Bartlett method of Power Spectrum Estimation.

This method aims at reducing the variance by averaging the periodogram, but at the expense of increased spectral width. (Increase in spectral width results in reduce in resolution i.e. minimum frequency that can be resolved increases)

Data frame length is taken as N.

Step1: The N-point sequence is devided into K number of non-overlapping segments of length M each. ($\therefore K = \frac{N}{M}$). Last segment may need zero-padding)

$$x_i(n) = x(n+iM) \quad ; \quad \begin{array}{l} n = 0, 1, \cdots M-1 \\ i = 0, 1, \cdots K-1 \end{array}$$

Step 2: For each segment, Periodogram is computed ~~ex~~ as $P_{xx}^{(i)}(f)$

$$P_{xx}^{(i)}(f) = \frac{1}{M} \left| \sum_{n=0}^{M-1} x_i(n)\, e^{-j2\pi f n} \right|^2 \quad ; \quad i = 0, 1, \cdots K-1$$

Step 3: These are averaged over all K segments to get Bartlett Power spectrum estimate $P_{xx}^{B}(f)$

$$P_{xx}^{(B)}(f) = \frac{1}{K} \sum_{i=0}^{K-1} P_{xx}^{(i)}(f)$$

FFT is used to calculate Fourier Transform.

Mean or expected value $E[P_{xx}^B]$, variance, quality factor Q and computational requirements (using FFT) are used to compare the performance of different methods

For Bartlett method

$$E[P_{xx}^B] = E[P_{xx}^{(i)}(f)]$$

$$var[P_{xx}^B] = \frac{1}{K} \cdot var[P_{xx}^{(i)}(f)]$$

thus we see that variance is divided by K and hence reduced. But frequency resolution is decreased (Δf increased) by K.

Note: The quality factor Q of $P_{xx}(f)$ is defined as

$$Q = \frac{\{E[P_{xx}(f)]\}^2}{Var[P_{xx}(f)]}$$

~~For Bartlett PS estimate, Q_B ex~~

For Bartlett method it can be proved that $Q_B = \frac{N}{M} = K$

Taking 3-dB spectral width of rectangular window as $\frac{0.9}{M}$, the freq. resolution $\Delta f = \frac{0.9}{M} \Rightarrow M = \frac{0.9}{\Delta f}$

$\therefore Q_B = 1.1 N \Delta f$

<u>Computational requirement</u>: K number of M-length FFT to be computed. No. of computation for M-length FFT is $\frac{M}{2}\log_2(M)$

$\therefore$ total Computation $= K \cdot \frac{M}{2} \cdot \log_2 M$

$$= \frac{N}{2}\log_2\left(\frac{0.9}{\Delta f}\right)$$

<u>Welch method</u>: This method is also aiming at reducing variance of the estimate by averaging the periodogram. Two modifications are made to the Bartlet method. The N-length signal is ~~segmente~~ Segmented into <u>overlapping</u> segments. The other modification is that these segments are further ~~window~~ multiplied by ~~tapered~~ tapered window prior to ~~estimatio~~ Computation of periodogram (This results in modified periodogram)

<u>Step I</u>: The N-length sequence is segmented into overlapping segments of length M. (N determines the frequency resolution Δf).

Let $x(n)$ be the N-length signal. The segmented sequence is

$$x_i(n) = x(n + iD) \quad , \quad n = 0, 1, \cdots M-1$$
$$i = 0, 1, \cdots L-1$$

iD is starting point of i^{th} segment. ~~If~~ Let $K = \frac{N}{M}$ = no. of segments if there is no overlap (same as K in Bartlet method). Then if $D = M$, no overlap, & $L = K$. If 50% overlap, then $D = \frac{M}{2}$ and $L = 2K$. ~~Usually~~ 50% overlap is commonly used.

<u>Step 2</u>: Each segment is multiplied by window $w(n)$. (commonly used window is triangular window). Periodogram is computed ~~to~~ resulting in modified periodogram $\tilde{P}^{(i)}_{xx}(f)$:

$$\tilde{P}^{(i)}_{xx}(f) = \frac{1}{MU} \left| \sum_{n=0}^{M-1} x_i(n)\, w(n)\, e^{-j2\pi f n} \right|^2 ; \quad i = 0, 1, \cdots L-1$$

U is a normalization factor for the spectral power of the window such that $\int_{-1/2}^{1/2} W(f)\, df = 1$

For this $U = \frac{1}{M} \sum_{n=0}^{M-1} w^2(n)$.

<u>Step 3</u>: These modified periodograms are averaged to get Welch Power spectrum estimate

$$P^W_{xx}(f) = \frac{1}{L} \sum_{i=0}^{L-1} \tilde{P}^{(i)}_{xx}(f)$$

The performance of this method is summarized below.

i) The mean or expected value of this estimate is

$$E[P_{xx}^{W}(f)] = E[\tilde{P}_{xx}^{(i)}(f)]$$

ii) Variance is $\quad Var[P_{xx}^{W}(f)] \simeq \frac{1}{L}\Gamma_{xx}^{2}(f)$ for no overlap

$$\simeq \frac{9}{8L}\Gamma_{xx}^{2}(f) \text{ for } 50\% \text{ overlap \& Triangular window.}$$

($\Gamma_{xx}(f)$ = true PSD of $x(n)$).

