IIR filter
Spectral Transformation
+
Direct design of IIR filters

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To verify the designed filter

Q. Design a digital Chebyshev filter that satisfies the constraints

$$0.707 \le |H(e^{j\omega})| \le 1, \quad 0 \le \omega \le 0.2\pi = \omega_p$$

$= A_p$

$$|H(e^{j\omega})| \le 0.1 = A_s, \quad 0.5\pi \le \omega \le \pi$$

$0.5\pi = \omega_s$

Use bilinear transf and assume T = 1sec.

$$H(z) = \frac{0.04 + 0.08z^{-1} + 0.04z^{-2}}{1 - 1.44z^{-1} + 0.67z^{-2}} \; //$$

- Put Z = $e^{j\omega}$ and verify

LPF

Design of IIR filters

↓

Design in analog domain → Butterworth, Chebyshev, Elliptic

↓

Map from analog to digital → Imp invar., Bilinear transf, Matched z-transform

↓

Desired digital LPF $\xrightarrow{\text{spectral transf.}}$ HPF ✓, BPF ✓, BEF ✓

Spectral Transformation : / Frequency   HPF, BPF, BSF

1) In analog domain

Design analog LPF → Apply freq band transformation → Digitize filter

2) In digital domain

Design analog LPF → Digitize filter → Apply freq band transformation

Analog domain – Frequency transf:

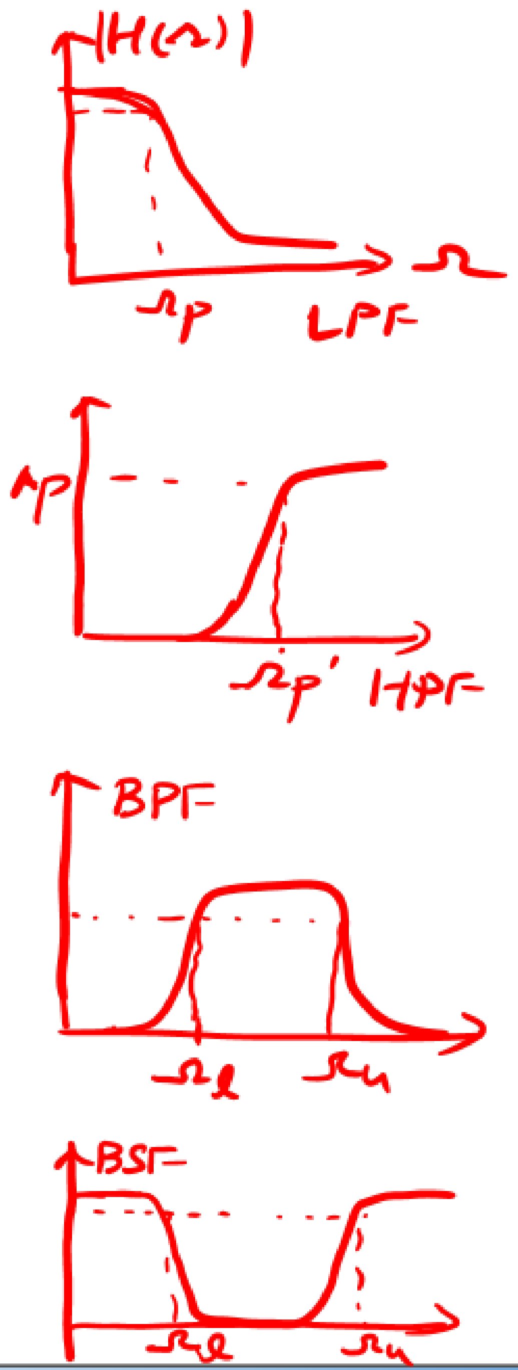
Prototype LPF $\rightarrow \Omega_p \longrightarrow$ LPF' $\rightarrow \Omega_p'$

H(s)

$$H'(s)\Big|_{s \rightarrow \frac{\Omega_p}{\Omega_p'} \cdot s}$$

Ω_l – lower band

Ω_u – upperband



Desired filters	Transformation	Bandedge freq
LPF	$s \rightarrow \frac{\Omega_p}{\Omega_p'} \cdot s$	Ω_p'
HPF	$s \rightarrow \frac{\Omega_p \cdot \Omega_p'}{s}$	Ω_p'
BPF	$s \rightarrow \Omega_p \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}$	
BSF	$s \rightarrow \Omega_p \cdot \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_u \Omega_l}$	

Ω_l – lower band

Ω_u – upperband

Q. A prototype LPF has s/m fn, $H(s) = \frac{1}{s^2+3s+2}$ with $\Omega_p = 1$ rad/s, Obtain a BPF with centre freq $\Omega_0 = 3$ rad/s, & quality factor = 12.

