Functions of One dimensional random variables

If X is a discrete random variable and $Y=H(X)$ is a continuous function of X, then Y is also a Discrete Random Variable.

Eg:

X	-1	0	1
$P(x)$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

Suppose $Y=3X+1$,then pmf of Y is given by

Y	-2	1	4
$P(y)$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

Suppose $Y=X^2$, then pmf of Y is

Y	1	0
$P(y)$	$\frac{1}{2}$	$\frac{1}{2}$

Suppose X is a continuous random variable with pdf $f(x)$ and $H(X)$ is a continuous function of X. Then Y is a continuous random variable. To obtain pdf of Y we follow the following steps.

1. Obtain cdf of Y, i.e., $G(y)=P(Y\leq y)$.
2. Differentiate $G(y)$ with respect to y to get pdf of y i.e., $g(y)$.
3. Determine the range space of Y such that $g(y)>0$.

Problems:

1. If $f(x)=\begin{cases} 2x; & 0<x<1 \\ 0; & Otherwise \end{cases}$, and $Y=3X+1$, find pdf of Y.

Soln: $G(y)=P(Y\leq y)=P(3X+1\leq y)=P\left(X\leq\frac{y-1}{3}\right)$

$$G(y)=\int_0^{\frac{y-1}{3}} 2x dx=\left(\frac{y-1}{3}\right)^2.$$

$$g(y)=G'(y)=\frac{2(y-1)}{9}.$$

$$0<x<1 \Longrightarrow 0<\frac{y-1}{3}<1 \Longrightarrow 1<y<4.$$

Therefore, $g(y)=\begin{cases} \frac{2(y-1)}{9}; & 1<y<4 \\ 0; & Otherwise \end{cases}$.

2. If $f(x)=\begin{cases} 2x; & 0<x<1 \\ 0; & Otherwise \end{cases}$, and $Y=e^{-X}$, find pdf of Y.

Soln: $G(y)=P(Y\le y)=P(e^{-X}\le y)=P\left(\log_e \frac{1}{y}\le X\right)$

$G(y)=\int_{\log_e \frac{1}{y}}^{1} 2x dx=1-\left(\log_e \frac{1}{y}\right)^2$.

$g(y)=G'(y)=\frac{2}{y}\log_e \frac{1}{y}$.

$0<x<1 \Longrightarrow 0<\log_e \frac{1}{y}<1 \Longrightarrow \frac{1}{e}<y<1.$

Therefore, $g(y)=\begin{cases} \frac{2}{y}\log_e \frac{1}{y}; & \frac{1}{e}<y<1 \\ 0; & Otherwise \end{cases}$.

Result: Let X be a continuous random variable with pdf $f(x)$. Let $Y=X^2$. Then pdf of Y is $g(y)=\frac{1}{2\sqrt{y}}\left(f(\sqrt{y})+f(-\sqrt{y})\right)$

Example 1: Suppose $f(x)=\begin{cases} 2x e^{-x^2}; & 0<x<\infty \\ 0; & Otherwise \end{cases}$. Find pdf of $Y=X^2$.

Soln:

$g(y)=\frac{1}{2\sqrt{y}}\left(f(\sqrt{y})+f(-\sqrt{y})\right)=\frac{1}{2\sqrt{y}}\left(2\sqrt{y}e^{-y}+0\right)=e^{-y}; 0<x<\infty$.

Example 2: Suppose $f(x)=\begin{cases} \frac{2}{9}(x+1); & -1<x<1 \\ 0; & Otherwise \end{cases}$. Find pdf of $Y=X^2$.

Soln:

$g(y)=\frac{1}{2\sqrt{y}}\left(f(\sqrt{y})+f(-\sqrt{y})\right)=\frac{1}{2\sqrt{y}}\left(\frac{2(\sqrt{y}+1)}{9}+\frac{2(-\sqrt{y}+1)}{9}\right)=\frac{2}{9\sqrt{y}}; 0<x<1.$

Theorem: Let X be a continuous random variable with pdf $f(x)$. Suppose $Y=H(X)$ is a strictly monotone (increasing or decreasing) function of X, then pdf of Y is given by

$g(y)=f(x)\left|\frac{dx}{dy}\right|$ where $x=H^{-1}(y)$.

Example:

1. Suppose X is uniformly distributed over (0,1), find pdf of $Y=\frac{1}{X+1}$.

   Soln: We know that Y is strictly monotone.

   $f(x)=\begin{cases}1; & 0<x<1\\ 0; & Otherwise\end{cases}$.

   Note that $X=\frac{1}{Y}-1$. $\Rightarrow f(x)=f\left(\frac{1}{Y}-1\right)=1.$

   $\left|\frac{dx}{dy}\right|=\frac{1}{y^2}$.

   Therefore, $g(y)=\frac{1}{y^2}; \frac{1}{2}<y<1.$

2. If X is uniformly distributed over $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$, find the pdf of $Y=tanX$.(Or show that $Y=tanX$ follows Cauchy's distribution).

   Soln: Given $f(x)=\begin{cases}\frac{1}{\pi}; & \frac{-\pi}{2}<x<\frac{\pi}{2}\\ 0; & Otherwise\end{cases}$.

   We know that Y is strictly monotone.

   Then $X=\tan^{-1}Y \Rightarrow f$ ¿¿ And $\left|\frac{dx}{dy}\right|=\frac{1}{1+y^2}$.

   Therefore, $g(y)=\frac{1}{\pi}\frac{1}{1+y^2}; -\infty<y<\infty.$

3. If $X \;\; N(\mu,\sigma^2)$, then show that $Z=\frac{X-\mu}{\sigma} \;\; N(0,1)$ and $Y=Z^2 \;\; \chi^2(1)$.

   Soln: $G(z)=P(Z\leq z)=P\left(\frac{X-\mu}{\sigma}\leq z\right)=P(\sigma z+\mu\geq x)$

   $$G(z)=F(\sigma z+\mu).$$

   $$g(z)=G'(z)=F'(\sigma z+\mu)\sigma=f(\sigma z+\mu)\sigma=\frac{1}{\sqrt{2\pi}}e^{\frac{-z^2}{2}} \;\; N(0,1).$$

   Now, $g(y)=\frac{1}{2\sqrt{y}}\left(f(\sqrt{y})+f(-\sqrt{y})\right)=\frac{1}{2\sqrt{y}}\left(\frac{1}{\sqrt{2\pi}}e^{\frac{-y}{2}}+\frac{1}{\sqrt{2\pi}}e^{\frac{-y}{2}}\right)$

   $$g(y)=\frac{1}{\sqrt{y}\sqrt{2\pi}}e^{\frac{-y}{2}}.$$

   Hence, $g(y) \;\; \chi^2(1)$.

   Extra Problem:

1. A random variable X having Cauchy distribution. Show that $1/X$ also has Cauchy distribution.