Analog filter design
Butterworth filter

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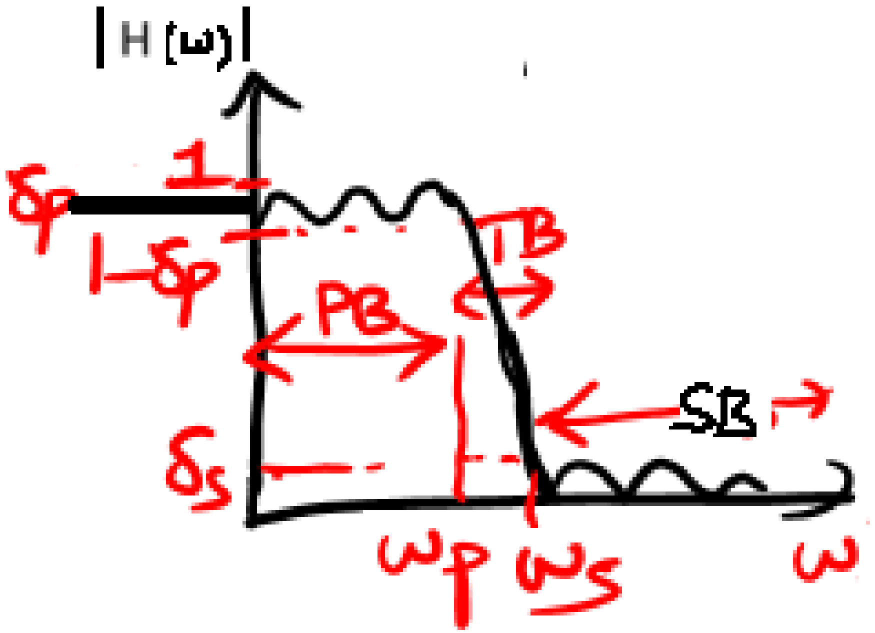
Analog filter design

- **Butterworth filter** (discussed in this ppt)
- Chebyshev filter
- Elliptic filter

$H(s)$ analog $\longrightarrow$ $H(z)$ digital

LPF – Ideal

Practical



Butterworth filter design - LPF

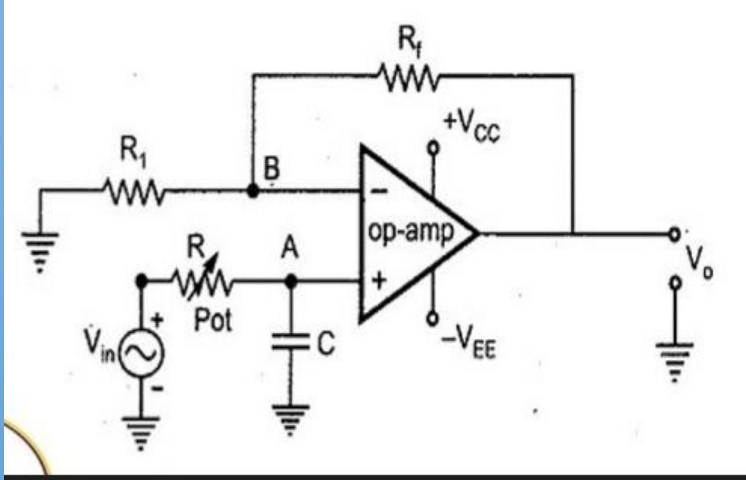
All pole.

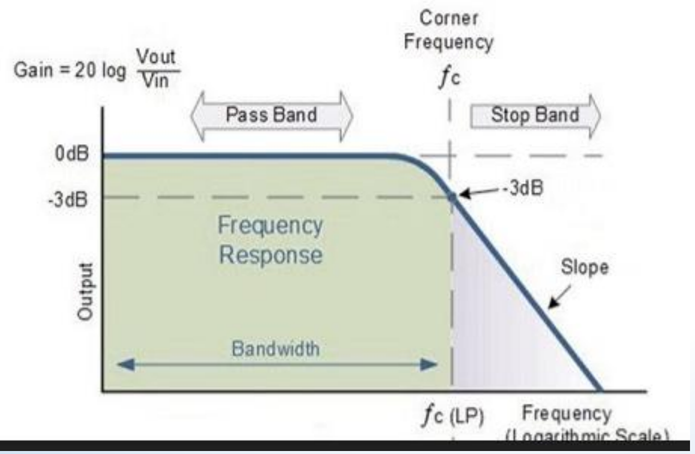
$$|H(\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

