Examples: $x^2+y^2-z^2-xy+yz, xyz^4, xy+x^3y^2$ etc.

If every point (x,y,z) of a region R in space there corresponds a vector $\vec{F}(x,y,z)$ then $\vec{F}$ is called a vector point function or the vector field R.

Example: $xi - yj + zk,\ x^2yi + 2y^2zk - xy^3zk\ etc$

Vector differential operator:

The differential operator ∇ is defined as

$$\nabla = \frac{\partial}{\partial x}\,\boldsymbol{i} + \frac{\partial}{\partial y}\,\boldsymbol{j} + \frac{\partial}{\partial z}\,\boldsymbol{k}.$$

The vector operator possesses properties analogous to those of ordinary vectors. It is useful in defining three quantities that appear in applications and which are known as the gradient, divergence and the curl. The operator ∇ is also known as nabla.

Gradient:

Let $\phi = \phi(x,y,z)$ is the scalar function defined and differentiable at each point (x,y,z) in a certain region of space. Then the gradient of ϕ written as $\nabla\phi$ or $grad\ \phi$ and defined as follows: $grad\phi = \nabla\phi = \left(\frac{\partial}{\partial x}\,\boldsymbol{i} + \frac{\partial}{\partial y}\,\boldsymbol{j} + \frac{\partial}{\partial z}\,\boldsymbol{k}\right)\phi = \frac{\partial\phi}{\partial x}\,\boldsymbol{i} + \frac{\partial\phi}{\partial y}\,\boldsymbol{j} + \frac{\partial\phi}{\partial z}\,\boldsymbol{k} = \sum \frac{\partial\phi}{\partial x}\,\boldsymbol{i}.$

Always, gradient of a scalar is a vector field.

Example: If $\phi = x^3y^2z$ then $\nabla\phi = \frac{\partial}{\partial x}(x^3y^2z)\,\boldsymbol{i} + \frac{\partial}{\partial y}(x^3y^2z)\,\boldsymbol{j} + \frac{\partial}{\partial z}(x^3y^2z)\,\boldsymbol{k}$

$$\therefore \quad \nabla\phi = 3x^2y^2z\,i + 2x^3yz\,j + x^3y^2k.$$

Divergence:

Let $\vec{f} = f_1\boldsymbol{i} + f_2\,\boldsymbol{j} + f_3\,\boldsymbol{k}$ is a vector field defined and differentiable at each point (x,y,z) in a certain region of space. Then the divergence of $\vec{f}$ written as $div\vec{f}$ or $\nabla.\vec{f}$ and defined as

$$div\vec{f} = \nabla.\vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \sum \frac{\partial f_1}{\partial x}.$$

Always, divergence of a vector field is a scalar.

Example: Suppose $\vec{f} = x^2y^2\,i - 2y^2z^2j + 3xy^2z\,k$ then

$$\nabla.\vec{f} = \frac{\partial}{\partial x}(x^2y^2) + \frac{\partial}{\partial y}(-2y^2z^2) + \frac{\partial}{\partial z}(3xy^2)$$

$$\nabla.\ \overrightarrow{f} = 2xy^2 - 4yz^2 + 3xy^2$$

At the point $(1, 1, 1,)$

$$\nabla.\ \overrightarrow{f} = 1.$$

Curl:

Let $\vec{f} = f_1\boldsymbol{i} + f_2\boldsymbol{j} + f_3\boldsymbol{k}$ is a differentiable vector field, then the curl or rotation of $\vec{f}$ at the point (x, y, z) is written as $curl\,\vec{f}$ or $\nabla \times \vec{f}$ or $rot\,\vec{f}$ is defined as

$$curl\,\vec{f} = \nabla \times \vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}.$$

Always, curl of a vector field is a vector.

Example: Suppose $\overrightarrow{f} = x^2y^2\,i - 2y^2z^2j + 3xy^2z\,k$ then

$$\nabla \times \vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y^2 & -2y^2z^2 & 3xy^2z \end{vmatrix} = i(6xyz + 4y^2z) - j(3y^2z - 0) + k(0 - 2x^2y).$$

$$\nabla \times \vec{f} = (6xyz + 4y^2z)i - 3y^2zj - 2x^2yk.$$

Laplacian operator:

If $\phi(x, y, z)$ is a continuously differentiable scalar function and $\vec{A}$ is a continuously differentiable vector function then the Laplacian of ