Moment Generating Functions(mgf)

Definition:

Let X be a random variable. The moment generating function of X denoted by $M_X(t)$ and is defined as

$$M_X(t)=\begin{cases}\sum_{j=1}^{\infty} e^{t x_j} P\left(X=x_j\right) ; X \text{ is discrete} \\ \int_{-\infty}^{\infty} e^{tx} f(x) dx ; X \text{ is continuous}\end{cases}$$

Note:

1. $M_X(t)=E\left(e^{tx}\right)$
2. The nth derivative of $M_X(t)$ at t = 0 is $E\left(X^n\right)$. i.e., $M_X^{(n)}(0)=E\left(X^n\right)$.
3. $E\left(X^n\right)$ is the coefficient of $\frac{t^n}{n!}$ in the equation $M_X(t)=E\left(e^{tx}\right)$.
4. $V(X)=M_X''(0)-\left[M_X'(0)\right]^2$.

Properties:

1. $M_{ax}(t)=M_X(at)$
2. $M_{ax+b}(t)=e^{tb} M_X(at)$
3. If X and Y are independent random variables, then $M_{X+Y}(t)=M_X(t).M_Y(t)$. This can be generalized for 'n' random variables.
4. If two or more independent random variables having a certain distribution are added, the resulting random variable has a distribution of the same type as that of the random variables. This property is called reproductive property.
   Example: Suppose that X and Y are independent random variables with distributions $N\left(\mu_1, \sigma_1^2\right)$ and $N\left(\mu_2, \sigma_2^2\right)$ respectively.
   Let Z = X + Y.
   $M_X(t)=E\left(e^{zt}\right)=E\left(e^{(X+Y)t}\right)$

   $¿ E\left(e^{Xt}\right).E\left(e^{Yt}\right)=M_X(t).M_Y(t)$

   $= e^{t\mu_1+\frac{\sigma_1^2 t^2}{2}} . e^{t\mu_2+\frac{\sigma_2^2 t^2}{2}} = e^{\left[t(\mu_1+\mu_2)+\left(\sigma_1^2+\sigma_2^2\right)\frac{t^2}{2}\right]}$

   This represents the mgf of a normally distributed random variable with expected value ($\mu_1+\mu_2$¿ and variance ($\sigma_1^2+\sigma_2^2$¿.

Problems:

1. If X is the outcome obtained when a die is tossed, then find the moment generating function. Also find its mean and variance.

Solution: $P(X=x)=\frac{1}{6}; x=1,2,3,4,5,6$

We have $M_X(t)=E(e^{tx})$

¿ $\sum_{1}^{6} e^{tx}P(x)$

¿ $\frac{1}{6}$ ¿]

We have $E(X)=M_X^{'}(0)=\frac{1}{6}$ [1 + ….+ 6]

¿ $\frac{21}{6}$

$$M_X^{''}(t)=\frac{1}{6}\left[e^t+4e^{2t}+\ldots+36e^{6t}\right]$$

$$E(X^2)=M_X^{''}(0)=\frac{91}{6}$$

$$V(X)=M_X^{''}(0)-\left[M_X^{'}(0)\right]^2=2.91$$

2. If X is a random variable taking values 0,1,2,…. and $P(X)=ab^x$, where a and b are positive constants such that a+b = 1, then
   (i) Find mgf.
   (ii) If $E(X)=m_1, E(X^2)=m_2$, show that $m_2=m_1(2m_1+1)$.

Solution: (i): $M_X(t)=E(e^{tx})$

¿ $\sum_{x=0}^{\infty} e^{tx}p(x)=\frac{a}{1-be^t}$

$(ii): E(X)=M_X^{'}(0)=\frac{ab}{(1-b)^2}=\frac{b}{a}=m_1$

$$E(X^2)=m_2=M_X^{''}(0)$$

$$=\frac{b}{a}\left(\frac{1+b}{a}\right)$$

$$=m_1\frac{1}{a}[a+b+b]$$

$$=m_1(2m_1+1)$$

3. If X has pdf $f(x)=\lambda e^{-\lambda(x-a)}$ if $x\geq a$. Find its mgf and also find the mean and variance.
4. Suppose that X has pdf $f(x)=\frac{e^{-|x|}}{2}$, -∞ < x < ∞, find mean and variance using mgf.

Mgf of Binomial distribution:

If $p(x) = nCx\, p^x q^{n-x}; x = 0, 1, 2, \ldots, n$, then

$$M_X(t) = \sum_{x=0}^{n} e^{tx} nCx\, p^x q^{n-x}$$

$$Expanding,$$

$$M_X(t) = (pe^t + q)^n$$

Mgf of Poisson distribution:

If $p(x) = \frac{\alpha^x e^{-\alpha}}{x!}; x = 0, 1, 2, \ldots, \infty$, then

$$M_X(t) = \sum_{x=0}^{n} e^{tx} \frac{\alpha^x e^{-\alpha}}{x!}$$

¿ $e^{-\propto} e^{e^t \propto}$

¿ $e^{\propto(e^t - 1)}$

Mgf of Uniform distribution in (-a, a):

If $f(x) = \frac{1}{2a}$ in (-a, a), then

$M_X(t) = \int_{-a}^{a} \frac{1}{2a} e^{tx}$dx $= \frac{sinhat}{at}$

The value of $E(X^{2n})$:

We know that $E(X^{2n})$ is the coefficient of $\frac{t^{2n}}{(2n)!}$ in the equation $M_X(t) = E(e^{tx})$.

E¿) $= \frac{a^{2n} t^{2n}}{(2n+1)!} = \frac{a^{2n}}{2n+1}$

Mgf of Exponential distribution:

If $f(x) = \propto e^{-\propto x}, x > 0$, then

$$M_X(t) = \int_0^{\infty} \propto e^{-\propto x} e^{tx}\, dx$$

¿ $\propto \int_0^{\infty} e^{-(\propto - t)x}\, dx$

$$M_X(t) = \frac{\propto}{\propto - t}$$

Mgf of Gamma distribution:

If $f(x) = \frac{\alpha^r x^{r-1} e^{-\alpha x}}{\Gamma(r)}; \alpha > 0, r > 0, 0 < x < \infty$, then

$$M_X(t) = \int_0^{\infty} e^{tx} \frac{\alpha^r x^{r-1} e^{-\alpha x}}{\Gamma(r)}\, dx$$

Take $x(\propto -t)=v$, $$M_X(t)=\frac{\propto^r}{\Gamma(r)}\int_0^\infty\left(\frac{v}{\propto -t}\right)^{r-1}e^{-v}\frac{dv}{\propto -t}$$

$$=\frac{\propto^r}{(\propto -t)^r}\frac{1}{\Gamma(r)}\int_0^\infty e^{-v}v^{r-1}dv$$

$$=\frac{\propto^r}{(\propto -t)^r}$$

<u>Mgf of Chi-square distribution:</u>

By substituting $r=\frac{n}{2}$, $\propto=\frac{1}{2}$ in mgf of gamma distribution, we get chi-square distribution.

Therefore $M_X(t)=(1-2t)^{-n/2}$

<u>Mgf of Normal distribution:</u>

If $f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty<x<\infty$, then

$$M_X(t)=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{tx}e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}dx$$

Substitute $z=\frac{x-\mu}{\sigma}$,

$$M_X(t)=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{t(\mu+\sigma z)}e^{\frac{-z^2}{2}}\sigma dz$$

$$=\frac{e^{t\mu}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{\frac{-1}{2}¿¿}¿$$

$$=\frac{e^{t\mu}e^{\frac{\sigma^2t^2}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{\frac{-1}{2}(z-t\sigma)^2}dz$$

Substitute $y=z-t\sigma$ and use gamma function,

$$M_X(t)=e^{t\mu+\frac{\sigma^2t^2}{2}}$$

<u>Mgf of Standard normal distribution:</u>

By substituting $\mu=0,\sigma^2=1=¿\sigma=1$ in mgf of normal distribution, we get the mgf of Standard normal distribution.

i.e.

$M_X(t)=e^{t^2/2}$