1.3 Sufficient Condition for Constrained Optimization

Consider now the problem of maximizing $f(x_1, ..., x_n)$ on the constraint set

$$C_h = \{x \in R^n : \ h_i(x) = c_i, \ i = 1, ..., k\}.$$

As usual we consider the Lagrangian

$$L(x_1, ..., x_n, \mu_1, ..., \mu_k) = f(x_1, ..., x_n) - \sum_{i=1}^{k} \mu_i(h_i(x_1, ..., x_n) - c_i),$$

and the following bordered Hessian matrix

$$H = \begin{pmatrix} 0 & \dots & 0 & \frac{\partial h_1}{\partial x_1} & \dots & \frac{\partial h_1}{\partial x_n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \frac{\partial h_k}{\partial x_1} & \dots & \frac{\partial h_k}{\partial x_n} \\ \frac{\partial h_1}{\partial x_1} & \dots & \frac{\partial h_k}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \dots & \frac{\partial^2 L}{\partial x_n \partial x_1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{\partial h_1}{\partial x_n} & \dots & \frac{\partial h_k}{\partial x_n} & \frac{\partial^2 L}{\partial x_1 \partial x_n} & \dots & \frac{\partial^2 L}{\partial x_n^2} \end{pmatrix}.$$

This $(k+n) \times (k+n)$ matrix has $k+n$ leading principal minors

$$H_1, \ H_2, \ \dots, H_k, \ H_{k+1}, \ \dots, H_{2k-1}, \ H_{2k}, \ H_{2k+1}, \ \dots, H_{k+n} = H.$$

The first m matrices $H_1, \ \dots, H_k$ are zero matrices.

Next $k-1$ matrices $H_{k+1}, \ \dots, H_{2k-1}$ have zero determinant.

The determinant of the next minor H_{2k} is $\pm(det \ H')^2$ where H' is the upper $k \times k$ minor of H after block of zeros, so $det \ H_{2k}$ does not contain information about f.

And only the determinants of last $n-k$ leading principal minors

$$H_{2k+1},\ H_{2k+2},\ \dots,H_{2k+(n-k)=k+n} = H$$

carry information about both, the objective function f and the constraints h_i.

Exactly these minors are essential for the following sufficient condition for constraint optimization.

Theorem 1 *Suppose that $x^* = (x_1^*, ..., x_n^*) \in R^n$ satisfies the conditions*

(a) $x^* \in C_h$;

(b) there exists $\mu^* = (\mu_1^*, ..., \mu_k^*) \in R^k$ such that $(x_1^*, ..., x_n^*, \mu_1^*, ..., \mu_k^*)$ is a critical point of L;

(c) for the bordered Hessian matrix H the last $n-k$ leading principal minors

$$H_{2k+1},\ H_{2k+2},\ \dots,H_{n+k} = H$$

evaluated at $(x_1^*, ..., x_n^*, \mu_1^*, ..., \mu_k^*)$ alternate in sign where the last minor $H_{n+k} = H$ has the sign as $(-1)^n$.

Then x^* is a local max in C_h.

If instead of (c) we have the condition

(c') For the bordered hessian H all the last $n-k$ leading principal minors

$$H_{2k+1},\ H_{2k+2},\ \dots,H_{n+k} = H$$

evaluated at $(x_1^*, ..., x_n^*, \mu_1^*, ..., \mu_k^*)$ have the same sign as $(-1)^k$, then x^* is a local min on C_h.

Example 1. Find extremum of $F(x,y) = xy$ subject of $h(x,y) = x + y = 6$.

Solution. The Lagrangian here is

$$L(x,y) = xy - \mu(x + y - 6).$$

The first order conditions give the solution

$$x = 3, \;\; y = 3, \;\; \mu = 3$$

which needs to be tested against second order conditions before we can tell whether it is maximum, minimum or neither.

The bordered Hessian of our problem looks as

$$H = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Here $n = 2$, $k = 1$ so we have to check just $n - k = 2 - 1 = 1$ last leading principal minors, so just H itself. Calculation shows that $det\ H = 2 > 0$ has the sign $(-1)^2 = (-1)^n$ so our critical point $(x = 3, y = 3)$ is max.

Example 2. Find extremum of $F(x,y,z) = x^2+y^2+z^2$ subject of $h_1(x,y,z) = 3x + y + z = 5$, $\;\; h_2(x,y,z) = x + y + z = 1$.

Solution. The lagrangian here is

$$L(x,y,\mu_1,\mu_2) = x^2 + y^2 + z^2 - \mu_1(3x + y + z - 5) - \mu_2(x + y + z - 1).$$

The first order conditions give the solution

$$x = 2, \;\; y = -\frac{1}{2}, \;\; z = -\frac{1}{2}, \;\; \mu_1 = \frac{5}{2}, \;\; \mu_2 = -\frac{7}{2}.$$

Now it is time to switch to bordered hessian in order to tell whether it is maximum, minimum or neither

$$H = \begin{pmatrix} 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 3 & 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 & 2 \end{pmatrix}.$$

Here $n = 3,\ k = 2$ so we have to check just $n - k = 3 - 2 = 1$ leading principal minors, so just H itself. Calculation shows that $det\ H = 16 > 0$ has the sign as $(-1)^k = (-1)^2 = +1$, so our critical point is min.

Example 3. Find extremum of $F(x, y) = x+y$ subject of $h(x, y) = x^2+y^2 = 2$.

Solution. The lagrangian here is

$$L(x, y) = x + y - \mu(x^2 + y^2 - 2).$$

The first order conditions give two solutions

$$x = 1,\ y = 1, \mu = 0.5\ \ and\ \ x = -1,\ y = -1, \mu = -0.5$$

Now it is time to switch to bordered hessian

$$H = \begin{pmatrix} 0 & 2x & 2y \\ 2x & -2\mu & 0 \\ 2y & 0 & -2\mu \end{pmatrix}.$$

Here $n = 2,\ k = 1$ so we have to check just $n - k = 3 - 2 = 1$ leading principal minor $H_2 = H$.

Checking H for $(x = 1,\ y = 1,\ \mu = 0.5)$ we obtain $H = 4 > 0$, that is it has the sign of $(-1)^n = (-1)^2$, so this point is max.

Checking H for $(x = -1,\ y = -1,\ \mu = -0.5)$ we obtain $H = -4 < 0$, that is it has the sign of $(-1)^k = (-1)^1$, so this point is min.

Exercise

1. Find the extremum of x^2y^2 subject of $x^2 + y^2 = 2$.
2. Find the extremum of $x^2 + y^2$ subject of $x^2 + xy + y^2 = 3$.