Structure for FIR systems:
Frequency sampling structure

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Frequency sampling structure:

$$H_d(e^{j\omega}) \longrightarrow H_d(k) \longrightarrow h(n)$$

By freq^n sampling

$$\omega = \frac{2\pi k}{N}$$

By appling IDFT.

$$h(n) = \frac{1}{N} \sum_{k=0}^{N-1} H(k)\, W_N^{-nk}$$

where

$n = 0 - N-1$

Take Z-transform to obtain the structure

Frequency Sampling Structure

Alternative structure for FIR filter

Parameters that characterize the parameter are the values of the desired frequency response instead of impulse response

To derive the frequency sampling structure, we specify the desired frequency response at a set of equally spaced frequencies,

$$\omega_k = \frac{2\pi}{M}(k+\alpha), \quad k = 0, 1, \ldots \left(\frac{M-1}{2}\right) \text{ if } M \text{ odd}$$

$$k = 0, 1, \ldots \frac{M}{2} - 1 \quad \text{if } M \text{ even}$$

$$\alpha = 0 \text{ or } 1/2$$

To get the frequency sampling structure, we take F.T. of $h(n)$

$$H(\omega) = \sum_{n=0}^{M-1} h(n) e^{-j\omega n} \quad \text{——} (1)$$

The value of $H(\omega)$ at frequencies $\omega_k = \frac{2\pi(k+\alpha)}{M}$ are

$$H(k+\alpha) = H\left(\frac{2\pi}{M}(k+\alpha)\right) \quad \text{——} (2)$$

$$H(k+\alpha) = \sum_{n=0}^{M-1} h(n) e^{-j\frac{2\pi(k+\alpha)}{M} n} \quad k = 0, 1, \ldots M-1 \quad \text{——} (3)$$

The set of values $\{H(k+\alpha)\}$ are called frequency samples of $H(\omega)$

For $\alpha = 0$, $\{H(k)\}$ corresponds to M-pt DFT of $\{h(n)\}$

So we can get back $\{h(n)\}$ as follows:

$$h(n) = \frac{1}{M} \sum_{k=0}^{M-1} H(k+\alpha)\, e^{\frac{j2\pi(k+\alpha)n}{M}} \quad \text{—— (4)} \qquad n = 0, 1, \ldots M-1$$

Our intension is to get filter structure, which can be obtained using Z.T.

$$H(z) = \sum_{n=0}^{M-1} h(n) z^{-n} \quad \text{—— (5)}$$

$$= \sum_{n=0}^{M-1} \left[\frac{1}{M} \sum_{k=0}^{M-1} H(k+\alpha) e^{\frac{j2\pi(k+\alpha)}{M} \cdot n} \right] z^{-n} \quad \text{—— (6)}$$

$$H(z) = \sum_{k=0}^{M-1} H(k+\alpha) \frac{1}{M} \sum_{n=0}^{M-1} \left[e^{\frac{j2\pi(k+\alpha)}{M}} z^{-1} \right]^{n} = \sum_{k=0}^{M-1} \frac{H(k+\alpha)}{M} \frac{1 - \left[e^{\frac{j2\pi(k+\alpha)}{M}} z^{-1} \right]^{M}}{1 - e^{\frac{j2\pi(k+\alpha)}{M}} \cdot z^{-1}}$$

$$H(z) = \sum_{k=0}^{M-1} \frac{H(k+\alpha)}{M} \frac{1-\left[e^{\frac{j2\pi(k+\alpha)}{M}} z^{-1}\right]^{M}}{1-e^{\frac{j2\pi(k+\alpha)}{M}} \cdot z^{-1}} = \sum_{k=0}^{M-1} \frac{H(k+\alpha)}{M} \frac{1-e^{j2\pi\alpha} \cdot z^{-M}}{1-e^{\frac{j2\pi(k+\alpha)}{M}} z^{-1}}$$

$$H(z) = \frac{1-e^{j2\pi\alpha} z^{-M}}{M} \sum_{k=0}^{M-1} \frac{H(k+\alpha)}{1-e^{\frac{j2\pi(k+\alpha)}{M}} z^{-1}} \quad \text{——} (7)$$

