WATSAL JINDAL - 58

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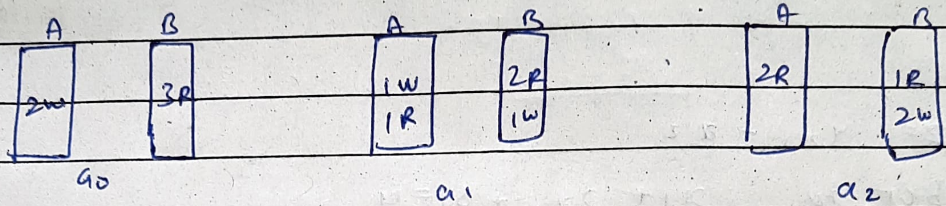
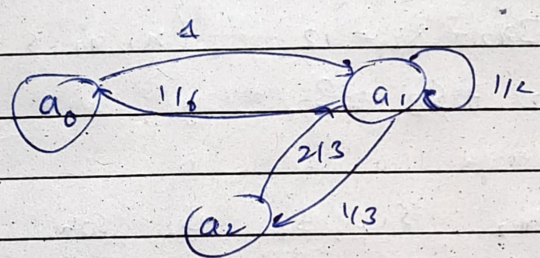
MATH I.A. - 4

Q1

Box A = 2W

Box B = 3R





(a) Transition matrix

$$P = \begin{array}{c c} & \begin{array}{ccc} a_0 & a_1 & a_2 \end{array} \\ \begin{array}{c} a_0 \\ a_1 \\ a_2 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix} \end{array}$$

(b) Let $x_0 = [1 \ 0 \ 0]$

$x_3 = x_0 P^3 \quad \rightarrow \quad x_3 = [1/2 \quad 23/36 \quad 5/18]$

$\rightarrow$ 2 marbles in Box A after 3 steps $= 5/18 \doteq \underline{0.277}$

(c) Let $X = [x \ y \ z]$

$x + y + z = d$

$XP = X$

$\Rightarrow [x \ y \ z] \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix} = [x \ y \ z]$

$\left[y/6 \quad x + y/2 + 2z/3 \quad \frac{y}{3} + \frac{z}{3}\right] = [x \ y \ z]$

Vat

$a = \frac{y}{6}$, $x + \frac{y}{2} + \frac{2z}{3} = y$; $\frac{y}{3} + \frac{z}{3} = 2$

$x = \left[\frac{1}{10} \quad \frac{3}{3} \quad \frac{3}{10} \right]$

→ long run probability of 2 red marbles in A $= \frac{3}{10} = 0.3$

Q2

$z = 4x_1 - x_2$

Subject to $x_1 + 2x_2 + S_1 = 4$

$2x_1 + 3x_2 + S_2 = 12$ $(x_1, x_2, S_1, S_2 \geq 0)$

$x_1 - x_2 + S_3 = 3$

CB	Basic Variable	x_1	x_2	S_1	S_2	S_3	Soln	Ratio
	c_j	4	-1	0	0	0		
0	S_1	1	2	1	0	0	4	4
0	S_2	2	3	0	1	0	12	6
0	S_3	1	-1	0	0	1	3	3
	Z_j	0	0					
	$Z_j - C_j$	-4	-1					

CB	Basic Variable	x_1	x_2	S_1	S_2	S_3	Soln	Ratio
0	S_1	0	3	1	0	-1	1	1/3
0	S_2	0	5	0	1	-2	6	6/5
4	x_1	1	-1	0	0	1	3	-3
	Z_j	4	-4					
	$Z_j - C_j$	0	-3					

