Fast Fourier Transform Algorithms - DIFFFT

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Decimation In Frequency FFT (DIFFFT) Algorithm

- Decompose the computation into successively smaller DFT computations.
- We exploit both the symmetry and periodicity property of the complex exponential $W_N^{nk} = e^{-j\left(\frac{2\pi}{N}\right)nk}$
- In this algorithm the decomposition is based on the output sequence $X(k)$ into successively smaller subsequences

Let N be equal to 2^{ν}, where ν is an integer

Consider a finite length sequence $x(n)$

Let $L \le N$ be the length of the signal $x(n)$

Then we can define N-pt DFT of $x(n)$ is

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk}, \quad k = 0, 1, \ldots N-1 \qquad (1)$$

Even numbered frequency points can be written as

$$X(2r) = \sum_{n=0}^{N-1} x(n) W_N^{n2r}, \quad k = 0, 1, \ldots N/2-1 \qquad (2)$$

Illy odd frequency points can be written as

$$X(2r+1) = \sum_{n=0}^{N-1} x(n) W_N^{(2r+1)n} \quad k = 0, 1, 2, \ldots \frac{N}{2}-1 \qquad (3)$$

consider eqn (2), $X(2r) = \sum_{n=0}^{N-1} x(n) W_N^{2rn}$

We can expand it as follows

$$X(2r) = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{2nr} + \sum_{n=\frac{N}{2}}^{N-1} x(n) W_N^{2nr} \quad \text{—— (4)}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{2nr} + \sum_{n=0}^{\frac{N}{2}-1} x(n+\tfrac{N}{2}) W_N^{2r(n+\frac{N}{2})} \quad \text{—— (5)}$$

$$W_N^{2r(n+\frac{N}{2})} = W_N^{(2rn+rN)} = W_N^{2rn} \cdot W_N^{rN} = W_N^{2rn} = W_{N/2}^{rn} \quad \text{—— (6)}$$

$\therefore$ Eqn (5) can be written as

$$X(2r) = \sum_{n=0}^{N/2-1} x(n) \cdot W_{N/2}^{rn} + \sum_{n=0}^{\frac{N}{2}-1} x(n+N/2) W_{N/2}^{rn}$$

$$X(2r) = \sum_{n=0}^{N/2-1} \Big[\underbrace{x(n)+x(n+\tfrac{N}{2})}_{g(n)}\Big] W_{N/2}^{rn} \quad \text{—— (7)}$$

Consider now the odd numbered frequency points

$$X(2r+1) = \sum_{n=0}^{N-1} x(n)\, W_N^{(2r+1)n} \qquad r = 0, 1, \cdots \tfrac{N}{2}-1$$

$$X(2r+1) = \sum_{n=0}^{\frac{N}{2}-1} x(n)\, W_N^{(2r+1)n} + \sum_{n=N/2}^{N-1} x(n)\, W_N^{(2r+1)n} \quad \text{—— (8)}$$

Consider the second term in (8)

$$\sum_{n=N/2}^{N-1} x(n)\, W_N^{(2r+1)n} = \sum_{n=0}^{N/2-1} x(n+\tfrac{N}{2})\, W_N^{(2r+1)(n+\frac{N}{2})}$$

$$= \sum_{n=0}^{N/2-1} x(n+\tfrac{N}{2})\, W_N^{2rn} \cdot W_N^{2r\frac{N}{2}} \cdot W_N^{n} \cdot W_N^{N/2}$$

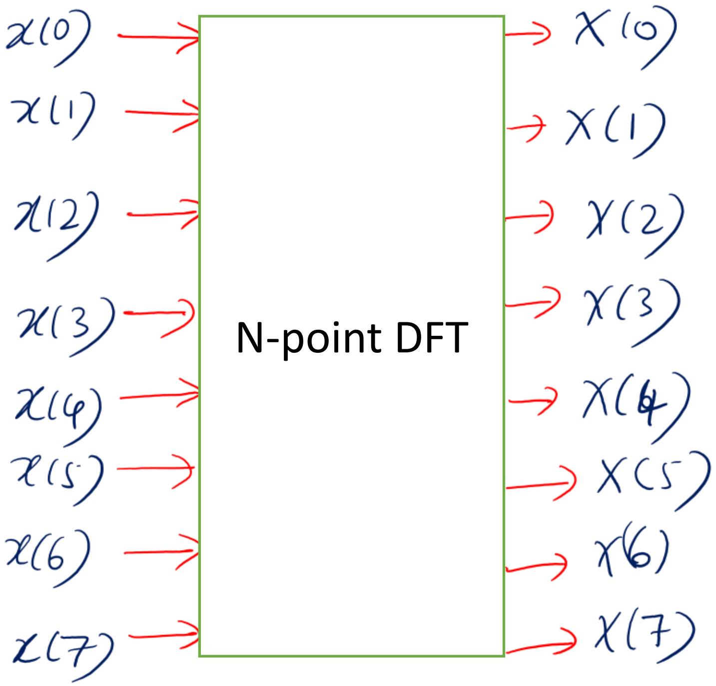
$$= \underbrace{W_N^{2rN/2}}_{} \cdot \underbrace{W_N^{N/2}}_{} \cdot \sum_{n=0}^{\frac{N}{2}-1} x(n+\tfrac{N}{2})\, W_N^{n} \cdot W_{N/2}^{rn}$$

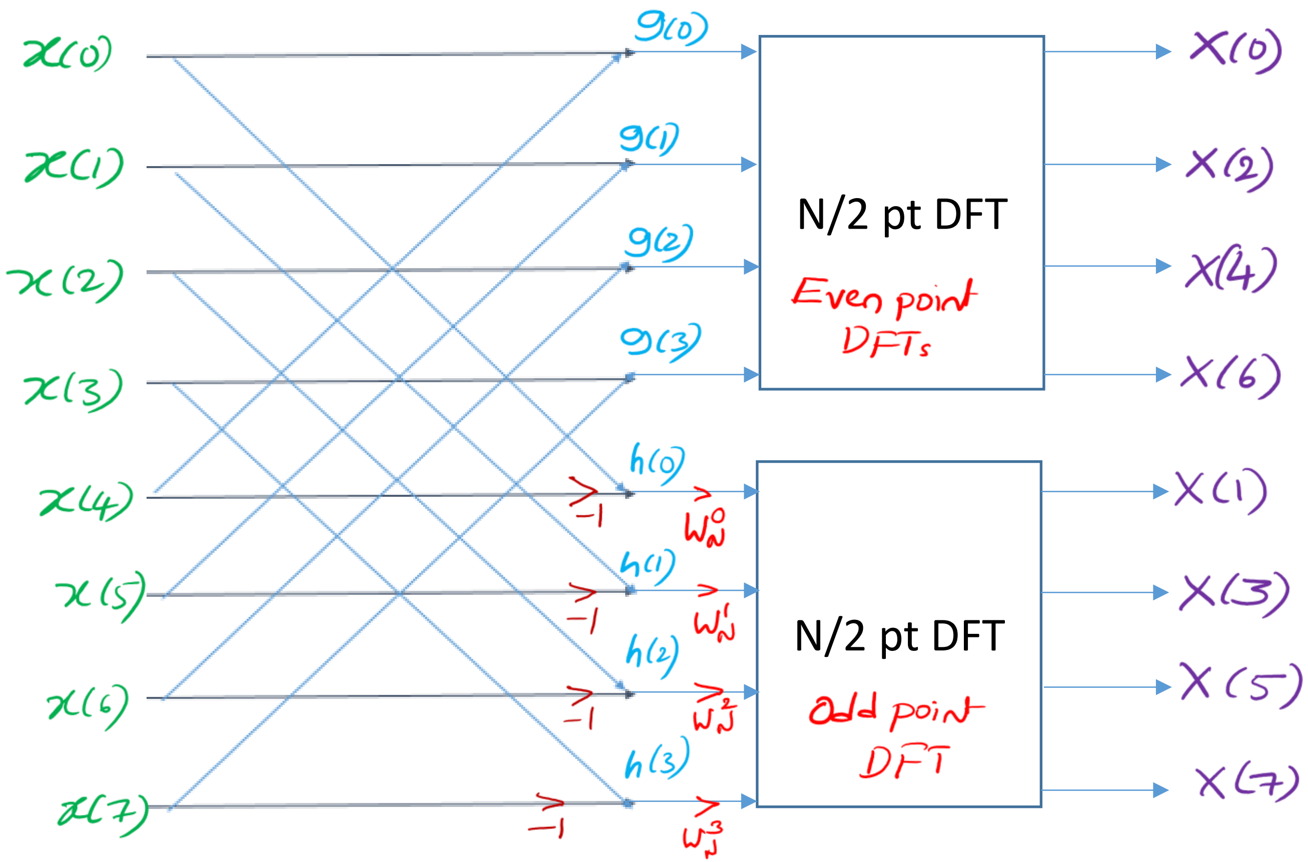
$$= (1) \cdot (-1) \sum_{n=0}^{N/2-1} x(n+N/2)\, W_N^{n} \cdot W_{N/2}^{rn} \quad \text{—— (9)}$$

