Linear Phase FIR Filter Window method

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Methods for design of FIR filters:

1) using window functions

2) using frequency sampling technique.

$H_d(\omega)$ is the desired freq resp.

$\downarrow$ IDTFT

$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega$, is the desired sample response.

$\downarrow$ To make it finite, (FIR) truncate at $n = M-1$

$h(n) = h_d(n) \cdot w(n)$ ← window function $(0, \dots M-1)$

eg. Rectangular window:

$$\begin{cases} h_d(n) & , n = 0, \dots M-1 \\ 0 & \text{otherwise} \end{cases}$$

$\downarrow$

$H(\omega)$ is the designed filter freq response

$H(\omega)$, $H_d(\omega)$, $W(\omega)$

$$H(\omega) = H_d(\omega) * W(\omega)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\theta) \cdot W(\omega-\theta)\, d\omega$$

Rectangular window function:

$$W(\omega) = \sum_{n=0}^{M-1} w(n)\, e^{j\omega n}$$

$$= \sum_{n=0}^{M-1} e^{-j\omega n}$$

$$= \frac{1 - e^{-j\omega M}}{1 - e^{-j\omega}}$$

$$= \frac{e^{-j\omega\frac{M}{2}}\left(e^{j\omega\frac{M}{2}} - e^{-j\omega\frac{M}{2}}\right)}{e^{-j\frac{\omega}{2}}\left(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right)}$$

$$= \frac{e^{-j\omega\frac{M}{2}}\left(\cancel{2j}\,\sin(\omega M/2)\right)}{e^{-j\frac{\omega}{2}}\left(\cancel{2j}\,\sin(\omega/2)\right)}$$

$$W(\omega) = e^{-j\omega\left(\frac{M-1}{2}\right)} \cdot \frac{\sin(\omega M/2)}{\sin(\omega/2)}$$

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Rectangular window fn:

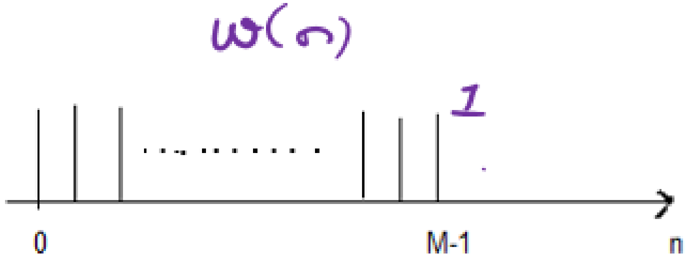
- width of mainlobe $= \frac{4\pi}{M}$

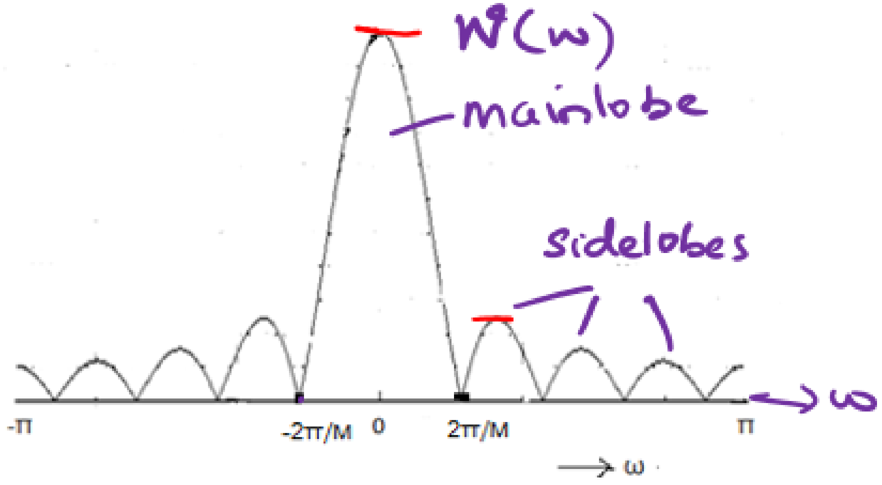
$M\uparrow$, width of mainlobe $\downarrow$

height " $\uparrow$

sidelobe freq $\uparrow$

- No. of sidelobes depends on M

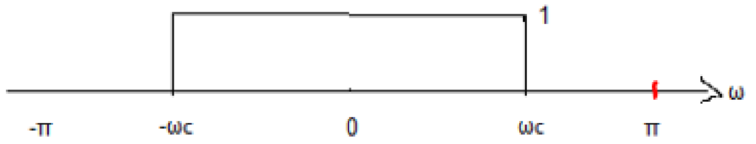




−13dB

eg. LPF with cut-off freq at ω_c :

$$H_d(\omega) = \begin{cases} 1 \cdot e^{-j\omega\left(\frac{M-1}{2}\right)}, & 0 \le \omega \le \omega_c \\ 0 & \text{otherwise} \end{cases}$$



