# Review of Z transform and DFT

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# Transforms:

- Continuous time – Laplace transform
- Stable and periodic – Fourier Series
- Stable and non-periodic – Fourier Transform

- Discrete time – Z transform
- Stable and periodic – Discrete Time Fourier Series
- Stable and non-periodic – Discrete Time Fourier Transform

# Why z-Transform?

- A generalization of Fourier transform
- Why generalize it?
  - FT does not converge on all sequence
  - Simplifies analysis – convolution becomes multiplication
  - Characterizes the LTI System – Transfer function
  - Its response to various signals – locating poles and zeros

# Definition

- The $z$-transform of sequence $x(n)$ is defined by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

- Let $z = e^{-j\omega}$.

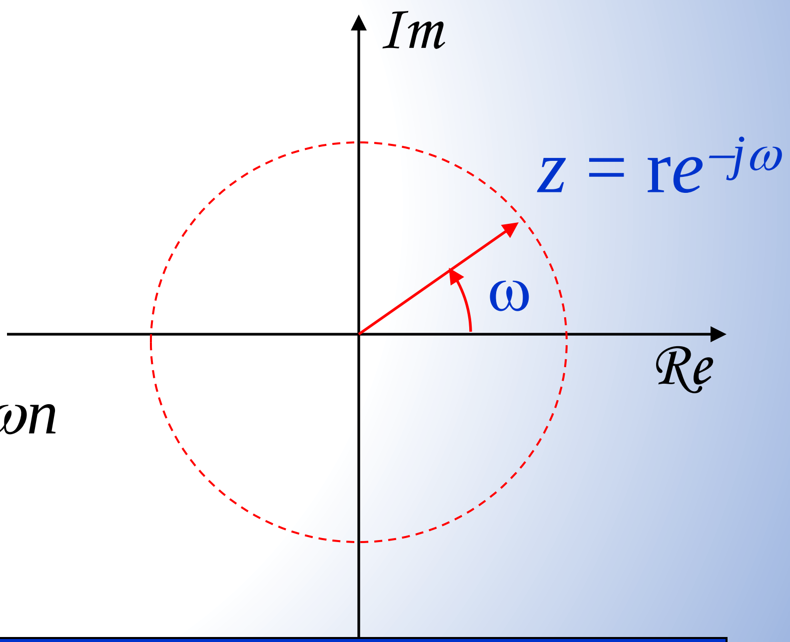
Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

# z-Plane

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$




Fourier Transform is to *evaluate z-transform on a unit circle* (r = 1)

# Definition

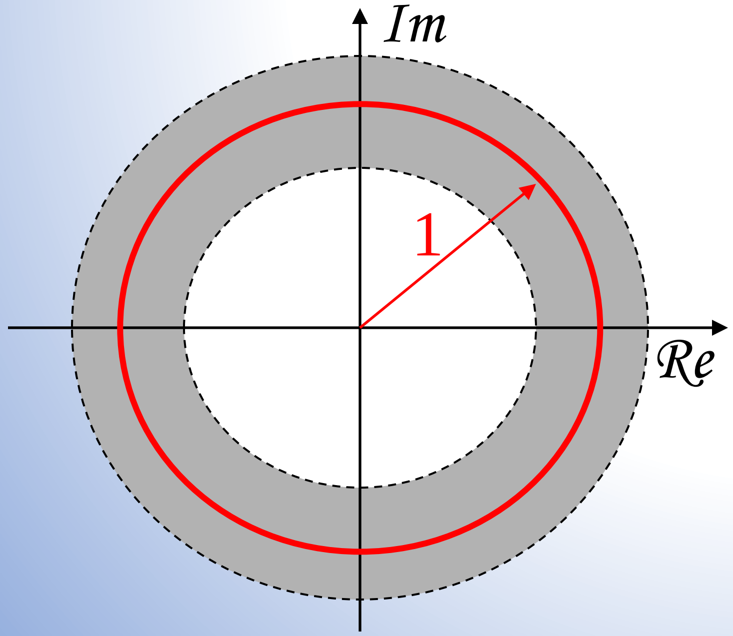
- Give a sequence, the set of values of $z$ for which the $z$-transform converges, i.e., $|X(z)|<\infty$, is called the region of convergence.