$Q \uparrow$ PBW $\downarrow$

Soln: $\Omega_0 = \sqrt{\Omega_u \cdot \Omega_l}$

$$Q = \frac{\Omega_0}{\Omega_u - \Omega_l}$$

$$s \to \Omega_p \cdot \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}$$

$$= \Omega_p \cdot \frac{s^2 + \Omega_0^2}{s(\Omega_0/Q)}$$

$$= 1 \cdot \frac{s^2+3^2}{s(\cancel{3}/\cancel{12}_4)} = \frac{4(s^2+9)}{s}$$

$$H(s) = \frac{1}{s^2+3s+2}$$

$$H'(s)\Big|_{s \to \frac{4}{s}(s^2+9)}$$

$$H'(s) = \frac{1}{\left[\frac{4}{s}(s^2+9)\right]^2 + 3\times\frac{4}{s}(s^2+9) + 2}$$

$$= \frac{s^2}{16\,(s^2+9)^2 + 12s(s^2+9) + 2s^2}$$

$$H'(s) = \frac{1}{16} \times \frac{s^2}{s^4+0.75s^3+18.125s^2+6.75s+81}$$

2) Freq transf in digital domain :-

1) Mapping $z^{-1} \longrightarrow g(z^{-1})$ must map pts inside unit circle in z-plane to itself

2) Unit circle must map to itself.

Type of transformation	Transformation	Parameters
Lowpass	$z^{-1} \longrightarrow \dfrac{z^{-1} - a}{1 - az^{-1}}$	ω'_p = band edge frequency of new filter $a = \dfrac{\sin[(\omega_p - \omega'_p)/2]}{\sin[(\omega_p + \omega'_p)/2]}$
Highpass	$z^{-1} \longrightarrow -\dfrac{z^{-1} + a}{1 + az^{-1}}$	ω'_p = band edge frequency new filter $a = -\dfrac{\cos[(\omega_p + \omega'_p)/2]}{\cos[(\omega_p - \omega'_p)/2]}$

Bandpass $\quad z^{-1} \longrightarrow -\dfrac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}$

$$\omega_l = \text{lower band edge frequency}$$
$$\omega_u = \text{upper band edge frequency}$$
$$a_1 = -2\alpha K/(K+1)$$
$$a_2 = (K-1)/(K+1)$$
$$\alpha = \frac{\cos[(\omega_u + \omega_l)/2]}{\cos[(\omega_u - \omega_l)/2]}$$
$$K = \cot\frac{\omega_u - \omega_l}{2}\tan\frac{\omega_p}{2}$$

Bandstop $\quad z^{-1} \longrightarrow \dfrac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}$

$$\omega_l = \text{lower band edge frequency}$$
$$\omega_u = \text{upper band edge frequency}$$
$$a_1 = -2\alpha/(K+1)$$
$$a_2 = (1-K)/(1+K)$$
$$\alpha = \frac{\cos[(\omega_u + \omega_l)/2]}{\cos[(\omega_u - \omega_l)/2]}$$
$$K = \tan\frac{\omega_u - \omega_l}{2}\tan\frac{\omega_p}{2}$$

Qn. Convert the single pole lowpass Butterworth filter with system function, $H(z) = \frac{0.245(1+z^{-1})}{1-0.509z^{-1}}$ into a bandpass filter with upper and lower cutoff frequencies, $\omega_u = \frac{3\pi}{5}$ and $\omega_l = \frac{2\pi}{5}$. The LPF has 3dB bandwidth, $\omega_p = 0.2\pi$

Soln:

$$K = \cot\left(\frac{\omega_u - \omega_l}{2}\right) \cdot \tan\left(\frac{\omega_p}{2}\right)$$

$$= \cot\left(\frac{\frac{3\pi}{5} - \frac{2\pi}{5}}{2}\right) \cdot \tan\left(\frac{0.2\pi}{2}\right)$$

$$= \cot\left(\frac{\pi}{10}\right) \cdot \tan\left(\frac{\pi}{10}\right) = \underline{\underline{1}}$$

$$\alpha = \frac{\cos\left(\frac{\omega_u + \omega_l}{2}\right)}{\cos\left(\frac{\omega_u - \omega_l}{2}\right)} = 0$$

Qn. Convert the single pole lowpass Butterworth filter with system function, $H(z) = \frac{0.245(1+z^{-1})}{1-0.509z^{-1}}$ into a bandpass filter with upper and lower cutoff frequencies, $\omega_u = \frac{3\pi}{5}$ and $\omega_l = \frac{2\pi}{5}$.

