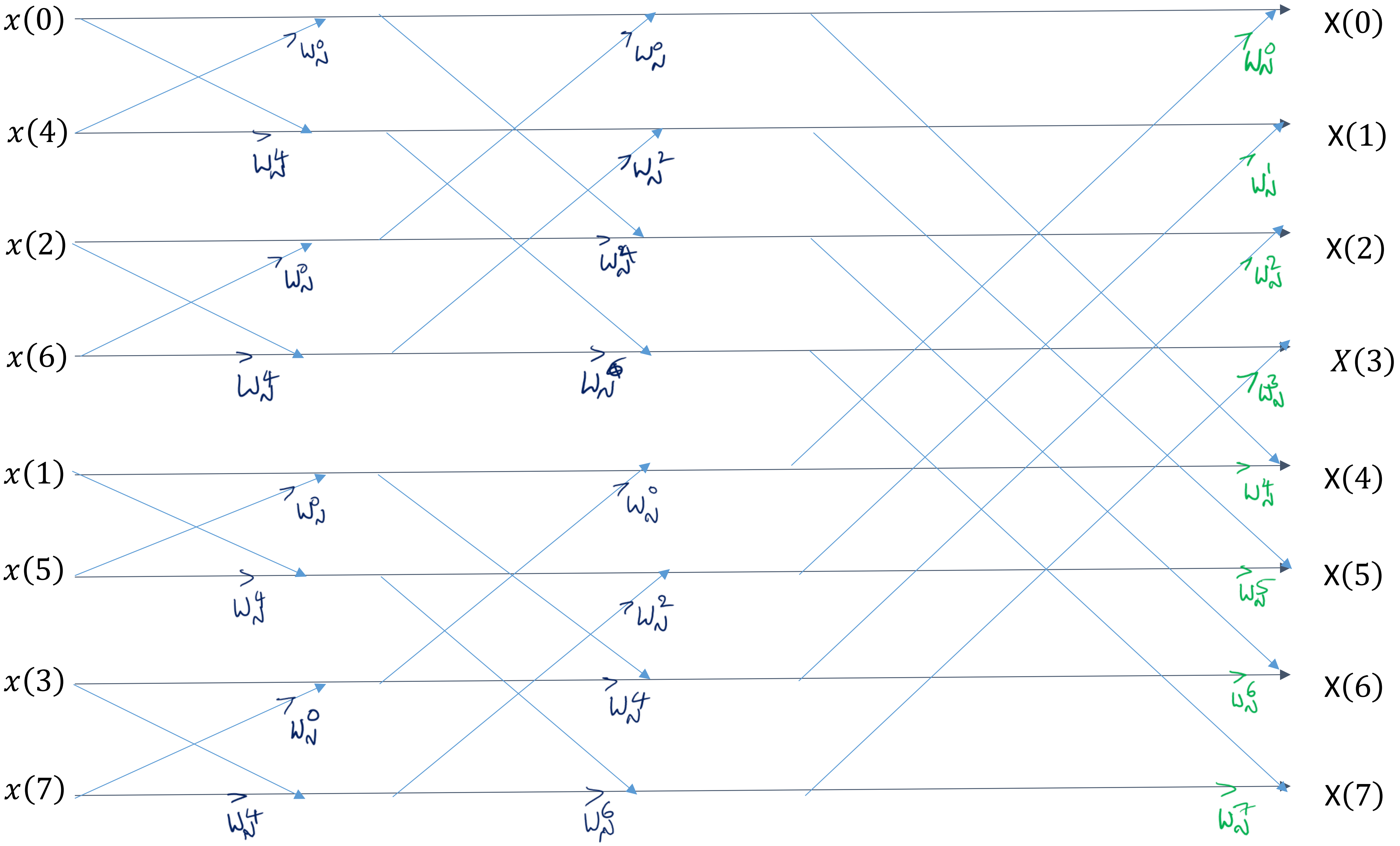
So far we have seen the decomposition of $N=8$ -pt DFT

and we have arrived at $\frac{N}{4}$-pt DFT i.e, 2-pt DFT computation

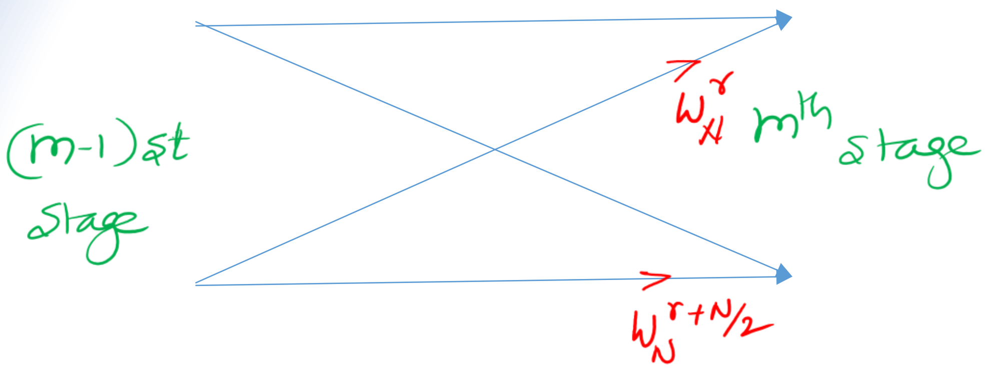


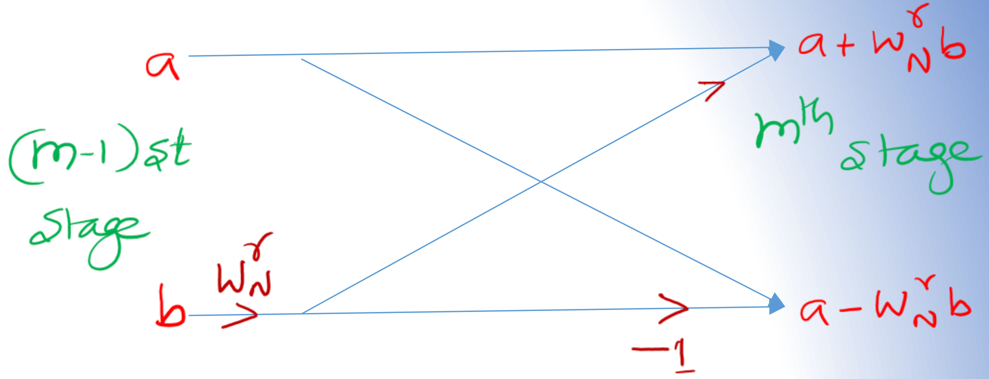



Simplified signal flow graph for N=8-point DFT



# Butterfly Computation



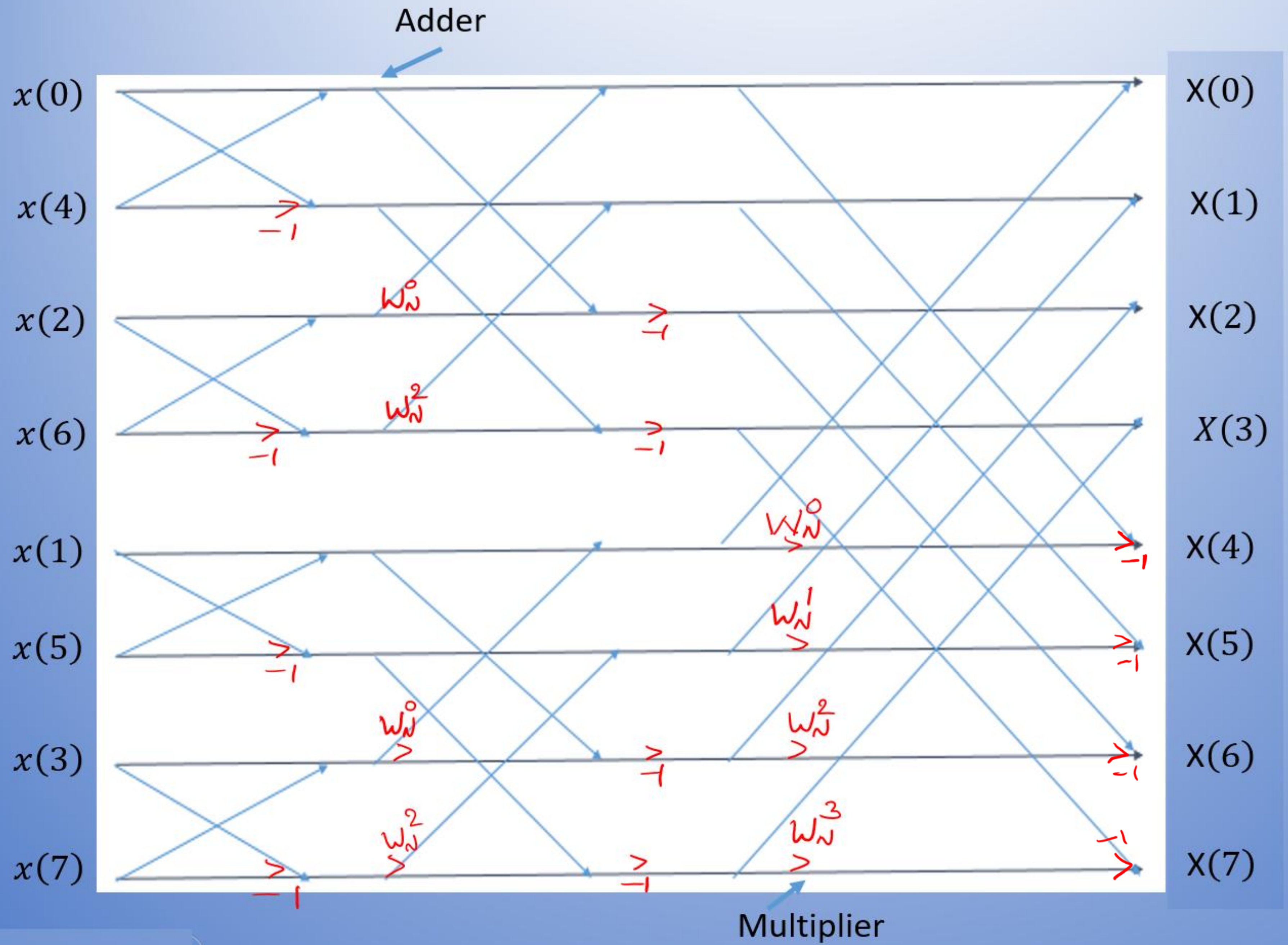

We can write $W_N^{r+N/2} = W_N^{N/2} \cdot W_N^r$

$$= e^{-j2\pi \frac{N/2}{N}} \cdot W_N^r$$

$$= e^{-j\pi} W_N^r$$

$$\therefore W_N^{r+N/2} = -W_N^r$$

Complete signal flow graph for DIT FFT Algorithm for 8-pt DFT




Thank
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# Fast Fourier Transform Algorithms - DITFFT

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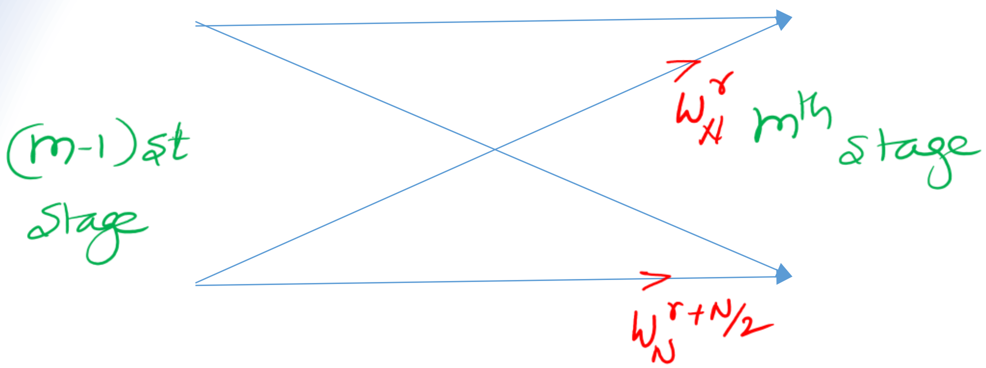
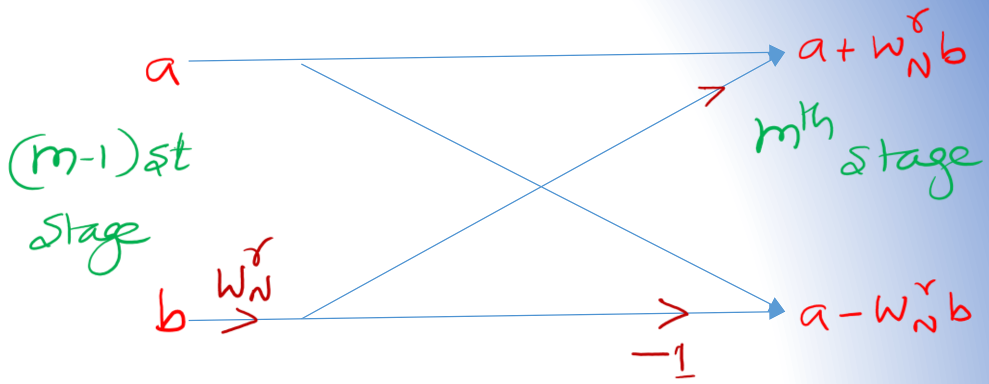
# Butterfly Computation



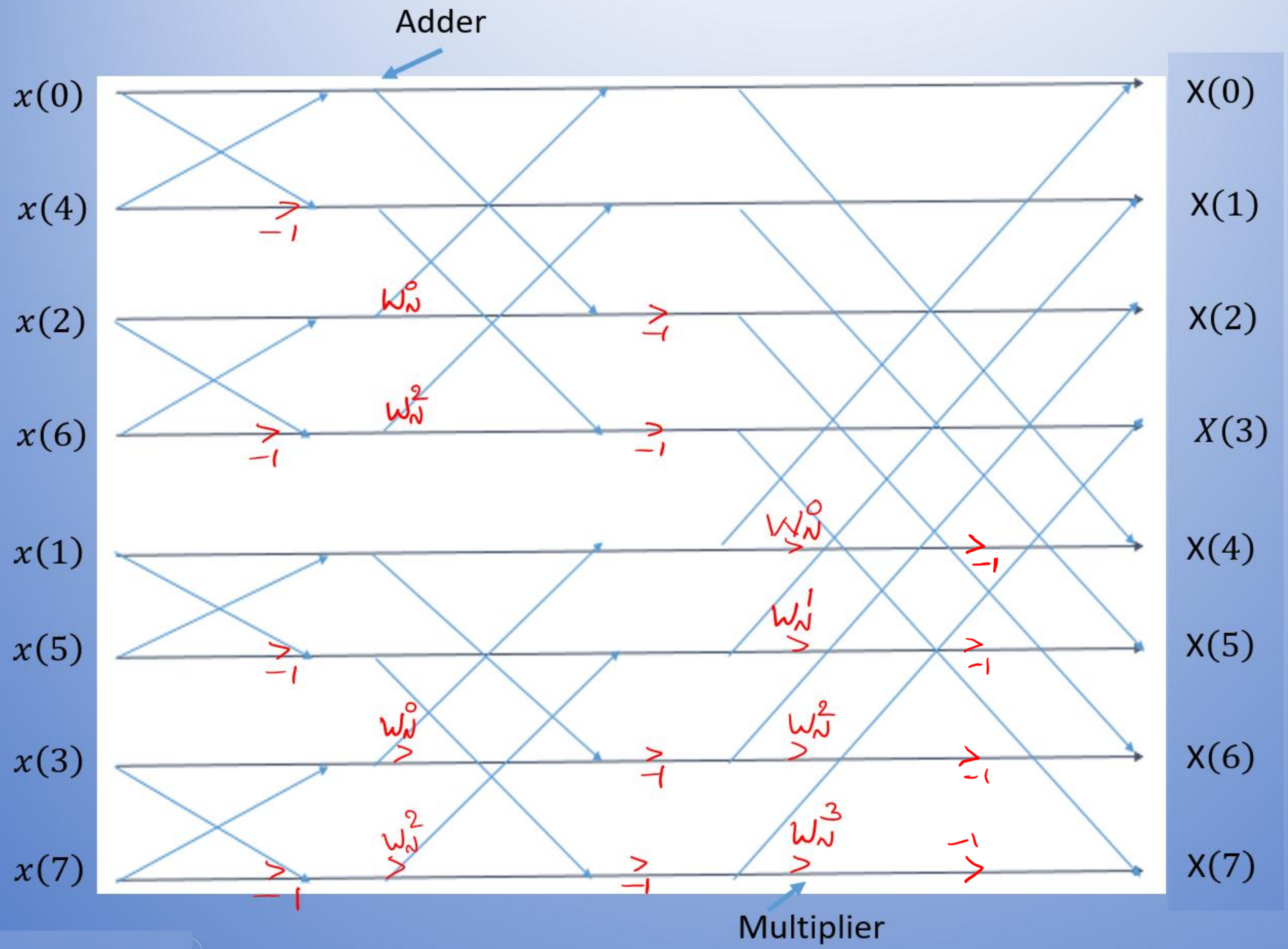

We can write $W_N^{r+N/2} = W_N^{N/2} \cdot W_N^r$

$$= e^{-j2\pi \frac{N/2}{N}} \cdot W_N^r$$

$$= e^{-j\pi} W_N^r$$

$$\therefore \; W_N^{r+N/2} = -W_N^r$$

Complete signal flow graph for DIT FFT Algorithm for 8-pt DFT




| Binary | Bit Reverse | Decimal |
|---|---|---|
| 000 | 000 | 0 |
| 001 | 100 | 4 |
| 010 | 010 | 2 |
| 011 | 110 | 6 |
| 100 | 001 | 1 |
| 101 | 101 | 5 |
| 110 | 011 | 3 |
| 111 | 111 | 7 |

Complex phase factors for $N = 8$

$$W_N^0 = e^{-j\frac{2\pi \times 0}{N}} = 1$$

$$W_N^1 = e^{-j\frac{\pi}{4}} = \cos(\pi/4) - j\sin(\tfrac{\pi}{4}) = \frac{1-j}{\sqrt{2}} = \underline{\underline{0.707 - j0.707}}$$

$$W_N^2 = e^{-j\pi/2} = -j$$

$$W_N^3 = e^{-j3\pi/4} = \cos(\tfrac{3\pi}{4}) - j\sin(\tfrac{3\pi}{4}) = \frac{-1-j}{\sqrt{2}}$$

$$= \underline{\underline{-0.707 - j0.707}}$$

$$W_N^4 = e^{-j\pi} = -1$$

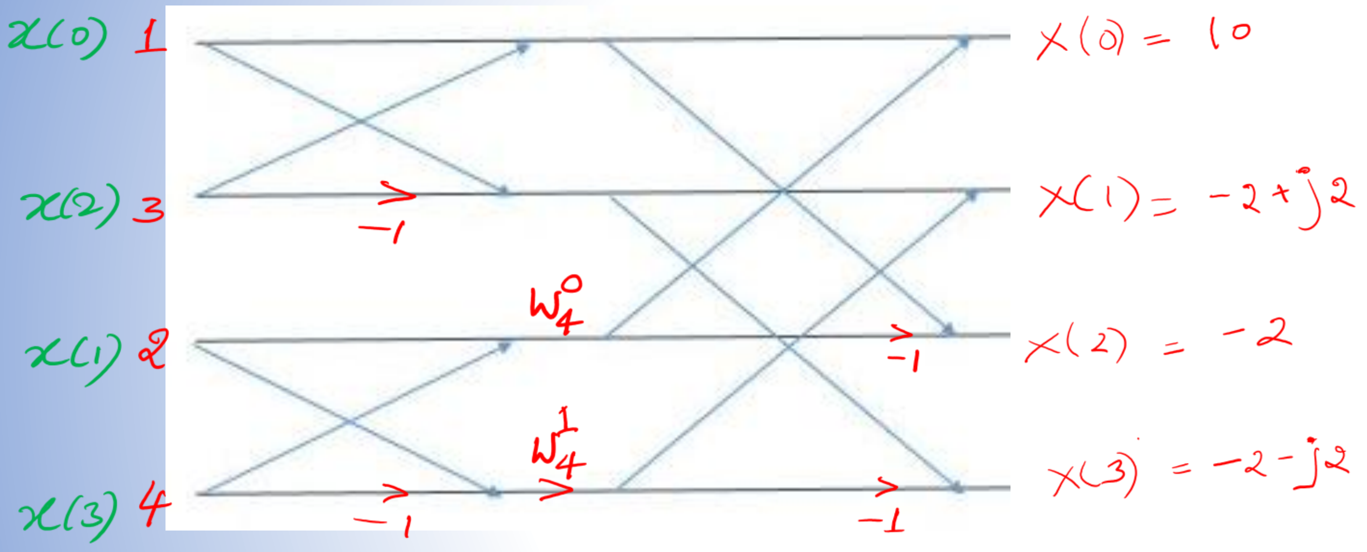
for $N = 4$

$$W_4^0 = 1$$

$$W_4^1 = e^{-j\frac{\pi}{2}} = -j$$

$$W_4^2 = e^{-j\pi} = -1$$

Problem: Compute 4-pt DFT of a sequence $x(n) = \{1, 2, 3, 4\}$




$W_4^0 = 1$

$W_4^1 = e^{-j\frac{\pi}{2}} = -j$

Try {2,1,2,1}.

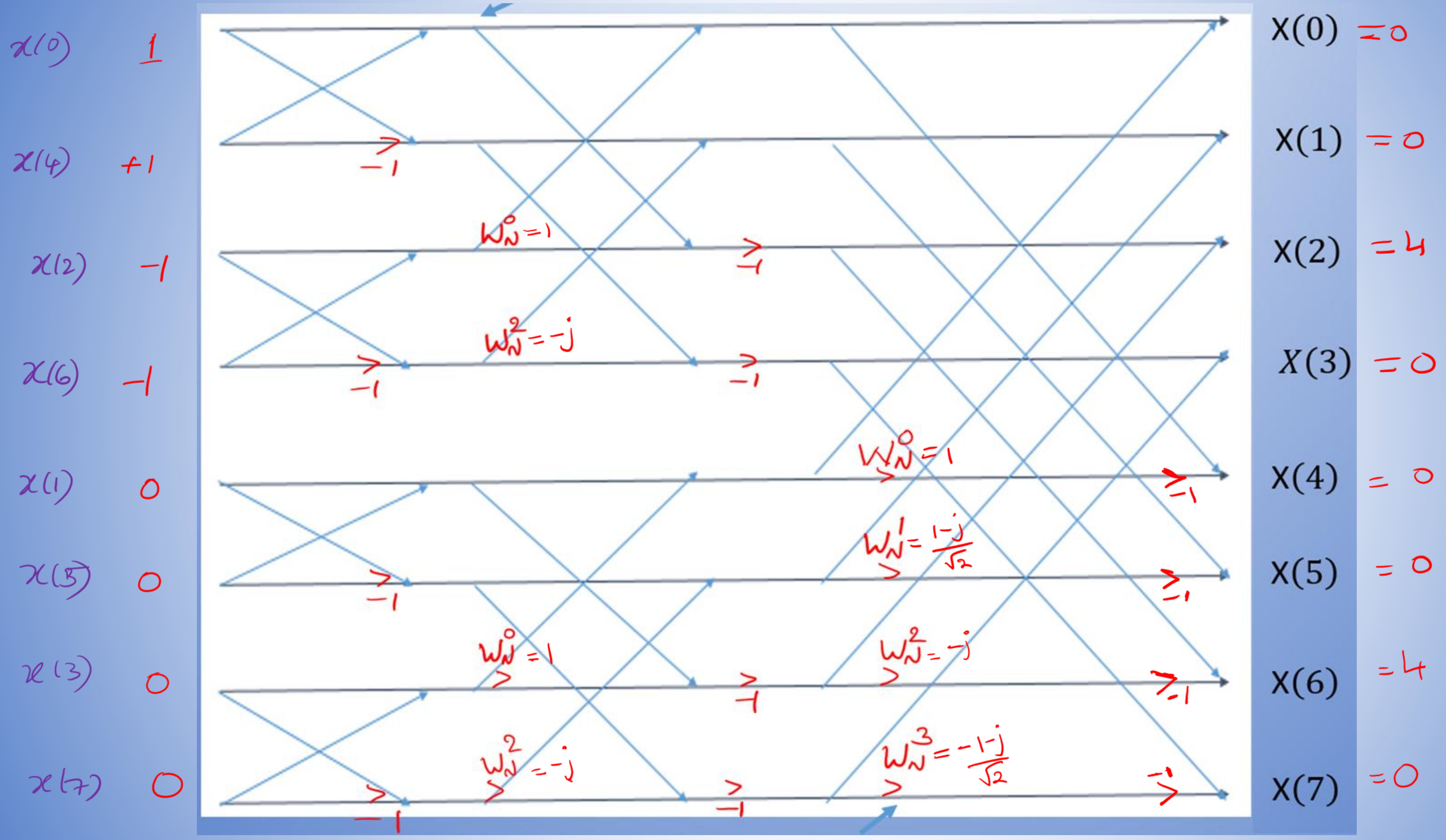
Ans:{6,0,2,0}

Compute 8-pt DFT of the following sequence $x(n) = \cos\frac{n\pi}{2} \quad 0 \le n \le 7$

$= 0$ elsewhere

$$\begin{array}{ccccccccc} & x(0) & & x(2) & & x(4) & & x(6) & \\ x(n) = \{ & 1, & 0, & -1, & 0, & 1, & 0 & -1, & 0 \} \\ n \Rightarrow & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array}$$

$x(n) = \{1, 0, -1, 0, 1, 0, -1, 0\}$



x(0) 1
x(4) +1
x(2) -1
x(6) -1
x(1) 0
x(5) 0
x(3) 0
x(7) 0
-1
W_N^0 = 1
W_N^2 = -j
W_N^1 = (1-j)/√2
W_N^3 = (-1-j)/√2
X(0) = 0
X(1) = 0
X(2) = 4
X(3) = 0
X(4) = 0
X(5) = 0
X(6) = 4
X(7) = 0

# Verify this using DIT-FFT algorithm

$x(n) = \{1,1,1,0,0,0,0,0\}$

$X(k) = \{3,\ 1.71-j1.71,\ -j,\ 0.29+j0.29,\ 1,\ 0.29-j0.29,\ j,\ 1.71+j1.71\}$


Thank you

# Fast Fourier Transform Algorithms - DIFFFT

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# Decimation In Frequency FFT (DIFFFT) Algorithm

- Decompose the computation into successively smaller DFT computations.
- We exploit both the symmetry and periodicity property of the complex exponential $W_N^{nk} = e^{-j\left(\frac{2\pi}{N}\right)nk}$
- In this algorithm the decomposition is based on the output sequence $X(k)$ into successively smaller subsequences

Let $N$ be equal to $2^\nu$, where $\nu$ is an integer

Consider a finite length sequence $x(n)$

Let $L \leq N$ be the length of the signal $x(n)$

Then we can define N-pt DFT of $x(n)$ is

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk}, \quad k = 0, 1, \ldots N-1 \qquad (1)$$

Even numbered frequency points can be written as

$$X(2r) = \sum_{n=0}^{N-1} x(n) W_N^{n2r}, \quad k = 0, 1, \ldots N/2 - 1 \qquad (2)$$

Ily odd frequency points can be written as

$$X(2r+1) = \sum_{n=0}^{N-1} x(n) W_N^{(2r+1)n} \quad k = 0, 1, 2, \ldots \frac{N}{2} - 1 \qquad (3)$$

consider eqn (2), $X(2r) = \sum_{n=0}^{N-1} x(n) W_N^{2rn}$

We can expand it as follows

$$X(2r) = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{2nr} + \sum_{n=\frac{N}{2}}^{N-1} x(n) W_N^{2nr} \quad \text{--- (4)}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{2nr} + \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\frac{N}{2}\right) W_N^{2r\left(n+\frac{N}{2}\right)} \quad \text{--- (5)}$$

$$W_N^{2r\left(n+\frac{N}{2}\right)} = W_N^{(2rn+rN)} = W_N^{2rn} \cdot W_N^{rN} = W_N^{2rn} = W_{N/2}^{rn} \quad \text{--- (6)}$$

$\therefore$ Eqn (5) can be written as

$$X(2r) = \sum_{n=0}^{N/2-1} x(n) \cdot W_{N/2}^{rn} + \sum_{n=0}^{\frac{N}{2}-1} x(n+N/2)\, W_{N/2}^{rn}$$

$$X(2r) = \sum_{n=0}^{N/2-1} \underbrace{\left[x(n) + x\left(n+\frac{N}{2}\right)\right]}_{g(n)} W_{N/2}^{rn} \quad \text{--- (7)}$$

Consider now the odd numbered frequency points

$$X(2r+1) = \sum_{n=0}^{N-1} x(n)\, W_N^{(2r+1)n} \qquad r = 0, 1, \cdots \tfrac{N}{2}-1$$

$$X(2r+1) = \sum_{n=0}^{\frac{N}{2}-1} x(n)\, W_N^{(2r+1)n} + \sum_{n=N/2}^{N-1} x(n)\, W_N^{(2r+1)n} \quad \text{—— (8)}$$

Consider the second term in (8)

$$\sum_{n=N/2}^{N-1} x(n)\, W_N^{(2r+1)n} = \sum_{n=0}^{N/2-1} x(n+\tfrac{N}{2})\, W_N^{(2r+1)(n+\frac{N}{2})}$$

$$= \sum_{n=0}^{N/2-1} x(n+\tfrac{N}{2})\, W_N^{2rn} \cdot W_N^{2r\frac{N}{2}} \cdot W_N^{n} \cdot W_N^{N/2}$$

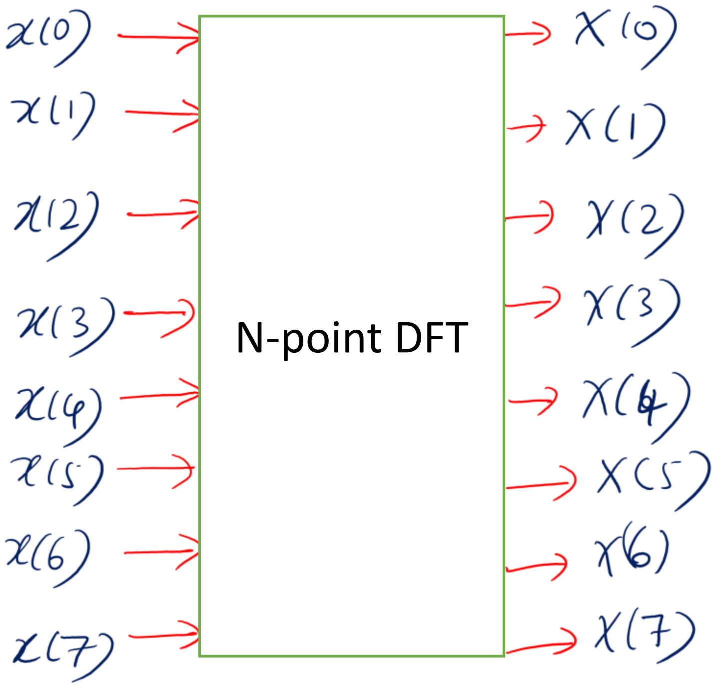
$$= \underbrace{W_N^{2rN/2}} \cdot \underbrace{W_N^{N/2}} \cdot \sum_{n=0}^{\frac{N}{2}-1} x(n+\tfrac{N}{2})\, W_N^{n} \cdot W_{N/2}^{rn}$$

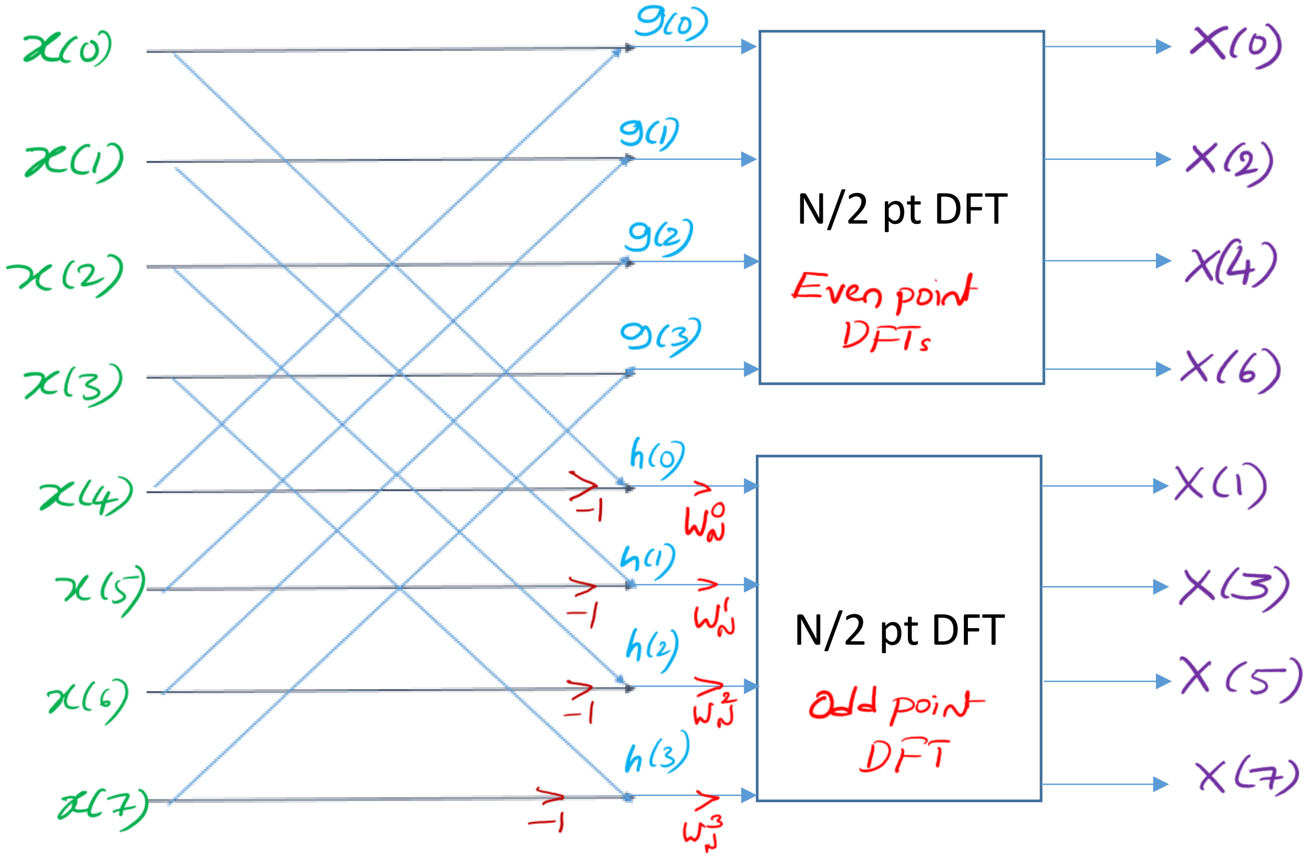
$$= (1) \cdot (-1) \sum_{n=0}^{N/2-1} x(n+N/2)\, W_N^{n} \cdot W_{N/2}^{rn} \quad \text{—— (9)}$$

Therefore eqn (8) can be rewritten as

$$X(2r+1) = \sum_{n=0}^{N/2-1} x(n)\, W_N^{(2r+1)n} + (-1) \sum_{n=0}^{N/2-1} x\left(n+\frac{N}{2}\right) W_N^{(2r+1)n} \qquad (10)$$

$$X(2r+1) = \sum_{n=0}^{N/2-1} \underbrace{\left[x(n) - x(n+N/2)\right]}_{h(n)} W_N^{n} \cdot W_{N/2}^{r} \qquad (11)$$



Equations (7) and equation (11) can be re written as

$$G(r) = X(2r) = \sum_{n=0}^{N/2-1} g(n) W_{N/2}^{rn} \quad \text{—— (12)}$$

&

$$X(2r+1) = \sum_{n=0}^{N/2-1} h(n) W_N^n \cdot W_{N/2}^{rn} \quad \text{—— (13)}$$

Now we consider eqn (12)

$$G(l) = \sum_{n=0}^{N/2-1} g(n) W_{N/2}^{ln}$$

Then even DFT points canbe

$$G(2r) = \sum_{n=0}^{N/2-1} g(n) W_{N/2}^{2rn} \qquad r = 0, 1, \ldots \frac{N}{4}-1 \quad \text{—— (14)}$$

Odd DFT points are

$$G(2r+1) = \sum_{n=0}^{N/2-1} g(n) W_{N/2}^{(2r+1)n} \qquad r = 0, 1, \ldots N/4-1 \quad \text{—— (15)}$$

Consider now eqn (14)

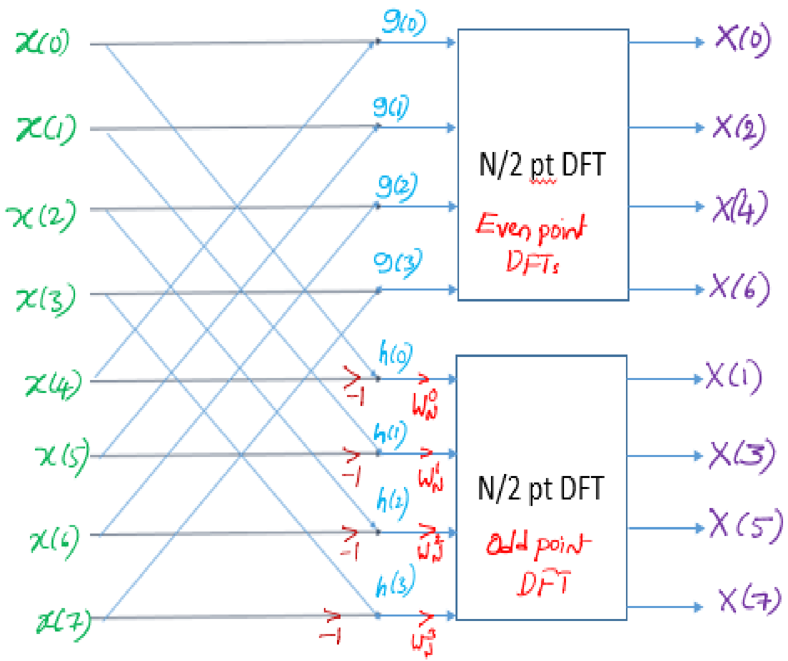
$$G(2r) = \sum_{n=0}^{N/2-1} g(n)\, W_{N/2}^{2rn} = \sum_{n=0}^{N/4-1} g(n)\, W_{N/2}^{2rn} + \sum_{n=N/4-1}^{N/2-1} g(n)\, W_{N/2}^{2rn}$$

$$= \sum_{n=0}^{N/4-1} g(n)\, W_{N/4}^{rn} + \sum_{n=0}^{N/4-1} g(n+N/4) \cdot W_{N/2}^{2r(n+N/4)}$$

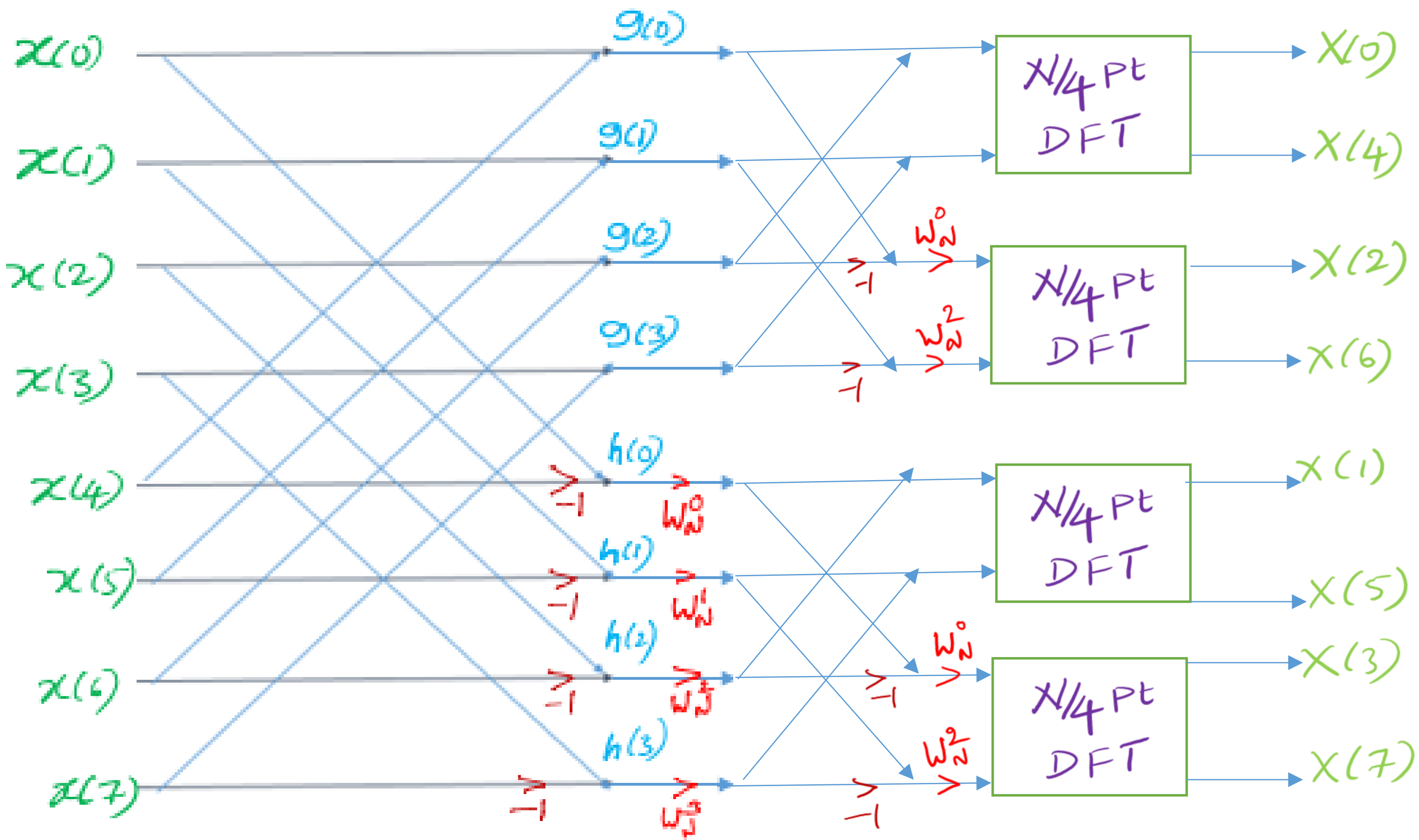
$$= \sum_{n=0}^{N/4-1} g(n)\, W_{N/4}^{rn} + \sum_{n=0}^{N/4-1} g(n+N/4)\, W_{N/4}^{rn} \cdot \underbrace{W_{N/2}^{2rN/4}}_{1}$$

$$G(2r) = \sum_{n=0}^{N/4-1} \left[ g(n) + g\left(n+\frac{N}{4}\right) \right] W_{N/4}^{rn}$$

Similarly $G(2r+1) = \sum_{n=0}^{(N/4-1)} \left[ g(n) - g\left(n-\frac{N}{4}\right) \right] W_{N/2}^{n} \cdot W_{N/4}^{rn}$



x(0)
x(1)
x(2)
x(3)
x(4)
x(5)
x(6)
x(7)
g(0)
g(1)
g(2)
g(3)
h(0)
h(1)
h(2)
h(3)
N/2 pt DFT
Even point DFTs
N/2 pt DFT
Odd point DFT
X(0)
X(2)
X(4)
X(6)
X(1)
X(3)
X(5)
X(7)

x(0)
x(1)
x(2)
x(3)
x(4)
x(5)
x(6)
x(7)
g(0)
g(1)
g(2)
g(3)
h(0)
h(1)
h(2)
h(3)
N/4 pt DFT
N/4 pt DFT
N/4 pt DFT
N/4 pt DFT
X(0)
X(4)
X(2)
X(6)
X(1)
X(5)
X(3)
X(7)

$$G(r) = \sum_{n=0}^{N/4-1} \underbrace{\left[g(n) + g\left(n+\frac{N}{4}\right)\right]}_{x(n)} W_{N/4}^{rn} \rightarrow r = 0, 1, \cdots N/4 - 1$$

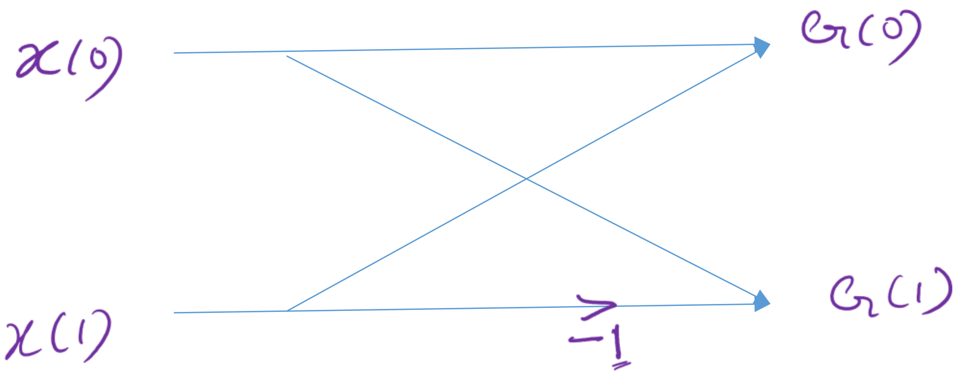
For $N = 8$, $\quad G(r) = \sum_{n=0}^{1} x(n) W_2^{rn} \quad r = 0, 1$