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x(n) z^{-n} \right| = \sum_{n=-\infty}^{\infty} |x(n)||z|^{-n} < \infty$$

ROC is centered on origin and consists of a set of rings.

# Stable Systems

- A stable system requires that its Fourier transform is uniformly convergent.




- Fact: Fourier transform is to evaluate $z$-transform on a unit circle.
- A stable system requires the ROC of $z$-transform to include the unit circle.

# Review of Fourier representation of discrete signals

$$x[n] \longleftrightarrow X(e^{j\omega})$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

– Analysis Equation
– DTFT

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$$

– Synthesis Equation
– Inverse DTFT

- The DTFT and inverse DTFT are not symmetric. One is integration over a finite interval (2π), and the other is summation over infinite terms
- The signal, x[n] is aperiodic, and hence, the transform is a continuous function of frequency
- Not practical for (real-time) computation on a digital computer.

# Examples

1) $x[n] = \delta[n]$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n} = 1$$

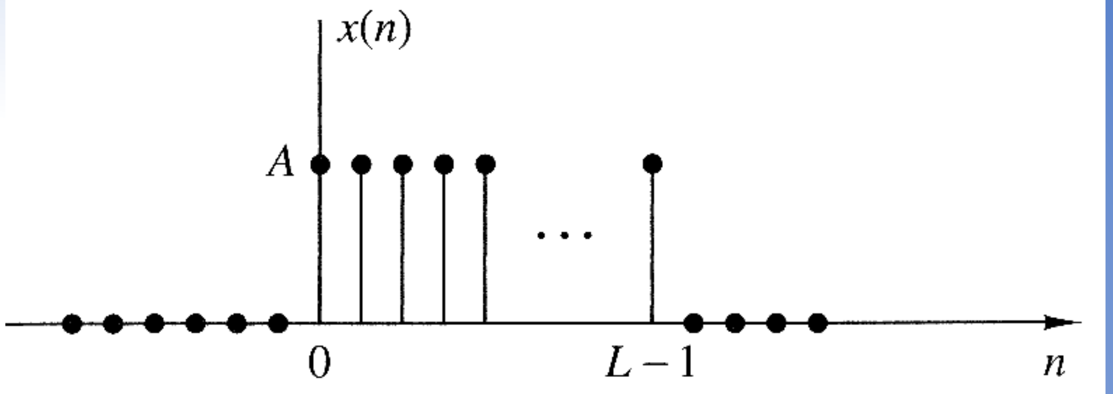
2) $x[n] = \delta[n - n_0]$ - shifted unit sample

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n - n_0] e^{-j\omega n} = e^{-j\omega n_0}$$

Same amplitude (=1) as above,
but with a *linear* phase $-\omega n_0$

Determine the Fourier transform of the sequence

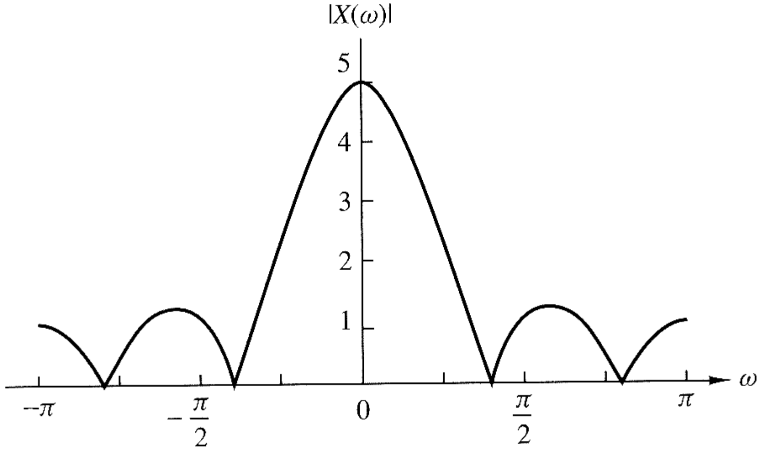
$$x(n) = \begin{cases} A, & 0 \le n \le L-1 \\ 0, & \text{otherwise} \end{cases}$$