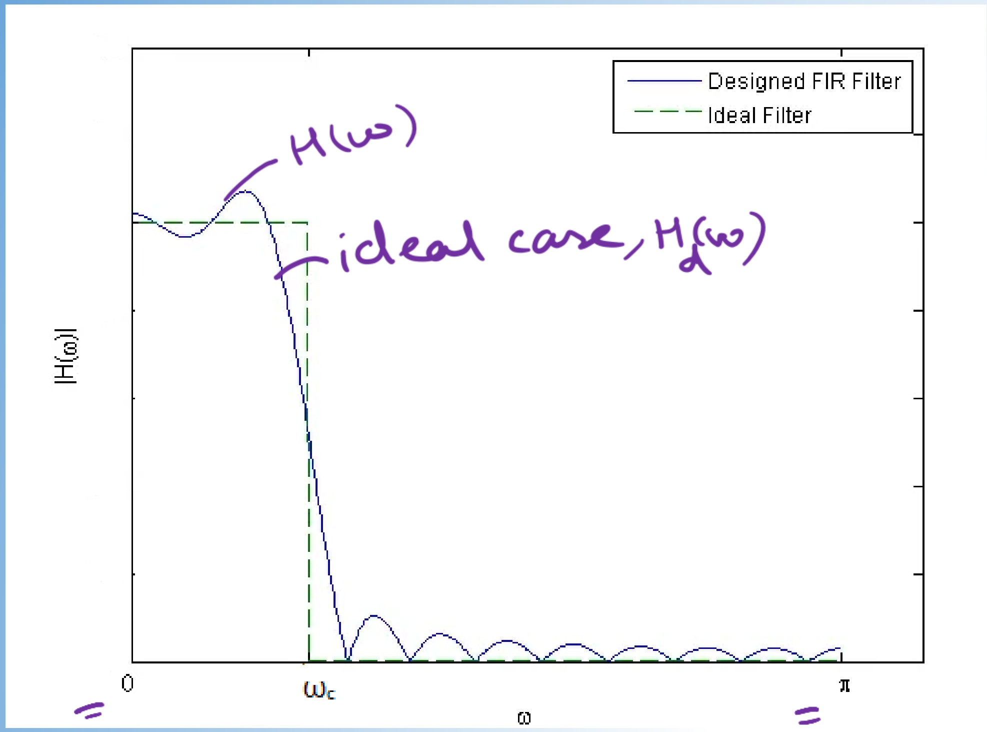
Ideal case.

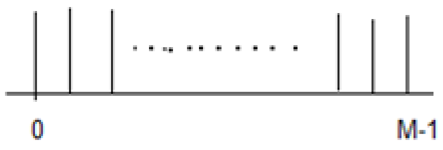
$$\Big\downarrow \text{IDTFT}$$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega$$

$$= \begin{cases} \dfrac{\sin \omega_c\left(n - \frac{M-1}{2}\right)}{\pi\left(n - \frac{M-1}{2}\right)} & \text{for } n \ne \frac{M-1}{2} \\[2ex] \dfrac{\omega_c}{\pi} & \text{for } n = \frac{M-1}{2} \;\; \& \;\; M = \text{odd} \end{cases}$$

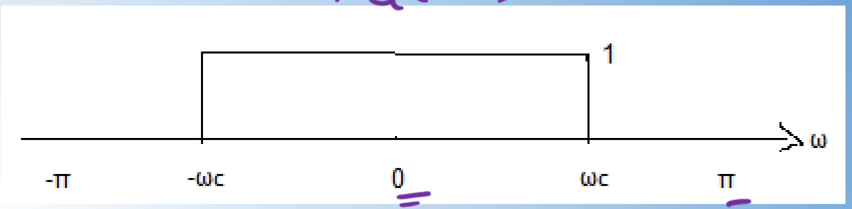
This function has infinite duration. To make it finite, multiply with rectangular window function.