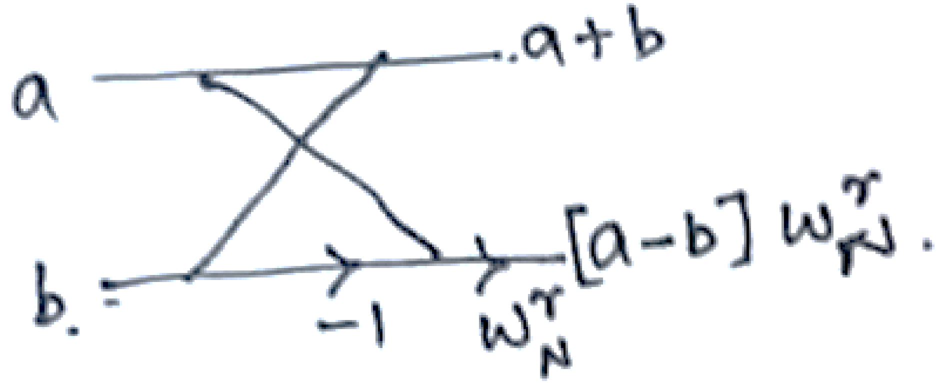

$$G(r) = x(0) W_2^0 + x(1) W_2^r$$

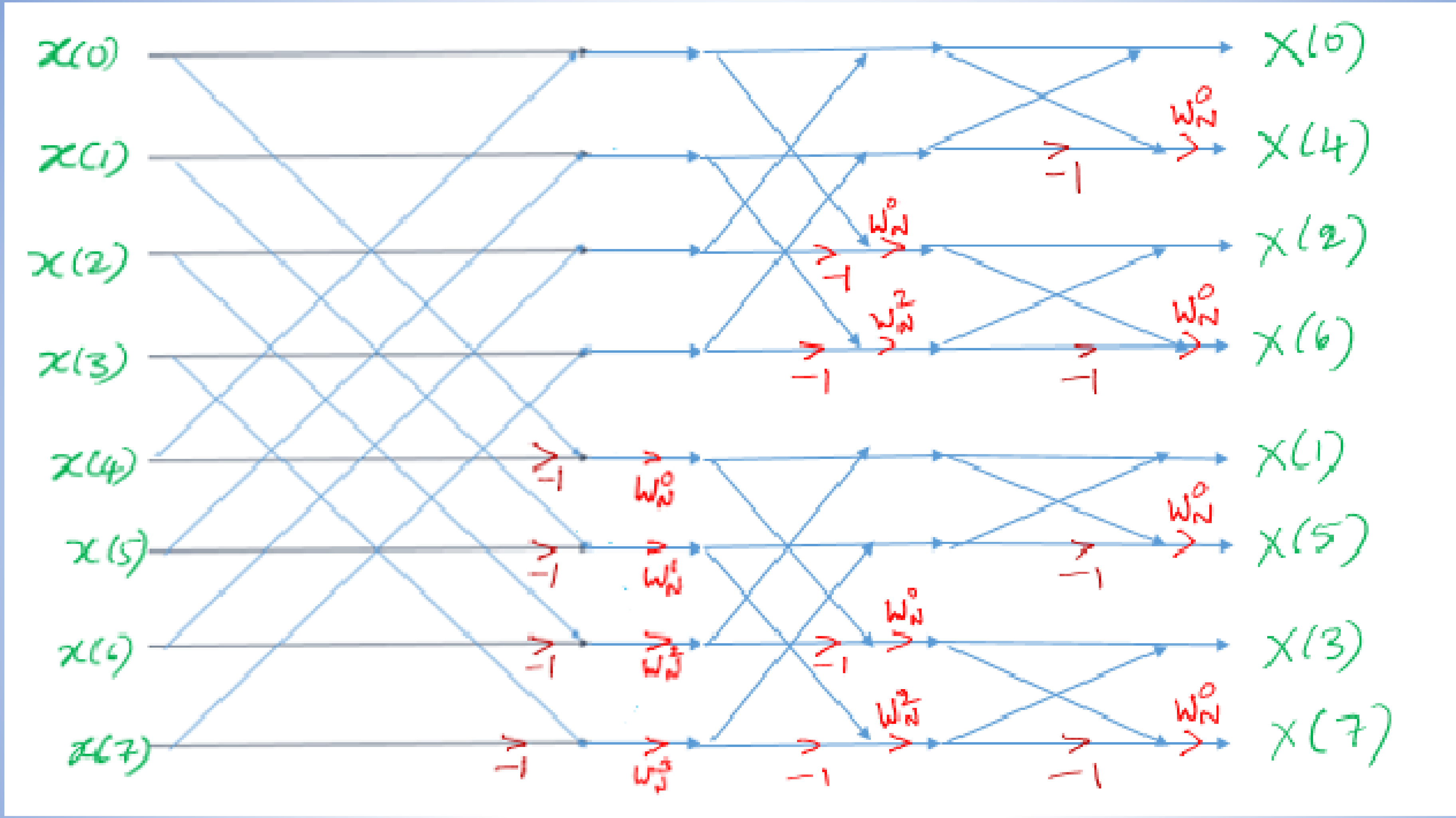
$$G(0) = x(0) + x(1)$$

$$G(1) = x(0) + x(1) W_2^1$$

$$= x(0) - x(1)$$

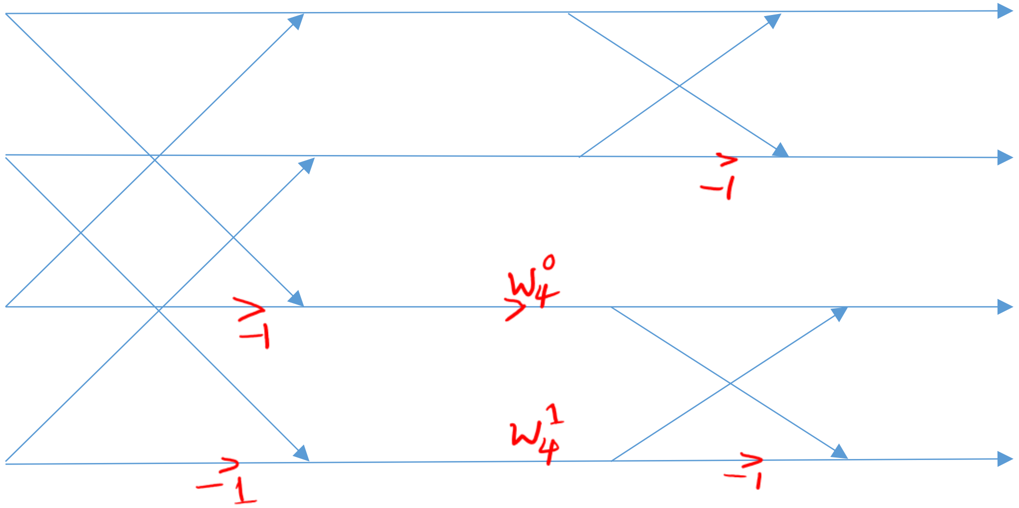


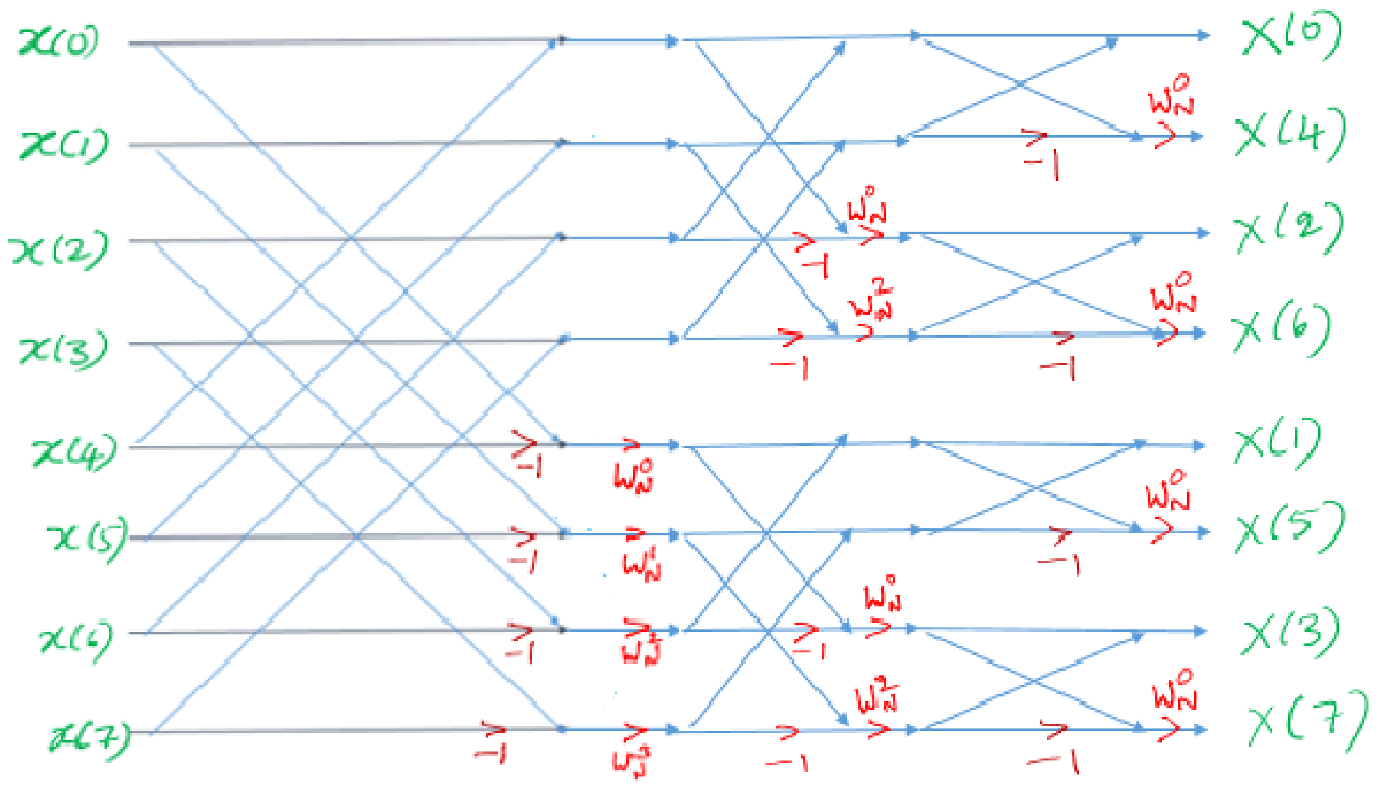





x(0)
x(1)
x(2)
x(3)
x(4)
x(5)
x(6)
x(7)
X(0)
X(4)
X(2)
X(6)
X(1)
X(5)
X(3)
X(7)

Compute 4-pt DFT of a sequence $x(n) = \{1, 2, 3, 4\}$ using DIFFFT algorithm






Number of stages in the signal flow graph = $\log_2 N$

Number of Butterflies in the signal flow graph = $\frac{N}{2}\log_2 N$

Number of complex additions in each butterfly = 2

Number of complex multiplications in each butterfly = 1

Total number of complex additions and multiplications = ? $\left(N\log_2 N + \frac{N}{2}\log_2 N\right)$


Thank
you

# Implementation of Discrete Time Systems (Realization): Structure for FIR systems

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# FIR and IIR systems

- A LTI system is classified based on impulse response as below
  - FIR system
  - IIR system

⇒ FIR system:

- Finite impulse response system.
- impulse response is defined only for finite number of samples.

$$h(n) = \begin{cases} 1 \; ; & n = -1, 2 \\ 2 \; ; & n = 1 \\ 3 \; ; & n = 0, 3 \\ 0 \; ; & \text{otherwise} \end{cases}$$

⇒ IIR system:

- infinite impulse response system
- impulse response have infinite number of samples.

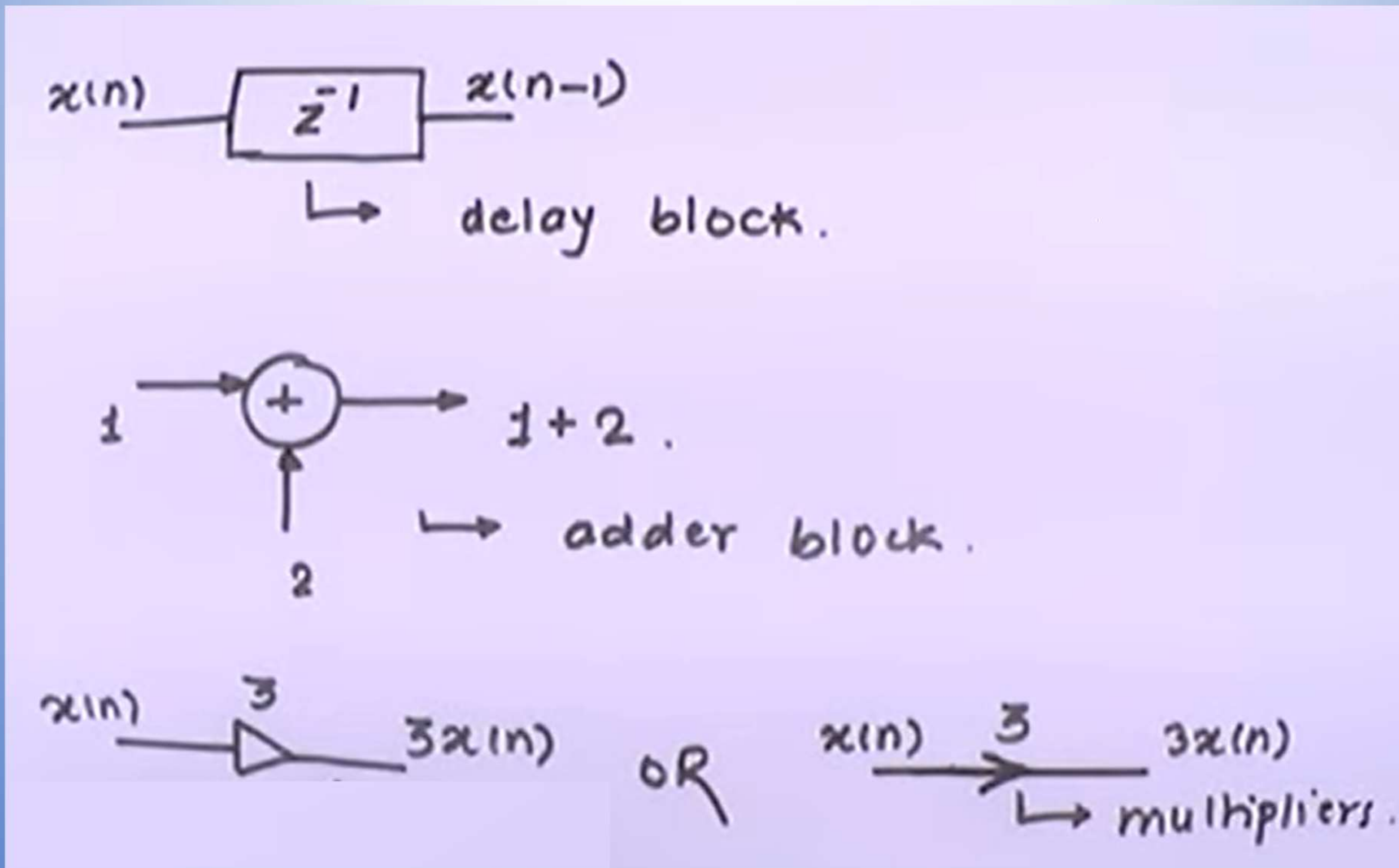
$$h(n) = a^n u(n)$$

| | |
|---|---|
| IIR filters are less powerful than FIR filters, & require less processing power and less work to set up the filters | FIR filters are more powerful than IIR filters, but also require more processing power and more work to set up the filters |
| It cannot implement linear-phase filtering. | It can implement linear-phase filtering. |
| It cannot be used to correct frequency-response errors in a loudspeaker | It can be used to correct frequency-response errors in a loudspeaker to a finer degree of precision than using IIRs |
| Usage is generally more easier than FIR filters. | Usage is generally more complicated and time-consuming than IIR filters |
| IIR filter uses current input sample value, past input and output samples to obtain current output sample value. | FIR filter uses only current and past input digital samples to obtain a current output sample value. It does not utilize past output samples. |

| | |
|---|---|
| Simple IIR equation is mention below.,y(n)= b(0)x(n) + b(1)x(n-1) + b(2)x(n-2) + b(3)x(n-3) + a(1)y(n-1) + a(2)y(n-2) + a(3)y(n-3) | Simple FIR equation is mention below. $\boxed{y(n)= h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + h(3)x(n-3) + h(4)x(n-4)}$ |
| Transfer function of IIR filter will have both zeros and poles and will require less memory than FIR counterpart | Transfer function of FIR filter will have only zeros, need more memory |

| | |
|---|---|
| IIR filters are not stable as they are recursive in nature and feedback is also involved in the process of calculating output sample values. | FIR filters are preferred due to its linear phase response and also they are non-recursive. Feedback is not involved in FIR, hence they are stable |

- Realization – hardware – adder, multiplier and delay


x(n)
z^-1
x(n-1)
delay block.
1
1+2.
2
adder block.
x(n)
3
3x(n)
OR
x(n)
3
3x(n)
multipliers.

## Structures for FIR systems

FIR systems can be described by $y(n) = \sum_{k=0}^{M-1} b_k \cdot x(n-k)$

or $Y(z) = \sum_{k=0}^{M-1} b_k z^{-k} X(z)$

$$\frac{Y(z)}{X(z)} = H(z) = \sum_{k=0}^{M-1} b_k z^{-k}$$

The unit sample response (impulse response), $h(n) = \begin{cases} b_n & 0 \le n \le M-1 \\ 0 & \text{otherwise} \end{cases}$

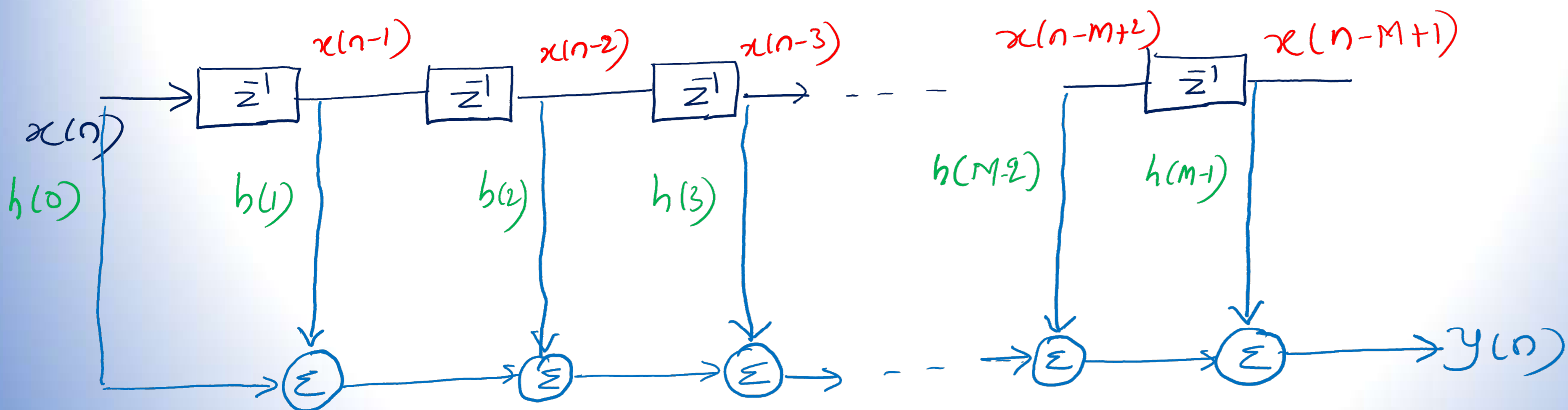
Structures:

1. Direct form
2. Cascaded form
3. Frequency Sampling
4. Lattice

Direct Form Structure

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k)$$

$$= h(0)\cdot x(n) + h(1)x(n-1) + \cdots + h(M-1)\cdot x(n-(M-1))$$




(M) multiplications and (M-1) additions

Tapped delay line structure

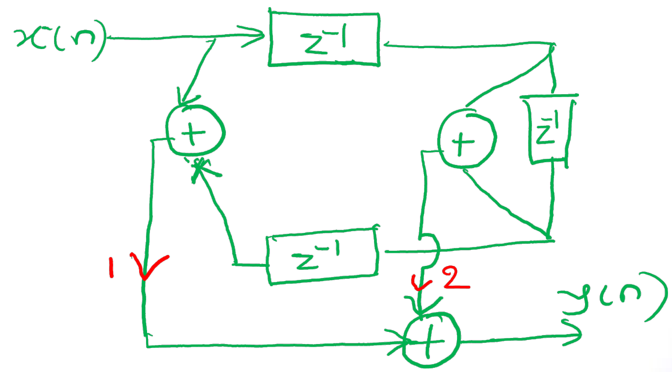
## Linear Phase FIR filters:

An FIR filter is said to have linear phase if its impulse response satisfies either symmetry or asymmetry conditions

$$h(n) = \pm h(M-1-n)$$

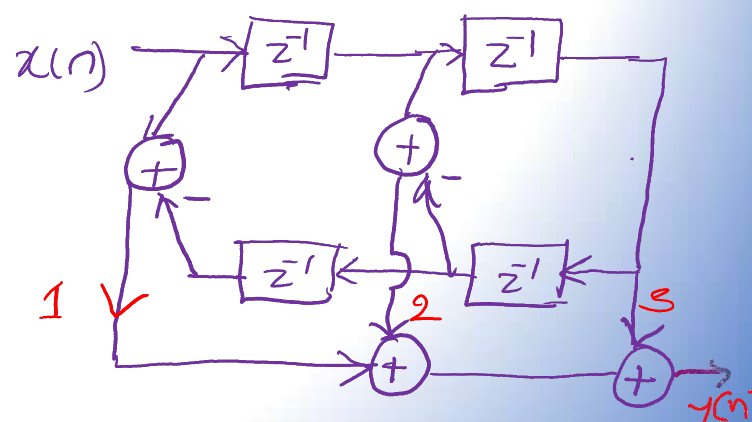
Ex: $h(n) = \{1, 2, 2, 1\}$ $\quad M = 4$

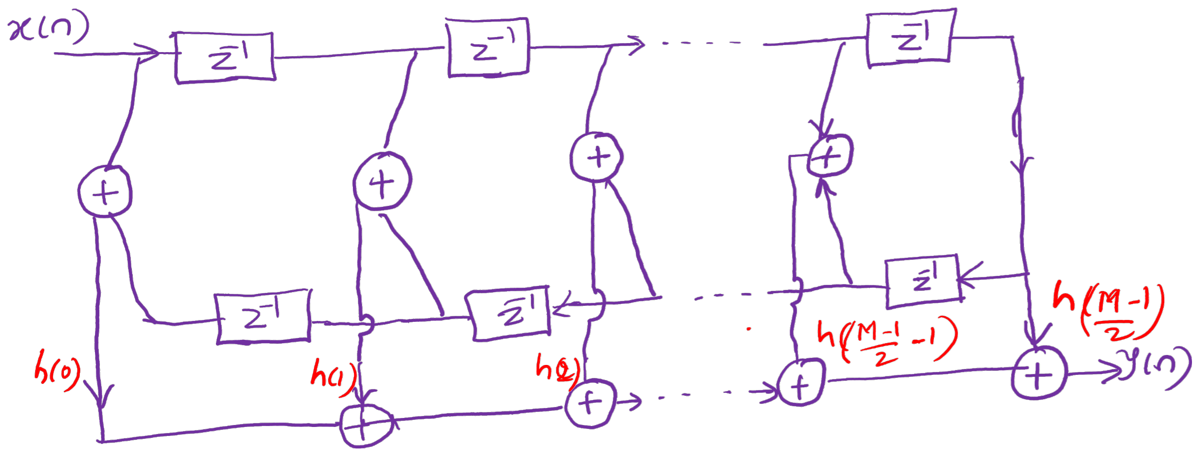
$$y(n) = x(n) + 2x(n-1) + 2x(n-2) + x(n-3)$$



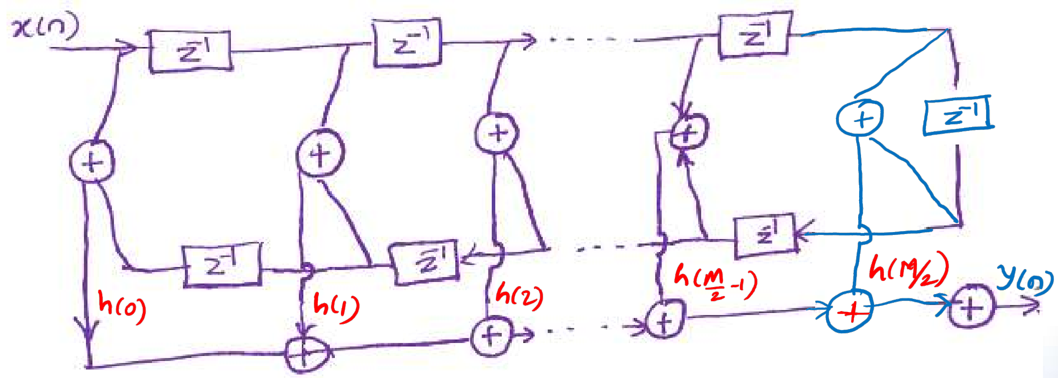

$$h(n) = \{1, 2, 3, -2, -1\} \quad M = 5$$

$$y(n) = x(n) + 2x(n-1) + 3x(n-2) - 2x(n-3) - x(n-4)$$






Odd M

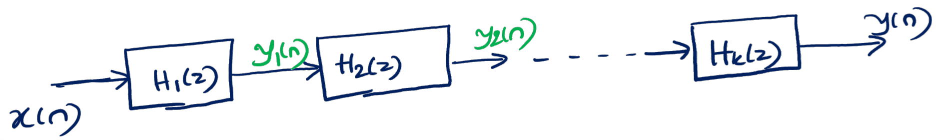



Even M

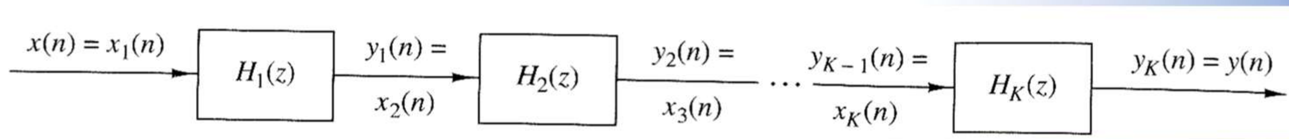
Cascade- Form structures

$$H(z) = \prod_{k=1}^{K} H_k(z)$$

$$H_k(z) = b_{k0} + b_{k1} z^{-1} + b_{k2} z^{-2} \quad , \quad k = 1, 2, \cdots K$$



Cascade- Form structures



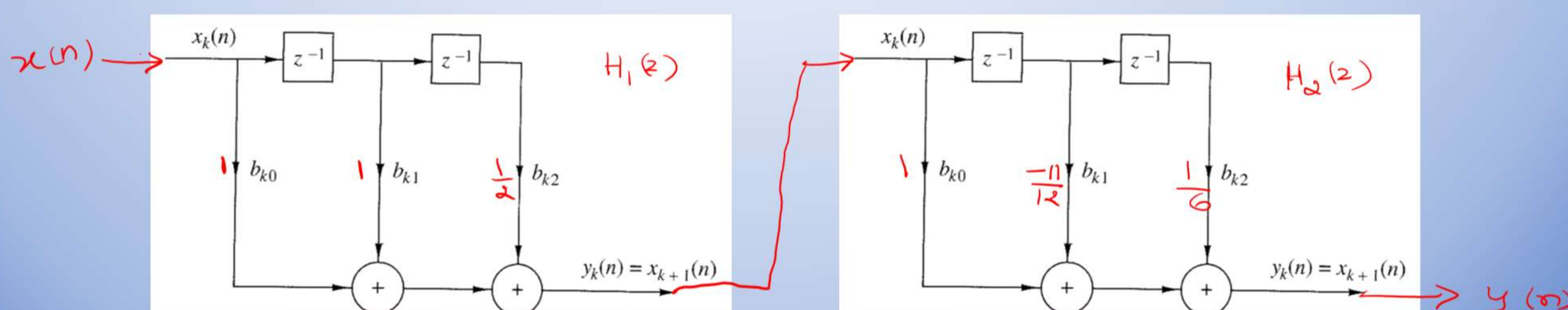

Example: Obtain the cascaded realization for

$$H(z) = \left[1+\left(\tfrac{1}{2}+j\tfrac{1}{2}\right)z^{-1}\right]\left[1+\left(\tfrac{1}{2}-j\tfrac{1}{2}\right)z^{-1}\right]\left[1-\tfrac{2}{3}z^{-1}\right]\left[1-\tfrac{1}{4}z^{-1}\right]$$

$$= \left[1+z^{-1}+\tfrac{1}{2}z^{-2}\right]\left[1-\tfrac{11}{12}z^{-1}+\tfrac{1}{6}z^{-2}\right]$$

$H_1(z) = \left[1+z^{-1}+\tfrac{1}{2}z^{-2}\right]$ $\qquad \dfrac{Y_1(z)}{X(z)} = H_1(z)$

$H_2(z) = \left[1-\tfrac{11}{12}z^{-1}+\tfrac{1}{6}z^{-2}\right]$ $\qquad \dfrac{Y_2(z)}{Y_1(z)} = H_2(z)$




Thank
you

# Structure for IIR systems

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## Structure for IIR systems:

LTI discrete time systems can be characterized by Linear constant coefficient difference equation(LCCDE)

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

Zeros

Poles

Direct Form Structures:

- Direct Form I
- Direct Form II

Transposed

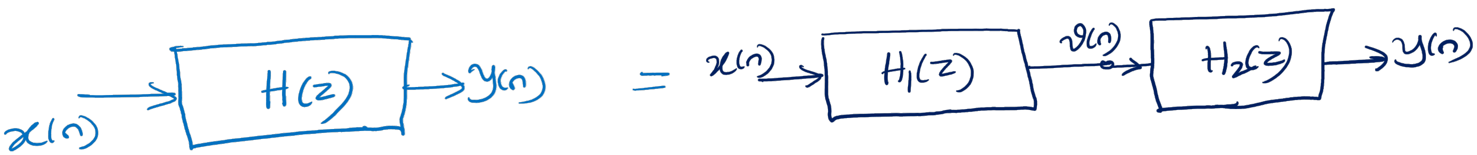
Cascaded

Parallel

Lattice-Ladder structure

## Direct Form I structure realization:

$$H(z)=\frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}} = \sum_{k=0}^{M} b_k z^{-k} \times \frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}} = H_1(z)\, H_2(z) \quad \text{—} (1)$$




i.e, $$H(z)=\frac{Y(z)}{X(z)} = \frac{V(z)}{X(z)}\cdot\frac{Y(z)}{V(z)} \quad \text{—} (2)$$

$$= H_1(z)\cdot H_2(z)$$

$$H_1(z) = \frac{V(z)}{X(z)} = \sum_{k=0}^{M} b_k z^{-k} \quad \text{—} (3)$$

and

$$H_2(z) = \frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}} \quad \text{—} (4)$$

Consider eqn (3)

$$H_1(z) = \frac{V(z)}{X(z)} = \sum_{k=0}^{M} b_k z^{-k} \quad \text{--- (3)}$$

$$V(z) = \sum_{k=0}^{M} b_k X(z) z^{-k}$$

Taking inverse Z.T. we get

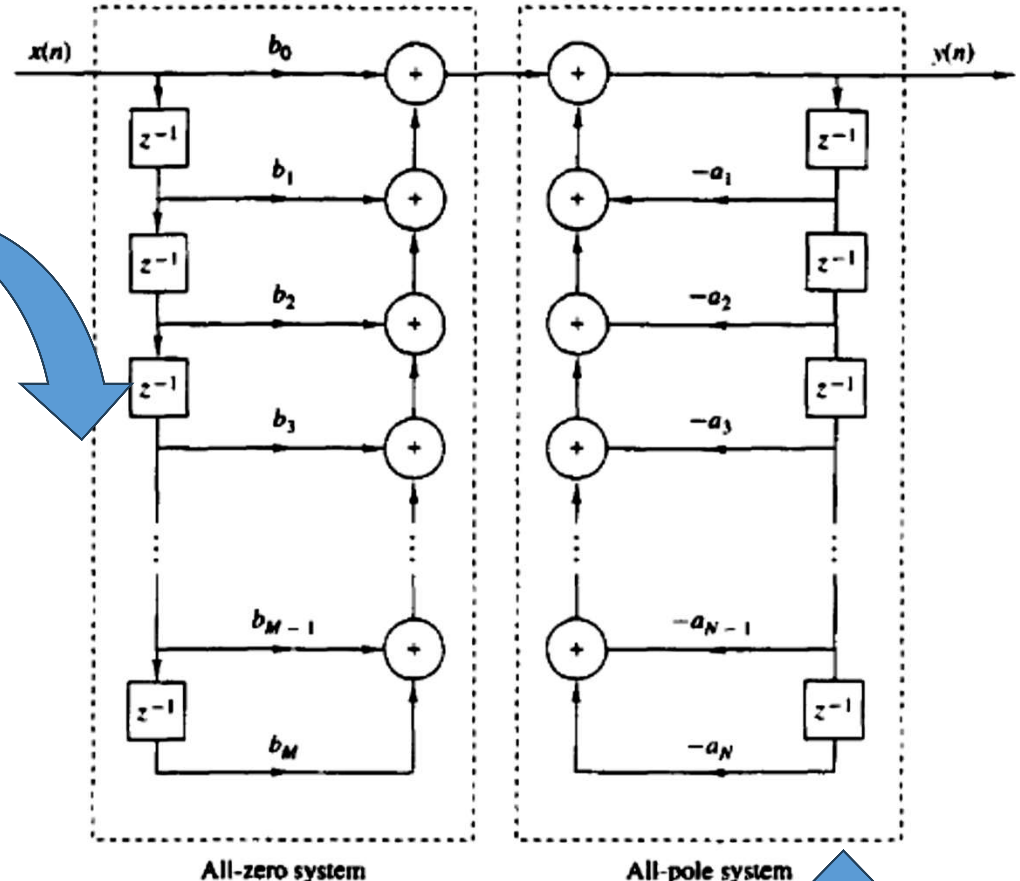
$$v(n) = \sum_{k=0}^{M} b_k x(n-k) \quad \text{--- (5)}$$

Consider eqn (4) $\quad H_2(z) = \dfrac{Y(z)}{V(z)} = \dfrac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}}$

$$Y(z) + \sum_{k=1}^{N} a_k z^{-k} Y(z) = V(z)$$

$$Y(z) = V(z) - \sum_{k=1}^{N} a_k z^{-k} \cdot Y(z)$$

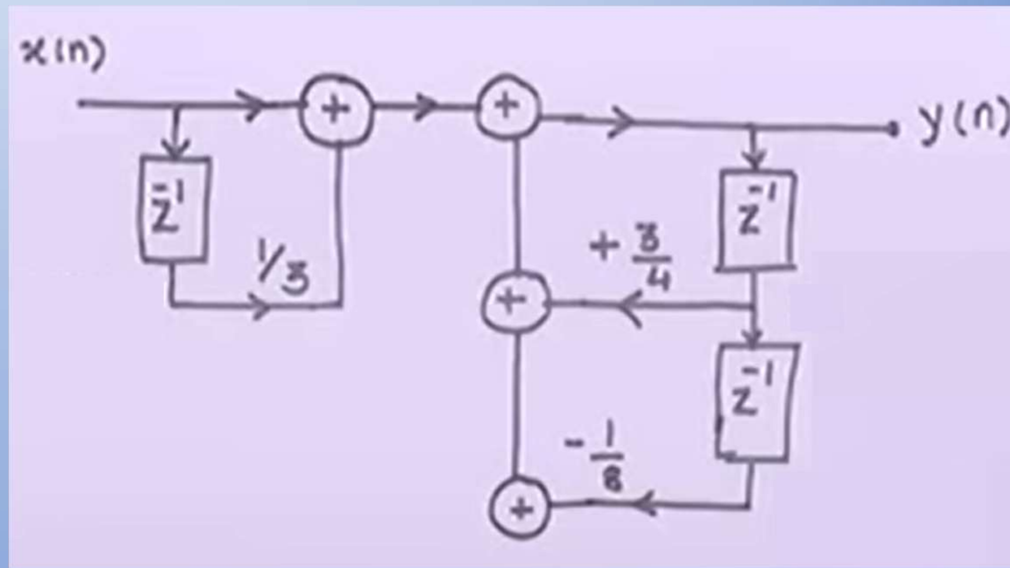
or $\quad y(n) = v(n) - \sum_{k=1}^{N} a_k y(n-k) \quad \text{--- (6)}$



Find Direct form I structure

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n) + \frac{1}{3}x(n-1)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}.$$



Direct Form II structure realization:

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}} = \frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}} \text{ X } \sum_{k=0}^{M} b_k z^{-k} = H_1(z)\, H_2(z)$$

$$H_1(z) = \frac{V(z)}{X(z)} = \frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}}$$

and

$$H_2(z) = \frac{Y(z)}{V(z)} = \sum_{k=0}^{M} b_k z^{-k}$$

$$V(z) + \sum_{k=1}^{N} a_k z^{-k} V(z) = X(z)$$

$$Y(z) = \sum_{k=0}^{M} b_k z^{-k} V(z)$$

Taking inverse Z.T.

$$v(n) = x(n) - \sum_{k=1}^{N} a_k v(n-k)$$

$$y(n) = \sum_{k=0}^{M} b_k v(n-k)$$

Let us rewrite

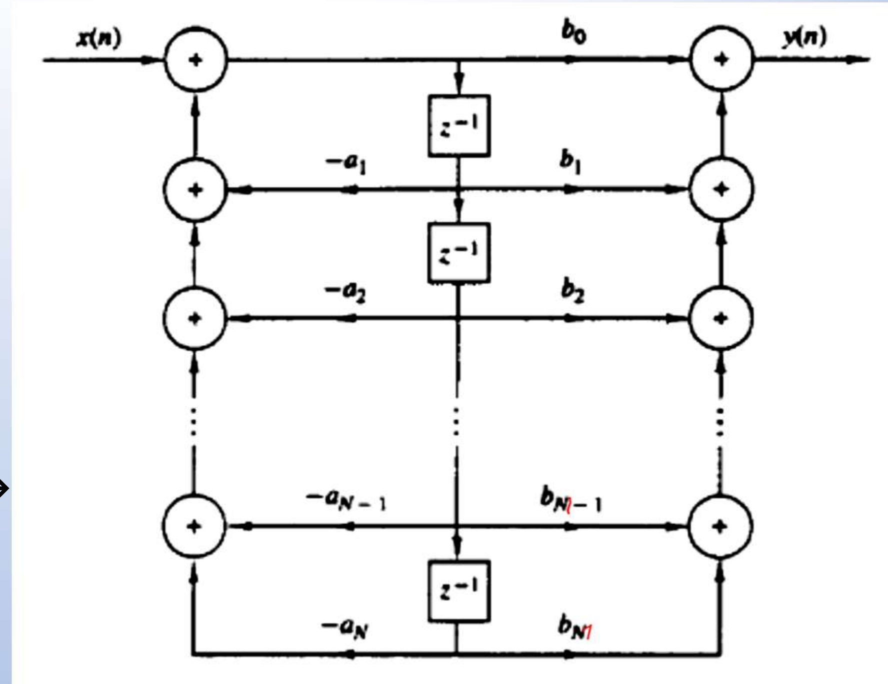
$$v(n) = x(n) - \sum_{k=1}^{N} a_k v(n-k)$$

$$v(n) = x(n) - a_1 v(n-1) - a_2 v(n-2) \cdots a_N v(n-N)$$

$$y(n) = \sum_{k=0}^{M} b_k v(n-k)$$

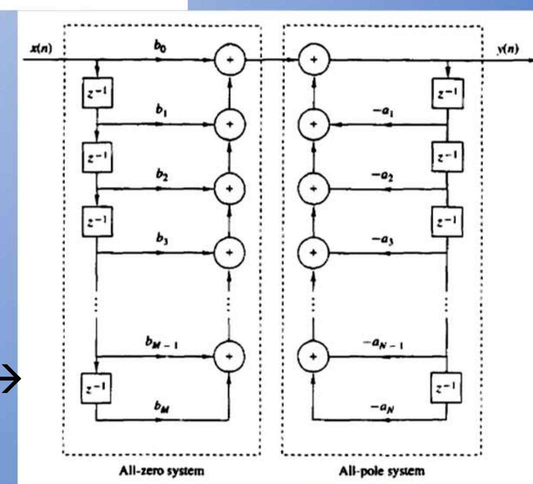
$$y(n) = b_0 v(n) + b_1 v(n-1) + \cdots b_M v(n-M)$$

DF 2→



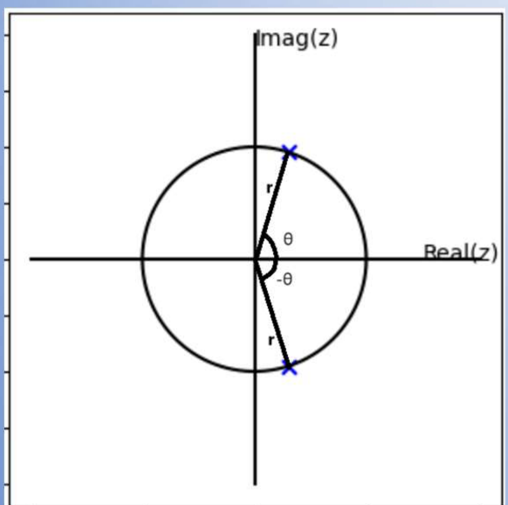

**Delay blocks are reduced – Canonic form**

DF 1 →



# Two pole structures:

- If zero/pole is complex – difficult to implement (complex coefficients)
- This can be converted into structure with real coefficients – 2 pole structures




$$H(z) = \frac{1}{(1 - p_1 z^{-1})(1 - p_1^* z^{-1})}$$

$$H(z) = \frac{1}{(1 - re^{j\theta} z^{-1})(1 - re^{-j\theta} z^{-1})}$$

$$= \frac{1}{1 - (re^{j\theta} + re^{-j\theta}) z^{-1} + r^2 z^{-2}}$$

$$H(z) = \frac{1}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}}$$

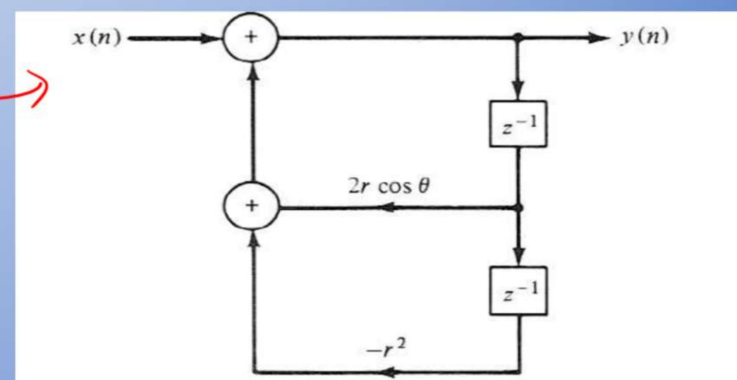



**FIGURE 7.44** Realization of a two-pole IIR filter.

# Recap – Goertzel algorithm:

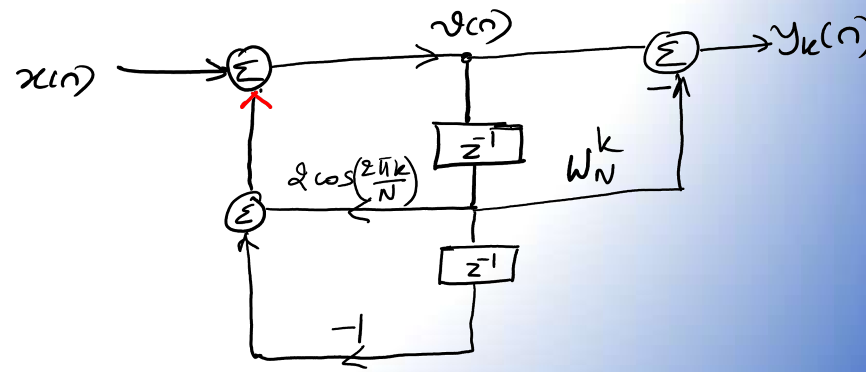
$$h(n) = W_N^{-nk} u(n) \Longleftrightarrow H(z) = \frac{1}{1 - W_N^{-k} z^{-1}} \quad \text{—— (5)}$$

Now we can realize the structure for equation (5) as follows:

$$H_k(z) = \frac{1}{1 - W_N^{-k} z^{-1}} \frac{(1 - W_N^{k} z^{-1})}{(1 - W_N^{k} z^{-1})} = \frac{1 - W_N^{k} z^{-1}}{1 - (W_N^{k} + W_N^{-k}) z^{-1} + z^{-2}}$$

$$H_k(z) = \frac{1 - W_N^{k} z^{-1}}{1 - 2\cos\left(\frac{2\pi k}{N}\right) z^{-1} + z^{-2}} \quad \text{—— (6)}$$

Let $H_k(z) = \dfrac{Y_k(z)}{X(z)} = \dfrac{1 - W_N^{k} z^{-1}}{1 - 2\cos\left(\frac{2\pi k}{N}\right) z^{-1} + z^{-2}}$




Direct form II realization for computing $k^{th}$ DFT point


Thank you

# Structure for IIR systems - 2

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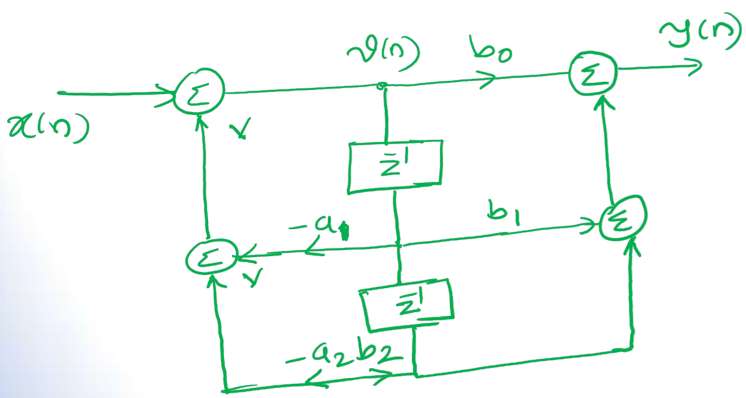
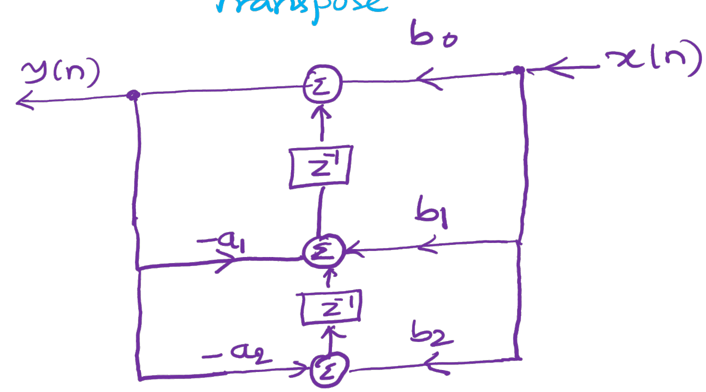
# Transpose structure

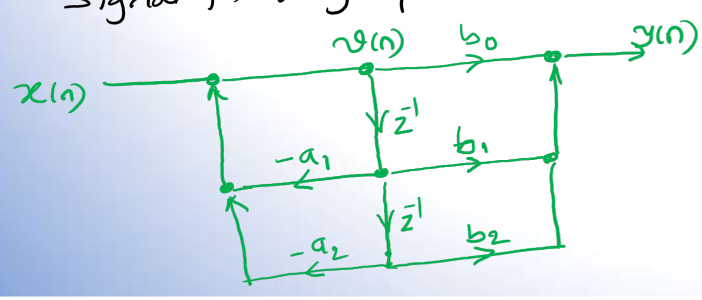
- First draw DF 2 structure and form SFG
- Interchange i/p and o/p
- Reverse the direction of all branches
- Summing points become branching points and branching points become summing points



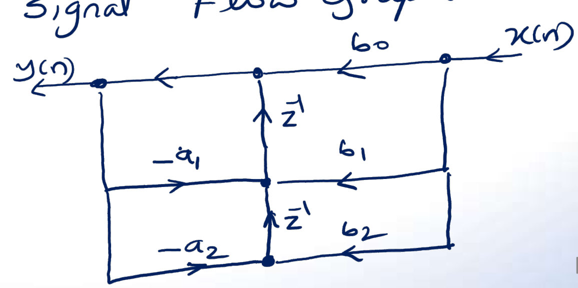

Transpose




Signal flow graph




Signal Flow graph



## Cascaded form structures

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

Assume $N \geq M$

This system canbe cascaded into second order subsystems

$$H(z) = \prod_{k=1}^{K} H_k(z)$$

$$x(n) \rightarrow \boxed{H_1(z)} \rightarrow \boxed{H_2(z)} \rightarrow \cdots \rightarrow \boxed{H_K(z)} \rightarrow y(n)$$

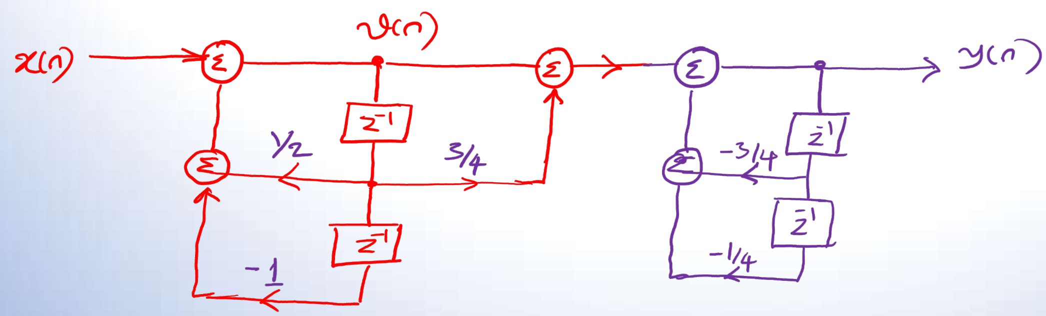
Where each $H_k(z)$ to have denominator polynomial of degree 2.

Ex: Consider the system function $H(z) = \dfrac{1+\frac{3}{4}z^{-1}}{\left(1-\frac{1}{2}z^{-1}+z^{-2}\right)\left(1+\frac{3}{4}z^{-1}+\frac{1}{4}z^{-2}\right)}$

Realize the given function in cascaded form.

$$\text{Let } H(z) = \frac{1+\frac{3}{4}z^{-1}}{\left(1-\frac{1}{2}z^{-1}+z^{-2}\right)\left(1+\frac{3}{4}z^{-1}+\frac{1}{4}z^{-2}\right)} = \underbrace{\frac{1+\frac{3}{4}z^{-1}}{\left(1-\frac{1}{2}z^{-1}+z^{-2}\right)}}_{H_1(z)} \times \underbrace{\frac{1}{\left(1+\frac{3}{4}z^{-1}+\frac{1}{4}z^{-2}\right)}}_{H_2(z)}$$

$$H(z) = H_1(z) \cdot H_2(z)$$



# Parallel Form structures

Parallel form structures can be obtained by performing partial fraction of H(z)

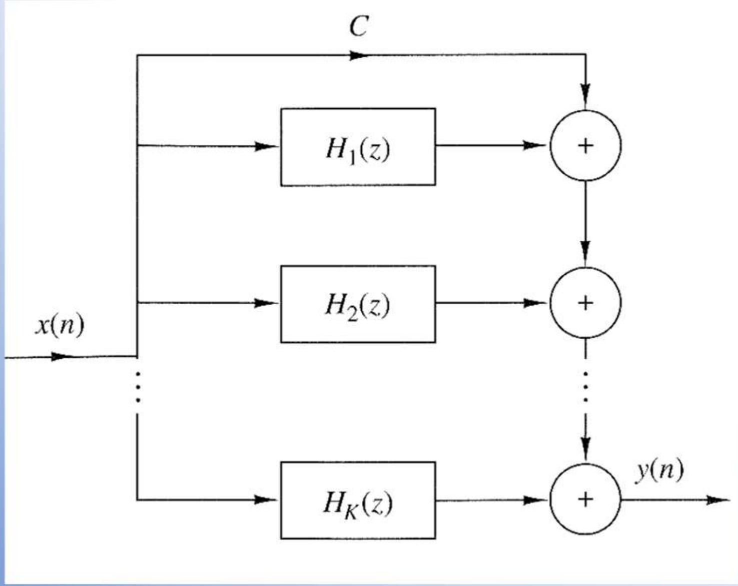
Assume $N \geq M$

constant $= \frac{b_N}{a_N}$

residues in the partial fraction

$$H(z) = C + \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}}$$

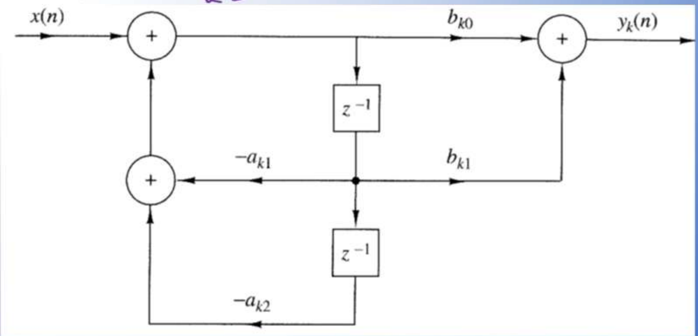
pole (Distinct)




Note:
For complex valued poles corresponding $A_k$ are also complex.
To avoid multiplication by complex numbers we can form two pole subsystem by combining pair of conjugate poles.