$\leftarrow$ order = N

$\leftarrow$ cutoff freq

$$s = j\Omega \Rightarrow \Omega = s/j$$





$$|H(\Omega)|^2 = \frac{1}{1+\left(\frac{s/j}{\Omega_c}\right)^{2N}}$$

$$= \frac{1}{1+\left(\frac{-s^2}{\Omega_c^2}\right)^{N}}$$

$$H(s) \cdot H(-s) \big|_{s=j\Omega}$$

$$H(s)H(-s) = \frac{1}{1+\left(\frac{-s^2}{\Omega_c^2}\right)^{N}}$$

To find poles:

$$1 + \left(\frac{-s^2}{\Omega_c^2}\right)^N = 0$$

$$\left(\frac{-s^2}{\Omega_c^2}\right)^N = -1 = e^{j(2k+1)\pi} \qquad k = \text{integer}$$

$$\frac{-s^2}{\Omega_c^2} = e^{j(2k+1)\frac{\pi}{N}}$$

$$s^2 = -\Omega_c^2 \, e^{j(2k+1)\frac{\pi}{N}}$$

$$s = j\Omega_c \, e^{j(2k+1)\frac{\pi}{2N}}$$

$$\boxed{s_k = \Omega_c \, e^{j(\pi/2)} \cdot e^{j(2k+1)\frac{\pi}{2N}}} \qquad k = 0, 1, \ldots 2N-1$$

eg. $N = 2$ (even) & $\Omega_c = 1$ (normalized)

$S_k = e^{j\pi/2} \cdot e^{j(2k+1)\frac{\pi}{4}}$ $k = \underline{\underline{0, 1}}, 2, 3$

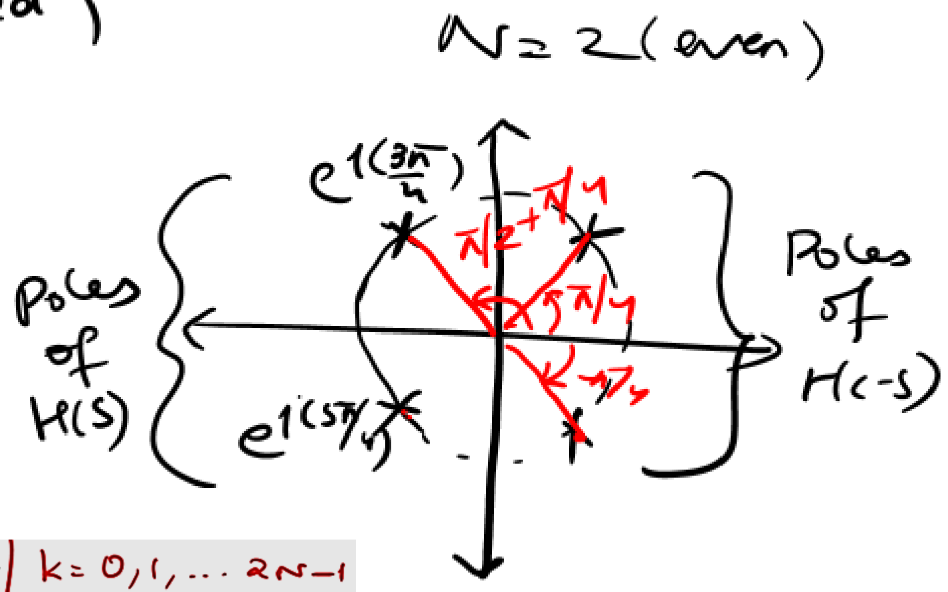
$\underline{\underline{S_0}} = e^{j(\pi/2+\pi/4)} = e^{j(3\pi/4)} = -0.707 + j0.707$

$S_1 = e^{j(5\pi/4)-2\pi} = e^{j(-3\pi/4)} = -0.707 - j0.707$

$S_2 = e^{j(7\pi/4)-2\pi} = -\frac{\pi}{4}$

$S_3 = e^{j(9\pi/4)-2\pi} = \frac{\pi}{4}$

$\boxed{S_k = \Omega_c \, e^{j(\pi/2)} \cdot e^{j(2k+1)\frac{\pi}{2N}}}$ $k = 0, 1, \ldots 2N-1$



Stable poles: $N = \text{even}$, $k = \underline{\underline{0}}, 1, \ldots \frac{N}{2} - 1$