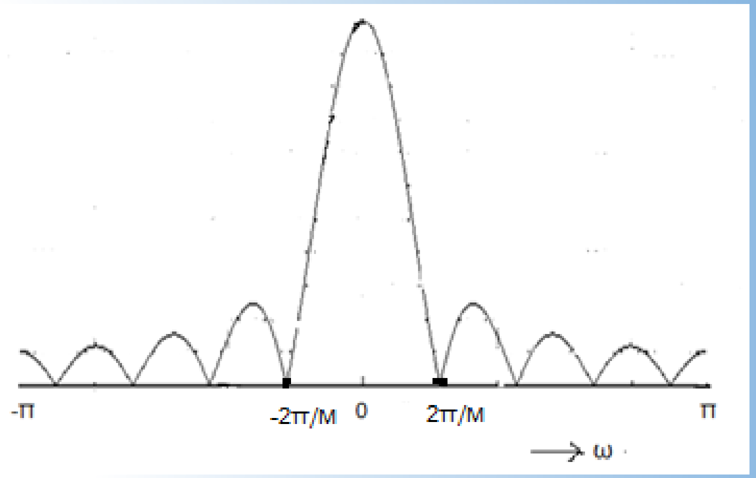




$w(n)$

$H_d(\omega)$





$W(\omega)$

$$h(n) = h_d(n) \cdot w(n)$$

$$H(\omega) = H_d(\omega) * W(\omega)$$

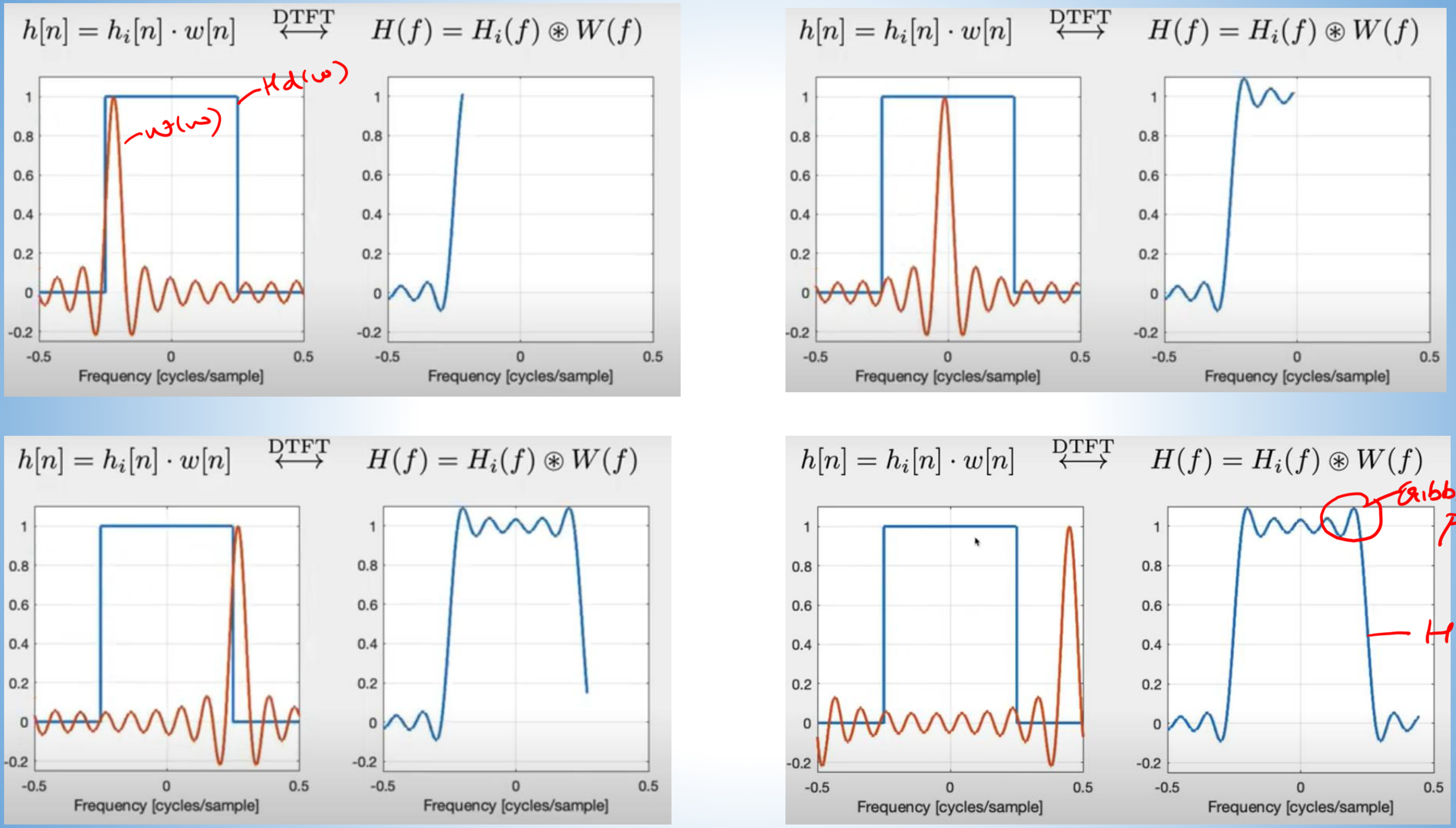
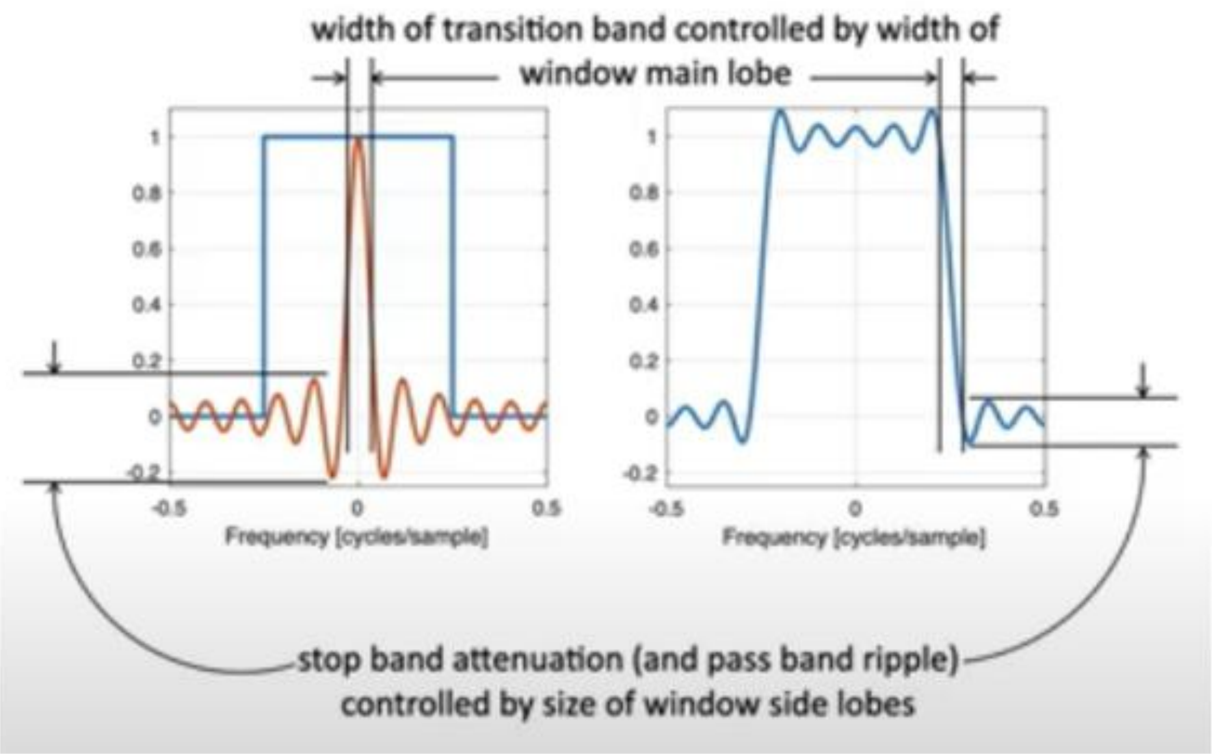


Figure Source: https://www.youtube.com/watch?v=2tmilbi4L0o

- Width of mainlobe causes the transition band in the designed filter response $H(\omega)$