Such subsystems are of the form

$$H_k(z) = \frac{b_{k0} + b_{k1} z^{-1}}{1 + a_{k1} z^{-1} + a_{k2} z^{-2}}$$



Example: Realize the given function in parallel form.

$$H(z) = \frac{1+2z^{-1}+z^{-2}}{1-0.75z^{-1}+0.125z^{-2}}$$

$$\begin{array}{r} 8 \\ 1-0.75z^{-1}+0.125z^{-2} \;\big)\; 1+2z^{-1}+z^{-2} \\ 8-6z^{-1}+z^{-2} \\ \hline -7+8z^{-1} \end{array}$$

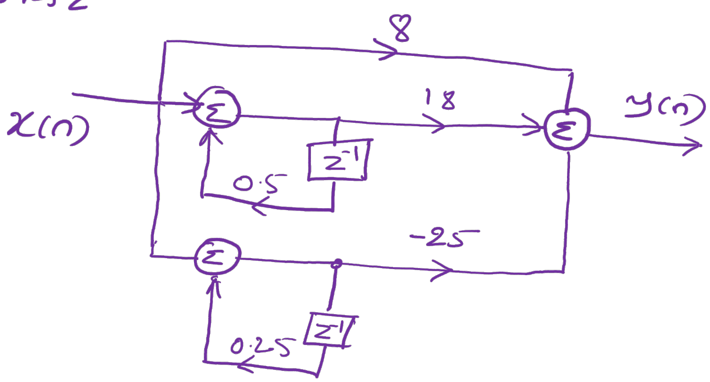
$$\therefore\; H(z) = \frac{1+2z^{-1}+z^{-2}}{1-0.75z^{-1}+0.125z^{-2}} = 8 + \frac{-7+8z^{-1}}{1-0.75z^{-1}+0.125z^{-2}}$$

$$\frac{-7+8z^{-1}}{1-0.75z^{-1}+0.125z^{-2}} = \frac{A}{(1-0.5z^{-1})} + \frac{B}{(1-0.25z^{-1})}$$

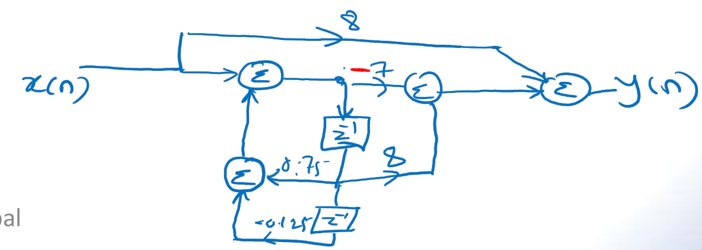
$$\therefore\; (-7+8z^{-1}) = A(1-0.25z^{-1}) + B(1-0.5z^{-1})$$

$A = 18,\; B = -25$

$$H(z) = 8 + \frac{18}{1-0.5z^{-1}} - \frac{25}{1-0.25z^{-1}}$$

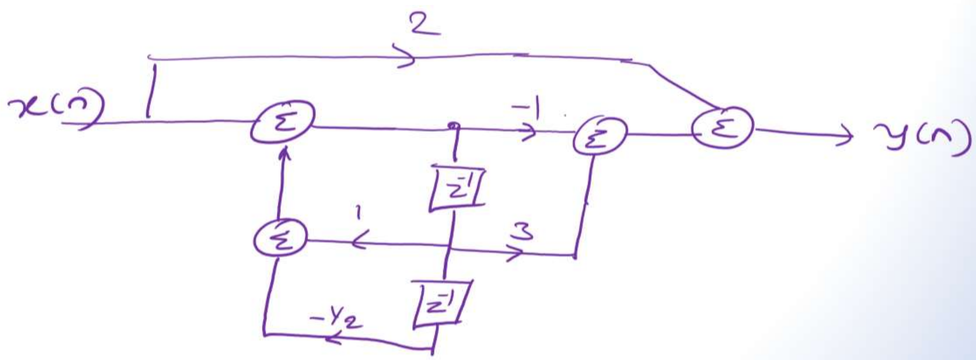



OR



Ex: Realize the given function in parallel form: $H(z) = \dfrac{1+z^{-1}+z^{-2}}{1-z^{-1}+\frac{1}{2}z^{-2}}$

$$H(z) = 2 + \frac{-1+3z^{-1}}{1-z^{-1}+\frac{1}{2}z^{-2}}$$




H.W.

$$H(z) = \frac{1+\frac{1}{4}z^{-1}}{\left(1+\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}\right)} = \frac{A}{\left(1+\frac{1}{2}x\right)} + \frac{Bx+C}{\left(1+\frac{1}{2}x+\frac{1}{4}x^{2}\right)}$$


Thank you

# Structure for FIR systems: Frequency sampling structure

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# Frequency sampling structure:

$$H_d(e^{jw}) \xrightarrow[\text{By freq}^n\text{ sampling}]{} H_d(k) \xrightarrow[\text{By appling IDFT.}]{} h(n) \longrightarrow$$

Take Z-transform to obtain the structure

By freq$^n$ sampling

$$\omega = \frac{2\pi k}{N}$$

By appling IDFT.

$$h(n) = \frac{1}{N} \sum_{k=0}^{N-1} H(k)\, W_N^{-nk}$$

where

$n = 0 - N-1$

# Frequency Sampling Structure

Alternative structure for FIR filter

Parameters that characterize the parameter are the values of the desired frequency response instead of impulse response

To derive the frequency sampling structure, we specify the desired frequency response at a set of equally spaced frequencies,

$$\omega_k = \frac{2\pi}{M}(k+\alpha), \quad k = 0, 1, \ldots \left(\frac{M-1}{2}\right) \text{ if } M \text{ odd}$$

$$k = 0, 1, \ldots \frac{M}{2} - 1 \quad \text{if } M \text{ even}$$

$$\alpha = 0 \text{ or } 1/2$$

To get the frequency sampling structure, we take F.T. of h(n)

$$H(\omega) = \sum_{n=0}^{M-1} h(n) e^{-j\omega n} \quad \text{——— (1)}$$

The value of $H(\omega)$ at frequencies $\omega_k = \frac{2\pi(k+\alpha)}{M}$ are

$$H(k+\alpha) = H\left(\frac{2\pi}{M}(k+\alpha)\right) \quad \text{——— (2)}$$

$$H(k+\alpha) = \sum_{n=0}^{M-1} h(n) e^{-j\frac{2\pi(k+\alpha)}{M} n} \quad k = 0, 1, \ldots M-1 \quad \text{——— (3)}$$

The set of values $\{H(k+\alpha)\}$ are called frequency samples of $H(\omega)$

For $\alpha = 0$, $\{H(k)\}$ corresponds to M-pt DFT of $\{h(n)\}$

So we can get back $\{h(n)\}$ as follows:

$$h(n) = \frac{1}{M} \sum_{k=0}^{M-1} H(k+\alpha)\, e^{\frac{j2\pi(k+\alpha)n}{M}} \quad \text{— (4)} \qquad n = 0, 1, \ldots M-1$$

Our intension is to get filter structure, which can be obtained using Z.T.

$$H(z) = \sum_{n=0}^{M-1} h(n) z^{-n} \quad \text{— (5)}$$

$$= \sum_{n=0}^{M-1} \left[ \frac{1}{M} \sum_{k=0}^{M-1} H(k+\alpha) e^{\frac{j2\pi(k+\alpha)}{M} \cdot n} \right] z^{-n} \quad \text{— (6)}$$

$$H(z) = \sum_{k=0}^{M-1} H(k+\alpha) \frac{1}{M} \sum_{n=0}^{M-1} \left[ e^{\frac{j2\pi(k+\alpha)}{M}} z^{-1} \right]^{n} = \sum_{k=0}^{M-1} \frac{H(k+\alpha)}{M} \frac{1 - \left[ e^{\frac{j2\pi(k+\alpha)}{M}} z^{-1} \right]^{M}}{1 - e^{\frac{j2\pi(k+\alpha)}{M}} \cdot z^{-1}}$$

$$H(z) = \sum_{k=0}^{M-1} \frac{H(k+\alpha)}{M} \frac{1 - \left[e^{j\frac{2\pi(k+\alpha)}{M}} z^{-1}\right]^M}{1 - e^{j\frac{2\pi(k+\alpha)}{M}} \cdot z^{-1}} = \sum_{k=0}^{M-1} \frac{H(k+\alpha)}{M} \frac{1 - e^{j2\pi\alpha} \cdot z^{-M}}{1 - e^{j\frac{2\pi(k+\alpha)}{M}} z^{-1}}$$

$$H(z) = \frac{1 - e^{j2\pi\alpha} z^{-M}}{M} \sum_{k=0}^{M-1} \frac{H(k+\alpha)}{1 - e^{j\frac{2\pi(k+\alpha)}{M}} z^{-1}} \quad \text{—— (7)}$$

Now the system function $H(z)$ is characterized by frequency samples $\{H(k+\alpha)\}$

We can view this FIR filter realization as a cascade of two filters

$$H(z) = H_1(z) \cdot H_2(z)$$

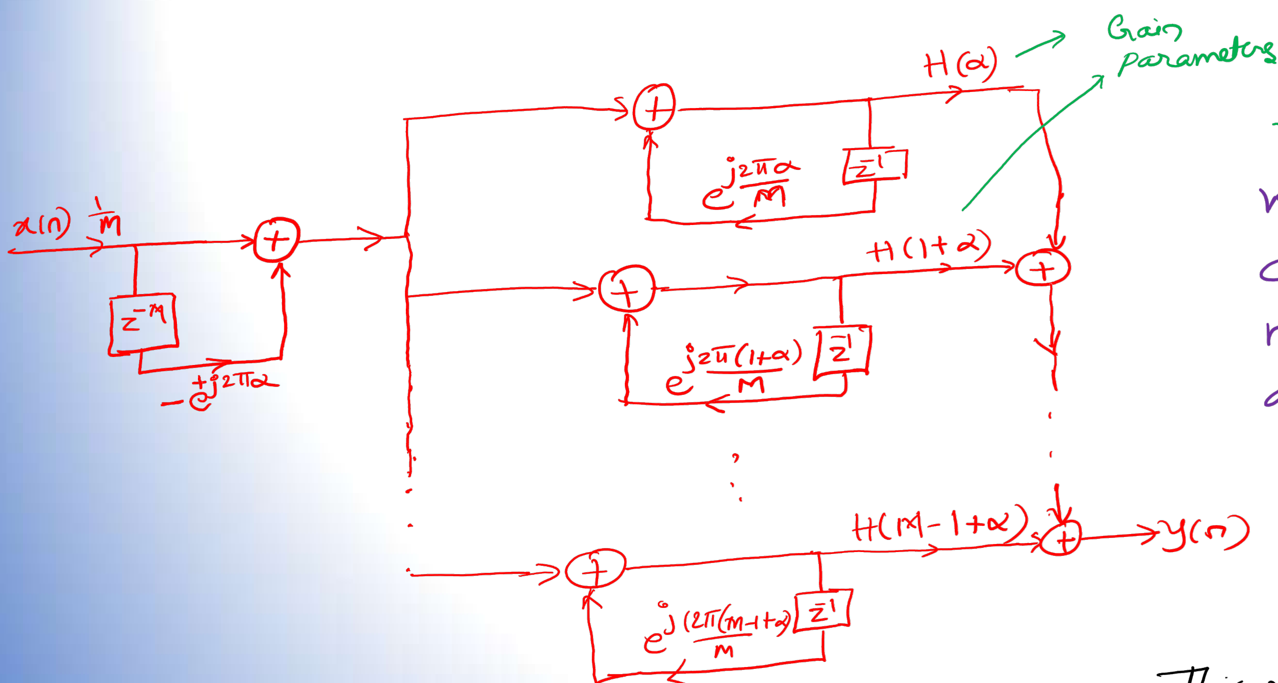
$$H_1(z) = \frac{1 - e^{j2\pi\alpha} z^{-M}}{M} \quad \text{—— (8)}$$

, zeros are located at equally spaced points on the unit circle, $z_k = e^{j\frac{2\pi(k+\alpha)}{M}}$, $k = 0, 1, \cdots M-1$

$$H_2(z) = \sum_{k=0}^{M-1} \frac{H(k+\alpha)}{1 - e^{j\frac{2\pi(k+\alpha)}{M}} z^{-1}} \quad \text{—— (9)}$$

Parallel bank of single pole filters with resonant frequencies $p_k = e^{j\frac{2\pi(k+\alpha)}{M}}$, $k = 0, 1, 2, \cdots M-1$

$$H(z) = \frac{1 - e^{j2\pi\alpha} z^{-M}}{M} \sum_{k=0}^{M-1} \frac{H(k+\alpha)}{1 - e^{j2\pi(k+\alpha)/M} z^{-1}}$$ ← IIR structure




**Advantage:**

When desired frequency response characteristic is narrow band, most of the gain parameters $\{H(k+\alpha)\}$ are zero. Consequently, the corresponding resonant filters can be eliminated. and only the filters with nonzero gain need to be retained.

This results in a filter that requires fewer computations

Further simplification of frequency sampling structure

We know that

$$H(k) = H^*(M-k) \quad \text{for } \alpha = 0 \qquad \text{(symmetry)}$$

$$H(k+\alpha) = H^*(M-k-\tfrac{1}{2}) \quad \text{for } \alpha = \tfrac{1}{2}$$

$$H_2(z) = \sum_{k=0}^{M-1} \frac{H(k+\alpha)}{1 - e^{j2\pi(k+\alpha)/M} \cdot z^{-1}}$$

For $\alpha = 0$

$$H_2(z) = \frac{H(0)}{1-z^{-1}} + \sum_{k=1}^{\frac{M-1}{2}} \frac{A(k) - z^{-1}B(k)}{1 - 2\cos\left(\frac{2\pi k}{M}\right)z^{-1} + z^{-2}} \qquad M \text{ odd}$$

$$= \frac{H(0)}{1-z^{-1}} + \frac{H(M/2)}{1+z^{-1}} + \sum_{k=1}^{\frac{M}{2}-1} \frac{A(k) - z^{-1}B(k)}{1 - 2\cos\left(\frac{2\pi k}{M}\right)z^{-1} + z^{-2}} \qquad M \text{ even}$$

$$A(k) = H(k) + H(M-k)$$

$$B(k) = H(k)e^{-j\frac{2\pi k}{M}} + H(M-k)e^{j\frac{2\pi k}{M}}$$

Ex: Realize a frequency sampling structure for an FIR causal system whose impulse response is $h(n) = \{1,2,3,4\}$ $(M=4)$

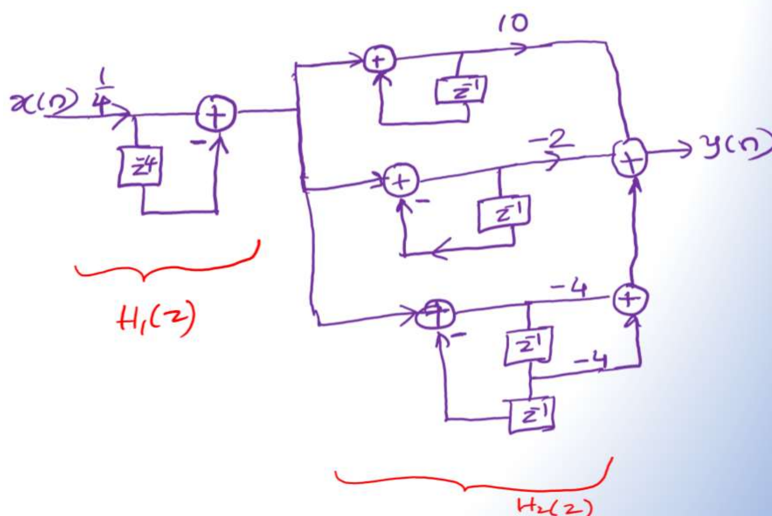
$H(k) = \{10,\ -2+j2,\ -2,\ -2-j2\}$

$$H_1(z) = \frac{1-z^{-4}}{4},$$

$$H_2(z) = \sum_{k=0}^{3} \frac{H(k)}{1-e^{\frac{j2\pi k}{4}} z^{-1}} = \frac{H(0)}{1-z^{-1}} + \frac{-2+j2}{1-jz^{-1}} + \frac{-2}{1+z^{-1}} + \frac{-2-j2}{1+jz^{-1}}$$

$$= \frac{10}{1-z^{-1}} + \frac{(-2+j2)(1+jz^{-1}) + (-2-j2)(1-jz^{-1})}{(1-jz^{-1})(1+jz^{-1})} + \frac{-2}{1+z^{-1}}$$

$$= \frac{10}{1-z^{-1}} + \frac{-2}{1+z^{-1}} + \frac{-4-4z^{-1}}{1+z^{-2}}$$



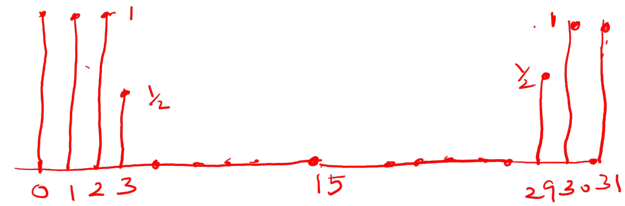

$$H_2(z) = \frac{H(0)}{1-z^{-1}} + \frac{H(M/2)}{1+z^{-1}} + \sum_{k=1}^{\frac{M}{2}-1} \frac{A(k) - z^{-1}B(k)}{1-2\cos\left(\frac{2\pi k}{M}\right)z^{-1} + z^{-2}} \quad M \text{ even}$$

$A(k) = H(k) + H(M-k)$

$B(k) = H(k)e^{-\frac{j2\pi k}{M}} + H(M-k)e^{\frac{j2\pi k}{M}}$

Sketch the block diagram for the direct-form realization and the frequency-sampling realization of the $M = 32$, $\alpha = 0$, linear-phase (symmetric) FIR filter which has frequency samples

$$H\left(\frac{2\pi k}{32}\right) = \begin{cases} 1, & k = 0, 1, 2 \\ \frac{1}{2}, & k = 3 \\ 0, & k = 4, 5, \ldots, 15 \end{cases}$$




$$H(z) = \frac{1-e^{j2\pi\alpha}z^{-M}}{M}\sum_{k=0}^{M-1}\frac{H(k+\alpha)}{1-e^{j\frac{2\pi(k+\alpha)}{M}}z^{-1}}$$

$$H(z) = H_1(z)\cdot H_2(z)$$

$$H_1(z) = \frac{1-z^{-32}}{32}, \quad H_2(z) = \sum_{k=0}^{M-1}\frac{H(k)}{1-e^{j\frac{2\pi k}{M}}z^{-1}} = \frac{H(0)}{1-z^{-1}} + \sum_{k=1}^{M-1}\frac{H(k)}{1-e^{j\frac{2\pi k}{M}}z^{-1}}$$

$$H_2(z) = \frac{H(0)}{1-z^{-1}} + \frac{H(1)}{1-e^{j\frac{2\pi k}{32}}z^{-1}} + \frac{H(31)}{1-e^{j\frac{2\pi 31}{32}}z^{-1}} + \frac{H(2)}{1-e^{j\frac{2\pi\times 2}{32}}z^{-1}} + \frac{H(30)}{1-e^{j\frac{2\pi\times 30}{32}}z^{-1}} + \frac{H(3)}{1-e^{j\frac{2\pi\times 3}{32}}\cdot z^{-1}} + \frac{H(29)}{1-e^{j\frac{2\pi\times 29}{32}}z^{-1}}$$

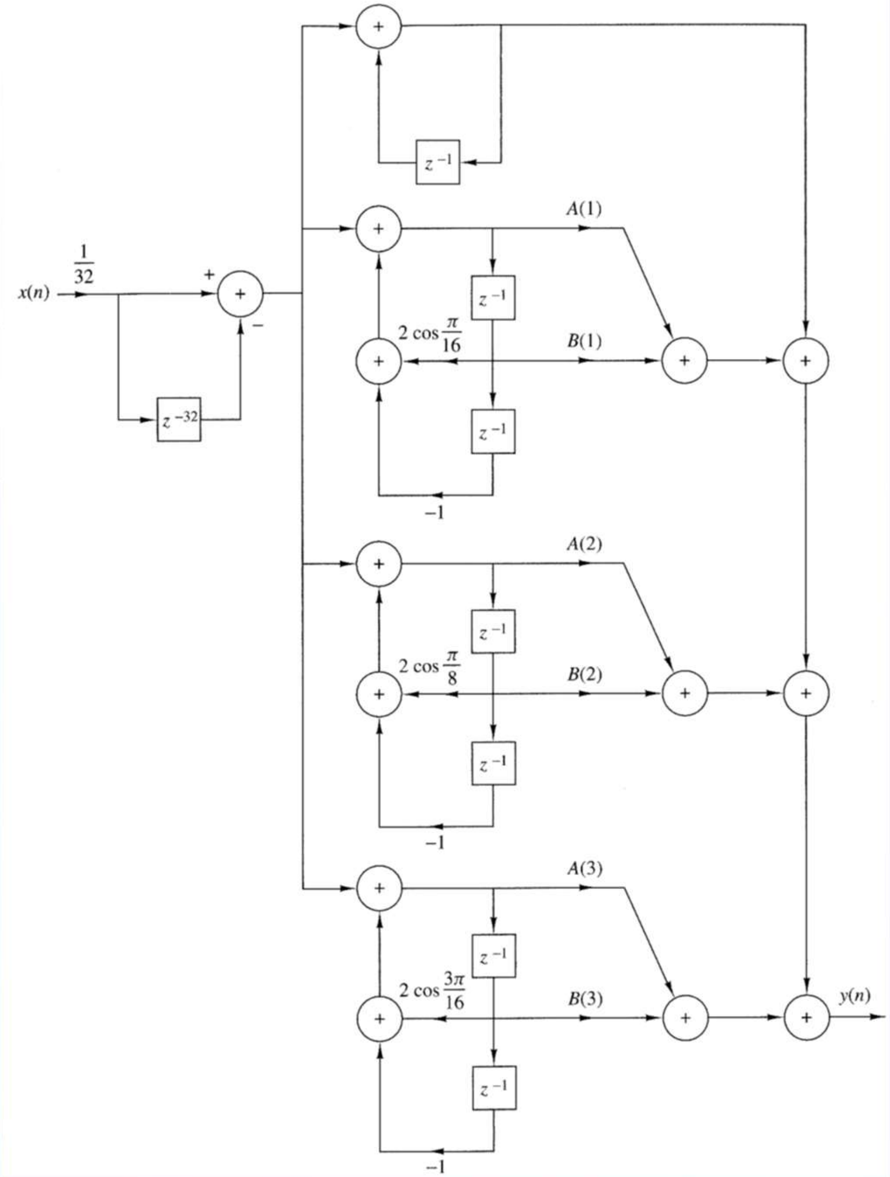
$$H_2(z) = \frac{H(0)}{1-z^{-1}} + \underbrace{\frac{H(1)}{1-e^{j\frac{2\pi}{32}}z^{-1}} + \frac{H(31)}{1-e^{-j\frac{2\pi}{32}}z^{-1}}} + \underbrace{\frac{H(2)}{1-e^{j\frac{2\pi\times 2}{32}}z^{-1}} + \frac{H(30)}{1-e^{-j\frac{2\pi\times 2}{32}}z^{-1}}} + \underbrace{\frac{H(3)}{1-e^{j\frac{2\pi\times 3}{32}}z^{-1}} + \frac{H(29)}{1-e^{-j\frac{2\pi\times 3}{32}}z^{-1}}}$$

$$H_2(z) = \frac{1}{1-z^{-1}} + \frac{2-2\cos\frac{2\pi}{32}z^{-1}}{1-2\cos\frac{\pi}{16}z^{-1}+z^{-2}} + \frac{2-2\cos\frac{4\pi}{32}z^{-1}}{1-2\cos\frac{\pi}{8}z^{-1}+z^{-2}} + \frac{1-\cos\frac{6\pi}{32}z^{-1}}{1-2\cos\frac{3\pi}{16}z^{-1}+z^{-2}}$$

$$H_1(z) = \frac{1 - z^{-32}}{32}$$

$$H_2(z) = \frac{1}{1-z^{-1}} + \frac{2 - 2\cos\frac{8\pi}{32} z^{-1}}{1 - 2\cos\frac{\pi}{16} z^{-1} + z^{-2}} + \frac{2 - 2\cos\frac{4\pi}{32} z^{-1}}{1 - 2\cos\frac{\pi}{8} z^{-1} + z^{-2}} + \frac{1 - \cos\frac{6\pi}{32} z^{-1}}{1 - 2\cos\frac{3\pi}{16} z^{-1} + z^{-2}}$$

$$A(k) = H(k) + H(M-k)$$
$$B(k) = H(k)\, e^{-j\frac{2\pi k}{M}} + H(M-k)\, e^{j\frac{2\pi k}{M}}$$




Thank you

# Lattice Structure

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# Lattice Structure

We know that $y(n) = \sum_{k=0}^{M-1} h(k)\, x(n-k)$ for FIR systems

$$H(z) = \sum_{k=0}^{M-1} b_k z^{-k}$$

$$h(n) = \begin{cases} b_n & 0 \le n \le M-1 \\ 0 & \text{otherwise} \end{cases} \longrightarrow b_0 = 1 \Rightarrow h_m(0) = 1$$

Let us define the FIR filter in the following form

$$H_m(z) = A_m(z) \quad , \quad m = 0, 1, 2, \cdots M-1$$

Then $A_m(z) = 1 + \sum_{k=1}^{m} \alpha_m(k) z^{-k} \quad m \geq 1$ —— (1)

$A_0(z) = 1$, $\quad \alpha_m(k) = h_m(k)$

Let $\{x(n)\}$ be the input sequence to the filter $A_m(z)$ and $\{y(n)\}$ be the output

- Lattice filter implementation is widely used in adaptive filtering. Assume that we have a filter with transfer function H(z). We can write,

$$H_m(z) = A_m(z), \qquad m = 0, 1, 2, \ldots, M-1$$

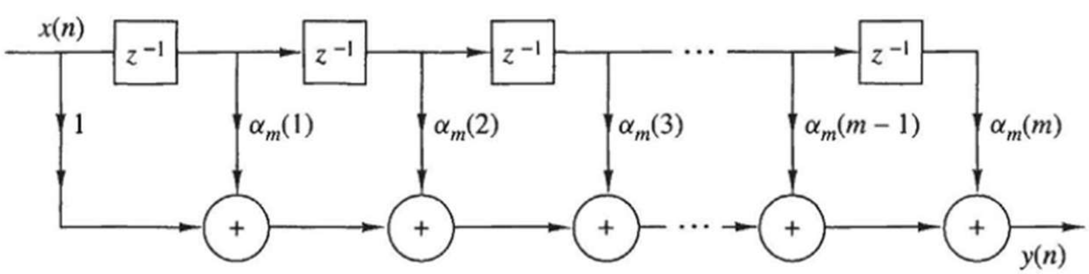
- where $A_m(z)$ is a polynomial with $A_0(z) = 1$

$$A_m(z) = 1 + \sum_{k=1}^{m} \alpha_m(k) z^{-k}, \qquad m \geq 1$$

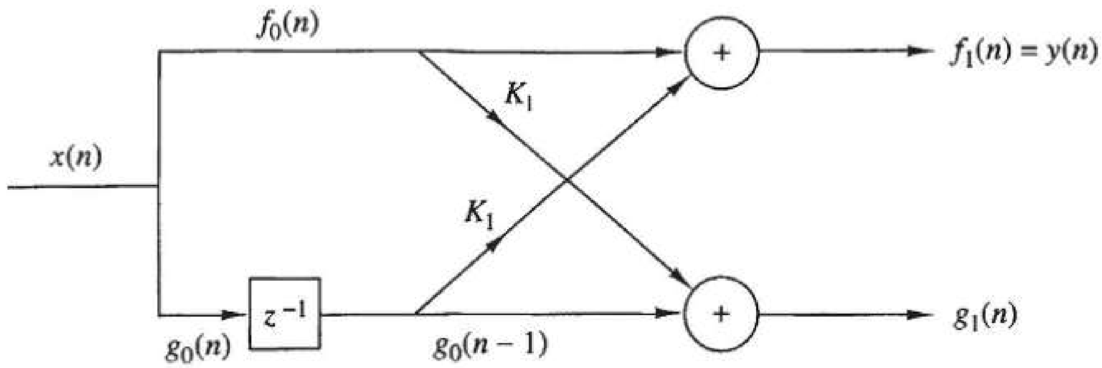
- Y[n] can be written as

$$y(n) = x(n) + \sum_{k=1}^{m} \alpha_m(k) x(n-k)$$

- The direct form implementation can be expressed as



- Let's consider a first order FIR filter, i.e., m=1: $y(n) = x(n) + \alpha_1(1)x(n-1)$
- Let the reflection coefficient $K_1 = \alpha_1(1)$. to get:




$$f_0(n) = g_0(n) = x(n)$$
$$f_1(n) = f_0(n) + K_1 g_0(n-1) = x(n) + K_1 x(n-1)$$
$$g_1(n) = K_1 f_0(n) + g_0(n-1) = K_1 x(n) + x(n-1)$$

- Now consider m=2:

$$y(n) = x(n) + \alpha_2(1)x(n-1) + \alpha_2(2)x(n-2)$$

- We cascade two lattice stages:
- The out put of the first stage is,

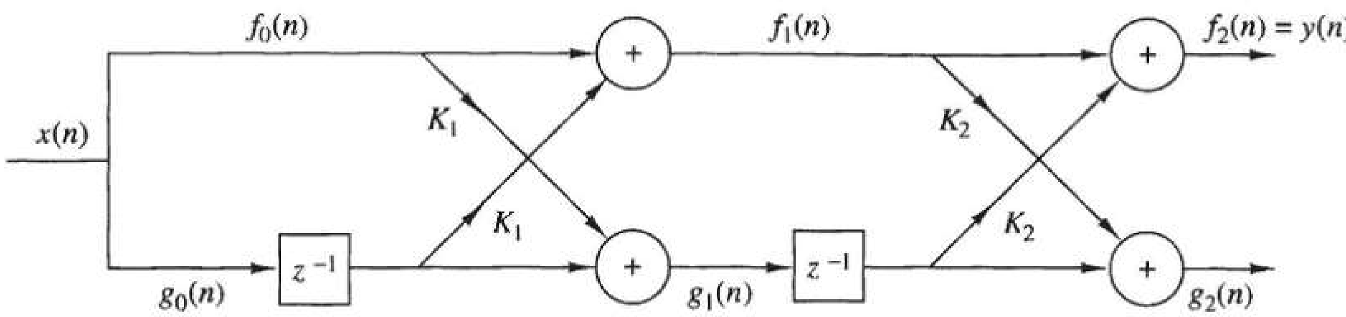
$$f_1(n) = x(n) + K_1 x(n-1)$$

$$g_1(n) = K_1 x(n) + x(n-1)$$

- And the output of the second stage is:

$$f_2(n) = f_1(n) + K_2 g_1(n-1)$$

$$g_2(n) = K_2 f_1(n) + g_1(n-1)$$




Two-stage lattice filter.

- Let's consider on $f_2[n]$:

$$f_2(n) = x(n) + K_1 x(n-1) + K_2[K_1 x(n-1) + x(n-2)]$$
$$= x(n) + K_1(1+K_2)x(n-1) + K_2 x(n-2)$$

- $f_2[n]$ will be y[n] if:

$$\alpha_2(2) = K_2, \qquad \alpha_2(1) = K_1(1+K_2)$$

- or, equivalently if:

$$K_2 = \alpha_2(2), \qquad K_1 = \frac{\alpha_2(1)}{1+\alpha_2(2)}$$

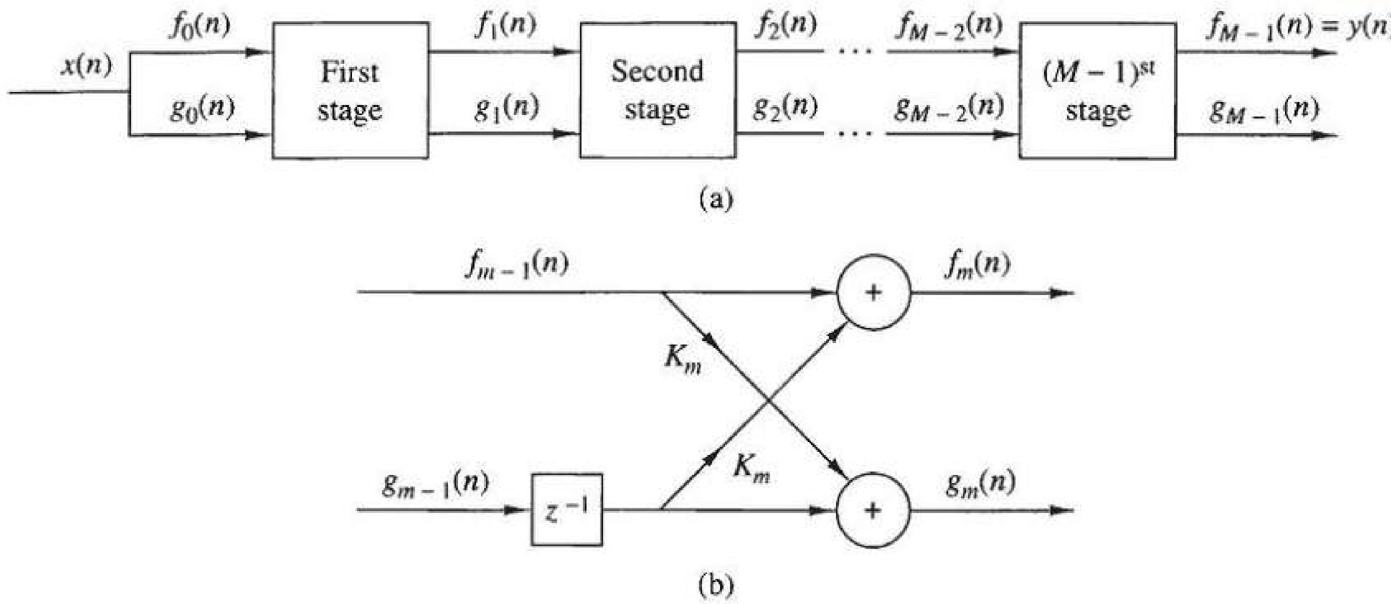
- In general: $f_0(n) = g_0(n) = x(n)$

$$f_m(n) = f_{m-1}(n) + K_m g_{m-1}(n-1), \qquad m = 1, 2, \ldots, M-1$$

$$g_m(n) = K_m f_{m-1}(n) + g_{m-1}(n-1), \qquad m = 1, 2, \ldots, M-1$$

- Then:

$y(n) = f_{M-1}(n)$




$(M-1)$-stage lattice filter.

# Conversion of FIR taps to Lattice coefficients

$$\alpha_{m-1}(k) = \frac{\alpha_m(k) - \alpha_m(m)\,\alpha_m(m-k)}{1 - \alpha_m^2(m)}$$

Determine the lattice coefficients corresponding to the FIR filter with system function

$$y(n) = x(n) + \frac{1}{3}x(n-1) + \frac{1}{2}x(n-2) + \frac{1}{3}x(n-3)$$

$\alpha_3(0) = 1$
$\alpha_3(1) = 1/3$
$\alpha_3(2) = 1/2$
$\alpha_3(3) = 1/3$

$K_m = \alpha_m(m)$
$K_1 = \alpha_1(1)$
$K_2 = \alpha_2(2)$
$K_3 = \alpha_3(3)$.

$$\alpha_{m-1}(k) = \frac{\alpha_m(k) - \alpha_m(m)\,\alpha_m(m-k)}{1 - \alpha_m^2(m)}$$

$K_3 = \alpha_3(3) = 1/3$.

$K_2 = \alpha_2(2)$, $\underline{m = 3, k = 2}$

$$\alpha_2(2) = \frac{\alpha_3(2) - \alpha_3(3)\,\alpha_3(1)}{1 - \alpha_3^2(3)}$$

$K_2 = \frac{7}{16}$

$\alpha_2(1) =$ $m = 3$ $k = 1$

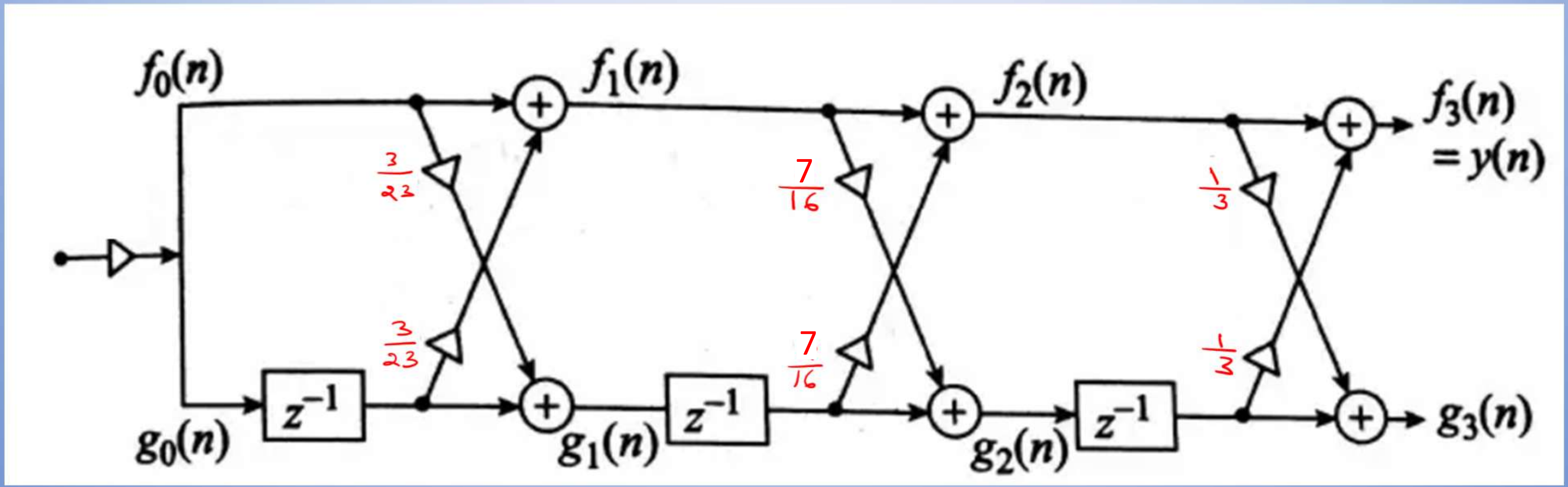
$$\alpha_2(1) = \frac{\alpha_3(1) - \alpha_3(3)\,\alpha_3(2)}{1 - \alpha_3^2(3)}$$

$\boxed{\alpha_2(1) = \frac{3}{16}}$

$K_1 = \alpha_1(1)|\ \underline{m = 2, k = 1}$

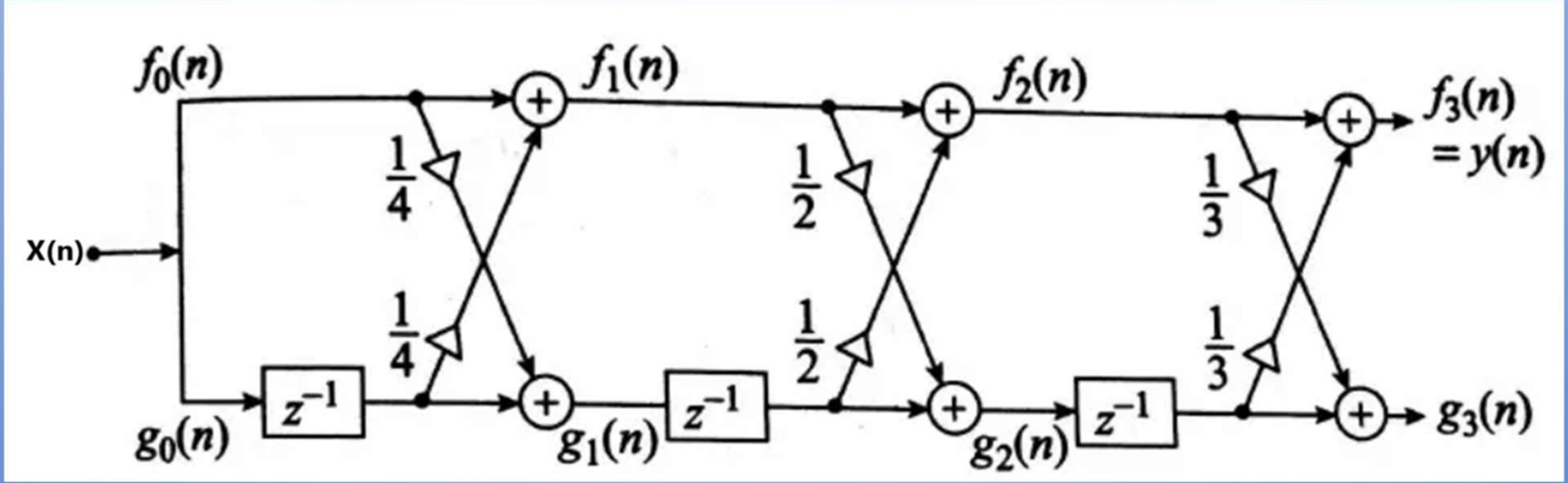
$$\alpha_1(1) = \frac{\alpha_2(1) - \alpha_2(2)\,\alpha_2(1)}{1 - \alpha_2^2(2)}$$

$= \frac{3}{23}$


$f_0(n)$
$f_1(n)$
$f_2(n)$
$f_3(n) = y(n)$
$g_0(n)$
$g_1(n)$
$g_2(n)$
$g_3(n)$
$z^{-1}$
$z^{-1}$
$z^{-1}$
$\frac{3}{23}$
$\frac{3}{23}$
$\frac{7}{16}$
$\frac{7}{16}$
$\frac{1}{3}$
$\frac{1}{3}$

Determine the lattice coefficients corresponding to the FIR filter with system function

$$H(z) = A_3(z) = 1 + \frac{13}{24}z^{-1} + \frac{5}{8}z^{-2} + \frac{1}{3}z^{-3}$$



Consider a three – stage FIR lattice structure having the coefficients

$k1 = 0.65,\ k2 = 0.5\ \&\ k3 = 0.9$

Find its impulse response and direct form structure.

