Table 3 THE STANDARD NORMAL DISTRIBUTION

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt = P(X \le x)$$

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5369
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9131	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767

The Standard Normal Distribution (Continued)

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9989	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Optimization:
Linear programming problem deals with the optimization (maximization or minimization) of a linear function of n variables (called decision variables) subject to linear constraints.The general structure of a LPP is as follows:
Optimize $z = \sum_{j=1}^{n} c_j x_j$ (objective function)
subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq \quad (or \ \geq) \ b_i, \quad i = 1, \ldots, m \quad \text{(constraints)}$$
$$x_j \geq 0, \quad j = 1, 2, \ldots, n \ \text{(non-negative constraints).}$$

Any set $X = \{x_1, x_2, x_3, ..., x_n\}$ of variables is called a *feasible solution* of L.P. problem, if it satisfies the set of constraints as well as non-negativity restrictions. For a system of m simultaneous linear equations in n variables $(n > m)$, a solution obtained by setting $(n - m)$ variables equal to zero and solving for the remaining variables is called a *basic solution.* Such m variables are called basic variables and remaining $(n - m)$ zero-valued variables are called *non-basic variables. Basic feasible solution* is a basic solution which also satisfies the non-negativity restrictions. A basic feasible solution is said to be optimum, if it also optimizes (maximizes or minimizes) the objective function.

In the standard form of a LPP.

a) The objective function may be a maximization or minimization type.

b) All constraints are equations.

c) The R.H,S. terms of all the constraint equations are non-negative.

d) All variables associated with the system are non-negative.

If the inequality in a constraint is less than or equal, then we add a *slack variable* +S to change it to equality. If the inequality is greater than or equal, then we can add a *surplus variable* $-S$ to change it to equality.

Graphical Method: A LPP involving only two variables (say x and y) can be solved graphically by plotting the given constraints as equality on XY plane and identifying the feasible region that satisfies all the constraints. In this case the optimum solution occurs at the corner points of the feasible region.

Simplex method is the most popular method used for the solution of LPP. It is the search procedure that shifts through the set of basic feasible solutions, one at a time until the optimal basic feasible solution is identified. The following is the simplex algorithm:

Step 1: Formulate the linear programming model. Maximize or minimize objective function subject to the constraints.

Step 2: Express LP problem in standard form by adding slack variables/surplus variables to the contraints and assign zero coefficient to these variables in objective function.

Step 3: Determine a starting basic feasible solution that is by setting $x_1 = x_2 = x_3, \cdots, x_n = 0$.

Step 4: Construct the starting simplex tableau. Select an entering variable using the optimality conditions. Stopif there is no variable; the last solution is optimal. Else go to the step 5.

Step 5: Select a leaving variable using the feasibility condition.

Step 6: Determine the new basic solution by using the appropraite Gauss-Jordan computations, then go to Step 4.

Optimality condition: The entering variable in a maximization problem is the non-basic variable having the most negative coefficient in the Z-row. The optimum is reached at the iteration where all the Z-row coefficient of the non-basic variables are non-negative (non-positive).

Feasibility condition: For both maximization and minimization problems the leaving variable is the basic associated with the smallest non-negative ratio (with strictly positive denominator).

Steepest descent method or gradient method determines a minimum of real valued function $f(X)$, $X = (x_1, x_2, \ldots, x_n)$ by repeatedly computing minima of a function $g(t)$ of a single variable t, as follows. Suppose that $f(X)$ has a minimum at X_0 and we start at a point X. Then we look for a minimum of f closest to X along the straight line in the direction of $-\nabla f(X)$ which is the direction of steepest descent (direction of maximum decrease) of f at X. That is, we determine the value of t and the corresponding point $z(t) = X - t\nabla f(X)$ at which the function $g(t) = f(z(t))$ has a minimum. We take this as our next approximation.

Lagrange's multiplier method

Sufficient condition for constrained optimization

Theorem 1 *Suppose that* $x^* = (x_1^*, ..., x_n^*) \in R^n$ *satisfies the conditions*

(a) $x^* \in C_h$*;*

(b) there exists $\mu^* = (\mu_1^*, ..., \mu_k^*) \in R^k$ *such that* $(x_1^*, ..., x_n^*, \mu_1^*, ..., \mu_k^*)$ *is a critical point of* L*;*

(c) for the bordered Hessian matrix H *the last* $n - k$ *leading principal minors*

$$H_{2k+1},\ H_{2k+2},\ \dots\ , H_{n+k} = H$$

evaluated at $(x_1^*, ..., x_n^*, \mu_1^*, ..., \mu_k^*)$ *alternate in sign where the last minor* $H_{n+k} = H$ *has the sign as* $(-1)^n$.