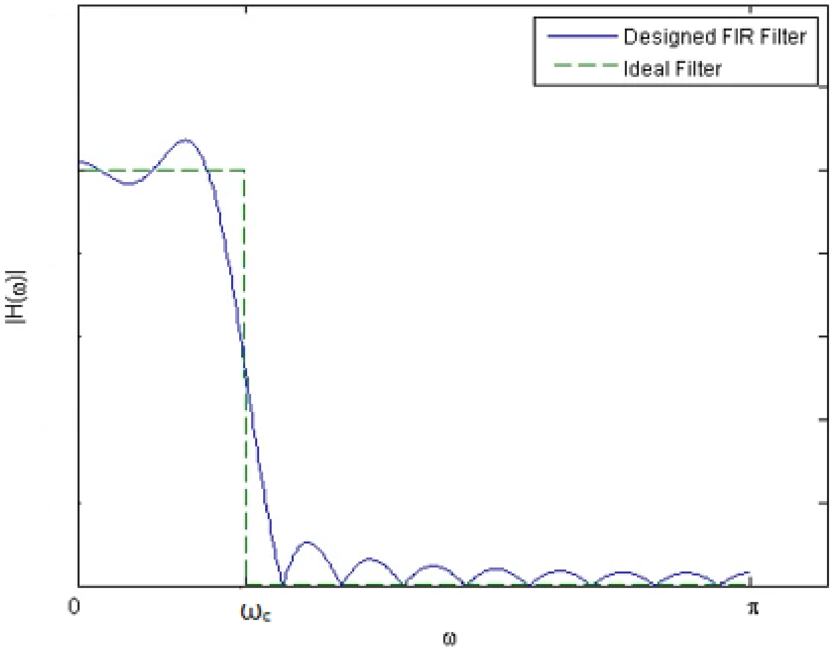
- Transition bandwidth depends on M

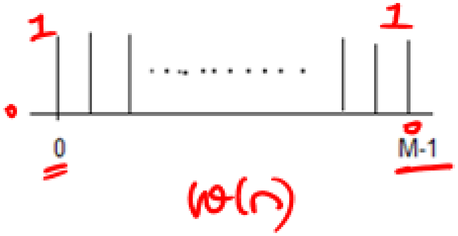
$M\uparrow$ $TB\downarrow$

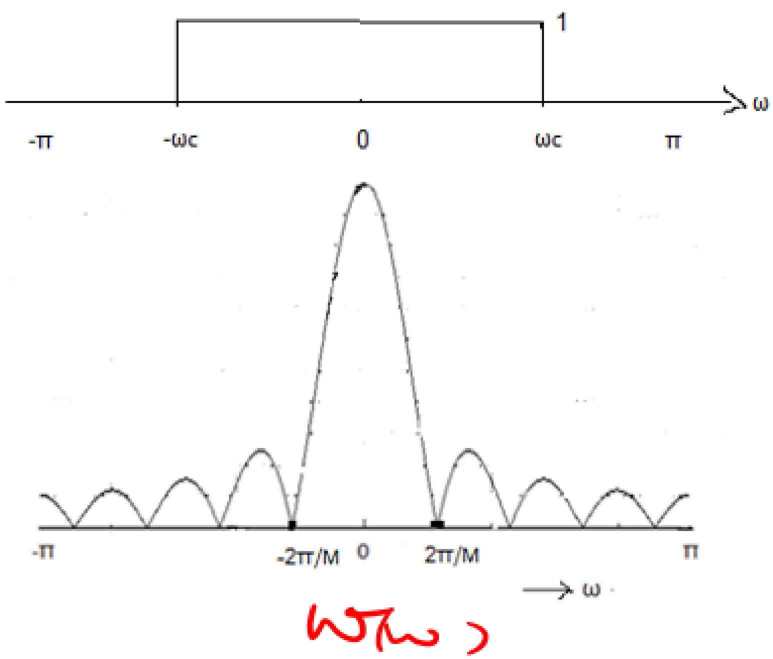


- Ripples in passband & stopband are due to significant sidelobes in $W(\omega)$

- Relatively large oscillations occur near the transition band edge. This is called Gibb's phenomenon.







- We desire the mainlobe width to be narrow and amplitude of sidelobe to be small.

- Sidelobes must be suppressed to reduce the ripples in passband & stopband

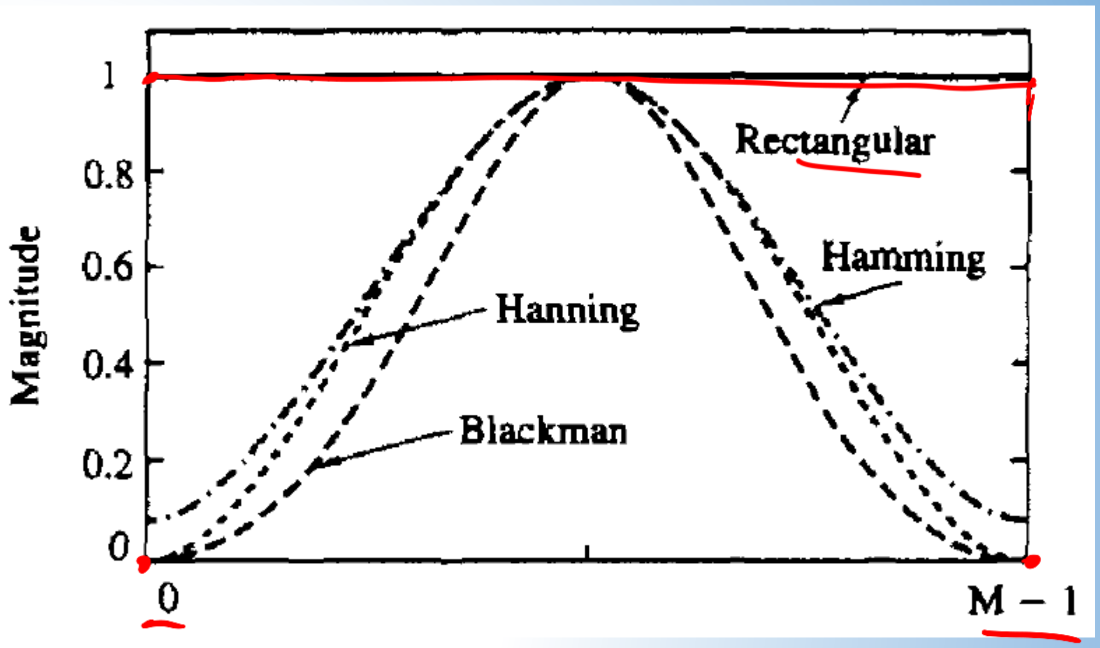
- Sidelobes are significant in rectangular window due to sharp transitions at $n = 0$ & $M-1$.

