SET THEORY

A set is a collection of well-defined objects. The objects comprising the set are called elements. If x is an element of a set A, then we write $x \in A$. If x is not an element, then we write $x \notin A$.

Subset: Let A & B be two sets. A is said to be a subset of B, if $x \in A \Rightarrow x \in B$ & is denoted by $A \subset B$.

Equality of sets: Two sets A and B are said to be equal if $A \subseteq B$ & $B \subseteq A$ and we write $A = B$.

Universal set: All sets under consideration are taken to be subsets of a fixed set. This set is called universal set & is denoted by U.

Null set: A set containing no elements is called a null set and is denoted by ϕ.

Singleton set: A set containing a single element is called a singleton set.

SET OPERATIONS:

Let A and B be two sets.

Union: Union of A and B is denoted by $A \cup B$ and is defined as $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.

Intersection: Intersection of A and B is defined as $A \cap B = \{x \mid x \in A \;\&\; x \in B\}$. If $A \cap B = \phi$, then A & B are disjoint sets.

Difference: The difference $A - B$ is defined as $A - B = \{x \mid x \in A \;\&\; x \notin B\} = A \cap B'$

Complement: Complement of the set A is denoted by A' and is defined as $A' = \{x \mid x \in U \text{ but } x \notin A\}$.

Laws of Set Algebra:

1. $A \cap B = B \cap A$, $A \cup B = B \cup A$ **(Commutative laws)**
2. $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$ **(Associative laws)**
3. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ **(Distributive laws)**
4. $A \cup A = A$, $A \cap A = A$ **(Idempotent laws)**
5. $A \cup U = U$, $A \cap U = A$, $A \cup \phi = A$, $A \cap \phi = \phi$
6. $A \cap A' = \phi$, $A \cup A' = U$, $U' = \phi$, $\phi' = U$
7. $(A \cup B)' = A' \cap B'$, $(A \cap B)' = A' \cup B'$ **(De Morgan's laws)**

Cardinality: The number of elements in a set A is called cardinality of A and is denoted by n (A).

1. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
2. $n(A - B) = n(A) - n(A \cap B)$
3. $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

Methods of Enumeration:

Multiplication Principle: Suppose that a procedure, say procedure A can be done in n different ways and another procedure say B can be done in m different ways. Also suppose that any way of doing A can be followed by any way of doing B. Then, the procedure consisting of 'A followed by B' can be performed in mn ways.

Addition Principle: The number ways in which either A or B, but not both, can be performed is m + n.

Permutation: (Arrangement of given objects; Order is important)

The number of permutations of n distinct objects taken r at a time is

- ${}^{n}P_{r} = \frac{n!}{(n-r)!}$, $r \leq n$, if repetition is not allowed.
- n^r, if repetition is allowed.
- $\frac{n!}{k_1!k_2!\ldots k_m!}$ where, of the n objects, k_1 are of one kind, k_2 are of a second kind, …,k_m are of m^{th} kind,

  $k_1 + k_2 + \ldots + k_m = n$. (all objects are taken).
- (n – 1)!, when arranged along a circle.
- (n – 1)!/2, when clockwise and anticlockwise arrangements are indistinguishable

Combination: (Selection of objects; Order is not important)

- ${}^{n}C_{r} = \frac{{}^{n}P_{r}}{r!} = \frac{n!}{r!\,(n-r)!}$, if repetition is not allowed.
- ${}^{n+r-1}C_r$, if repetition is allowed.

Probability Theory

Random experiment: If the repetition of an experiment under identical conditions results in different possible outcomes, that experiment is called a random experiment.
Examples: Tossing of a coin, rolling of a die.

Sample Space: The set of all possible outcomes of a random experiment.
Examples:

- In tossing of a coin: S = {H, T}
- In tossing of two coins: S= {HH, HT, TH, TT}
- In tossing of two identical coins: S = {HH, HT, TT}
- In rolling of a die: S = {1,2,3,4,5,6}

If a sample space has finite number of elements, then it is called a finite sample space. Otherwise, the sample space is said to be an infinite sample space.
Example:

- Consider rolling of a die till a 5 appears: S = {5, 15,25,..65, 115, 125,…,215,225,….}

Event: An event is a subset of the sample space.
Null Event: An event, which does not contain any element, is called a null event or an impossible event, denoted by ϕ.
Certain or sure Event: If the event contains all the elements of the sample space, then it is called a certain event.

Definition: We say that an event has occurred if the outcome of the experiment belongs to the event. An event has not occurred if the outcome of the experiment is not in the event.

Mutually Exclusive Events: Two events A and B are said to be mutually exclusive if both of them cannot occur simultaneously. i.e., if occurrence of one event prevents the occurrence of the other, then the events are said to be mutually exclusive. A and B are mutually exclusive if $A \cap B = \phi$.

- ✓ In tossing of a coin head and tail are mutually exclusive
- ✓ In rolling of a die all six faces are mutually exclusive

Independent events:
Are **those events whose occurrence is not dependent on any other event**. For example, if we flip a coin in the air and get the outcome as Head, then again if we flip the coin but this time, we get the outcome as Tail. In both cases, the occurrence of both events is independent of each other.
If A and B are independent events, then $P(A \cap B) = P(A). P(B)$

Equally likely outcomes: If all outcomes of a random experiment have equal chances of occurrence, then the outcomes are said to be equally likely.

- ✓ In tossing of an unbiased coin, head and tail are equally likely.
- ✓ In rolling of an honest die, all six faces are equally likely.

Exhaustive cases: The total number of possible outcomes of a random experiment is called exhaustive cases for that experiment.

- In tossing of a coin, exhaustive cases = 2
- In tossing of 2 coins, exhaustive cases = 4
- In tossing n coins, exhaustive cases = 2^n
- In rolling of two dice, exhaustive cases = 6

Favourable cases: An outcome x is said to be favourable to an event A, if x belongs to A. The total number of outcomes favourable to A is called favourable cases to A.

- In tossing of two coins, favourable cases for getting 2 heads is 1, for getting exactly one head is 2 and for getting at least 2 heads is 3.
- In drawing a card from a pack, there are 4 cases favouring a king, 2 cases favouring a red queen and 26 cases favouring a black card.

Probability is a quantitative measure of chances of occurrence. There are 3 approaches to the study of probability.

1. Classical approach
2. Statistical or empirical approach
3. Axiomatic approach

Classical Definition of Probability: If an event A can occur in m different ways out of a total of n ways all of which are **equally likely** and mutually exclusive, then the probability of the event A is given by

$$P(A) = \frac{m}{n} = \frac{\text{favourable cases}}{\text{Exhaustive cases}}.$$

- For a null set, m = 0. Hence $P(\phi) = 0$
- For the sample space m = n. Hence P(S) = 1
- $0 \le m \le n$. Hence $0 \le m/n \le 1$ i.e. $0 \le P(A) \le 1$
- m outcomes are favourable to A $\Rightarrow$ remaining n – m are favourable to A′.
  Hence $P(A') = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - P(A)$, i.e. $P(A) + P(A') = 1$

Statistical Definition of Probability:

If an experiment is repeated several times under essentially homogeneous and identical conditions, then the limiting value of the ratio of the number of times the event occurs to the number of trials, as the number of trials become indefinitely large, is called the probability of that event.

i.e. if an event A occurs m times in n trials then $P(A) = \lim_{n\to\infty} \frac{m}{n}$

Axiomatic approach:

1. For any event E, Probability P (E) ≥ 0.
2. When S is the sample space of an experiment; that is the set of all possible outcomes, P(S) = 1.
3. If A and B are mutually exclusive outcomes, then P (A ∪ B) = P (A) + P (B).

THOREMS:

Theorem 1.1:
If ϕ is the empty set, then P (ϕ) = 0.
Theorem 1.2:
If $\bar{A}$ is the complementary even of A, then P(A) = 1- P($\bar{A}$).
Theorem 1.3:
If A and B are any two events, then P$(A \cup B)$ = P(A) + P(B) – P $(A \cap B)$
Theorem 1.4: SAME as above for three events.
Theorem 1.5:
If $A \subset B$ then P(A) ≤ P(B).
Theorem 1.6:
Probability that exactly one of the event A or B occur i.e., P $(\{A \cap \bar{B}\} \cup \{\bar{A} \cap B\})$ = P(A)+P(B)-2 P $(A \cap B)$

Problems:

1. A pair of dice is rolled. What is the probability of getting a sum greater than 6? A pair of dice is rolled. What is the probability of rolling a sum neither 5 nor 10?

2. There are 8 positive numbers and 6 negative numbers. 4 numbers are chosen at random and multiplied. What is the probability that the product is a positive number?

3. A number is chosen between 1 and 50. What is the probability that it is divisible by 8?

4. An urn contains 5 red and 10 black balls. 8 of them are placed in another urn. What is the chance that the later then contains 2 red and 6 black balls?

5. A bag contains 8 white and 6 red balls. What is the probability of drawing two balls of the same color?

6. The coefficient A, b, c of the quadratic equation $ax^2 + \mathrm{bx} + \mathrm{c} = 0$ are determined by throwing a die 3 times find the probability that 1. Roots are real 2. Roots are complex.

7. Three group of children contain respectively 3 girls 1 boy, 2 girls 2 boys, 1 girl 3 boys. One child is selected at random from each group. Show that the chance that the 3 selected consist of 1 girl and 2 boys is 13/32.

8. A committee of 4 person is to be appointed from 3 offices of production department, 4 officers from purchase department, 2 officer from the sales department and 1 charted accountant. Find the probability of

1. There must be one from each category.
2. It should have at least one from the purchase department.
3. The CA must be in the committee.

9. What is the probability that a randomly selected year contains 53 Sundays?

10. Each of 2 person A and B tosses 3 fair coins. Find the probability that they get the same number of heads.

11. A and B throw a die alternatively till one of them gets a ‘6’ and wins the game. Find their respective probabilities of winning if A starts first.

    Solution: Let S denote the success (getting a ‘6’) and F denote the failure (not getting a ‘6’).
    Thus, P(S) =1/6, P(F) =5/6
    P(A wins in the first throw) = P(S) = 1/6
    A gets the third throw, when the first throw by A and second throw by B result into failures.
    Therefore, P(A wins in the 3rd throw) = P(FFS) = P(F)P(F)P(S)= (5/6)(5/6)(1/6)
    P(A wins in the 5th throw) = P (FFFFS)= (5/6)(5/6) (5/6)(5/6)(1/6)
    Hence, P(A wins) =1/6 + (5/6)(5/6)(1/6) + (5/6)(5/6) (5/6)(5/6)(1/6) + ... = 6/11 (G.P infinite sum)
    P(B wins) = 1 – P (A wins) = 1- (6/11)= 5/11.

12. A and B throw alternatively a pair of die. A wins if he throws sum 6 before B throws sum 7 and B wins the other way. If A begins, find his chances of winning the game.

13. Six people toss a fair coin one by one. The game is win by the player who throws head. Find the probability of success of the 4^{th} player.

Answers:
1. 7/12, 29/36
2. 505/1001
3. 6/25
4. 140/429
5. 43/91
6. 43/216, 173/216
7. 13/32
8. 4/35, 195/210, 0.4
9. 1/7 and 2/7(leap year)
10. 5/16
11. Solution
12. For A: 30/61 For B:31/61
13. 4/63

Conditional Probability

Until now in probability, we have discussed the methods of finding the probability of events. If we have two events from the same sample space, does the information about the occurrence of one of the events affect the probability of the other event? Let us try to answer this question by taking up a random experiment in which the outcomes are equally likely to occur. Consider the experiment of tossing three fair coins. The sample space of the experiment is

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

Since the coins are fair, we can assign the probability 1/8 to each sample point. Let E be the event 'at least two heads appear' and F be the event 'first coin shows tail'.

Then, E = {HHH, HHT, HTH, THH} and F = {THH, THT, TTH, TTT}
Therefore P(E)=P ({HHH})+P({HHT}+P ({HTH})+P ({THH})
=1/8+1/8+1/8+1/8=1/2
and
P(F) = P ({THH})+P ({THT}) + P ({TTH}) + P ({TTT}) =1/8+1/8+1/8+1/8=1/2

Also E ∩ F = {THH} with P(E ∩ F) = P({THH}) =1/8

Now, suppose we are given that the first coin shows tail, i.e. F occurs, then what is the probability of occurrence of E? With the information of occurrence of F, we are sure that the cases in which first coin does not result into a tail should not be considered while finding the probability of E. This information reduces our sample space from the set S to its subset F for the event E. In other words, the additional information really amounts to telling us that the situation may be considered as being that of a new random experiment for which the sample space consists of all those outcomes only which are favourable to the occurrence of the event F.

Now, the sample point of F which is favourable to event E is THH.
Thus, Probability of E considering F as the sample space =1/4
or Probability of E given that the event F has occurred =1/4
This probability of the event E is called the conditional probability of E given that F has already occurred and is denoted by P (E|F).
Thus P(E|F) =1/4
Note that the elements of F which favour the event E are the common elements of E and F, i.e., the sample points of E ∩ F.
Thus, we can also write the conditional probability of E given that F has occurred as

$$P(E|F) = \frac{Number\ of\ elementary events favourable\ to\ E\cap F}{Number\ of\ elementary events\ which\ are favourable\ to\ F} = \frac{n(E\cap F)}{n(F)}$$

Dividing the numerator and the denominator by total number of elementary events of the sample space, we see that P(E|F) can also be written as

$$P(E|F) = \frac{n(E\cap F)/n(s)}{n(F)/n(s)} = \frac{P(E\cap F)}{P(F)}$$

If E and F are two events associated with the same sample space of a random experiment, the conditional probability of the event E given that F has occurred, i.e. P (E|F) is given by

$P(E|F) = \dfrac{P(E\cap F)}{P(F)}$, *provided* $P(F) \neq 0$.

Example: A die is tossed if the number is odd on the face, what is the probability that it is a prime?
S = {1,2,3,4,5,6}
A= {1,3,5} reduction in the sample space because of the additional information that it is odd.
B= {3,5}

$$P(B|A) = \frac{P(B\cap A)}{P(A)} = \frac{2}{3}$$

NOTE:

If A and B are independent events, then $P(B|A) = \frac{P(B\cap A)}{P(A)} = \frac{P(B).P(A)}{P(A)} = P(B)$.

Theorem:

If A and B are 2 independent events of S then prove that A& $\bar{B}$, *B &* $\bar{A}$ *and* $\bar{A}$ *&* $\bar{B}$ *are also independent.*

Problems:

1. *If A and B are 2 independent events of S such that* $P(\bar{A} \cap B)=2/15$, $P(A \cap \bar{B})=1/6$ *then find P(B). Answer= 1\6 or 4\5.*
2. *If A and B are 2 independent events of S such that P(A)= 1/3, P(B)=1/4,* $P(A\cup B) =1/2$ *then find i) P(A|B) ii) P(B|A) iii)* $P(A\cap \bar{B})$ *iv)* $P(A|\bar{B})$. *Answer: i) 1/3 ii) 1/4 iii)1/4 iv) 1/3*
3. *In a certain town 40% have brown hair, 25% have brown eyes, 15% have both brown hair and brown eyes. A person is selected at random.*
   i. *If he has brown hair, then what is the probability that he has brown eyes also.*
   ii. *If he has brown eyes, then what is the probability that he has not have brown hair.*
   iii. *Determine the probability that he neither have brown hair nor brown eyes.*
   Answer: 3/8, 0.4 and 0.5

4. *A bag contains 10 gold coins and 8 silver coins. Two successive drawings of 4 coins are made such that.*
    i. *The coins are replaced before the second trial.*
    ii. *The coins are not replaced before the second trial.*

Find the probability that the first drawing will give 4 gold coins and second drawing will give 4 silver coins.

i) coins are replaced before the second trial

Gold = 10

Silver = 8

first drawing will give 4 gold = ${}^{10}C_4/{}^{18}C_4$

second drawing will give 4 silver = ${}^{8}C_4/{}^{18}C_4$

probability that the first drawing will give 4 gold and the second 4 silver coins. = $({}^{10}C_4/{}^{18}C_4) * ({}^{8}C_4/{}^{18}C_4)$

= $\mathbf{{}^{10}C_4 * \ {}^{8}C_4 /({}^{18}C_4)^2}$

ii) the coins are not replaced before the second trial.

first drawing will give 4 gold = ${}^{10}C_4/{}^{18}C_4$

second drawing will give = ${}^{8}C_4/{}^{14}C_4$

probability that the first drawing will give 4 gold and the second 4 silver coins. = $({}^{10}C_4/{}^{18}C_4) * ({}^{8}C_4/{}^{14}C_4)$

$\mathbf{=({}^{10}C_4 * \ {}^{8}C_4) /({}^{18}C_4 . {}^{14}C_4)}$

5. *Two defective tubes get mixed up with 4 good ones. The tubes are tested one by one, until both defective are found. What is the probability that the last defective tube is obtained on a) 2^{nd} test, b)3^{rd} test, c)6^{th} test.*

Total probability theorem, Bayes' theorem

Partition of a set:

The family of sets $C_1, C_2, \ldots, C_n$ is said to be a partition of S, if

i. $\bigcup_{i=1}^{n} C_i = S$ Collectively exhaustive

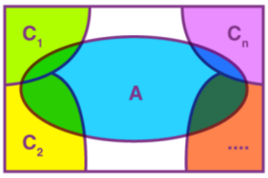
ii. $C_i \cap C_j = \phi, i \neq j, \forall i, j$ Mutually Exclusive

Law of Total Probability Statement

Let events $C_1, C_2 \ldots C_n$ form partitions of the sample space S, where all the events have a non-zero probability of occurrence. For any event, A associated with S, according to the total probability theorem,

$$P(A) = \sum_{k=0}^{n} P(C_k)P(A|C_k)$$

Proof:



$S = C_1 \cup C_2 \cup \ldots \ldots \cup C_n$

For any event A,

$A = A \cap S$

$= A \cap (C_1 \cup C_2 \cup \ldots \ldots \cup C_n)$

$= (A \cap C_1) \cup (A \cap C_2) \cup \ldots \cup (A \cap C_n) \ldots \ldots (1)$

We know that $A \cap C_i$ and $A \cap C_k$ are the subsets of C_i and C_k. Here, C_i and C_k are disjoint for $i \neq k$. since they are mutually events which implies that $A \cap C_i$ and $A \cap C_k$ are also disjoint for all $i \neq k$. Thus,

$P(A) = P[(A \cap C_1) \cup (A \cap C_2) \cup \ldots \cup (A \cap C_n)]$

$= P(A \cap C_1) + P(A \cap C_2) + \ldots + P(A \cap C_n) \ldots \ldots \ldots (2)$

We know that,

$P(A \cap C_i) = P(C_i) P(A|C_i)$(By multiplication rule of probability) . . . . (3)

Using (2) and (3), (1) can be rewritten as,

$P(A) = P(C_1)P(A|C_1) + P(C_2)P(A|C_2) + P(C_3)P(A|C_3) + \ldots \ldots + P(C_n)P(A|C_n)$

Hence, the theorem can be stated in form of equation as,

$$P(A) = \sum_{k=0}^{n} P(C_k)P(A|C_k)$$

Bayes Theorem Statement

Let $E_1, E_2, \ldots, E_n$ be a set of events associated with a sample space S, where all the events $E_1, E_2, \ldots, E_n$ have nonzero probability of occurrence and they form a partition of S. Let A be any event associated with S, then according to Bayes theorem,

$$P(E_i \mid A) = \frac{P(E_i)P(A \mid E_i)}{\sum_{k=1}^{n} P(E_k)P(A|E_k)}$$

for any k = 1, 2, 3, ..., n

Bayes Theorem Proof

According to the conditional probability formula,

$$P(E_i \mid A) = \frac{P(E_i \cap A)}{P(A)} \ldots (1)$$

Using the multiplication rule of probability,

$$P(E_i \cap A) = P(E_i)P(A \mid E_i) \ldots (2)$$

Using total probability theorem,

$$P(A) = \sum_{k=1}^{n} P(E_k)P(A|E_k) \ldots (3)$$

Putting the values from equations (2) and (3) in equation 1, we get

$$P(E_i \mid A) = \frac{P(E_i)P(A \mid E_i)}{\sum_{k=1}^{n} P(E_k)P(A|E_k)}$$

Problems:

1. A person has undertaken a construction job. The probabilities are 0.65 that there will be strike, 0.80 that the construction job will be completed on time if there is no strike, and 0.32 that the construction job will be completed on time if there is a strike. Determine the probability that the construction job will be completed on time.