Then x^* *is a local max in* C_h.

If instead of (c) we have the condition

(c') For the bordered hessian H *all the last* $n-k$ *leading principal minors*

$$H_{2k+1},\ H_{2k+2},\ \dots\ , H_{n+k} = H$$

evaluated at $(x_1^*, ..., x_n^*, \mu_1^*, ..., \mu_k^*)$ *have the same sign as* $(-1)^k$, *then* x^* *is a local min on* C_h.

Probability

Addition rule

If A and B are two events of an experiment having sample space S, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

The conditional probability of an event B, given that the event A already taken place is

$$P(B / A) = \frac{P(A \cap B)}{P(A)}, \quad P(A) > 0.$$

Baye's Theorem

Let $B_1, B_2, \ldots B_k$ are the partitions of S with $P(B_i) \neq 0, i = 1, 2, \ldots k$ and A be any event of S, then

$$P(B_i / A) = \frac{P(A / B_i)P(B_i)}{\sum_{i=1}^{k} P(A / B_i)P(B_i)}.$$

The multiplicative rule of probability : $P(A \cap B) = \begin{cases} P(A)P(B|A), if\ P(A) \neq 0 \\ P(B)P(A|B), if\ P(B) \neq 0 \end{cases}$

If $P(A \cap B) = P(A)P(B)$, then A and B are independent.

Random Variable: Let S be the sample of space of a random experiment. Suppose with each element s of S, a unique real number X is associated according to some rule then X is called random variable. There are two types of random variable, i) Discrete and ii) Continuous.

Discrete Random Variable: A random variable X is said to be discrete, if the number of possible values of X is finite or countably infinite. The probability distribution function (pdf) is named as probability mass function (PMF). The Probability mass function is defined as, let X be a random variable, hence the range space R_X of consists of atmost a countably infinite number of values. The probability mass function is defined as

$p(x_i) = \Pr\{X = x_i\}$, satisfying the conditions i) $p(x_i) \geq 0 \; for \; all \; i$

ii) $\sum_{i=1}^{k} p(x_i) = 1.$

Continuous Random Variable: A random variable X is said to be continuous if it can take all possible values between certain limits, here the range space of X is infinite. Therefore the probability distribution function named for such random variable is Probability density function (PDF), which is defined as the pdf of X is a function $f(x)$ satisfying the following properties i) $f(x) \geq 0$

ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

iii) $\Pr\{a \leq X \leq b\} = \int_{a}^{b} f(x)dx$ for any a, b such that $-\infty < a < b < \infty$.

Note: 1. If X is a continuous random variable with pdf f(x), then

$$P(a < X < b) = P(a \leq X < b) = P(a < X \leq b) = P(a \leq X \leq b) = \int_{a}^{b} f(x)dx.$$

2. $P(X = a) = 0$, if X is a continuous random variable.

Cumulative distribution function: Let X be random variable (discrete or continuous), we define F to be the cumulative distribution function of a random variable X given by $F(x) = \Pr\{X \leq x\}$.

Case i) If X is discrete random variable then

$$F(t) = \Pr\{X \leq t\} = P(x_1) + P(x_2) + \cdots + P(t)$$

Case ii) If x is a continuous random variable then $F(x) = \Pr\{X \leq x\} = \int_{-\infty}^{x} f(x)dx$.

Two dimensional random variable: Let E be an experiment and S be a sample space associated with E. Let X=X(s) and Y=Y(s) be two functions each assigning a real number to each outcome s of S. We call (X, Y) to be two dimensional random variable.

Discrete 2D: If the possible values of (X, Y) are finite or countably infinite then (X, Y) is called discrete and it is defined as $P(x_i, y_j)$ satisfying the following condition,

i) $P(x_i, y_j) \geq 0$ and
ii) $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} P(x_i, y_j) = 1$. The function $P(x_i, y_j)$ defined is called as Joint probability distribution function (Jpdf).

Continuous 2D: If (X, Y) is a continuous random variable assuming all values in some region R of the Euclidean plane, then the Joint probability density function $f(x, y)$ is a function satisfying the following conditions

i) $f(x, y) \geq 0$ for all (x, y)ϵR
ii) $\iint f(x, y)dx\, dy = 1$ over the region R.

Marginal Probability distribution: The marginal probability distribution is defined as

Case i) In the discrete (X, Y), it is defined as $p(x_i) = P\{X = x_i\} = \sum_{j=1}^{\infty} P(x_i, y_j)$ is the marginal probability distribution of X. Similarly $q(y_j) = P\{Y = y_j\} = \sum_{i=1}^{\infty} P(x_i, y_j)$ is the marginal probability distribution of Y.

Case ii) In the continuous (X, Y), it is defined as the marginal probability function of X is defined as $g(x) = \int_{-\infty}^{\infty} f(x, y)dy$ and the marginal probability function of Y is defined as $h(y) = \int_{-\infty}^{\infty} f(x, y)dx$.

To calculate the conditional probability:

Case i) Discrete: Probability of x_i given y_j is defined as $= \frac{P(x_i, y_j)}{q(y_j)}$, $q(y_j) > 0$

Probability of y_j given x_i is defined as $= \frac{P(x_i, y_j)}{p(x_i)}$, $p(x_i) > 0$

Case ii) Continuous: The pdf of X for given Y=y is $= \frac{f(x,y)}{h(y)}, h(y) > 0$

The pdf off Y for given X=x is $= \frac{f(x,y)}{g(x)}, g(x) > 0.$

Independent Random variable: If X and Y are independent random variable then two dimensional random variable in case of discrete is defined as $P(x_i, y_j) = p(x_i).q(y_j)$ for all the values of i and j. In case of Continuous it is defined as $f(x, y) = g(x).h(y)$.

Mathematical Expectation: If X is a discrete random variable with pmf p(x), then the expectation of X is given by $E(X) = \sum_x xp(x)$, provided the series is absolutely convergent.

If X is continuous with pdf f(x), then the expectation of X is given by $E(X) = \int xf(x)dx$, provided $\int |x|f(x)dx < \infty$.

Variance of X is given by $V(X) = E(X - E(X))^2 = E(X^2) - \left(E(X)\right)^2$.

Distribution	PMF/PDF	Mean	Variance
Binomial distribution $X \sim B(n,p)$	$P(x) = {}^{n}C_{k} p^{k}(1-p)^{n-k} \quad ,k = 0,1,2,\ldots,n$	$E(x) = np$	$V(x) = np(1-p)$
Poisson's Distribution $X \sim P(\alpha)$	$P(x) = \dfrac{e^{-\alpha}\alpha^{k}}{k!},\ k = 0,1,2,\ldots,\ \alpha > 0$	$E(x) = \alpha = np$	$V(x) = \alpha = np$
Uniform Distribution $X \sim U(a,b)$	$f(x) = \begin{cases} \dfrac{1}{b-a}, & a < x < b \\ 0, & otherwise \end{cases}$	$E(x) = \dfrac{b+a}{2}$	$V(x) = \dfrac{(b-a)^{2}}{12}$
Normal Distribution $X \sim N(\mu,\sigma^{2})$	$f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\frac{(x-\mu)^{2}}{\sigma^{2}}} \quad , \quad -\infty < x,\mu < \infty,\ \sigma > 0$	$E(x) = \mu$	$V(x) = \sigma^{2}$
Exponential Distribution $X \sim E(\lambda)$	$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & otherwise \end{cases}$	$E(x) = \dfrac{1}{\lambda}$	$V(X) = \dfrac{1}{\lambda^{2}}$
Chi-square Distribution $X \sim \chi^{2}(n)$	$f(x) = \begin{cases} \dfrac{x^{\frac{n}{2}-1} e^{-\frac{x}{2}}}{\Gamma(n/2)2^{\frac{n}{2}}}, & x > 0 \\ 0, & elsewhere \end{cases}$	$E(x) = n$	$V(x) = 2n$

Uniform distribution on a two dimensional set: If R is a set in the two-dimensional plane, and R has a finite area, then we may consider the density function equal to the reciprocal of the area of R inside R, and equal to 0 otherwise:

$$f(x,y) = \begin{cases} \frac{1}{area\ R}; & if\ (x,y) \in R \\ 0 & Otherwise \end{cases}.$$

Covariance:

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

Correlation coefficient:

$$\rho_{xy} = \rho = \frac{E(XY) - E(X)E(Y)}{\sqrt{V(X)V(Y)}}$$

Properties:

1. $E(c) = c$, where c is a constant.
2. $V(c) = 0$, where c is a constant.
3. If $E(XY) = 0$ then X and Y are orthogonal.
4. $V(AX+B) = A^2V(X)$ when AX+B is linear function of X.
5. If $\rho_{xy} = 0$ then X and Y are un correlated.
6. $V(AX+BY) = A^2V(X) + B^2V(Y) + 2ABCOV(X,Y)$