- Sidelobes can be suppressed by tapering the edges of the window function.

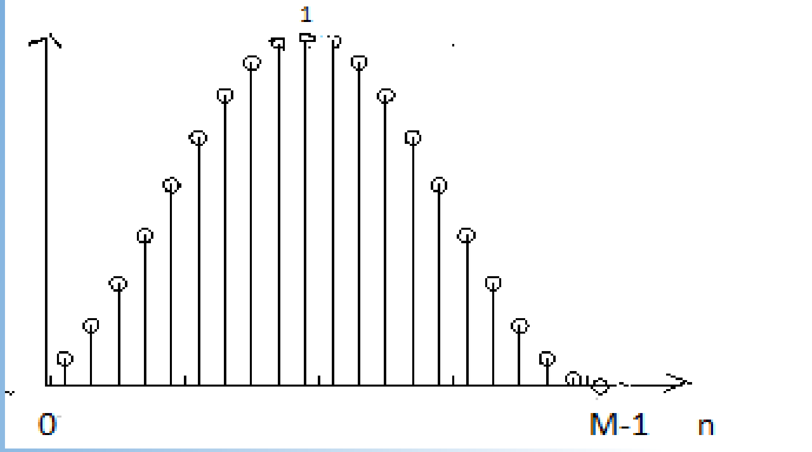
- Different window functions exist, depending on the way in which band edges are tapered.

1) Hanning window
2) Hamming window
3) Blackman window

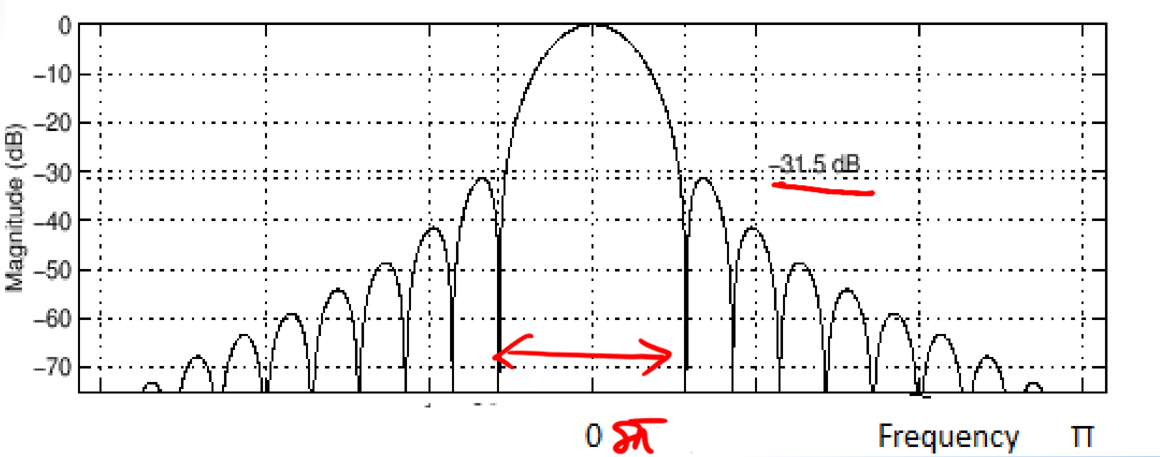
etc.



Hanning window, w(n)

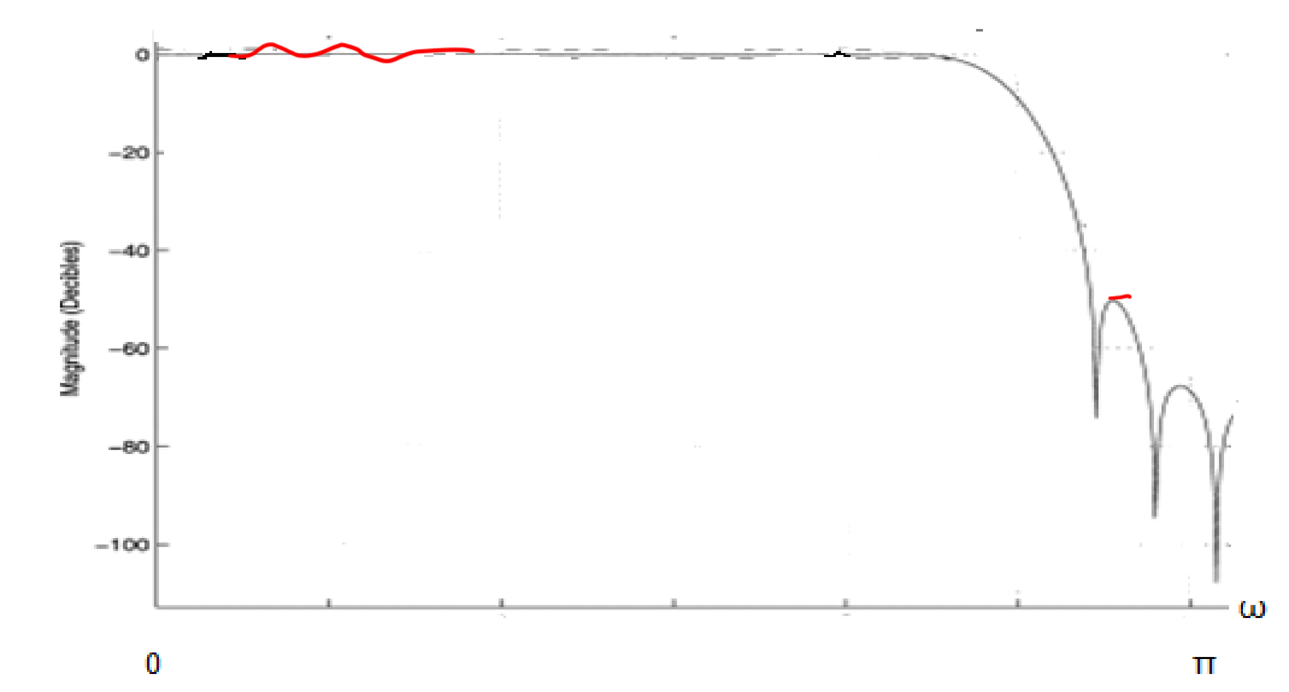


W(ω)



$\frac{8\pi}{M}$

Sidelobe ≈ −32dB .