Therefore eqn (8) can be rewritten as

$$X(2r+1) = \sum_{n=0}^{N/2-1} x(n)\, W_N^{(2r+1)n} + (-1) \sum_{n=0}^{N/2-1} x\left(n+\frac{N}{2}\right) W_N^{(2r+1)n} \quad \text{——— (10)}$$

$$X(2r+1) = \sum_{n=0}^{N/2-1} \Big[\underbrace{x(n) - x(n+N/2)}_{h(n)}\Big]\, W_N^{n} \cdot W_{N/2}^{r} \quad \text{——— (11)}$$





Equations (7) and equation (11) can be re written as

$$G(r) = X(2r) = \sum_{n=0}^{N/2-1} g(n)\, W_{N/2}^{rn} \quad \text{---} (12)$$

&

$$X(2r+1) = \sum_{n=0}^{N/2-1} h(n)\, W_N^{n} \cdot W_{N/2}^{rn} \quad \text{---} (13)$$

Now we consider eqn (12)

$$G(l) = \sum_{n=0}^{N/2-1} g(n)\, W_{N/2}^{ln}$$

Then even DFT points can be

$$G(2r) = \sum_{n=0}^{N/2-1} g(n)\, W_{N/2}^{2rn} \qquad r = 0, 1, \ldots \frac{N}{4}-1 \quad \text{---} (14)$$

Odd DFT points are

$$G(2r+1) = \sum_{n=0}^{N/2-1} g(n)\, W_{N/2}^{(2r+1)n} \qquad r = 0, 1, \ldots N/4-1 \quad \text{---} (15)$$

Consider now eqn (14)