$N = \text{odd}$, $k = 0, 1, \ldots \frac{N-1}{2}$

$$H(s) = \frac{1}{[s - (-0.707 + j0.707)][s - (-0.707 - j0.707)]} \quad (\Omega_c = 1)$$

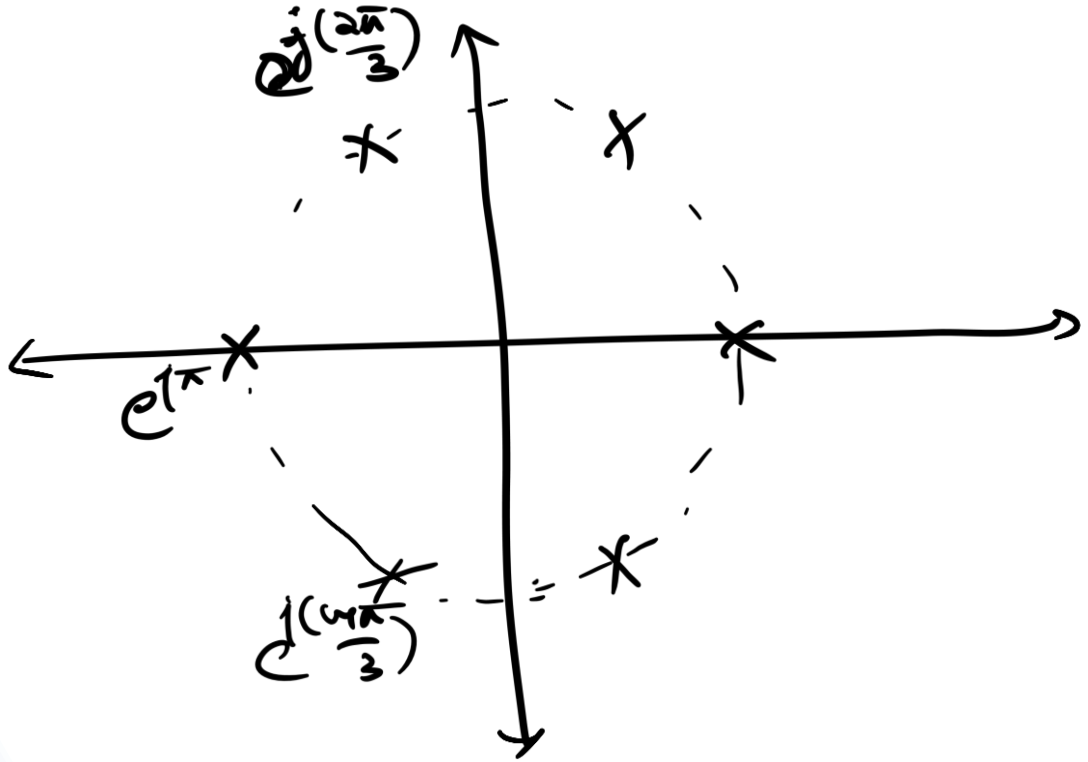
$$(s + 0.707)^2 - (j0.707)^2 = s^2 + 1.414s + 1$$

$N = 3$ (odd), $\Omega_c = 1$

$s_k = e^{j\pi/2} e^{j(2k+1)\frac{\pi}{6}}$, $k = 0, 1, \ldots 2N-1$

$\phi_k = \underbrace{\frac{2\pi}{3}, \pi, \frac{4\pi}{3}}_{\text{stable.}}, \frac{5\pi}{3}, 2\pi, \frac{7\pi}{3}$

$H(s) = \dfrac{1}{(s+1)(s^2+s+1)}$



Freq resp.

N↑, TB↓

$$A_p = 1 - \delta_p = \frac{1}{\sqrt{1+\varepsilon^2}}$$

$$A_s = \delta_s$$

$$A_p \le |H(\Omega)| \le 1 \quad , \quad 0 \le \Omega \le \Omega_p.$$

$$|H(\Omega)| \le A_s \quad , \quad \Omega_s \le \Omega.$$

$$|H(\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

$\Omega = \Omega_p$

$\Omega = \Omega_s$

N

$$\varepsilon = \left(\frac{\Omega_p}{\Omega_c}\right)^N$$

$$\Omega_c^{2N} = \frac{\Omega_p^{2N}}{\varepsilon^2}$$

$$\boxed{|H(\Omega)|^2 = \frac{1}{1+\varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}}}$$