- Ripples in the passband and Gibb's phenomenon has reduced.

- Stopband attenuation has increased.

Some of the commonly used window functions are given below

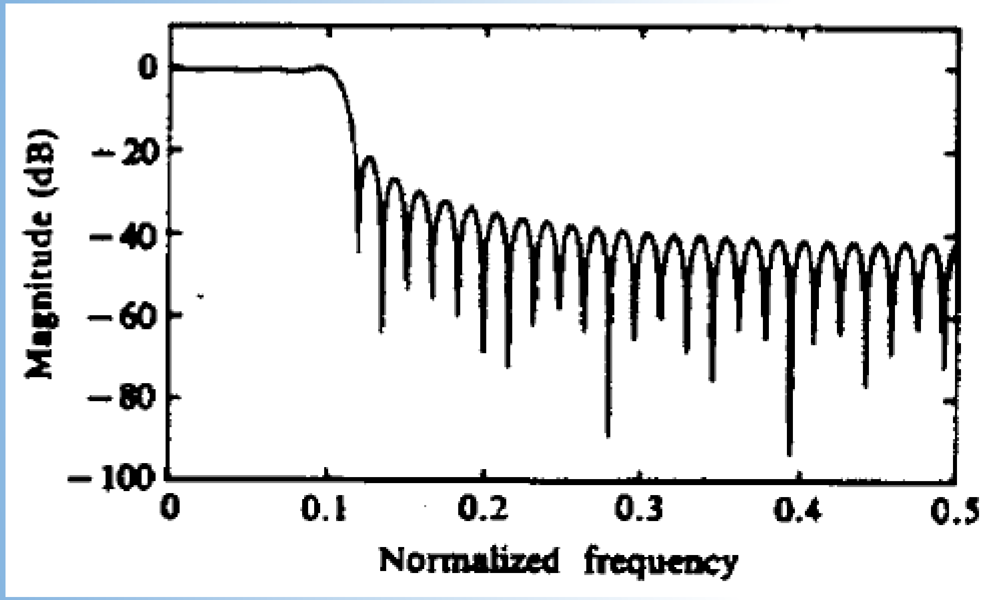
Name of the window	Window function $0 \le n \le M-1$	Main lobe width	Peak side lobe (dB)	Normalised transition width #	Stop band attenuation (dB)
Rectangular	1	$4\pi/M$	-13	0.9/(M-1)	21
Hanning	$0.5 - 0.5cos\left(\frac{2\pi n}{M-1}\right)$	$8\pi/M$	-32	3.1/(M-1)	44
Hamming	$0.54 - 0.46cos\left(\frac{2\pi n}{M-1}\right)$	$8\pi/M$	-43	3.3/(M-1)	53
Blackman	$0.42 - 0.5cos\left(\frac{2\pi n}{M-1}\right) + 0.08cos\left(\frac{4\pi n}{M-1}\right)$	$12\pi/M$	-58	5.5/(M-1)	75
Keiser*	$\dfrac{I_0\left[\beta\sqrt{\left(\frac{M-1}{2}\right)^2 - \left(n - \frac{M-1}{2}\right)^2}\right]}{I_0\left[\beta\left(\frac{M-1}{2}\right)\right]}$				> 70

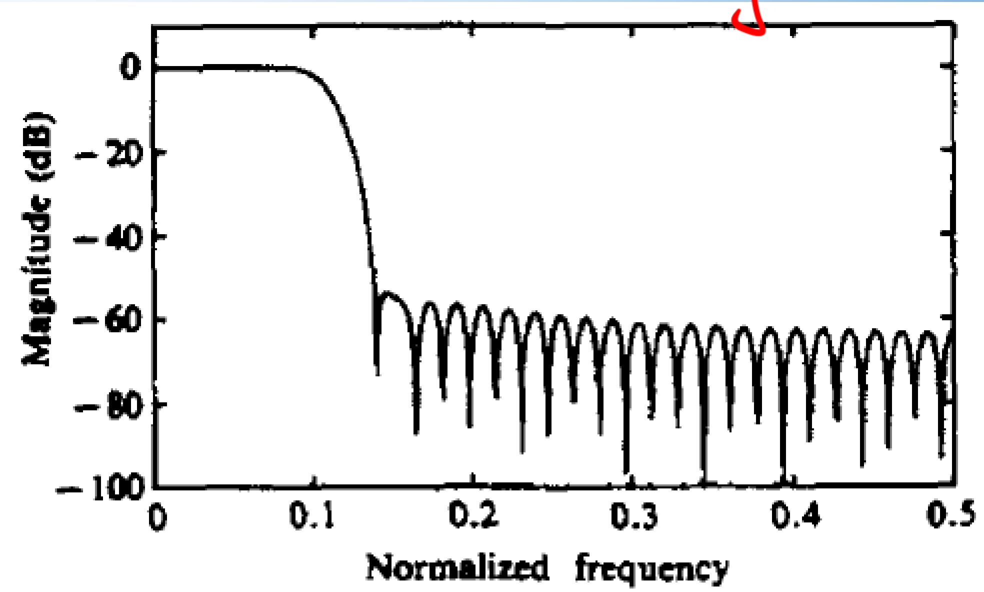
*Keiser window parameters can be controlled by β. $\boldsymbol{I_0}[.]$ is modified Bessel function of first kind and order 0.