Now the system function $H(z)$ is characterized by frequency samples $\{H(k+\alpha)\}$

We can view this FIR filter realization as a cascade of two filters

$$H(z) = H_1(z) \cdot H_2(z)$$

$$H_1(z) = \frac{1-e^{j2\pi\alpha} z^{-M}}{M} \quad \text{——} (8)$$

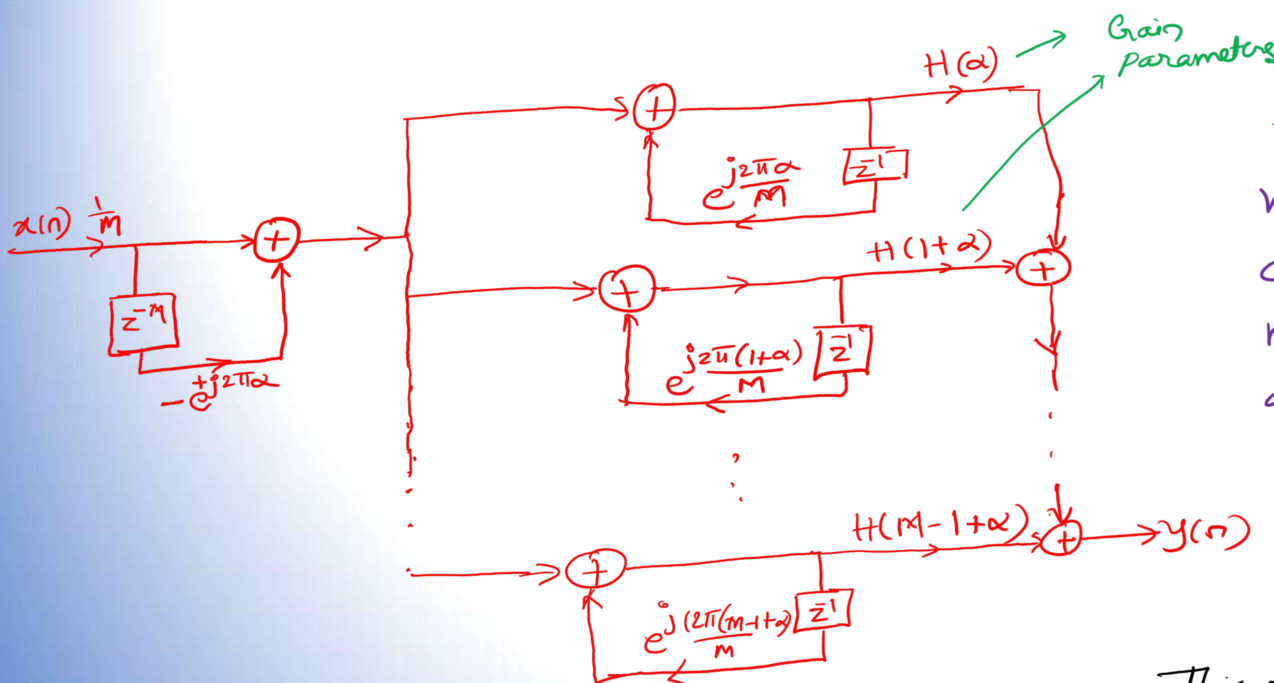
, zeros are located at equally spaced points on the unit circle, $z_k = e^{\frac{j2\pi(k+\alpha)}{M}}$, $k=0,1,\cdots M-1$

$$H_2(z) = \sum_{k=0}^{M-1} \frac{H(k+\alpha)}{1-e^{\frac{j2\pi(k+\alpha)}{M}} z^{-1}} \quad \text{——} (9)$$

Parallel bank of single pole filters with resonant frequencies $p_k = e^{\frac{j2\pi(k+\alpha)}{M}}$, $k=0, 1, 2, \cdots M-1$

$$H(z) = \frac{1 - e^{j2\pi\alpha} z^{-M}}{M} \sum_{k=0}^{M-1} \frac{H(k+\alpha)}{1 - e^{j2\pi(k+\alpha)/M} z^{-1}}$$

IIR structure



Advantage:

When desired frequency response characteristic is narrow band, most of the gain parameters $\{H(k+\alpha)\}$ are zero. Consequently, the corresponding resonant filters can be eliminated and only the filters with nonzero gain need to be retained.

This results in a filter that requires fewer computations

Further simplification of frequency sampling structure

We know that

$$H(k) = H^*(M-k) \quad \text{for } \alpha = 0 \qquad \text{(symmetry)}$$

$$H(k+\alpha) = H^*(M-k-\tfrac{1}{2}) \quad \text{for } \alpha = \tfrac{1}{2}$$

$$H_2(z) = \sum_{k=0}^{M-1} \frac{H(k+\alpha)}{1 - e^{j2\pi(k+\alpha)/M} \cdot z^{-1}}$$

For $\alpha = 0$

$$H_2(z) = \frac{H(0)}{1-z^{-1}} + \sum_{k=1}^{\frac{M-1}{2}} \frac{A(k) - z^{-1}B(k)}{1 - 2\cos\left(\frac{2\pi k}{M}\right)z^{-1} + z^{-2}} \qquad M \text{ odd}$$

$$= \frac{H(0)}{1-z^{-1}} + \frac{H(M/2)}{1+z^{-1}} + \sum_{k=1}^{\frac{M}{2}-1} \frac{A(k) - z^{-1}B(k)}{1 - 2\cos\left(\frac{2\pi k}{M}\right)z^{-1} + z^{-2}} \qquad M \text{ even}$$

$$A(k) = H(k) + H(M-k)$$

$$B(k) = H(k)\,e^{-j\frac{2\pi k}{M}} + H(M-k)\,e^{j\frac{2\pi k}{M}}$$

Ex: Realize a frequency sampling structure for an FIR causal system whose impulse response is $h(n)=\{1,2,3,4\}$ $(M=4)$

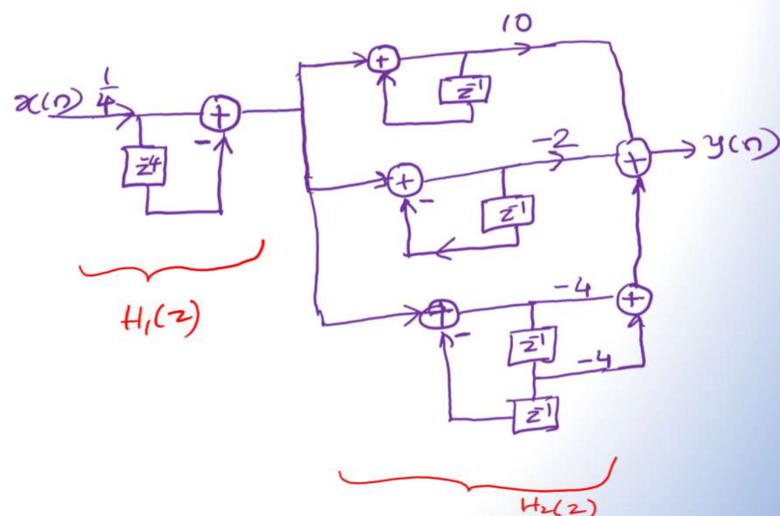
$H(k)=\{10,\ -2+j2,\ -2,\ -2-j2\}$

$H_1(z)=\frac{1-z^{-4}}{4}$,

$$H_2(z)=\sum_{k=0}^{3}\frac{H(k)}{1-e^{\frac{j2\pi k}{4}}z^{-1}}=\frac{H(0)}{1-z^{-1}}+\frac{-2+j2}{1-jz^{-1}}+\frac{-2}{1+z^{-1}}+\frac{-2-j2}{1+jz^{-1}}$$

$$=\frac{10}{1-z^{-1}}+\frac{(-2+j2)(1+jz^{-1})+(-2-j2)(1-jz^{-1})}{(1-jz^{-1})(1+jz^{-1})}+\frac{-2}{1+z^{-1}}$$