   Solution: Let A be the event that the construction job will be completed on time, and B be the event that there will be a strike.
   We must find P(A).
   We have P(B) = 0.65, P(no strike) = P(B′) =1 − 0.65 = 0.35
   P(A|B) = 0.32, P(A|B′) = 0.80
   Since events B and B′ form a partition of the sample space S, therefore, by theorem on total probability, we have.
   P(A) = P(B) P(A|B) + P(B′) P(A|B′)
   = 0.65 × 0.32 + 0.35 × 0.8 = 0.208 + 0.28 = 0.488

2. suppose 3 companies x y z produce TVs x produces twice as many as y while y and z produce same number. It is known that 2% of x, 2% of y, 4% of z are defected. All the TVs are produced are put into 1 shop and then 1 tv is chosen at random what is the probability that the tv is defecated. Suppose a tv chosen is defective what is the probability that this tv is produced by company x.

Solution:

Let X denote the event "tv is produced by company x."

Let Y denote the event "tv is produced by company y."

Let Z denote the event "tv is produced by company z."

Let D denote the event "tv is defected".

Given

$$P(X)=0.5, P(Y)=P(Z)=0.25$$
$$P(D|X)=0.02, P(D|Y)=0.02, P(D|Z)=0.04$$

a) Theorem of total probability

$$P(D)=P(X)P(D|X)+P(Y)P(D|Y)$$
$$+P(Z)P(Z|D)$$
$$=0.5(0.02)+0.25(0.02)+0.25(0.04)=0.025$$

The probability that the tv is defected is 0.025.

ii) By the Bayes' Rule

$$P(X|D)$$
$$=\frac{P(X)P(D|X)}{P(X)P(D|X)+P(Y)P(D|Y)+P(Z)P(Z|D)}$$
$$=\frac{0.5(0.02)}{0.5(0.02)+0.25(0.02)+0.25(0.04)}$$
$$=0.4$$

The probability that this tv is produced by company x is 0.4.

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3. There are 3 boxes, the first one containing 1 white, 2 red and 3 black balls: the second one containing 2 white, 3 red and 1 black ball and the third one containing 3 white, 1 red and 2 black balls. A box is chosen at random and from it two balls are drawn at random. One ball is red and the other, white. What is the probability that they come from the second box? Ans: 6/11

4. A randomly selected year has 53 Sundays. Find the probability that it is a leap year. Ans: 0.4

5. Two factories produce identical clocks. The production of the first factory consists of 10,000 clocks of which 100 are defective. The second factory produces 20,000 clocks of which 300 are defective. What is the probability that a particular defective clock was produced in the first factory? Ans:0.25

6. One percent of the population suffers from a certain disease. There is blood test for this disease, and it is **99%** accurate, in other words, the probability that is gives the correct answer is **0.99**, regardless of whether the person is sick or healthy. A person takes the blood test, and the result says that he has the disease. The probability that he actually has the disease, is?

Solution:

A be the event of having the disease.

B be the event of testing positive.

$P(A) = 0.01$

$P(B) = P(B/A)P(A) + P(B/notA)P(notA)$

$P(B) = 0.01 * 0.99 + 0.01 * 0.99 = 0.0198$

$P(A \cap B) = 0.99 * 0.01$

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$\Rightarrow \frac{0.99 * 0.01}{0.0198}$$

=0.5 or 50%

7. If a machine is correctly set up, it produces 90% acceptable items. If it is incorrectly set up, it produces only 40% acceptable items. Past experience shows that 80% of the set ups are correctly done. If after a certain set up, the machine produces 2 acceptable items, find the probability that the machine is correctly setup.

Solution:
Let A be the event that the machine produces 2 acceptable items.
Also let B_1 represent the event of correct set up and B_2 represent the event of incorrect setup.
Now $P(B_1) = 0.8$, $P(B_2) = 0.2$
$P(A|B_1) = 0.9 \times 0.9$ and $P(A|B_2) = 0.4 \times 0.4$
Therefore $P(B_1|A) = 0.95$

8. It is suspected that a patient has one of the diseases A_1, A_2, A_3. Suppose that the population suffering from this illness are in the ratio 2:1:1. The patient is given a test which turns out to be positive in 25%of the cases of A_1, 50% of the cases of A_2 and 90% of the cases of A_3. Given that out of 3 tests taken by the patient two are positive, then find the probability for each of the diseases. Ans: 0.3128, 0.4170, 0.2703
Solution:
$A_i \rightarrow$ the patient has the illness A_i
$B \rightarrow$ two test results are positive.
$P(A_1) = 2/4$ $P(A_2) = 1/4$ $P(A_3) = 1/4$

$P(B|A_1) = [PPN+PNP+NPP]$
$= {}^3C_2(1/4)^2(3/4)$

$P(B|A_2) = {}^3C_2(1/2)^2(1/2)$

$P(B|A_3) = {}^3C_2(9/10)^2(1/10)$

$P(A_1|B) = ?$ $P(A_2|B) = ?$, $P(A_3|B) = ?$

9. An anchor with an accuracy of 75% fires 3 arrows at one target. The probability of the target falling is 0.6 if he hit once, 0.7 if he his twice. 0.8 if he hits thrice. Given that, the target has fallen find the probability that it was hit twice. Ans:0.411
Solution:
$B_i \rightarrow$ target is hit the i^{th} time.
$B \rightarrow$ target falls
P (anchor hits) =0.75
$P(B|B_1) = 0.6$, $P(B|B_2) = 0.7$, $P(B|B_3) = 0.8$

$P(B_1) = {}^3C_1(1/4)^2(3/4)$ $P(B_2) = {}^3C_2(1/4)(3/4)^2$ $P(B_3) = {}^3C_3(1/4)^0(3/4)^3$

$P(B_2|B) = ?$

Expectation and Variance

Given a random variable, we often compute the expectation and variance, two important summary statistics. The expectation describes the average value and the variance describes the spread (amount of variability) around the expectation.

Let X be a random variable whose possible values $x_1, x_2, x_3, \ldots, x_n$ occur with probabilities $p_1, p_2, p_3, \ldots, p_n$, respectively. The mean of X, denoted by μ, is the number $\sum x_i p_i$ i.e. the **mean of X is the weighted average of the possible values of X**, each value being weighted by its probability with which it occurs. The mean of a random variable X is also called the expectation of X.
Thus, $\boldsymbol{\mu = E(X) = x_1 p_1 + x_2 p_2 + \ldots + x_n p_n}$.

Definition: Let X be a **continuous** random variable with p.d.f. $f_X(x)$. The expected value of X is

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f_X(x)\, dx.$$

Definition: Let X be a **discrete** random variable with probability function $f_X(x)$. The expected value of X is

$$\mathbb{E}(X) = \sum_x x f_X(x) = \sum_x x \mathbb{P}(X = x).$$

Properties of Expectation:

1. E(c) = c
   i.e E(c) = $\sum cP(X = c) = c.1 = c$
2. E(aX+b)= $\sum(aX + b)P(X = x) = a\sum xP(X = x) + b\sum P(X = x)$
   = aE(x)+b.1= aE(X)+b
3. E(E(X))= E(X)

Example Let a pair of dice be thrown and the random variable X be the sum of the numbers that appear on the two dice. Find the mean or expectation of X.
Solution The sample space of the experiment consists of 36 elementary events in the form of ordered pairs (x_i , y_i), where x_i = 1, 2, 3, 4, 5, 6 and y_i = 1, 2, 3, 4, 5, 6. The random variable X i.e. the sum of the numbers on the two dice takes the values 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 or 12.

X or x_i	2	3	4	5	6	7	8	9	10	11	12
P(X) or p_i	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Now $\mu = 7$ Thus, the mean of the sum of the numbers that appear on throwing two fair dice is 7.

Variance

The mean of the random variable does not give us information about the variability in the values of the random variable. In fact, if the variance is small, then the values of the random variable are close to the mean. Also random variables with different probability distributions can have equal means.

$$s_x = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}}$$

$$\sigma_x^{\,2} = \mathrm{Var}(X) = \sum_{i=1}^{n}(x_i - \mu)^2\, p(x_i)$$

$$= \sum_{i=1}^{n} x_i^{\,2}\, p(x_i) + \sum_{i=1}^{n} \mu^2\, p(x_i) - \sum_{i=1}^{n} 2\mu x_i\, p(x_i)$$

$$= \sum_{i=1}^{n} x_i^{\,2}\, p(x_i) + \mu^2 \sum_{i=1}^{n} p(x_i) - 2\mu \sum_{i=1}^{n} x_i\, p(x_i)$$

$$= \sum_{i=1}^{n} x_i^{\,2}\, p(x_i) + \mu^2 - 2\mu^2 \left[\text{since} \sum_{i=1}^{n} p\ (x_i) = 1 \text{ and } \mu = \sum_{i=1}^{n} x_i\, p(x_i)\right]$$

$$= \sum_{i=1}^{n} x_i^{\,2}\, p(x_i) - \mu^2$$

$$\mathrm{Var}\ (X) = \sum_{i=1}^{n} x_i^{\,2}\, p(x_i) - \left(\sum_{i=1}^{n} x_i\, p(x_i)\right)^2$$

$$\mathrm{Var}\ (X) = E(X^2) - [E(X)]^2, \text{ where } E(X^2) = \sum_{i=1}^{n} x_i^{\,2}\, p(x_i)$$

Example Find the variance of the number obtained on a throw of an unbiased die.

Solution The sample space of the experiment is S = {1, 2, 3, 4, 5, 6}.

Let X denote the number obtained on the throw. Then X is a random variable which can take values 1, 2, 3, 4, 5, or 6.

$$E(X^2) = 1^2 \times \frac{1}{6} + 2^2 \times \frac{1}{6} + 3^2 \times \frac{1}{6} + 4^2 \times \frac{1}{6} + 5^2 \times \frac{1}{6} + 6^2 \times \frac{1}{6} = \frac{91}{6}$$

$$\mathrm{Var}\ (X) = E\ (X^2) - (E(X))^2$$

$$= \frac{91}{6} - \left(\frac{21}{6}\right)^2 = \frac{91}{6} - \frac{441}{36} \quad \frac{35}{12}$$

Properties of Variance:

1. **V(c) =0**
   i.e V(c) = E(c^2) –[E(c)]^2= $c^2 - [c]^2 = 0$
2. **V(aX+b) = $a^2 V(X)$**
   WKT, V(X)= E(X^2) –[E(X)]^2
   V(aX+b)= E({aX + b}^{2}) –[E(aX + b)]^2
   $$= E(a^2x^2 + b^2 + 2abX) - \{aE(X) + b\}^2$$
   $$= a^2E(X^2) + b^2 + 2abE(X) - [a^2\{E(X)\}^2 + b^2 + 2abE(X)]$$
   $$= a^2\{E(X^2) - [E(X)]^2\} - 0$$
   $$= a^2V(X)$$

Example: Let X be a continuous random variable with p.d.f.

$$f_X(x) = \begin{cases} 2x^{-2} & \text{for } 1 < x < 2, \\ 0 & \text{otherwise.} \end{cases}$$

Find $\mathbb{E}(X)$ and $\text{Var}(X)$.

$$\begin{aligned} E(X) = \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \int_1^2 x \times 2x^{-2}\, dx \;&=\; \int_1^2 2x^{-1}\, dx \\ &= \Big[2\log(x)\Big]_1^2 \\ &= 2\log(2) - 2\log(1) \\ &= 2\log(2). \end{aligned}$$

For Var(X), we use

$$\text{Var}(X) = \mathbb{E}(X^2) - \{\mathbb{E}(X)\}^2 .$$

Now

$$\begin{aligned} E(X^2) = \int_{-\infty}^{\infty} x^2\, f_X(x)\, dx = \int_1^2 x^2 \times 2x^{-2}\, dx \;&=\; \int_1^2 2\, dx \\ &= \Big[2x\Big]_1^2 \\ &= 2 \times 2 - 2 \times 1 \\ &= 2. \end{aligned}$$

Thus

$$\begin{aligned} \text{Var}(X) &= \mathbb{E}(X^2) - \{\mathbb{E}(X)\}^2 \\ &= 2 - \{2\log(2)\}^2 \\ &= 0.0782. \end{aligned}$$

Mean, Median and MODE

If X is A CRV then median M, $\int_{-\infty}^{M} f(x)dx = \int_{M}^{\infty} f(x)\,dx = \frac{1}{2}$
And MODE of X for which f(x) is maximum, $f^{|} = 0$ and $f^{||} < 0$.

Problems:

1. Find mean, Median and mode also variance of a random variable X having pdf f(x)= $\begin{cases} 6x(1-x) & 0 \le x \le 1 \\ 0 & otherwise \end{cases}$.
   Ans: E(x)= ½ , E(x^2)= 3/10, V(x)= 1/20, M=½, MODE= ½

2. Find pdf, mean, Median and mode also variance of a random variable X having cdf F(x)= $\begin{cases} 1-e^{-x} - xe^{-x} & x \ge 0 \\ 0 & otherwise \end{cases}$.
   Ans: E(x)= 2 , E(x^2)= 6, V(x)= 2, M= 5/3, MODE=1

   Formulas: MODE= 3M-2E(X) AND $\int UV$ PARTS: $\int UV = U \int V - \int U' \int V$

3. If $F(x) = \begin{cases} -e^{-\frac{x^2}{2}} & x > 0 \\ 0 & x \le 0 \end{cases}$ then find V(X).

   Solution:

   The pdf of X is $f(x) = \begin{cases} xe^{-\frac{x^2}{2}} & x > 0 \\ 0 & x \le 0 \end{cases}$,
   E(X) = $\int_0^{\infty} x^2 \; e^{-\frac{x^2}{2}} dx = \sqrt{2} \int_0^{\infty} \sqrt{t}\, e^{-t}\, dt$ using $\frac{x^2}{2} = t, \; x\,dx = dt$

   WKT, $\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t}\, dt$ and $\Gamma(n+1) = n\,\Gamma(n)\,, \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

   E(X)= $\sqrt{2}\,\Gamma(3/2) = \sqrt{2}\,.\,\frac{1}{2}.\,\sqrt{\pi} = \frac{\sqrt{\pi}}{\sqrt{2}}$

   E(X^2)= $\int_0^{\infty} x^3 \; e^{-\frac{x^2}{2}} dx = 2\int_0^{\infty} te^{-t}\, dt$ using $\frac{x^2}{2}$ =t
   = 2 $\Gamma(2)$ =2

   V(X) = 2 - $\left(\sqrt{\pi}\frac{1}{\sqrt{2}}\right)^2$ = $4 - \frac{\pi}{2}$

Uniform distribution

Let X be a continuous random variable assuming all values in the interval [a, b] where a and b are finite. If the pdf of X is given by

$$f(x) = \begin{cases} \frac{1}{(b-a)} & a \leq x \leq b \\ 0 & else\ where \end{cases}$$

then we say that X has uniform distribution defined over [a, b].

Note that,   for any sub interval [c, d],

$$\mathrm{P}\ (\mathrm{c}\ <\ \mathrm{X}\ < \mathrm{d}) = \int_c^d f(x)dx = \int_c^d \frac{1}{(b-a)}dx = \frac{(d-c)}{(b-a)}$$

Cdf = $F(x) = \begin{cases} 0 & x \le a \\ \frac{x-a}{(b-a)} & a < x < b \\ 1 & x \ge b \end{cases}$

Mean E(X) = $\int_a^b x f(x) dx = \frac{1}{(b-a)} \left\{ \frac{x^2}{2} \right\} \begin{matrix} b \\ a \end{matrix} = \frac{(b-a)(a+b)}{2(b-a)} = \frac{(a+b)}{2}$

E(X^2) = $\int_a^b x^2 f(x) dx = \frac{1}{(b-a)} \left\{ \frac{x^3}{3} \right\} \begin{matrix} b \\ a \end{matrix} = \frac{(b^3 - a^3)}{3(b-a)} = \frac{(a^2 + b^2 + ab)}{3}$

Variance V(X) = E(X^2) – $[E(X)]^2 = \frac{(a^2 + b^2 + ab)}{3} - \{ \frac{(a+b)}{2} \}^2 = \frac{(b-a)^2}{12}$

Problems:

1. If X is uniformly distributed over (-2, 2) then find i) P(X<1) ii) $P\left(|X-1| \geq \frac{1}{2}\right)$.

Solution: Given that $X \in U(-2,2)$.

Therefore, $f(x) = \begin{cases} \frac{1}{4} & -2 \leq x \leq 2 \\ 0 & else\ where \end{cases}$

i) $P(X<1) = \int_{-\infty}^{1} f(x)dx = \int_{-2}^{1} \frac{1}{4}\ dx = \frac{3}{4}$

ii) $P\left(|X-1| \geq \frac{1}{2}\right) = 1 - P\left(|X-1| < \frac{1}{2}\right)$

$= 1 - P\left(-\frac{1}{2} < X - 1 < \frac{1}{2}\right)$

$= 1 - P\left(\frac{1}{2} < X < \frac{3}{2}\right)$

$= 1 - \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{4}\ dx = 1 - \frac{1}{4} = \frac{3}{4}$

2. If K is uniformly distributed over (0, 5) then what is the probability that the roots of the equation $4x^2 + 4xK + K + 2 = 0$ are real ?

Solution: Given that K$\in U(0,5)$.

Therefore, $f(k) = \begin{cases} \frac{1}{5} & 0 \le k \le 5 \\ 0 & else\ where \end{cases}$

P { Roots are real} = P{ $(4K)^2 - 4.4(k+2) \geq 0$ }

= P{ $K^2 - K - 2 \geq 0$ } = P{ $(K+1)(K-2) \geq 0$ }

= P{ $(K+1) \geq 0, (K-2) \geq 0$ or $(K+1) \leq 0, (K-2) \leq 0$}

= P{ $K \geq -1, K \geq 2$ or $K \leq -1, K \leq 2$ }

= P{ $K \geq 2$ or $K \leq -1$ }

= P{ $K \geq 2$ } + P{ $K \leq -1$ }

$= \int_2^5 \frac{1}{5} dx + \int_{-\infty}^{-1} 0\, dx = 3/5$

Problems on Variance and Expectation:

1. A student takes a multiple choice test consisting of 2 problems. The first one has 3 possible answers and the second one has 5. The student chosen at random, one answer as the right answer for each of the 2 problems. Let X denote the number of right answers of student. Find V(X).

Solution: Let X : the number of right answers

X:	0	1	2
P(X=x):	8/15	6/15	1/15

{P(X =0) = 2/3 . 4/5 and P(X =1) = 1/3 . 4/5 + 2/3.1/5 = 6/15 and P(X =2) = 1/3 . 1/5 =1/15}

E(X) = 0 + 1.(6/15) + 2. (1/15) = 8/15, E(X^2) = 0 + 6/15 + 4.(1/15) =10/15 = 2/3 and

V(X) = E(X^2) – $\{E(X)\}^2$ = 2/3 – $\{8/15\}^2$ = 86/225=0.3822

2. Three balls are randomly selected from an urn containing 3 white, 3 red, 5 black balls. The person who selects the ball wins \$ 1.00 for each white ball selected and lose \$1.00 for each red ball selected. Let X be the total winnings from the experiment. Find the Probability distribution of X and V(X).

Solution: Let X : total winnings

X:	-3	-2	-1	0	1	2	3
P(X = x):	1/165	15/165	39/165	55/165	39/165	15/165	1/165

{P(X =0) = P{Selection of 3B or 1W, 1B, 1R balls} = $\frac{5C_3}{11C_3}$ + $\frac{3C_1 \times 3C_1 \times 5C_1}{11C_3}$ = 55/165

P(X =1) = P(X = -1) = P{Selection of 2B, 1W or 2W, 1R balls} = $\frac{3C_1 \times 3C_2}{11C_3}$ + $\frac{3C_1 \times 5C_2}{11C_3}$ = 39/165

P(X =2) = P(X = - 2) = P{Selection of 1B, 2W balls} = $\frac{5C_1 \times 3C_2}{11C_3}$ = 15/165

and P(X =3) =P(X = -3) = P{Selection of 3W balls} = $\frac{3C_3}{11C_3}$ = 1/165}

E(X) = 0 , E(X^2) = 2{9/165 + 15X4/165 + 39/165 }=216/165 and V(X)= 216/165

3. A coin is tossed till head appears then find the probability distribution on number of tosses. Let X denote the number of tosses. Find E(X).

Solution:

X denote the number of tosses

P(H)= p

P(T)= q

X	**1**	**2**	**3**	**…**
P(X)	p	qp	$\mathrm{q}^2 p$	…

Therefore, P(X=k)= pq^{k-1}

Hence, E(X) = $\sum_1^\infty k\ pq^{k-1}$ = p $\sum_1^\infty k\ q^{k-1}$= $\frac{p}{(1-q)^2} = \frac{1}{p}$

4. Suppose that an electronic device has a life length X (in units' of 1000 hours) which is considered as a continuous random variable with the following pdf: f(x) = $e^{-x}, x > 0$, Suppose that the cost of manufacturing one such item is 2 rupees. The manufacturer sells the item for 5 rupees but guarantees a total refund if $x \leq 0.9$. What is the manufacturer's expected profit per item?

Solution:

Y- expected profit

Profit, p= $\begin{cases} 3 & x > 0.9 \\ -2 & x \leq 0.9 \end{cases}$

E(Y)= 3 (probability of x>0.9) +(-2) (probability of $x \leq 0.9$)

E(Y)= 3 ($\int_{0.9}^{\infty} e^{-x}\, dx$) +(-2) ($\int_{-\infty}^{0.9} e^{-x}\, dx$)

E(Y)= 3 ($\int_{0.9}^{\infty} e^{-x}\, dx$) +(-2) (1-$\int_{0.9}^{\infty} e^{-x}\, dx$)

E(Y)= 3 ($\int_{0.9}^{\infty} e^{-x}\, dx$) +(-2) (1-$\int_{0.9}^{\infty} e^{-x}\, dx$)

E(Y)= 3 (0.4066) +(-2) (1-0.466)

E(Y)=0.033

5. Let X be a random variable with probability function

$$P(X=k)=p\,(1-p)^{k-1}\ ,k=\ \ 1,2,3\dots n.$$ Find V(X).

Solution: Given $P(X=k)=p\,(1-p)^{k-1}\ ,k=1,2,3\dots n$

$E(X)=\sum_1^n x\,p(x)\ =\sum_1^n k\,p(1-p)^{k-1}$

$= p\{\ 1 + 2\ (1-p) + 3\ (1-p)^2+\dots\}$

$= p\ .\ \frac{1}{(1-(1-p))^2} = 1/p$

P(H)is not equal to P(T). P(H) =p, P(T) = 1-p

$1+x+(x)^2+\dots\dots=1/1-x$

$1+2x+3(x)^2+\ \dots= 1/(1-x)^2$

X	1	2	3	...	N
P(X=k)	p	P(1-p)	(1-p)(1-p)p	...	(1-p) (1-p)..... (1-p) (1-p)p

$\sum_1^n p(x)$ =p $[1+(1-p)+(1-p)^2+\dots]$ =p/1-(1-p)=1

E(X^2)= $\sum_1^n x^2\, p(x)$ $= \sum_1^n k^2\, p(1-p)^{k-1}$ = p{ 1 + 4 (1-p) + 9 $(1-p)^2$+….} =pS

Where S = { 1 + 4 (1-p) + 9 $(1-p)^2$+….} ……………..(1)

Multi ply (1) by 1-p we get,

S (1-p) = { 1-p + 4 $(1-p)^2$+ 9 $(1-p)^3$+….} …………(2)

Eq(1)- Eq (2) simplifies to Sp = {1 +3(1-p) + 5 $(1-p)^2$ +………..} …………………..(3)

Multi ply (3) by 1-p we get,

Sp (1-p) = {1-p +3$(1-p)^2$ + 5 $(1-p)^3$ +………..} …………………..(4)

Eq(3)- Eq (4) simplifies to

Sp^2 = {1 +2(1-p) + 2 $(1-p)^2$ +………..} …………………..(3)

Therefore, Sp^2 = {1 +2(1-p) { 1+ (1-p) +$(1-p)^2$ +………..} }

S = $\frac{1}{p^2}$ [1+ 2(1-p) $\frac{1}{1-(1-p)}$] =$\frac{2}{p^3} - \frac{1}{p^2}$

E(X^2)= p($\frac{2}{p^3} - \frac{1}{p^2}$) = $\frac{2}{p^2} - \frac{1}{p}$

V(X) = $\frac{1-p}{p^2}$

Two-Dimensional Random Variable

Let S be the sample space associated with a random experiment E. Let X=X(S) and Y=Y(S) be two functions each assigning a real number to each s∈S. Then (X,Y) is called a two dimensional random variable.

Joint Probability distribution function

Definition. (a) Let (X, Y) be a two-dimensional discrete random variable. With each possible outcome (x_i, y_j) we associate a number $p(x_i, y_j)$ representing $P(X = x_i, Y = y_j)$ and satisfying the following conditions:

$$(1)\quad p(x_i, y_j) \geq 0 \quad \text{for all} \quad (x, y),$$

$$(2)\quad \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} p(x_i, y_j) = 1. \tag{6.1}$$

Joint Probability density function

(b) Let (X, Y) be a continuous random variable assuming all values in some region R of the Euclidean plane. The *joint probability density function* f is a function satisfying the following conditions:

$$(3)\quad f(x, y) \geq 0 \quad \text{for all} \quad (x, y) \in R,$$

$$(4)\quad \iint_R f(x, y)\, dx\, dy = 1. \tag{6.2}$$

Joint Cumulative distribution function

For two dimentional random variable (X, Y) the CDF F(x,y) is defined as

$$\mathbf{F(x, y) = P(X \leq x, Y \leq y)}$$

For Discrete RV: $F(x, y) = \sum_{x_i \leq x} \sum_{y_j \leq y} P(x_i, y_j)$

For Continuous RV: $F(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(x, y)\, dy\, dx$

Note:

If F is the cdf of a two-dimensional random variable with joint pdf f, then

$$\partial^2 F(x, y)/\partial x\, \partial y = f(x, y)$$

The Marginal probability functions.

For Discrete RV:

In the *discrete* case we proceed as follows: Since $X = x_i$ must occur with $Y = y_j$ for some j and can occur with $Y = y_j$ for only one j, we have

$$p(x_i) = P(X = x_i) = P(X = x_i, Y = y_1 \text{ or } X = x_i, Y = y_2 \text{ or } \cdots)$$
$$= \sum_{j=1}^{\infty} p(x_i, y_j).$$

$$q(y_j) = P(Y = Y_j) = \textstyle\sum_{i=1}^{\infty} p(x_i, y_j)$$

Y \ X	0	1	2	3	4	5	Sum
0	0	0.01	0.03	0.05	0.07	0.09	0.25
1	0.01	0.02	0.04	0.05	0.06	0.08	0.26
2	0.01	0.03	0.05	0.05	0.05	0.06	0.25
3	0.01	0.02	0.04	0.06	0.06	0.05	0.24
Sum	0.03	0.08	0.16	0.21	0.24	0.28	1.00

For Continuous RV:

In the *continuous* case we proceed as follows: Let f be the joint pdf of the continuous two-dimensional random variable (X, Y). We define g and h, the *marginal probability density functions* of X and Y, respectively, as follows:

$$g(x) = \int_{-\infty}^{+\infty} f(x, y)\, dy; \qquad h(y) = \int_{-\infty}^{+\infty} f(x, y)\, dx.$$

These pdf's correspond to the basic pdf's of the one-dimensional random variables X and Y, respectively. For example

$$P(c \leq X \leq d) = P[c \leq X \leq d, -\infty < Y < \infty]$$
$$= \int_c^d \int_{-\infty}^{+\infty} f(x, y)\, dy\, dx$$
$$= \int_c^d g(x)\, dx.$$

EXAMPLE 6.1. Two production lines manufacture a certain type of item. Suppose that the capacity (on any given day) is 5 items for line I and 3 items for line II. Assume that the number of items actually produced by either production line is a random variable. Let (X, Y) represent the two-dimensional random variable yielding the number of items produced by line I and line II, respectively. Table 6.1 gives the joint probability distribution of (X, Y). Each entry represents

$$p(x_i, y_j) = P(X = x_i, Y = y_j).$$

TABLE 6.1

$Y \backslash X$	0	1	2	3	4	5
0	0	0.01	0.03	0.05	0.07	0.09
1	0.01	0.02	0.04	0.05	0.06	0.08
2	0.01	0.03	0.05	0.05	0.05	0.06
3	0.01	0.02	0.04	0.06	0.06	0.05

Thus $p(2, 3) = P(X = 2, Y = 3) = 0.04$, etc. Hence if B is defined as

$$B = \{\text{More items are produced by line I than by line II}\}$$

we find that

$$\begin{aligned} P(B) &= 0.01 + 0.03 + 0.05 + 0.07 + 0.09 + 0.04 + 0.05 + 0.06 \\ &\qquad + 0.08 + 0.05 + 0.05 + 0.06 + 0.06 + 0.05 \\ &= 0.75. \end{aligned}$$

EXAMPLE 6.5. Two characteristics of a rocket engine's performance are thrust X and mixture ratio Y. Suppose that (X, Y) is a two-dimensional continuous random variable with joint pdf:

$$\begin{aligned} f(x, y) &= 2(x + y - 2xy), \qquad 0 \leq x \leq 1, \quad 0 \leq y \leq 1, \\ &= 0, \qquad \text{elsewhere.} \end{aligned}$$

(The units have been adjusted in order to use values between 0 and 1.) The marginal pdf of X is given by

$$\begin{aligned} g(x) = \int_0^1 2(x + y - 2xy)\, dy &= 2(xy + y^2/2 - xy^2)\big|_0^1 \\ &= 1, \qquad 0 \leq x \leq 1. \end{aligned}$$

That is, X is uniformly distributed over $[0, 1]$.

The marginal pdf of Y is given by

$$\begin{aligned} h(y) = \int_0^1 2(x + y - 2xy)\, dx &= 2(x^2/2 + xy - x^2y)\big|_0^1 \\ &= 1, \qquad 0 \leq y \leq 1. \end{aligned}$$

Hence Y is also uniformly distributed over $[0, 1]$.

Definition. We say that the two-dimensional continuous random variable is *uniformly distributed* over a region R in the Euclidean plane if

$$f(x, y) = \text{const} \quad \text{for } (x, y) \in R,$$
$$= 0, \quad \text{elsewhere.}$$

Because of the requirement $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y)\, dx\, dy = 1$, the above implies that the constant equals $1/\text{area}(R)$. We are assuming that R is a region with finite, nonzero area.

Note: This definition represents the two-dimensional analog to the one-dimensional uniformly distributed random variable.

EXAMPLE 6.6. Suppose that the two-dimensional random variable (X, Y) is *uniformly distributed* over the shaded region R indicated in Fig. 6.5. Hence

$$f(x, y) = \frac{1}{\text{area}(R)}, \quad (x, y) \in R.$$

We find that

$$\text{area}(R) = \int_0^1 (x - x^2)\, dx = \tfrac{1}{6}.$$

Therefore the pdf is given by

$$f(x, y) = 6, \quad (x, y) \in R$$
$$= 0, \quad (x, y) \notin R.$$

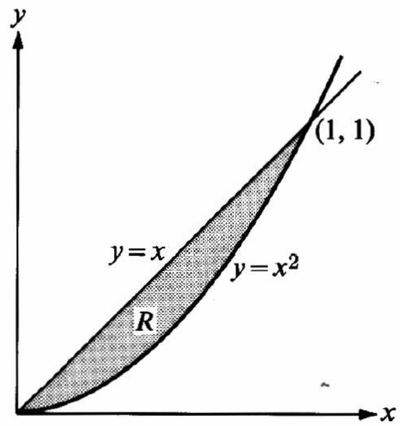


FIGURE 6.5

In the following equations we find the marginal pdf's of X and Y.

$$g(x) = \int_{-\infty}^{+\infty} f(x, y)\, dy = \int_{x^2}^{x} 6\, dy$$
$$= 6(x - x^2), \quad 0 \le x \le 1;$$
$$h(y) = \int_{-\infty}^{+\infty} f(x, y)\, dx = \int_{y}^{\sqrt{y}} 6\, dx$$
$$= 6(\sqrt{y} - y), \quad 0 \le y \le 1.$$

Conditional probability function:

$$p(x_i \mid y_j) = P(X = x_i \mid Y = y_j) = \frac{p(x_i, y_j)}{q(y_j)} \quad \text{if} \quad q(y_j) > 0,$$

$$q(y_j \mid x_i) = P(Y = y_j \mid X = x_i) = \frac{p(x_i, y_j)}{p(x_i)} \quad \text{if} \quad p(x_i) > 0.$$

Y \ X	0	1	2	3	4	5	Sum
0	0	0.01	0.03	0.05	0.07	0.09	0.25
1	0.01	0.02	0.04	0.05	0.06	0.08	0.26
2	0.01	0.03	0.05	0.05	0.05	0.06	0.25
3	0.01	0.02	0.04	0.06	0.06	0.05	0.24
Sum	0.03	0.08	0.16	0.21	0.24	0.28	1.00

evaluate the conditional probability $P(X = 2 \mid Y = 2)$. According to the definition of conditional probability we have

$$P(X = 2 \mid Y = 2) = \frac{P(X = 2, Y = 2)}{P(Y = 2)} = \frac{0.05}{0.25} = 0.20.$$

Definition. Let (X, Y) be a continuous two-dimensional random variable with joint pdf f. Let g and h be the marginal pdf's of X and Y, respectively.

The *conditional* pdf of X for given $Y = y$ is defined by

$$g(x \mid y) = \frac{f(x, y)}{h(y)}, \qquad h(y) > 0. \tag{6.7}$$

The *conditional* pdf of Y for given $X = x$ is defined by

$$h(y \mid x) = \frac{f(x, y)}{g(x)}, \qquad g(x) > 0. \tag{6.8}$$

EXAMPLE 6.3. Suppose that the two-dimensional continuous random variable (X, Y) has joint pdf given by

$$f(x, y) = x^2 + \frac{xy}{3}, \qquad 0 \le x \le 1, \quad 0 \le y \le 2,$$
$$= 0, \qquad \text{elsewhere.}$$

$$g(x) = \int_0^2 \left(x^2 + \frac{xy}{3}\right) dy = 2x^2 + \frac{2}{3}x,$$

$$h(y) = \int_0^1 \left(x^2 + \frac{xy}{3}\right) dx = \frac{y}{6} + \frac{1}{3}.$$

Hence,

$$g(x \mid y) = \frac{x^2 + xy/3}{1/3 + y/6} = \frac{6x^2 + 2xy}{2 + y}, \qquad 0 \le x \le 1, \quad 0 \le y \le 2;$$

$$h(y \mid x) = \frac{x^2 + xy/3}{2x^2 + 2/3(x)} = \frac{3x^2 + xy}{6x^2 + 2x} = \frac{3x + y}{6x + 2},$$

$$0 \le y \le 2, \quad 0 \le x \le 1.$$

To check that $g(x \mid y)$ is a pdf, we have

$$\int_0^1 \frac{6x^2 + 2xy}{2 + y}\, dx = \frac{2 + y}{2 + y} = 1 \qquad \text{for all } y.$$

A similar computation can be carried out for $h(y \mid x)$.

Independent Random Variable

Just as we defined the concept of independence between two events A and B, we shall now define *independent random variables*. Intuitively, we intend to say that X and Y are independent random variables if the outcome of X, say, in no way influences the outcome of Y. This is an extremely important notion and there are many situations in which such an assumption is justified.

Definition. (a) Let (X, Y) be a two-dimensional discrete random variable. We say that X and Y are independent random variables if and only if $p(x_i, y_j) = p(x_i)q(y_j)$ for all i and j. That is, $P(X = x_i, Y = y_j) = P(X = x_i)P(Y = y_j)$, for all i and j.

(b) Let (X, Y) be a two-dimensional continuous random variable. We say that X and Y are independent random variables if and only if $f(x, y) = g(x)h(y)$ for all (x, y), where f is the joint pdf, and g and h are the marginal pdf's of X and Y, respectively.

EXAMPLE 6.10. Suppose that a machine is used for a particular task in the morning and for a different task in the afternoon. Let X and Y represent the number of times the machine breaks down in the morning and in the afternoon, respectively. Table 6.3 gives the joint probability distribution of (X, Y).