$\alpha_m(0) = 1$ —— I

$\alpha_m(m) = k_m$ —— II

$\alpha_m(k) = \alpha_{m-1}(k) + k_m \alpha_{m-1}(m-k)$ — III

$\boxed{m=3}$ $\boxed{\alpha_m(0) = 1}$ $\boxed{\alpha_1(1) = 0.65}$ $\boxed{\alpha_2(2) = 0.5}$ $\boxed{\alpha_3(3) = 0.9}$

To realize direct form structure we need to find

$\alpha_3(1)$ $\alpha_3(2)$ $\alpha_3(3)$

$$\alpha_m(k) = \alpha_{m-1}(k) + k_m \alpha_{m-1}(m-k) \quad \text{--- (III)}$$

put $m=3$, $k=1$ in Eqn (III),

$$\alpha_3(1) = \alpha_2(1) + k_3 \times \alpha_2(2) \quad \text{--- (A)}$$

Now calculate $\alpha_2(1)$

put $m=2$, $k=1$ in Eqn (III)

$$\alpha_2(1) = \alpha_1(1) + k_2 \times \alpha_1(1) = 0.65 + 0.5 \times 0.65 = 0.975$$

Substitute $\alpha_2(1)$ in Eqn (A)

$$\therefore \quad \alpha_3(1) = 0.975 + 0.9 \times 0.5 = 1.425$$

Now substitute $m=3$, $k=2$ in Eqn (III)

$$\alpha_3(2) = \alpha_2(2) + k_3 \times \alpha_2(1) = 0.5 + 0.9 \times 0.975 = 1.3775$$

$\alpha_3(1) = 1.425 \qquad \alpha_3(2) = 1.3775 \qquad \alpha_3(3) = 0.9$

$\therefore$ w.k.t

$$H(z) = 1 + \sum_{k=1}^{m} \alpha_m(k)\, z^{-k}$$

$$H(z) = 1 + \sum_{k=1}^{3} \alpha_m(k)\, z^{-k}$$

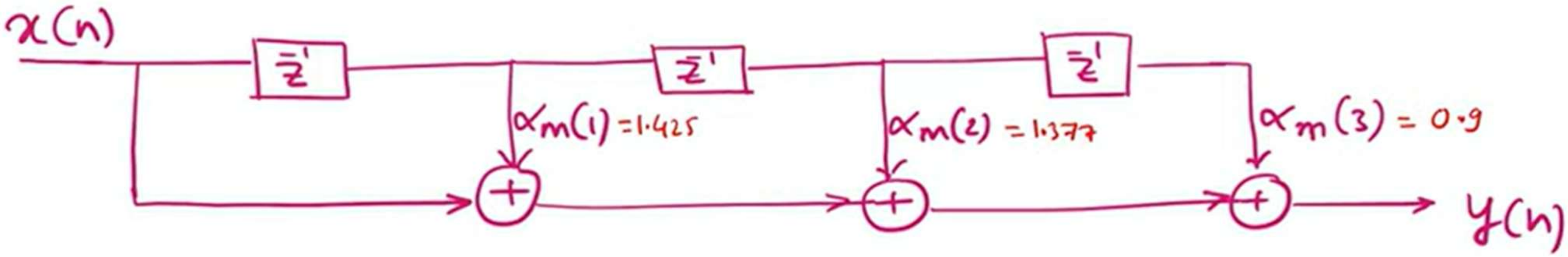
$$H(z) = 1 + \alpha_3(1) z^{-1} + \alpha_3(2) z^{-2} + \alpha_3(2) z^{-3}$$

$$\therefore H(z) = 1 + 1.425 z^{-1} + 1.3775 z^{-2} + 0.9 z^{-3}$$

$$\frac{Y(z)}{X(z)} = 1 + 1.425 z^{-1} + 1.375 z^{-2} + 0.9 z^{-3}$$

$$\boxed{y(n) = x(n) + 1.425\, x(n-1) + 1.375\, x(n-2) + 0.9\, x(n-3)}$$

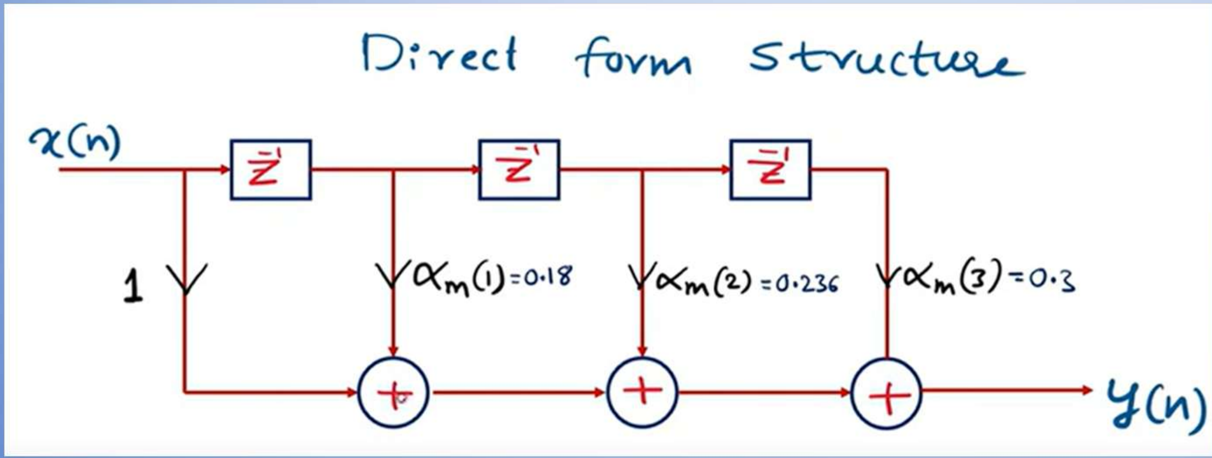
Direct form structure is shown below



**Let the coefficients of a three stage FIR lattice structure be**

**$K1=0.1$, $K2=0.2$ and $K3=0.3$.**

**Find the coefficients of the direct form 1 FIR filter and draw its block diagram**




Thank you

# Lattice-Ladder structures

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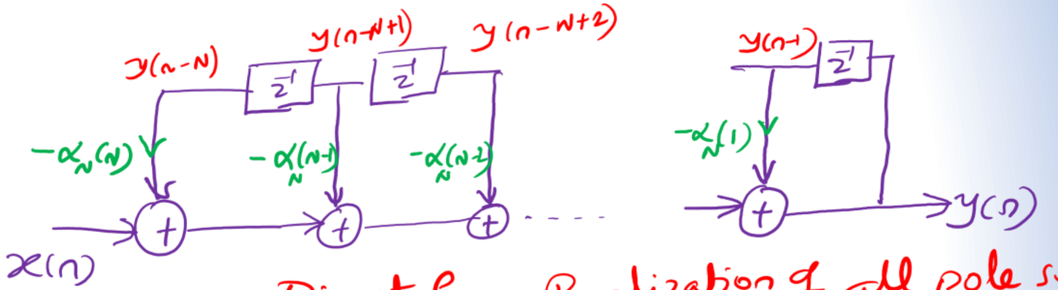
# Lattice and Lattice-Ladder structures for IIR Systems

Consider an all poles system $H(z) = \dfrac{1}{1+\sum_{k=1}^{N} a_N(k) z^{-k}} = \dfrac{1}{A_N(z)}$ — 1

$$Y(z) + \sum_{k=1}^{N} a_N(k) z^{-k} Y(z) = X(z) \quad \text{— 2}$$

$$y(n) = -\sum_{k=1}^{N} a_N(k) y(n-k) + x(n) \quad \text{— 3}$$




Direct form Realization of all pole system

If we interchange the roles of input and output

$$x(n) = -\sum_{k=1}^{N} a_N(k) x(n-k) + y(n)$$

OR

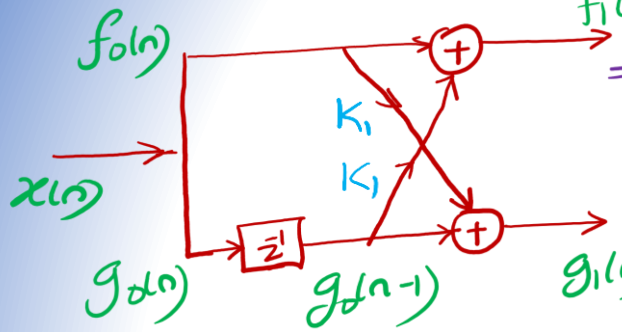
$$y(n) = x(n) + \sum_{k=1}^{N} a_N(k) x(n-k) \quad \text{— 4}$$

Eq(n) 4 describes an FIR system with system fn.

$$H(z) = 1 + \sum_{k=1}^{N} a_N(k) z^{-k} = A_N(z) \quad \text{— 5}$$

$\therefore$ One system can be obtained from the other simply by interchanging the roles of input and output

We shall use FIR lattice to to obtain all-pole IIR lattice

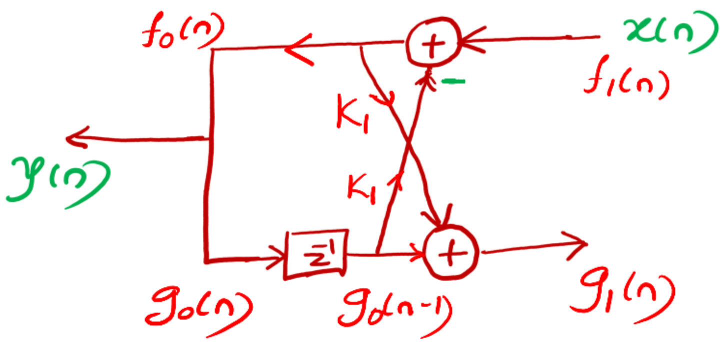


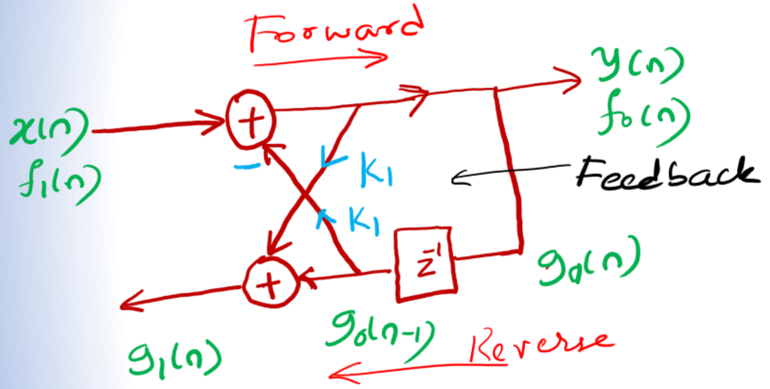

$f_1(n) = f_0(n) + K_1 g_0(n-1)$

$= y(n) = x(n) + K_1 x(n-1)$

$g_1(n) = K_1 f_0(n) + g_0(n-1)$

Let us now reverse the roles of input and output

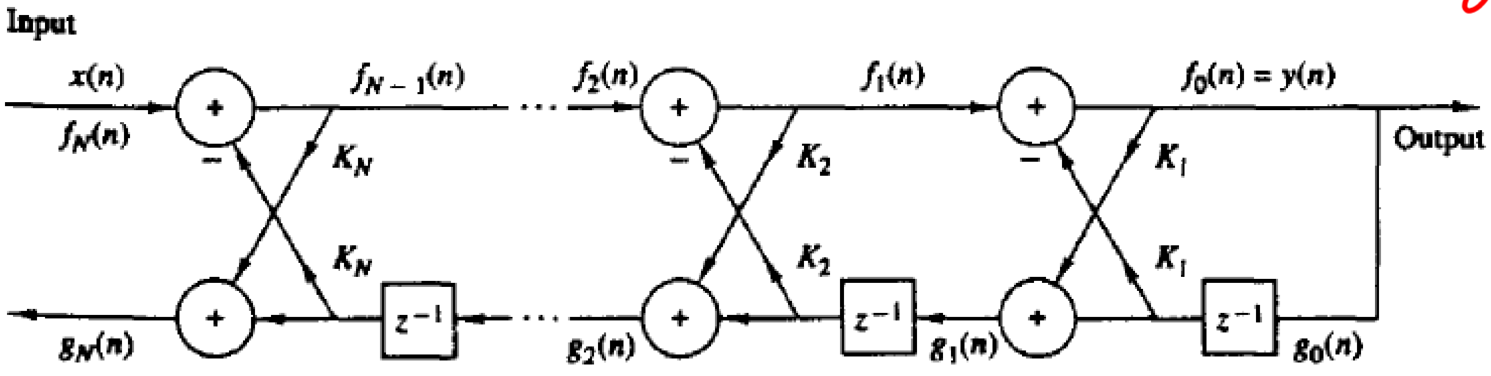



$$\boxed{f_0(n) = y(n) = f_1(n) - K_1 g_0(n-1)} \quad (7)$$

$$y(n) = x(n) - K_1 y(n-1)$$

$$g_1(n) = K_1 y(n) + y(n-1)$$




Lattice structure for an all-pole IIR system.

$$\boxed{f_{m-1}(n) = f_m(n) - K_m g_{m-1}(n-1) \quad (8) \qquad m = N, N-1, \cdots 1}$$

$$g_m(n) = K_m f_{m-1}(n) + g_{m-1}(n-1) \quad (9)$$

$y(n) = x(n) - K_1[1+K_2]\, y(n-1) - K_2\, y(n-2) \rightarrow$ Two pole IIR system $\Rightarrow [\,1,\ -K_1[1+K_2]\ \ -K_2\,]$

$g_2(n) = K_2\, y(n) + K_1[1+K_2]\, y(n-1) + y(n-2) \rightarrow$ Two zero FIR system $\Rightarrow [\,K_2\ \ K_1[1+K_2]\ \ 1\,]$

The system function for the all pole IIR system is

$$H_a(z) = \frac{Y(z)}{X(z)} = \frac{F_0(z)}{F_m(z)} = \frac{1}{A_m(z)} \qquad m = N, N-1, \cdots 1 \qquad (16)$$

Similarly for all zero FIR, the system fn is

$$H_b(z) = \frac{G_m(z)}{Y(z)} = \frac{G_m(z)}{G_0(z)} = B_m(z) = z^{-m} A_m(z^{-1}) \qquad (17)$$

$$m = N, N-1, \cdots 1$$

Note: The all pole lattice structure has a all zero path with $g_0(n)$ as input and $g_N(n)$ as output

$B_m(z)$ represents the system function of all-zero path common to both all-zero and all-pole systems. (latice structure)

For all pole lattice $B_m(z)$ is backword system function

All zero and all pole lattice structures are characterized by the same set of lattice parameters, $K_1, K_2, \cdots K_N$

We know that, in Lattice structure

$g_m(n)$ is a linear combination of present and Past output

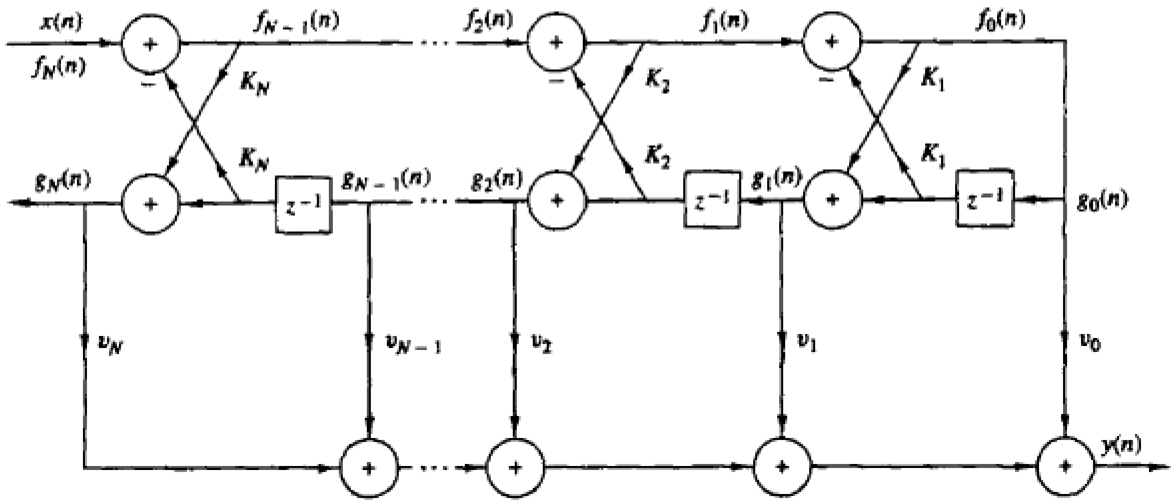
for $N=2$, $g_2(n) = K_2 y(n) + K_1(1+K_2) y(n-1) + y(n-2)$

$$\therefore \frac{G_2(z)}{Y(z)} = K_2 + K_1(1+K_2)z^{-1} + z^{-2} = B_2(z)$$

Or In general

$$H_b(z) = \frac{G_m(z)}{Y(z)} = B_m(z) \rightarrow \text{Is an all zero system.}$$

Thus we begin with an all pole lattice structure to realize pole-zero IIR structure called Lattice-Ladder.




Lattice Ladder for pole zero IIR $M=N$

$$y(n) = \sum_{m=0}^{M} v_m g_m(n)$$

$\{v_m\}$ are the parameters that determine the zeros of the system.

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$$y(n) = \sum_{m=0}^{M} v_m \, g_m(n)$$

$$Y(z) = \sum_{m=0}^{M} v_m G_m(z)$$

We know that $X(z) = F_N(z)$ & $F_0(z) = G_0(z)$

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{m=0}^{M} v_m \frac{G_m(z)}{X(z)}$$

$$H(z) = \sum_{m=0}^{M} v_m \frac{G_m(z)}{G_0(z)} \cdot \frac{F_0(z)}{F_N(z)}$$

$$= \sum_{m=0}^{M} v_m B_m(z) \cdot \frac{1}{A_N(z)}$$

$$= \frac{\sum_{m=0}^{M} v_m B_m(z)}{A_N(z)} \qquad (3)$$

We know

$$H(z) = \frac{\sum_{k=0}^{M} c_M(k) z^{-k}}{1 + \sum_{k=1}^{N} a_N(k) z^{-k}} = \frac{C_M(z)}{A_N(z)}$$

$$\therefore \; C_M(z) = \sum_{m=0}^{M} v_m B_m(z) \qquad (4)$$

Coefficients of the numerator polynomial, $C_M(z)$ determine the ladder parameters $\{v_m\}$

Coefficients of denominator polynomial, $A_N(z)$ determine the lattice parameters $\{K_m\}$

# Procedure to realize Lattice-Ladder Structure

For a given system function $H(z)$ with $N \geq M$,

1. Determine lattice parameters of the all-pole lattice parameters using the equation

$$A_{m-1}(z) = \frac{A_m(z) - K_m B_m(z)}{1 - K_m^2}$$ [Alternate method]

2. The ladder parameters can be determined from the equation

$$C_M(z) = \sum_{m=0}^{M} v_m B_m(z) \quad \text{--- (4)}$$

$$C_m(z) = \sum_{k=0}^{m-1} v_k B_k(z) + v_m B_m(z) \quad \text{--- (5)}$$

$$C_m(z) = C_{m-1}(z) + v_m B_m(z) \quad \text{--- (6)}$$

Thus $C_m(z)$ can be computed recursively from $B_m(z)$ with $m = 1, 2, \cdots M$

Since $B_m(m) = 1$ for all $m$, the parameters $v_m$, $m = 0, 1, \cdots M$ can be determined by

$$v_m = C_m(m), \quad m = 0, 1, \cdots M$$

and then

$$C_{m-1}(z) = C_m(z) - v_m B_m(z) \quad \text{--- (7)}$$

→ running backward to obtain $C_m(m)$

Sketch the Lattice - ladder realization for

$$\boxed{H(z) = \frac{1-0.8z^{-1}+0.15z^{-2}}{1+0.1z^{-1}-0.72z^{-2}} = \frac{C_2(z)}{A_2(z)}}$$

Soln: To find $\{K_m\}$ $K_1$ $K_2$ [Alternate method]

$K_2 = -0.72$

$A_2(z) = 1+0.1z^{-1}-0.72z^{-2}$

$\boxed{B_2(z) = -0.72+0.1z^{-1}+z^{-2}}$

$$A_1(z) = \frac{A_2(z)-K_2B_2(z)}{1-K_2^2} = 1+0.36z^{-1}$$

$\boxed{K_1 = 0.36}$

Lattice parameters $K = 0.36$, $K_2 = -0.72$

Ladder parameters $v_0, v_1, v_2$

$v_m = C_m(m)$

$C_m(z) = C_{m-1}(z) + v_m B_m(z)$

$C_2(z) = C_1(z) + v_2 B_2(z)$

$\boxed{v_2 = 0.15}$ ✓

$C_1(z) = C_2(z) - v_2 B_2(z)$

$= 1-0.8z^{-1}+0.15z^{-2} - 0.15[-0.72+0.1z^{-1}+z^{-2}]$

$C_1(z) = 1.108 - 0.815z^{-1}$

$\boxed{v_1 = -0.815}$ ✓

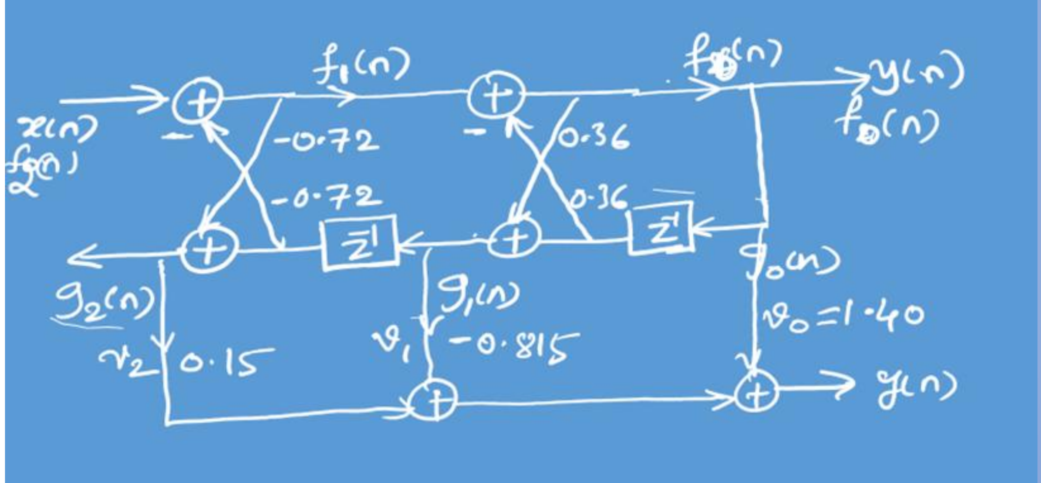
$$\boxed{C_2(z) = \sum_{m=0}^{2} v_m B_m(z)}$$

$A_0(z) = B_0(z) = 1$

$= v_0 B_0(z) + v_1 B_1(z) + v_2 B_2(z)$

$1-0.8z^{-1}+0.15z^{-2} = v_0 1 + v_1[0.36+z^{-1}] + v_2[-0.72+0.1z^{-1}+z^{-2}]$

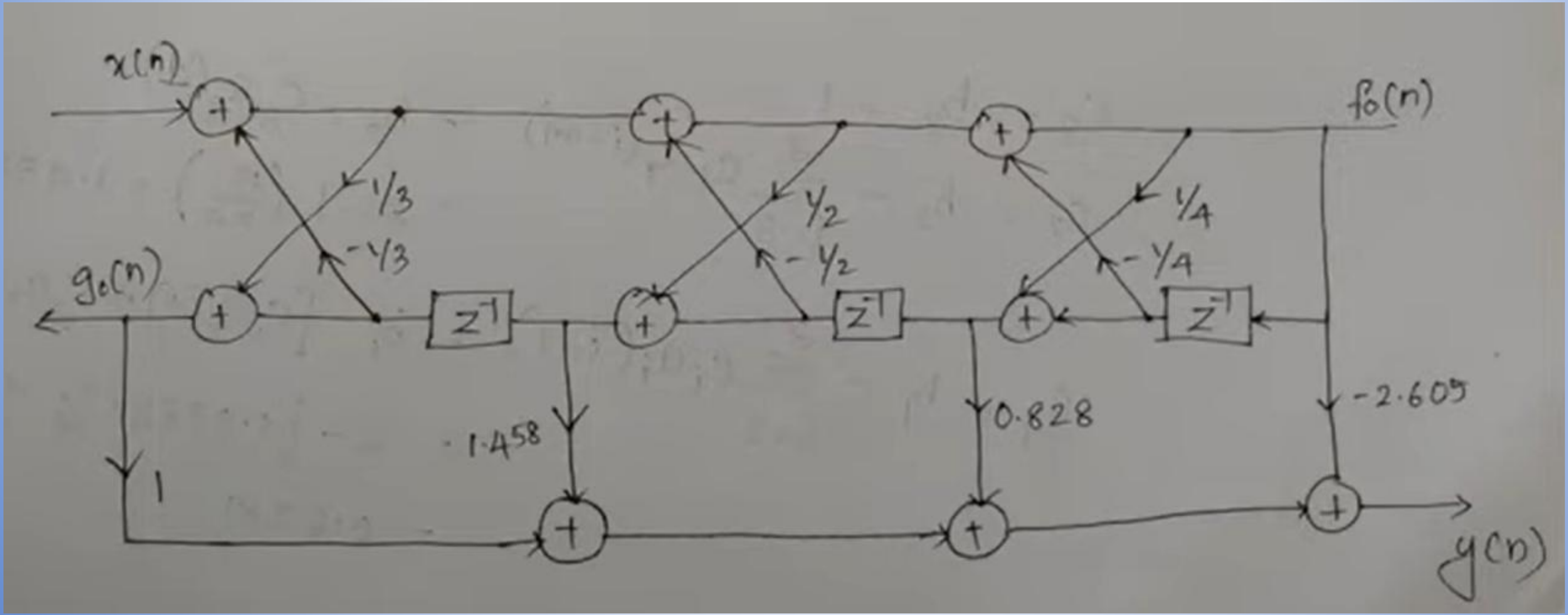
$v_2 = 0.15$, $v_1 = -0.815$ $\boxed{v_0 = 1.4014}$




Note: If value of all the lattice coefficients are less than 1, then the system is stable

Convert the following pole-zero IIR filter into a lattice-ladder structure.

$$H(z) = \frac{1+2z^{-1}+2z^{-2}+z^{-3}}{1+\frac{13}{24}z^{-1}+\frac{5}{8}z^{-2}+\frac{1}{3}z^{-3}}$$



# Alternate method for finding lattice coefficients:

Consider an FIR filter with system function

$$H(z) = 1 + 2.8z^{-1} + 3.404z^{-2} + 1.74z^{-3} + 0.4z^{-4}$$

Sketch the direct form and Lattice realizations of the filter and determine in detail the corresponding input-output equations.

Soln: $H(z) = 1 + 2.8z^{-1} + 3.404z^{-2} + 1.72z^{-3} + 0.4z^{-4}$

$K_1\ K_2\ K_3\ \&\ K_4$ $\qquad \boxed{K_4 = 0.4}$

$$A_{m-1}(z) = \frac{A_m(z) - K_m B_m(z)}{1 - K_m^2}$$

$$A_3(z) = \frac{A_4(z) - K_4 B_4(z)}{1 - 0.4^2}$$

$$A_3(z) = \frac{1 + 2.8z^{-1} + 3.404z^{-2} + 1.72z^{-3} + 0.4z^{-4} - 0.4[0.4 + 1.74z^{-1} + 3.404z^{-2} + 2.8z^{-3} + z^{-4}]}{0.84}$$

$$A_3(z) = 1 + 2.5142z^{-1} + 2.43z^{-2} + 0.73z^{-3}$$

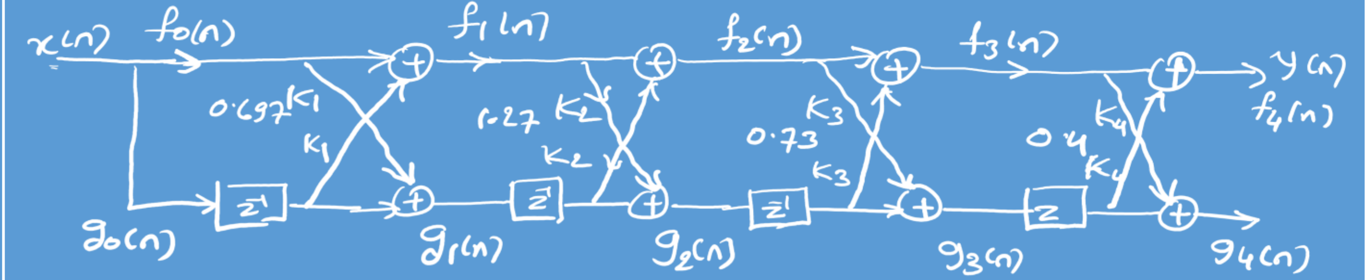
$$\boxed{K_3 = 0.73}$$

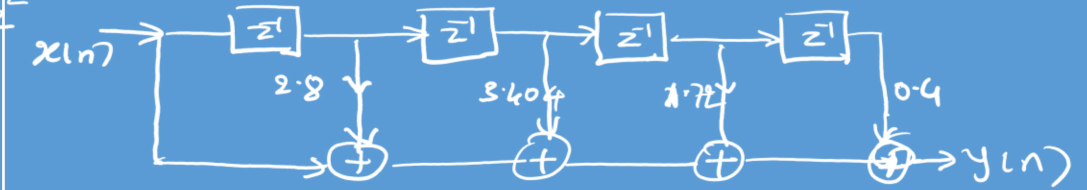
$$A_2(z) = \frac{A_3(z) - K_3 B_3(z)}{1 - 0.73^2}$$

$$A_2(z) = \frac{1 + 2.5142z^{-1} + 2.43z^{-2} + 0.73z^{-3} - 0.73[0.73 + 2.43z^{-1} + 2.514z^{-2} + z^{-3}]}{0.4671}$$

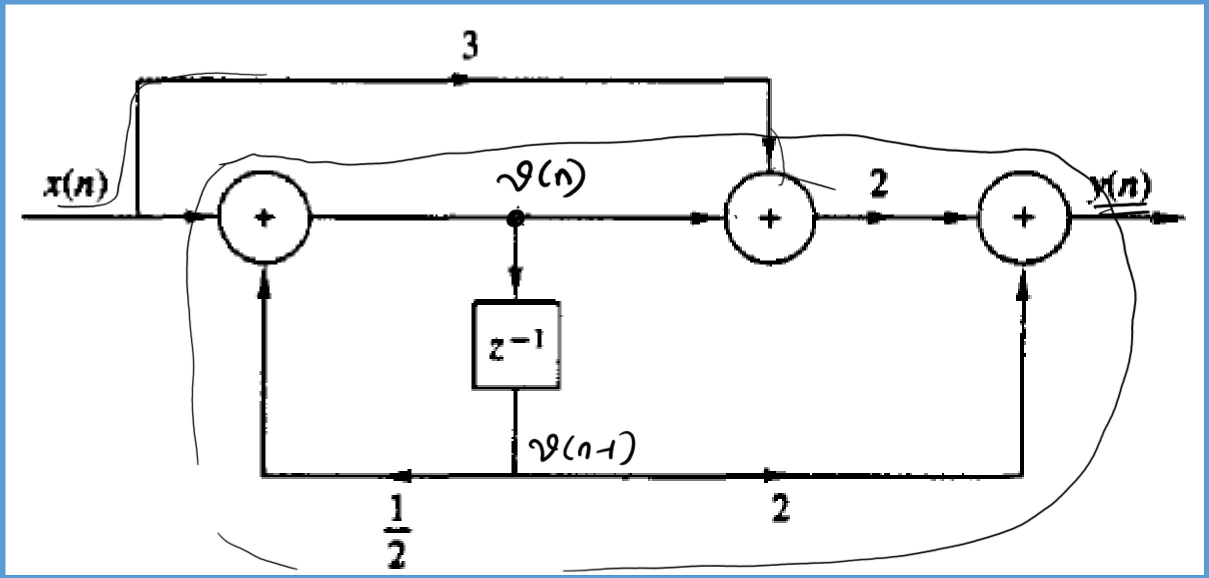
$$A_2(z) = 1 + 1.58z^{-1} + 1.27z^{-2} \qquad \boxed{K_2 = 1.27}$$

$$A_1(z) = \frac{A_2(z) - K_2 B_2(z)}{1 - 1.27^2} = 1 + 0.697z^{-1}, \quad \boxed{K_1 = 0.697}$$



Determine the system function and the impulse response
Of the system shown below




$$y(n) = [3x(n) + v(n)] \times 2 + 2v(n-1)$$

$$\boxed{y(n) = 6x(n) + 2v(n) + 2v(n-1)}$$

$$v(n) = x(n) + \tfrac{1}{2}v(n-1)$$

$$v(n) - \tfrac{1}{2}v(n-1) = x(n)$$

$$V(z) - \tfrac{1}{2}z^{-1}V(z) = X(z)$$

$$\frac{V(z)}{X(z)} = \frac{1}{1-\frac{1}{2}z^{-1}} \Rightarrow V(z) = \frac{1}{1-\frac{1}{2}z^{-1}}X(z)$$

$$y(n) = 6x(n) + 2v(n) + 2v(n-1)$$

$$Y(z) = 6X(z) + 2\cdot V(z) + 2z^{-1}V(z)$$

$$Y(z) = 6X(z) + 2\cdot\frac{X(z)}{1-\frac{1}{2}z^{-1}} + \frac{2z^{-1}}{1-\frac{1}{2}z^{-1}}X(z)$$

$$\frac{Y(z)}{X(z)} = \left[6 + \frac{2}{1-\frac{1}{2}z^{-1}} + \frac{2z^{-1}}{1-\frac{1}{2}z^{-1}}\right]$$

$$\boxed{z^{-1}X(z) \to x(n-1)}$$

$$= \frac{6 - 3z^{-1} + 2 + 2z^{-1}}{1-\frac{1}{2}z^{-1}}$$

$$\boxed{\frac{Y(z)}{X(z)} = \frac{8 - z^{-1}}{1-\frac{1}{2}z^{-1}}}$$

System function

$$\boxed{h(n) = 8\left(\tfrac{1}{2}\right)^n u(n) - \left(\tfrac{1}{2}\right)^{n-1} u(n-1)}$$


Thank you

# IIR filter design
# Impulse invariance technique

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# Design of IIR filters from Analog filters

Analog filter system function can be described

1) using filter coefficients

$$H_a(s) = \frac{B(s)}{A(s)} = \frac{\sum_{k=0}^{M} \beta_k s^k}{\sum_{k=0}^{N} \alpha_k s^k} = \frac{\beta_0 + \beta_1 s + \beta_2 s^2 + \cdots + \beta_M s^M}{\alpha_0 + \alpha_1 s + \alpha_2 s^2 + \cdots + \alpha_N s^N}$$

$\{\alpha_k\}$ and $\{\beta_k\}$ – analog filter coefficients

2) using impulse response

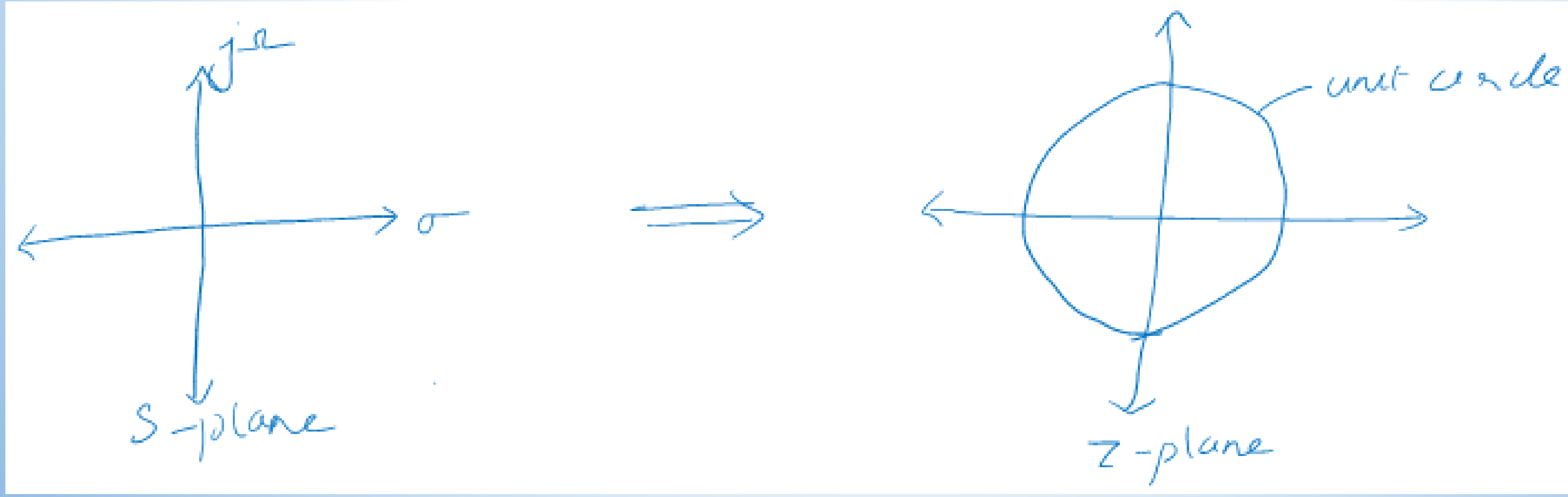
$$H_a(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt$$

3) using LCCDE

$$\sum_{k=0}^{N} \alpha_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} \beta_k \frac{d^k x(t)}{dt^k}$$

# Properties for effective conversion from Analog to Digital (mapping from s-plane to z-plane)

- Direct relationship between the frequency variable in the s-domain and z-domain
  - $j\Omega$ axis in s-plane should map into the unit circle in z-plane
- LHP of s-plane should map into the inside of the unit circle in z-plane.
  - A stable analog filter can be converted to a stable digital filter
  - Analog LTI system is stable if all its poles lie in the left half of s-plane



Linear phase filter must have a system function that satisfies the condition:

$$H(z) = \pm z^{-N} H(z^{-1})$$

Symmetric and antisymmetric response

$$h(n) = \pm h(N-n)$$

$\downarrow ZT$

$$H(z) = \sum_{n=0}^{N} h(n) z^{-n} = h(0) + h(1) z^{-1} + \dots + h_N z^{-N}$$

$$H(z^{-1}) = \sum_{k=0}^{N} h(k) z^{k} = h(0) + h(1) z + \dots + h(N) z^{N}$$

$$z^{-N} H(z^{-1}) = h(0) z^{-N} + h(1) z^{-N+1} + \dots h(N-1) z^{-1} + h(N) z^{0}$$

If condition $H(z) = \pm z^{-N} H(z^{-1})$ is met,

$h(0) = h(N)$, $h(1) = h(N-1)$, ...

This condition implies the roots of polynomial $H(z)$ are identical to the roots of polynomial $H(z^{-1})$

- Roots of $H(z)$ must occur in reciprocal pairs.

If $z_1$ is a root of $H(z)$, $\frac{1}{z_1}$ is also a root

For every pole inside a unit circle, there will be a mirror image pole outside the unit circle.

Therefore the system is unstable.

A causal and stable IIR system cannot have linear phase.

# Design of IIR filters:

- Two steps
- First design an analog filter – Butterworth, Chebyshev or Elliptic
- Convert this analog filter to digital using
  - Impulse invariance technique
  - Bilinear transformation
  - Matched Z – transform

# Impulse invariance transformation:

We design an IIR filter having unit sample response

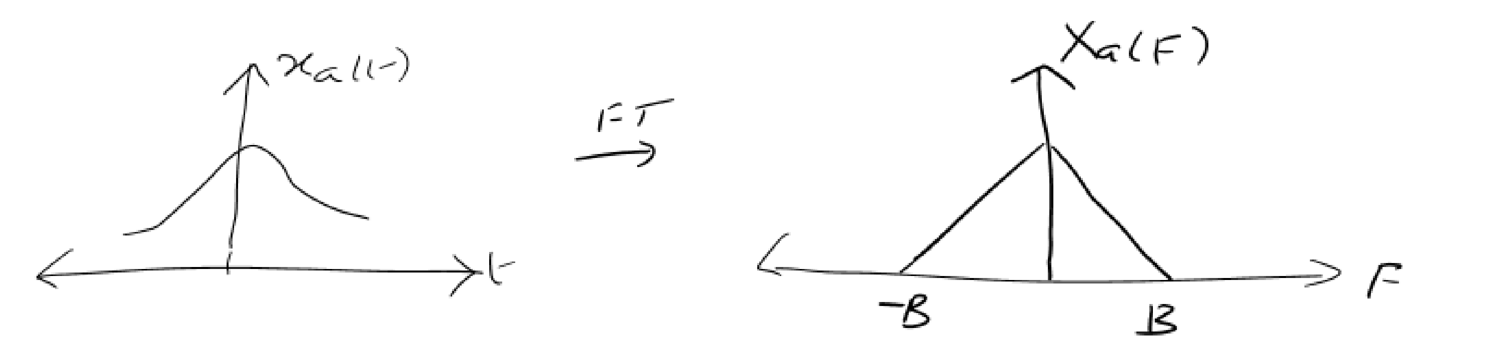
$h(n)$ = sampled version of IR of analog filter

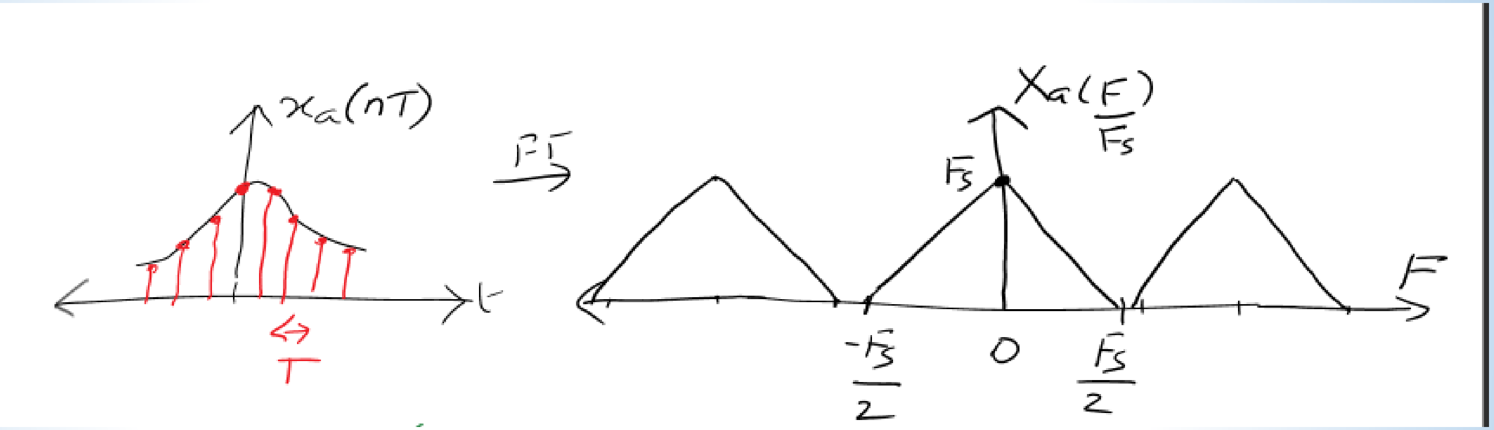
$h(n) \equiv h(nT)$ , $n = 0, 1, 2, \ldots$

$\uparrow$

$T$ = sampling interval

When a CT signal having a spectrum $X_a(F)$ is sampled at a rate $F_s = \frac{1}{T}$ samples per sec, the spectrum of the sampled signal $x_a(nT)$ is the periodic repetition of the scaled spectrum $F_s X_a(F)$, with period $F_s$.




$$\therefore X(f) = F_s \sum_{k=-\infty}^{\infty} X_a(f-k)F_s \quad , \quad f = \frac{F}{F_s}$$

$$= F_s\left[\cdots X_a\left(\frac{F}{F_s}-1\right)F_s + X_a\left(\frac{F}{F_s}\right)F_s + X_a\left(\frac{F}{F_s}+1\right)F_s + \cdots\right]$$

$$= F_s\left[\cdots X_a(F-F_s) + X_a(F) + X_a(F+F_s) + \cdots\right]$$

Aliasing occurs if the sampling rate

$F_s < 2F_{max}$ of $X_a(F)$

$T > 2T_{max}$

As sampling interval T increases,
the copies of the spectrum move closer
spectral overlap → aliasing

With respect to sampling the IR of analog filter
with frequency response $H_a(F)$,
the digital filter with unit-sample response
$h(n) \equiv h_a(nT)$ has the frequency response

$$H(f) = F_s \sum_{k=-\infty}^{\infty} H_a(f-k)F_s$$

$$H(f) = F_s \sum_{k=-\infty}^{\infty} H_a[(f-k) F_s]$$

$$H(\omega) = F_s \sum_{k=-\infty}^{\infty} H_a(\omega - 2\pi k) F_s]$$

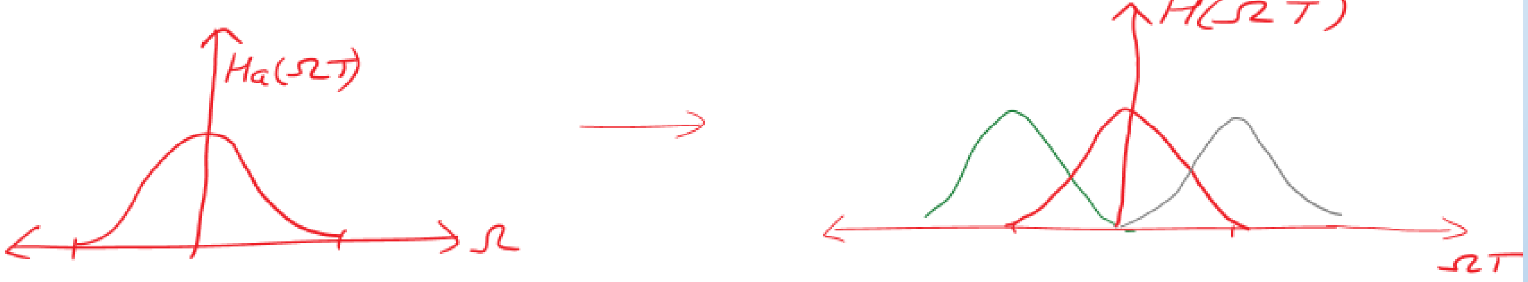
$\omega$ is periodic with $2\pi$

$\downarrow \Omega T$

$$H(\Omega T) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_a(\Omega T - 2\pi k) \frac{1}{T}]$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} H_a\left(\Omega - \frac{2\pi k}{T}\right)$$

Frequency response of analog filter :




Effects of aliasing can be minimized if the sampling interval $T$ is selected sufficiently small.

Mapping of points between z-plane & s-plane implied by the sampling process:

Relation between z-transform of $h(n)$ and Laplace transform of $h_a(t)$:

$$H_a(s) = \int_0^{\infty} h_a(t)\, e^{-st}\, dt$$

$$H_a(s)\Big|_{\text{sampled at } nT} = \int_0^{\infty} \sum_{n=0}^{\infty} h(n)\,\delta(t-nT)\, e^{-st}\, dt$$

$$= \sum_{n=0}^{\infty} h(n) \int_0^{\infty} \delta(t-nT)\, e^{-st}\, dt$$

$$= \sum_{n=0}^{\infty} h(n)\, e^{-snT} \quad \left[\because \mathcal{L}(\delta(t-a)) = e^{-sa}\right]$$

$$= H(z)\Big|_{z=e^{sT}} \quad \left[H(z) = \sum_{n=0}^{\infty} h(n)\, z^{-n}\right]$$

Frequency response of digital filter with unit sample resp. $h_a(nT)$:

$$H(\Omega T) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_a\left(\Omega - \frac{2\pi k}{T}\right)$$

$$\Rightarrow H(z)\Big|_{z=e^{sT}} = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_a\left(s - j\frac{2\pi k}{T}\right) \Leftarrow \text{Relation}$$

Mapping points from s-plane to z-plane:

$$z = e^{sT}$$

$$re^{j\omega} = e^{(\sigma + j\Omega)T} = e^{\sigma T} \cdot e^{j\Omega T}$$

$$\therefore \; r = e^{\sigma T}$$

$$\omega = \Omega T$$

$\sigma < 0$, $0 < r < 1$ $\Rightarrow$ LHP in s-plane is mapped inside unit-○ in z-plane

$\sigma > 0$, $r > 1$ $\Rightarrow$ RHP in s-plane is mapped outside the unit ○ in z-plane

$\sigma = 0$, $r = 0$ $\Rightarrow$ $j\Omega$ axis is mapped onto the unit ○ in z-plane.

Mapping of $j\Omega$ axis onto the unit circle is not one-to-one

Mapping $\omega = \Omega T$,

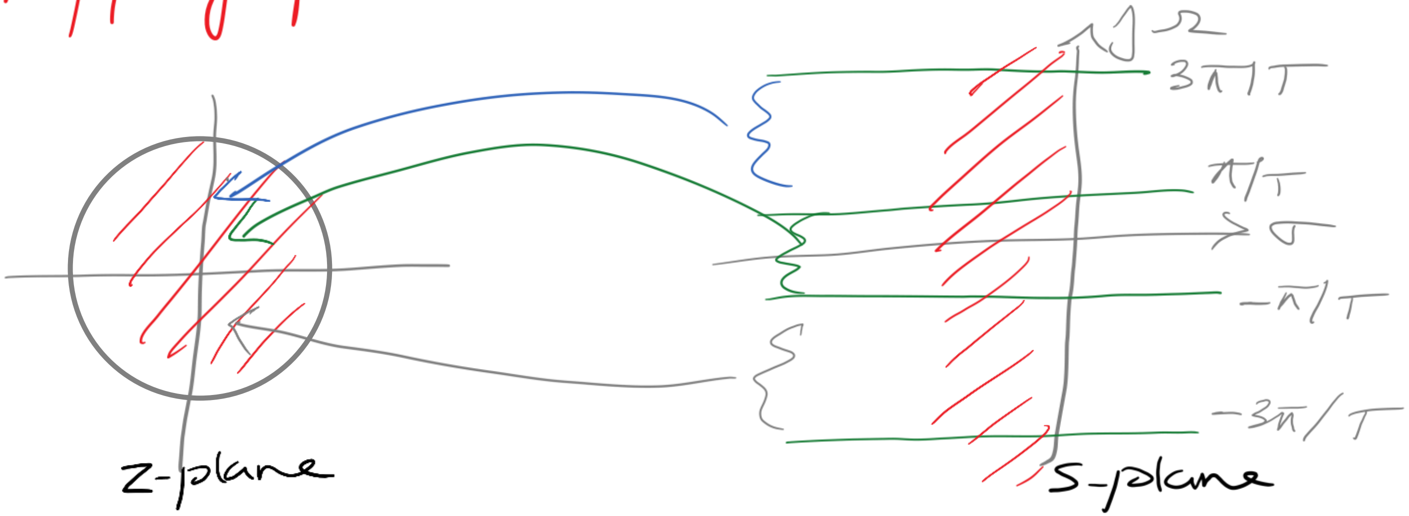
Analog freq interval maps into freq interval in digital

$$-\frac{\pi}{T} \le \Omega \le \frac{\pi}{T} \qquad\qquad -\pi \le \omega \le \pi$$

$$\frac{\pi}{T} \le \Omega \le \frac{3\pi}{T} \qquad\qquad -\pi \le \omega \le \pi$$

$$(2k-1)\frac{\pi}{T} \le \Omega \le (2k+1)\frac{\pi}{T} \qquad\qquad -\pi \le \omega \le \pi, \quad k = \text{integer}$$

$\therefore$ Mapping from $\Omega$ to $\omega$ is many-to-one



Let us express the system function of analog filter in partial fraction form.

Assuming distinct poles of analog filter:

coefficients in partial fraction

$$H_a(s) = \sum_{k=1}^{N} \frac{C_k}{s-p_k}$$

distinct poles

$$h_a(t) = \sum_{k=1}^{N} C_k e^{p_k t} \quad , t \geq 0 \qquad \left[ \mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at} \right]$$

Sampling $h_a(t)$ periodically at $t = nT$,

$$h(n) = h_a(nT)$$

$$= \sum_{k=1}^{N} C_k e^{p_k nT}$$

System function of resulting digital IIR filter,

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n}$$ , substitute $h(n)$

$$H(z) = \sum_{n=0}^{\infty} h(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} \sum_{k=1}^{N} C_k e^{p_k nT} . z^{-n}$$

$$= \sum_{k=1}^{N} C_k \sum_{n=0}^{\infty} \left(e^{p_k T} z^{-1}\right)^n$$

$$\frac{1}{1 - e^{p_k T} . z^{-1}}$$

Geometric Series exp

$$\sum_{i=0}^{\infty} a_i r^i = \frac{a_1}{1-r}$$

$$H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}}$$

We observe that the digital filter has poles at

$$z_k = e^{p_k T}, \qquad k = 1, 2, \ldots, N$$

# Impulse invariance transformation

$$H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}}$$

We observe that the digital filter has poles at

$$z_k = e^{p_k T}, \qquad k = 1, 2, \ldots, N$$

This equation holds for IIR filter having distinct poles.
Due to presence of aliasing, impulse invariance is suitable for LPF and BPF only.

Convert the analog filter with system function

$$H_a(s) = \frac{s + 0.1}{(s + 0.1)^2 + 9}$$

into a digital IIR filter by means of the impulse invariance method.

$$H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}}$$

**Solution.** We note that the analog filter has a zero at $s = -0.1$ and a pair of complex-conjugate poles at

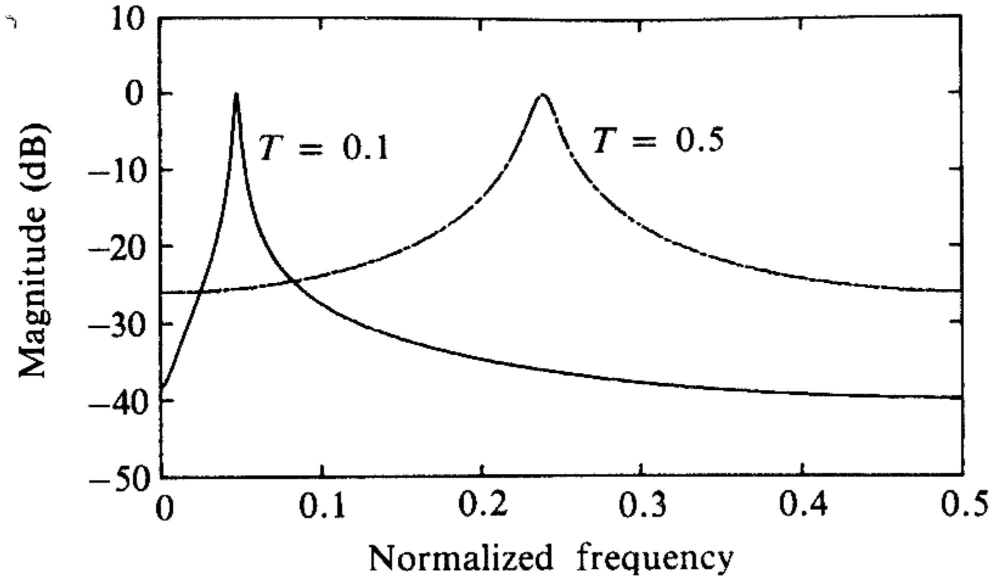
$$p_k = -0.1 \pm j3$$

from the partial-fraction expansion of $H_a(s)$

$$H(s) = \frac{\frac{1}{2}}{s + 0.1 - j3} + \frac{\frac{1}{2}}{s + 0.1 + j3}$$

$$H(z) = \frac{\frac{1}{2}}{1 - e^{-0.1T} e^{j3T} z^{-1}} + \frac{\frac{1}{2}}{1 - e^{-0.1T} e^{-j3T} z^{-1}}$$

$$H(z) = \frac{1 - (e^{-0.1T}\cos 3T)z^{-1}}{1 - (2e^{-0.1T}\cos 3T)z^{-1} + e^{-0.2T}z^{-2}}$$




We note that aliasing is significantly more prevalent when $T = 0.5$ than when $T = 0.1$. Also, note the shift of the resonant frequency as $T$ changes.