ε = Band edge value

Filter order

$$|H(\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

At $\Omega = \Omega_p$, $|H(\Omega_p)|^2 = A_p^2 = \frac{1}{1+\left(\frac{\Omega_p}{\Omega_c}\right)^{2N}} \Rightarrow \left(\frac{\Omega_p}{\Omega_c}\right)^{2N} = \frac{1}{A_p^2} - 1$

At $\Omega = \Omega_s$, $|H(\Omega_s)|^2 = A_s^2 = \frac{1}{1+\left(\frac{\Omega_s}{\Omega_c}\right)^{2N}} \Rightarrow \left(\frac{\Omega_s}{\Omega_c}\right)^{2N} = \frac{1}{A_s^2} - 1$

Now, $\left(\frac{\Omega_p}{\Omega_s}\right)^{2N} = \left(\frac{\Omega_p}{\Omega_c}\right)^{2N} \cdot \left(\frac{\Omega_c}{\Omega_s}\right)^{2N}$

$$k^{2N} = \frac{\frac{1}{A_p^2} - 1}{\frac{1}{A_s^2} - 1} = d^2$$

$2N \log(k) = 2\log d \Rightarrow \boxed{N = \frac{\log(d)}{\log(k)}}$ //

$$d = \sqrt{\frac{1/A_p^2 - 1}{1/A_s^2 - 1}}$$

discrimination factor

$$k = \frac{\Omega_p}{\Omega_s}$$

selectivity / transition ratio.

Filter order – 2nd eqn.

$$|H(\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

$\Omega = \Omega_s$,

$$N = \frac{\log\left(1/\delta_s^2 - 1\right)}{2\log\left(\Omega_s/\Omega_c\right)}$$

$$N = \frac{\log(\delta/\varepsilon)}{\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$\frac{1}{\delta_s^2} - 1 = \delta^2$$

Question :

Order & poles LPF Butterworth , BW = 500 Hz , att = 40 dB at 1000 Hz

Soln :

$-3dB$ BW $= f_c = 500\,Hz \Rightarrow \Omega_c = 2\pi f = 1000\pi$

$f_s = 1000\,Hz \Rightarrow \Omega_s = 2\pi f = 2000\pi$

$\delta_s^2 = A_s^2 = -40dB \Rightarrow \log \delta_s = -40$

$\delta_s = 10^{-2} = 0.01$

Filter order, $N = \dfrac{\log\left(1/\delta_s^2 - 1\right)}{2\log\left(\Omega_s/\Omega_c\right)} = \dfrac{4}{2 \times 0.03} = 6.64 \approx 7$

Pole : $S_k = \Omega_c\, e^{j(\pi/2)}\, e^{j(2k+1)\frac{\pi}{2N}}$

$= 1000\pi\, e^{j\pi/2}\, e^{j(2k+1)\frac{\pi}{14}}$ $\quad K = 0, 1, \ldots \frac{N-1}{2} = 3$

Pole : $S_k = \Omega_c \, e^{j(\pi/2)} \, e^{j(2k+1)\frac{\pi}{2N}}$

$= 1000\pi \, e^{j\pi/2} \, e^{j(2k+1)\frac{\pi}{14}}$ $\quad k = 0, 1, \cdots \frac{N-1}{2} = 3$

$S_0 = 1000\pi \, e^{j(\pi/2 + \pi/14)} = 1000\pi \, e^{j(8\pi/14)}$

$= 3141.59\,(-0.2225 \pm j0.9749) = -699 \pm j\,3062.74$

$S_1 = 1000\pi \, (e^{j\,\pi/2 + 3\pi/14}) = 1000\pi \, e^{j\,10\pi/14}$ ✓

$S_2 = 1000\pi \, e^{j(12\pi/14)}$ ✓

$S_3 = 1000\pi \, e^{j\pi} = -1000\pi$ ✓

Q.2) The specifications of desired LPF response is given by

$$A_p = \frac{1}{\sqrt{2}} \le |H(e^{j\omega})| \le 1 \quad , \quad 0 \le \omega \le 0.2\pi = \omega_p$$

$$|H(e^{j\omega})| \le 0.08 = A_s , \quad 0.4\pi = \omega_s \le \omega \le \pi$$

Design a Butterworth digital filter using bilinear transf

Assume a sampling interval of $T = 2$ sec.

