Functions of two dimensional random variables

Let (X,Y) be a continuous two dimensional random variable with pdf $f(x,y)$. If $Z=H_1(X,Y)$ is a continuous function of (X,Y), then Z will be a continuous (one-dimensional) random variable. In order to find the pdf of Z, we shall follow the procedure discussed below.

To find the pdf of $Z=H_1(X,Y)$, we first introduce a second random variable, say $W=H_2(X,Y)$, and obtain the joint pdf of Z and W, say $k(z,w)$. From the knowledge of $k(z,w)$, we can then obtain the desired pdf of Z, say $g(z)$, by simply integrating $k(z,w)$ with respect to w. That is, $g(z)=\int_{-\infty}^{\infty} k(z,w)\,dw$.

Two problems which arise here are

i. how to find the joint pdf $k(z,w)$ of Zand W
ii. how to choose the appropriate random variable $W=H_2(X,Y)$

To resolve these problems, let us simply state that we usually make the simplest possible choice for W as it plays only an intermediate role. In order to find the joint pdf $k(z,w)$, we need the following theorem.

Theorem:

Suppose that (X,Y) is a two-dimensional continuous random variable with joint pdf $f(x,y)$. Let $Z=H_1(X,Y)$ and $W=H_2(X,Y)$and assume that the functions H_1 and H_2 satisfy the following conditions:

i. The equations $z=H_1(x,y)$ and $w=H_2(x,y)$ may be uniquely solved for x and $y$in terms of z and w, say $x=G_1(z,w)$ and $y=G_2(z,w)$.
ii. The partial derivatives $\frac{\partial x}{\partial z}, \frac{\partial x}{\partial w}, \frac{\partial y}{\partial z}$ and $\frac{\delta y}{\delta w}$exist and are continuous.

Then the joint pdf (Z,W), say $k(z,w)$, is given by the following expression:

$k(z,w)=f[G_1(z,w),G_2(z,w)]\vee J(z,w)\vee ¿$,

where $J(z,w)$ is the following 2×2 determinant:

$$J(z,w)=\begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\delta y}{\delta w} \end{vmatrix}$$

This determinant is called the 'Jacobian' of the transformation $(x,y)\rightarrow(z,w)$ and is sometimes denoted by $\frac{\delta(x,y)}{\delta(z,w)}$. We note that $k(z,w)$ will be nonzero for those values of (z,w) corresponding to values of (x,y) for which $f(x,y)$ is nonzero.

Problems

1. Suppose that X and Y are two independent random variables having pdf $f(x)=e^{-x}, 0\le x\le\infty$ and $g(y)=2e^{-2y}, 0\le y\le\infty$. Find the pdf of X+Y

   Solution:

   Since X and Y are independent, the joint pdf of (X,Y) is given by,

$$f(x,y)=f(x)g(y)=2e^{-(x+2y)}, 0\le x,y\le\infty$$

   Let $Z=X+Y$ and $W=Z$, that is $Y=W$ and $X=Z-W$.

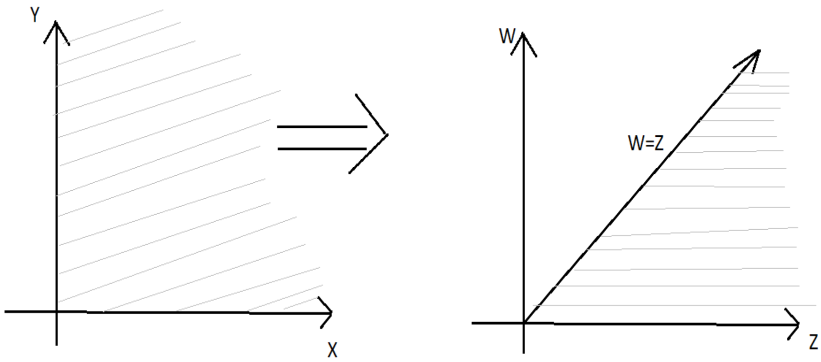
   The Jacobian $J=\begin{vmatrix}\frac{\partial x}{\partial z} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial z} & \frac{\delta y}{\delta w}\end{vmatrix}=\begin{vmatrix}1 & -1\\ 0 & 1\end{vmatrix}=1$

   Thus joint pdf of (W,Z) is,

$$k(z,w)=f(x,y)|J|=2e^{-(x+2y)}=2e^{-(z+w)}$$

$$0\le y\le\infty\Rightarrow 0\le w\le\infty$$

$$0\le x\le\infty\Rightarrow 0\le z-w\le\infty\Rightarrow w\le z\le\infty$$



(X,Y) (W,Z)

   Thus $k(w,z)=2e^{-(z+w)}, 0\le w\le z\le\infty$

   The required pdf of z, $h(z)=\int_{w=0}^{z} 2e^{-(z+w)}dw$

¿$2(e^{-z}-e^{-2z}), 0\le z\le\infty$.

2. If $X\ \ N(0,\sigma^2)$, $Y\ \ N(0,\sigma^2)$ and X, Y are independent. Find the pdf of $R=\sqrt{X^2+Y^2}$

   Solution:

   Pdf of $X: f(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-x^2/2\sigma^2}, -\infty\le x\le\infty$

   Pdf of $Y: g(y)=\frac{1}{\sqrt{2\pi}\sigma}e^{-y^2/2\sigma^2}, -\infty\le y\le\infty$

   Since X and Y are independent, the joint pdf of (X,Y) is given by,

$$f(x,y)=f(x)g(y)=\frac{1}{2\pi\sigma^2}e^{-\text{¿¿}}$$

   Let $R=\sqrt{X^2+Y^2}$ and $\theta=\tan^{-1}\left(\frac{X}{Y}\right)$, that is $X=R\cos\theta$ and $Y=R\sin\theta$ and the Jacobian $J=R$.