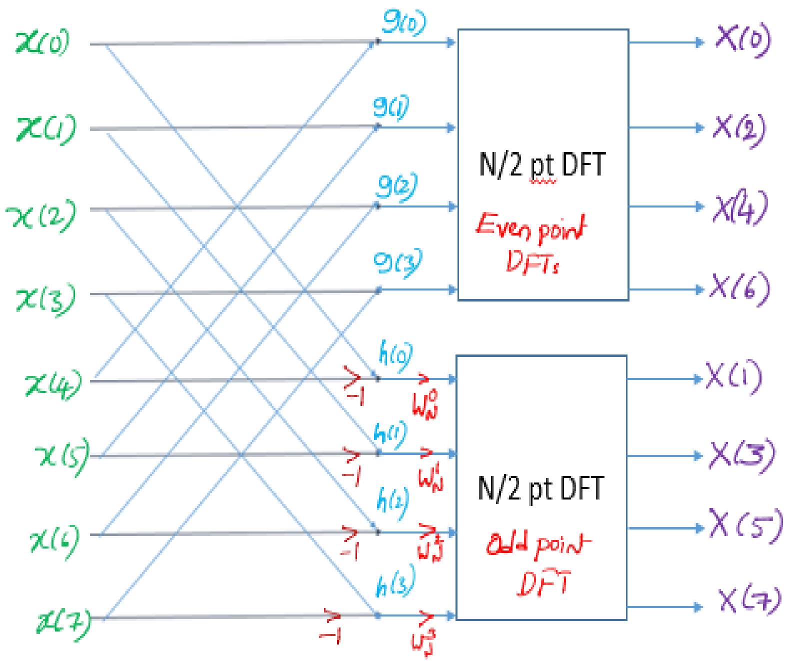
$$G(2r) = \sum_{n=0}^{N/2-1} g(n)\, W_{N/2}^{2rn} = \sum_{n=0}^{N/4-1} g(n)\, W_{N/2}^{2rn} + \sum_{n=N/4-1}^{N/2-1} g(n)\, W_{N/2}^{2rn}$$

$$= \sum_{n=0}^{N/4-1} g(n)\, W_{N/4}^{rn} + \sum_{n=0}^{N/4-1} g(n+N/4) \cdot W_{N/2}^{2r(n+N/4)}$$

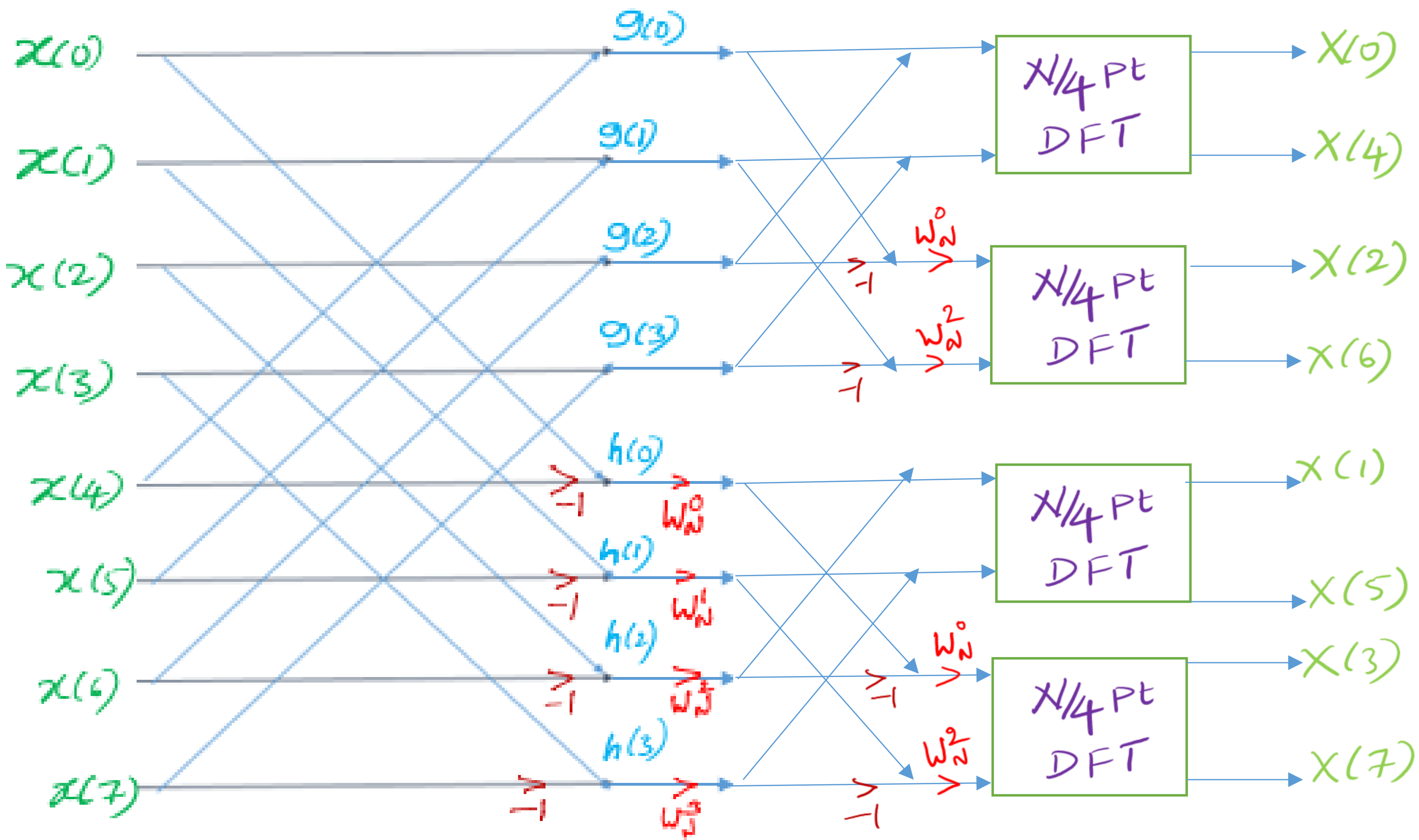
$$= \sum_{n=0}^{N/4-1} g(n)\, W_{N/4}^{rn} + \sum_{n=0}^{N/4-1} g(n+N/4)\, W_{N/4}^{rn} \cdot \underbrace{W_{N/2}^{2rN/4}}_{1}$$