FUNCTIONS OF ONE DIMENSIONAL RANDOM VARIABLES

Theorem: Let X be a continuous random variable with p.d.f. $f(x)$ where $f(x) > 0$ for $a < x < b$. Suppose that $Y = H(X)$ is strictly monotonic function on $[a, b]$. Then the p.d.f. of the random variable $Y = H(X)$ is given by

$$g(y) = f(x)\left|\frac{dx}{dy}\right|$$

If $Y = H(X)$ is strictly increasing then $g(y) > 0$ for $H(a) < y < H(b)$.

If $Y = H(X)$ is strictly decreasing then $g(y) > 0$ for $H(b) < y < H(a)$.

Theorem: Let X be a continuous random variable with p.d.f. $f(x)$. Let $Y = X^2$ then the p.d.f. of Y is

$$g(y) = \frac{1}{2\sqrt{y}}\left[f(\sqrt{y}) + f(-\sqrt{y})\right]$$

FUNCTIONS OF TWO DIMENSIONAL RANDOM VARIABLES

Let (X, Y) be a two dimensional continuous random variable. Let $Z = H(X, Y)$ be a continuous function of X and Y then $Z = H(X, Y)$ is a continuous one dimensional random variable.

To find the p.d.f. of Z, we introduce another suitable random variable say,

$W = G(X, Y)$ and obtain the joint p.d.f. of the two dimensional random variable (Z, W), say $k(z, w)$. From this distribution, the p.d.f. of Z can be obtained by integrating k with respect to w.

Theorem: Suppose (X, Y) is a two dimensional continuous random variable with joint p.d.f. $f(x, y)$ defined on a region R of the XY-plane. Let $Z = H_1(X, Y)$ and $W = H_2(X, Y)$. Suppose that H_1 and H_2 satisfies the following conditions;

(i) $z = H_1(x, y)$ and $w = H_2(x, y)$ may be uniquely solved for x, y in terms of z & w say, $x = G_1(z, w)$ and $y = G_2(z, w)$.

(ii) The partial derivatives $\frac{\partial x}{\partial z}, \frac{\partial x}{\partial w}, \frac{\partial y}{\partial z}, \frac{\partial y}{\partial w}$ exist and are continuous

Then the joint p.d.f. of (Z, W) say $k(z, w)$ is given by,

$$k(z, w) = f[G_1(z, w), G_2(z, w)]|J(z, w)|$$

where $J(z, w) = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix}$ is called the Jacobian of the transformation

$(x, y) \mapsto (z, w)$. Also, $k(z, w) > 0$ for those values of (z, w) corresponding to the values of (x, y) for which $f(x, y) > 0$.

MOMENT GENERATING FUNCTION (M.G.F.) OF ONE DIMENSIONAL RANDOM VARIABLES

Let X be any one dimensional random variable then the mathematical expectation $E(e^{tX})$ if exists then it is called the moment generating function (m.g.f.) of X.

i.e., $M_X(t) = E(e^{tX})$

In particular, if X is discrete then, $M_X(t) = \sum_{i=1}^{i=\infty} e^{tx_i} P(X = x_i)$.

If X is continuous then, $M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$.

MGF of some standard distributions:

1. **Binomial Distributions:** $\boldsymbol{M_X(t)} = M_X(t) = (pe^t + q)^n$
2. **Poisson Distributions:** $\boldsymbol{M_X(t)} = e^{\propto(e^t-1)}$
3. **Normal Distributions:** $\boldsymbol{M_X(t)} = e^{t\mu+\frac{\sigma^2 t^2}{2}}$
4. **Exponential Distributions:** $\boldsymbol{M_X(t)} = \frac{\propto}{\propto - t}$
5. **Gamma Distributions:** $\boldsymbol{M_X(t)} = \frac{\propto^r}{(\propto - t)^r}$
6. **Chi square Distributions:** $\boldsymbol{M_X(t)} = (1 - 2t)^{-n/2}$

Stochastic process is a family of random variables $\{X(t)|t \in T\}$ defined on a common sample space S and indexed by the parameter t , which varies on an index set T. The values assumed by the random variables $X(t)$ are called states, and the set of all possible values from the state space of the process. A Markov process is a stochastic process whose entire past history is summarized in its current (present) state. i.e., the "future" is independent of its "past ". Markov chain is a Markov process in which the state space is discrete (finite or countably infinite).