$$=\frac{10}{1-z^{-1}}+\frac{-2}{1+z^{-1}}+\frac{-4-4z^{-1}}{1+z^{-2}}$$



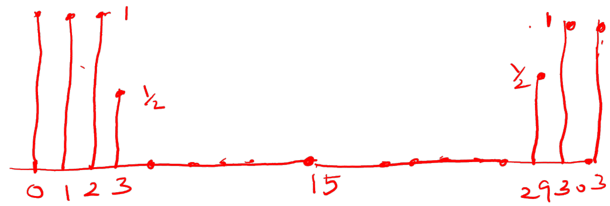
$$H_2(z)=\frac{H(0)}{1-z^{-1}}+\frac{H(M/2)}{1+z^{-1}}+\sum_{k=1}^{\frac{M}{2}-1}\frac{A(k)-z^{-1}B(k)}{1-2\cos\left(\frac{2\pi k}{M}\right)z^{-1}+z^{-2}}\quad M \text{ even}$$

$A(k)=H(k)+H(M-k)$

$B(k)=H(k)e^{-\frac{j2\pi k}{M}}+H(M-k)e^{\frac{j2\pi k}{M}}$

Sketch the block diagram for the direct-form realization and the frequency-sampling realization of the $M = 32$, $\alpha = 0$, linear-phase (symmetric) FIR filter which has frequency samples

$$H\left(\frac{2\pi k}{32}\right) = \begin{cases} 1, & k = 0, 1, 2 \\ \frac{1}{2}, & k = 3 \\ 0, & k = 4, 5, \ldots, 15 \end{cases}$$



$$H(z) = \frac{1 - e^{j2\pi\alpha} z^{-M}}{M} \sum_{k=0}^{M-1} \frac{H(k+\alpha)}{1 - e^{j\frac{2\pi(k+\alpha)}{M}} z^{-1}}$$

$$H(z) = H_1(z) \cdot H_2(z)$$

$$H_1(z) = \frac{1 - z^{-32}}{32}, \quad H_2(z) = \sum_{k=0}^{M-1} \frac{H(k)}{1 - e^{j\frac{2\pi k}{M}} z^{-1}} = \frac{H(0)}{1 - z^{-1}} + \sum_{k=1}^{M-1} \frac{H(k)}{1 - e^{j\frac{2\pi k}{M}} z^{-1}}$$

$$H_2(z) = \frac{H(0)}{1 - z^{-1}} + \frac{H(1)}{1 - e^{j\frac{2\pi k}{32}} z^{-1}} + \frac{H(31)}{1 - e^{j\frac{2\pi 31}{32}} z^{-1}} + \frac{H(2)}{1 - e^{j\frac{2\pi \times 2}{32}} z^{-1}} + \frac{H(30)}{1 - e^{j\frac{2\pi \times 30}{32}} z^{-1}} + \frac{H(3)}{1 - e^{j\frac{2\pi \times 3}{32}} \cdot z^{-1}} + \frac{H(29)}{1 - e^{j\frac{2\pi \times 29}{32}} z^{-1}}$$

$$H_2(z) = \frac{H(0)}{1 - z^{-1}} + \underbrace{\frac{H(1)}{1 - e^{j\frac{2\pi}{32}} z^{-1}} + \frac{H(31)}{1 - e^{-j\frac{2\pi}{32}} z^{-1}}} + \underbrace{\frac{H(2)}{1 - e^{j\frac{2\pi \times 2}{32}} z^{-1}} + \frac{H(30)}{1 - e^{-j\frac{2\pi \times 2}{32}} z^{-1}}} + \underbrace{\frac{H(3)}{1 - e^{j\frac{2\pi \times 3}{32}} z^{-1}} + \frac{H(29)}{1 - e^{-j\frac{2\pi \times 3}{32}} z^{-1}}}$$