An easy computation reveals that for *all* the entries in Table 6.3 we have

$$p(x_i, y_j) = p(x_i)q(y_j).$$

TABLE 6.3

Y \ X	0	1	2	$q(y_j)$
0	0.1	0.2	0.2	0.5
1	0.04	0.08	0.08	0.2
2	0.06	0.12	0.12	0.3
$p(x_i)$	0.2	0.4	0.4	1.0

EXAMPLE 6.11. Let X and Y be the life lengths of two electronic devices. Suppose that their joint pdf is given by

$$f(x, y) = e^{-(x+y)}, \qquad x \geq 0, \quad y \geq 0.$$

Since we can factor $f(x, y) = e^{-x}e^{-y}$, the independence of X and Y is established.

PROBLEMS

6.1. Suppose that the following table represents the joint probability distribution of the discrete random variable (X, Y). Evaluate all the marginal and conditional distributions.

Y \ X	1	2	3
1	$\frac{1}{12}$	$\frac{1}{6}$	0
2	0	$\frac{1}{9}$	$\frac{1}{5}$
3	$\frac{1}{18}$	$\frac{1}{4}$	$\frac{2}{15}$

6.2. Suppose that the two-dimensional random variable (X, Y) has joint pdf

$$\begin{aligned} f(x, y) &= kx(x - y), \qquad 0 < x < 2, \qquad -x < y < x, \\ &= 0, \qquad \text{elsewhere.} \end{aligned}$$

(a) Evaluate the constant k.
(b) Find the marginal pdf of X.
(c) Find the marginal pdf of Y.

6.3. Suppose that the joint pdf of the two-dimensional random variable (X, Y) is given by

$$\begin{aligned} f(x, y) &= x^2 + \frac{xy}{3}, \qquad 0 < x < 1, \qquad 0 < y < 2, \\ &= 0, \qquad \text{elsewhere.} \end{aligned}$$

Compute the following.

(a) $P(X > \frac{1}{2})$; (b) $P(Y < X)$; (c) $P(Y < \frac{1}{2} \mid X < \frac{1}{2})$.

1. Check f(x,y) is a pdf. 2. Marginal pdfs 3. P(X+Y >=1) 4. g(X/Y) 5. h(y/x)
Ans: 1. yes 2. 2x^2+(2/3)x and (1/3)+(y/6) 3. 65/72 4. 6x^2+2xy/(2+y) 5. 3x^2+xy/(6x^2+2x)

6.5. For what value of k is $f(x, y) = ke^{-(x+y)}$ a joint pdf of (X, Y) over the region $0 < x < 1,\ 0 < y < 1$?

6.6. Suppose that the continuous two-dimensional random variable (X, Y) is uniformly distributed over the square whose vertices are $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$. Find the marginal pdf's of X and of Y.

6.14. Suppose that the joint pdf of (X, Y) is given by

$$f(x, y) = \begin{cases} e^{-y}, & \text{for } x > 0, \quad y > x, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the marginal pdf of X.
(b) Find the marginal pdf of Y.
(c) Evaluate $P(X > 2 \mid Y < 4)$.

Answers:

6.2. (a) $k = \frac{1}{8}$ (b) $h(x) = x^3/4,\ 0 < x < 2$

(c) $g(y) = \begin{cases} \frac{1}{3} - y/4 + y^3/48, & 0 \le y \le 2 \\ \frac{1}{3} - y/4 + (5/48)y^3, & -2 \le y \le 0 \end{cases}$

6.3. (a) $\frac{5}{6}$ (b) $\frac{7}{24}$ (c) $\frac{5}{32}$ **6.5.** $k = 1/(1 - e^{-1})^2$

6.6. (a) $k = \frac{1}{2}$ (b) $h(x) = 1 - |x|,\ -1 < x < 1$

(c) $g(y) = 1 - |y|,\ -1 < y < 1$

6.14. (a) $g(x) = e^{-x},\ x > 0$ (b) $h(y) = ye^{-y},\ y > 0$

Problems:

1. Find C for which f(x,y)= Cx+Cy^2. $Ans: C = \frac{1}{37}$
2. If f(x,y)= $\begin{cases} \frac{2}{a^2} & 0 \le x \le y \le a \\ & elsewhere \end{cases}$. Find f(y/x) and f(x/y).

   Ans: $\frac{1}{a-x}\ and\ \frac{1}{y}$
3. If f(x,y)= $\begin{cases} 8xy & 0 < x < y < 1. \\ 0 & elsewhere \end{cases}$ Find the marginal pdf of X and Y. Check whether they are independent.

   Ans: 4x- $4x^3, 4\,y^3$ and are not independent.

EXAMPLE 6.2. Suppose that a manufacturer of light bulbs is concerned about the number of bulbs ordered from him during the months of January and February. Let X and Y denote the number of bulbs ordered during these two months, respectively. We shall assume that (X, Y) is a two-dimensional continuous random

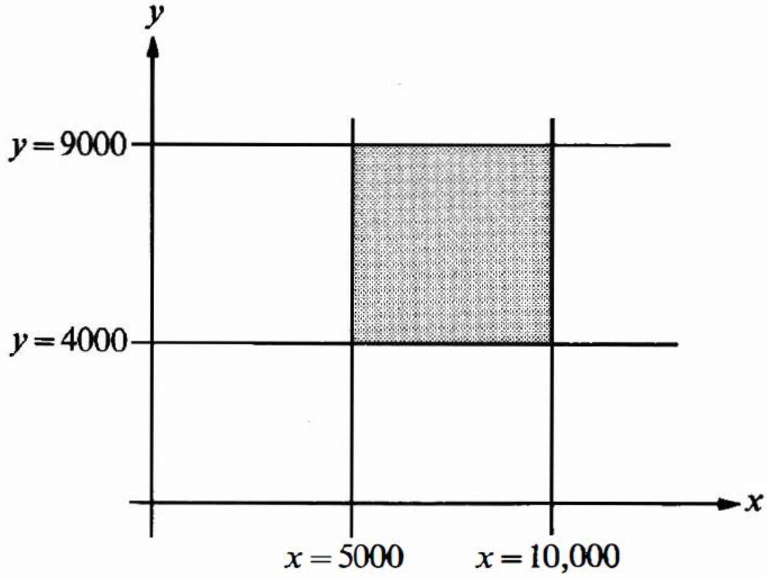


FIGURE 6.3

variable with the following joint pdf (see Fig. 6.3):

$$f(x, y) = c \quad \text{if } 5000 \le x \le 10{,}000 \text{ and } 4000 \le y \le 9000,$$
$$= 0, \quad \text{elsewhere.}$$

To determine c we use the fact that $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x, y)\,dx\,dy = 1$. Therefore

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x, y)\,dx\,dy = \int_{4000}^{9000}\int_{5000}^{10{,}000} f(x, y)\,dx\,dy = c[5000]^2.$$

Thus $c = (5000)^{-2}$. Hence if $B = \{X \ge Y\}$, we have

$$P(B) = 1 - \frac{1}{(5000)^2}\int_{5000}^{9000}\int_{5000}^{y} dx\,dy$$
$$= 1 - \frac{1}{(5000)^2}\int_{5000}^{9000} [y - 5000]\,dy = \frac{17}{25}.$$

EXAMPLE 6.3. Suppose that the two-dimensional continuous random variable (X, Y) has joint pdf given by

$$f(x, y) = x^2 + \frac{xy}{3}, \qquad 0 \le x \le 1, \quad 0 \le y \le 2,$$
$$= 0, \qquad \text{elsewhere.}$$

To check that $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y)\, dx\, dy = 1$:

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y)\, dx\, dy = \int_0^2 \int_0^1 \left(x^2 + \frac{xy}{3}\right) dx\, dy$$
$$= \int_0^2 \frac{x^3}{3} + \frac{x^2 y}{6} \Big|_{x=0}^{x=1} dy$$
$$= \int_0^2 \left(\frac{1}{3} + \frac{y}{6}\right) dy = \frac{1}{3} y + \frac{y^2}{12} \Big|_0^2$$
$$= \tfrac{2}{3} + \tfrac{4}{12} = 1.$$

Let $B = \{X + Y \ge 1\}$. (See Fig. 6.4.) We shall compute $P(B)$ by evaluating $1 - P(\overline{B})$, where $\overline{B} = \{X + Y < 1\}$. Hence

Let $B = \{X + Y \ge 1\}$. (See Fig. 6.4.) We shall compute $P(B)$ by evaluating $1 - P(\overline{B})$, where $\overline{B} = \{X + Y < 1\}$. Hence

$$P(B) = 1 - \int_0^1 \int_0^{1-x} \left(x^2 + \frac{xy}{3}\right) dy\, dx$$
$$= 1 - \int_0^1 \left[x^2(1 - x) + \frac{x(1 - x)^2}{6}\right] dx$$
$$= 1 - \tfrac{7}{72} = \tfrac{65}{72}.$$

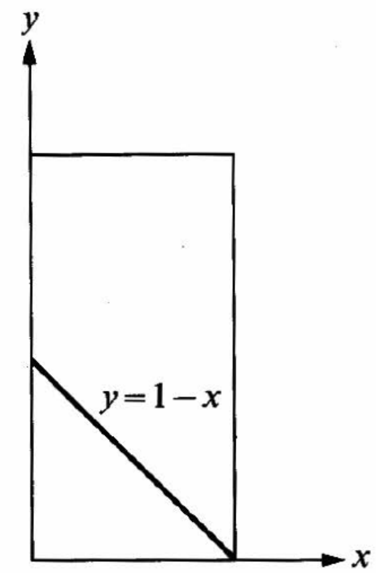


FIGURE 6.4

Expectation of 2D RV

For discrete RV:

E(X) = $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x_i\ P\ (x_i, y_i)$

$= \sum_{i=1}^{\infty} x_i\ \{\sum_{j=1}^{\infty}\ P\ (x_i, y_i)\}$

$= \sum_{i=1}^{\infty} x_i\ p(x_i)\}$ where $p(x_i)$ is marginal pmf of x.

E(X) $= \sum_{i=1}^{\infty} x_i\ p(x_i)\}$ and E(Y) $= \sum_{i=1}^{\infty} y_j\ q(y_i)\}$

For continuous RV:

E(X) $= \int_{-\infty}^{\infty} x$ g(x) dx and E(Y)= $\int_{-\infty}^{\infty} y$ h(y) dy

Properties:

1. E(c)= c
2. E(cX)= cE(x)
3. E(X+Y)= E(X)+E(Y)

Property 7.6. Let (X, Y) be a two-dimensional random variable and suppose that X and Y are *independent*. Then $E(XY) = E(X)E(Y)$.

Proof

$$E(XY) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xyf(x, y)\, dx\, dy$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xyg(x)h(y)\, dx\, dy$$

$$= \int_{-\infty}^{+\infty} xg(x)\, dx \int_{-\infty}^{+\infty} yh(y)\, dy = E(X)E(Y).$$

Property 7.9. If (X, Y) is a two-dimensional random variable, and if X and Y are *independent* then

$$V(X+Y) = V(X) + V(Y). \tag{7.15}$$

Proof

$$\begin{aligned} V(X+Y) &= E(X+Y)^2 - (E(X+Y))^2 \\ &= E(X^2 + 2XY + Y^2) - (E(X))^2 - 2E(X)E(Y) - (E(Y))^2 \\ &= E(X^2) - (E(X))^2 + E(Y^2) - (E(Y))^2 = V(X) + V(Y). \end{aligned}$$

Problem:

If f(x,y)= $\begin{cases} x^2 + \frac{xy}{3} & 0 \le x \le 1, 0 \le y \le 2 \\ 0 & elsewhere \end{cases}$. Find E(X), E(Y) and V(Y).

Solution:

E(X) $= \int_{-\infty}^{\infty} x$ g(x) dx

$= \int_{-\infty}^{\infty} x \left\{2x^2 + \frac{2}{3}x\right\} dx = \frac{13}{18}$

E(Y) = $\int_{-\infty}^{\infty} y$ h(y) dy= $\frac{10}{19}$

E(Y^2) = $\int_{-\infty}^{\infty} y^2$ h(y) dy= $\frac{14}{9}$

V(Y) = E(Y^2) – $\{E(Y)\}^2$= 0.3209

Conditional Expectation:

Definition. (a) If (X, Y) is a two-dimensional continuous random variable we define the *conditional expectation* of X for given $Y = y$ as

$$E(X \mid y) = \int_{-\infty}^{+\infty} xg(x \mid y)\, dx. \tag{7.23}$$

(b) If (X, Y) is a two-dimensional discrete random variable we define the conditional expectation of X for given $Y = y_j$ as

$$E(X \mid y_j) = \sum_{i=1}^{\infty} x_i p(x_i \mid y_j). \tag{7.24}$$

Chebyshev's inequality

There is a well-known inequality due to the Russian mathematician Chebyshev which will play an important role in our subsequent work. In addition, it will give us a means of understanding precisely how the variance measures variability about the, expected value of a random variable.

If we know the probability distribution of a random variable X (either the pdf in the continuous case or the point probabilities in the discrete case), we may then compute E(X) and V(X), if these exist. However, the converse is not true. That is, from a knowledge of E(X) and V(X) we cannot reconstruct the probability distribution of X.

Nonetheless, it turns out that although we cannot evaluate such probabilities [from a knowledge of E(X) and V(X)), we can give a very useful upper (or lower) bound to such probabilities. This result is contained in what is known as Chebyshev's inequality.

Chebyshev's inequality. Let X be a random variable with $E(X) = \mu$ and let c be any real number. Then, if $E(X - c)^2$ is finite and ϵ is any positive number, we have

$$P[|X - c| \geq \epsilon] \leq \frac{1}{\epsilon^2} E(X - c)^2. \tag{7.20}$$

NOTE:

(a) By considering the complementary event we obtain

$$P[|X - c| < \epsilon] \geq 1 - \frac{1}{\epsilon^2} E(X - c)^2. \tag{7.20a}$$

(b) Choosing $c = \mu$ we obtain

$$P[|X - \mu| \geq \epsilon] \leq \frac{\operatorname{Var} X}{\epsilon^2}. \tag{7.20b}$$

(c) Choosing $c = \mu$ and $\epsilon = k\sigma$, where $\sigma^2 = \operatorname{Var} X > 0$, we obtain

$$P[|X - u| \geq k\sigma] \leq k^{-2}. \tag{7.21}$$

NOTE:

$$P[|X - \mu| < k\sigma] < 1 - k^{-2}$$

Problems:

1. Apply Chebyshev's inequality to find (with $\mu = 10$ and $\sigma^2 = 4$)

i) P(5<X<15) Ans: $\geq \frac{21}{25}$ where, k=5/2

ii) P(|X-10|$\leq$ 3) Ans: $\geq \frac{5}{9}$ where, k= 3/2

iii) P(|X-10|<3) Ans: $\geq \frac{4}{9}$

2. A random variable has mean 3 and variance 2. Find an upper bound for

   i) P(|X-3|$\geq$ 2) Ans: $\leq \frac{1}{2}$

   ii) P(|X-3|$\geq$ 1) Ans: ≤ 2

3. Find a smallest value of k in Chebyshev's inequality for which the probability is at most 0.95.

   Solution:

   P(|x-μ|$\leq ka) \geq 1 - 1/k^2$

   $0.95 \geq \ 1 - 1/k^2$

   K= $\sqrt{20}$

Correlation Coefficient

Parameter which measures "degree of association" between X and Y.

Definition. Let (X, Y) be a two-dimensional random variable. We define ρ_{xy}, the *correlation coefficient*, between X and Y, as follows:

$$\rho_{xy} = \frac{E\{[X - E(X)][Y - E(Y)]\}}{\sqrt{V(X)V(Y)}}. \tag{7.22}$$

The numerator is called as covariance of X and Y and is denoted by σ_{xy} or COV(X,Y).

Theorem 7.9

$$\rho = \frac{E(XY) - E(X)E(Y)}{\sqrt{V(X)V(Y)}}.$$

Proof: Consider

$$\begin{aligned} E\{[X - E(X)][Y - E(Y)]\} &= E[XY - XE(Y) - YE(X) + E(X)E(Y)] \\ &= E(XY) - E(X)E(Y) - E(Y)E(X) + E(X)E(Y) \\ &= E(XY) - E(X)E(Y). \end{aligned}$$

Theorem 7.10. If X and Y are independent, then $\rho = 0$.

Proof: This follows immediately from Theorem 7.9, since

$$E(XY) = E(X)E(Y)$$

if X and Y are independent.

NOTE:
The converse of Theorem 7.10 is in general not true. That is, we may have ρ= 0, and yet X and Y need not be independent. If p = 0, we say that X and Y are uncorrelated. Thus, being uncorrelated and being independent are, in general, not equivalent.
Example: Consider the RV Y=X^2 where f(x)= ½ , -1≤ $x \leq\ 1$.
E(XY)- E(X)E(Y)
= E(X^3)- E(X)E(X^2)
=0
$\rho = 0$ but X and Y are not independent.