Soln :

Butterworth analog → order, pole locations, H(S)

↓ BLT

digital (H(z))

Given: $A_p = \frac{1}{\sqrt{2}} = 0.707$

$A_s = 0.08$

$\omega_p = 0.2\pi \Rightarrow \Omega_p = \tan\left(\frac{0.2\pi}{2}\right) = 0.324$

$\omega_s = 0.4\pi \Rightarrow \Omega_s = \tan\left(\frac{0.4\pi}{2}\right) = 0.726$

BLT, $\omega \to \Omega$: $\Omega = \frac{2}{T}\tan\frac{\omega}{2} = \tan\frac{\omega}{2}$

Order of desired LP Butterworth:

$$N = \frac{\log d}{\log k} = \frac{\log 0.08}{\log 0.446} = 3.12 \approx 4 \text{ (even)}$$

$$d = \sqrt{\frac{1/A_p^2 - 1}{1/A_s^2 - 1}} = \sqrt{\frac{(1/0.707^2) - 1}{(1/0.08^2) - 1}} = 0.08$$

$$k = \frac{\Omega_p}{\Omega_s} = \frac{0.324}{0.726} = 0.446$$

$$S_k = \Omega_c e^{j\pi/2} e^{j(2k+1)\frac{\pi}{8}}$$

$$\left(\frac{\Omega_p}{\Omega_c}\right)^N = \varepsilon = \sqrt{\frac{1}{A_p^2} - 1}$$

$$\Omega_c = 0.324$$

$$S_k = 0.324\, e^{j(\pi/2)} e^{j(2k+1)\frac{\pi}{8}}$$

$$k = 0, \ldots \frac{N}{2} - 1$$

$$S_0 = 0.324\, e^{j(\pi/2 + \pi/8)} = 0.324\, e^{j\frac{5\pi}{8}}$$

$$= 0.324(-0.382 + j\,0.923)$$

$$= -0.124 \pm j\,0.299$$

$$S_1 = 0.324\, e^{j(\pi/2 + 3\pi/8)} = 0.324\, e^{j\frac{7\pi}{8}}$$

$$= -0.29 \pm j\,0.124$$

Stable poles:

$$k = 0 \ldots \frac{N}{2} - 1 = 1$$

$$k = 0, 1$$

$N = 4$

$2N$ poles

$$H(S)_{\substack{\text{normalised} \\ \Omega_c=1)}} = \frac{1}{(S-e^{j5\pi/8})(s-e^{-j5\pi/8})(S-e^{j7\pi/8})(s-e^{-j7\pi/8})}$$

$$H(s)\Big|_{s\to \frac{s}{\Omega_c}} = \frac{1}{\left(\frac{S}{\Omega_c}-e^{j5\pi/8}\right)\left(\frac{S}{\Omega_c}-e^{-j5\pi/8}\right)\cdots}$$

$$= \frac{\Omega_c^4}{\frac{1}{\Omega_c^4}(s-\Omega_c e^{j5\pi/8})(s-\Omega_c e^{-j5\pi/8})\cdots}$$

$$H(s) = \frac{0.011}{(S+0.124-j0.299)(s+0.124+j0.299)(s+0.29-j0.124)(s+0.29+j0.124)}$$

$$= \frac{0.011}{((S+0.124)^2-j^2 0.299^2)((S+0.29)^2-j^2 0.124^2)}$$

$$= \frac{0.011}{(S^2+0.248s+0.105)(s^2+0.58s+0.09)}$$

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) = \frac{1-z^{-1}}{1+z^{-1}}$$

$$H(z) = \frac{0.011}{\left[\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 0.248\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.105\right]\left[\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 0.58\,\frac{1-z^{-1}}{1+z^{-1}} + 0.09\right]}$$

$$H(z) = \frac{0.011\,(1+z^{-1})^4}{\left[(1-z^{-1})^2 + 0.248\,(1-z^{-1})(1+z^{-1}) + 0.105\,(1+z^{-1})^2\right] \times \left[(1-z^{-1})^2 + 0.58\,(1-z^{-1})\,(1+z^{-1}) + 0.09\,(1+z^{-1})^2\right]}$$

$$= \frac{0.011\,(1+z^{-1})^4}{(1.353 - 1.79z^{-1} + 0.857z^{-2})\,(1.67 - 1.82z^{-1} + 0.51z^{-2})}$$

Reference

- Proakis J. G, Manolakis D. G. Mimitris D., “Introduction to Digital Signal Processing” Prentice Hall, India, 2007.


Thank you