CB	Basic Variable	x_1	x_2	S_1	S_2	S_3	Soln
-1	x_2	0	1	1/3	0	-1/3	1/3
0	S_2	0	0	5/3	-1	1/3	-13/3
4	x_1	1	0	1/3	0	2/3	10/3

max z @ $x_1 = 10/3$, $x_2 = 1/3$

$\boxed{\text{maximum } z = 13}$ ans

Vat

Q3 $f(x) = 9x_1^2 + x_2^2 + 18x_1 - 4x_2$; $x_0 = (2, 4)$

$$\nabla f(x) = \begin{bmatrix} \partial f / \partial x_1 \\ \partial f / \partial x_2 \end{bmatrix} = \begin{bmatrix} 18x_1 + 18 \\ 2x_2 - 4 \end{bmatrix}$$

$$\nabla f(x_0) = \begin{bmatrix} 54 \\ 4 \end{bmatrix}$$

$$\lambda_0 = \underset{\lambda \geq 0}{\arg\min}\ f(x_0 - \lambda \nabla f(x_0))$$

$$= f\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix} - \lambda \begin{bmatrix} 54 \\ 4 \end{bmatrix}\right)$$

$$= f\begin{bmatrix} 2-54\lambda \\ 4-4\lambda \end{bmatrix}$$

→ $\lambda_0 = \underset{\lambda \geq 0}{\arg\min}\left(9(4 - 216\lambda + 2916\lambda^2) + (16 + 16\lambda^2 - 32\lambda) + 18(2 - 54\lambda) - 4(4-4\lambda)\right)$

$$\lambda_0 = \underset{\lambda \geq 0}{\arg\min}\ (26260\lambda^2 - 2932\lambda + 72)$$

$$\lambda_0 = \left[\frac{d}{d\lambda}(26260\lambda^2 - 2932\lambda) = 0\right]$$

$$\lambda_0 = \frac{733}{13130} = \underline{0.0558}$$

→ $x_1 = x_0 - \lambda_0 \nabla f(x_0)$

$$= \begin{pmatrix} 2 \\ 4 \end{pmatrix} - 0.558 \begin{pmatrix} 54 \\ 4 \end{pmatrix} = \begin{bmatrix} -1.0146 \\ 3.7767 \end{bmatrix}$$

$$\nabla f(x_1) = \begin{bmatrix} -0.2628 \\ 3.5534 \end{bmatrix}$$

→ Similarly, $\lambda_1 = 0.4792$

$$x_2 = x_1 - \lambda_1 \nabla f(x_1) = \begin{pmatrix} -1.0146 \\ 3.7767 \end{pmatrix} - 0.4792 \begin{pmatrix} -0.2628 \\ 3.5534 \end{pmatrix}$$

$$x_2 = \begin{pmatrix} -0.887 \\ 2.0739 \end{pmatrix}$$

Q4

$\phi(x, y, z) = xyz$

$\psi(x, y, z) = xy^2 + yz^2 - 2zx^2 = 4$

$\nabla \phi = \frac{d\phi}{dx}\hat{i} + \frac{d\phi}{dy}\hat{j} + \frac{d\phi}{dz}\hat{k}$

$= yz\hat{i} + xz\hat{j} + xy\hat{k}$

$\nabla\phi(1, 1, -1) = \underline{-i - j + k}$

$\nabla \phi = \frac{d\phi}{dx} i + \frac{d\psi}{dy} j + \frac{d\psi}{dz} k$

$= i(y^2 - 4xz) i + (2xy + z^2) j + (2yz - 2x^2) k$

$\nabla\phi(1, 1, -1) = 5i + 3j - 4k$

$\hat{n} = \frac{5i + 3j - 4k}{\sqrt{25 + 9 + 16}} = \frac{5i + 3j - 4k}{\sqrt{50}}$

$\nabla\phi \hat{n} = (-i - j + k) \cdot \left(\frac{5i + 3j - 4k}{\sqrt{50}}\right) = \frac{-12}{\sqrt{50}} - \frac{6\sqrt{2}}{5}$

→ Equation of tangent

$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$

→ Equation of normal

$\frac{x - x_1}{A} + \frac{y - y_1}{B} = \frac{z - z_1}{C}$

$A = \frac{d\psi}{dx}$, $B = \frac{d\psi}{dy}$, $C = \frac{d\psi}{dz}$, at $(1, 1, -1)$

$A = 5$,

$B = 3$, $C = -4$.