Transition width is normalised to 2π or equivalently to sampling frequency F_s

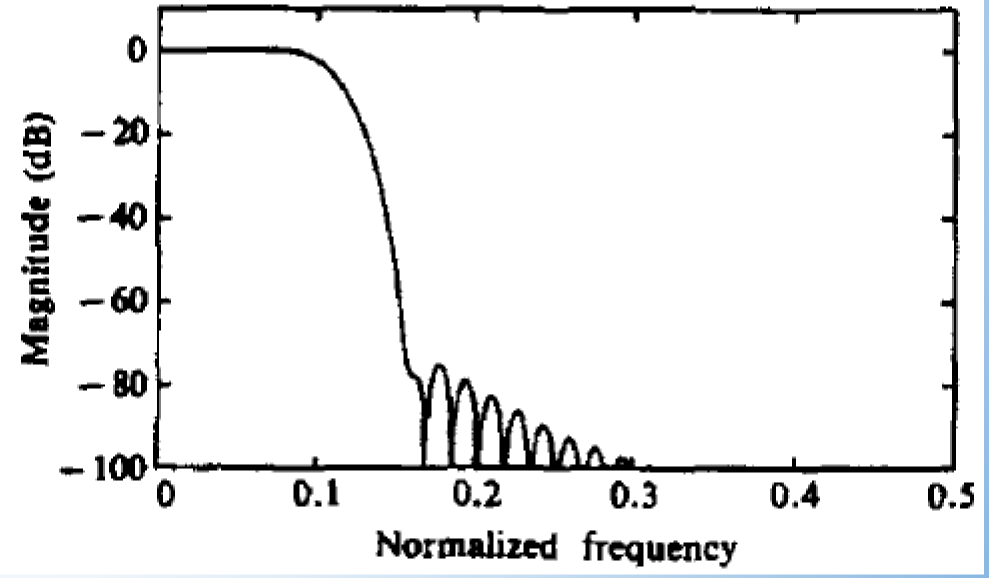
FIR LPF designed using rectangular window, Hamming window and Blackman window (M=61)

Hamming





Rectangular



Blackman

- More the tapering, wider is the normalized TB and better the stopband attenuation

Q. Design a symmetric FIR LPF with the desired freq resp

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha}, & |\omega| \le \omega_c \\ 0 & \text{otherwise} \end{cases}$$

The length of the FIR filter is to be 7, cutoff freq $\omega_c = 1$,
Use rectangular window function.

$$\alpha = \frac{M-1}{2}$$

Soln:

$H_d(\omega)$ — given

$\downarrow$

$h_d(n)$

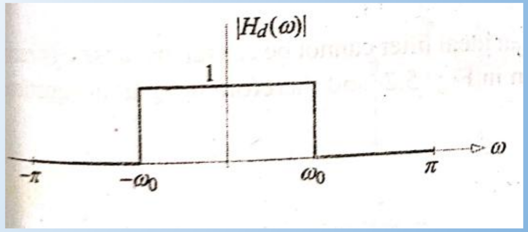
$\downarrow$

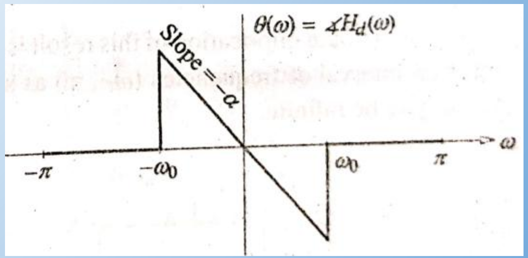
$h(n) = h_d(n) \cdot w(n) \leftarrow$ rect

$\downarrow$

$H(\omega)$ – filter resp of designed FIR filter

$$H_d(\omega) = \begin{cases} e^{-j\omega\alpha}, & |\omega| < \omega_0 \\ 0, & \omega_0 < |\omega| < \pi \end{cases}$$





$$H_d(\omega) = \begin{cases} e^{-j\omega\alpha} & |\omega| \le \omega_c \\ 0 & \text{otherwise} \end{cases}$$

$\omega_c = 1$

$M = 7$

$$h_d(n) = \begin{cases} \dfrac{\sin \omega_c \left(n - \frac{M-1}{2}\right)}{\pi \left(n - \frac{M-1}{2}\right)} & \text{for } n \ne \frac{M-1}{2} \\ \dfrac{\omega_c}{\pi} & \text{for } n = \frac{M-1}{2} \end{cases}$$

$$= \begin{cases} \dfrac{\sin(n-3)}{\pi(n-3)} & , \quad n \ne 3 \\ \dfrac{1}{\pi} & , \quad n = 3 \end{cases}$$

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\tau} & -\omega_c \le \omega \le \omega_c \\ 0 & Otherwise \end{cases}$$

By taking inverse Fourier transform

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega\tau}e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega(n-\tau)}d\omega \end{aligned}$$

$$\begin{aligned} h_d(n) &= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\tau)}}{j(n-\tau)}\right]_{-\omega_c}^{\omega_c} \\ &= \frac{1}{2j\pi(n-\tau)}\left[e^{j\omega_c(n-\tau)} - e^{-j\omega_c(n-\tau)}\right] \\ &= \frac{1}{\pi(n-\tau)}\left[\frac{e^{j\omega_c(n-\tau)} - e^{-j\omega_c(n-\tau)}}{2j}\right] \end{aligned}$$