The LPF has 3dB bandwidth, $\omega_p = 0.2\pi$

$$a_1 = \frac{2\alpha K}{K+1} = 0$$

$$a_2 = \frac{K-1}{K+1} = 0$$

$$z^{-1} \rightarrow -\frac{(z^{-2} - a_1 z^{-1} + a_2)}{a_1 z^{-2} - a_1 z^{-1} + 1} = -z^{-2}$$

$$H(z) = \frac{0.245(1-z^{-2})}{1+0.509z^{-2}} \quad //$$

Direct design of IIR filters:

1) Direct design of LPF :

Direct placement of poles near the unit circle in z-plane at the points corresponding to low freq (i.e. near $\omega = 0$)

eg. Pole at $z = a$,

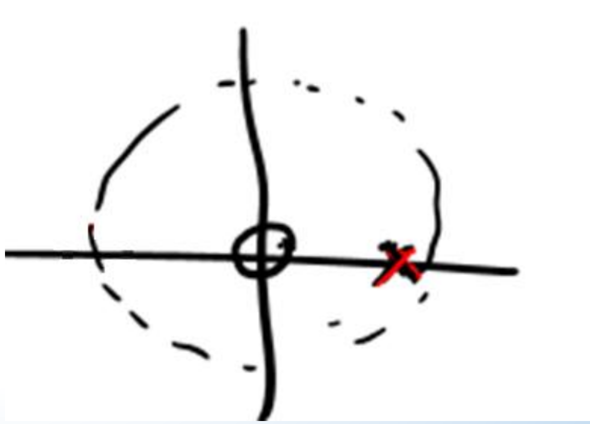
s/m fn, $H(z) = b_0 \cdot \frac{1}{1 - az^{-1}}$

At $\omega = 0$, $|H(0)| = 1$

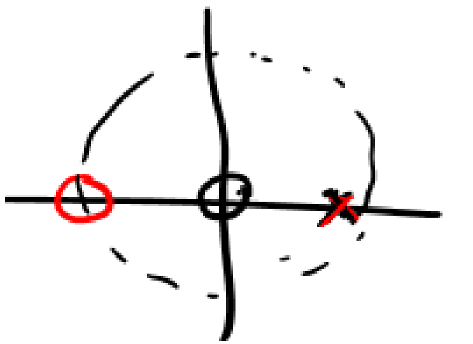
$$\left| \frac{b_0}{1-a} \right| = 1$$

$$b_0 = 1 - a$$

$$H_1(z) = \frac{1-a}{1 - az^{-1}}$$



② Placing a zero at $z = -1$ (i.e. $\omega = \pi$)



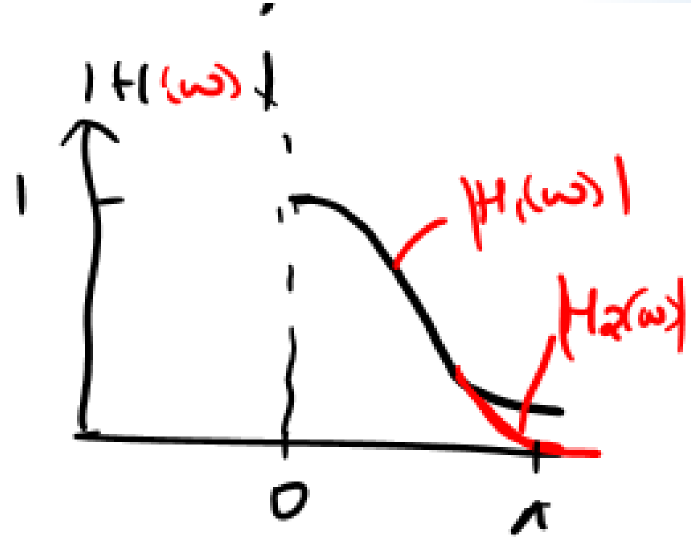
$$H_2(z) = b_0 \cdot \frac{1+z^{-1}}{1-az^{-1}}$$, single pole-zero

At $\omega = 0$, $|H(0)| = 1$

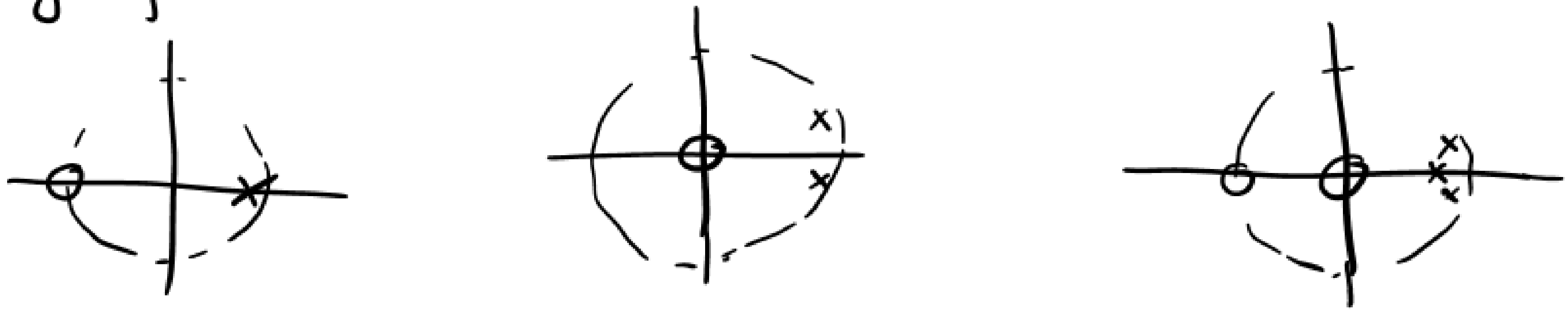
$$\left| b_0 \cdot \frac{1+1}{1-a} \right| = 1$$

$$\Rightarrow b_0 = \frac{1-a}{2}$$

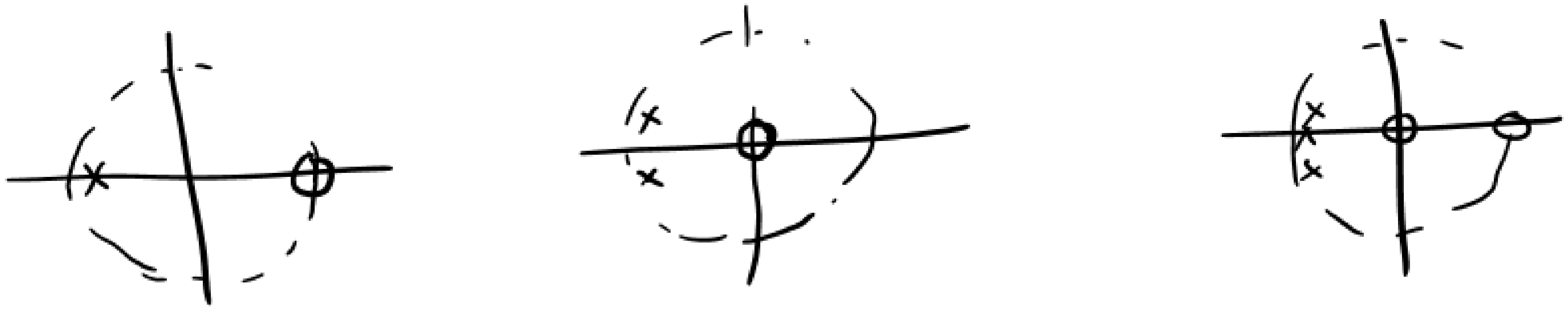
$$\therefore H_2(z) = \frac{1-a}{2} \cdot \frac{1+z^{-1}}{1-az^{-1}}$$