iii) ∴ Quality factor Q_W is given by

$$Q_W = L = \frac{N}{M} \text{ for no overlap}$$

$$= \frac{16N}{9M} \text{ for } 50\% \text{ overlap.}$$

For triangular window, $\quad \Delta f = \frac{1.28}{M}$

$$\therefore Q_W = 0.78 N \Delta f \text{ for no overlap}$$

$$= 1.39 N \Delta f \text{ for } 50\% \text{ overlap}$$

iv) Computational requirement :

$$\text{FFT length} = M = \frac{1.28}{\Delta f}$$

$$\text{No. of FFT} = 2\frac{N}{M} \quad (\text{for } 50\% \text{ overlap})$$

$$\therefore \text{No. of Computations} = \frac{2N}{M} \cdot \frac{M}{2} \log_2 M$$

$$= N \log_2\left(\frac{1.28}{\Delta f}\right)$$

In addition to this, since there is additional window multiplication where each segment needs M multiplications. There are L segments and with 50% overlap $L = 2\frac{N}{M}$. Hence total multiplications will be

$$2\frac{N}{M} \cdot M + N \log_2\left(\frac{1.28}{\Delta f}\right)$$

$$= N \log_2\left(\frac{5.12}{\Delta f}\right)$$

Blackmon – Tukey method

~~In~~ This method aims at improving the quality of PS estimate by smoothening the periodogram. Also, indirect method is used for periodogram. The periodogram is Fourier transform of autocorrelation of the signal.

The autocorrelation $r^*_{xx}(m)$ of M-length sequence is first ~~xx~~ multiplied by window to get biased autocorrelation ~~$r_{xx}(m)$~~. (Refer to page 6)

(The data record length is assumed to be N to get required resolution Δf. This is segmented into smaller segments of length M.)

~~The bias to $r_{xx}(m)$ ex ...~~. It is to be noted that, with $r'_{xx}(m)$ for large lag, the variance is high. This problem is solved by ~~multiplex~~ window multiplication (biasing) as for larger lag, acf is given lesser weightage. Thus the window smoothens the Power spectrum.

$$\cancel{P^{BT}_{xx}(f) = \sum_{m=-(M-1)}^{(M-1)} r^*_{xx}(m)\, w(m)\, e^{j}}$$

$$P^{BT}_{xx}(f) = \sum_{m=-(M-1)}^{(M-1)} r_{xx}(m)\, w(m)\, e^{-j2\pi fm}$$

The window function $w(m)$ has length $2M-1$ and is zero for $|m| \geq M$. The spectrum $W(f)$ should be such that $W(f) \geq 0,\ |f| \leq \frac{1}{2}$.

$$Q_{BT} = 2.34\, N \Delta f$$

No. of computation $= N \log_2\left(\frac{1.28}{\Delta f}\right)$

Advantages of non-parametric methods:

i) Low variance

ii) Simple and easy to compute using FFT.

Drawbacks:

i) Require availability of quite large data record (larger N) to get good frequency resolution

ii) Suffer from spectral leakage which may mask weaker signals. If we try to reduce leakage, it results in poor resolution.

iii) It is assumed that $r_{xx}(m) = 0, \quad m \geq N$ which severely limits the quality of the estimated Power spectrum.

Parametric methods: In these methods, only a short segment of data is used. (These methods are specifically useful when only short segment of data is available. (Note that even when large length record is available, because of non stationary conditions of random process, we are forced to use smaller segments).

The autocorrelation $r_{xx}(m)$ for $m \geq N$ is not assumed to be zero, but extrapolated by knowing some a-priori information on how data is generated. For this, a model for the data generation is constructed from certain parameters extracted from short length of the available data. From this model parameters, the power spectrum is estimated.

In these methods, we assume $x(n)$ be the output of linear system characterized by the system function

$$H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^{q} b_k z^{-k}}{1 + \sum_{k=1}^{p} a_k z^{-k}} \quad --- (*)$$

The corresponding difference equation is

$$x(n) = -\sum_{k=1}^{p} a_k x(n-k) + \sum_{k=0}^{q} b_k w(n-k)$$

where $w(n)$ is the input. Then input is actually not known, but taken as zero mean white ~~noise~~ noise process with variance σ_w^2.

The parameters (model parameters) a_k and b_k are obtained from short segment of available data. (construction of the signal model).

Then power spectrum is computed as

$$PSD = \sigma_w^2 \frac{|B(f)|^2}{|A(f)|^2} \quad --- (*)'$$

$\left(B(f) = B(z)\big|_{z=e^{j2\pi f}} \right.$ and similarly $\left. A(f). \right)$

Step 1: observe small data record $x(n)$; $0 \le n \le N-1$ (N is small here)

Step 2: Estimate model parameters a_k & b_k and get model $H(z)$ as in $(*)$

Step 3: Get PS using $(*)'$.

The model given in $(*)$ is known as Autoregressive Moving Average Model (ARMA) of order (p, q). If $q = 0$ & $b_0 = 1$, then $H(z) = \frac{1}{A(z)}$. This

is referred to as auto regressive (AR) model of order p. (This is all-pole model)
Similarly if $A(z) = 1$, then $H(z) = B(z)$. This model is moving average model (MA) which is an all-zero model of order q.

AR model is largely used. Some important AR model estimation methods are

i) Yule-Walker method
ii) Burg method
iii) Least-Square method
iv) Sequential estimation method.

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