Theorem:

$$-1 \leq \rho \leq 1$$

Proof:
We have E$(X) \geq 0$
Since $V(X) \geq 0$
E(X^2)- $[E(X)]^2 \geq 0$

$$E(X^2) \geq 0$$

$$E\left\{\frac{X - E(X)}{\sqrt{V(X)}} \pm \frac{Y - E(Y)}{\sqrt{V(Y)}}\right\}^2 \geq 0$$

Simplification

$$2 \pm 2\rho \geq 0$$

$$1 \pm \rho \geq 0$$

$1 + \rho \geq 0$ and $1 - \rho \geq 0$
$\rho \geq -1$ and $\rho \geq 1$
Hence, $$-1 \leq \rho \leq 1.$$

Theorem:
If X and Y are linearly related then $\rho = \ \pm 1$.
Proof:
Let X and Y are linearly related.
Y = a+bX

$$\rho = \frac{E(XY) - E(X)E(Y)}{\sqrt{V(X)V(Y)}}$$

Here: $E(XY) = E(X(a + bX)) = E(Xa + BX^2) = aE(X) + bE(X^2)$
Therefore,

$$\rho = \frac{aE(X) + bE(X^2) - E(X)E(a + bX)}{\sqrt{V(X)V(Y)}}$$

$$\rho = \frac{aE(X) + bE(X^2) - E(X)\big((a + bE(X)\big)}{\sqrt{V(X)V(Y)}}$$

$$\rho = \frac{aE(X) + bE(X^2) - aE(X) - bE(X)^2}{\pm b\, V(X)}$$

$$\rho = \frac{E(X^2) - E(X)^2}{\pm V(X)} = \pm 1$$

Problems:

1. With usual notation, prove that $\rho_{uv} = \pm \rho_{xy}$ where u=ax+b and v= c+dx.
2. The random variable (X, Y) has a joint pdf given by f(x, y)= x+y, $0 \le x \le 1, 0 \le y \le 1$compute correlation between X &Y.
   Solution:
   $$\rho = \frac{E(XY) - E(X)E(Y)}{\sqrt{V(X)V(Y)}}$$
   E(X)= $\int_0^1 \int_0^1 x(x+y)dx\, dy = \frac{7}{12}$
   E(Y)= $\int_0^1 \int_0^1 y(x+y)dx\, dy = \frac{7}{12}$
   E(XY)= $\int_0^1 \int_0^1 xy(x+y)dx\, dy = \frac{1}{3}$
   V(X)= 11/144
   V(Y)= 11/144
   $\rho = \frac{-1}{11}$
3. If (X, Y) has the joint density function f(x,y)= 2-x-y, , $0 \le x \le 1, 0 \le y \le 1$ compute the correlation between x and y. Ans: -1/11
4. Prove that V(aX+bY)=$a^2V(X) + b^2V(Y) + 2ab\ Cov\ (X, Y)$.
5. And when X and Y are independent V(aX+bY)=$a^2V(X) + b^2V(Y)$.
6. Two independent random variable $X_1\ and\ X_2$has mean 5 and 10 and variance 4 and 9 respectively. Find the covariance between u = $3x_1 + 4x_2$ v = $3x_1 - x_2$.
7. If $X_1, X_{2\,,} X_3$ be uncorrelated random variables having same standard deviation. Find the correlation coefficient between $X_1 + X_2$ and $X_3 + X_2$ Ans:1/2
8. Suppose that 2 dimensional random variable is uniformly distributed over the triangular region R={(x,y)/ 0<x<y <1}
   i) Find pdf — Ans: 2
   ii) Marginal pdf of X and Y — Ans: 2(1-x) and 2y
   iii) Find ρ. — Ans: E(x)= 1/3, E(Y)= 2/3,

   E(XY)= 1/4, E(X^2)=1/6, E(Y^2)= ½ , V(X)= 1/18, V(Y)= 1/8, $\rho = \frac{1}{2}$

9. The random Variable (X, Y) has a joint pdf by
   f(x,y)= x+y, $0 \le x \le 1, 0 \le y \le 1$ compute the correlation coefficient between X and Y.
   Ans: g(x)= x+1/2, h(y)= y+1/2 , E(X)= 7/12, E(Y)= 7/12, E(X^2)= 5/12, E(Y^2)= 5/12, V(X) = 11/144, V(Y) = 11/144, E(XY)=1/3 and $\rho = -1/11$
10. Given E(XY)=43, P(X=x_i)= 1/5 and P(Y=y_i)= 1/5. Find ρ.

X	1	3	4	6	8
Y	1	2	24	12	5

   Ans: E(X)=22/5, E(Y)= 44/5, E(X^2)= 126/5, E(Y^2)= 150, V(X)= 5.84, V(Y)= 7256 and $\rho = 0.20791$.

11. If X,Y and Z are uncorrelated random variable with standard deviation 5, 12, 9 respectively. If U=X+Y and V= Y+Z. Evaluate ρ between U and V.
   Ans: COV (X, Y) = COV (Y, Z) = COV (X, Z)= 0
   V(U) = 169, V(V) = 225 and $\rho = 0.73$.

Example 7.21. Suppose that the two-dimensional random variable (X, Y) is uniformly distributed over the triangular region

$$R = \{(x, y) \mid 0 < x < y < 1\}.$$

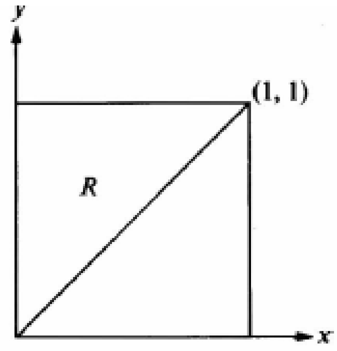


FIGURE 7.9

(See Fig. 7.9.) Hence the pdf is given as

$$f(x, y) = 2, \quad (x, y) \in R,$$
$$= 0, \quad \text{elsewhere.}$$

Thus the marginal pdf's of X and of Y are

$$g(x) = \int_x^1 (2)\, dy = 2(1 - x), \quad 0 \le x \le 1;$$

$$h(y) = \int_0^y (2)\, dx = 2y, \quad 0 \le y \le 1.$$

Therefore

$$E(X) = \int_0^1 x2(1 - x)\, dx = \tfrac{1}{3}, \qquad E(Y) = \int_0^1 y2y\, dy = \tfrac{2}{3};$$

$$E(X^2) = \int_0^1 x^2 2(1 - x)\, dx = \tfrac{1}{6}, \qquad E(Y^2) = \int_0^1 y^2 2y\, dy = \tfrac{1}{2};$$

$$V(X) = E(X^2) - [E(X)]^2 = \tfrac{1}{18}, \qquad V(Y) = E(Y^2) - [E(Y)]^2 = \tfrac{1}{18};$$

$$E(XY) = \int_0^1 \int_0^y xy2\, dx\, dy = \tfrac{1}{4}.$$

Hence

$$\rho = \frac{E(XY) - E(X)E(Y)}{\sqrt{V(X)V(Y)}} = \frac{1}{2}$$

Repeated trials

If a coin is tossed thrice, the probability of getting one head and two tails can be combined as HTT, THT and TTH. The probability of each one of these being $\left(\frac{1}{2}\right)^3$, their total probability shall be $3\left(\frac{1}{2}\right)^3$.

Similarly, if trial is repeated 'n' times and if p is the probability of success and q is that of failure, then probability of 'r' success and '$n-r$' failure is given by $p^r q^{n-r}$. But these 'r' success and '$n-r$' failure can occur in any of these n_{Cr} ways in each of which the probability is same.

Thus the probability of r success is $n_{Cr}\, p^r q^{n-r}$.

The probability of at least r success in n trials= sum of probabilities $r, r+1, \ldots, n$ success = $n_{C_r} p^r q^{n-r} + n_{C_{r+1}} p^{r+1} q^{n-r-1} + \ldots + n_{Cn} p^n$.

Binomial distribution:

It is concerned with trials of repetitive nature in which only the occurrence or non-occurrence i.e., success or failure of particular event is of interest.

A random experiment with only two types of outcomes, success or failure of a particular event is called a Bernoulli trial. A random variable X that takes the value either 0 or 1 is known as Bernoulli variable.

x_i	0	1
$P(x_i)$	$1-p$	p

Mean $\mu = \sum x_i P(x_i) = p$.

Variance $\sigma^2 = \sum (x_i - \mu)^2 P(x_i)$

$$¿(0-p)^2(1-p)+(1-p)^2 p = pq.$$

Standard deviation $\sigma = \sqrt{pq}$.

$f(x) = P(X = x) = {}^nC_x\, p^x (1-p)^{n-x}$, where the random variable X denotes the number of successes in n trials and $x = 0, 1, \ldots, n$.

The Random variable X given by $x=0,1,2,\ldots,n$ would have probability as

Number of success	Probability $P(x)$
0	q^n
1	$n_{C_1} p q^{n-1}$
2	$n_{C_2} p^2 q^{n-2}$
$\vdots$	$\vdots$
n	$n_{C_n} p_n q^{n-n}$

This table is in the form of frequency distribution. Where in it should be observed that the values of $P(x)$ for different values of $x=0,1,2,\ldots$ are the various terms expansion of $(p+q)^n$ and accordingly the frequency distribution is called as the Binomial distribution.

<u>Mean and variance of Binomial distribution (B.D):</u>

For various values of $x=0,1,2,\ldots,n$, we get the corresponding values of $P(x)=n_{C_x} p^x q^{n-x}$.

$\therefore$ Sum of the probability $=q^n+n_{C_1} p q^{n-1}+\ldots+p^n=(q+p)^n=1$.

$$\therefore \sum P(x_i)=1.$$

Mean $\mu=\sum x_i P(x_i)=\sum_{x=0}^{n} x\, n_{C_x} p^x q^{n-x}$

$$¿\sum_{x=0}^{n} x \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$¿\sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!} p^x q^{n-x}$$

$$¿n \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!¿¿} ¿$$

$$¿np(q+p)^{n-1}=np.$$

Variance $\sigma^2=\sum (x_i-\mu)^2 P(x_i)$

$$¿\sum_{x=0}^{n} (x-\mu)^2 P(x)$$

$$¿\sum_{x=0}^{n} (x^2-2x\mu+\mu^2) P(x)$$

$$¿\sum_{x=0}^{n} x^2 P(x)-2\sum_{x=0}^{n} x\mu P(x)+\sum_{x=0}^{n} \mu^2 P(x)$$

$$¿\sum_{x=0}^{n} x^2 n_{C_x} p^x q^{n-x} - 2\mu^2 + \mu^2$$

$$¿\sum_{x=0}^{n} [x(x-1)+x] n_{C_x} p^x q^{n-x} - \mu^2$$

$$¿\sum_{x=0}^{n} x(x-1) \frac{n!}{(n-x)!\,x!} p^x q^{n-x} + \mu - \mu^2$$

$$¿ n(n-1) p^2 \sum_{x=2}^{n} \frac{(n-2)!}{[n-2-(x-2)]!\,(x-2)!} p^{x-2} q^{n-2-(x-2)} + \mu - \mu^2$$

$$¿ n(n-1) p^2 (q+p)^{n-2} + \mu - \mu^2$$

$$¿ n(n-1) p^2 + \mu - \mu^2$$

$$¿ n^2 p^2 - n p^2 + np - n^2 p^2 = np(1-p) = npq.$$

Standard deviation $\sigma = \sqrt{npq}$.

<u>Problems on Binomial distribution:</u>

1. The probability that a pen manufactured by a company will be defective is $\frac{1}{10}$. If 12 such pens are manufactured, find the probability that a) exactly 2 will be defective b) at least 2 will be defective c) none will be defective.

<u>Solution:</u> The probability of a defective pen=0.1

$\therefore$ Probability of a non- defective pen $q = 1 - 0.1 = 0.9$

a) Exactly 2 will be defective $¿ 12_{C2} p^2 q^{n-2}$

$$¿ 12_{C_2} p^2 q^{12-2}$$

$$¿ 12_{C_2} p^2 q^{10} = 12_{C_2} (0.1)^2 (0.9)^{10} = 0.2301.$$

b) The probability that at least 2 will be defective=1-pobability that less than 2 are defective

=1-[$12_{C0} p^0 q^{12} + 12_{C_1} p^1 q^{11} ¿$

=1-$[(0.9)^{12} + 12(0.1)(0.9)^{11}]$

=$0.3412.$

c) The probability that none will be defective =$12_{C_0} p^0 q^{12} = (0.9)^{12} = 0.2824.$

2. 4 coins are tossed 100 times and the following results were obtained. Fit a binomial distribution for the data and calculate the theoretical frequency (corresponding probability frequencies).

No. of heads	0	1	2	3	4
Frequency	5	29	36	25	5

Solution: Let X be random variable denoting no. of heads. If p is the probability of head, then q represents the probability of tail.

From the given frequency distribution, mean $\mu=\frac{\sum f_i x_i}{\sum f_i}=\frac{196}{100}=1.96$.

But $\mu=np=1.96$

$$p=\frac{1.96}{n}=\frac{1.96}{4}=0.49$$

$$\therefore q=1-p=1-0.49=0.51.$$

$\therefore$ The binomial distribution fit for this data is $100(0.49+0.51)^4$.

Hence, the corresponding probability frequencies are obtained as $100\,4_{C_0}(0.49)^0(0.51)^4=7$

$$100\,4_{C_1}(0.49)^1(0.51)^3=26$$

$$100\,4_{C_2}(0.49)^2(0.51)^2=37$$

$$100\,4_{C_3}(0.49)^3(0.51)^1=24$$

$$100\,4_{C_4}(0.49)^4(0.51)^0=6$$

Thus the theoretical frequencies are 7, 26, 37, 24, 6.

3. Out of 800 families with 5 children each, how many would have expect to have a) 3 boys b) 5 girls c) either 2 or 3 boys? Assume equal probability for boys and girls.

Solution: Let $p=$¿ probability of a boy$=\frac{1}{2}$ and $q=$¿probability of girl¿ $\frac{1}{2}$.

a) Probability of a family having 3 boys$=5_{C_3}(0.5)^3(0.5)^2=0.3125$.
   $\therefore$Expected no. of families=800 $\times 0.3125$=250.

b) Probability of a family having 5 girls$=5_{C0}(0.5)^0(0.5)^5=0.03125$.
   $\therefore$ Expected no. of families=800$\times$0.03125=25.

c) Probability of a family having either 2 or 3 boys= $5_{C_2}(0.5)^2(0.5)^3+5_{C_3}(0.5)^3(0.5)^2$
   ¿ 0.625.

$\therefore$ Expected no. of families = 800 $\times$0.625=500.

4. The number of telephone lines busy at a particular time is a binomial variable with probability 0.1 that a line is busy. If 10 lines are selected at random, what is the probability that i) no line is busy ii) at least one line is busy iii) at most 2 lines are busy.

Solution: By data we have $p=0.1, q=1-p=0.9$. Also n=10.

$\therefore$ Out of 10 lines, the probability that r lines are busy is given by $10_{C_r}(0.1)^r(0.9)^{10-r}$.

i) No. line is busy $=10_{C_0}(0.1)^0(0.9)^{10}=0.3487$.

ii) Probability that at least one line is busy =1- probability that no. line is busy.

$$¿1-0.3487=0.6513.$$

iii) At most 2 lines are busy = $10_{C_0}(0.1)^0(0.9)^{10}+10_{C_1}(0.1)(0.9)^9+10_{C_2}(0.1)^2(0.9)^8$ =0.929830.

5. In a bombing action, there is 50% chance that any bomb will strike the target. Two direct hits are needed to destroy the target completely. How many bombs are required to be dropped to give 99% chance or better of destroying the target?

Solution: From data, $p=0.5$ and $q=0.5$.

Where p is the probability that bomb strike the target.

$P(x)=n_{C_x}(0.5)^x(0.5)^{n-x}=n_{C_x}(0.5)^n$ represents the probability that x bombs out of n strikes the target.

We need to find minimum value of n such that $2\leq k\leq n$ bombs destroy the target completely.

i.e, $P(2\leq k\leq n)\geq 0.99$

$$\sum n_{C_x}(0.5)^n\geq 0.99$$

$$n_{C_0}(0.5)^n+n_{C_1}(0.5)^n\leq 1-0.99$$

$$\frac{1+n}{2^n}\leq 0.01$$

i.e., $100(1+n)\leq 2^n$

till $n=10$, we get $100(1+n)>2^n$

i.e., for $n=10, 100(1+10)=1100>2^{10}$

for n=11, we get $100(1+n)=1200<2^{11}$

$$\therefore n=11$$

Minimum 11 bombs are required.

6. An airline knows that 5% of the people making reservation on a certain flight will not turn up. Consequently their policy is to sell 52 tickets for a flight that can hold only 50 passengers. What is the probability that there will be a seat available for every passengers who turns up.

Solution: Probability that passenger will not turn up=0.05.

i.e., $¿0.05$,$q=1-0.05=0.95$.

x=no. of passenger will not turn up, $n=52$.

$$P(x)=52_{C_x}(0.05)^x(0.95)^{52-x}$$

Required probability $¿P(x\geq 50)=1-P(x>50)$

$$¿1-\{P(x=51)+P(x=52)\}$$

$$¿1-\{52_{C_{51}}(0.05)^{51}(0.95)^1+52_{C_{52}}(0.05)^{52}(0.95)^0\}.$$

7. If 20% of the bolts produced by a machine are defective, determine the probability that out of 4 bolts chosen at random, (a) 1 (b) 0 (c) less than 2, bolts will be defective.

Solution: The probability of a defective bolt is $p = 0.2$, of a non - defective bolt is $q = 1 - p = 0.8$.

Let the random variable X be the number of defective bolts. Then

(a) $P(X = 1) = {}^4C_1 (0.2)^1 (0.8)^3 = 0.4096$

(b) $P(X = 0) = {}^4C_0 (0.2)^0 (0.8)^4 = 0.4096$

(c) $P(X < 2) = P(X = 0) + P(X = 1) = 0.4096 + 0.4096 = 0.8192$.

8. Let the probability that the birth weight (in grams) of babies in America is less than 2547 grams be 0.1. If X equals the number of babies that weigh less than 2547 grams at birth among 20 of these babies selected at random, then what is $P(X \le 3)$?

Solution: If a baby weighs less than 2547, then it is a success; otherwise it is a failure. Thus X is a binomial random variable with probability of success p and n = 20. We are given that p = 0.1. Hence $P(X \le 3) = \sum_{k=0}^{3} {}^{20}C_k (0.1)^k (0.9)^{20-k} = 0.867$

9. A gambler plays roulette at Monte Carlo and continues gambling, wagering the same amount each time on "Red", until he wins for the first time. If the probability of "Red" is $\frac{18}{38}$ and the gambler has only enough money for 5 trials, then (a) what is the probability that he will win before he exhausts his funds; (b) what is the probability that he wins on the second trial?

Solution: $p = P(\text{Red}) = \frac{18}{38}$.

(a) Hence the probability that he will win before he exhausts his funds is given by

$P(X \le 5) = 1 - P(X \ge 6) = 1 - (1 - p)^5 = 1 - \left(1 - \frac{18}{38}\right)^5 = 0.956$.

(b) Similarly, the probability that he wins on the second trial is given by

$P(X = 2) = (1 - p)\, p = (1 - \frac{18}{38}) \frac{18}{38} = 0.2493$.

10. A continuous random variable has pdf $f(x) = \begin{cases} \frac{x}{2}, & 0 < x < 2 \\ 0, & elsewhere \end{cases}$.

(a) If two independent determination of X are made, then what is the probability that both these determinations will be greater than 1.

(b) If three independent determinations are made, what is the probability that atleast 2 of these are greater than 1.

Solution:

$$f(x) = \begin{cases} \frac{x}{2}, & 0 < x < 2 \\ 0, & elsewhere \end{cases}$$

Z : Number of independent determination greater than 1.

$$p = P(X > 1) = \int_1^2 \frac{x}{2} dx = 0.75$$

(a) $P(Z = z) = {}^nC_z \, p^z \, (1 - p)^{n-z}$

$P(Z = 2) = \frac{9}{16}$.

(b) $n = 3, P(Z \geq 2) = P(Z = 2) + P(Z = 3)$

$= {}^3C_2 \, (0.75)^2 \, (0.25)^{3-2} + {}^3C_3 \, (0.75)^3 \, (0.25)^0$

$= \frac{27}{32}$.

The Poisson Distribution

It is a distribution related to the probabilities of events which are extremely rare, but which have large number of independent opportunities for occurrence.
Eg: Number of persons born blind per year in a large city, how many hits will websites get in a particular minute.
Poisson distribution is regarded as the limiting form of the binomial distribution when $n$is very large i.e., $n \to \infty$ and p, the probability of success is very small i.e., $p = 0$. So that the mean nptends to a fixed finite constant.

$\therefore$ Probability of x success in Poisson distribution is given by $P(x) = \frac{\lambda^x e^{-\lambda}}{x!}$.