eg. of LPF



eg. of HPF

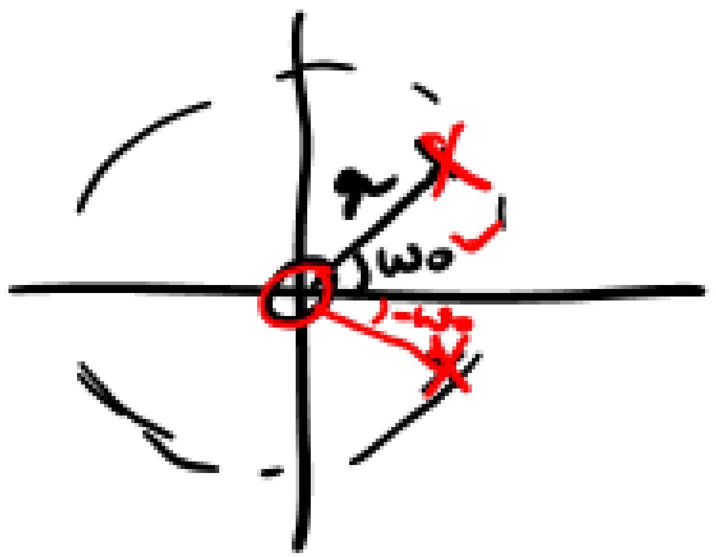


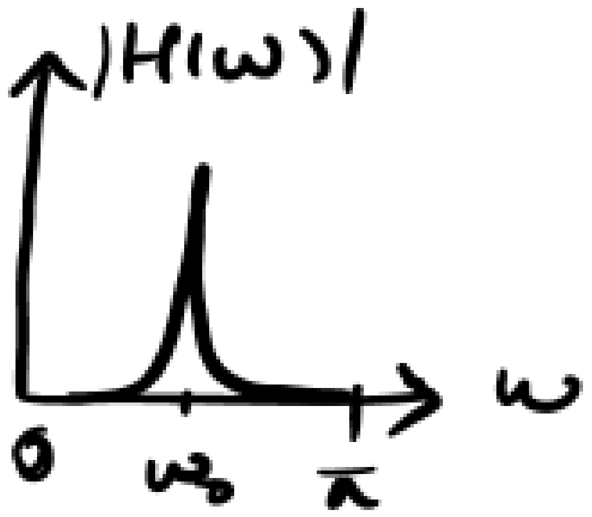
Digital resonator

2-pole BPF

ω_0 - resonant freq

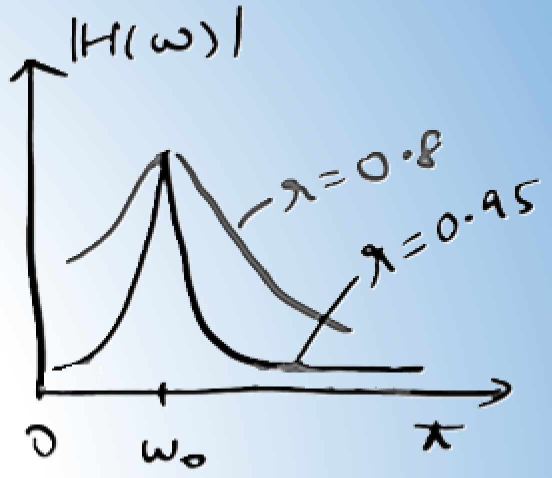
$p_{1,2} = r e^{\pm j\omega_0}$ $\quad 0 < r < 1$





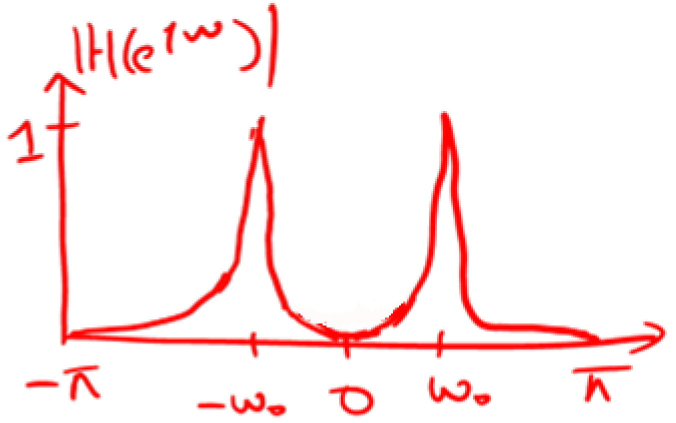
case 1) 1) zeros at origin

$$H(z) = \frac{b_0}{(1-r e^{j\omega_0} z^{-1})(1-r e^{-j\omega_0} z^{-1})}$$



2) zeros at $z = \pm 1$ ($\omega = 0$ & π)

$$H(z) = b_0 \cdot \frac{(1-z^{-1})(1+z^{-1})}{(1-r e^{j\omega_0} z^{-1})(1-r e^{-j\omega_0} z^{-1})}$$



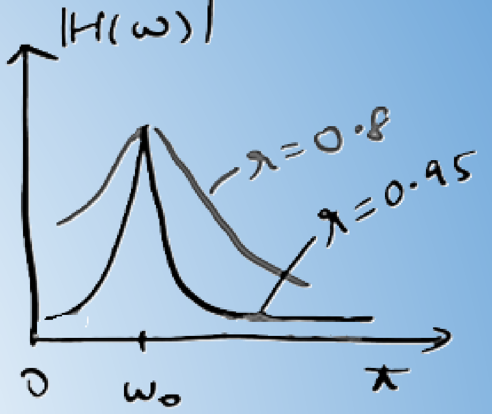
Case 1 : S/m fn of dg resonator with zeros at origin.

$$H(z) = \frac{b_0}{(1 - r e^{j\omega_0} z^{-1})(1 - r e^{-j\omega_0} z^{-1})} \quad \text{——— (1)}$$

$$= \frac{b_0}{1 - r e^{j\omega_0} z^{-1} - r e^{-j\omega_0} z^{-1} + r^2 e^{j\omega_0 - j\omega_0} z^{-2}}$$

$$= \frac{b_0}{1 - r(e^{j\omega_0} + e^{-j\omega_0}) z^{-1} + r^2 z^{-2}}$$

$$= \frac{b_0}{1 - 2r\cos\omega_0 z^{-1} + r^2 z^{-2}}$$

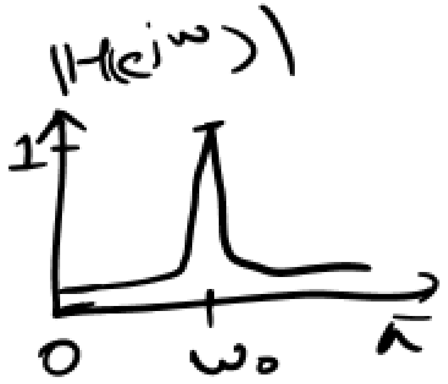


$$\frac{e^{j\theta} + e^{-j\theta}}{2} = \cos\theta$$

$$|H(e^{j\omega})|_{\omega=\omega_0} = 1$$

$$|H(e^{j\omega_0})| = \left|\frac{b_0}{(1-re^{j\omega_0}e^{-j\omega_0})(1-re^{-j\omega_0}e^{-j\omega_0})}\right| = 1$$

$$\Rightarrow \left|\frac{b_0}{(1-r)(1-re^{-2j\omega_0})}\right| = 1$$



$$z = e^{j\omega_0}$$

$$|H(e^{j\omega_0})| = \left|\frac{b_0}{(1-r)(1-r\cos 2\omega_0 + jr\sin 2\omega_0)}\right| = 1$$