Let $a_1, a_2, \ldots, a_n$ be the states of a Markov process. The transition matrix P of the Markov chain is the $n \times n$ matrix (p_{ij}), where p_{ij} is the probability that the system will change from state a_i to state a_j. The n-step transition matrix $P^{(n)}$ of the Markov chain is the $n \times n$ matrix $(p_{ij}^{(n)})$, where $p_{ij}^{(n)}$ is the probability that the system will change from state a_i to state a_j in n steps.

Note: • A vector v is said to be a fixed vector or a fixed point of a matrix A if $vA = v$ and $v \neq 0$.

• The transition matrix P is called regular if all the entries of some power P^m are positive.

• If P is the transition matrix of a Markov chain, then the n-step transition matrix $P^{(n)}$ is equal to the nth power of P , i.e., $P^{(n)} = P^n$.

• Let P be a regular transition matrix of a Markov chain. Then in the long run, the probability that any state a_j occurs is approximately equal to the component j of the unique fixed probability vector of P.

VECTOR CALCULUS

1) The differential operator ∇ is defined as $\nabla = \frac{\partial}{\partial x}\,\boldsymbol{i} + \frac{\partial}{\partial y}\,\boldsymbol{j} + \frac{\partial}{\partial z}\,\boldsymbol{k}$

2) Let $\emptyset = \emptyset(x, y, z)$ is the scalar field, the gradient of $\emptyset$ at the point (x, y, z) is $grad\emptyset = \nabla\emptyset = \frac{\partial\emptyset}{\partial x}\,\boldsymbol{i} + \frac{\partial\emptyset}{\partial y}\,\boldsymbol{j} + \frac{\partial\emptyset}{\partial z}\,\boldsymbol{k} = \sum \frac{\partial\emptyset}{\partial x}\,\boldsymbol{i}$

3) Let $\vec{f} = f_1\boldsymbol{i} + f_2\,\boldsymbol{j} + f_3\,\boldsymbol{k}$ is the vector field, the divergence of $\vec{f}$ at the point (x, y, z) is, $div\vec{f} = \nabla.\,\vec{\boldsymbol{f}} = \frac{\partial f1}{\partial x} + \frac{\partial f2}{\partial y} + \frac{\partial f3}{\partial z} = \sum \frac{\partial f1}{\partial x}$

4) If $\phi_1(x, y, z)$ and $\phi_2(x, y, z)$ are the two surfaces, then the angle between their surfaces at (x_1, y_1, z_1) is

$$\cos\theta = \frac{\nabla\phi_1.\nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}.$$

5) If $\phi(x, y, z)$ is a scalar function and $\vec{d}$ is the given direction vector, then the directional derivative of ϕ along $\hat{n}$ is $\nabla\phi.\,\hat{n}$, where $\hat{n} = \frac{\vec{d}}{|\vec{d}|}$.

6) If $\phi(x, y, z) = c$ be the equation of a surface and $P(x_1, y_1, z_1)$ is a point on it then
   a) equation of tangent plane at P is $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$.
   b) Equation of normal line at P is $\frac{x - x_1}{A} = \frac{y - y_1}{B} = \frac{z - z_1}{C}$,

   where $A = \left(\frac{\partial\phi}{\partial x}\right)_{(x_1, y_1, z_1)}, B = \left(\frac{\partial\phi}{\partial y}\right)_{(x_1, y_1, z_1)}, C = \left(\frac{\partial\phi}{\partial z}\right)_{(x_1, y_1, z_1)}$.

7) The directional derivative of a scalar function ϕ at any point is maximum along $\nabla\phi$.

8) The Laplacian operator ∇^2 is defined as $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

9) The Laplacian of a scalar function $\emptyset$ as, $\nabla^2\emptyset = \frac{\partial^2\emptyset}{\partial x^2} + \frac{\partial^2\emptyset}{\partial y^2} + \frac{\partial^2\emptyset}{\partial z^2}$

10) Let $\vec{f} = f_1\boldsymbol{i} + f_2\boldsymbol{j} + f_3\boldsymbol{k}$ is the vector field, the curl of $\vec{f}$ at the point (x, y, z) is $curl\,\vec{f} = \nabla \times \vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$

11) If $\nabla \times \vec{f} = \mathbf{0}$, then the vector field $\vec{f}$ is irrotational

12) If $\nabla . \vec{f} = \mathbf{0}$, then the vector field $\vec{f}$ is solenoidal