# Remember this

$$\frac{s+a}{(s+a)^2 + b^2} \Rightarrow \frac{1 - e^{-aT}\cos bT\, z^{-1}}{1 - 2e^{-aT}\cos bT\, z^{-1} + e^{-2aT} z^{-2}}$$

$$\frac{b}{(s+a)^2 + b^2} \Rightarrow \frac{e^{-aT}\sin bT\, z^{-1}}{1 - 2e^{-aT}\cos bT\, z^{-1} + e^{-2aT} z^{-2}}$$

Convert the analog filter in to its equivalent digital filter whose system function is given by $H(s) = \dfrac{s+0.4}{s^2+0.8s+25.16}$ using impulse invariance technique. Assume sampling frequency of 10Hz.

4B)

$$H(s) = \frac{s+0.4}{(s+0.4)^2 + 5^2}$$

$$T = \frac{1}{f_s} = \frac{1}{10} = 0.1$$

$$\frac{s+a}{(s+a)^2 + b^2} \Longrightarrow \frac{1 - e^{-aT}\cos bT\, z^{-1}}{1 - 2e^{-aT}\cos bT\, z^{-1} + e^{-2aT} z^{-2}}$$

(or)

They can solve by Partial fraction method

(1M)

$$H(z) = \frac{1 - e^{-0.4\times0.1}\cos(0.5)\, z^{-1}}{1 - 2e^{-0.4\times0.1}\cos(0.5)\, z^{-1} + e^{-2\times0.4\times0.1} z^{-2}}$$

$$H(z) = \frac{1 - \cancel{0.5\ldots}\ 0.843\, z^{-1}}{1 - 1.686\, z^{-1} + 0.923\, z^{-2}}$$

(2M)


Thank
you

# IIR filter design
# Bilinear Transformation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# Bilinear transformation method for IIR filter design

This method overcomes the limitation of the previous method.

Transforms the $j\Omega$ axis into unit ⊙ only once.

Avoids aliasing of frequency components

All points in LHP are mapped inside unit ⊙

" RHP " outside "

Consider an analog filter with system function

$$H(s) = \frac{b}{s+a} = \frac{Y(s)}{X(s)}$$

$$Y(s)(s+a) = X(s)\, b.$$

This system can also be characterized by the differential equation,

$$\frac{dy}{dt} + a y(t) = b x(t) \text{ ——— } (1)$$

Integrate the derivative and approximate the integral using trapezoidal formula,

$$y(t) = \int_{t_0}^{t} y'(\tau)\, d\tau + y(t_0) \quad \text{——— ②}$$

Approximate this integral by trapezoidal formula, at $t = nT$ and $t_0 = nT - T$

$$\int_{t_0}^{t} y'(\tau)\, d\tau = \frac{t - t_0}{2}\left[y'(t_0) + y'(t)\right]$$

Substitute for $t$ and $t_0$,

$$\int_{t_0}^{t} y'(\tau)\, d\tau = \frac{nT - nT + T}{2}\left[y'(nT) + y'(nT - T)\right]$$

Substitute in equation ②,

$$y(nT) = \frac{T}{2}\left[y'(nT) + y'(nT - T)\right] + y(nT - T) \quad \text{——— ③}$$

Evaluate the differential eqn in ① at $t = nT$,

$$① \Rightarrow \frac{dy(t)}{dt} + ay(t) = bx(t)$$

$$y'(nT) + ay(nT) = bx(nT)$$

$$y'(nT) = -ay(nT) + bx(nT)$$

Substitute this in eqn ③,

$$③ \Rightarrow y(nT) = \frac{T}{2}\left[y'(nT) + y'(nT-T)\right] + y(nT-T)$$

$$= \frac{T}{2}\left[-ay(nT) + bx(nT) - ay(nT-T) + bx(nT-T)\right] + y(nT-T)$$

Take $y(nT) \equiv y(n)$ and $x(nT) \equiv x(n)$,
and $nT - T = n-1$,

$$y(n) = \frac{T}{2}\left[-ay(n) + bx(n) - ay(n-1) + bx(n-1)\right] + y(n-1)$$

Grouping all output & input terms,

$$\left(1+\frac{aT}{2}\right)y(n)-\left(1-\frac{aT}{2}\right)y(n-1)=\frac{bT}{2}\left[x(n)+x(n-1)\right]$$

Thus we obtain a difference equation

Taking z-transform,

$$\left(1+\frac{aT}{2}\right)Y(z)-\left(1-\frac{aT}{2}\right)z^{-1}Y(z)=\frac{bT}{2}\left[X(z)+z^{-1}X(z)\right]$$

$$H(z)=\frac{Y(z)}{X(z)}=\frac{\frac{bT}{2}\left[1+z^{-1}\right]}{1+\frac{aT}{2}-z^{-1}+\frac{aT}{2}z^{-1}}$$

$$=\frac{\frac{bT}{2}\left[1+z^{-1}\right]}{(1-z^{-1})+\frac{aT}{2}(1+z^{-1})}$$

$$\Rightarrow H(z) = \frac{b}{\frac{2}{T}\frac{(1-z^{-1})}{(1+z^{-1})} + \frac{2}{T}\cdot\frac{aT}{2}\frac{(1+z^{-1})}{(1+z^{-1})}}$$

$$= \frac{b}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + a}$$

We know, $H(s) = \frac{b}{s+a}$

Therefore the mapping from s-plane to z-plane is

$$\boxed{s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

This is called bilinear transformation

This transformation holds in general for $N^{th}$ order differential equation too.

Characteristics

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$= \frac{2}{T}\left(\frac{z-1}{z+1}\right)$$

$$= \frac{2}{T}\,\frac{re^{j\omega}-1}{re^{j\omega}+1}$$

$$= \frac{2}{T}\left[\frac{r\cos\omega + jr\sin\omega - 1}{r\cos\omega + jr\sin\omega + 1}\right]$$

$$= \frac{2}{T}\left[\frac{r\cos\omega - 1 + jr\sin\omega}{r\cos\omega + 1 + jr\sin\omega}\right]\left[\frac{r\cos\omega + 1 - jr\sin\omega}{r\cos\omega + 1 - jr\sin\omega}\right]$$

$$z = re^{j\omega}$$

$$s = \sigma + j\Omega$$

$$e^{j\omega} = \cos\omega + j\sin\omega$$

$$s = \frac{2}{T}\frac{\left[r^2\cos^2\omega + r\cos\omega - jr^2\cos\omega\sin\omega - r\cos\omega - 1 - jr\sin\omega + jr^2\cos\omega\sin\omega + jr\sin\omega - j^2r^2\sin^2\omega\right]}{(r\cos\omega+1)^2 - (jr\sin\omega)^2}$$

$$= \frac{2}{T} \times \frac{r^2 + 2jr\sin\omega - 1}{r^2 + 2r\cos\omega + 1}$$

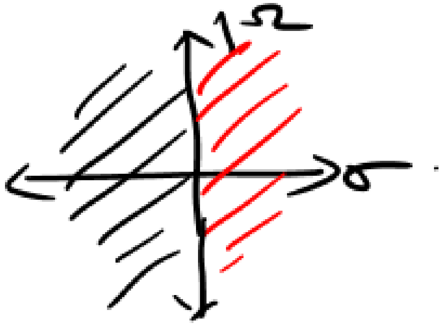
$$s = \frac{2}{T}\left[\frac{r^2-1}{r^2+2r\cos\omega+1}\right] + j\frac{2}{T}\frac{2r\sin\omega}{r^2+2r\cos\omega+1}$$

$$s = \sigma + j\Omega$$

$r<1$, $\sigma<0$

$r>1$, $\sigma>0$

$\boxed{r=1,\ \sigma=0}$




Splane


z-plane.

$$\Omega = \frac{2}{T} \times \frac{\sin \omega}{1+\cos \omega}$$

$$= \frac{2}{T} \cdot \frac{2 \sin \omega/2 \cos \omega/2}{2 \cos^2 \omega/2}$$

$$\boxed{\Omega = \frac{2}{T} \tan \frac{\omega}{2}}$$

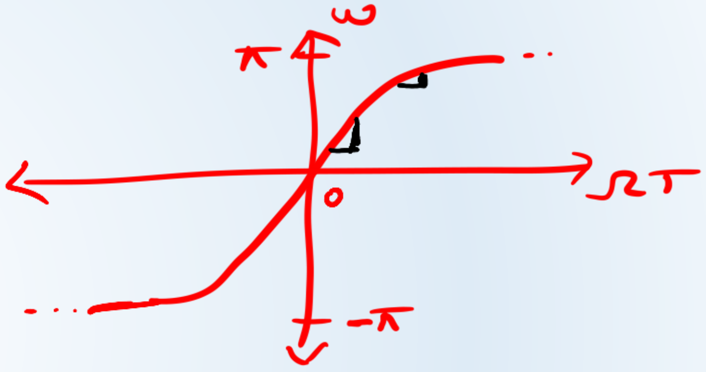
$$\Rightarrow \omega = 2 \tan^{-1}\left(\frac{\Omega T}{2}\right)$$

$\Omega = \pm\infty$, $\omega = \pm\pi$

one to one mapping

$\sin 2\theta = 2 \sin\theta \cos\theta$

$\cos 2\theta = 2\cos^2\theta - 1$




freq compression
freq warping

nonlinear

Q. Analog filter, $H_a(s) = \dfrac{s+0.1}{(s+0.1)^2+16}$

Digital filter, using bilinear transf.

↓ resonant freq $\omega_r = \pi/2$.

Soln: $H(s) = \dfrac{s+0.1}{(s+0.1)^2-(j4)^2} = \dfrac{s+0.1}{(s+0.1-j4)(s+0.1+j4)}$

Poles, $s_k = -0.1 \pm j4$

$(s = \sigma + j\Omega)$

$\Omega_r = 4$

$\omega_r = \pi/2$

$\Omega_r = \dfrac{2}{T}\tan\dfrac{\omega_r}{2}$

$4 = \dfrac{2}{T}\tan\dfrac{\pi/2}{2} \Rightarrow T = 1/2$

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = 4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$= \frac{s+0.1}{(s+0.1)^2+16}\Big|_{s=}$$

$$= \frac{4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+0.1}{\left(4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+0.1\right)^2+16}$$

$$H(z) = \frac{0.128 + 0.006z^{-1} - 0.122z^{-2}}{1+0.995z^{-2}}$$

$$= 0.128\frac{(z^2+0.046z-0.953)}{z^2+0.995}$$

Poles @ $0.987e^{\pm j\pi/2}$

Zeros@ $-1, 0.953$

3) Matched z-transform

Direct mapping of poles & zeros

$$H(s) = \frac{\prod_{k=1}^{M} (s - z_k)}{\prod_{k=1}^{N} (s - p_k)} \xrightarrow{\text{map}} H(z) = \frac{\prod_{k=1}^{M} \left(1 - e^{z_k T} z^{-1}\right)}{\prod_{k=1}^{N} \left(1 - e^{p_k T} z^{-1}\right)}$$


Thank
you

# Analog filter design
# Butterworth filter

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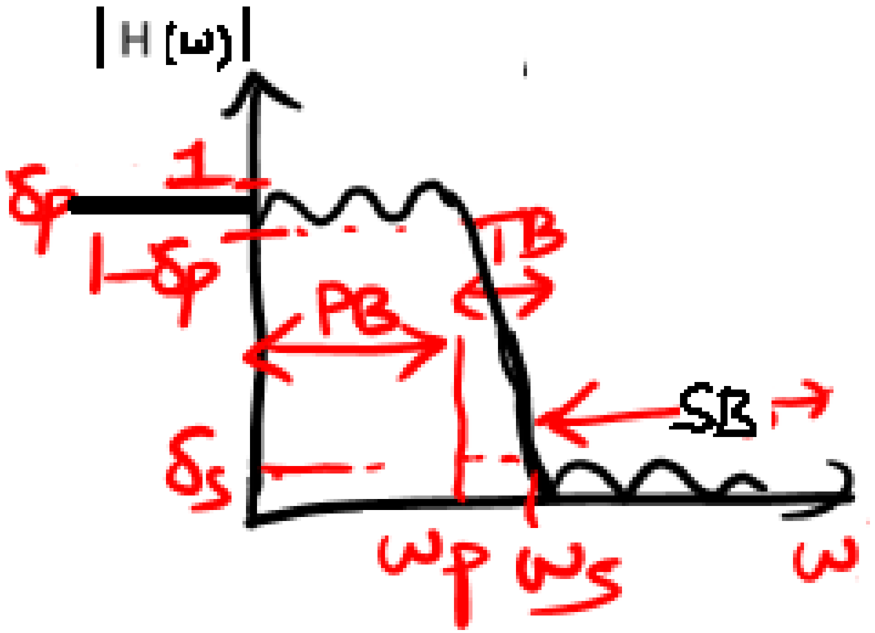
# Analog filter design

- **Butterworth filter** (discussed in this ppt)
- Chebyshev filter
- Elliptic filter

$H(s)$ analog → $H(z)$ digital.

LPF – Ideal.


Practical.



# Butterworth filter design – LPF

All pole.

$$|H(\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

$\leftarrow$ order = N

$\leftarrow$ cutoff freq

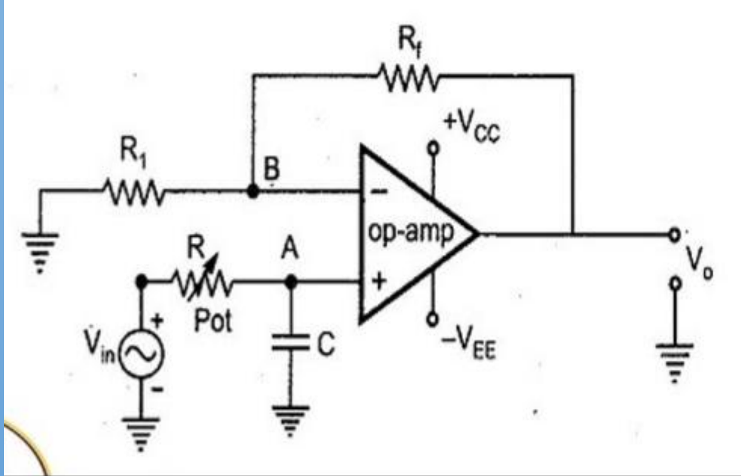
$s = j\Omega \Rightarrow \Omega = s/j$

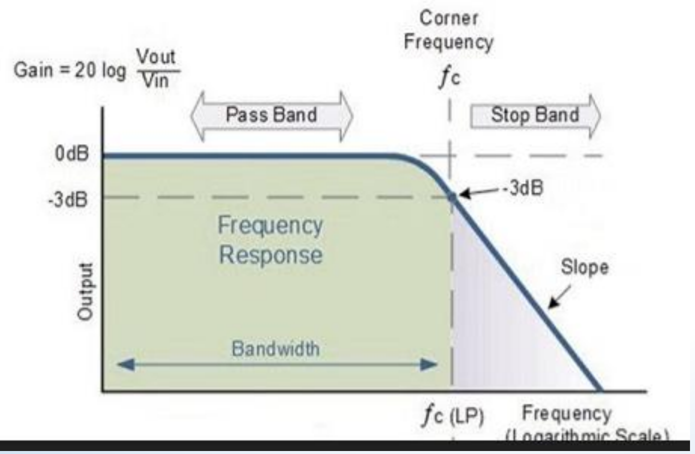
$$|H(\Omega)|^2 = \frac{1}{1+\left(\frac{s/j}{\Omega_c}\right)^{2N}}$$

$$= \frac{1}{1+\left(\frac{-s^2}{\Omega_c^2}\right)^{N}}$$

$$H(s)\cdot H(-s)\Big|_{s=j\Omega}$$

$$H(s)H(-s) = \frac{1}{1+\left(\frac{-s^2}{\Omega_c^2}\right)^{N}}$$



To find poles:

$$1 + \left(\frac{-s^2}{\Omega_c^2}\right)^N = 0$$

$$\left(\frac{-s^2}{\Omega_c^2}\right)^N = -1 = e^{j(2k+1)\pi} \qquad k = \text{integer}$$

$$\frac{-s^2}{\Omega_c^2} = e^{j(2k+1)\frac{\pi}{N}}$$

$$s^2 = -\Omega_c^2 \, e^{j(2k+1)\frac{\pi}{N}}$$

$$s = j\Omega_c \, e^{j(2k+1)\frac{\pi}{2N}}$$

$$\boxed{s_k = \Omega_c \, e^{j(\pi/2)} \cdot e^{j(2k+1)\frac{\pi}{2N}}} \qquad k = 0, 1, \ldots 2N-1$$

eg. $N = 2$ (even) & $\Omega_c = 1$ (normalized)

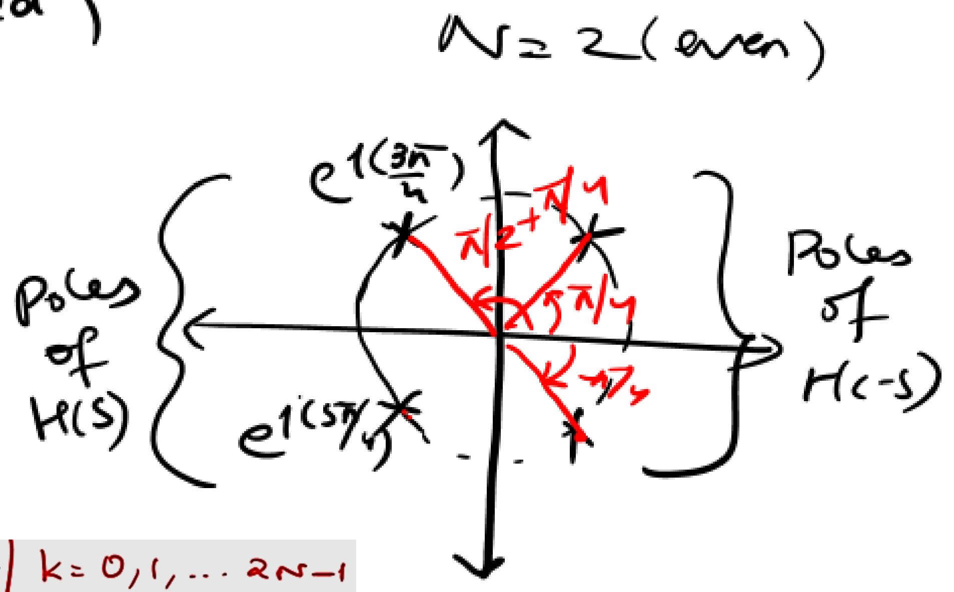
$S_k = e^{j\pi/2} \cdot e^{j(2k+1)\frac{\pi}{4}}$ $\quad k = \underline{\underline{0, 1}}, 2, 3$

$S_0 = e^{j(\pi/2 + \pi/4)} = e^{j(3\pi/4)} = -0.707 + j0.707$

$S_1 = e^{j(5\pi/4) - 2\pi} = e^{j(-3\pi/4)} = -0.707 - j0.707$

$S_2 = e^{j(7\pi/4) - 2\pi} = \frac{-\pi}{4}$

$S_3 = e^{j(9\pi/4) - 2\pi} = \frac{\pi}{4}$




$\boxed{S_k = \Omega_c \, e^{j(\pi/2)} \cdot e^{j(2k+1)\frac{\pi}{2N}}}$ $\quad k = 0, 1, \ldots 2N-1$

Stable poles : $N =$ even , $k = \underline{\underline{0}}, 1, \ldots \frac{N}{2} - 1$
$N =$ odd , $k = 0, 1, \ldots \frac{N-1}{2}$

$(\Omega_c = 1)$

$$H(s) = \frac{1}{[s - (-0.707 + j0.707)][s - (-0.707 - j0.707)]}$$

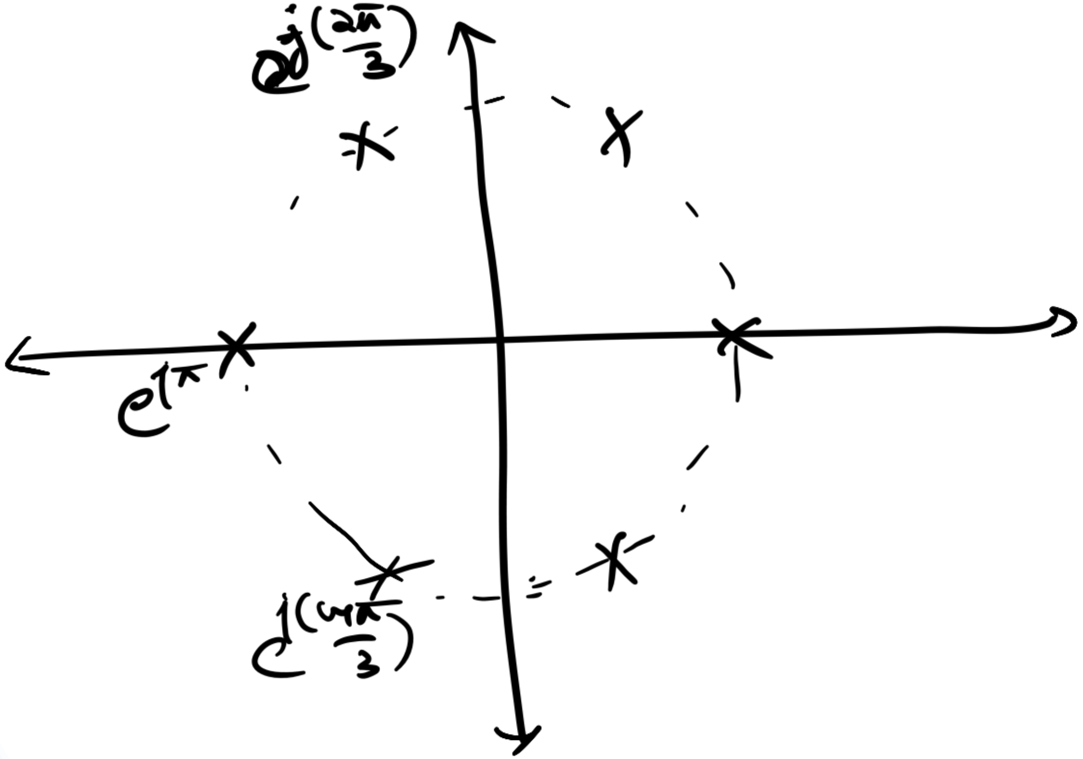
$$\rightarrow (s + 0.707)^2 - (j0.707)^2 = s^2 + 1.414s + 1$$

$N = 3$ (odd), $\Omega_c = 1$

$$s_k = e^{j\pi/2} e^{j(2k+1)\frac{\pi}{6}} \quad , \quad k = 0, 1, \ldots 2N-1$$

$$\phi_k = \underbrace{\frac{2\pi}{3}, \pi, \frac{4\pi}{3}}_{\text{stable}}, \frac{5\pi}{3}, 2\pi, \frac{7\pi}{3}$$

$$H(s) = \frac{1}{(s+1)(s^2+s+1)}$$



Freq resp.


$N\uparrow, \quad TB\downarrow$

$$A_p = 1-\delta_p = \frac{1}{\sqrt{1+\varepsilon^2}}$$

$$A_s = \delta_s$$

$$A_p \le |H(\Omega)| \le 1, \quad 0 \le \Omega \le \Omega_p.$$

$$|H(\Omega)| \le A_s, \quad \Omega_s \le \Omega.$$

$$|H(\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

$$\varepsilon = \left(\frac{\Omega_p}{\Omega_c}\right)^N$$

$$\Omega_c^{2N} = \frac{\Omega_p^{2N}}{\varepsilon^2}$$

$\Omega = \Omega_p$

$\Omega = \Omega_s$

$N$

$$\boxed{|H(\Omega)|^2 = \frac{1}{1+\varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}}}$$

ε = Band edge value

## Filter order

$$|H(\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

At $\Omega = \Omega_p$, $|H(\Omega_p)|^2 = A_p^2 = \frac{1}{1+\left(\frac{\Omega_p}{\Omega_c}\right)^{2N}}$ $\Rightarrow \left(\frac{\Omega_p}{\Omega_c}\right)^{2N} = \frac{1}{A_p^2} - 1$

At $\Omega = \Omega_s$, $|H(\Omega_s)|^2 = A_s^2 = \frac{1}{1+\left(\frac{\Omega_s}{\Omega_c}\right)^{2N}}$ $\Rightarrow \left(\frac{\Omega_s}{\Omega_c}\right)^{2N} = \frac{1}{A_s^2} - 1$

Now, $\left(\frac{\Omega_p}{\Omega_s}\right)^{2N} = \left(\frac{\Omega_p}{\Omega_c}\right)^{2N} \cdot \left(\frac{\Omega_c}{\Omega_s}\right)^{2N}$

$$k^{2N} = \frac{\frac{1}{A_p^2} - 1}{\frac{1}{A_s^2} - 1} = d^2$$

$2N \log(k) = 2\log d \Rightarrow \boxed{N = \frac{\log(d)}{\log(k)}}$ //

$$d = \sqrt{\frac{1/A_p^2 - 1}{1/A_s^2 - 1}}$$

discrimination factor

$$k = \frac{\Omega_p}{\Omega_s}$$

selectivity / transition ratio.

Filter order – 2nd eqn.

$$|H(\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

$\Omega = \Omega_s$,

$$\boxed{N = \frac{\log\left(1/\delta_s^2 - 1\right)}{2\log\left(\Omega_s/\Omega_c\right)}}$$

$$\boxed{N = \frac{\log(\delta/\varepsilon)}{\log\left(\frac{\Omega_s}{\Omega_p}\right)}}$$

$$\frac{1}{\delta_s^2} - 1 = \delta^2$$

Question:

Order & poles LPF Butterworth, BW = 500 Hz, att = 40 dB at 1000 Hz

Soln:

$-3dB$ BW $= f_c = 500\,Hz \Rightarrow \Omega_c = 2\pi f = 1000\pi$

$f_s = 1000\,Hz \Rightarrow \Omega_s = 2\pi f = 2000\pi$

$\delta_s^2 = A_s^2 = -40\,dB \Rightarrow \log \delta_s = -40$

$\delta_s = 10^{-2} = 0.01$

$$\text{Filter order, } N = \frac{\log\left(1/\delta_s^2 - 1\right)}{2\log\left(\Omega_s/\Omega_c\right)} = \frac{4}{2 \times 0.03} = 6.64 \approx 7$$

$$\text{Pole: } S_k = \Omega_c\, e^{j(\pi/2)}\, e^{j(2k+1)\frac{\pi}{2N}}$$

$$= 1000\pi\, e^{j\pi/2}\, e^{j(2k+1)\frac{\pi}{14}} \qquad k = 0, 1, \ldots \frac{N-1}{2} = 3$$

$$\text{Pole}: \; S_k = \Omega_c \, e^{j(\pi/2)} \, e^{j(2k+1)\frac{\pi}{2N}}$$

$$= 1000\pi \, e^{j\pi/2} \, e^{j(2k+1)\frac{\pi}{14}} \qquad k = 0, 1, \ldots \frac{N-1}{2} = 3$$

$$S_0 = 1000\pi \, e^{j(\pi/2 + \pi/14)} = 1000\pi \, e^{j(8\pi/14)}$$

$$= 3141.59\,(-0.2225 \pm j0.9749) = -699 \pm j\,3062.74$$

$$S_1 = 1000\pi \left(e^{j\,\pi/2 + 3\pi/14}\right) = 1000\pi \, e^{j\,10\pi/14}$$

$$S_2 = 1000\pi \, e^{j(12\pi/14)}$$

$$S_3 = 1000\pi \, e^{j\pi} = -1000\pi$$

Q.2) The specifications of desired LPF response is given by

$$\underset{A_p}{\frac{1}{\sqrt{2}}} \le |H(e^{j\omega})| \le 1 \quad , \quad 0 \le \omega \le \underset{=\omega_p}{0.2\pi}$$

$$|H(e^{j\omega})| \le \underset{=A_s}{0.08}, \quad \underset{=\omega_s}{0.4\pi} \le \omega \le \pi$$

Design a Butterworth digital filter using bilinear transf

Assume a sampling interval of T = 2 sec.

Soln :

Butterworth analog → order, pole locations, H(s)

↓ BLT

digital (H(z))

Given: $A_p = \frac{1}{\sqrt{2}} = 0.707$

$A_s = 0.08$

$\omega_p = 0.2\pi \Rightarrow \Omega_p = \tan\left(\frac{0.2\pi}{2}\right) = 0.324$

$\omega_s = 0.4\pi \Rightarrow \Omega_s = \tan\left(\frac{0.4\pi}{2}\right) = 0.726$

BLT, $\omega \to \Omega$ : $\Omega = \frac{2}{T} \tan\frac{\omega}{2} = \tan\frac{\omega}{2}$

Order of desired LP Butterworth:

$$N = \frac{\log d}{\log k} = \frac{\log 0.08}{\log 0.446} = 3.12 \approx 4 \text{ (even)}$$

$$d = \sqrt{\frac{1/A_p^2 - 1}{1/A_s^2 - 1}} = \sqrt{\frac{(1/0.707^2) - 1}{(1/0.08^2) - 1}} = 0.08$$

$$k = \frac{\Omega_p}{\Omega_s} = \frac{0.324}{0.726} = 0.446$$

$$S_k = \Omega_c e^{j\pi/2} e^{j(2k+1)\frac{\pi}{8}}$$

$$\left(\frac{\Omega_p}{\Omega_c}\right)^N = \varepsilon = \sqrt{\frac{1}{A_p^2}-1}$$

$$\Omega_c = 0.324$$

$$S_k = 0.324\, e^{j(\pi/2)} e^{j(2k+1)\frac{\pi}{8}}$$

$$k = 0, \ldots \frac{N}{2}-1$$

$$S_0 = 0.324\, e^{j(\pi/2+\pi/8)} = 0.324\, e^{j\frac{5\pi}{8}}$$
$$= 0.324(-0.382 + j0.923)$$
$$= -0.124 \pm j0.299$$

$$S_1 = 0.324\, e^{j(\pi/2+3\pi/8)} = 0.324\, e^{j\frac{7\pi}{8}}$$
$$= -0.29 \pm j0.124$$

Stable poles:

$$k = 0 \ldots \frac{N}{2}-1 = 1$$

$$k = 0, 1$$

$N = 4$

$2N$ poles

$$H(S)_{\left(\substack{\text{normalised} \\ \Omega_c=1}\right)} = \frac{1}{(S-e^{j5\pi/8})(s-e^{-j5\pi/8})(S-e^{j7\pi/8})(S-e^{-j7\pi/8})}$$

$$H(s)\Big|_{s\to\frac{s}{\Omega_c}} = \frac{1}{\left(\frac{S}{\Omega_c}-e^{j5\pi/8}\right)\left(\frac{S}{\Omega_c}-e^{-j5\pi/8}\right)\cdots}$$

$$= \frac{\Omega_c^4}{\frac{1}{\Omega_c^4}(s-\Omega_c e^{j5\pi/8})(s-\Omega_c e^{-j5\pi/8})\cdots}$$

$$H(s) = \frac{0.011}{(S+0.124-j0.299)(s+0.124+j0.299)(s+0.29-j0.124)(s+0.29+j0.124)}$$

$$= \frac{0.011}{((S+0.124)^2-j^2 0.299^2)((S+0.29)^2-j^2 0.124^2)}$$

$$= \frac{0.011}{(S^2+0.248s+0.105)(s^2+0.58s+0.09)}$$

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) = \frac{1-z^{-1}}{1+z^{-1}}$$

$$H(z) = \frac{0.011}{\left[\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 0.248\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.105\right]\left[\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 0.58\,\frac{1-z^{-1}}{1+z^{-1}} + 0.09\right]}$$

$$H(z) = \frac{0.011\,(1+z^{-1})^4}{\left[(1-z^{-1})^2 + 0.248\,(1-z^{-1})(1+z^{-1}) + 0.105\,(1+z^{-1})^2\right] \times \left[(1-z^{-1})^2 + 0.58\,(1-z^{-1})\,(1+z^{-1}) + 0.09\,(1+z^{-1})^2\right]}$$

$$= \frac{0.011\,(1+z^{-1})^4}{(1.353 - 1.79z^{-1} + 0.857z^{-2})\,(1.67 - 1.82z^{-1} + 0.51z^{-2})}$$

# Reference


- Proakis J. G, Manolakis D. G. Mimitris D., “Introduction to Digital Signal Processing” Prentice Hall, India, 2007.


Thank
you

# Analog filter design
# Chebyshev Filter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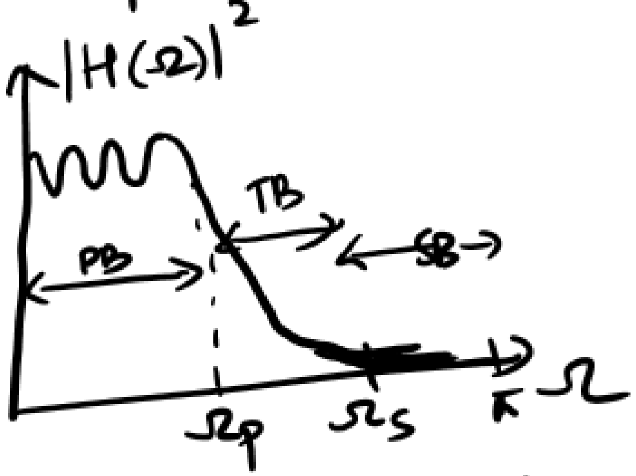
# Chebyshev filter design

## Type I Chebyshev

- all pole filters

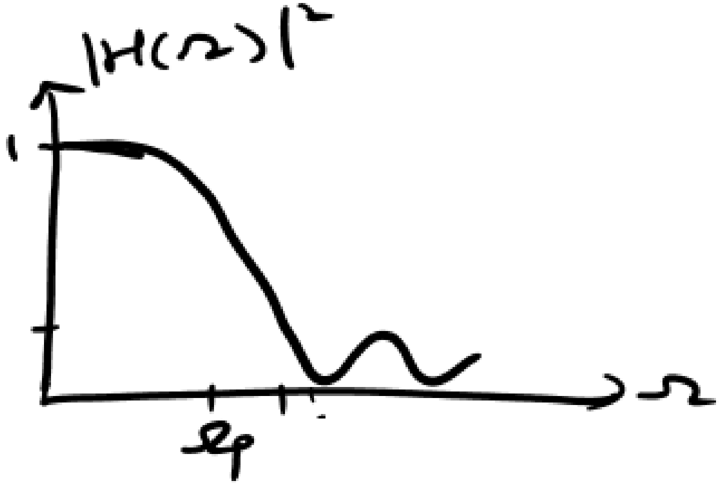



- equiripple behaviour in passband
- monotonic behaviour in stopband

✓

## Type II Chebyshev

- both poles & zeros




- monotonic - PB
- equiripples - SB

X

## Chebyshev Type I filter:

$$|H(\Omega)|^2 = \frac{1}{1+\varepsilon^2 T_N^2\left(\frac{\Omega}{\Omega_p}\right)}$$

$$A_p = \frac{1}{\sqrt{1+\varepsilon^2}} = 1-\delta_p$$

$T_N(x)$ – Chebyshev polynomial.

$$T_N(x) = \cos(N\cos^{-1}x), \quad |x| \le 1$$
$$= \cosh(N\cosh^{-1}x), \quad |x| > 1$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

$T_N(x)$, $T_{N-1}(x)$, $T_{N+1}(x)$

$x = \cos\theta$

$$T_N(x) = \cos(N\cos^{-1}x) = \cos(N\cos^{-1}\cos\theta) = \cos(N\theta)$$

$$T_{N+1}(x) = \cos\left[(N+1)\cos^{-1}x\right] = \cos\left[(N+1)\theta\right]$$
$$= \cos N\theta \cos\theta - \sin N\theta \sin\theta.$$

$T_N(x) = \cos(N\theta)$

$T_{N+1}(x) = \cos[(N+1)\theta] = \cos N\theta \cos\theta - \sin N\theta \sin\theta$.

$T_{N-1}(x) = \cos[(N-1)\theta] = \cos N\theta \cos\theta + \sin N\theta \sin\theta$.

$T_{N+1}(x) + T_{N-1}(x) = 2\cos N\theta \cos\theta$.

$= 2T_N(x) \cdot x$

$T_{N+1}(x) = 2T_N(x) \cdot x - T_{N-1}(x)$

when $N=0$, $T_0(x) = \cos(0 \cdot \cos^{-1}x) = 1$

$N=1$, $T_1(x) = \cos(1 \cdot \cos^{-1}x) = x$

$N=2$, $T_2(x) = 2T_1(x) \cdot x - T_0(x) = 2x^2 - 1$

$N=3$, $T_3(x) = 2T_2(x) \cdot x - T_1(x) =$

$= 2x(2x^2-1) - x = 4x^3 - 3x$.

Properties of Cheb polynomials :

1) $|T_N(x)| \le 1$ for all $|x| \le 1$

2) $|T_N(x)| > 1$ for all $|x| > 1$

3) $N =$ odd, $T_N(x)$ odd fn of $x$.

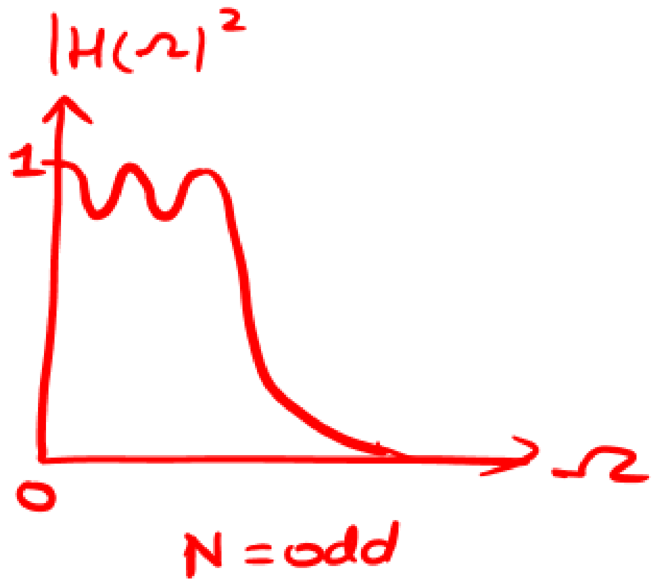
$N =$ even, $T_N(x)$ even "

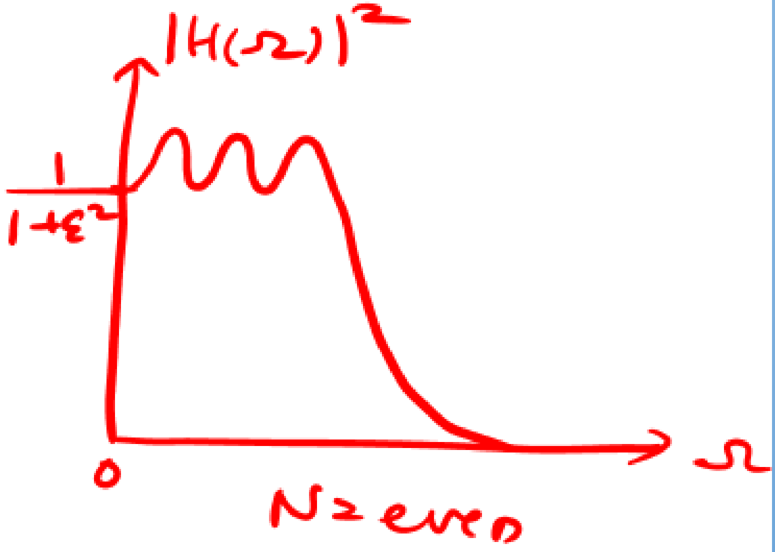
4) $T_N(1) = 1$, for all $N$

5) roots of $T_N(x)$ $\pm 1$, $-1 \le x \le 1$

6) $N =$ odd, $T_N(0) = 0 \Rightarrow |H(0)|^2 = 1$

$N =$ even, $T_N(0) = \pm 1 \Rightarrow |H(0)|^2 = \frac{1}{1+\epsilon^2}$

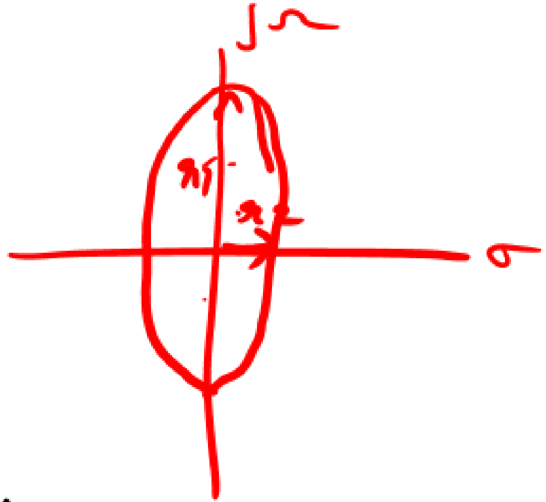


## Location of poles of Type I cheb filter:

$$|H(\Omega)|^2 = \frac{1}{1+\varepsilon^2 T_N^2\left(\frac{j\Omega}{j\Omega_p}\right)} = \frac{1}{1+\varepsilon^2 T_N^2\left(\frac{s}{j\Omega_p}\right)} = H(s)\cdot H(-s)$$

$$\text{dnr} = 0, \quad 1+\varepsilon^2 T_N^2\left(\frac{s}{j\Omega_p}\right) = 0$$




$$s_k = \sigma_k + j\Omega_k.$$

Eqn of ellipse : $\dfrac{\sigma_k^2}{r_2^2} + \dfrac{\Omega_k^2}{r_1^2} = 1$

$$r_1 = \Omega_p\left(\frac{\beta^2+1}{2\beta}\right)$$

$$r_2 = \Omega_p\left(\frac{\beta^2-1}{2\beta}\right)$$

$$\beta = \left[\frac{\sqrt{1+\varepsilon^2}+1}{\varepsilon}\right]^{1/N}$$

$(x_k, y_k)$

$$x_k = r_2 \cos\phi_k$$

$$y_k = r_1 \sin\phi_k.$$

$$\phi_k = \frac{\pi}{2} + (2k+1)\frac{\pi}{2N}$$

$$x_k + jy_k = r_2\cos\phi_k + j\,r_1\sin\phi_k$$

## Order of Type 1 Chebyshev filter.

$$|H(\Omega)|^2 = \frac{1}{1+\varepsilon^2 T_N^2\left(\frac{\Omega}{\Omega_p}\right)}$$

$A_p^2 = \frac{1}{1+\varepsilon^2}$

At $\Omega = \Omega_s$, $|H(\Omega_s)|^2 = A_s^2 = \frac{1}{1+\varepsilon^2 T_N^2\left(\frac{\Omega_s}{\Omega_p}\right)}$

$$\Rightarrow 1+\varepsilon^2 T_N^2\left(\frac{\Omega_s}{\Omega_p}\right) = \frac{1}{A_s^2}$$

$$T_N^2\left(\frac{\Omega_s}{\Omega_p}\right) = \frac{\frac{1}{A_s^2}-1}{\varepsilon^2} = \frac{\frac{1}{A_s^2}-1}{\frac{1}{A_p^2}-1} = \frac{1}{d^2}$$

$$A_p^2 = \frac{1}{1+\varepsilon^2}$$

$$d = \sqrt{\frac{1/A_p^2 - 1}{1/A_s^2 - 1}}$$

$$k = \frac{\Omega_p}{\Omega_s}$$

$$T_N^2\left(\frac{1}{k}\right) = \frac{1}{d^2}$$

$$T_N\left(\frac{1}{k}\right) = \frac{1}{d}$$

$|x|>1$, $\cosh\left(N \cosh^{-1}(1/k)\right) = \frac{1}{d}$

$$\boxed{N = \frac{\cosh^{-1}(1/d)}{\cosh^{-1}(1/k)}} \quad \text{—— (1)}$$

$$N = \frac{\cosh^{-1}(1/d)}{\cosh^{-1}(1/k)}$$

$$\frac{1}{d} = \sqrt{\frac{\frac{1}{A_s^2} - 1}{\frac{1}{A_p^2} - 1}} = \frac{\sqrt{\frac{1}{\delta_s^2} - 1}}{\varepsilon} = \frac{\delta}{\varepsilon}$$

$$\delta_s = \frac{1}{\sqrt{1+\delta^2}}$$

$$\boxed{N = \frac{\cosh^{-1}(\delta/\varepsilon)}{\cosh^{-1}(\Omega_s/\Omega_p)}} \quad \text{— } ②$$

Q. Design a digital Chebyshev filter that satisfies the constraints

$$\underset{=A_p}{0.707} \le |H(e^{j\omega})| \le 1 \quad , \quad 0 \le \omega \le 0.2\pi = \omega_p$$

$$|H(e^{j\omega})| \le 0.1 = A_s \quad , \quad \underset{=\omega_s}{0.5\pi} \le \omega \le \pi$$

Use bilinear transf and assume $T = 1\,\text{sec}$.