$$X(\omega) = \sum_{n=0}^{L-1} A e^{-j\omega n}$$

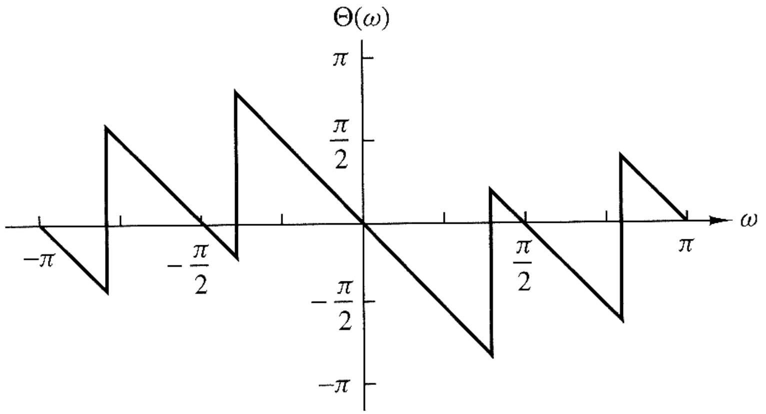
$$= A \frac{1 - e^{-j\omega L}}{1 - e^{-j\omega}}$$

$$= A e^{-j(\omega/2)(L-1)} \frac{\sin(\omega L/2)}{\sin(\omega/2)}$$

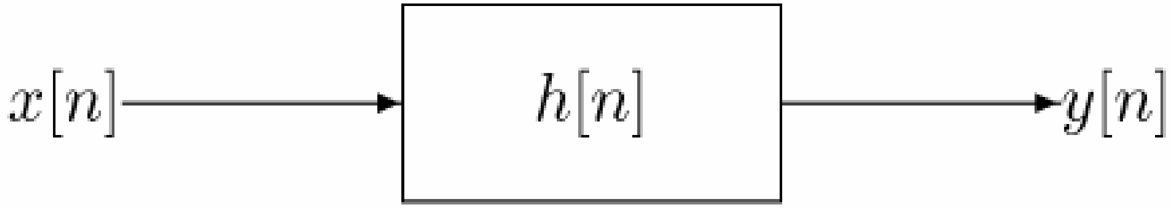



$$|X(\omega)| = \begin{cases} |A|L, & \omega = 0 \\ |A| \left| \dfrac{\sin(\omega L/2)}{\sin(\omega/2)} \right|, & \text{otherwise} \end{cases}$$

$$\measuredangle X(\omega) = \measuredangle A - \frac{\omega}{2}(L-1) + \measuredangle \frac{\sin(\omega L/2)}{\sin(\omega/2)}$$



# Convolution Property




$$y[n] = h[n] * x[n]$$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

$$H(e^{j\omega}) = \text{DTFT of } h[n]$$

$$\text{Frequency response} = \text{DTFT of the unit sample response}$$

# Multiplication Property

$$y[n] = x_1[n] \cdot x_2[n]$$

$$\begin{aligned}
Y(e^{j\omega}) &= \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta \\
&= \frac{1}{2\pi} X_1(e^{j\omega}) \otimes X_2(e^{j\omega}) \\
&\hookrightarrow \text{ Periodic Convolution}
\end{aligned}$$

# Find DTFT of $x[n] = (0.5)^n u[n-4]$

$$a^n u(n) \xleftrightarrow{DTFT} \frac{1}{1 - a e^{-j\omega}}, \quad |a| < 1$$

Time shifting

$$x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$$

$$x(n-n_0) \xleftrightarrow{DTFT} e^{-j\omega n_0} X(e^{j\omega})$$

$$X(e^{j\omega}) = \frac{(0.5)^4 e^{-j4\omega}}{1 - 0.5 e^{-j\omega}}$$

Magnitude Spectrum

$$|X(e^{j\omega})| = \frac{(0.5)^4}{\sqrt{(1 - 0.5\cos\omega)^2 + (0.5\sin\omega)^2}} \quad |e^{-j4\omega}| = 1$$

Phase Spectrum

$$\angle X(e^{j\omega}) = -4\omega - \tan^{-1}\left[\frac{0.5\sin\omega}{1 - 0.5\cos\omega}\right]$$


Thank
you