→ Tangent equation $= 5(x - 1) + 3(y - 1) - 4(z + 1) = 0$

$= 5x + 3y - 4z = 12$

Normal line →

$\cancel{\frac{x - 1}{5} = y}$

$$\frac{x - 1}{5} = \frac{y - 1}{3} = \frac{z + 1}{4}$$

Q5 $\phi_1 = x^2+y^2+z^2 = 9$

$\phi_2 = x^2+y^2-3 = z$ @ $(2,-1,2)$

$\nabla\phi_1 = 2xi + 2yj + 2zk$

$= 4i - 2j + 4k$

$\nabla\phi_2 = 2xi + 2yj$

$= 4i - 2j - k$

$\therefore \cos\theta = \dfrac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$

$\cos\theta = \dfrac{(4i-2j+4k)(4i-2j-k)}{(\sqrt{16+4+16})(\sqrt{4+16+1})}$

$\therefore \cos\theta = \dfrac{16}{6\sqrt{21}}$

$\therefore \theta = \cos^{-1}\left(\dfrac{8}{3\sqrt{21}}\right)$

Q6 we have $r^n \vec{r} = r^n \sum xi = \sum(r^n x)i$

$\nabla(r^n \cdot \vec{r}) = \left(\sum \frac{\partial}{\partial x} i\right) \cdot \sum(r^n x)i$

$= \sum \frac{\partial}{\partial x}(r^n x) = \sum\left(r^n + nr^{n-1}\frac{\partial r}{\partial x} x\right)$

$= \sum\left(r^n + nr^{n-1}\frac{x}{r}x\right) = \sum(r^n + nr^{n-2}x^2)$

→ on expanding the summation we get

$\nabla(r^n \cdot \vec{r}) = (r^n + nr^{n-2}x^2) + (r^n + nr^{n-2}y^2) + (r^n + nr^{n-2}z^2)$

$= 3r^n + nr^{n-2}(x^2+y^2+z^2)$

$= 3r^n + nr^{n-2}r^2 = 3r^n + nr^n$

$\boxed{\nabla(r^n \vec{r}) = (n+3)r^n}$ hence proved

Q2 Let $\vec{A} = a_1 i + a_2 j + a_3 k$ be a vector point function of x, y, z.

$$\nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix} = \sum i \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z} \right)$$

$$\nabla \times (\nabla \times A) = \begin{vmatrix} i & j & k \\ \partial / \partial x & \partial / \partial y & \partial / \partial z \\ \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right) & \left(\frac{\partial a_1}{\partial z} - \frac{\partial a_3}{\partial x}\right) & \left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y}\right) \end{vmatrix}$$

$$= \sum i \left\{ \frac{\partial}{\partial y}\left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y} \right) - \frac{\partial}{\partial z}\left(\frac{\partial a_1}{\partial z} - \frac{\partial a_3}{\partial x} \right) \right\}$$

$$= \sum i \left(\frac{\partial^2 a_2}{\partial x \partial y} + \frac{\partial^2 a_3}{\partial z \partial x} \right) - \sum i \left(\frac{\partial^2 a_1}{\partial y^2} + \frac{\partial^2 a_1}{\partial z^2} \right)$$

=> Adding and subtracting $\sum i \frac{\partial^2 a_1}{\partial x^2}$ we get

$$= \sum i \frac{\partial}{\partial x}\left(\frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z} \right) - \sum a_1 i \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)$$

$$= \sum i \frac{\partial}{\partial x} (\operatorname{div} \vec{A}) - \nabla^2 \sum a_1 i = \nabla \operatorname{div}(\vec{A}) - \nabla^2 \vec{A}$$

$$\boxed{\nabla \times (\nabla \times A) = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}}$$ Hence proved