Soln: Given, $A_p$, $A_s$, $\omega_p$, $\omega_s$

$$\Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2} = 2\tan\frac{0.2\pi}{2} = 2\tan 0.1\pi = 0.649$$

$$\Omega_s = \frac{2}{T}\tan\frac{\omega_s}{2} = 2\tan\frac{0.5\pi}{2} = 2$$

Order of filter, $N = \dfrac{\cosh^{-1}(1/d)}{\cosh^{-1}(1/k)} = \dfrac{\cosh^{-1}(1/0.1)}{\cosh^{-1}(1/0.32)} = 1.65 \approx 2$

$$d = \sqrt{\frac{\frac{1}{A_p^2} - 1}{\frac{1}{A_s^2} - 1}} = \sqrt{\frac{\frac{1}{0.707^2} - 1}{\frac{1}{0.1^2} - 1}} = 0.1 \quad , \quad k = \frac{\Omega_p}{\Omega_s} = \frac{0.649}{2} = 0.32$$

$$x_k = r_2 \cos \phi_k$$

$$y_k = r_1 \sin \phi_k \,.$$

$$r_1 = \Omega_p \left( \frac{\beta^2+1}{2\beta} \right)$$

$$r_2 = \Omega_p \left( \frac{\beta^2-1}{2\beta} \right)$$

$$\beta = \left[ \frac{1+\sqrt{1+\varepsilon^2}}{\varepsilon} \right]^{1/N}$$

$$\varepsilon = \sqrt{\frac{1}{A_p^2} - 1}$$

$N = 2$

Poles : $r_1$, $r_2$, $\phi_k$

$$\varepsilon = \sqrt{\frac{1}{A_p^2} - 1} = \sqrt{\frac{1}{0.707^2} - 1} = 1$$

$$\beta = \left[ \frac{1+\sqrt{1+1^2}}{1} \right]^{1/2} = 1.553$$

$$r_1 = 0.649 \left( \frac{1.553^2+1}{2 \times 1.553} \right) = 0.7139$$

$$r_2 = 0.649 \left( \frac{1.553^2-1}{2 \times 1.553} \right) = 0.295$$

$$\phi_k = \frac{\pi}{2} + (2k+1)\frac{\pi}{2N} \quad , \; k = 0, 1$$

$$\phi_0 = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$$

$$\phi_1 = \frac{\pi}{2} + \frac{3\pi}{4} = \frac{5\pi}{4} = -\frac{3\pi}{4}$$

$$\phi_k = \pm \frac{3\pi}{4}$$

$$x_k + j y_k = r_2 \cos\phi_k + j\, r_1 \sin\phi_k$$

$$= 0.2949 \cos\left(\frac{3\pi}{4}\right) + j\, 0.7139 \sin\left(\frac{3\pi}{4}\right)$$

$$= -0.209 \pm j\, 0.5048$$

To find the s/m fn:

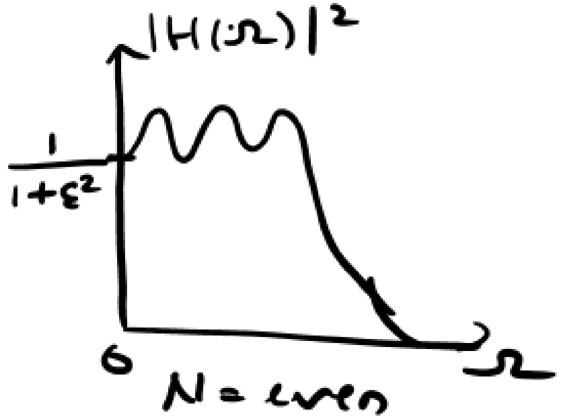
$$H(s) = \frac{\text{Constant} = C}{[s-(-0.209+j0.5048)][s-(-0.209-j0.5048)]}$$

$$= \frac{C}{(\underbrace{s+0.209}_{a} - j\underbrace{0.5048}_{b})(\underbrace{s+0.209}_{a} + j\underbrace{0.5048}_{b})}$$

$(a+b)(a-b)$
$= a^2 - b^2$

$$= \frac{C}{(s+0.209)^2 - (j0.5048)^2}$$

$$H(s) = \frac{C}{s^2 + 0.418s + 0.299}$$




At $\Omega = 0$, $|H(0)| = A_p = \frac{1}{\sqrt{1+\varepsilon^2}}$ if $N$ = even

$$H(s)\Big|_{s=0} = 0.707$$

$$\frac{C}{0.299} = 0.707 \Rightarrow C = 0.211$$

$$H(s) = \frac{0.211}{s^2 + 0.418s + 0.299}$$

$$H(z) = \frac{0.211}{4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 0.418 \times 2\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.299}$$

$$H(z) = \frac{0.211\,(1+z^{-1})^2}{4(1-z^{-1})^2 + 0.836(1-z^{-1})(1+z^{-1}) + 0.299\,(1+z^{-1})^2}$$

$$= \frac{0.211 + 0.422\,z^{-1} + 0.211\,z^{-2}}{5.135 - 7.402\,z^{-1} + 3.463\,z^{-2}}$$

$$= \frac{5.135\,(0.04 + 0.08\,z^{-1} + 0.04\,z^{-2})}{5.135\,(1 - 1.44\,z^{-1} + 0.67\,z^{-2})}$$

$$= \frac{0.04 + 0.08\,z^{-1} + 0.04\,z^{-2}}{1 - 1.44\,z^{-1} + 0.67\,z^{-2}}$$

HW

- Solve the above problem using impulse invariance method


Thank you

# IIR filter
# Spectral Transformation
# +
# Direct design of IIR filters

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# To verify the designed filter

Q. Design a digital Chebyshev filter that satisfies the constraints

$$\underbrace{0.707}_{=A_p} \le |H(e^{j\omega})| \le 1 \;, \quad 0 \le \omega \le \underbrace{0.2\pi}_{=\omega_p}$$

$$|H(e^{j\omega})| \le \underbrace{0.1}_{=A_s} \;, \quad \underbrace{0.5\pi}_{=\omega_s} \le \omega \le \pi$$

Use bilinear transf and assume $T = 1$ sec.

$$H(z) = \frac{0.04 + 0.08 z^{-1} + 0.04 z^{-2}}{1 - 1.44 z^{-1} + 0.67 z^{-2}} \quad //$$

- Put $Z = e^{j\omega}$ and verify

LPF

Design of IIR filters

↓

Design in analog domain → Butterworth, Chebyshev, Elliptic

↓

Map from analog to digital → Imp invar., Bilinear transf, Matched z-transform

↓

Desired digital LPF $\xrightarrow[\text{transf.}]{\text{spectral}}$ HPF ✓, BPF ✓, BEF ✓

## Spectral Transformation : / Frequency HPF, BPF, BSF

1) In analog domain ✓

Design analog LPF → Apply freq band transformation → Digitize filter

2) In digital domain ✓

Design analog LPF → Digitize filter → Apply freq band transformation

Analog domain – Frequency transf:

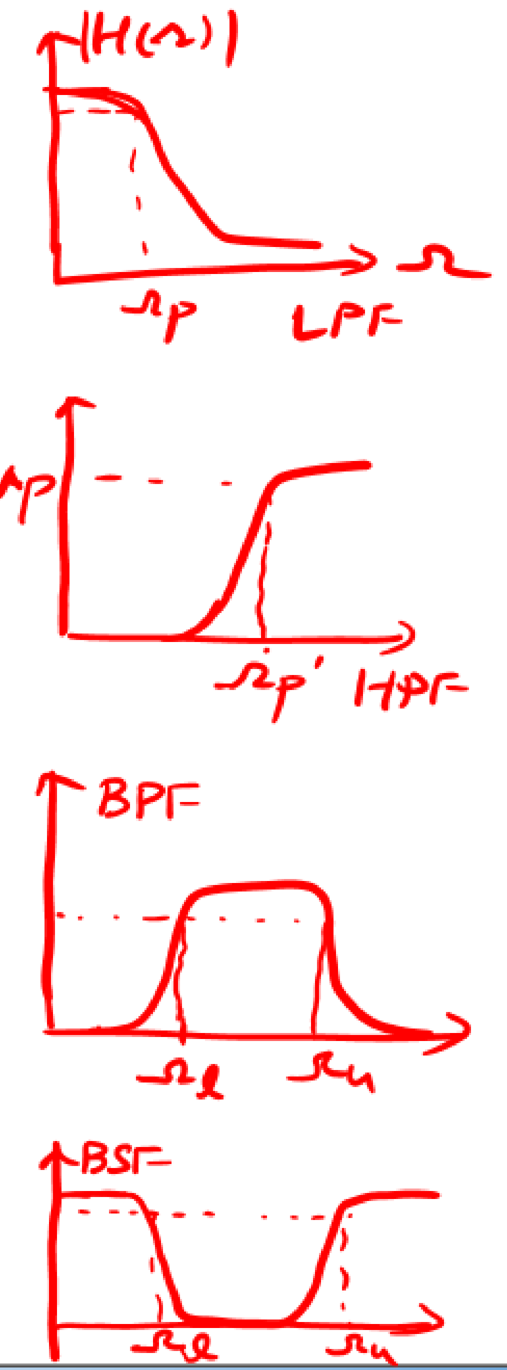
Prototype LPF $\rightarrow \Omega_p$ $\longrightarrow$ LPF' $\rightarrow \Omega_p'$

$H(s)$

$H'(s) \Big|_{s \rightarrow \frac{\Omega_p}{\Omega_p'} \cdot s}$

$\Omega_l$ – lower band

$\Omega_u$ – upperband



| Desired filters | Transformation | Bandedge freq |
| --- | --- | --- |
| LPF | $s \rightarrow \frac{\Omega_p}{\Omega_p'} \cdot s$ | $\Omega_p'$ |
| HPF | $s \rightarrow \frac{\Omega_p \cdot \Omega_p'}{s}$ | $\Omega_p'$ |
| BPF | $s \rightarrow \Omega_p \frac{s^2+\Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}$ | |
| BSF | $s \rightarrow \Omega_p \cdot \frac{s(\Omega_u - \Omega_l)}{s^2+\Omega_u \Omega_l}$ | |

$\Omega_l$ – lower band

$\Omega_u$ – upperband

Q. A prototype LPF has s/m fn, $H(s) = \frac{1}{s^2+3s+2}$ with $\Omega_p = 1$ rad/s, Obtain a BPF with centre freq $\Omega_0 = 3$ rad/s, & quality factor = 12.

Q ↑ PBW ↓

Soln: $\Omega_0 = \sqrt{\Omega_u \cdot \Omega_l}$

$$Q = \frac{\Omega_0}{\Omega_u - \Omega_l}$$

$$s \rightarrow \Omega_p \cdot \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}$$

$$= \Omega_p \cdot \frac{s^2 + \Omega_0^2}{s(\Omega_0/Q)}$$

$$= 1 \cdot \frac{s^2 + 3^2}{s(\cancel{3}/\cancel{12}_{4})} = \frac{4(s^2+9)}{s}$$

$$H(s) = \frac{1}{s^2+3s+2}$$

$$H'(s)\Big|_{s \to \frac{4}{s}(s^2+9)}$$

$$H'(s) = \frac{1}{\left[\frac{4}{s}(s^2+9)\right]^2 + 3\times\frac{4}{s}(s^2+9) + 2}$$

$$= \frac{s^2}{16(s^2+9)^2 + 12s(s^2+9) + 2s^2}$$

$$H'(s) = \frac{1}{16} \times \frac{s^2}{s^4+0.75s^3+18.125s^2+6.75s+81}$$

2) Freq transf in digital domain :-

1) Mapping $z^{-1} \longrightarrow g(z^{-1})$ must map pts inside unit circle in z-plane to itself

2) Unit circle must map to itself.

| Type of transformation | Transformation | Parameters |
| --- | --- | --- |
| Lowpass | $z^{-1} \longrightarrow \dfrac{z^{-1} - a}{1 - az^{-1}}$ | $\omega'_p$ = band edge frequency of new filter<br>$a = \dfrac{\sin[(\omega_p - \omega'_p)/2]}{\sin[(\omega_p + \omega'_p)/2]}$ |
| Highpass | $z^{-1} \longrightarrow -\dfrac{z^{-1} + a}{1 + az^{-1}}$ | $\omega'_p$ = band edge frequency new filter<br>$a = -\dfrac{\cos[(\omega_p + \omega'_p)/2]}{\cos[(\omega_p - \omega'_p)/2]}$ |

Bandpass

$$z^{-1} \longrightarrow -\frac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}$$

$$\begin{aligned}
\omega_l &= \text{lower band edge frequency} \\
\omega_u &= \text{upper band edge frequency} \\
a_1 &= -2\alpha K/(K+1) \\
a_2 &= (K-1)/(K+1) \\
\alpha &= \frac{\cos[(\omega_u + \omega_l)/2]}{\cos[(\omega_u - \omega_l)/2]} \\
K &= \cot\frac{\omega_u - \omega_l}{2} \tan\frac{\omega_p}{2}
\end{aligned}$$

Bandstop

$$z^{-1} \longrightarrow \frac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}$$

$$\begin{aligned}
\omega_l &= \text{lower band edge frequency} \\
\omega_u &= \text{upper band edge frequency} \\
a_1 &= -2\alpha/(K+1) \\
a_2 &= (1-K)/(1+K) \\
\alpha &= \frac{\cos[(\omega_u + \omega_l)/2]}{\cos[(\omega_u - \omega_l)/2]} \\
K &= \tan\frac{\omega_u - \omega_l}{2} \tan\frac{\omega_p}{2}
\end{aligned}$$

Qn. Convert the singlepole lowpass Butterworth filter with system function, $H(z) = \dfrac{0.245(1+z^{-1})}{1-0.509z^{-1}}$

into a bandpass filter with upper and lower cut off frequencies, $\omega_u = \frac{3\pi}{5}$ and $\omega_l = \frac{2\pi}{5}$.

The LPF has 3dB bandwidth, $\omega_p = 0.2\pi$

Soln:

$$K = \cot\left(\frac{\omega_u - \omega_l}{2}\right) \cdot \tan\left(\frac{\omega_p}{2}\right)$$

$$= \cot\left(\frac{\frac{3\pi}{5} - \frac{2\pi}{5}}{2}\right) \cdot \tan\left(\frac{0.2\pi}{2}\right)$$

$$= \cot\left(\frac{\pi}{10}\right) \cdot \tan\left(\frac{\pi}{10}\right) = \underline{\underline{1}}$$

$$\alpha = \frac{\cos\left(\frac{\omega_u + \omega_l}{2}\right)}{\cos\left(\frac{\omega_u - \omega_l}{2}\right)} = 0$$

Qn. Convert the single pole lowpass Butterworth filter with system function, $H(z) = \frac{0.245\,(1+z^{-1})}{1-0.509\,z^{-1}}$ into a bandpass filter with upper and lower cut off frequencies, $\omega_u = \frac{3\pi}{5}$ and $\omega_l = \frac{2\pi}{5}$.

The LPF has 3dB bandwidth, $\omega_p = 0.2\pi$

$$a_1 = \frac{2\alpha K}{K+1} = 0$$

$$a_2 = \frac{K-1}{K+1} = 0$$

$$z^{-1} \rightarrow -\frac{(z^{-2} - a_1 z^{-1} + a_2)}{a_1 z^{-2} - a_1 z^{-1} + 1} = -z^{-2}$$

$$H(z) = \frac{0.245\,(1-z^{-2})}{1+0.509\,z^{-2}}$$ //

# Direct design of IIR filters:

1) Direct design of LPF :

Direct placement of poles near the unit circle in z-plane at the points corresponding to low freq (i.e. near $w=0$)

eg. Pole at $z=a$,

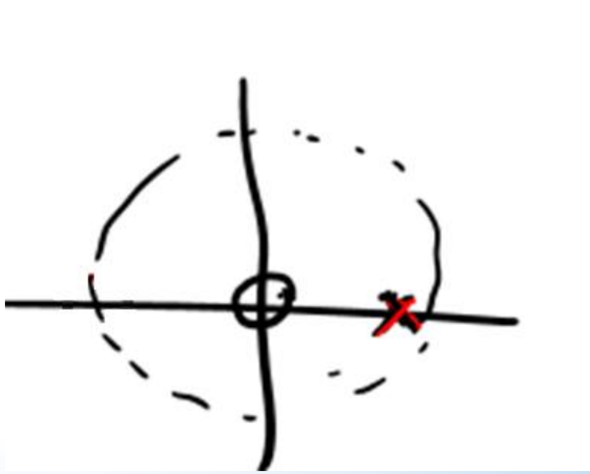
s/m fn, $H(z) = b_0 \cdot \frac{1}{1-az^{-1}}$

At $w=0$, $|H(0)| = 1$

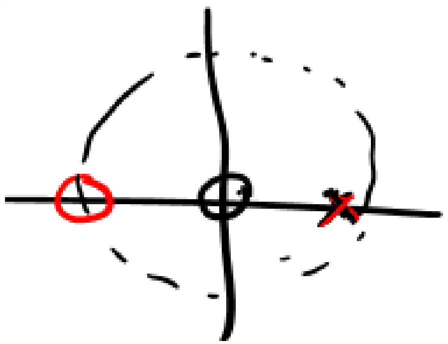
$$\left|\frac{b_0}{1-a}\right| = 1$$

$$b_0 = 1-a$$

$$H_1(z) = \frac{1-a}{1-az^{-1}}$$



② Placing a zero at
$z = -1$ (i.e. $\omega = \pi$)



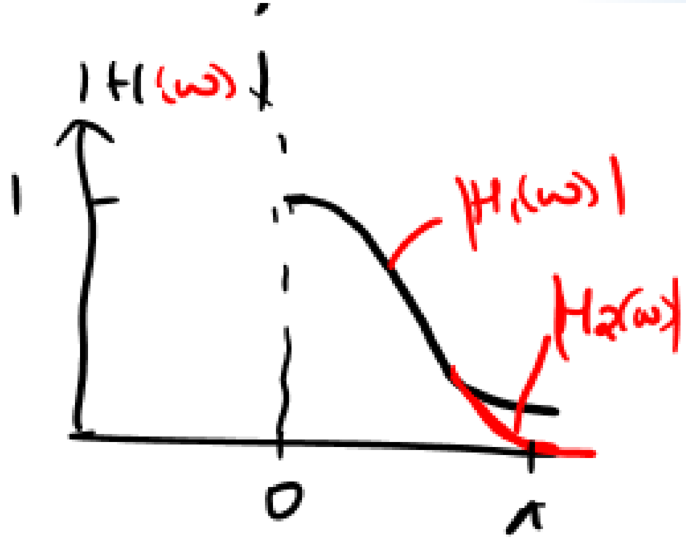
$$H_2(z) = b_0 \cdot \frac{1+z^{-1}}{1-az^{-1}}$$ , single pole-zero

At $\omega = 0$, $|H(0)| = 1$

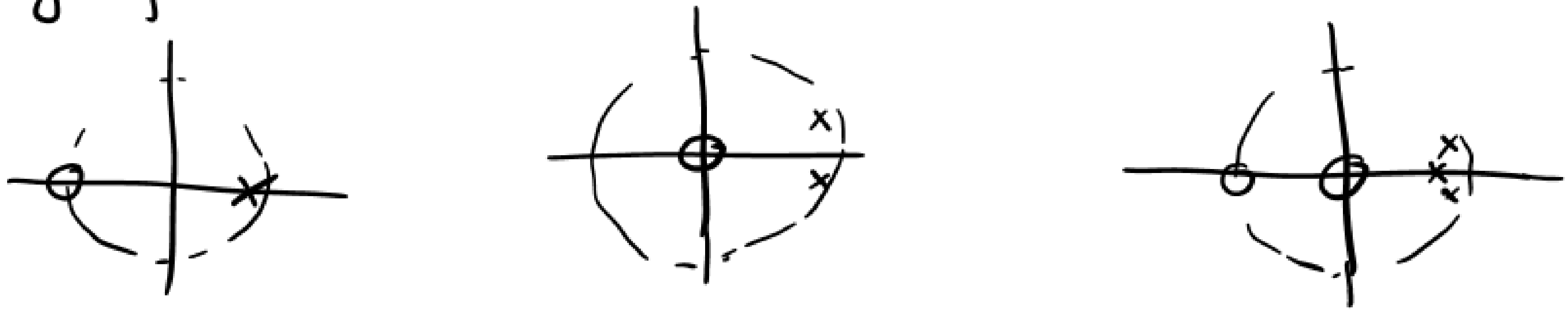
$$\left| b_0 \cdot \frac{1+1}{1-a} \right| = 1$$

$$\Rightarrow b_0 = \frac{1-a}{2}$$

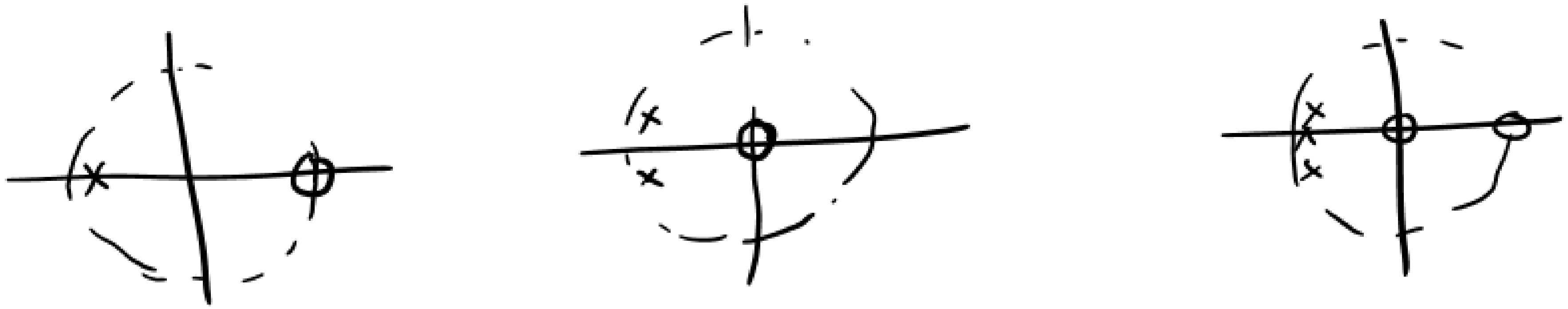
$$\therefore H_2(z) = \frac{1-a}{2} \cdot \frac{1+z^{-1}}{1-az^{-1}}$$



eg. of LPF



eg. of HPF

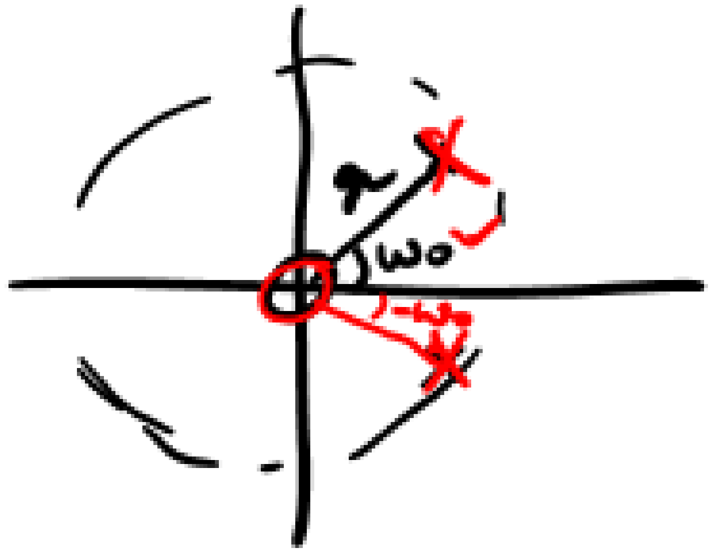


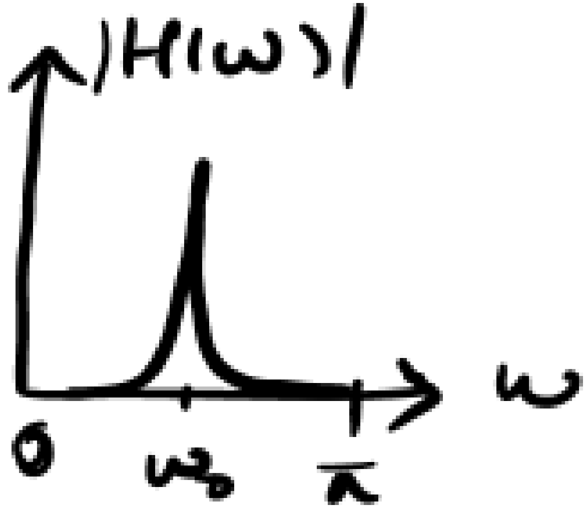
## Digital resonator

2-pole BPF

$\omega_0$ - resonant freq

$p_{1,2} = r e^{\pm j\omega_0}$  $0 < r < 1$

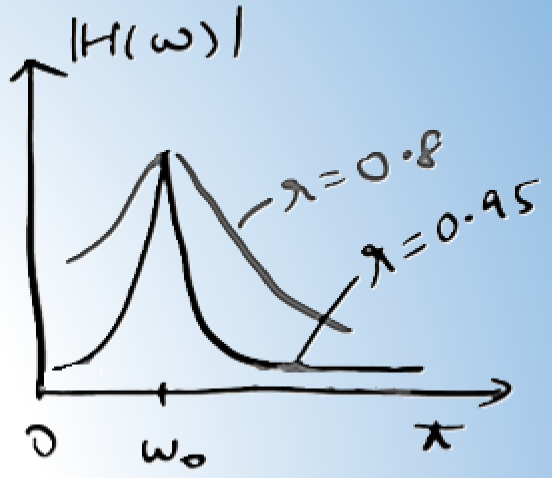


case 1)

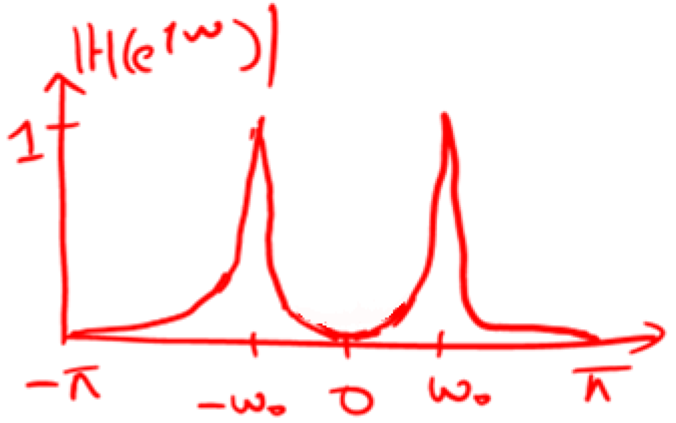
1) zeros at origin

$$H(z) = \frac{b_0}{(1-re^{j\omega_0}z^{-1})(1-re^{-j\omega_0}z^{-1})}$$




2) zeros at $z = \pm 1$ ($\omega = 0$ & $\pi$)

$$H(z) = b_0 \cdot \frac{(1-z^{-1})(1+z^{-1})}{(1-re^{j\omega_0}z^{-1})(1-re^{-j\omega_0}z^{-1})}$$



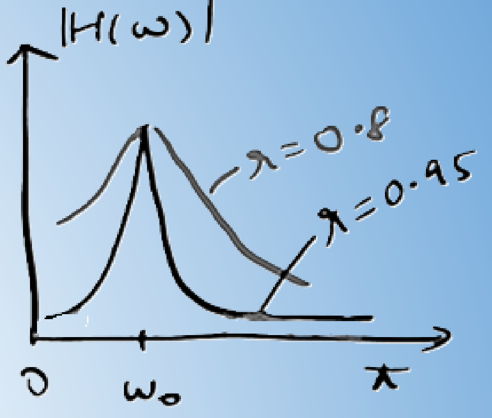
Case 1 : S/m fn of dig resonator with zeros at origin.

$$H(z) = \frac{b_0}{(1 - re^{j\omega_0}z^{-1})(1 - re^{-j\omega_0}z^{-1})} \qquad \text{--------- (1)}$$

$$= \frac{b_0}{1 - re^{j\omega_0}z^{-1} - re^{-j\omega_0}z^{-1} + r^2 e^{j\omega_0 - j\omega_0} z^{-2}}$$

$$= \frac{b_0}{1 - r(e^{j\omega_0} + e^{-j\omega_0})z^{-1} + r^2 z^{-2}}$$

$$= \frac{b_0}{1 - 2r\cos\omega_0 z^{-1} + r^2 z^{-2}}$$

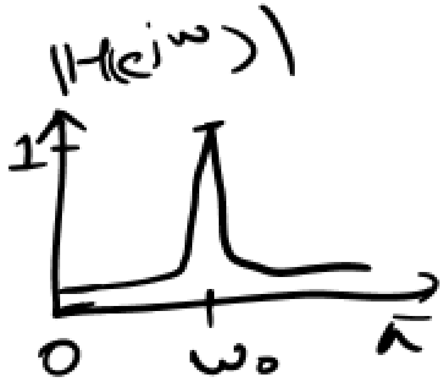



$$\frac{e^{j\theta} + e^{-j\theta}}{2} = \cos\theta$$

$$|H(e^{j\omega})|_{\omega=\omega_0} = 1$$

$$|H(e^{j\omega_0})| = \left| \frac{b_0}{(1-re^{j\omega_0}e^{-j\omega_0})(1-re^{-j\omega_0}e^{-j\omega_0})} \right| = 1$$

$$\Rightarrow \left| \frac{b_0}{(1-r)(1-re^{-2j\omega_0})} \right| = 1$$




$$z = e^{j\omega_0}$$

$$|H(e^{j\omega_0})| = \left| \frac{b_0}{(1-r)(1-r\cos 2\omega_0 + jr\sin 2\omega_0)} \right| = 1$$

$$|a+ib| = \sqrt{a^2+b^2}$$

$$\Rightarrow \frac{b_0}{(1-r)\sqrt{(1-r\cos 2\omega_0)^2 + (r\sin 2\omega_0)^2}} = 1$$

$$\Rightarrow \frac{b_0}{(1-r)\sqrt{1-2r\cos 2\omega_0 + r^2\cos^2 2\omega_0 + r^2\sin^2 2\omega_0}} = 1 \qquad [\cos^2\theta + \sin^2\theta = 1]$$

$$\Rightarrow \underbrace{b_0}_{\text{normalized filter gain}} = (1-r)\sqrt{1-2r\cos 2\omega_0 + r^2}$$

Magnitude response : evaluate $H(z)$ at $z = e^{j\omega}$

$$|H(e^{j\omega})| = \left| \frac{b_0}{(1 - r e^{j\omega_0} e^{-j\omega})(1 - r e^{-j\omega_0} e^{-j\omega})} \right| \quad \text{---} \quad (2)$$

$$\text{dnr} = (1 - r e^{j(\omega_0 - \omega)})(1 - r e^{-j(\omega_0 + \omega)})$$

$e^{j\theta} = \cos\theta + j\sin\theta$

$e^{-j\theta} = \cos\theta - j\sin\theta$

$$= [1 - r\cos(\omega_0 - \omega) - jr\sin(\omega_0 - \omega)][1 - r\cos(\omega_0 + \omega) + jr\sin(\omega_0 + \omega)]$$

$$|\text{dnr}| = \sqrt{(1 - r\cos(\omega_0 - \omega))^2 + (r\sin(\omega_0 - \omega))^2} \cdot \sqrt{(1 - r\cos(\omega_0 + \omega))^2 + (r\sin(\omega_0 + \omega))^2}$$

$$= \sqrt{1 - 2r\cos(\omega_0 - \omega) + r^2\cos^2(\omega_0 - \omega) + r^2\sin^2(\omega_0 - \omega)} \cdot \sqrt{1 - 2r\cos(\omega_0 + \omega) + r^2\cos^2(\omega_0 + \omega) + r^2\sin^2(\omega_0 + \omega)}$$

$$= \sqrt{1 - 2r\cos(\omega_0 - \omega) + r^2} \cdot \sqrt{1 - 2r\cos(\omega_0 + \omega) + r^2}$$

$$(2) \Rightarrow |H(e^{j\omega})| = \frac{b_0}{\underbrace{\sqrt{1 - 2r\cos(\omega_0 - \omega) + r^2}}_{u_1(\omega)} \underbrace{\sqrt{1 - 2r\cos(\omega_0 + \omega) + r^2}}_{u_2(\omega)}} = \frac{b_0}{u_1(\omega) \cdot u_2(\omega)}$$

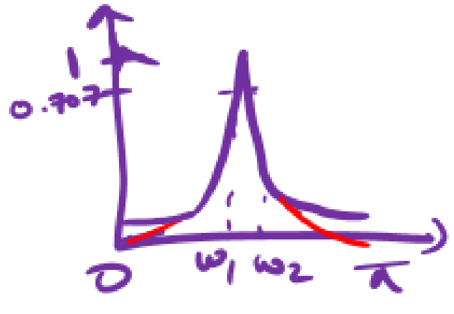
The product term $U_1(\omega)\cdot U_2(\omega)$ reaches minimum value at

$$\omega_r = \cos^{-1}\left(\frac{1+r^2}{2r}\cos\omega_0\right)$$, is the resonance freq.

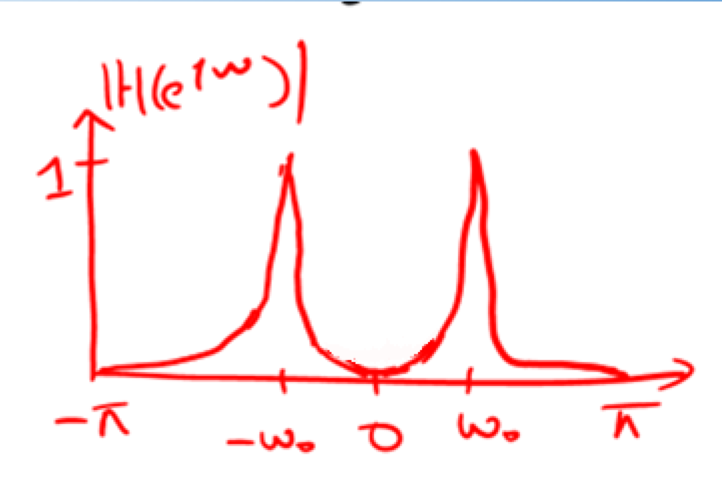
$r\to 1$, sharper peak, BW $\downarrow$

$\omega_r \to \omega_0$

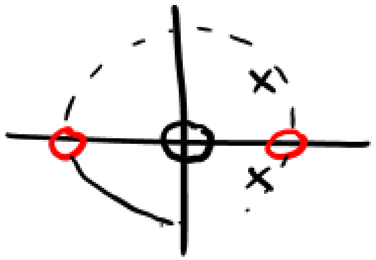
$r \to 1$, 3dB BW $= \Delta\omega \approx 2(1-r)$




$\Delta\omega = \omega_2 - \omega_1$




Case 2: When zeros are placed at $z = \pm 1$:




$$H(z) = \frac{b_0 \cdot (1-z^{-1})(1+z^{-1})}{(1-re^{j\omega_0}z^{-1})(1-re^{-j\omega_0}z^{-1})}$$

$$= \frac{b_0\,(1-z^{-1})\,(1+z^{-1})}{1-2r\cos\omega_0\, z^{-1}+r^2 z^{-2}}$$

Magnitude response of case 2:

$H(z)\big|_{z=e^{j\omega}}$

$$|H(e^{j\omega})| = \left| \frac{b_0 (1-e^{-j\omega})(1+e^{-j\omega})}{(1-r e^{j\omega_0} e^{-j\omega})(1-r e^{-j\omega_0} e^{-j\omega})} \right|$$

$$= \left| \frac{b_0 (1-e^{-j2\omega})}{(1-r e^{j\omega_0} e^{-j\omega})(1-r e^{-j\omega_0} e^{-j\omega})} \right|$$

$$= \frac{b_0 \underbrace{|1-e^{-j2\omega}|}_{N(\omega)}}{\sqrt{\underbrace{1-2r\cos(\omega_0-\omega)+r^2}_{U_1(\omega)}}\sqrt{\underbrace{1-2r\cos(\omega_0+\omega)+r^2}_{U_2(\omega)}}}$$

$$= \frac{b_0 \cdot N(\omega)}{U_1(\omega) \cdot U_2(\omega)}$$

$$N(\omega) = \left| 1 - e^{-j2\omega} \right|$$

$$= \left| 1 - \cos 2\omega + j \sin 2\omega \right|$$

$$= \sqrt{(1 - \cos 2\omega)^2 + \sin^2 2\omega} = \sqrt{1 - 2\cos 2\omega + \underbrace{\cos^2 2\omega + \sin^2 2\omega}_{1}}$$

$$= \sqrt{2(1 - \cos 2\omega)} \quad //$$

Qn. Design a digital resonator with peak gain of unity at 50Hz and a 3dB BW of 6Hz. Assume a sampling freq of 300Hz.

Resonant freq, $\omega_0 = 2\pi \frac{f}{F_s} = 2\pi \times \frac{50}{300} = \pi/3$

3dB BW, $\Delta\omega = 2\pi \times \frac{6}{300} = \pi/25$

$\Delta\omega \simeq 2(1-r) \Rightarrow \frac{\pi}{25} = 2(1-r) \Rightarrow r = 0.937$

Assume zeros at origin:

$$H(z) = \frac{b_0}{1 - 2r\cos\omega_0 \, z^{-1} + r^2 z^{-2}}$$

$$= \frac{b_0}{1 - 2\times 0.937\cos(\pi/3) z^{-1} + 0.937^2 z^{-2}} = \frac{b_0}{1 - 0.937 z^{-1} + 0.877 z^{-2}}$$

To find $b_0$:

$$|H(e^{j\omega})|_{\omega=\omega_0} = 1 \quad , \omega_0 = \pi/3$$

$$\Rightarrow \left| \frac{b_0}{1-0.937e^{-j\omega}+0.877e^{-2j\omega}} \right|_{\omega=\pi/3} = 1$$

$$\Rightarrow \left| \frac{b_0}{1-0.937\left(\cos\frac{\pi}{3} - j\sin\frac{\pi}{3}\right) + 0.877\left(\cos\frac{2\pi}{3} - j\sin\frac{2\pi}{3}\right)} \right| = 1$$

$$\Rightarrow \left| \frac{b_0}{0.0925+j0.0522} \right| = 1$$

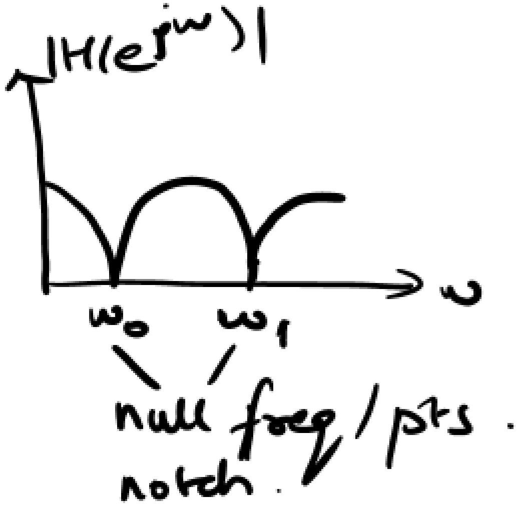
$$\Rightarrow b_0 = \sqrt{0.0925^2+0.0522^2} = 0.105 \qquad \therefore H(z) = \frac{0.105}{1-0.937z^{-1}+0.877z^{-2}}$$

## Notch filter:

$$z_{1,2} = e^{\pm j\omega_0} \quad , r = 1$$

$$H(z) = b_0 \left(1 - e^{j\omega_0} z^{-1}\right)\left(1 - e^{-j\omega_0} z^{-1}\right)$$

$$= b_0 \left(1 - 2\cos\omega_0 \, z^{-1} + z^{-2}\right)$$

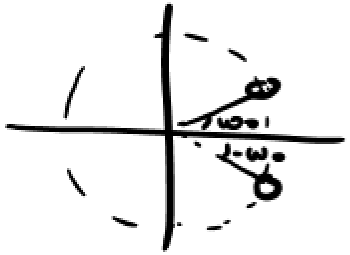


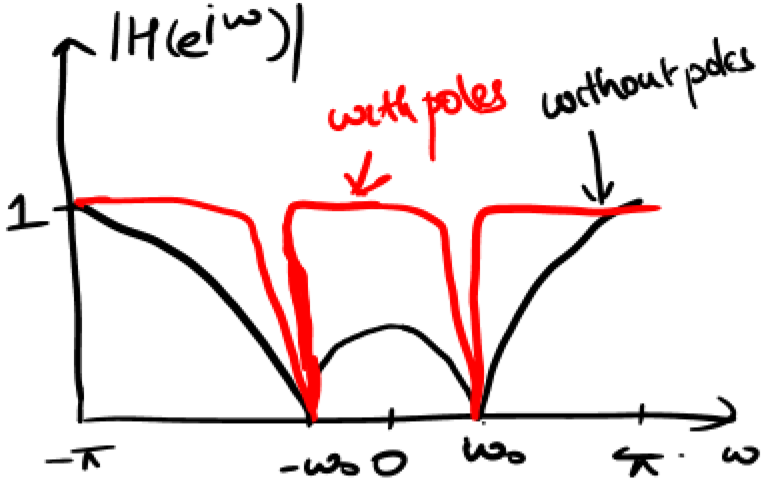

By placing poles near unit-circle at centre freq $\omega_0$

$$p_{1,2} = r e^{\pm j\omega_0}$$

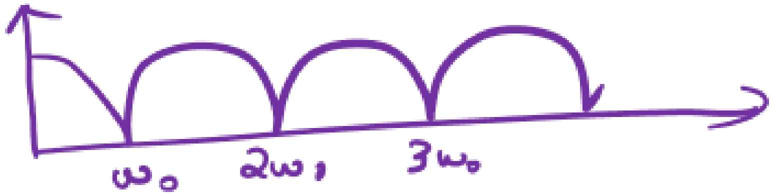
$$H(z) = \frac{b_0 \left(1 - e^{j\omega_0} z^{-1}\right)\left(1 - e^{-j\omega_0} z^{-1}\right)}{\left(1 - r e^{j\omega_0} z^{-1}\right)\left(1 - r e^{-j\omega_0} z^{-1}\right)}$$

$$= \frac{b_0 \left(1 - 2\cos\omega_0 \, z^{-1} + z^{-2}\right)}{1 - 2r\cos\omega_0 \, z^{-1} + r^2 z^{-2}}$$




## Comb filter.



Q A digital notch filter is reqd to remove an undesired 60Hz hum associated with a power supply in a ECG recording. The sampling freq used is 500 samples per second.

(a) Design a 2nd order FIR notch filter.

(b) Design a 2nd order pole-zero notch filter.

In both cases, choose gain $b_0$ s.t. $|H(e^{j\omega})| = 1$ for $\omega = 0$.

Soln: Notch freq, $\omega_0 = 2\pi \frac{f}{F_s} = \frac{2\pi \times 60}{500} = \frac{6\pi}{25} = 0.754$

(a) Pair of complex conjugate zeros at $e^{\pm j\omega_0}$ where $\omega_0 = 0.754$

$$H(z) = b_0 \left(1 - 2\cos\omega_0 z^{-1} + z^{-2}\right)$$

$$= b_0 \left(1 - 1.4579 z^{-1} + z^{-2}\right)$$

To find $b_0$: $|H(e^{j\omega})|_{\omega=0} = 1$

$\Rightarrow |b_0(1 - 1.4579 e^0 + e^0)| = 1 \Rightarrow b_0 = 1.845$

$\therefore H(z) = 1.845\left(1 - 1.4579 z^{-1} + z^{-2}\right)$

Case (b): $H(z) = \dfrac{b_0(1-2\cos\omega_0 z^{-1}+z^{-2})}{(1-2r\cos\omega_0 z^{-1}+r^2 z^{-2})}$

$0<r<1$, $r\to 1$, Assume $r=0.95$,

$$H(z) = \frac{b_0(1-2\cos(0.754)z^{-1}+z^{-2})}{1-2\times 0.95\cos(0.754)z^{-1}+0.95^2 z^{-2}}$$

To find $b_0$: $|H(e^{j\omega})|_{\omega=0} = 1$

$b_0 = 0.9546$

$$H(z) = \frac{0.9546(1-1.4579z^{-1}+z^{-2})}{1-1.385z^{-1}+0.9025z^{-2}}$$

# Reference


- Proakis J. G, Manolakis D. G. Mimitris D., “Introduction to Digital Signal Processing” Prentice Hall, India, 2007.


Thank you

# FIR filter design
# Linear Phase FIR Filter Characteristics

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# Linear Phase FIR Filters


$h(n)$ - length $M$, $0 \dots M-1$

$$H(z) = \sum_{n=0}^{M-1} h(n) z^{-n}$$

$$= h(0) + h(1) z^{-1} + \dots h(M-1) z^{-(M-1)}$$

$$H(\omega) = |H(\omega)| e^{j\phi(\omega)}$$

Group delay, $\tau_g = -\frac{d(\phi(\omega))}{d\omega}$

$$h(n) = \pm h(M-1-n)$$

$= +h(M-1-n)$ - symmetric ✓

$= -h(M-1-n)$ - antisymm. ✓

Linear phase property delays the input signal but preserves the signal shape with no distortion. Watch demo @ https://youtu.be/zCdV9IUCSy8

$$H(\omega) = H_r(\omega) \cdot e^{-j\omega\alpha}, \quad \alpha = \frac{M-1}{2} \quad - \text{symm}$$

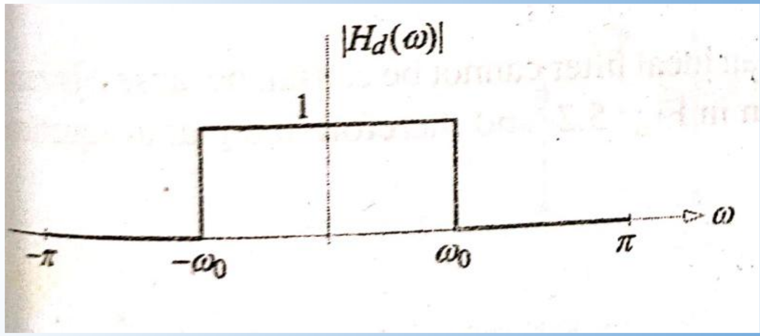
$$H(\omega) = H_r(\omega) \cdot e^{-j(\omega\alpha - \pi/2)}$$

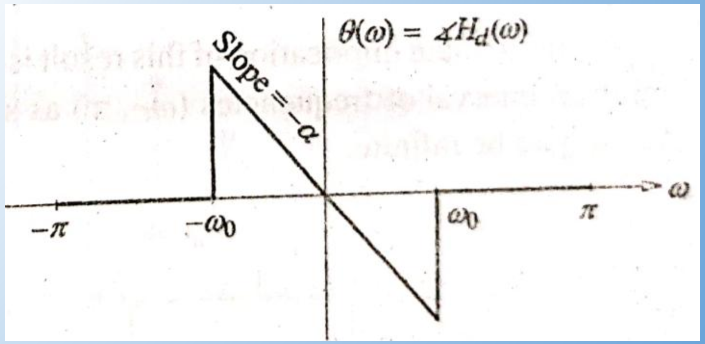
$\Rightarrow$ antisymm.

$$\tau_g = -\frac{d\phi(\omega)}{d\omega}$$

$$= \alpha = \frac{M-1}{2}$$

$$H_d(\omega) = \begin{cases} e^{-j\omega\alpha}, & |\omega| < \omega_0 \\ 0, & \omega_0 < |\omega| < \pi \end{cases}$$




Linear phase filters have same group delay

4 cases of linear phase FIR:

1) M = even → symm.
→ antisymm.

2) M = odd → symm
→ antisymm.

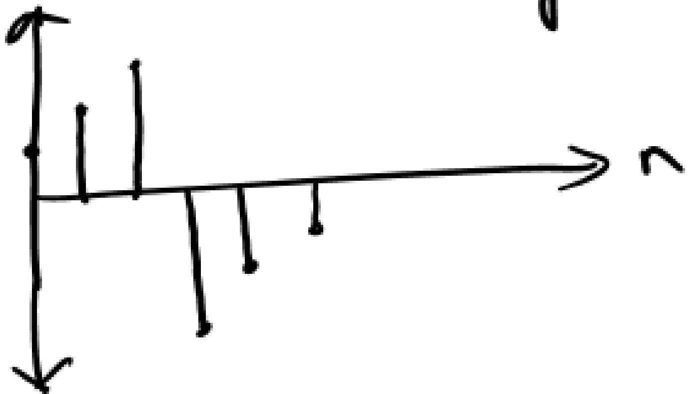
$$h(n) = +h(M-1-n)$$

$$h(n) = -h(M-1-n)$$

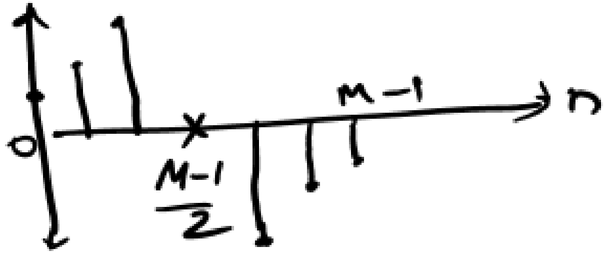
M = even, h(n) = symm


M = odd, h(n) = symm


M = even, antisymm




M = odd, h(n) - antisymm



Case 1 : M = even

$$h(n) = \pm h(M-1-n)$$

$$H(z) = \sum_{n=0}^{M-1} h(n) z^{-n}$$

$$= \sum_{n=0}^{\frac{M}{2}-1} h(n) z^{-n} + \sum_{n=\frac{M}{2}}^{M-1} h(n) z^{-n}$$

$$= \sum_{n=0}^{\frac{M}{2}-1} h(n) z^{-n} \pm \sum_{n=\frac{M}{2}}^{M-1} h(\underline{M-1-n}) z^{-n}$$

substitute $\underline{\underline{k}} = M-1-n$, $n = \frac{M}{2} \Rightarrow k = M-1-\frac{M}{2} = \frac{M}{2}-1$

$n = M-1 \Rightarrow k = M-1-(M-1) = 0$

$$\therefore H(z) = \sum_{n=0}^{\frac{M}{2}-1} h(n) z^{-n} \pm \sum_{k=0}^{\frac{M}{2}-1} h(k) z^{-(M-1-k)}$$

$$k \to n \Rightarrow H(z) = \sum_{n=0}^{\frac{M}{2}-1} h(n) z^{-n} \pm \sum_{n=0}^{\frac{M}{2}-1} h(n) z^{-(M-1-n)}$$

$$H(z) = \sum_{n=0}^{\frac{M}{2}-1} h(n)\left[z^{-n} \pm z^{-(M-1-n)}\right]$$

$$= z^{-\left(\frac{M-1}{2}\right)} \sum_{n=0}^{\frac{M}{2}-1} h(n)\left[z^{\left(\frac{M-1}{2}-n\right)} \pm z^{-\left(M-1-n-\frac{M-1}{2}\right)}\right]$$

$$= z^{-\left(\frac{M-1}{2}\right)} \sum_{n=0}^{\frac{M}{2}-1} h(n)\left[z^{\left(\frac{M-1}{2}-n\right)} \pm z^{-\left(\frac{M-1}{2}-n\right)}\right] \longrightarrow (1)$$

$$H(e^{j\omega}) \to H(z)\big|_{z=e^{j\omega}}$$

(a) M = even, symm. imp response:

$$(1) \Rightarrow H(e^{j\omega}) = e^{-j\omega\left(\frac{M-1}{2}\right)} \sum_{n=0}^{\frac{M}{2}-1} h(n)\Big[\underbrace{e^{j\omega\left(\frac{M-1}{2}-n\right)} + e^{-j\omega\left(\frac{M-1}{2}-n\right)}}_{e^{j\theta}+e^{-j\theta} = 2\cos\theta}\Big]$$

$$= e^{-j\omega\underbrace{\left(\frac{M-1}{2}\right)}_{\text{phase, } \phi(\omega)}} \underbrace{\sum_{n=0}^{\frac{M}{2}-1} h(n) \cdot 2\cos\omega\left(\frac{M-1}{2}-n\right)}_{H_R(\omega)}$$

$$H(e^{j\omega}) = H_r(\omega) \cdot e^{-j\omega\left(\frac{M-1}{2}\right)}$$

Magn resp, $H_r(\omega) = \sum_{n=0}^{M/2-1} h(n) \left[2\cos\omega\left(\frac{M-1}{2}-n\right)\right]$

Phase resp, $\phi(\omega) = -\omega\left(\frac{M-1}{2}\right)$ if $H_r(\omega) > 0$

$= -\omega\left(\frac{M-1}{2}\right) + \pi$ if $H_r(\omega) < 0$.

(b) $M$ = even, ~~antisymm~~.