$$|a+ib| = \sqrt{a^2+b^2}$$

$$\Rightarrow \frac{b_0}{(1-r)\sqrt{(1-r\cos 2\omega_0)^2 + (r\sin 2\omega_0)^2}} = 1$$

$$\Rightarrow \frac{b_0}{(1-r)\sqrt{1-2r\cos 2\omega_0 + r^2\cos^2 2\omega_0 + r^2\sin^2 2\omega_0}} = 1 \qquad [\cos^2\theta + \sin^2\theta = 1]$$

$$\Rightarrow b_0 = (1-r)\sqrt{1-2r\cos 2\omega_0 + r^2}$$

↓ normalized filter gain.

Magnitude response : evaluate $H(z)$ at $z = e^{j\omega}$

$$|H(e^{j\omega})| = \left| \frac{b_0}{(1 - r e^{j\omega_0} e^{-j\omega})(1 - r e^{-j\omega_0} e^{-j\omega})} \right| \quad \text{---} \quad (2)$$

$$\text{dnr} = (1 - r e^{j(\omega_0 - \omega)})(1 - r e^{-j(\omega_0 + \omega)})$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$
$$e^{-j\theta} = \cos\theta - j\sin\theta$$

$$= [1 - r\cos(\omega_0 - \omega) - j r\sin(\omega_0 - \omega)][1 - r\cos(\omega_0 + \omega) + j r\sin(\omega_0 + \omega)]$$

$$|\text{dnr}| = \sqrt{(1 - r\cos(\omega_0 - \omega))^2 + (r\sin(\omega_0 - \omega))^2} \cdot \sqrt{(1 - r\cos(\omega_0 + \omega))^2 + (r\sin(\omega_0 + \omega))^2}$$

$$= \sqrt{1 - 2r\cos(\omega_0 - \omega) + r^2\cos^2(\omega_0 - \omega) + r^2\sin^2(\omega_0 - \omega)} \cdot \sqrt{1 - 2r\cos(\omega_0 + \omega) + r^2\cos^2(\omega_0 + \omega) + r^2\sin^2(\omega_0 + \omega)}$$

$$= \sqrt{1 - 2r\cos(\omega_0 - \omega) + r^2} \cdot \sqrt{1 - 2r\cos(\omega_0 + \omega) + r^2}$$

$$(2) \Rightarrow |H(e^{j\omega})| = \frac{b_0}{\underbrace{\sqrt{1 - 2r\cos(\omega_0 - \omega) + r^2}}_{U_1(\omega)} \underbrace{\sqrt{1 - 2r\cos(\omega_0 + \omega) + r^2}}_{U_2(\omega)}} = \frac{b_0}{U_1(\omega) \cdot U_2(\omega)}$$

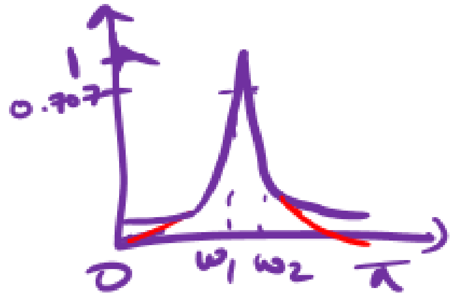
The product term $U_1(\omega) \cdot U_2(\omega)$ reaches minimum value at

$$\omega_r = \cos^{-1}\left(\frac{1+r^2}{2r}\cos\omega_0\right)$$, is the resonance freq.

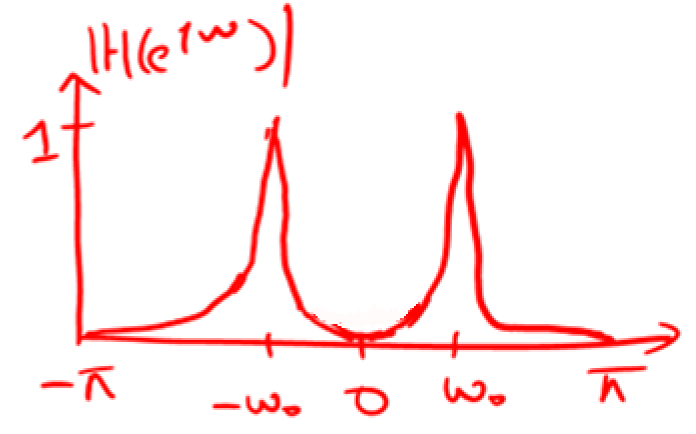
$r \to 1$, sharper peak, BW ↓

$\omega_r \to \omega_0$

$r \to 1$, 3dB BW $= \Delta\omega \approx 2(1-r)$



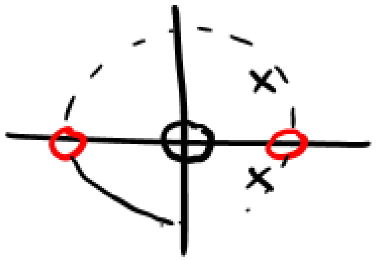
$\Delta\omega = \omega_2 - \omega_1$



Case 2: When zeros are placed at $z = \pm 1$:

$$H(z) = \frac{b_0 \cdot (1-z^{-1})(1+z^{-1})}{(1-re^{j\omega_0}z^{-1})(1-re^{-j\omega_0}z^{-1})}$$

$$= \frac{b_0 (1-z^{-1})(1+z^{-1})}{1-2r\cos\omega_0 z^{-1} + r^2 z^{-2}}$$



$p_{1,2} = re^{\pm j\omega_0}$

Magnitude response of case 2:

$H(z)\big|_{z=e^{j\omega}}$.

$$|H(e^{j\omega})| = \left| \frac{b_0 (1-e^{-j\omega})(1+e^{-j\omega})}{(1-r e^{j\omega_0} e^{-j\omega})(1-r e^{-j\omega_0} e^{-j\omega})} \right|$$

$$= \left| \frac{b_0 (1-e^{-j2\omega})}{(1-r e^{j\omega_0} e^{-j\omega})(1-r e^{-j\omega_0} e^{-j\omega})} \right|$$

$$= \frac{b_0 \underbrace{|1-e^{-j2\omega}|}_{N(\omega)}}{\underbrace{\sqrt{1-2r\cos(\omega_0-\omega)+r^2}}_{U_1(\omega)}\ \underbrace{\sqrt{1-2r\cos(\omega_0+\omega)+r^2}}_{U_2(\omega)}}$$

$$= \frac{b_0 \cdot N(\omega)}{U_1(\omega) \cdot U_2(\omega)}$$

$$N(\omega) = \left|1 - e^{-j2\omega}\right|$$

$$= \left|1 - \cos 2\omega + j\sin 2\omega\right|$$

$$= \sqrt{(1-\cos 2\omega)^2 + \sin^2 2\omega} = \sqrt{1 - 2\cos 2\omega + \underbrace{\cos^2 2\omega + \sin^2 2\omega}_{1}}$$

$$= \sqrt{2(1-\cos 2\omega)} \quad //$$

Qn. Design a digital resonator with peak gain of unity at 50Hz and a 3dB BW of 6 Hz. Assume a sampling freq of 300Hz.

Resonant freq, $\omega_0 = 2\pi \frac{f}{F_s} = \frac{2\pi \times 50}{300} = \pi/3$

3dB BW, $\Delta\omega = \frac{2\pi \times 6}{300} = \pi/25$

$\Delta\omega \simeq 2(1-r) \Rightarrow \frac{\pi}{25} = 2(1-r) \Rightarrow r = 0.937$

Assume zeros at origin:

$$H(z) = \frac{b_0}{1 - 2r\cos\omega_0 z^{-1} + r^2 z^{-2}}$$

$$= \frac{b_0}{1 - 2\times 0.937\cos(\pi/3)z^{-1} + 0.937^2 z^{-2}} = \frac{b_0}{1 - 0.937z^{-1} + 0.877z^{-2}}$$

To find b_0:

$$|H(e^{j\omega})|_{\omega=\omega_0} = 1 \quad , \omega_0 = \pi/3$$

$$\Rightarrow \left| \frac{b_0}{1-0.937e^{-j\omega}+0.877e^{-2j\omega}} \right|_{\omega=\pi/3} = 1$$

$$\Rightarrow \left| \frac{b_0}{1-0.937\left(\cos\frac{\pi}{3}-j\sin\frac{\pi}{3}\right)+0.877\left(\cos\frac{2\pi}{3}-j\sin\frac{2\pi}{3}\right)} \right| = 1$$

$$\Rightarrow \left| \frac{b_0}{0.0925+j0.0522} \right| = 1$$

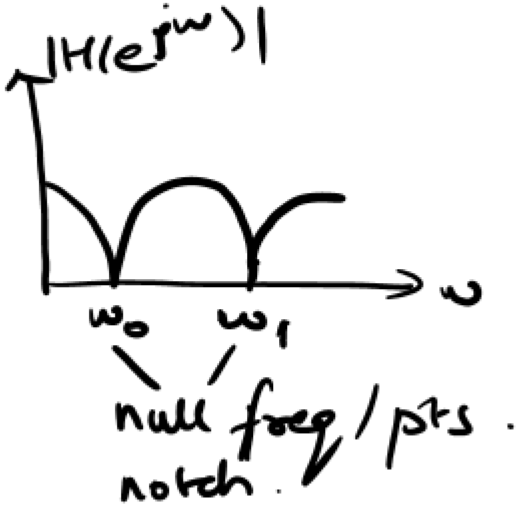
$$\Rightarrow b_0 = \sqrt{0.0925^2+0.0522^2} = 0.105 \qquad \therefore H(z) = \frac{0.105}{1-0.937z^{-1}+0.877z^{-2}}$$