$$G(2r) = \sum_{n=0}^{N/4-1} \left[g(n) + g\left(n+\frac{N}{4}\right)\right] W_{N/4}^{rn}$$

Similarly $G(2r+1) = \sum_{n=0}^{(N/4-1)} \left[g(n) - g\left(n-\frac{N}{4}\right)\right] W_{N/2}^{n} \cdot W_{N/4}^{rn}$



x(0)
x(1)
x(2)
x(3)
x(4)
x(5)
x(6)
x(7)
g(0)
g(1)
g(2)
g(3)
h(0)
h(1)
h(2)
h(3)
-1
N/2 pt DFT
Even point DFTs
N/2 pt DFT
Odd point DFT
X(0)
X(2)
X(4)
X(6)
X(1)
X(3)
X(5)
X(7)



x(0)
x(1)
x(2)
x(3)
x(4)
x(5)
x(6)
x(7)
g(0)
g(1)
g(2)
g(3)
h(0)
h(1)
h(2)
h(3)
-1
N/4 pt DFT
X(0)
X(4)
X(2)
X(6)
X(1)
X(5)
X(3)
X(7)

$$G(r) = \sum_{n=0}^{N/4-1} \underbrace{\left[g(n) + g\left(n+\tfrac{N}{4}\right)\right]}_{x(n)} W_{N/4}^{rn} \rightarrow r = 0, 1, \dots N/4-1$$

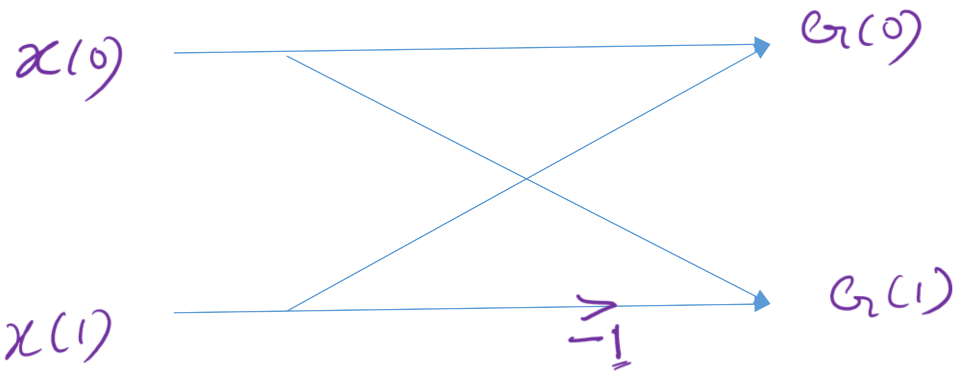
For $N = 8$, $$G(r) = \sum_{n=0}^{1} x(n)\, W_2^{rn} \qquad r = 0, 1$$