Mean and standard deviation of Poisson distribution:

Mean $\mu = \sum x_i P(x_i)$

$¿ \sum_{x=0}^{\infty} \frac{\lambda^x e^{-\lambda}}{x!} x$

$¿ e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{(x-1)!}$

$¿ e^{-\lambda} \lambda \sum_{x=0}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$

$¿ e^{-\lambda} \lambda \left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \ldots \right] = e^{-\lambda} \lambda e^{\lambda} = \lambda.$

Variance $\sigma^2 = \sum (x - \mu)^2 P(x)$

$¿ \sum (x^2 - 2\mu x + \mu^2) P(x)$

¿ $\sum_{x=0}^{\infty} x^2 P(x) - 2\mu \sum_{x=0}^{\infty} x\, P(x) + \mu^2$

¿ $\sum_{x=0}^{\infty} [x(x-1)+x] P(x) - \lambda^2$

¿ $\sum_{x=0}^{\infty} x(x-1) P(x) + \sum xP(x) - \lambda^2$

¿ $\sum_{x=0}^{\infty} \frac{x(x-1)\lambda^x e^{-\lambda}}{x!} + \lambda - \lambda^2$

¿ $\sum_{x=2}^{\infty} \frac{\lambda^x e^{-\lambda}}{(x-2)!} + \lambda - \lambda^2$

$$\sigma^2 = \lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} e^{-\lambda} + \lambda - \lambda^2$$

¿ $\lambda^2 e^{-\lambda} e^{\lambda} + \lambda - \lambda^2 = \lambda$

Standard deviation $\sigma = \sqrt{\lambda}$.

Note: Mean and Variance are equal in Poisson distribution.

Theorem : Prove that Poisson distribution is the limiting case of binomial distribution.
Proof: If X is binomial distributed, then

$$f(x) = P(X = x) = {}^nC_x\, p^x (1-p)^{n-x} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where E(x) = np.

Let λ = np so that $p = \frac{\lambda}{n}$.

$$P(X = x) = {}^nC_x \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{x!\,n^x} \lambda^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{x-1}{n}\right)}{x!} \lambda^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

As n $\rightarrow \infty$

$$\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{x-1}{n}\right) \rightarrow 1$$

While $\left(1-\frac{\lambda}{n}\right)^{n-x} = \left(1-\frac{\lambda}{n}\right)^{n}\left(1-\frac{\lambda}{n}\right)^{-x} = e^{-\lambda}$

using the well-known result from calculus that $\lim_{n\to\infty}\left(1+\frac{u}{n}\right)^n = e^u$

It follows that when n $\to \infty$, $P(X=x) = \dfrac{\lambda^x e^{-\lambda}}{x!}$
which is the Poisson distribution.

Problems on Poisson distribution:

1. X is a Poisson variable and it is found that the probability that $X=2$ is two third of probability that $X=1$. Find the probability that $X=0$ and $X=3$. What is the probability that X exceeds 3?

Solution: $P(x)=\dfrac{\lambda^x e^{-\lambda}}{x!}, x=0,1,2,\ldots$

$$\therefore P(X=2)=\frac{2}{3}P\{X=1\}$$

i.e., $\dfrac{3\lambda^2 e^{-\lambda}}{2!}=\dfrac{2\lambda e^{-\lambda}}{1!}$

$3\lambda^2-4\lambda=0$

$\lambda(3\lambda-4)=0$

$\lambda=0$ or $\lambda=\dfrac{4}{3}$

If $\lambda=0$, then $P(x)=0$.

$$\therefore \lambda\neq 0 \wedge \lambda=\frac{4}{3}=¿P(x)=\frac{\left(\frac{4}{3}\right)^x e^{\frac{-4}{3}}}{x!}$$

$$P(X=0)=\left(\frac{4}{3}\right)^0 e^{\frac{-4}{3}}=e^{\frac{-4}{3}}=0.26359$$

$$P(X=3)=\left(\frac{4}{3}\right)^0 e^{\frac{-4}{3}}=0.10413714$$

$$P(X>3)=1-P(X\leq 3)=1-¿$$

$$¿1-\left[\frac{\left(\frac{4}{3}\right)^0 e^{\frac{-4}{3}}}{0!}+\frac{\left(\frac{4}{3}\right)^1 e^{\frac{-4}{3}}}{1!}+\frac{\left(\frac{4}{3}\right)^2 e^{\frac{-4}{3}}}{2!}+\frac{\left(\frac{4}{3}\right)^3 e^{\frac{-4}{3}}}{3!}\right]$$

$$¿1-0.953505=0.046494.$$

2. A manufacturer knows that the condensers that he makes contain on the average 1% defective. He packs them in box of 100. What is the probability that a box picked at random will contain 3 or more faulty condensers?

Solution: Let p=probability of a defective=0.01

$$n=100, mean\,\lambda=np=100\times 0.01=1.$$

$\therefore$ Probability that a box will contain 3 or more faulty condensers

¿ $1-P\{X<3\}$

¿ $1-\left[\frac{\lambda^0 e^{-\lambda}}{0!}+\frac{\lambda^1 e^{-\lambda}}{1!}+\frac{\lambda^2 e^{-\lambda}}{2!}\right]$

¿ $1\left[e^{-1}+e^{-1}+\frac{e^{-1}}{2!}\right]$

$= 1-\frac{5e^{-1}}{2}=0.0803013$

3. The number of accidents in a year to taxi drivers in a city follows a Poisson distribution with mean 3 out of 1000 taxi drivers. Find approximately the number of the drivers with i) no accident in a year ii) more than 3 accidents in a year.

<u>**Solution**</u>: $P(x)=\frac{3^x e^{-3}}{x!}$ gives the probability of accidents to taxi drivers.

Approximate number of drivers out of 1000 with x accidents¿ $1000\,P(x)$.

i) No. of drivers with no accidents ¿ $1000\times P(0)$

¿ $1000\,\frac{3^0 e^{-3}}{0!}=49.7870\approxeq 50.$

ii) No. of drivers with more than 3 accidents in a year=$1000-$¿no. of drivers with less than or equal to 3 accidents in a year.

¿ $1000-1000[P(0)+P(1)+P(2)+P(3)]$

¿ $1000-1000\left[\frac{3^0 e^{-3}}{0!}+\frac{3e^{-3}}{1!}+\frac{3^2 e^{-3}}{2!}+\frac{3^3 2^{-3}}{3!}\right]$

¿ 353.

4. A bag contains 1 red and 7 white marbles. A marble is drawn from the bag and its color is observed. Then the marble is put back into the bag and the contents are thoroughly mixed. Find the probability that in 8 such drawings, a red ball is selected exactly 3 times?

<u>**Solution:**</u> Let X be a random variable denoting the number of times red ball is selected in 8 drawings.

By data, mean $\lambda=np=8\times\frac{1}{8}=1.$

$P(x)=\frac{1^x e^{-1}}{x!},x=0,1,2,3,\ldots$

$\therefore$Probability of selecting a red ball exactly 3 times ¿ $P(X=3)=\frac{1^3 e^{-1}}{3!}=0.06131.$

5. Suppose that 0.01% of the population of the city with population 10,000 suffers from certain disease. Find the probability that there is at least two persons, who suffer from the disease. If there are 10 such cities in state, what is the probability that at least one city will have at least one person who suffer from the disease.

Solution: $n=10,000\,,p=\frac{0.01}{100}=0.0001\,,\lambda=np=10,000\times0.0001=1$

$$\therefore P(X\geq2)=1-P(X<2)$$

$$¿1-[P(X=0)+P(X=1)]$$

$$¿1-\left(\frac{e^{-1}\lambda^0}{0!}+\frac{e^{-1}\lambda^1}{1!}\right)=1-\frac{2}{e}=0.2642.$$

No. of cities, $n=10$.

For each city P(Person suffer from disease)$=P(X\geq1)=1-P(X<1)$

$¿1-P(X=0)$=0.6321.

Probability of at least one suffer from each city = 0.6321.

$$p=0.6321,n=10\,,q=1-p=1-0.6321=0.3679.$$

Using binomial distribution,

Probability that at least one city will have at least one person who suffer from disease $¿P(X\geq1)=1-P(X=0)=1-10_{C_0}p^0q^{10}=0.999.$

6. An insurance company has discovered that only about 0.1% of the population is involved in a certain type of accident each year. If its 10,000 policy holders were randomly selected from the population, what is the probability that not more than 5 of its clients are involved in such an accident next year?

Solution: Let X be a number of clients involved in accidents.

$$n=10,000\,,p=0,0001\,,\lambda=np=10$$

$$P(X=x)=\frac{e^{-\lambda}\lambda^x}{x!}\,,\lambda=10$$

$$P(X\leq5)=e^{-10}¿$$

$¿0.0671.$

7. The number of traffic accidents per week in a small city has a Poisson distribution with mean equal to 3. What is the probability of exactly 2 accidents occur in 2 weeks?

Solution: The mean traffic accident is 3. Thus, the mean accidents in two weeks are λ = (3) (2) = 6.

Since $f(x)=\frac{\lambda^x e^{-\lambda}}{x!}$

$$f(2)=\frac{6^2e^6}{2!}=18e^6.$$

8. The distributor of been seeds determines from extensive test that 5% of large batch of seeds won't germinates. He sells seeds in packets of 50 and guarantees 90% germination. Determine the probability that particular packet violate the guaranty.

Solution: p = 0.05, n = 50, λ = 50 × 0.05 = 2.5.

X = Number of seeds that do not germinate

P(X > 10% of 50) = P(X > 5) = 1 – P(X ≤ 5) = $1 - \sum_{0}^{5} \frac{2.5^x e^{-2.5}}{x!}$ = 0.042

The Normal Distribution

One of the most important examples of a continuous probability distribution is the normal distribution, sometimes called the Gaussian distribution. The density function for this distribution is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}, \; -\infty < x < \infty \quad -\infty<\mu<\infty, \sigma>0.$$

,

Such a variable X following the normal law is expressed as $X \quad N(\mu, \sigma^2)$.

If $z = \frac{x-\mu}{\sigma}$, then $f(z) = \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}}, \; -\infty < z < \infty$. This is often referred to as the standard normal density function.

This $Z = \frac{X-\mu}{\sigma}$ is the standard normal variate with $E(Z)=0$ and $var(Z)=1$and we write $Z \quad N(0,1)$.

The probability of X lying between x_1 and x_2 is given by the area under normal curve from x_1 to x_2.

$$P(x_1 \le X \le x_2) = \frac{1}{\sigma\sqrt{2\pi}} \int_{x_1}^{x_2} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$$

$$¿ \frac{1}{\sqrt{2\pi}} \int_{z_1}^{z_2} e^{\frac{-z^2}{2}} dz \text{ when } z = \frac{x-\mu}{\sigma}$$

$$¿ \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{z_2} e^{\frac{-z^2}{2}} dz - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_1} e^{\frac{-z^2}{2}} dz \right]$$

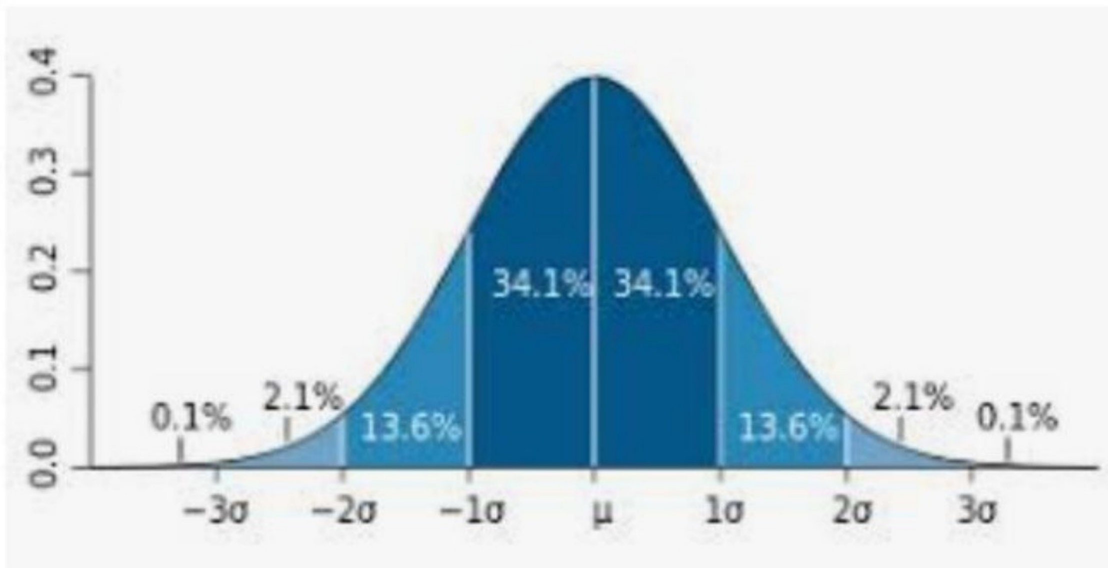
$$¿ \phi(z_2) - \phi(z_1)$$

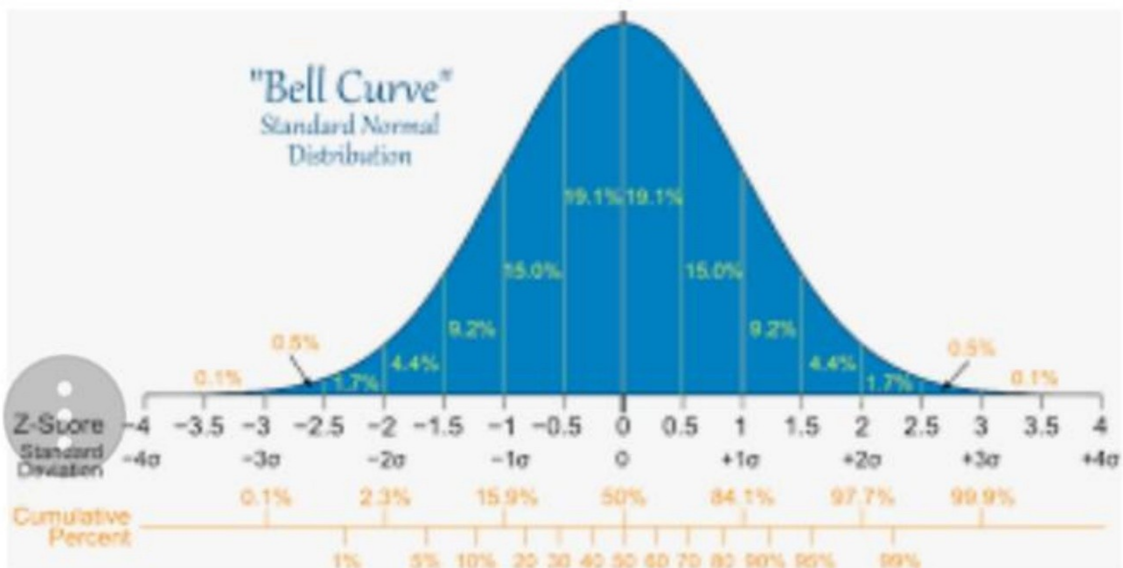
Where $\phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{\frac{-\mu^2}{2}} dx$for various values of $z$gives the corresponding standard normal function.

The cumulative function for z is given by

$$\phi(z)=P(Z\le z)=\int_{-\infty}^{z}\phi(\mu)\,dx=\text{¿}\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z}e^{\frac{-\mu^2}{2}}\,d\mu.\text{¿}$$

A graph of the density function f(z) sometimes called the standard normal curve, is shown in Figure.





In this graph $P(-1\le z\le 1)=0.6827$, $P(-2\le z\le 2)=0.9545$ and $P(-3\le z\le 3)=0.9973$.
Two important result on $\phi(z)$ are given by

1) $\phi(-z)=P\{Z\leq -z\}=P\{Z\geq z\}(by\, symmetry)$

¿ $1-P\{Z\leq z\}$

¿ $1-\phi(z)$

2) $P\{a\leq X\leq b\}=P\left\{\frac{a-\mu}{\sigma}\leq z\leq\frac{b-\mu}{\sigma}\right\}=P\left\{z\leq\frac{b-\mu}{\sigma}\right\}\;\; P\left\{z-\frac{a-\mu}{\sigma}\right\}$

¿ $\phi\left(\frac{b-\mu}{\sigma}\right)-\phi\left(\frac{a-\mu}{\sigma}\right)$

Expectation of normal distribution $E(X)=\mu$.

Variance of normal distribution $var(X)=\sigma^2$.

Note: 1) If $X\;\; N(\mu,\sigma^2)$ and $Z\;\; N(0,1)$, then X is called a normal variation with parameters μ and σ^2 while Z is the standard normal variate with the parameters 0 and 1.

1) To find $P\{a\leq X\leq b\}$,we change the variable X to the standard normal variable Z and hence find the area under the standard normal curve $\varphi(z)=\frac{1}{\sqrt{2\pi}}e^{\frac{-z^2}{2}}$.

i.e, $\phi(z)=P\{Z\leq z\}=\int_{-\infty}^{z}\frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}dx$.

Examples 1:

$P(z\geq 1.66)=1-P(z<1.66)=1-0.9515=0.0485.$ (Using standard distribution table).

Examples 2:

$P(-1.96\leq z\leq 1.96)=F(1.96)-F(-1.96)=F(1.96)-[1-F(1.96)]=0.9750-(1-0.9750)=0.95$ (Using standard distribution table).

Problems on Normal distribution:

1. In a normal distribution, 31% of the items are under 45 and 8% are over 64. Find the mean and variance of distribution.

Solution: Given $P\{X<45\}=31\%$ and $P\{X>64\}=8\%$

i.e., $P\left\{\frac{X-\mu}{\sigma}<\frac{45-\mu}{\sigma}\right\}=0.31$

$$P\left\{Z<\frac{45-\mu}{\sigma}\right\}=0.31$$

$$\phi\left(\frac{45-\mu}{\sigma}\right)=0.31$$

$$1-P\left(\frac{-45+\mu}{\sigma}\right)=0.31$$

i.e., $\phi\left(\frac{\mu-45}{\sigma}\right)=0.69$.

$$\frac{\mu-45}{\sigma}=0.5$$

$\mu-0.5\sigma=45$--------------------(1)

$P\{X>64\}=8\%$

i.e., $P>\left\{\frac{X-\mu}{\sigma}>\frac{64-\mu}{\sigma}\right\}$

$$P\left\{Z>\frac{64-\mu}{\sigma}\right\}=0.08$$

i.e., $1-\phi\left(\frac{64-\mu}{\sigma}\right)=0.08$

$$\phi\left(\frac{64-\mu}{\sigma}\right)=0.92$$

$$\frac{64-\mu}{\sigma}=1.4$$

$\mu+1.4\sigma=64$-------------------(2)

From (1) and (2), we get $\mu=50$ and $\sigma=10$.

2. Suppose X has the distribution $N(\mu,\sigma^2)$. Determine c as function of μ and σ such that $P\{X\le c\}=2P\{X>C\}$.