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{M-1}{2}\right)} \sum_{n=0}^{M/2-1} h(n) \left[\underbrace{e^{j\omega\left(\frac{M-1}{2}-n\right)} - e^{-j\omega\left(\frac{M-1}{2}-n\right)}}\right]$$

$$= e^{-j\omega\left(\frac{M-1}{2}\right)} \sum_{n=0}^{\frac{M}{2}-1} h(n) \left[2\cancel{j} \sin\omega\left(\frac{M-1}{2}-n\right)\right]$$

$j = e^{j\pi/2}$

$$= e^{-j\omega\left(\frac{M-1}{2}\right)} \sum_{n=0}^{\frac{M}{2}-1} h(n) \cdot 2 \cdot e^{j\frac{\pi}{2}} \cdot \sin\omega\left(\frac{M-1}{2}-n\right)$$

$$= \underbrace{e^{-j\omega\left(\frac{M-1}{2}\right)+j\frac{\pi}{2}}}_{\phi(\omega)} \cdot \underbrace{\sum_{n=0}^{\frac{M}{2}-1} h(n) \cdot 2 \cdot \sin\omega\left(\frac{M-1}{2}-n\right)}_{H_r(\omega)}$$

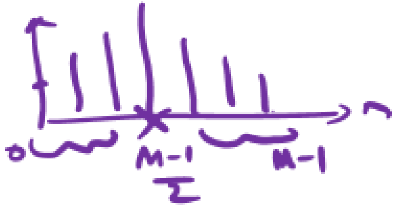
$$= \underbrace{e^{-j\omega\left(\frac{M-1}{2}\right)+j\frac{\pi}{2}}}_{\phi(\omega)} \cdot \underbrace{\sum_{n=0}^{\frac{M}{2}-1} h(n)\cdot 2\cdot \sin\omega\left(\frac{M-1}{2}-n\right)}_{H_R(\omega)}$$

$$H_R(\omega) = \sum_{n=0}^{M/2-1} h(n)\cdot 2\sin\omega\left(\frac{M-1}{2}-n\right)$$

$$\phi(\omega) = \frac{\pi}{2} - \omega\left(\frac{M-1}{2}\right) \quad \text{if } H_R(\omega) > 0$$

$$= \frac{\pi}{2} - \omega\left(\frac{M-1}{2}\right) + \pi = \frac{3\pi}{2} - \omega\left(\frac{M-1}{2}\right) \quad \text{if } H_R(\omega) < 0$$

## Case 2 : M = odd




$$H(z) = \sum_{n=0}^{M-1} h(n)\, z^{-n}$$

$$= \sum_{n=0}^{\frac{M-1}{2}-1} h(n) z^{-n} + h\left(\frac{M-1}{2}\right) z^{-\left(\frac{M-1}{2}\right)} + \sum_{n=\frac{M-1}{2}+1}^{M-1} h(n) z^{-n}$$

$$= h\left(\frac{M-1}{2}\right) z^{-\left(\frac{M-1}{2}\right)} + \sum_{n=0}^{\frac{M-3}{2}} h(n) z^{-n} + \sum_{n=\frac{M+1}{2}}^{M-1} h(n) z^{-n} \qquad \boxed{h(n) = \pm h(M-1-n)}$$

$$= h\left(\frac{M-1}{2}\right) z^{-\left(\frac{M-1}{2}\right)} + \sum_{n=0}^{\frac{M-3}{2}} h(n) z^{-n} \pm \sum_{n=\frac{M+1}{2}}^{M-1} h(\underline{M-1-n})\, z^{-n}$$

Subs. $k = M-1-n$ ; $n = \frac{M+1}{2} \Rightarrow k = M-1-\left(\frac{M+1}{2}\right) = \frac{M-3}{2}$

$n = M-1 \Rightarrow k = M-1-(M-1) = 0$

$$H(z) = h\left(\frac{M-1}{2}\right) z^{-\left(\frac{M-1}{2}\right)} + \sum_{n=0}^{\frac{M-3}{2}} h(n) z^{-n} \pm \sum_{\overset{n}{\cancel{k}}=0}^{\frac{M-3}{2}} h(\overset{n}{\cancel{k}})\, z^{-(M-1-\overset{n}{\cancel{k}})}$$

$$H(z) = h\left(\frac{M-1}{2}\right) z^{-\left(\frac{M-1}{2}\right)} + \sum_{n=0}^{\frac{M-3}{2}} h(n)\left[z^{-n} \pm z^{-(M-1-n)}\right]$$

$$= z^{-\left(\frac{M-1}{2}\right)}\left[h\left(\frac{M-1}{2}\right) + \sum_{n=0}^{\frac{M-3}{2}} h(n)\left\{z^{\left(\frac{M-1}{2}-n\right)} \pm z^{-\left(\frac{M-1}{2}-n\right)}\right\}\right] \quad (2)$$

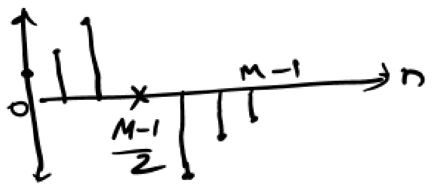
$$H(e^{j\omega}) = e^{-j\omega\left(\frac{M-1}{2}\right)}\left[h\left(\frac{M-1}{2}\right) + \sum_{n=0}^{\frac{M-3}{2}} h(n)\left\{\underbrace{e^{j\omega\left(\frac{M-1}{2}-n\right)} \pm e^{-j\omega\left(\frac{M-1}{2}-n\right)}}\right\}\right]$$

(a) M = odd, symm imp. resp:

$$H_r(\omega) = h\left(\frac{M-1}{2}\right) + \sum_{n=0}^{\frac{M-3}{2}} h(n)\cdot 2\cdot \cos\omega\left(\frac{M-1}{2}-n\right)$$

$$\left.\begin{aligned} \phi(\omega) &= -\omega\left(\frac{M-1}{2}\right) && \text{if } H_r(\omega) > 0 \\ &= -\omega\left(\frac{M-1}{2}\right) + \pi && \text{if } H_r(\omega) < 0 \end{aligned}\right\}$$

M = odd, h(n) - antisymm




(b) M = odd, antisymm.

$$H_r(\omega) = \underbrace{h\left(\frac{M-1}{2}\right)}_{=0} + \sum_{n=0}^{\frac{M-3}{2}} h(n)\cdot 2\cdot \sin\omega\left(\frac{M-1}{2}-n\right)$$

(b) M = odd , antisymm

$$H_r(\omega) = \sum_{n=0}^{\frac{M-3}{2}} h(n) \cdot 2 \sin \omega \left( \frac{M-1}{2} - n \right)$$

$$\phi(\omega) = \frac{\pi}{2} - \omega \left( \frac{M-1}{2} \right) \quad \text{if } H_r(\omega) > 0$$

$$= \frac{3\pi}{2} - \omega \left( \frac{M-1}{2} \right) \quad \text{if } H_r(\omega) < 0.$$

| | |
|---|---|
| Linear phase FIR filter frequency response | i) Symmetric impulse response, odd length<br>$H(\omega) = e^{-j\omega\left(\frac{M-1}{2}\right)}\left[2\sum_{n=0}^{\frac{M-3}{2}} h(n)cos\left\{\omega\left(\frac{M-1}{2}-n\right)\right\}\right]$<br>ii) Symmetric impulse response, even length<br>$H(\omega) = e^{-j\omega\left(\frac{M-1}{2}\right)}\left[2\sum_{n=0}^{\frac{M}{2}-1} h(n)cos\left\{\omega\left(\frac{M-1}{2}-n\right)\right\}\right]$<br>iii) Anti-symmetric impulse response, odd length<br>$H(\omega) = je^{-j\omega\left(\frac{M-1}{2}\right)}\left[2\sum_{n=0}^{\frac{M-3}{2}} h(n)sin\left\{\omega\left(\frac{M-1}{2}-n\right)\right\}\right]$<br>iv) Anti-symmetric impulse response, even length<br>$H(\omega) = je^{-j\omega\left(\frac{M-1}{2}\right)}\left[2\sum_{n=0}^{\frac{M}{2}-1} h(n)sin\left\{\omega\left(\frac{M-1}{2}-n\right)\right\}\right]$ |

Choice of filters:

1) Antisymm, odd : $H_r(\omega) = 0$ when $\omega = 0$ & $\omega = \pi$
LPF ✗, HPF ✗

2) Antisymm, even : $H_r(\omega) = 0$ when $\omega = 0$
LPF ✗

3) Symmetric imp resp is used for design of LPF, HPF, BPF, BSF.

## Zero location symmetry of linear phase FIR:

$$H(z) = \sum_{k=0}^{M-1} h(k) z^{-k}$$

$$= h(0) + h(1) z^{-1} + \ldots + h(M-2) z^{-(M-2)} + h(M-1) z^{-(M-1)}$$

Linear phase: $= h(0) + h(1) z^{-1} + \ldots \pm h(1) z^{-(M-2)} \pm h(0) z^{-(M-1)}$

$$= h(0)\left[1 \pm z^{-(M-1)}\right] + h(1)\left[z^{-1} \pm z^{-(M-2)}\right] + \ldots \qquad (1)$$

$$H(z^{-1}) = h(0)\left[1 \pm z^{(M-1)}\right] + h(1)\left[z^{1} \pm z^{+(M-2)}\right] + \ldots$$

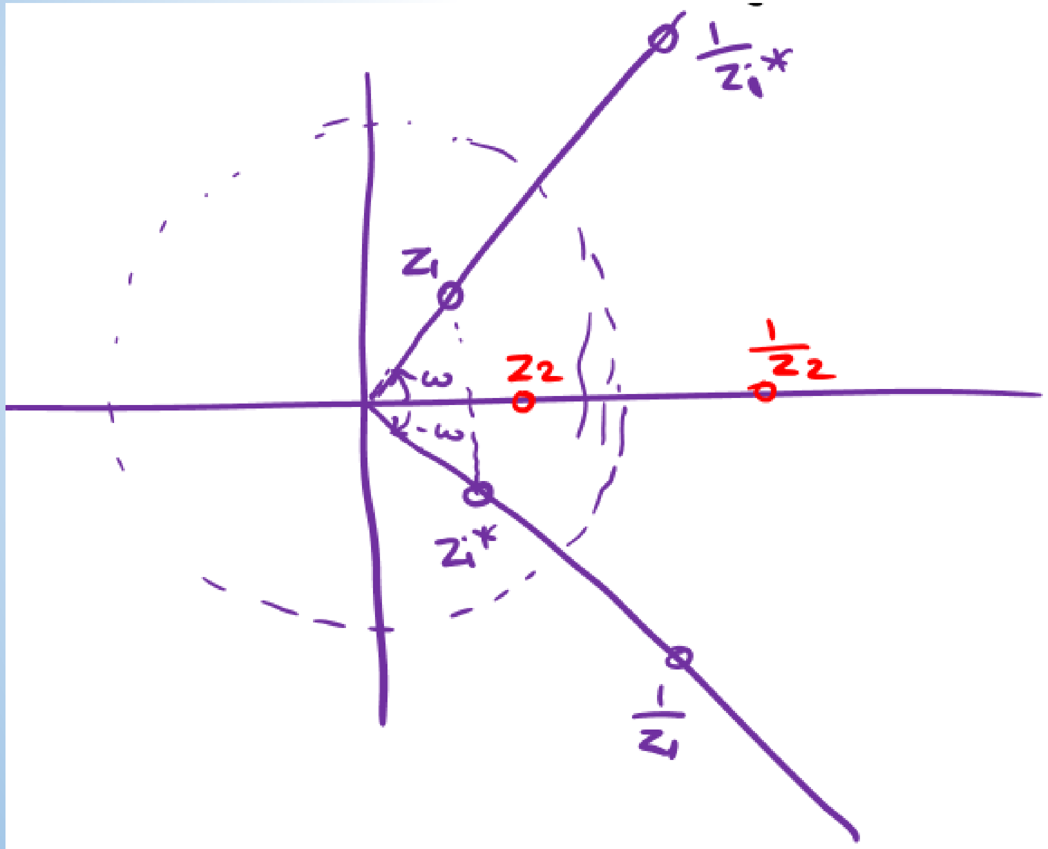
$$= z^{M-1} \cdot \left\{ h(0)\left[z^{-(M-1)} \pm z^{0}\right] + h(1)\left[z^{-(M-2)} \pm z^{-1}\right] + \ldots \right\}$$

$$\boxed{z^{-(M-1)} H(z^{-1}) = \pm H(z)}$$

Roots of $H(z)$ & $H(z^{-1})$ are identical.

$H(z) \downarrow z_i$, $H(z^{-1}) \downarrow \frac{1}{z_i}$ → reciprocal pairs

- If $z_i$ is a zero of $H(z)$, $\frac{1}{z_i}$ is also a zero (reciprocal pair)
- If $h(n)$ is real, roots – complex conjugate pairs $(z_i^*, \frac{1}{z_i^*})$



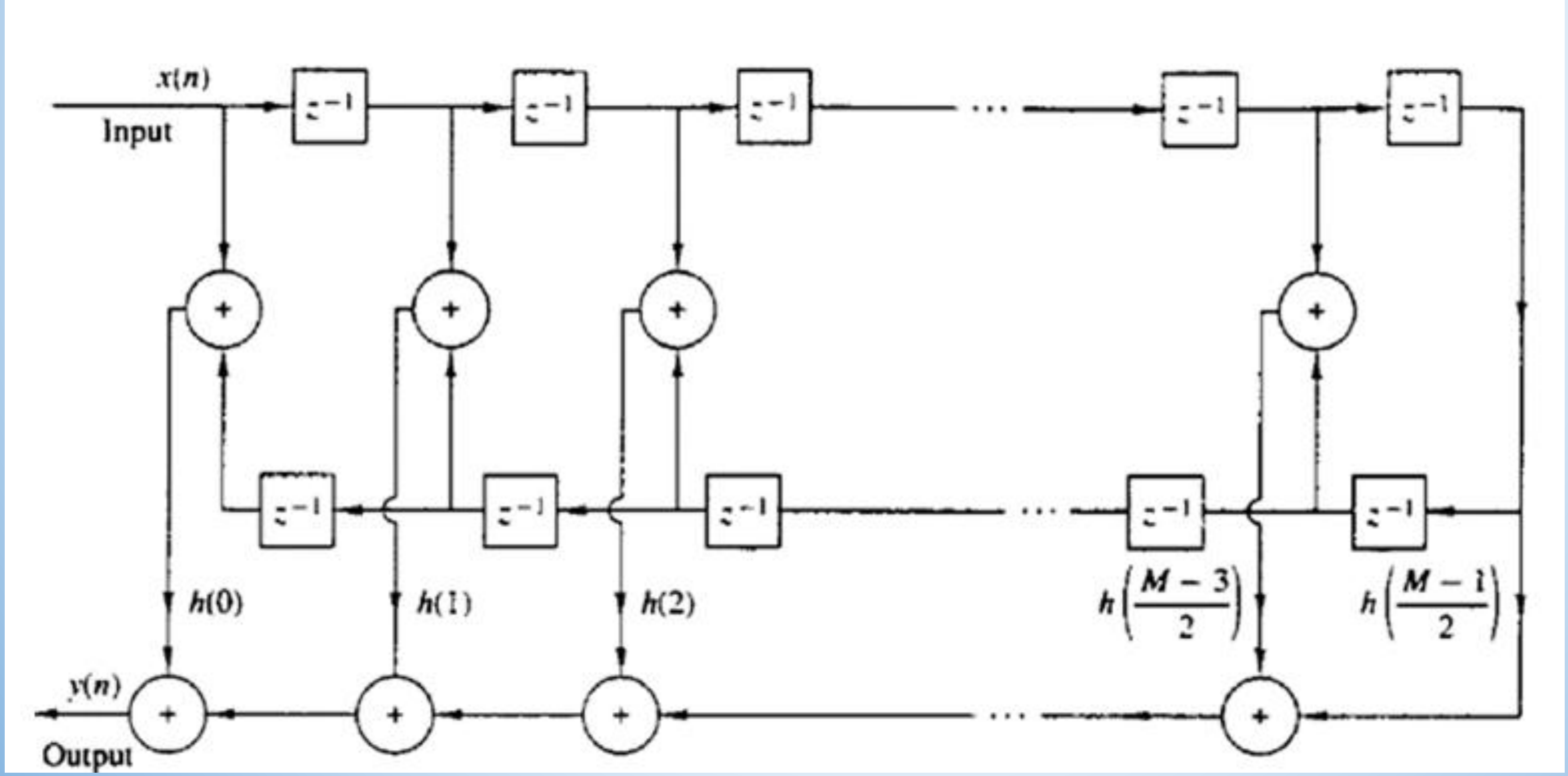

$z_i = r_i e^{j\phi_i}$

$\frac{1}{z_i} = \frac{1}{r_i} e^{-j\phi_i}$

HW: Draw an efficient tapped delay line structure for linear phase FIR with $M =$ odd & symm imp resp.

Soln:

$$H(z) = h\left(\frac{M-1}{2}\right) z^{-\left(\frac{M-1}{2}\right)} + \sum_{n=0}^{\frac{M-3}{2}} h(n)\left[z^{-n} + z^{-(M-1-n)}\right]$$




Thank you

# Linear Phase FIR Filter Window method

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Methods for design of FIR filters:

1) using window functions

2) using frequency sampling technique.

$H_d(\omega)$ is the desired freq resp.

$\downarrow$ IDTFT

$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega$, is the desired sample response.

To make it finite, (FIR)
truncate at $n = M-1$

$\downarrow$

$h(n) = h_d(n) \cdot w(n)$ ← window function $(0, \ldots M-1)$

eg. Rectangular window:

$$\begin{cases} h_d(n) & , n = 0, \ldots M-1 \\ 0 & \text{otherwise} \end{cases}$$

$\downarrow$

$H(\omega)$ is the designed filter freq response

$H(\omega)$, $H_d(\omega)$, $W(\omega)$

$$H(\omega) = H_d(\omega) * W(\omega)$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\vartheta) \cdot W(\omega - \vartheta)\, d\omega$$

Rectangular window function:


$$W(\omega) = \sum_{n=0}^{M-1} w(n)\, e^{j\omega n}$$

$$= \sum_{n=0}^{M-1} e^{-j\omega n}$$

$$= \frac{1 - e^{-j\omega M}}{1 - e^{-j\omega}}$$

$$= \frac{e^{-j\omega\frac{M}{2}}\left(e^{j\omega\frac{M}{2}} - e^{-j\omega\frac{M}{2}}\right)}{e^{-j\frac{\omega}{2}}\left(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right)}$$

$$= \frac{e^{-j\omega\frac{M}{2}}\left(\cancel{2j}\,\sin(\omega M/2)\right)}{e^{-j\frac{\omega}{2}}\left(\cancel{2j}\,\sin(\omega/2)\right)}$$

$$W(\omega) = e^{-j\omega\left(\frac{M-1}{2}\right)} \cdot \frac{\sin(\omega M/2)}{\sin(\omega/2)}$$

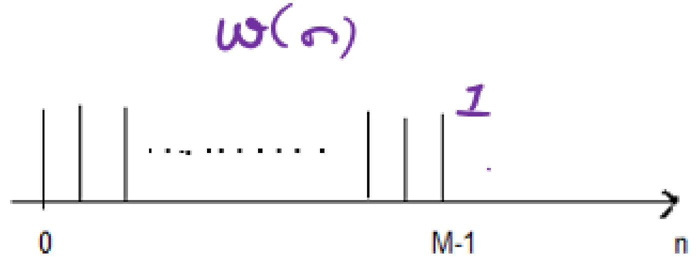
$$W(\omega) = e^{-j\omega\left(\frac{M-1}{2}\right)} \cdot \frac{\sin(\omega M/2)}{\sin(\omega/2)}$$

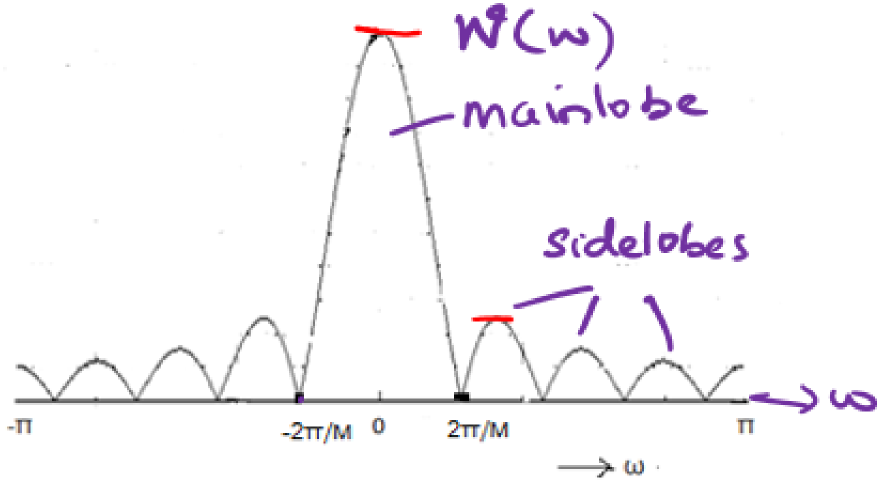
<u>Rectangular window fn:</u>

- width of mainlobe $= \frac{4\pi}{M}$

$M\uparrow$, width of mainlobe $\downarrow$
height " $\uparrow$
sidelobe freq $\uparrow$

- No. of sidelobes depends on M

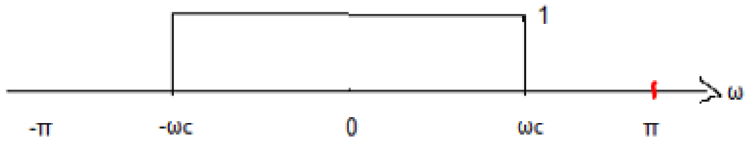



$-13\,dB$

eg. LPF with cut-off freq at $\omega_c$:

$$H_d(\omega) = \begin{cases} 1 \cdot e^{-j\omega\left(\frac{M-1}{2}\right)}, & 0 \le \omega \le \omega_c \\ 0 & \text{otherwise} \end{cases}$$




Ideal case.

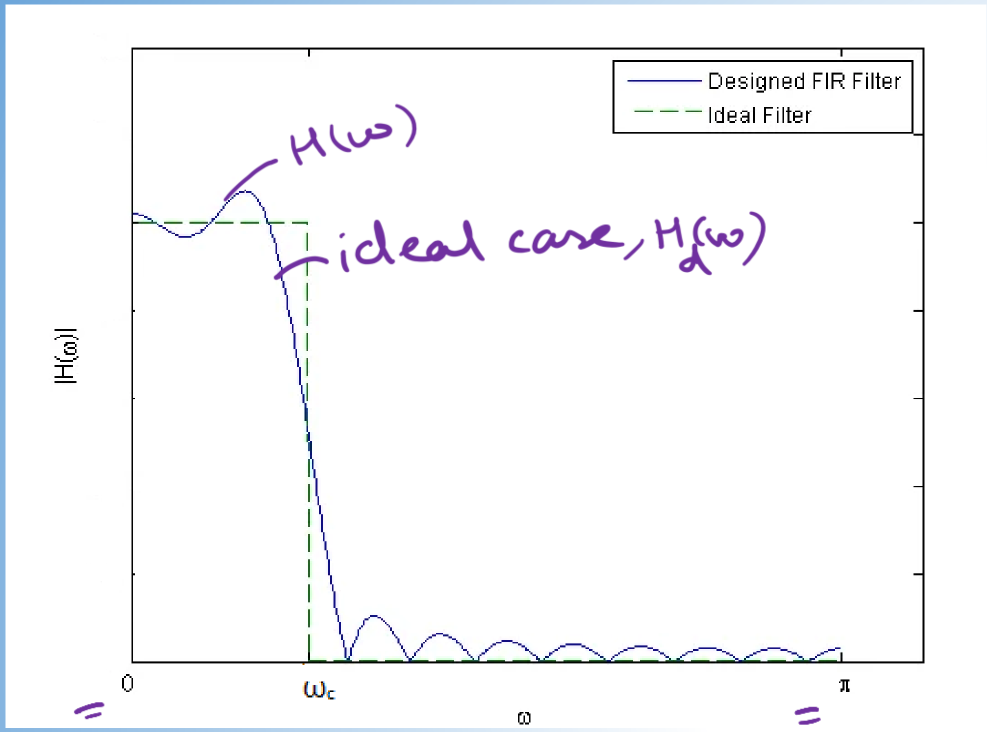
$$\Big\downarrow \text{IDTFT}$$

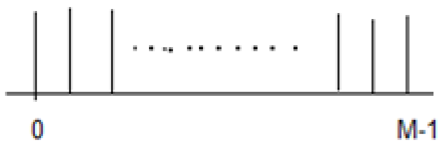
$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega.$$

$$= \begin{cases} \dfrac{\sin \omega_c\left(n - \frac{M-1}{2}\right)}{\pi\left(n - \frac{M-1}{2}\right)} & \text{for } n \ne \frac{M-1}{2} \\[2ex] \dfrac{\omega_c}{\pi} & \text{for } n = \frac{M-1}{2}. \end{cases}$$

& $M = \text{odd}$

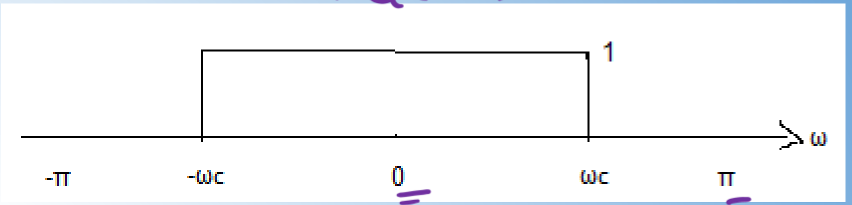
This function has infinite duration. To make it finite, multiply with rectangular window function.

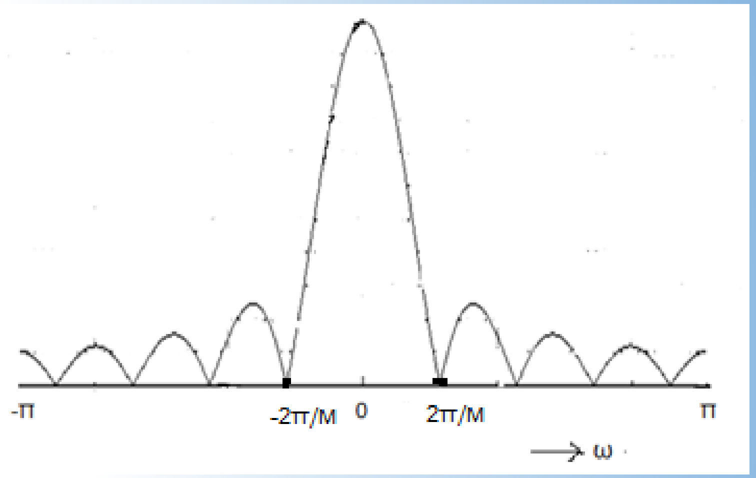



$\omega(n)$

$H_d(\omega)$




$W(\omega)$

$$h(n) = h_d(n) \cdot \omega(n)$$

$$H(\omega) = H_d(\omega) * W(\omega)$$

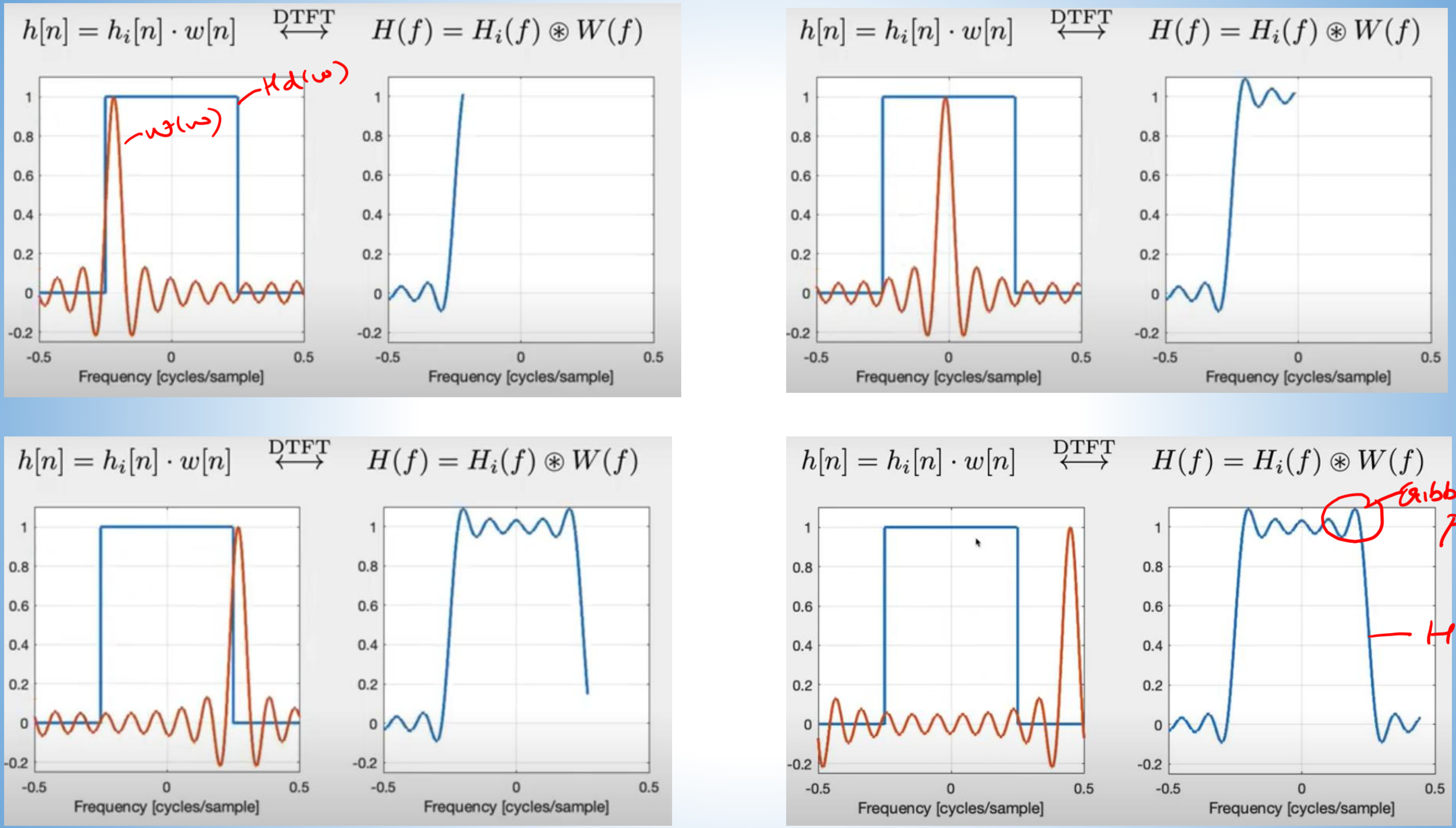
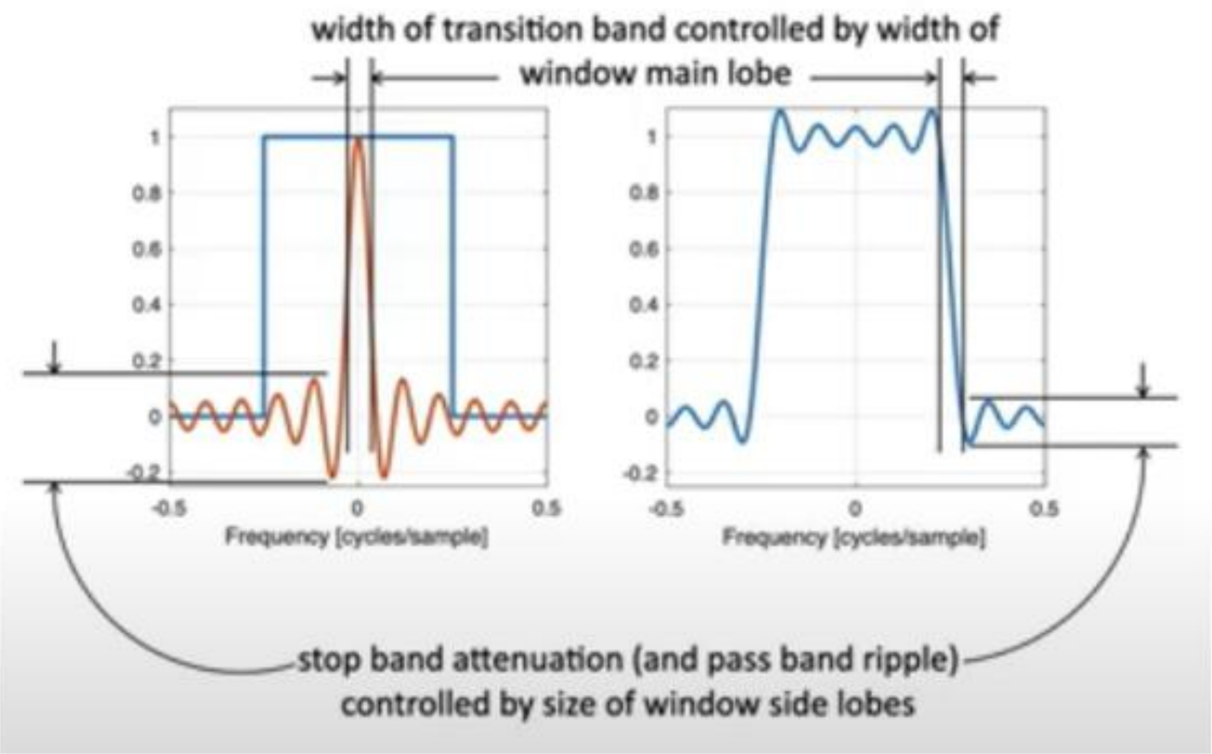


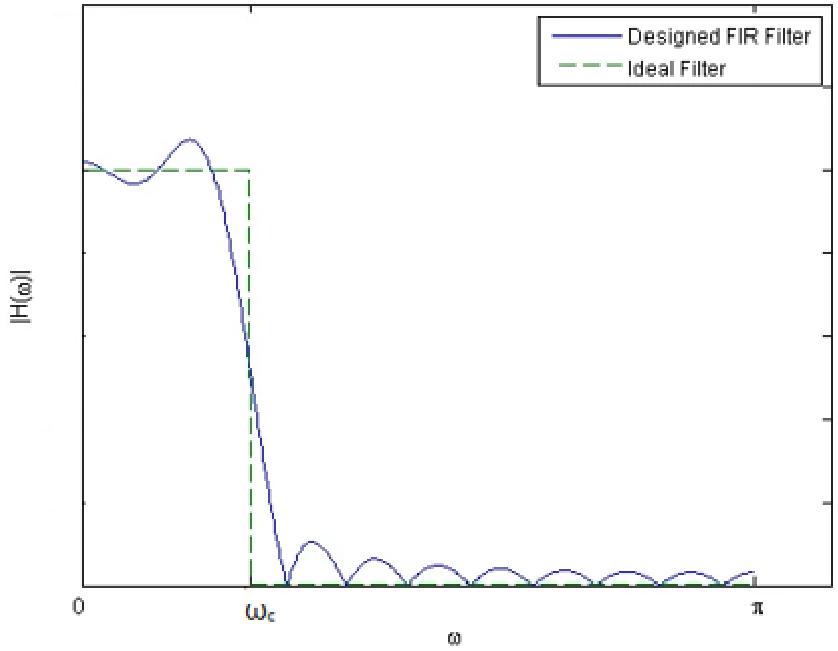

Figure Source: https://www.youtube.com/watch?v=2tmilbi4L0o

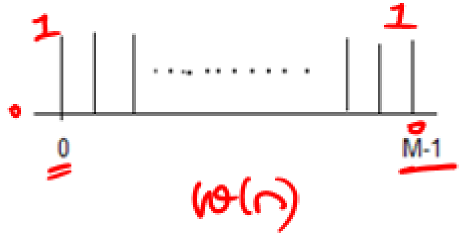
- Width of mainlobe causes the transition band in the designed filter response $H(\omega)$

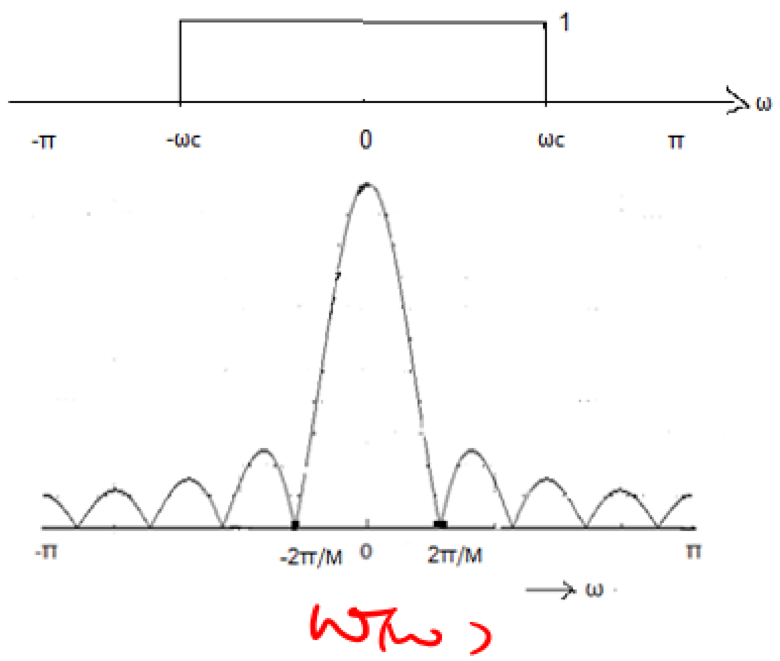
- Transition bandwidth depends on M
  M ↑ TB ↓




- Ripples in passband & stopband are due to significant sidelobes in $W(\omega)$
- Relatively large oscillations occur near the transition band edge. This is called Gibb's phenomenon.




- We desire the mainlobe width to be narrow and amplitude of sidelobe to be small.

- Sidelobes must be suppressed to reduce the ripples in passband & stopband
- Sidelobes are significant in rectangular window due to sharp transitions at $n = 0$ & $M-1$.

- Sidelobes can be suppressed by tapering the edges of the window function.

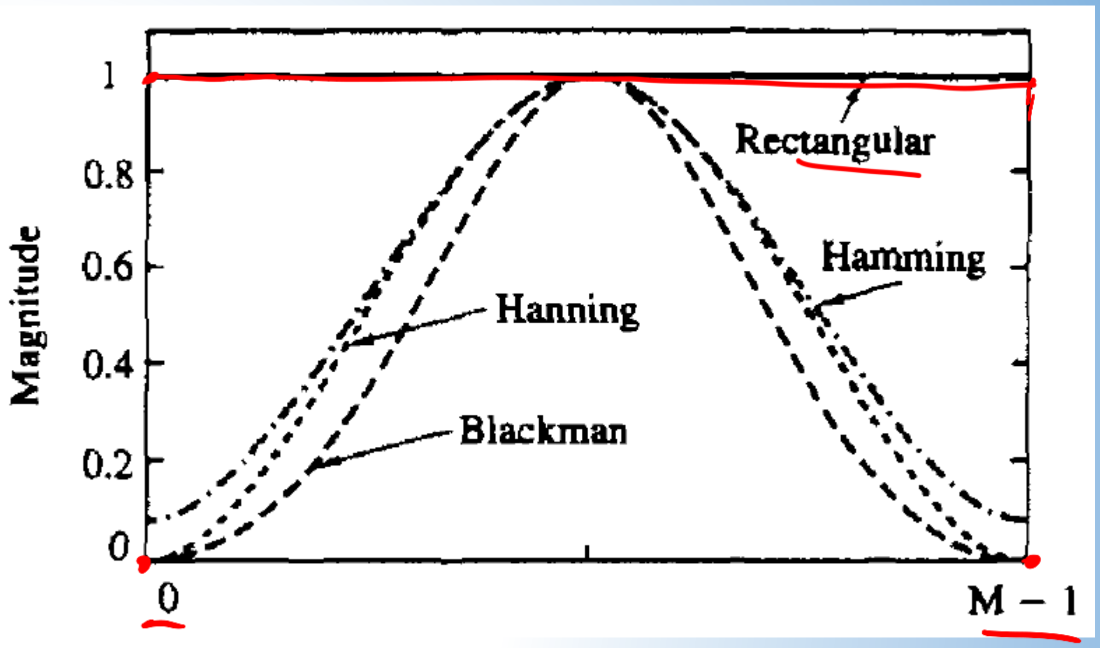
- Different window functions exist, depending on the way in which band edges are tapered.

1) Hanning window

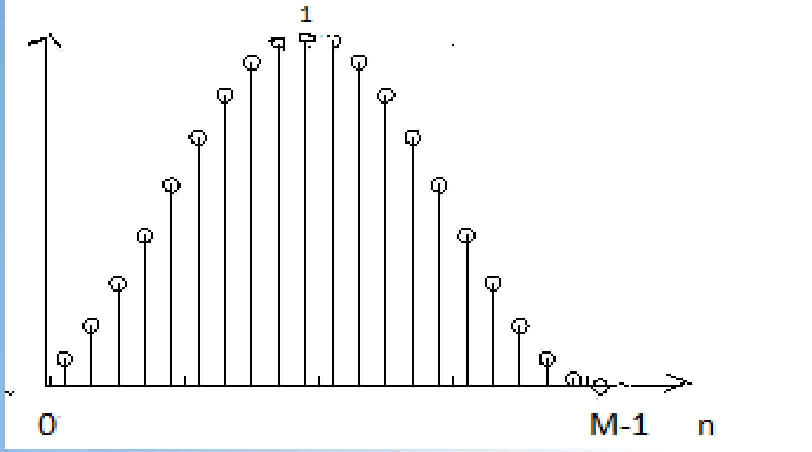
2) Hamming window

3) Blackman window

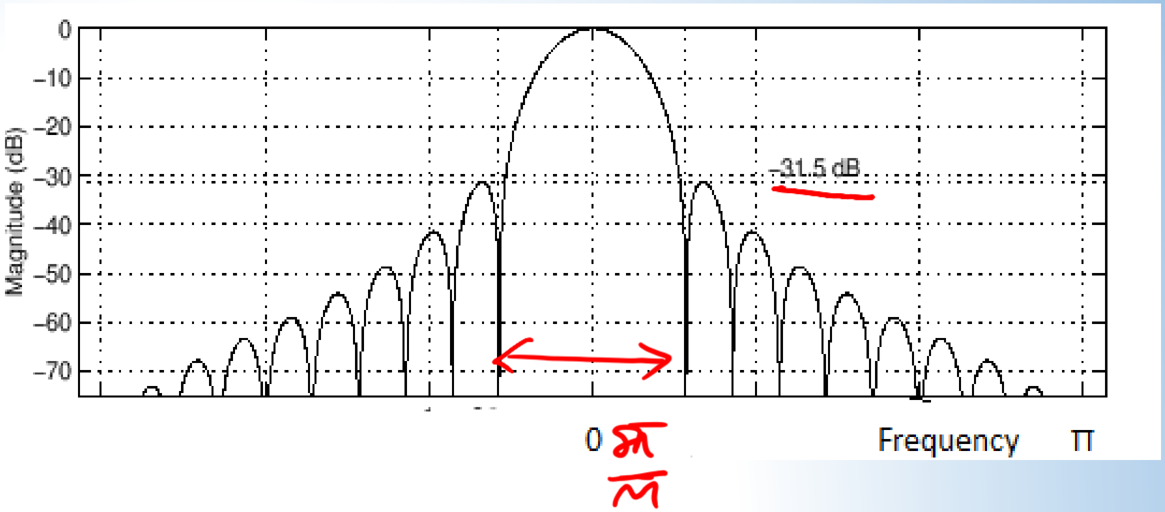
etc.

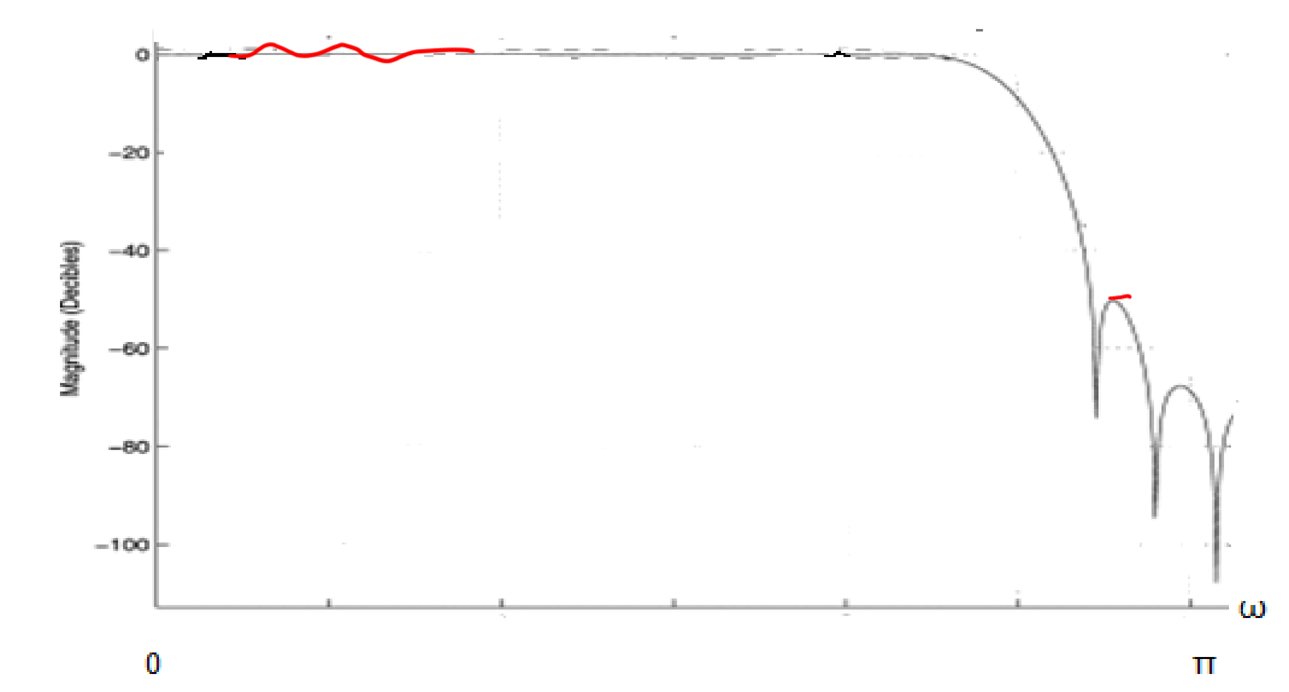


Hanning window, $w(n)$




$W(\omega)$




Sidelobe ≈ −32dB .

- Ripples in the passband and Gibbs' phenomenon has reduced.
- Stopband attenuation has increased.

Some of the commonly used window functions are given below

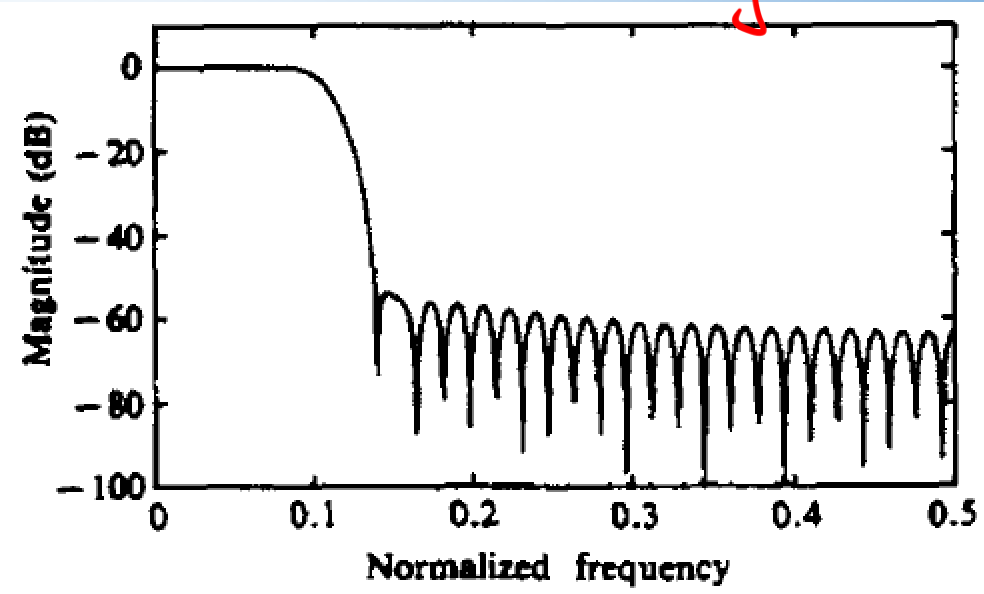
| Name of the window | Window function $0 \le n \le M-1$ | Main lobe width | Peak side lobe (dB) | Normalised transition width # | Stop band attenuation (dB) |
|---|---|---|---|---|---|
| Rectangular | 1 | $4\pi/M$ | -13 | 0.9/(M-1) | 21 |
| Hanning | $0.5 - 0.5\cos\left(\frac{2\pi n}{M-1}\right)$ | $8\pi/M$ | -32 | 3.1/(M-1) | 44 |
| Hamming | $0.54 - 0.46\cos\left(\frac{2\pi n}{M-1}\right)$ | $8\pi/M$ | -43 | 3.3/(M-1) | 53 |
| Blackman | $0.42 - 0.5\cos\left(\frac{2\pi n}{M-1}\right) + 0.08\cos\left(\frac{4\pi n}{M-1}\right)$ | $12\pi/M$ | -58 | 5.5/(M-1) | 75 |
| Keiser* | $\dfrac{I_0\left[\beta\sqrt{\left(\frac{M-1}{2}\right)^2 - \left(n - \frac{M-1}{2}\right)^2}\right]}{I_0\left[\beta\left(\frac{M-1}{2}\right)\right]}$ | | | | > 70 |

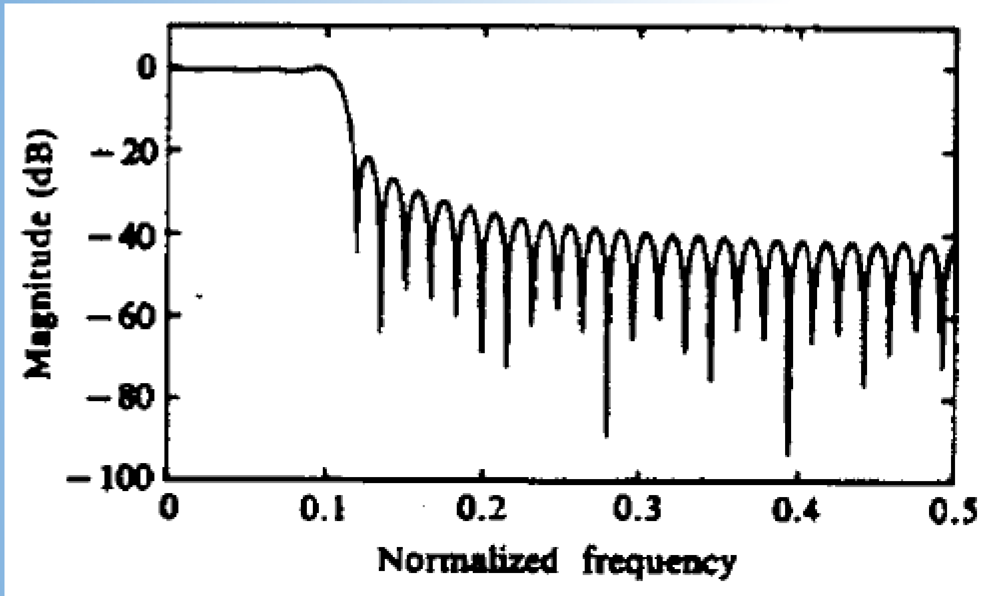
*Keiser window parameters can be controlled by $\beta$. $I_0[.]$ is modified Bessel function of first kind and order 0.