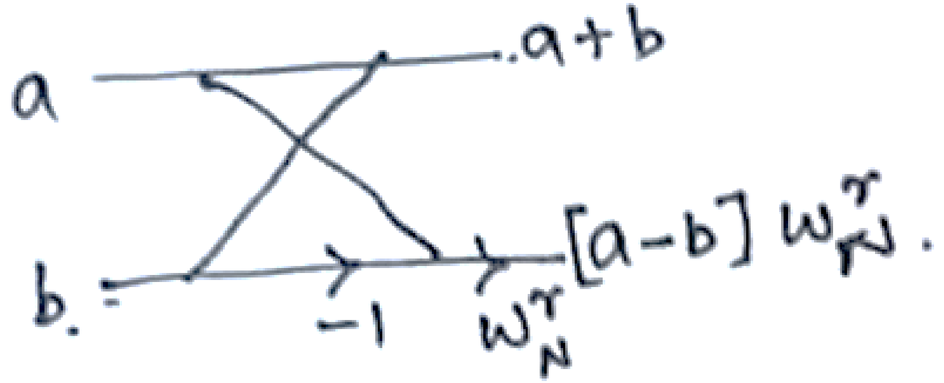
$$G(r) = x(0)\, W_2^0 + x(1)\, W_2^r$$

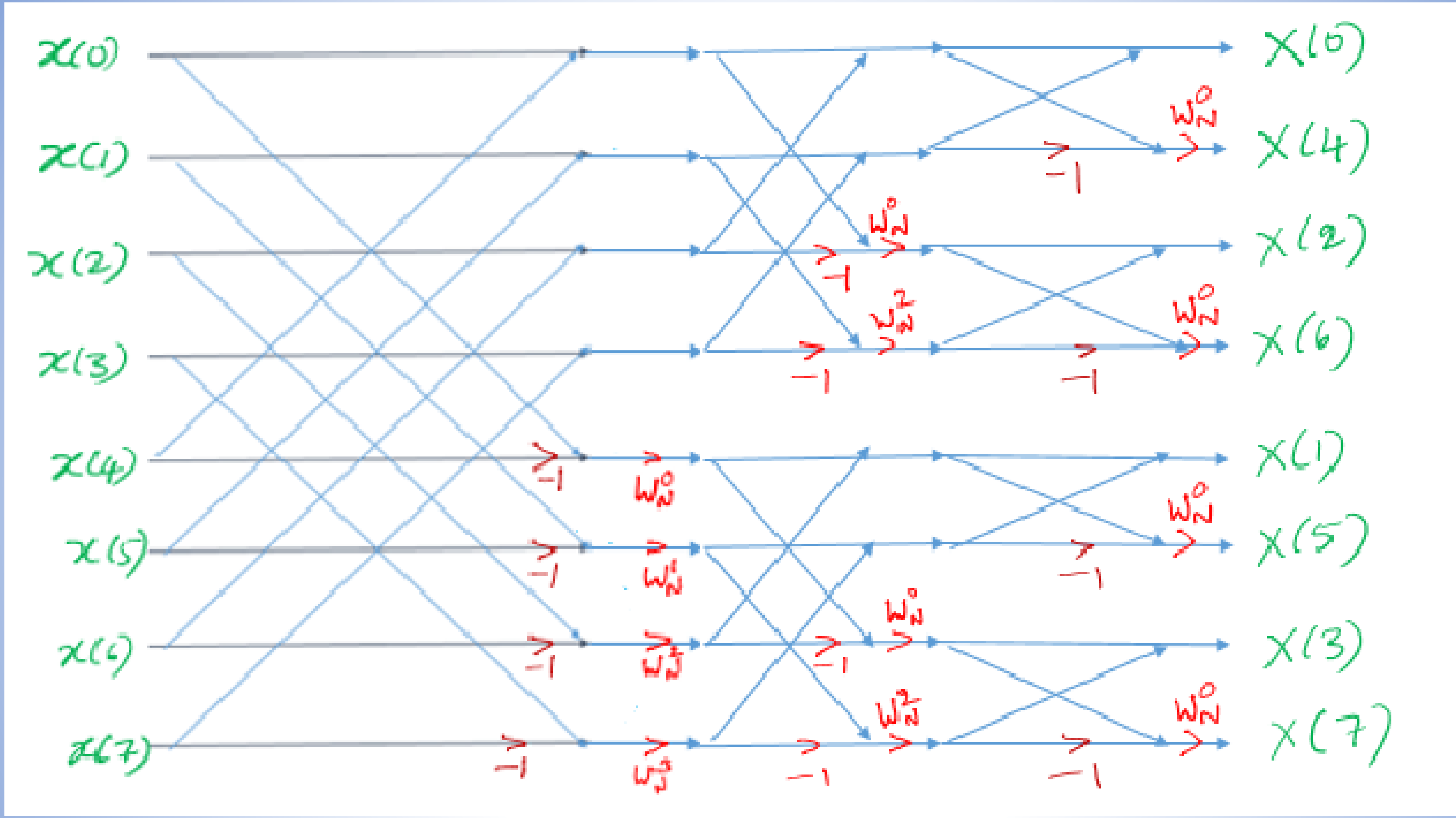
$$G(0) = x(0) + x(1)$$

$$G(1) = x(0) + x(1)\, W_2^1$$

$$= x(0) - x(1)$$

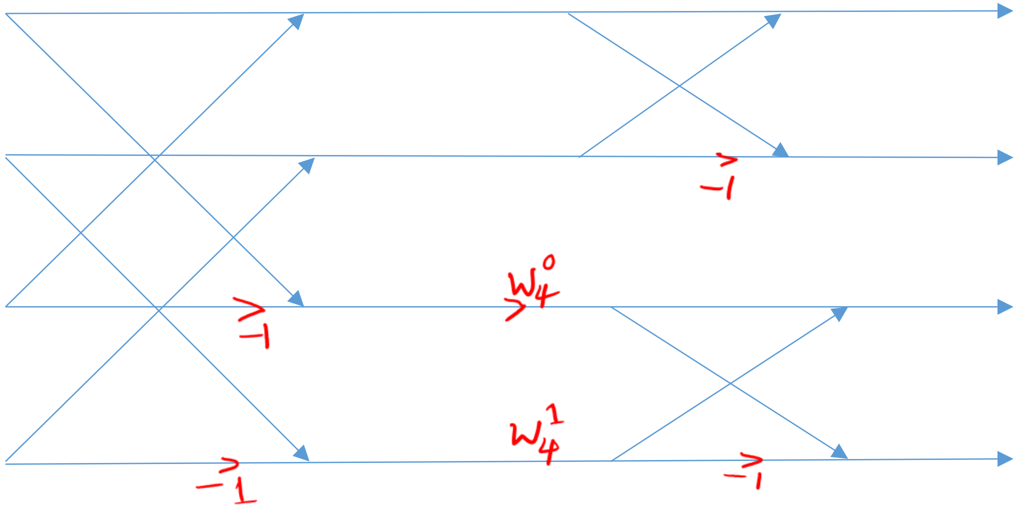


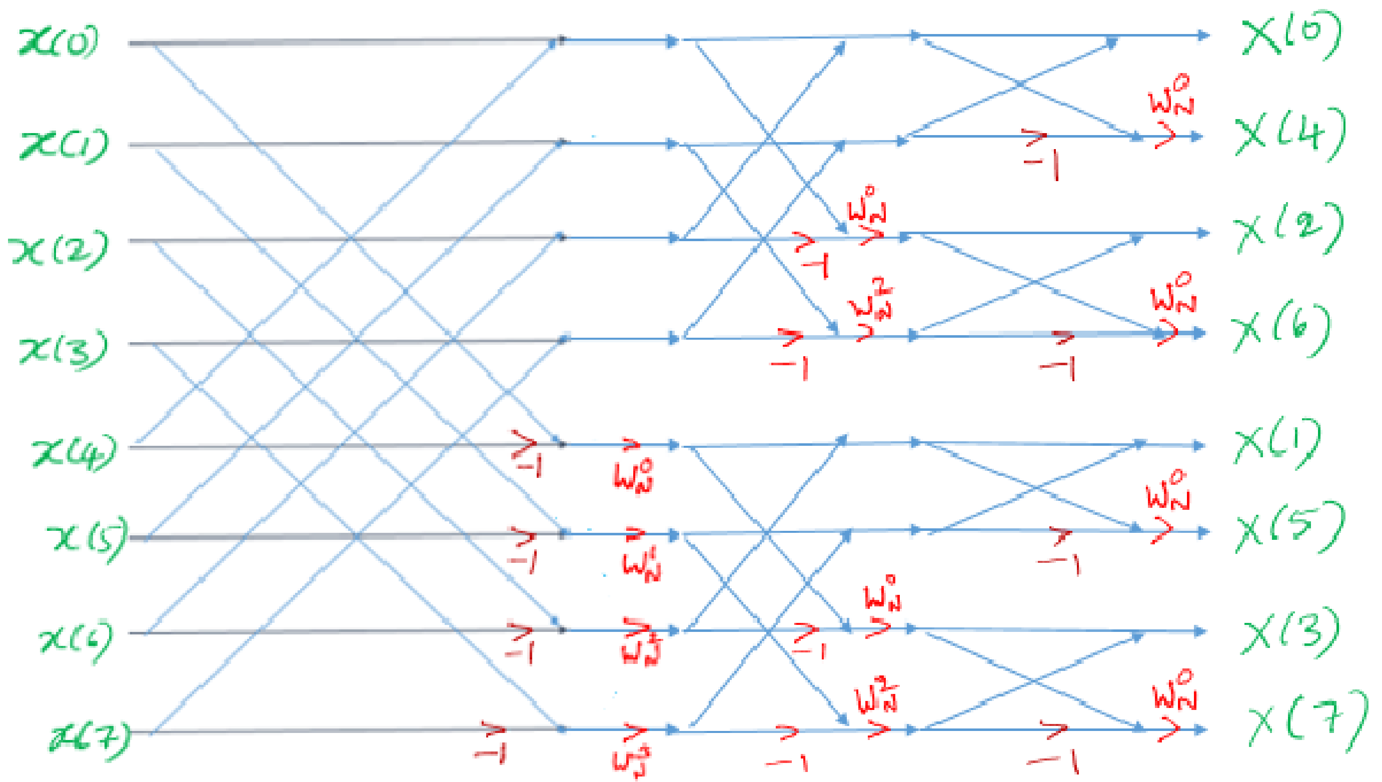




x(0)
x(1)
x(2)
x(3)
x(4)
x(5)
x(6)
x(7)
X(0)
X(4)
X(2)
X(6)
X(1)
X(5)
X(3)
X(7)

Compute 4-pt DFT of a sequence $x(n) = \{1, 2, 3, 4\}$ using DIFFFT algorithm





Number of stages in the signal flow graph = $\log_2 N$

Number of Butterflies in the signal flow graph = $\frac{N}{2}\log_2 N$

Number of complex additions in each butterfly = 2

Number of complex multiplications in each butterfly = 1

Total number of complex additions and multiplications = ? $\left(N\log_2 N + \frac{N}{2}\log_2 N\right)$


Thank
you