<u>Solution:</u> $P\{X\le c\}=2P\{X>C\}$

i.e., $P\{X\le c\}=2(1-P\{X\le c\})$

$P\{X\le c\}=2-2P\{X\le c\}$

$3P\{X\le c\}=2$

$$P\{X\le c\}=\frac{2}{3}=0.666$$

i.e., $P\left\{\frac{X-\mu}{\sigma} \leq \frac{c-\mu}{\sigma}\right\} = 0.666$

$P\left\{Z \leq \frac{c-\mu}{\sigma}\right\} = 0.666$

$\phi\left(\frac{c-\mu}{\sigma}\right) = 0.666$

$\frac{c-\mu}{\sigma} = 0.43$

$c = \mu + 0.43$.

3. Suppose that life length of two electronic device say D_1 and D_2 have distributions $N(40,36)$ and $N(45,9)$ respectively. If the electronic device is to be used for 45hours period, which device is to be preferred? If it is to be used for 48 hours period, which device is to be preferred?

Solution: Let X represents the life lengths of the electronic device.

For device D_1,

(i) For a period of 45 hours,

$P\{X \geq 45\} = 1 - P\{X < 45\} = 1 - P\{Z - 0.8333\} = 1 - \phi(0.8333)$

¿ $1 - 0.7967 = 0.2053$.

(ii) For a period of 48 hours,

$P\{X \geq 48\} = 1 - P\{X < 48\} = 1 - P\{Z < 1.333\} = 1 - \phi(1.333)$

¿ $1 - 0.9082 = 0.0918$.

For device D_2,

(i) For a period of 45 hours,

$P\{X \geq 45\} = 1 - P\{X < 45\} = 1 - P\{Z < 0\} = 1 - \phi(0) = 1 - 0.5$

¿ 0.5.

(ii) For a period of 48 hours,

$P\{X \geq 48\} = 1 - P\{X < 48\} = 1 - P\{Z < 1\} = 1 - \phi(1)$

¿ $1 - 0.8413 = 0.1587$.

For a period of 45 hours, device D_2 is to be preferred whereas for a period of 48 hours, device D_2 is preferred.

4. For a normally distributed variate with mean 1 and standard deviation 3. Find the probabilities that i) $3.43 \leq X \leq 6.19$ ii) $-1.43 \leq X \leq 6.19$

Solution:

(i) $P\{3.43 \leq X \leq 6.19\} = P\left\{\frac{3.43-1}{3} \leq Z \leq \frac{6.19-1}{3}\right\}$

¿ $P\{0.81 \leq Z \leq 1.73\}$

¿ $\phi(1.73) - \phi(0.81)$

¿ $0.9582 - 0.7910 = 0.1672$.

$$\text{(ii)}P(-1.43 \le X \le 6.19) = P\left\{\frac{-1.43-1}{3} \le Z \le \frac{6.19-1}{3}\right\}$$

$$¿P(-0.81 \le Z \le 1.73)$$

$$¿\phi(1.73) - \phi(-0.81)$$

$$¿\phi(1.73) - (1 - \phi(0.81))$$

$$¿0.9582 + 0.7910 - 1 = 0.7492.$$

5. A fair coin is tossed 500 times. Find the probabilities that the number of heads will not differ from 250 by (a) more than 10 (b) more than 30.

<u>Solution</u>: Let X represents the number of heads in 500 tosses.

$\therefore p = ¿$probability of head turning up$¿\frac{1}{2}$ and $n = 500$.

Mean$¿np = 500 \times \frac{1}{2} = 250$ and $\sigma = \sqrt{npq} = \sqrt{500 \times \frac{1}{2} \times \frac{1}{2}} = 11.18033.$

Since n is very large and p is not small. We can use normal approximation to binomial.

a) Probability that number of heads will not differ from 250 by more than 1

$$¿P(240 \le X \le 260) = P\left\{\frac{240-250}{11.18033} \le Z \le \frac{260-250}{11.18033}\right\}$$

$$¿P(-0.894427191 \le Z \le 0.894427191)$$

$$¿2\phi(0.894427191) - 1$$

$$¿2(0.8133) - 1 = 0.6266.$$

b) Probability that number of heads will not differ from 250 by more than 30=

$$P(220 \le X \le 280) = P\left\{\frac{220-250}{11.180339} \le Z \le \frac{280-250}{11.180339}\right\}$$

$$¿P\{-2.68328172 \le Z \le 2.68328172\}$$

$$¿2\phi(2.68328172) - 1$$

$$¿2 \times 0.9963 - 1 = 0.9920.$$

6. The weekly wages of workers in a certain factory was found to be normally distributed with mean $Rs.500$ and standard deviation $Rs.50$. There are 228 persons getting at least $Rs.600$. Find the number of workers in the factory.

<u>Solution</u>: For $P(X \le 600) = P\left\{Z \le \frac{600-500}{50}\right\} = P(Z \le 2) = 1 - P\{Z \le 2\}$

$$P(X \ge 600) = 1 - \phi(Z) = 0.0228.$$

If n is total number of workers then $nP(X \ge 600) = 228.$

i.e.,$n \times 0.0228 = 228.$

$$n = \frac{228}{0.0228} = 10{,}000.$$

There are 10,000 workers in the factory.

7. If X is normally distributed with N(1, 4). Find P(|X| > 4).

Solution: If $z = \frac{x - \mu}{\sigma}$,
then,

$$P(|X| > 4) = 1 - P(|X| \le 4) = 1 - P(-4 \le X \le 4)$$
$$= 1 - P\left(\frac{-4-1}{2} \le Z \le \frac{4-1}{2}\right) = 1 - \left[F(1.5) - F(-2.5)\right]$$
$$= 2 - 0.9332 - 0.9938 = 0.073$$

8. In a normal distribution 31% of items rare under 45 and 48% are over 64. Find mean and standard deviation.

Solution:

X = 45, area = 31%

$$\Rightarrow Z = -0.5$$
$$Z = \frac{X - \mu}{\sigma} \Rightarrow -0.5 = \frac{45 - \mu}{\sigma}$$
$$\Rightarrow -0.5\sigma = 45 - \mu \quad \ldots. \quad (1)$$

X = 64 , area = 48%

$$\Rightarrow Z = 0.05$$
$$Z = \frac{X - \mu}{\sigma} \Rightarrow 0.05 = \frac{64 - \mu}{\sigma}$$
$$\Rightarrow 0.05\sigma = 64 - \mu \quad \ldots. \quad (2)$$

Solving (1) and (2),
$\mu = 62.27, \ \sigma = 34.54.$

9. If mean marks is 60 and standard deviation is 10, 70% failed in examination. What is the grace marks given to obtain 70% pass the examination?

Solution:

$$\mu = 60,\ \sigma = 10.$$

$$P(Z \le a) = 0.7 \Rightarrow Z = 0.525$$

$$Z = \frac{X - \mu}{\sigma} \Rightarrow 0.525 = \frac{X - 60}{10}$$

$$\Rightarrow X = 65.25$$

$$P(Z \le a) = 0.3 \Rightarrow Z = -0.525$$

$$Z = \frac{X - \mu}{\sigma} \Rightarrow -0.525 = \frac{X - 60}{10}$$

$$\Rightarrow X = 54.75$$

Therefore, grace mark = 65.25 – 54.75 = 10.5

10. Local authorities in a certain city install 10000 electric lamps in streets of city. If these lamps have average life of 1000 burning hours with standard deviation of 200 hours. What is the number of lamps might be expected to fail (i) in first 800 hours (ii) between 800 and 1200 hours.

Solution: $\mu = 1000,\ \sigma = 200.$

(a)

$$P(Z \le 800) = P\left(Z \le \frac{800 - 1000}{200}\right)$$
$$= P(Z \le -1)$$
$$= 1 - P(Z \le 1)$$
$$= 0.1587$$

Number of lamps = 0.1587 X 10000 = 1587.

(b)

$$P(800 \le Z \le 1200) = P\left(\frac{800 - 1000}{200} \le Z \le \frac{1200 - 1000}{200}\right)$$
$$= P(-1 \le Z \le -1)$$
$$= 0.6826$$

Number of lamps = 0.6826 X 10000 = 6826

11. Annual rainfall at a place is known to be normal distributed with $\mu = 29.5$ inches and $\sigma = 2.5$ inches. How many inches of rain is expected to exceeded about 5% of the time?

Solution:

$$\mu = 29.5, \sigma = 2.5.$$
$$P(Z > a) = 0.05$$
$$\Rightarrow P(Z \le a) = 0.95$$
$$\Rightarrow Z = 1.645$$
$$X = \mu + \sigma Z = 33.61$$

Therefore, amount of rain exceeded = 33.61 - 29.5 = 4.11 inches.

Exponential distributions

A continuous random variable X is said to have exponential distribution with parameter $\alpha > 0$ if its pdf is given by

$$f(x) = \begin{cases} \alpha e^{-\alpha x}, x > 0 \\ 0, elsewhere \end{cases}$$

and we write $X \sim \exp(\alpha)$

Mean:

$$E(X) = \int_0^\infty x\alpha e^{-\alpha x} dx$$

Put $\alpha x = z$

Then $E(X) = \frac{1}{\alpha}\int_0^\infty z e^{-z} dz = \frac{1}{\alpha}\int_0^\infty e^{-z} z^{2-1} dz = \frac{1}{\alpha}\Gamma(2) = \frac{1}{\alpha}$

Variance:

$$V(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = \int_0^\infty x^2 \alpha e^{-\alpha x} dx$$

Put $\alpha x = z$

$$E\left(X^2\right)=\frac{1}{\alpha^2}\int_0^\infty z^2 e^{-z}\,dz=\frac{1}{\alpha^2}\int_0^\infty e^{-z} z^{3-1}\,dz=\frac{1}{\alpha}\Gamma(3)=\frac{2!}{\alpha^2}=\frac{2}{\alpha^2}$$

$$V(X)=\frac{1}{\alpha^2}$$

Gamma Distribution

A continuous random variable X is said to have Gamma distribution with parameter

$r>0 \wedge \alpha>0$ if its pdf is given by

$$f(x)=\begin{cases}\frac{\alpha}{\Gamma(r)}e^{-\alpha x}(\alpha x)^{r-1}, x>0\\ 0, elsewhere\end{cases}$$

and we write $X \;\; G(r,\alpha)$

Note: In Gamma distribution if r = 1 we get exponential distribution. Therefore, exponential distribution is a special case of Gamma distribution.

Mean:

$$E(X)=\int_0^\infty x\frac{\alpha}{\Gamma(r)}e^{-\alpha x}(\alpha x)^{r-1}dx$$

Put $\alpha x=z$

Then $$E(X)=\frac{1}{\alpha}\frac{1}{\Gamma(r)}\int_0^\infty e^{-z}z^r\,dz=\frac{1}{\alpha}\frac{1}{\Gamma(r)}\Gamma(r+1)=\frac{1}{\alpha}\frac{1}{\Gamma(r)}r\Gamma(r)=\frac{r}{\alpha}$$

Variance:

$$V(X)=E\left(X^2\right)-[E(X)]^2$$

$$E\left(X^2\right)=\int_0^\infty x^2\frac{\alpha}{\Gamma(r)}e^{-\alpha x}(\alpha x)^{r-1}dx=\frac{r^2+r}{\alpha^2}$$

$$V(X)=\frac{r}{\alpha^2}$$

Chi-square distribution

A continuous random variable X is said to have Chi-square distribution if its pdf is given by

$$f(x)=\begin{cases}\dfrac{1}{2^{n/2}\Gamma\left(\dfrac{n}{2}\right)}e^{\frac{-x}{2}}x^{\frac{n}{2}-1}, x>0, n>0\\ 0, elsewhere\end{cases}$$

We write $X \quad \lambda^2(n)$ where n is called the number of degrees of freedom.

Mean:

$$E(X)=\int_0^\infty x\frac{1}{2^{n/2}\Gamma\left(\frac{n}{2}\right)}e^{\frac{-x}{2}}x^{\frac{n}{2}-1}dx$$

Put $\frac{x}{2}=z$

Then

$$E(X)=\frac{1}{2^{n/2}\Gamma\left(\frac{n}{2}\right)}\int_0^\infty e^{-z}¿¿$$

Variance:

$$V(X)=E(X^2)-[E(X)]^2$$

$$E(X^2)=\int_0^\infty x^2\frac{1}{2^{n/2}\Gamma\left(\frac{n}{2}\right)}e^{\frac{-x}{2}}x^{\frac{n}{2}-1}dx=n^2+2n$$

$$V(X)=2n$$

Functions of One dimensional random variables

If X is a discrete random variable and $Y=H(X)$ is a continuous function of X, then Y is also a Discrete Random Variable.

Eg:

X	-1	0	1
$P(x)$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

Suppose $Y=3X+1$,then pmf of Y is given by

Y	-2	1	4
$P(y)$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

Suppose $Y=X^2$, then pmf of Y is

Y	1	0
$P(y)$	$\frac{1}{2}$	$\frac{1}{2}$

Suppose X is a continuous random variable with pdf $f(x)$ and $H(X)$ is a continuous function of X. Then Y is a continuous random variable. To obtain pdf of Y we follow the following steps.

1. Obtain cdf of Y, i.e., $G(y)=P(Y\le y)$.
2. Differentiate $G(y)$ with respect to y to get pdf of y i.e., $g(y)$.
3. Determine the range space of Y such that $g(y)>0$.

Problems:

1. If $f(x)=\begin{cases} 2x; & 0<x<1 \\ 0; & Otherwise \end{cases}$, and $Y=3X+1$, find pdf of Y.

   Soln: $G(y)=P(Y\le y)=P(3X+1\le y)=P\left(X\le\frac{y-1}{3}\right)$

   $$G(y)=\int_0^{\frac{y-1}{3}} 2x dx=\left(\frac{y-1}{3}\right)^2.$$

   $$g(y)=G'(y)=\frac{2(y-1)}{9}.$$

   $$0<x<1 \Longrightarrow 0<\frac{y-1}{3}<1 \Longrightarrow 1<y<4.$$

Therefore, $g(y)=\begin{cases} \dfrac{2(y-1)}{9}; & 1<y<4 \\ 0; & Otherwise \end{cases}$.

2. If $f(x)=\begin{cases} 2x; & 0<x<1 \\ 0; & Otherwise \end{cases}$, and $Y=e^{-X}$, find pdf of Y.

Soln: $G(y)=P(Y\leq y)=P(e^{-X}\leq y)=P\left(\log_e\frac{1}{y}\leq X\right)$

$$G(y)=\int_{\log_e\frac{1}{y}}^{1} 2x\,dx=1-\left(\log_e\frac{1}{y}\right)^2.$$

$$g(y)=G'(y)=\frac{2}{y}\log_e\frac{1}{y}.$$

$$0<x<1 \Longrightarrow 0<\log_e\frac{1}{y}<1 \Longrightarrow \frac{1}{e}<y<1.$$

Therefore, $g(y)=\begin{cases} \dfrac{2}{y}\log_e\dfrac{1}{y}; & \dfrac{1}{e}<y<1 \\ 0; & Otherwise \end{cases}$.

Result: Let X be a continuous random variable with pdf $f(x)$. Let $Y=X^2$. Then pdf of Y is $g(y)=\frac{1}{2\sqrt{y}}\left(f(\sqrt{y})+f(-\sqrt{y})\right)$

Example 1: Suppose $f(x)=\begin{cases} 2x\,e^{-x^2}; & 0<x<\infty \\ 0; & Otherwise \end{cases}$. Find pdf of $Y=X^2$.

Soln:

$$g(y)=\frac{1}{2\sqrt{y}}\left(f(\sqrt{y})+f(-\sqrt{y})\right)=\frac{1}{2\sqrt{y}}\left(2\sqrt{y}\,e^{-y}+0\right)=e^{-y};\,0<x<\infty.$$

Example 2: Suppose $f(x)=\begin{cases} \dfrac{2}{9}(x+1); & -1<x<1 \\ 0; & Otherwise \end{cases}$. Find pdf of $Y=X^2$.

Soln:

$$g(y)=\frac{1}{2\sqrt{y}}\left(f(\sqrt{y})+f(-\sqrt{y})\right)=\frac{1}{2\sqrt{y}}\left(\frac{2(\sqrt{y}+1)}{9}+\frac{2(-\sqrt{y}+1)}{9}\right)=\frac{2}{9\sqrt{y}};0<x<1.$$

Theorem: Let X be a continuous random variable with pdf $f(x)$. Suppose $Y=H(X)$ is a strictly monotone (increasing or decreasing) function of X, then pdf of Y is given by

$g(y)=f(x)\left|\frac{dx}{dy}\right|$ where $x=H^{-1}(y)$.

Example:

1. Suppose X is uniformly distributed over (0,1), find pdf of $Y=\frac{1}{X+1}$.

   Soln: We know that Y is strictly monotone.

   $f(x)=\begin{cases}1; & 0<x<1\\ 0; & Otherwise\end{cases}$.

   Note that $X=\frac{1}{Y}-1$. $\Rightarrow f(x)=f\left(\frac{1}{Y}-1\right)=1.$

   $\left|\frac{dx}{dy}\right|=\frac{1}{y^2}$.

   Therefore, $g(y)=\frac{1}{y^2}; \frac{1}{2}<y<1.$

2. If X is uniformly distributed over $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$, find the pdf of $Y=tanX$.(Or show that $Y=tanX$ follows Cauchy's distribution).

   Soln: Given $f(x)=\begin{cases}\frac{1}{\pi}; & \frac{-\pi}{2}<x<\frac{\pi}{2}\\ 0; & Otherwise\end{cases}$.

   We know that Y is strictly monotone.

   Then $X=\tan^{-1}Y \Rightarrow f$ ¿¿ And $\left|\frac{dx}{dy}\right|=\frac{1}{1+y^2}$.

   Therefore, $g(y)=\frac{1}{\pi}\frac{1}{1+y^2}; -\infty<y<\infty.$

3. If $X \quad N(\mu,\sigma^2)$, then show that $Z=\frac{X-\mu}{\sigma} \quad N(0,1)$ and $Y=Z^2 \quad \chi^2(1)$.

   Soln: $G(z)=P(Z\le z)=P\left(\frac{X-\mu}{\sigma}\le z\right)=P(\sigma z+\mu\ge x)$

   $$G(z)=F(\sigma z+\mu).$$

   $$g(z)=G'(z)=F'(\sigma z+\mu)\sigma=f(\sigma z+\mu)\sigma=\frac{1}{\sqrt{2\pi}}e^{\frac{-z^2}{2}} \quad N(0,1).$$

   Now, $g(y)=\frac{1}{2\sqrt{y}}\left(f(\sqrt{y})+f(-\sqrt{y})\right)=\frac{1}{2\sqrt{y}}\left(\frac{1}{\sqrt{2\pi}}e^{\frac{-y}{2}}+\frac{1}{\sqrt{2\pi}}e^{\frac{-y}{2}}\right)$

   $$g(y)=\frac{1}{\sqrt{y}\sqrt{2\pi}}e^{\frac{-y}{2}}.$$

   Hence, $g(y) \quad \chi^2(1)$.