# Transition width is normalised to $2\pi$ or equivalently to sampling frequency $F_s$

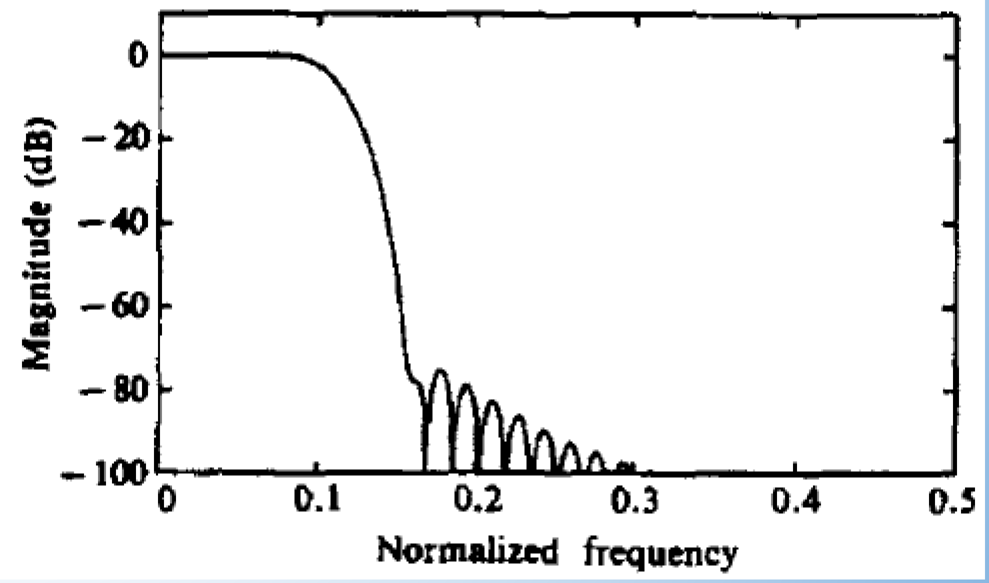
FIR LPF designed using rectangular window, Hamming window and Blackman window (M=61)

Hamming




Rectangular




Blackman

- More the tapering, wider is the normalized TB and better the stopband attenuation

Q. Design a symmetric FIR LPF with the desired freq resp

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha} & , |\omega| \le \omega_c \\ 0 & \text{otherwise} \end{cases}$$

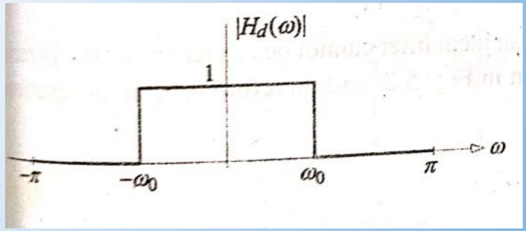
The length of the FIR filter is to be 7, cutoff freq $\omega_c = 1$,
Use rectangular window function.

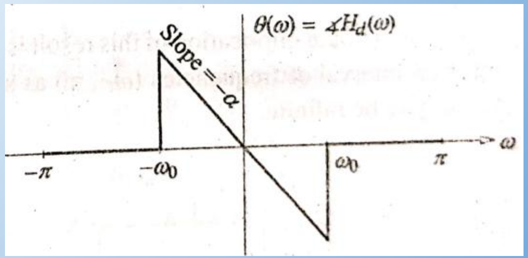
$$\alpha = \frac{M-1}{2}$$

Soln:

$H_d(\omega)$ — given
$\downarrow$
$h_d(n)$
$\downarrow$
$h(n) = h_d(n) \cdot w(n) \leftarrow$ rect
$\downarrow$
$H(\omega)$ – filter resp of designed FIR filter

$$H_d(\omega) = \begin{cases} e^{-j\omega\alpha}, & |\omega| < \omega_0 \\ 0, & \omega_0 < |\omega| < \pi \end{cases}$$



$$H_d(\omega) = \begin{cases} e^{-j\omega\alpha} & |\omega| \le \omega_c \\ 0 & \text{otherwise} \end{cases}$$

$\omega_c = 1$

$M = 7$

$$h_d(n) = \begin{cases} \dfrac{\sin \omega_c \left(n - \frac{M-1}{2}\right)}{\pi \left(n - \frac{M-1}{2}\right)} & \text{for } n \ne \frac{M-1}{2} \\ \dfrac{\omega_c}{\pi} & \text{for } n = \frac{M-1}{2} \end{cases}$$

$$= \begin{cases} \dfrac{\sin (n-3)}{\pi (n-3)} & , \quad n \ne 3 \\ \dfrac{1}{\pi} & , \quad n = 3 \end{cases}$$

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\tau} & -\omega_c \le \omega \le \omega_c \\ 0 & Otherwise \end{cases}$$

By taking inverse Fourier transform

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega\tau}e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega(n-\tau)}d\omega \end{aligned}$$

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\tau)}}{j(n-\tau)}\right]_{-\omega_c}^{\omega_c} \\ &= \frac{1}{2j\pi(n-\tau)}\left[e^{j\omega_c(n-\tau)} - e^{-j\omega_c(n-\tau)}\right] \\ &= \frac{1}{\pi(n-\tau)}\left[\frac{e^{j\omega_c(n-\tau)} - e^{-j\omega_c(n-\tau)}}{2j}\right] \end{aligned}$$

$h(n) = h_d(n) \cdot w(n)$

$M = 7$, $h(0) = h(6)$, $h(1) = h(5)$, $h(2) = h(4)$ ⇒ symmetric

| $n$ | $h_d(n)$ | $w(n)$ | $h(n) = h_d(n) \cdot w(n)$ |
|---|---|---|---|
| 0, 6 | 0.0149 | 1 | 0.0149 |
| 1, 5 | 0.1447 | 1 | 0.1447 |
| 2, 4 | 0.2678 | 1 | 0.2678 |
| 3 | 0.3183 | 1 | 0.3183. |

$$H(z) = \sum_{n=0}^{6} h(n) z^{-n}$$

$$= 0.0149(z^0 + z^{-6}) + 0.1447(z^{-1} + z^{-5}) + 0.2678(z^{-2} + z^{-4}) + 0.3183 z^{-3}.$$

| | |
|---|---|
| Linear phase FIR filter frequency response | i) Symmetric impulse response, odd length<br>$$H(\omega) = e^{-j\omega\left(\frac{M-1}{2}\right)}\left[2\sum_{n=0}^{\frac{M-3}{2}} h(n)cos\left\{\omega\left(\frac{M-1}{2}-n\right)\right\}\right]$$<br>ii) Symmetric impulse response, even length<br>$$H(\omega) = e^{-j\omega\left(\frac{M-1}{2}\right)}\left[2\sum_{n=0}^{\frac{M}{2}-1} h(n)cos\left\{\omega\left(\frac{M-1}{2}-n\right)\right\}\right]$$<br>iii) Anti-symmetric impulse response, odd length<br>$$H(\omega) = je^{-j\omega\left(\frac{M-1}{2}\right)}\left[2\sum_{n=0}^{\frac{M-3}{2}} h(n)sin\left\{\omega\left(\frac{M-1}{2}-n\right)\right\}\right]$$<br>iv) Anti-symmetric impulse response, even length<br>$$H(\omega) = je^{-j\omega\left(\frac{M-1}{2}\right)}\left[2\sum_{n=0}^{\frac{M}{2}-1} h(n)sin\left\{\omega\left(\frac{M-1}{2}-n\right)\right\}\right]$$ |

$$H(\omega) = e^{-j\omega\left(\frac{M-1}{2}\right)} \cdot H_r(\omega)$$

Given, $M = 7$ (odd), $h(n)$ - symmetric

$$H_r(\omega) = h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{\frac{M-3}{2}} h(n) \cos\omega\left(\frac{M-1}{2} - n\right)$$

$$= h(3) + 2\sum_{n=0}^{2} h(n) \cos\omega(3-n)$$

$$= 0.3183 + 2\left[0.0149 \cos 3\omega + 0.1447 \cos 2\omega + 0.2678 \cos\omega\right]$$

$$H(e^{j\omega}) = e^{-j3\omega} \cdot \left[0.3183 + 0.0298 \cos 3\omega + 0.2894 \cos 2\omega + 0.5356 \cos\omega\right]$$

Q. Use Hanning window for previous qn:

$$w(n) = 0.5 - 0.5\cos\left(\frac{2\pi n}{M-1}\right)$$

| n | hd(n) | w(n) | h(n) = hd(n)· w(n) |
|---|---|---|---|
| 0,6 | 0.0149 | 0 | 0 |
| 1,5 | 0.1447 | 0.25 | 0.0361 |
| 2,4 | 0.2678 | 0.75 | 0.2008 |
| 3 | 0.3183 | 1 | 0.3183. |

odd symm, $H_r(\omega) = h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{\frac{M-3}{2}} h(n)\cos\omega\left(\frac{M-1}{2}-n\right)$

$$H(e^{j\omega}) = e^{-j3\omega}\left[0.3183 + 0.0722\cos 2\omega + 0.4016\cos\omega\right]$$

Obtain the co.eff. of FIR LPF to meet the specification given below. use suitable window.

Passband edge frequency = 1.5 kHz

Transition width = 0.5 kHz

Stop band attenuation ≥ 50 dB

Sampling frequency = 8 kHz.

| Name of the window | Window function 0≤n≤M-1 | Main lobe width | Peak side lobe (dB) | Normalised transition width # | Stop band attenuation (dB) |
|---|---|---|---|---|---|
| **Rectangular** | 1 | 4π/M | -13 | 0.9/(M-1) | 21 |
| **Hanning** | $0.5 - 0.5cos\left(\frac{2\pi n}{M-1}\right)$ | 8π/M | -32 | 3.1/(M-1) | 44 |
| **Hamming** | $0.54 - 0.46cos\left(\frac{2\pi n}{M-1}\right)$ | 8π/M | -43 | 3.3/(M-1) | 53 |
| **Blackman** | $0.42 - 0.5cos\left(\frac{2\pi n}{M-1}\right) + 0.08cos\left(\frac{4\pi n}{M-1}\right)$ | 12π/M | -58 | 5.5/(M-1) | 75 |
| **Keiser*** | $\frac{I_0\left[\beta\sqrt{\left(\frac{M-1}{2}\right)^2 - \left(n - \frac{M-1}{2}\right)^2}\right]}{I_0\left[\beta\left(\frac{M-1}{2}\right)\right]}$ | | | | > 70 |

The window to be selected is hamming since the stopband attn is 53dB.

$$\Delta f = 0.5\ KHz.$$

$$f_{sl} = 8\ KHz.$$

$$\Delta F = \frac{3.3}{M}$$

$$\frac{\Delta f}{f_{sl}} = \frac{0.5\ KHz}{8\ KHz} = \frac{3.3}{M}$$

$$M = 52.8 \simeq 53.$$

$$\alpha = \frac{M-1}{2}$$

$$\boxed{\alpha = 26}$$

cutoff freq = Passband edge freq + $\frac{\text{Transition freq}}{2}$

$$f_c = 1.5\,\text{kHz} + \frac{0.5\,\text{kHz}}{2}$$

$$\boxed{f_c = 1.75\,\text{kHz}}$$

$$F_c = \frac{1.75\,\text{kHz}}{8\,\text{kHz}} = 0.21875$$

$$\omega_c = 2\pi F_c = 1.37744 \text{ rads.}$$

$$H_d(e^{j\omega}) = \begin{cases} 1\, e^{-j26\omega} & -1.3744 \le \omega \le 1.3744 \\ 0 & \text{Otherwise.} \end{cases}$$

$$\boxed{h_d(n) = \frac{1}{\pi(n-26)}\left[\sin(1.3744)(n-26)\right]} \rightarrow \text{Infinite}$$

Step 4 :- $h'(n) = h_d(n)\ \omega(n)$

$$0.54 - 0.46 \cos\left(\frac{2\pi n}{M-1}\right) \quad 0 \le n \le M-1$$

$$M = 53$$

$$0.54 - 0.46 \cos\left(\frac{2\pi n}{52}\right) \quad 0 \le n \le 52$$

| n | hd(n) | w(n) | h(n) = hd(n) · w(n) |
|---|---|---|---|

# Reference


- Proakis J. G, Manolakis D. G. Mimitris D., “Introduction to Digital Signal Processing” Prentice Hall, India, 2007.


Thank you

# Linear Phase FIR Filter Frequency sampling method

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Methods for design of FIR filters:

1) using window functions

2) using frequency sampling technique.

# Frequency-sampling method

$H_d(\omega)$ - desired frequency response

sampled at M frequency points, $\omega_k = \frac{2\pi}{M} k$ , $k = 0, 1, \cdots M-1$

$$H(k) = H_d(\omega_k) = H_d\left(\frac{2\pi}{M} k\right)$$

Take IDFT to find M filter coefficients

$$h(n) = \frac{1}{M} \sum_{k=0}^{M-1} H(k)\, e^{j\frac{2\pi}{M}kn} \qquad \text{———} \quad (1)$$

In case of linear phase FIR filter,

$$\omega_k = \frac{2\pi}{M} k \quad , \begin{cases} k = 0, 1, \dots \frac{M-1}{2} & \text{for } M = \text{odd} \\ k = 0, 1, \dots \frac{M}{2} - 1 & \text{for } M = \text{even} \end{cases}$$

$$H_d(\omega) = H_r(\omega) \cdot e^{-j\omega\left(\frac{M-1}{2}\right)}$$

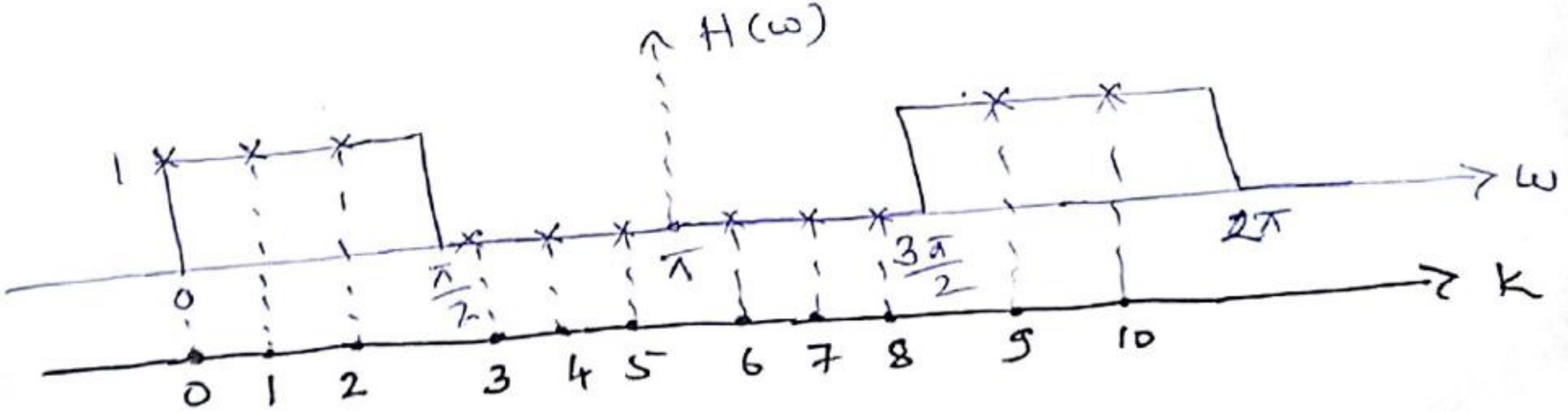
$$H(k) = H_d(\omega_k) = H_d\left(\frac{2\pi}{M} k\right) = H_r\left(\frac{2\pi}{M} k\right) \cdot e^{-j\frac{2\pi}{M} k\left(\frac{M-1}{2}\right)}$$

Substitute in eq(1),

$$h(n) = \frac{1}{M} \sum_{k=0}^{M-1} H_r\left(\frac{2\pi}{M} k\right) \cdot e^{j\frac{2\pi}{M} k\left(n - \frac{M-1}{2}\right)} \quad \text{——— (2)}$$

Example:

Consider an ideal LPF with cutoff freq at $\frac{\pi}{2}$ and $M=11$.




$$H(k) = H_d(k) = \begin{cases} 1 & , k = 0, 1, 2, 9, 10 \\ 0 & , k = 3, 4, 5, 6, 7, 8 \end{cases}$$

Imposing linear phase condition,

$$H(k) = \begin{cases} 1 \cdot e^{-j\frac{2\pi}{M}k\left(\frac{M-1}{2}\right)} = e^{-j\frac{2\pi}{11}\times 5k} & , \text{for } k = 0, 1, 2, 9, 10 \\ 0 & , \text{otherwise} \end{cases}$$

Q1. Determine the coefficients of linear phase FIR filter of length 15, which has symmetric impulse response h(n), that satisfies the condition,

$$H_r(k) = H_r\left(\frac{2\pi}{15}k\right) = \begin{cases} 1, & k = 0, 1, 2, 3 \\ 0.4, & k = 4 \\ 0, & k = 5, 6, 7 \end{cases}$$

Soln: Since h(n) is symmetric $H_r(k) = H_r(M-k)$

i.e. $$H_r\left(\frac{2\pi}{15}k\right) = \begin{cases} 1, & k = 12, 13, 14 \\ 0.4, & k = 11 \\ 0, & k = 8, 9, 10 \end{cases}$$

From eqn (2),

$$h(n) = \frac{1}{M} \sum_{k=0}^{M-1} H_r\left(\frac{2\pi}{M}k\right) \cdot e^{j\frac{2\pi}{M}k\left(n - \frac{M-1}{2}\right)}$$

$$= \frac{1}{15} \sum_{k=0}^{14} H_r\left(\frac{2\pi}{15}k\right) \cdot e^{j\frac{2\pi}{15}k(n-7)}$$

$$= \frac{1}{15}\Big[1 + 1\cdot e^{j\frac{2\pi}{15}(n-7)} + 1\cdot e^{j\frac{4\pi}{15}(n-7)} + 1\cdot e^{j\frac{6\pi}{15}(n-7)}$$
$$+ 0.4\, e^{j\frac{8\pi}{15}(n-7)} + 0.4\, e^{j\frac{\overset{-8\pi}{\cancel{22\pi}}}{15}(n-7)} + 1\cdot e^{j\frac{\overset{-6\pi}{\cancel{24\pi}}}{15}(n-7)}$$
$$+ 1\cdot e^{j\frac{\overset{-4\pi}{\cancel{26\pi}}}{15}(n-7)} + 1\cdot e^{j\frac{\overset{-2\pi}{\cancel{28\pi}}}{15}(n-7)}\Big]$$

We know, $\frac{22\pi}{15} - 2\pi = -\frac{8\pi}{15}$

Grouping $e^{j\theta} + e^{-j\theta}$ terms and subs as $2\cos\theta$,

$$\therefore h(n) = \frac{1}{15}\left[1 + 2\left\{\cos\frac{2\pi}{15}(n-7) + \cos\frac{4\pi}{15}(n-7) + \cos\frac{6\pi}{15}(n-7) + 0.4\cos\frac{8\pi}{15}(n-7)\right\}\right]$$

Substitute for $n = 0$ to $7$, to get values of $h(n)$

$h(0) = -0.014$

$h(1) = -0.002$

$h(2) = 0.04$

$h(3) = 0.012$

$h(4) = -0.091$

$h(5) = -0.018$

$h(6) = 0.313$

$h(7) = 0.52$

Since symm, $h(M-1-n) = h(n)$

$h(8) = h(6) = 0.313$

$h(9) = h(5) = -0.018$

$h(10) = h(4) = -0.091$

$h(11) = h(3) = 0.012$

$h(12) = h(2) = 0.04$

$h(13) = h(1) = -0.002$

$h(14) = h(0) = -0.014$

Another equation for linear phase condition,

$$H(k) = H_r(k) \cdot e^{-j\frac{2\pi}{M}k\left(\frac{M-1}{2}\right)}$$

$$= H_r(k) \cdot e^{-j\frac{2\pi}{M}k\frac{M}{2}} \cdot e^{j\frac{2\pi}{M}k\frac{1}{2}}$$

$$= \underbrace{H_r(k)\,(-1)^k}_{G(k)}\, e^{j\frac{\pi k}{M}} \qquad , k = 0, 1, \dots M-1$$

Imposing symmetric condition, $H(k) = H^*(M-k)$

$$h(n) = \begin{cases} \frac{1}{M}\left[G(0) + 2\sum_{k=1}^{\frac{M}{2}-1} G(k)\cos\frac{2\pi}{M}k\left(n+\frac{1}{2}\right)\right] & \text{for } M = \text{even} \\ \frac{1}{M}\left[G(0) + 2\sum_{k=1}^{\frac{M-1}{2}} G(k)\cos\frac{2\pi}{M}k\left(n+\frac{1}{2}\right)\right] & \text{for } M = \text{odd} \end{cases} \qquad (3)$$

Some important points:

1. Similar analysis can be done for antisymm case too.

2. In Q1, we obtained $H_r(\omega_k) = 0.4$ at $k = 4$. There is a transition band introduced.


3. Since $H_r(\omega)$ is sampled, the designed filter will have ripples in the passband and stopband.

Ripples can be suppressed by increasing filter length, M.

4. The main advantage of frequency-sampling method is that an efficient frequency sampling structure can be realized using this.

HW Qn: Obtain and draw the frequency sampling structure for the filter considered in Q1.

Frequency-sampling method

Type I sampling

$$\omega_k = \frac{2\pi}{M} k$$

$$\alpha = 0$$

Type II sampling

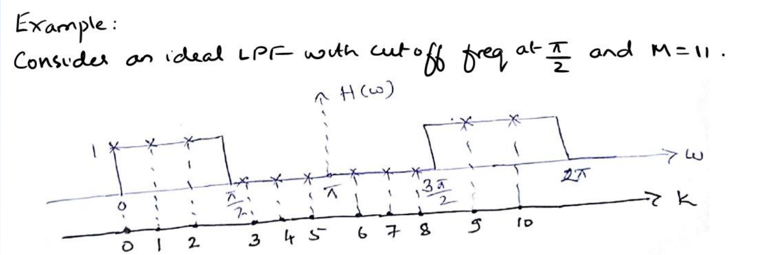
$$\omega_k = \frac{2\pi}{M} (k+\alpha)$$

$$\alpha = 1/2$$

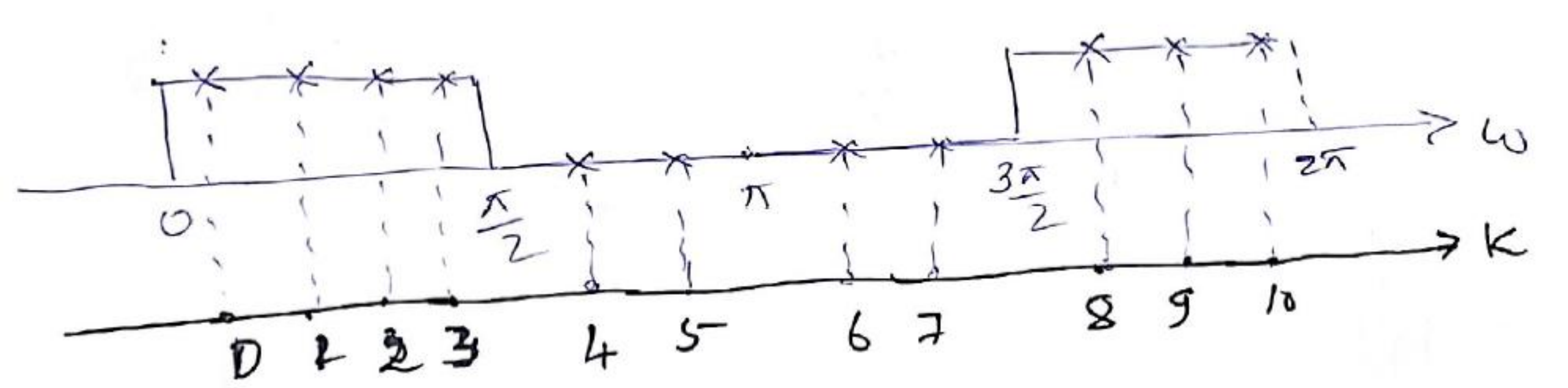
For Type II sampling,

$$H(k+\alpha) = H_r\left(\frac{2\pi}{M}(k+\alpha)\right) e^{-j\frac{2\pi}{M}(k+\alpha)\left(\frac{M-1}{2}\right)} \quad , k=0,1,\ldots M-1$$

In example 1, if we take $\alpha = 1/2$, we get




$$H(k+\tfrac{1}{2}) = \begin{cases} 1 & , k = 0,1,2,3,8,9,10 \\ 0 & , k = 4,5,6,7 \end{cases}$$




Similar analysis to find h(n) values can be done here.

# Reference


- Proakis J. G, Manolakis D. G. Mimitris D., “Introduction to Digital Signal Processing” Prentice Hall, India, 2007.


Thank
you

# Power Spectrum Estimation

The estimation of spectral characteristics of signals (more specifically random signals) is an important ~~issue~~ aspect in signal processing.

First, let us consider finite energy signal $x_a(t)$.

Then energy $E = \int_{-\infty}^{\infty} |x_a(t)|^2 \, dt < \infty$. Fourier transform exists and is $X_a(F) = \int_{-\infty}^{\infty} x_a(t) e^{-j2\pi Ft} dt$

From Parseval relation, $E = \int_{F=-\infty}^{\infty} |X_a(F)|^2 dF$

The quantity $|X_a(F)|^2 = S_{xx}(F) \equiv$ Energy spectrum density (ESD) --- (1)

This quantity represent distribution of signal energy as a function of frequency. Total energy in any frequency band can be obtained by integrating ESD.

If $R_{xx}(\tau)$ is autocorrelation function i.e

$R_{xx}(\tau) = \int_{t=-\infty}^{\infty} x_a^*(t)\, x_a(t+\tau)\, dt$, then it follows

that $S_{xx}(F) = FT[R_{xx}(\tau)] = \int_{\tau=-\infty}^{\infty} R_{xx}(\tau) e^{-j2\pi F\tau} d\tau$.

--- (2)

Signal $x_a(t)$ is bandlimited to $B$ and sampled at rate $F_s \geq 2B$ to get sampled data $x(n)$.

With normalised frequency $f = \frac{F}{F_s}$,

$$X(f) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi fn}$$

From sampling theory with $f = \frac{F}{F_s}$, we know that

$$X(f) = F_s \cdot \sum_{k=-\infty}^{\infty} X_a(F - kF_s)$$

With no aliasing, $X(f) = F_s \cdot X_a(F)$ ; $|F| \leq F_s/2$

$$\therefore \quad S_{xx}(f) = |X(f)|^2 = F_s^2 \, |X_a(f)|^2 \quad \text{---} \; (3)$$

If $r_{xx}(m)$ is autocorrelation, then

$$S_{xx}(f) = \sum_{m=-\infty}^{\infty} r_{xx}(m) \, e^{-j 2\pi f m} \quad \text{---} \; (4)$$

Similarly for random signals (Power signals) the power spectrum density (PSD) $= \Gamma_{xx}(F)$ is

$$\Gamma_{xx}(F) = |X_a(F)|^2 = \int_{-\infty}^{\infty} \gamma_{xx}(\tau) \, e^{-j 2\pi F \tau} \, d\tau \quad \text{where} \; \text{---} \; (5)$$

$\gamma_{xx}(\tau)$ is the autocorrelation function.

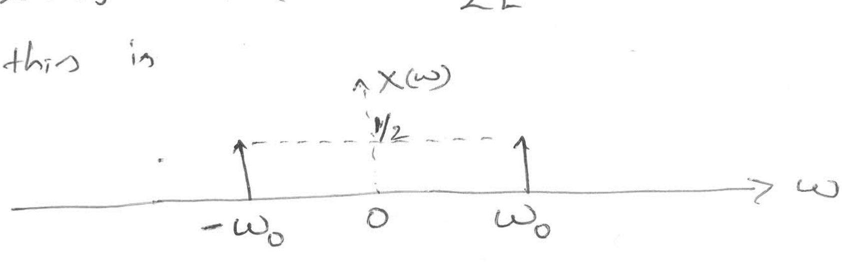
In all these definitions it is assumed that the signal is of infinite duration. But in practice we need to estimate ESD or PSD from the observation of signal over a finite time interval. The finite record length of the data sequence is a major limitation on the quality of the spectrum. The effects are i) Spectral leakage ii) Resolution problem (Resolution reduces). This is illustrated below.

consider $x(n) = \cos(\omega_0 n)$

Then ~~$X(\omega)$ = DTFT of $x(n)$~~ = Spectrum of $x(n)$ i.e

$$\text{DTFT}[x(n)] = X(\omega) = \frac{1}{2}\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right]$$

Plot of this is



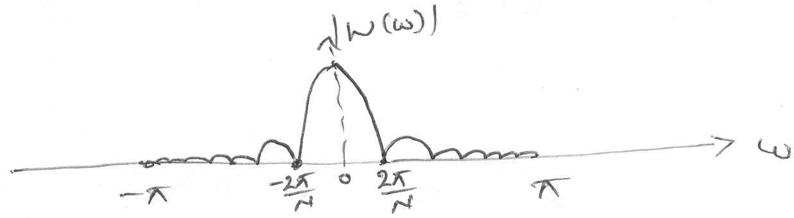

When we have a finite observation of this signal i.e

$\tilde{x}(n) = x(n) \cdot w(n)$ where $w(n)$ is rectangular window. If we take finite data length $= N$, then

$$w(n) = 1, \quad 0 \le n \le N-1$$
$$= 0 \quad \text{elsewhere.}$$

DTFT of $w(n) = W(\omega) = \dfrac{\sin(\omega N/2)}{\sin(\omega/2)} e^{-j\omega(N-1)/2}$

Plot of $|W(\omega)|$ is as follows



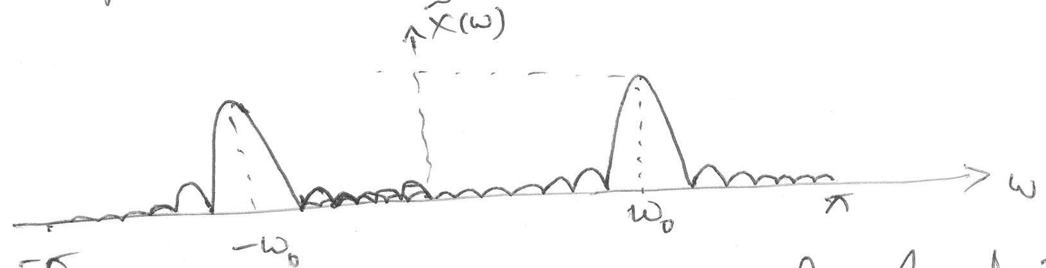

Now from DTFT property

$\tilde{X}(\omega) = X(\omega) * W(\omega)$

( Multiplication in time $\equiv$ convolution in Fourier domain)

$\therefore\ \tilde{X}(\omega) = \frac{1}{2}\left[W(\omega-\omega_0) + W(\omega+\omega_0)\right]$

This spectrum is plotted below




We note that the spectrum $X(\omega)$ is localised to single frequency whereas spectrum $\tilde{X}(\omega)$ is not; instead it is spread out ~~to~~ over whole frequency range. This means that power is "leaked out" into the entire frequency range. This is "spectral leakage" and due to windowing. ~~Leakage~~ Tapering the window will reduce the leakage.

The second effect of windowing (to get finite length data) that it reduces spectral resolution.

Let $x(n) = \cos(\omega_1 n) + \cos \omega_2(n)$. The difference b/w successive frequency is $\Delta\omega = (\omega_2 - \omega_1)$.

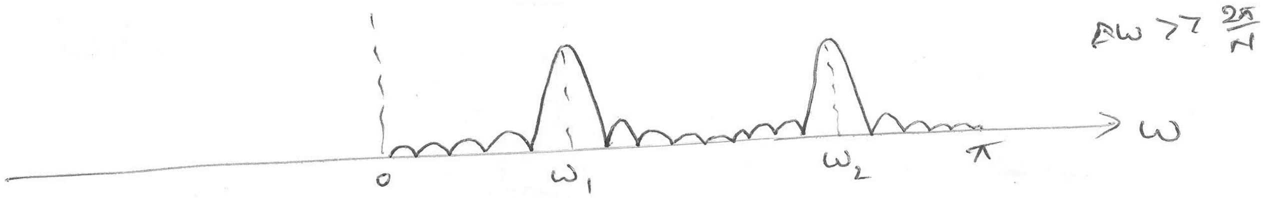
Minimum $\Delta\omega$ that can be resolved is spectral resolution.

As before let $\bar{x}(n) = x(n) \cdot w(n)$

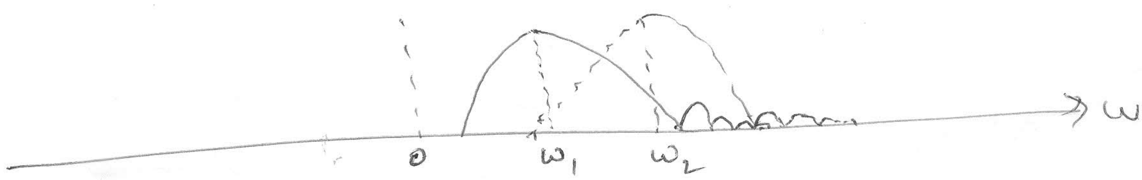
Then $\tilde{X}(\omega) = \frac{1}{2}\left[ W(\omega-\omega_1) + W(\omega-\omega_2) + W(\omega+\omega_1) + W(\omega+\omega_2) \right]$

Let window length $= N$.

First let us take $\Delta\omega \geq \frac{2\pi}{N}$. The plot of one-sided spectrum ($0$ to $\pi$) is




If $\Delta\omega < \frac{2\pi}{N}$, two peaks overlap. Hence not resolved




Thus to get better resolution (small $\Delta\omega$) we need to have large $N$. Tapering the window will also result in poor resolution.

Note:

In practice DFT is used to get the spectrum. $M$ point DFT is considered where $M \geq N$. Note that taking $M > N$ ~~doe~~ does not improve the resolution, but we are able to get the spectrum at more interpolated points. The resolution is decided by data length $N$, not DFT length $M$.

Coming back to our analysis of PSD estimation: we observe that finite record length of random data distorts the calculated PSD ~~from~~ using DFT or DFT of autocorrelation (due to windowing)

Why can't we have ~~to~~ infinite data length?

One problem is storage. Other problem is out signals of interest are ~~not~~ non stationary process.

If signal is stationary process we could have taken large $N$ and hence get better quality spectrum. But this is not possible with non-stationary random signals. Hence in power spectrum estimation, our goal is to select as short data record as possible that still allows us to resolve the spectral characteristics with good quality. (good resolution, low variance of the estimate etc)

There are different techniques for estimating PSD. They are broadly classified into

1. Non parametric methods where measured data is directly used, no assumptions are made about how data are generated.
2. Parametric methods where some assumption on data generation are used and a model for the data generator is first obtained.

## Estimation of autocorrelation & Power spectrum of random signals : "Periodogram"

We consider a single realization of the random process. Let $x_a(t)$ be such random signal. It is sampled at $F_s \geq 2B$ where $B$ is the highest frequency contained in the spectrum of $x_a(t)$. This results in $N$-length random signal $x(n)$ ; $0 \leq n \leq N-1$.

PSD can be obtained from autocorrelation function

The time averaged autocorrelation sequence of $x(n)$ is ( $x(n)$ has length $N$, $m$ is "lag" )

$$r'_{xx}(m) = \frac{1}{N-m} \sum_{n=0}^{N-m-1} x^*(n)\, x(n+m) \; ; \quad 0 \leq m \leq N-1 \qquad \text{---- (6)}$$

Random process can be best characterised by mean (expected value) and variance. It can be proved that expected value of $r'_{xx}(m)$ given by (6) gives true autocorrelation (not ~~aver~~ biased) and

$\lim_{N\to\infty} \mathrm{var}(r'_{xx}(m)) = 0$. Such estimate is referred to as unbiased and consistant estimate.

However the $\mathrm{var}[r'_{xx}(m)]$ is poor. i.e. it is large for large lag or when $m$ reaches $N-1$. This results in poor estimate of PSD. Alternate to this is to use biased autocorrelation given by

$$r_{xx}(m) = \frac{1}{N}\sum_{n=0}^{N-m-1} x^*(n)\, x(n+m) \quad ; \quad 0 \le m \le N-1 \quad \text{--- (7)}$$

(This has a bias $|m|\, \gamma_{xx}(m)/N$ where $\gamma_{xx}(m)$ is true autocorrelation). It can be still shown that $\cancel{\lim_{N\to\infty} \mathrm{var}[r_{xx}(m)]}$ $\lim_{N\to\infty} \mathrm{var}[r_{xx}(m)] = 0$

or $r_{xx}(m)$ is also consistant.

Now PSD is Fourier transform of $r_{xx}(m)$

$$\therefore \text{PSD} = P_{xx}(f) = \sum_{m=-(N-1)}^{(N-1)} r_{xx}(m)\, e^{-j2\pi fm} \quad \text{--- (8)}$$

(Note that ACF length is twice that of the signal and it is even symmetric)

If we substitute for $r_{xx}(m)$ in (8) from (7), we will get PSD as

$$P_{xx}(f) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi fn}\right|^2 = \frac{1}{N}|X(f)|^2 \quad \text{--- (9)}$$

where $X(f)$ is DTFT of $x(n)$.

This representation for PSD is referred to as "Periodogram".

Note that this is only an estimate of PSD but not exactly the true PSD of original signal.

It can be proved that

$\lim_{N \to \infty} \text{var}[\text{Periodogram}] = \Gamma_{xx}^{2}(f)$ where $\Gamma_{xx}(f)$ is True Power spectrum density. Since $\text{var}[P_{xx}(f)] \neq 0$, the periodogram estimate of PSD is not consistant. This is another poor quality indication of Periodogram method of PSD estimation apart from spectral leakage and poor resolution problem. These are the main drawbacks of Periodogram.

Other methods are developed to improve the quality. They are classified as nonparametric and parametric methods

<u>Non-parametric methods</u>:

<u>Principle</u>: These methods make no assumption about how the data were generated. N-length data record is used to first get periodogram. Then processing is done further to reduce the variance and hence to improve the quality of PSD estimate. N is selected to get the required frequency resolution $\Delta f$. (Note that the frequency resolution is, at best, equal to spectral width of rectangular window of length N. This is equal to $\frac{1}{N}$ (as $\Delta\omega = \frac{2\pi}{N}$, first zero crossing of $W(\omega)$ of rectangular window $w(n)$))

The three methods are

1. ~~Bartlett~~ Bartlett method
2. Welch method
3. Blackman & Tukey method

## Bartlett method of Power Spectrum Estimation.

This method aims at reducing the variance by averaging the periodogram, but at the expense of increased spectral width. (Increase in spectral width results in reduce in resolution i.e. minimum frequency that can be resolved increases)

Data frame length is taken as N.

Step 1: The N-point sequence is devided into K number of non-overlapping segments of length M each. ($\therefore K = \frac{N}{M}$). Last segment may need zero-padding)

$$x_i(n) = x(n + iM) \quad ; \quad \begin{array}{l} n = 0, 1, \cdots M-1 \\ i = 0, 1, \cdots K-1 \end{array}$$

Step 2: For each segment, Periodogram is computed as $P_{xx}^{(i)}(f)$

$$P_{xx}^{(i)}(f) = \frac{1}{M} \left| \sum_{n=0}^{M-1} x_i(n)\, e^{-j2\pi f n} \right|^2 \quad ; \quad i = 0, 1, \cdots K-1$$

Step 3: These are averaged over all K segments to get Bartlett Power spectrum estimate $P_{xx}^{B}(f)$

$$P_{xx}^{(B)}(f) = \frac{1}{K} \sum_{i=0}^{K-1} P_{xx}^{(i)}(f)$$

FFT is used to calculate Fourier Transform.

Mean or expected value $E[P_{xx}^{B}]$, variance, quality factor Q and computational requirements (using FFT) are used to compare the performance of different methods

For Bartlett method

$$E[P_{xx}^{B}] = E[P_{xx}^{(i)}(f)]$$

$$var[P_{xx}^{B}] = \frac{1}{K} \cdot var[P_{xx}^{(i)}(f)]$$

thus we see that variance is divided by K and hence reduced. But frequency resolution is decreased ( $\Delta f$ increased) by K.

Note: The quality factor Q of $P_{xx}(f)$ is defined as

$$Q = \frac{\{E[P_{xx}(f)]\}^2}{Var[P_{xx}(f)]}$$

~~For Bartlett PS estimate, $Q_B$ ex~~

For Bartlett method it can be proved that $Q_B = \frac{N}{M} = K$

Taking 3-dB spectral width of rectangular window as $\frac{0.9}{M}$, the freq. resolution $\Delta f = \frac{0.9}{M} \Rightarrow M = \frac{0.9}{\Delta f}$

$\therefore Q_B = 1.1 N \Delta f$

<u>Computational requirement</u>: K number of M-length FFT to be computed. No. of computation for M-length FFT is $\frac{M}{2}\log_2(M)$

$\therefore$ total computation $= K \cdot \frac{M}{2} \cdot \log_2 M$

$$= \frac{N}{2}\log_2\left(\frac{0.9}{\Delta f}\right)$$

<u>Welch method</u>: This method is also aiming at reducing variance of the estimate by averaging the periodogram. Two modifications are made to the Bartlet method. The N-length signal is ~~segmente~~ segmented into <u>overlapping</u> segments. The other modification is that these segments are further ~~window~~ multiplied by ~~tapered~~ tapered window prior to ~~estimatio~~ computation of periodogram (This results in modified periodogram)

Step I: The N-length sequence is segmented into overlapping segments of length M. (N determines the frequency resolution $\Delta f$).

Let $x(n)$ be the N-length signal. The segmented sequence is

$$x_i(n) = x(n + iD), \quad n = 0, 1, \cdots M-1$$
$$i = 0, 1, \cdots L-1$$

$iD$ is starting point of $i^{th}$ segment. ~~If~~ Let $K = \frac{N}{M}$ = no. of segments if there is no overlap (same as K in Bartlet method). Then if $D = M$, no overlap, & $L = K$. If 50% overlap, then $D = \frac{M}{2}$ and $L = 2K$. ~~Usually~~ 50% overlap is commonly used.

Step 2: Each segment is multiplied by window $w(n)$. (commonly used window is triangular window). Periodogram is computed ~~to~~ resulting in modified periodogram $\tilde{P}^{(i)}_{xx}(f)$:

$$\tilde{P}^{(i)}_{xx}(f) = \frac{1}{MU} \left| \sum_{n=0}^{M-1} x_i(n)\, w(n)\, e^{-j2\pi fn} \right|^2 ; \quad i = 0, 1, \cdots L-1$$

U is a normalization factor for the spectral power of the window such that $\int_{-1/2}^{1/2} W(f)\, df = 1$

For this $U = \frac{1}{M} \sum_{n=0}^{M-1} w^2(n)$.

Step 3: These modified periodograms are averaged to get Welch Power spectrum estimate

$$P^{W}_{xx}(f) = \frac{1}{L} \sum_{i=0}^{L-1} \tilde{P}^{(i)}_{xx}(f)$$

The performance of this method is summarized below.

i) The mean or expected value of this estimate is

$$E[P_{xx}^{W}(f)] = E[\tilde{P}_{xx}^{(i)}(f)]$$

ii) Variance is $Var[P_{xx}^{W}(f)] \simeq \frac{1}{L}\Gamma_{xx}^{2}(f)$ for no overlap

$$\simeq \frac{9}{8L}\Gamma_{xx}^{2}(f) \text{ for } 50\% \text{ overalap}$$

& Triangular window.

( $\Gamma_{xx}(f)$ = true PSD of $x(n)$ ).

iii) ∴ Quality factor $Q_W$ is given by

$$Q_W = L = \frac{N}{M} \text{ for no overlap}$$

$$= \frac{16N}{9M} \text{ for } 50\% \text{ overlap.}$$

For triangular window, $\Delta f = \frac{1.28}{M}$

$$\therefore Q_W = 0.78 N \Delta f \text{ for no overlap}$$

$$= 1.39 N \Delta f \text{ for } 50\% \text{ overlap}$$

iv) Computational requirement:

$$\text{FFT length} = M = \frac{1.28}{\Delta f}$$

$$\text{No. of FFT} = 2\frac{N}{M} \quad (\text{for } 50\% \text{ overlap})$$

$$\therefore \text{No. of Computations} = \frac{2N}{M}\cdot\frac{M}{2}\log_2 M$$

$$= N\log_2\left(\frac{1.28}{\Delta f}\right)$$

In addition to this, since there is additional window multiplication where each segment needs $M$ multiplications. There are $L$ segments and with 50% overlap $L = 2\frac{N}{M}$. Hence total multiplications will be

$$2\frac{N}{M}\cdot M + N\log_2\left(\frac{1.28}{\Delta f}\right)$$

$$= N\log_2\left(\frac{5.12}{\Delta f}\right)$$

## Blackmon - Tukey method

~~In~~ This method aims at improving the quality of PS estimate by smoothening the periodogram. Also, indirect method is used for periodogram. The periodogram is Fourier transform of autocorrelation of the signal.

The autocorrelation $r^*_{xx}(m)$ of M-length sequence is first ~~xx~~ multiplied by window to get biased autocorrelation ~~$r_{xx}(m)$~~. ( Refer to page 6 )

( The data record length is assumed to be $N$ to get required resolution $\Delta f$. This is segmented into smaller segments of length $M$.) :

~~The bias to $r_{xx}(m)$ ex ensures~~. It is to be noted that, with $r'_{xx}(m)$ for large lag, the variance is high. This problem is solved by ~~multiplica~~ window multiplication (biasing) as for larger lag, acf is given lesser weightage. Thus the window smoothens the Power spectrum.

$$\cancel{P^{BT}_{xx}(f) = \sum_{m=-(M-1)}^{(M-1)} r^*_{xx}(m)\, w(m)\, e^{-j}}$$

$$P^{BT}_{xx}(f) = \sum_{m=-(M-1)}^{(M-1)} r_{xx}(m)\, w(m)\, e^{-j2\pi fm}$$

The window function $w(m)$ has length $2M-1$ and is zero for $|m| \geq M$. The spectrum $W(f)$ should be such that $W(f) \geq 0$, $|f| \leq \frac{1}{2}$.

$$Q_{BT} = 2.34\, N \Delta f$$

No. of computation $= N \log_2\left(\frac{1.28}{\Delta f}\right)$

Advantages of non-parametric methods:

i) Low variance

ii) Simple and easy to compute using FFT.

Drawbacks:

i) Require availability of quite large data record (larger $N$) to get good frequency resolution

ii) Suffer from spectral leakage which may mask weaker signals. If we try to reduce leakage, it results in poor resolution.

iii) It is assumed that $r_{xx}(m) = 0,\ m \geq N$ which severely limits the quality of the estimated Power spectrum.

Parametric methods: In these methods, only a short segment of data is used. (These methods are specifically useful when only short segment of data is available. (Note that even when large length record is available, because of non stationary conditions of random process, we are forced to use smaller segments).

The autocorrelation $r_{xx}(m)$ for $m \geq N$ is not assumed to be zero, but extrapolated by knowing some a-priori information on how data is generated. For this, a model for the data generation is constructed from certain parameters extracted from short length of the available data. From this model parameters, the power spectrum is estimated.

In these methods, we assume $x(n)$ be the output of linear system characterized by the system function

$$H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^{q} b_k z^{-k}}{1+\sum_{k=1}^{p} a_k z^{-k}} \quad \text{---} \; (*)$$

The corresponding difference equation is

$$x(n) = -\sum_{k=1}^{p} a_k x(n-k) + \sum_{k=0}^{q} b_k w(n-k)$$

where $w(n)$ is the input. The input is actually not known, but taken as zero mean white ~~noise~~ noise process with variance $\sigma_w^2$.

The parameters (model parameters) $a_k$ and $b_k$ are obtained from short segment of available data. (construction of the signal model).

Then power spectrum is computed as

$$PSD = \sigma_w^2 \frac{|B(f)|^2}{|A(f)|^2} \quad \text{---} \; (*)'$$

($B(f) = B(z)\big|_{z=e^{j2\pi f}}$ and similarly $A(f)$.)

Step 1: observe small data record $x(n)$; $0 \le n \le N-1$ ($N$ is small here)

Step 2: Estimate model parameters $a_k$ & $b_k$ and get model $H(z)$ as in $(*)$

Step 3: Get PS using $(*)'$.

The model given in $(*)$ is known as Autoregressive Moving Average Model (ARMA) of order $(p, q)$. If $q = 0$ & $b_0 = 1$, then $H(z) = \frac{1}{A(z)}$. This

is referred to as auto regressive (AR) model of order p. (This is all-pole model)
Similarly if $A(z) = 1$, then $H(z) = B(z)$. This model is moving average model (MA) which is an all-zero model of order q.

AR model is largely used. Some important AR model estimation methods are

i) Yule-Walker method
ii) Burg method
iii) Least-square method
iv) Sequential estimation method.

—— x ——