Thus joint pdf of (R,θ) is,

$k(r,\theta)=f(x,y)|J|=\frac{R}{2\pi\sigma^2}e^{-R^2/2\sigma^2}, R\geq 0, 0\leq\theta\leq 2\pi$

The required pdf of z, $h(z)=\int_{\theta=0}^{2\pi}\frac{R}{2\pi\sigma^2}e^{-R^2/2\sigma^2}d\theta$

¿ $\frac{R}{\sigma^2}e^{-R^2/2\sigma^2}, R\geq 0.$

3. If X_1, X_2 are independent and have standard normal distribution $X_1, X_2\ \ N(0,1)$. Find the pdf of $\frac{X_1}{X_2}$.

   Solution:

   Pdf of $X_1 : f(x_1)=\frac{1}{\sqrt{2\pi}}e^{-x_1^2/2}, -\infty\leq x_1\leq\infty$

   Pdf of $X_2 : g(x_2)=\frac{1}{\sqrt{2\pi}}e^{-y_1^2/2}, -\infty\leq y_1\leq\infty$

   Since X_1, X_2 are independent, the joint pdf of $(X$¿¿$1, X_2)$¿is given by,

   $$f(x_1,x_2)=\frac{1}{2\pi}e^{-(x_1^2+x_2^2)/2}, -\infty\leq x_1, y_1\leq\infty$$

   Let $Z=\frac{X_1}{X_2}$and $W=X_2$, that is $X_2=W$ and $X_1=ZW$.

   The Jacobian $J=\begin{vmatrix} w & z \\ 0 & 1 \end{vmatrix}=w$

   Thus joint pdf of (W,Z) is,

   $k(z,w)=$¿$w\vee\frac{¿}{2\pi}e^{-w^2(1+z^2)/2}$,¿ $-\infty\leq w, z\leq\infty.$

   The required pdf of Z, $h(z)=\int_{-\infty}^{\infty}$¿$w\vee\frac{¿}{2\pi}e^{-w^2(1+z^2)/2}dw$¿

   $$¿\frac{2}{2\pi}\int_0^{\infty}|w|e^{-w^2(1+z^2)/2}dw$$

   On substitution: $-w^2(1+z^2)/2=t$

   $-w(1+z^2)dw=dt$

   We get, $h(z)=\frac{1}{\pi}\int_0^{\infty}\frac{e^{-t}}{1+z^2}dt=\frac{1}{\pi(1+z^2)}, -\infty\leq z\leq\infty.$

4. The joint pdf of the random variable(X,Y) is given by

   $$f(x,y)=\frac{x}{2}e^{-y}, 0<x<2, y>0$$

   Find the pdf of $X+Y$

   Solution:

   Let $Z=X+Y$ and $W=Z$, that is $Y=W$ and $X=Z-W$.

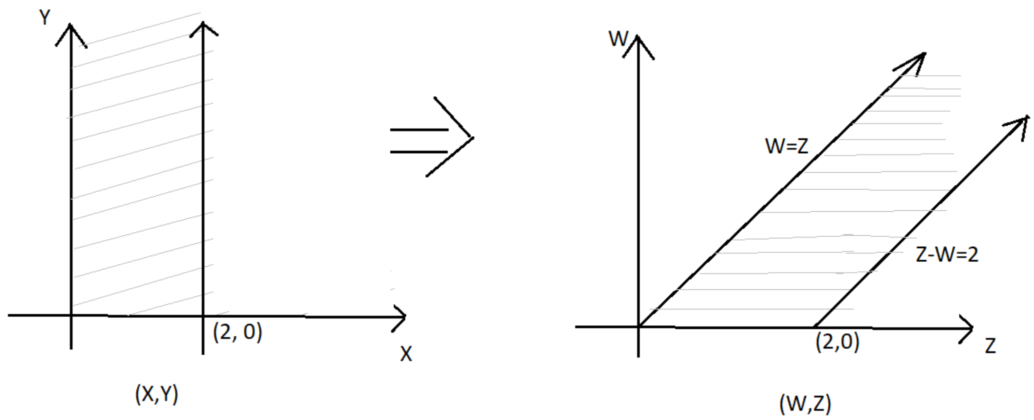
The Jacobian $J=1$

Thus joint pdf of (W, Z) is, $k(z,w)=f(x,y)|J|=\frac{z-w}{2}e^{-w}$

$$0 \le y \le \infty \Rightarrow 0 \le w \le \infty$$

$$0 \le x \le 2 \Rightarrow 0 \le z-w \le 2 \Rightarrow w \le z \le 2+w$$

$k(z,w)=\frac{z-w}{2}e^{-w}, 0 \le w \le z \le 2+w$



The required pdf of z, $h(z)=\begin{cases} \int_0^z \left(\frac{z-w}{2}\right)e^{-w}\,dw, when\, 0<z<2 \\ \int_{z-2}^z \left(\frac{z-w}{2}\right)e^{-w}\,dw, when\, 2<z<\infty \end{cases}$

$$h(z)=\begin{cases} \frac{1}{2}\left(z+e^{-z}-1\right), when\, 0<z<2 \\ \frac{1}{2}\left(e^{z}+e^{2-z}\right), when\, 2<z<\infty \end{cases}$$

5. Let $f(x)=\begin{cases} \alpha^{-2}e^{-\left(\frac{x+y}{2}\right)} & ;x,y>0,\alpha>0 \\ 0\,; elsewhere \end{cases}$. Find the distribution of $\frac{x-y}{2}$.

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