   Extra Problem:

1. A random variable X having Cauchy distribution. Show that $1/X$ also has Cauchy distribution.

Functions of two dimensional random variables

Let (X,Y) be a continuous two dimensional random variable with pdf $f(x,y)$. If $Z=H_1(X,Y)$ is a continuous function of (X,Y), then Z will be a continuous (one-dimensional) random variable. In order to find the pdf of Z, we shall follow the procedure discussed below.

To find the pdf of $Z=H_1(X,Y)$, we first introduce a second random variable, say $W=H_2(X,Y)$, and obtain the joint pdf of Z and W, say $k(z,w)$. From the knowledge of $k(z,w)$, we can then obtain the desired pdf of Z, say $g(z)$, by simply integrating $k(z,w)$ with respect to w. That is, $g(z)=\int_{-\infty}^{\infty} k(z,w)\,dw$.

Two problems which arise here are

i. how to find the joint pdf $k(z,w)$ of Zand W
ii. how to choose the appropriate random variable $W=H_2(X,Y)$

To resolve these problems, let us simply state that we usually make the simplest possible choice for W as it plays only an intermediate role. In order to find the joint pdf $k(z,w)$, we need the following theorem.

Theorem:

Suppose that (X,Y) is a two-dimensional continuous random variable with joint pdf $f(x,y)$. Let $Z=H_1(X,Y)$ and $W=H_2(X,Y)$and assume that the functions H_1 and H_2 satisfy the following conditions:

i. The equations $z=H_1(x,y)$ and $w=H_2(x,y)$ may be uniquely solved for x and $y$in terms of z and w, say $x=G_1(z,w)$ and $y=G_2(z,w)$.
ii. The partial derivatives $\frac{\partial x}{\partial z}, \frac{\partial x}{\partial w}, \frac{\partial y}{\partial z}$ and $\frac{\delta y}{\delta w}$exist and are continuous.

Then the joint pdf (Z,W), say $k(z,w)$, is given by the following expression:

$k(z,w)=f[G_1(z,w),G_2(z,w)] \vee J(z,w) \vee ¿$,

where $J(z,w)$ is the following 2×2 determinant:

$$J(z,w)=\begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\delta y}{\delta w} \end{vmatrix}$$

This determinant is called the 'Jacobian' of the transformation $(x,y)\to(z,w)$ and is sometimes denoted by $\frac{\delta(x,y)}{\delta(z,w)}$. We note that $k(z,w)$ will be nonzero for those values of (z,w) corresponding to values of (x,y) for which $f(x,y)$ is nonzero.

Problems

1. Suppose that X and Y are two independent random variables having pdf $f(x)=e^{-x}, 0\le x\le\infty$ and $g(y)=2e^{-2y}, 0\le y\le\infty$. Find the pdf of X+Y

Solution:

Since X and Y are independent, the joint pdf of (X,Y) is given by,

$$f(x,y)=f(x)g(y)=2e^{-(x+2y)}, 0\le x,y\le\infty$$

Let $Z=X+Y$ and $W=Z$, that is $Y=W$ and $X=Z-W$.

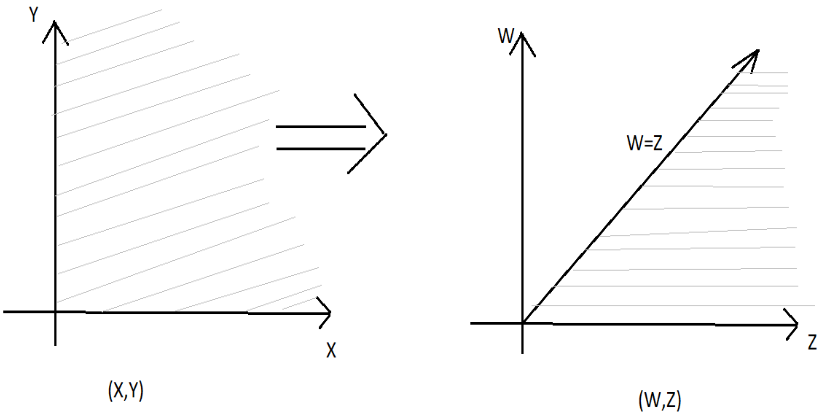
The Jacobian $J=\begin{vmatrix}\frac{\partial x}{\partial z} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial z} & \frac{\delta y}{\delta w}\end{vmatrix}=\begin{vmatrix}1 & -1\\ 0 & 1\end{vmatrix}=1$

Thus joint pdf of (W,Z) is,

$$k(z,w)=f(x,y)|J|=2e^{-(x+2y)}=2e^{-(z+w)}$$

$$0\le y\le\infty \Rightarrow 0\le w\le\infty$$

$$0\le x\le\infty\Rightarrow 0\le z-w\le\infty\Rightarrow w\le z\le\infty$$



Thus $k(w,z)=2e^{-(z+w)}, 0\le w\le z\le\infty$

The required pdf of z, $h(z)=\int_{w=0}^{z} 2e^{-(z+w)}dw$

$¿2(e^{-z}-e^{-2z}), 0\le z\le\infty$.

2. If $X \;\; N(0,\sigma^2)$, $Y \;\; N(0,\sigma^2)$ and X,Y are independent. Find the pdf of $R=\sqrt{X^2+Y^2}$

Solution:

Pdf of $X: f(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-x^2/2\sigma^2}, -\infty\le x\le\infty$

Pdf of $Y: g(y)=\frac{1}{\sqrt{2\pi}\sigma}e^{-y^2/2\sigma^2}, -\infty\le y\le\infty$

Since X and Y are independent, the joint pdf of (X,Y) is given by,

$$f(x,y)=f(x)g(y)=\frac{1}{2\pi\sigma^2}e^{-¿¿}$$

Let $R=\sqrt{X^2+Y^2}$ and $\theta=\tan^{-1}\left(\frac{X}{Y}\right)$, that is $X=R\cos\theta$ and $Y=R\sin\theta$ and the Jacobian $J=R$.

Thus joint pdf of (R,θ) is,

$k(r,\theta)=f(x,y)|J|=\frac{R}{2\pi\sigma^2}e^{-R^2/2\sigma^2}, R\geq 0, 0\leq\theta\leq 2\pi$

The required pdf of z, $h(z)=\int_{\theta=0}^{2\pi}\frac{R}{2\pi\sigma^2}e^{-R^2/2\sigma^2}d\theta$

$¿\frac{R}{\sigma^2}e^{-R^2/2\sigma^2}, R\geq 0.$

3. If X_1, X_2 are independent and have standard normal distribution $X_1, X_2\ \ N(0,1)$. Find the pdf of $\frac{X_1}{X_2}$.

Solution:

Pdf of $X_1 : f(x_1)=\frac{1}{\sqrt{2\pi}}e^{-x_1^2/2}, -\infty\leq x_1\leq\infty$

Pdf of $X_2 : g(x_2)=\frac{1}{\sqrt{2\pi}}e^{-y_1^2/2}, -\infty\leq y_1\leq\infty$

Since X_1, X_2 are independent, the joint pdf of $(X¿¿1, X_2)¿$ is given by,

$$f(x_1,x_2)=\frac{1}{2\pi}e^{-(x_1^2+x_2^2)/2}, -\infty\leq x_1, y_1\leq\infty$$

Let $Z=\frac{X_1}{X_2}$ and $W=X_2$, that is $X_2=W$ and $X_1=ZW$.

The Jacobian $J=\begin{vmatrix} w & z \\ 0 & 1 \end{vmatrix}=w$

Thus joint pdf of (W,Z) is,

$k(z,w)=¿w\vee\frac{¿}{2\pi}e^{-w^2(1+z^2)/2}, ¿\quad -\infty\leq w, z\leq\infty.$

The required pdf of Z, $h(z)=\int_{-\infty}^{\infty}¿w\vee\frac{¿}{2\pi}e^{-w^2(1+z^2)/2}dw¿$

$$¿\frac{2}{2\pi}\int_0^{\infty}|w|e^{-w^2(1+z^2)/2}dw$$

On substitution: $-w^2(1+z^2)/2=t$

$-w(1+z^2)dw=dt$

We get, $h(z)=\frac{1}{\pi}\int_0^{\infty}\frac{e^{-t}}{1+z^2}dt=\frac{1}{\pi(1+z^2)}, -\infty\leq z\leq\infty.$

4. The joint pdf of the random variable (X,Y) is given by

$$f(x,y)=\frac{x}{2}e^{-y}, 0<x<2, y>0$$

Find the pdf of $X+Y$

Solution:

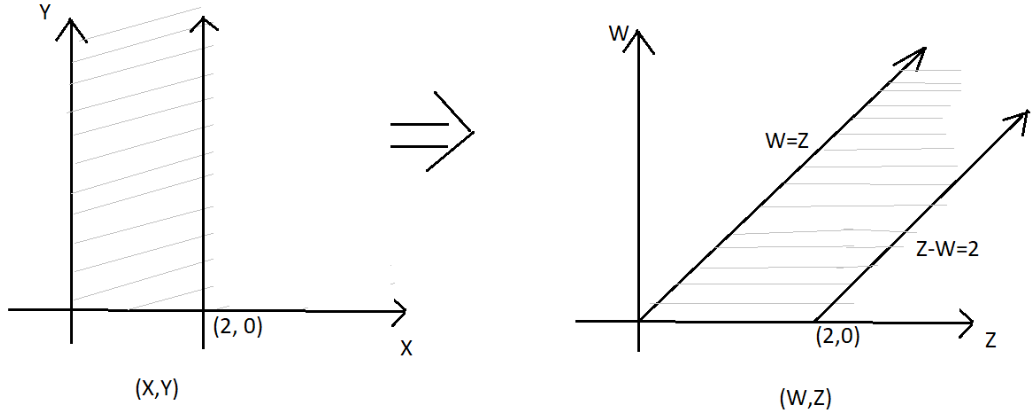
Let $Z=X+Y$ and $W=Z$, that is $Y=W$ and $X=Z-W$.

The Jacobian $J=1$

Thus joint pdf of (W,Z) is, $k(z,w)=f(x,y)|J|=\frac{z-w}{2}e^{-w}$

$$0\le y\le\infty\Rightarrow 0\le w\le\infty$$
$$0\le x\le 2\Rightarrow 0\le z-w\le 2\Rightarrow w\le z\le 2+w$$

$k(z,w)=\frac{z-w}{2}e^{-w}, 0\le w\le z\le 2+w$



The required pdf of z, $h(z)=\begin{cases}\int_0^z\left(\frac{z-w}{2}\right)e^{-w}dw, when\,0<z<2\\ \int_{z-2}^z\left(\frac{z-w}{2}\right)e^{-w}dw, when\,2<z<\infty\end{cases}$

$$h(z)=\begin{cases}\frac{1}{2}(z+e^{-z}-1), when\,0<z<2\\ \frac{1}{2}(e^{z}+e^{2-z}), when\,2<z<\infty\end{cases}$$

5. Let $f(x)=\begin{cases}\alpha^{-2}e^{-\left(\frac{x+y}{2}\right)} & ;x,y>0,\alpha>0\\ 0;elsewhere\end{cases}$. Find the distribution of $\frac{x-y}{2}$.

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Moment Generating Functions(mgf)

Definition:

Let X be a random variable. The moment generating function of X denoted by $M_X(t)$ and is defined as

$$M_X(t)=\begin{cases}\sum_{j=1}^{\infty} e^{t x_j} P\left(X=x_j\right); X \text{ is discrete}\\ \int_{-\infty}^{\infty} e^{tx} f(x) dx ; X \text{ is continuous}\end{cases}$$

Note:

1. $M_X(t)=E\left(e^{tx}\right)$
2. The nth derivative of $M_X(t)$ at t = 0 is $E\left(X^n\right)$. i.e., $M_X^{(n)}(0)=E\left(X^n\right)$.
3. $E\left(X^n\right)$ is the coefficient of $\frac{t^n}{n!}$ in the equation $M_X(t)=E\left(e^{tx}\right)$.
4. $V(X)=M_X''(0)-\left[M_X'(0)\right]^2$.

Properties:

1. $M_{ax}(t)=M_X(at)$
2. $M_{ax+b}(t)=e^{tb} M_X(at)$
3. If X and Y are independent random variables, then $M_{X+Y}(t)=M_X(t).M_Y(t)$. This can be generalized for ‘n’ random variables.
4. If two or more independent random variables having a certain distribution are added, the resulting random variable has a distribution of the same type as that of the random variables. This property is called reproductive property.
   Example: Suppose that X and Y are independent random variables with distributions $N\left(\mu_1, \sigma_1^2\right)$ and $N\left(\mu_2, \sigma_2^2\right)$ respectively.
   Let Z = X + Y.
   $M_X(t)=E\left(e^{zt}\right)=E\left(e^{(X+Y)t}\right)$

   $$¿ E\left(e^{Xt}\right).E\left(e^{Yt}\right)=M_X(t).M_Y(t)$$

   $$=e^{t\mu_1+\frac{\sigma_1^2 t^2}{2}}.e^{t\mu_2+\frac{\sigma_2^2 t^2}{2}}=e^{\left[t(\mu_1+\mu_2)+\left(\sigma_1^2+\sigma_2^2\right)\frac{t^2}{2}\right]}$$

   This represents the mgf of a normally distributed random variable with expected value ($\mu_1+\mu_2$¿ and variance ($\sigma_1^2+\sigma_2^2$¿.

Problems:

1. If X is the outcome obtained when a die is tossed, then find the moment generating function. Also find its mean and variance.

Solution: $P(X=x)=\frac{1}{6}; x=1,2,3,4,5,6$

We have $M_X(t)=E(e^{tx})$

$$¿\sum_{1}^{6} e^{tx} P(x)$$

$$¿\frac{1}{6}¿]$$

We have $E(X)=M_X'(0) = \frac{1}{6}$ [1 + ….+ 6]

$$¿\frac{21}{6}$$

$$M_X''(t)=\frac{1}{6}\left[e^t+4e^{2t}+\ldots+36e^{6t}\right]$$

$$E(X^2)=M_X''(0)=\frac{91}{6}$$

$$V(X)=M_X''(0)-\left[M_X'(0)\right]^2=2.91$$

2. If X is a random variable taking values 0,1,2,…. and $P(X)=ab^x$, where a and b are positive constants such that a+b = 1, then
   (i) Find mgf.
   (ii) If $E(X)=m_1, E(X^2)=m_2$, show that $m_2=m_1(2m_1+1)$.

Solution: (i): $M_X(t)=E(e^{tx})$

$$¿\sum_{x=0}^{\infty} e^{tx} p(x) = \frac{a}{1-be^t}$$

$$(ii): E(X)=M_X'(0)=\frac{ab}{(1-b)^2} = \frac{b}{a}=m_1$$

$$E(X^2)=m_2=M_X''(0)$$

$$=\frac{b}{a}\left(\frac{1+b}{a}\right)$$

$$=m_1\frac{1}{a}[a+b+b]$$

$$= m_1(2m_1+1)$$

3. If X has pdf $f(x)=\lambda e^{-\lambda(x-a)}$ if $x\geq a$. Find its mgf and also find the mean and variance.

4. Suppose that X has pdf $f(x)=\frac{e^{-|x|}}{2}$, $-\infty < x < \infty$, find mean and variance using mgf.

Mgf of Binomial distribution:

If $p(x)=nCx\,p^x q^{n-x}; x=0,1,2,\ldots,n$, then

$$M_X(t)=\sum_{x=0}^{n} e^{tx} nCx\,p^x q^{n-x}$$
$$Expanding,$$
$$M_X(t)=(pe^t+q)^n$$

Mgf of Poisson distribution:

If $p(x)=\frac{\alpha^x e^{-\alpha}}{x!}; x=0,1,2,\ldots,\infty$, then

$$M_X(t)=\sum_{x=0}^{n} e^{tx}\frac{\alpha^x e^{-\alpha}}{x!}$$
$$¿e^{-\alpha}e^{e^t\alpha}$$
$$¿e^{\alpha(e^t-1)}$$

Mgf of Uniform distribution in (-a, a):

If $f(x)=\frac{1}{2a}$ in (-a, a), then

$M_X(t)=\int_{-a}^{a}\frac{1}{2a}e^{tx}$dx = $\frac{sinhat}{at}$

The value of $E(X^{2n})$:

We know that $E(X^{2n})$ is the coefficient of $\frac{t^{2n}}{(2n)!}$ in the equation $M_X(t)=E(e^{tx})$.

$E¿$) = $\frac{a^{2n}t^{2n}}{(2n+1)!}$ = $\frac{a^{2n}}{2n+1}$

Mgf of Exponential distribution:

If $f(x)=\alpha e^{-\alpha x}, x>0$, then

$$M_X(t)=\int_0^\infty \alpha e^{-\alpha x}e^{tx}\,dx$$

$$¿\alpha\int_0^\infty e^{-(\alpha-t)x}\,dx$$

$$M_X(t)=\frac{\alpha}{\alpha-t}$$

Mgf of Gamma distribution:

If $f(x)=\frac{\alpha^r x^{r-1}e^{-\alpha x}}{\Gamma(r)}; \alpha>0, r>0, 0<x<\infty$, then

$$M_X(t)=\int_0^\infty e^{tx}\frac{\alpha^r x^{r-1}e^{-\alpha x}}{\Gamma(r)}\,dx$$

Take $x(\propto - t) = v$, $$M_X(t) = \frac{\propto^r}{\Gamma(r)} \int_0^\infty \left(\frac{v}{\propto - t}\right)^{r-1} e^{-v} \frac{dv}{\propto - t}$$

$$= \frac{\propto^r}{(\propto - t)^r} \frac{1}{\Gamma(r)} \int_0^\infty e^{-v} v^{r-1} dv$$

$$= \frac{\propto^r}{(\propto - t)^r}$$

Mgf of Chi-square distribution:

By substituting $r = \frac{n}{2}$, $\propto = \frac{1}{2}$ in mgf of gamma distribution, we get chi-square distribution.

Therefore $M_X(t) = (1 - 2t)^{-n/2}$

Mgf of Normal distribution:

If $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty < x < \infty$, then

$$M_X(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Substitute $z = \frac{x-\mu}{\sigma}$,

$$M_X(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\mu+\sigma z)} e^{\frac{-z^2}{2}} \sigma dz$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-1}{2}¿¿} ¿$$

$$= \frac{e^{t\mu} e^{\frac{\sigma^2 t^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-1}{2}(z - t\sigma)^2} dz$$

Substitute $y = z - t\sigma$ and use gamma function,

$$M_X(t) = e^{t\mu + \frac{\sigma^2 t^2}{2}}$$

Mgf of Standard normal distribution:

By substituting $\mu = 0, \sigma^2 = 1 = ¿ \sigma = 1$ in mgf of normal distribution, we get the mgf of Standard normal distribution.

i.e.

$M_X(t) = e^{t^2/2}$