Notch filter:

$z_{1,2} = e^{\pm j\omega_0}$, $r = 1$

$$H(z) = b_0 \left(1 - e^{j\omega_0} z^{-1}\right)\left(1 - e^{-j\omega_0} z^{-1}\right)$$

$$= b_0 \left(1 - 2\cos\omega_0 \, z^{-1} + z^{-2}\right)$$

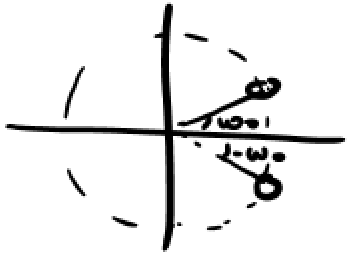


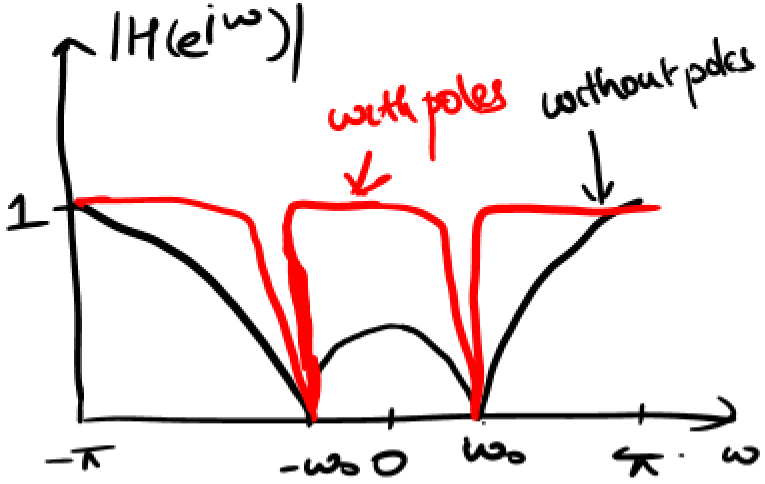
By placing poles near unit-circle at centre freq ω_0

$P_{1,2} = re^{\pm j\omega_0}$

$$H(z) = \frac{b_0 \left(1 - e^{j\omega_0} z^{-1}\right)\left(1 - e^{-j\omega_0} z^{-1}\right)}{\left(1 - re^{j\omega_0} z^{-1}\right)\left(1 - re^{-j\omega_0} z^{-1}\right)}$$

$$= \frac{b_0 \left(1 - 2\cos\omega_0 \, z^{-1} + z^{-2}\right)}{1 - 2r\cos\omega_0 \, z^{-1} + r^2 z^{-2}}$$





Comb filter.

Q A digital notch filter is reqd to remove an undesired 60Hz hum associated with a power supply in a ECG recording. The sampling freq used is 500 samples per second.

(a) Design a 2^{nd} order FIR notch filter.

(b) Design a 2^{nd} order pole-zero notch filter.

In both cases, choose gain b_0 s.t. $|H(e^{j\omega})| = 1$ for $\omega = 0$.

Soln: Notch freq, $\omega_0 = 2\pi \frac{f}{F_s} = \frac{2\pi \times 60}{500} = \frac{6\pi}{25} = 0.754$

(a) Pair of complex conjugate zeros at $e^{\pm j\omega_0}$ where $\omega_0 = 0.754$

$$H(z) = b_0 \left(1 - 2\cos\omega_0 z^{-1} + z^{-2}\right)$$
$$= b_0 \left(1 - 1.4579 z^{-1} + z^{-2}\right)$$

To find b_0: $|H(e^{j\omega})|_{\omega=0} = 1$

$$\Rightarrow \left|b_0(1 - 1.4579 e^{0} + e^{0})\right| = 1 \Rightarrow b_0 = 1.845$$

$$\therefore H(z) = 1.845\left(1 - 1.4579 z^{-1} + z^{-2}\right)$$

Case (b): $$H(z) = \frac{b_0 (1 - 2\cos\omega_0 z^{-1} + z^{-2})}{(1 - 2r\cos\omega_0 z^{-1} + r^2 z^{-2})}$$

$0 < r < 1$, $r \to 1$, Assume $r = 0.95$,

$$H(z) = \frac{b_0 (1 - 2\cos(0.754) z^{-1} + z^{-2})}{1 - 2 \times 0.95 \cos(0.754) z^{-1} + 0.95^2 z^{-2}}$$

To find b_0: $|H(e^{j\omega})|_{\omega=0} = 1$

$b_0 = 0.9546$

$$H(z) = \frac{0.9546 (1 - 1.4579 z^{-1} + z^{-2})}{1 - 1.385 z^{-1} + 0.9025 z^{-2}}$$ //.

Reference

- Proakis J. G, Manolakis D. G. Mimitris D., “Introduction to Digital Signal Processing” Prentice Hall, India, 2007.


Thank you