$h(n) = h_d(n) \cdot w(n)$

$M = 7$, $h(0) = h(6)$, $h(1) = h(5)$, $h(2) = h(4)$ $\Rightarrow$ symmetric

n	$h_d(n)$	$w(n)$	$h(n) = h_d(n) \cdot w(n)$
0, 6	0.0149	1	0.0149
1, 5	0.1447	1	0.1447
2, 4	0.2678	1	0.2678
3	0.3183	1	0.3183.

$$H(z) = \sum_{n=0}^{6} h(n) z^{-n}$$

$$= 0.0149(z^0 + z^{-6}) + 0.1447(z^{-1} + z^{-5}) + 0.2678(z^{-2} + z^{-4}) + 0.3183 z^{-3}.$$

Linear phase FIR filter frequency response	i) Symmetric impulse response, odd length $$H(\omega) = e^{-j\omega\left(\frac{M-1}{2}\right)}\left[2\sum_{n=0}^{\frac{M-3}{2}} h(n)cos\left\{\omega\left(\frac{M-1}{2} - n\right)\right\}\right]$$ ii) Symmetric impulse response, even length $$H(\omega) = e^{-j\omega\left(\frac{M-1}{2}\right)}\left[2\sum_{n=0}^{\frac{M}{2}-1} h(n)cos\left\{\omega\left(\frac{M-1}{2} - n\right)\right\}\right]$$ iii) Anti-symmetric impulse response, odd length $$H(\omega) = je^{-j\omega\left(\frac{M-1}{2}\right)}\left[2\sum_{n=0}^{\frac{M-3}{2}} h(n)sin\left\{\omega\left(\frac{M-1}{2} - n\right)\right\}\right]$$ iv) Anti-symmetric impulse response, even length $$H(\omega) = je^{-j\omega\left(\frac{M-1}{2}\right)}\left[2\sum_{n=0}^{\frac{M}{2}-1} h(n)sin\left\{\omega\left(\frac{M-1}{2} - n\right)\right\}\right]$$

$$H(\omega) = e^{-j\omega\left(\frac{M-1}{2}\right)} \cdot H_r(\omega)$$

Given, $M = 7$ (odd), $h(n)$ - symmetric

$$H_r(\omega) = h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{\frac{M-3}{2}} h(n) \cos\omega\left(\frac{M-1}{2} - n\right)$$

$$= h(3) + 2\sum_{n=0}^{2} h(n) \cos\omega(3-n)$$

$$= 0.3183 + 2\left[0.0149 \cos 3\omega + 0.1447 \cos 2\omega + 0.2678 \cos\omega\right]$$

$$H(e^{j\omega}) = e^{-j3\omega} \cdot \left[0.3183 + 0.0298 \cos 3\omega + 0.2894 \cos 2\omega + 0.5356 \cos\omega\right]$$

Q. Use Hanning window for previous qn:

$$w(n) = 0.5 - 0.5\cos\left(\frac{2\pi n}{M-1}\right)$$

n	$hd(n)$	$w(n)$	$h(n) = hd(n) \cdot w(n)$
0, 6	0.0149	0	0
1, 5	0.1447	0.25	0.0361
2, 4	0.2678	0.75	0.2008
3	0.3183	1	0.3183.

odd symm, $H_r(\omega) = h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{\frac{M-3}{2}} h(n)\cos\omega\left(\frac{M-1}{2} - n\right)$

$$H(e^{j\omega}) = e^{-j3\omega}\left[0.3183 + 0.0722\cos 2\omega + 0.4016\cos\omega\right]$$

Obtain the co.eff. of FIR LPF to meet the specification given below. use suitable window.

Passband edge frequency = 1.5 kHz

Transition width = 0.5 kHz

Stop band attenuation ≥ 50 dB

Sampling frequency = 8 kHz.

Name of the window	Window function $0 \le n \le M-1$	Main lobe width	Peak side lobe (dB)	Normalised transition width #	Stop band attenuation (dB)
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Hanning	$0.5 - 0.5cos\left(\frac{2\pi n}{M-1}\right)$	$8\pi/M$	-32	3.1/(M-1)	44
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The window to be selected is hamming since the stopband attn is 53dB.

$$\Delta f = 0.5\,KHz.$$

$$f_{sr} = 8\,KHz.$$

$$\Delta F = \frac{3.3}{M}$$

$$\frac{\Delta f}{f_{sr}} = \frac{0.5\,KHz}{8\,KHz} = \frac{3.3}{M}$$

$$M = 52.8 \simeq 53.$$

$$\alpha = \frac{M-1}{2}$$

$$\boxed{\alpha = \underline{\underline{26}}}$$

$$\text{cutoff freq} = \text{Passband edge freq} + \frac{\text{Transition freq}}{2}$$

$$f_c = 1.5\,\text{kHz} + \frac{0.5\,\text{kHz}}{2}$$

$$\boxed{f_c = 1.75\,\text{kHz}}$$

$$F_c = \frac{1.75\,\text{kHz}}{8\,\text{kHz}} = 0.21875$$

$$\omega_c = 2\pi F_c = 1.37744\ \text{rads}.$$

$$H_d(e^{j\omega}) = \begin{cases} 1\, e^{-j26\omega} & -1.3744 \le \omega \le 1.3744 \\ 0 & \text{otherwise.} \end{cases}$$

$$\boxed{h_d(n) = \frac{1}{\pi(n-26)}\left[\sin(1.3744)(n-26)\right]} \rightarrow \text{Infinite}$$

Step 4 :- $h'(n) = h_d(n)\, w(n)$

$$0.54 - 0.46\cos\left(\frac{2\pi n}{M-1}\right) \quad 0 \le n \le M-1$$

$$M = 53$$

$$0.54 - 0.46\cos\left(\frac{2\pi n}{52}\right) \quad 0 \le n \le 52$$

n	hd(n)	w(n)	h(n) = hd(n) · w(n)

Reference

- Proakis J. G, Manolakis D. G. Mimitris D., “Introduction to Digital Signal Processing” Prentice Hall, India, 2007.


Thank
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