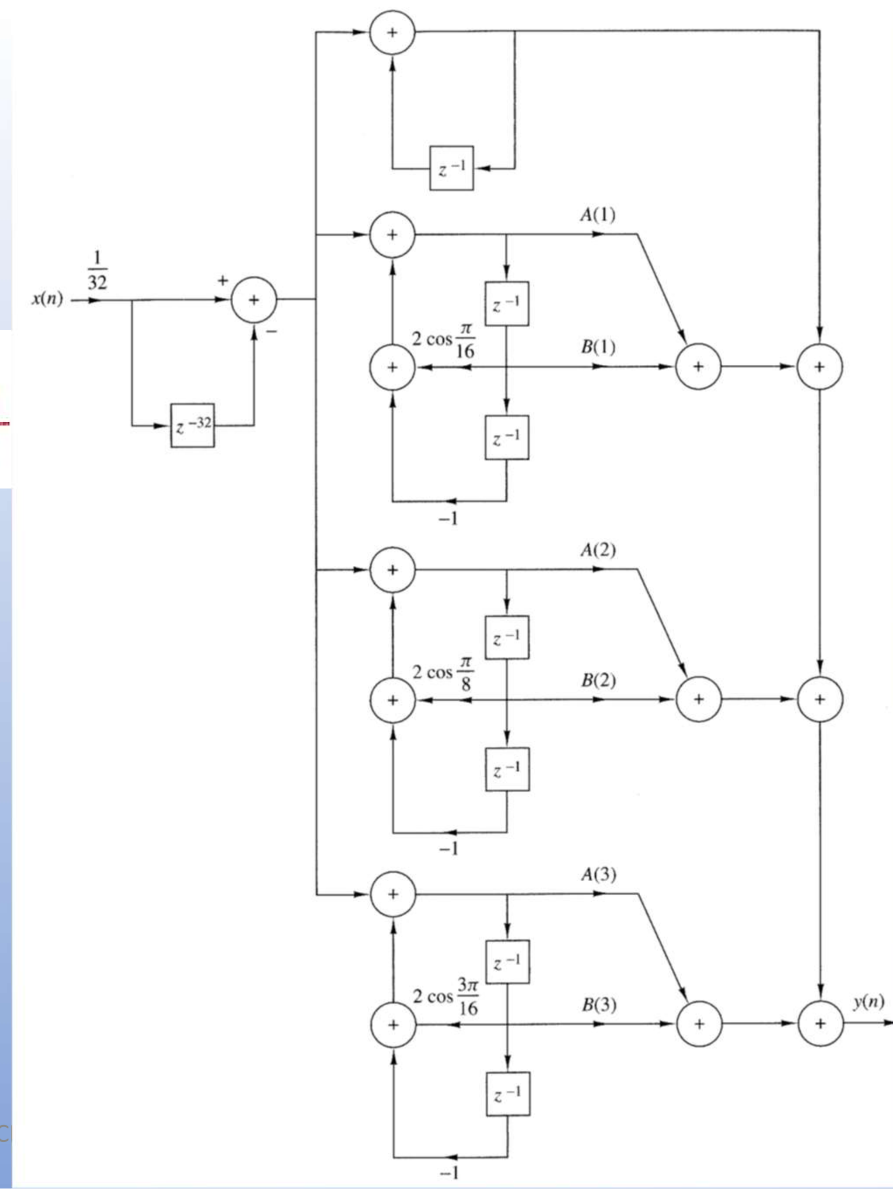
$$H_2(z) = \frac{1}{1 - z^{-1}} + \frac{2 - 2\cos\frac{2\pi}{32} z^{-1}}{1 - 2\cos\frac{\pi}{16} z^{-1} + z^{-2}} + \frac{2 - 2\cos\frac{4\pi}{32} z^{-1}}{1 - 2\cos\frac{\pi}{8} z^{-1} + z^{-2}} + \frac{1 - \cos\frac{6\pi}{32} z^{-1}}{1 - 2\cos\frac{3\pi}{16} z^{-1} + z^{-2}}$$

$$H_1(z) = \frac{1 - z^{-32}}{32}$$

$$H_2(z) = \frac{1}{1-z^{-1}} + \frac{2 - 2\cos\frac{2\pi}{32} z^{-1}}{1 - 2\cos\frac{\pi}{16} z^{-1} + z^{-2}} + \frac{2 - 2\cos\frac{4\pi}{32} z^{-1}}{1 - 2\cos\frac{\pi}{8} z^{-1} + z^{-2}} + \frac{1 - \cos\frac{6\pi}{32} z^{-1}}{1 - 2\cos\frac{3\pi}{16} z^{-1} + z^{-2}}$$

$$A(k) = H(k) + H(M-k)$$

$$B(k) = H(k)\, e^{-j\frac{2\pi k}{M}} + H(M-k)\, e^{j\frac{2\pi k}{M}}$$




Thank you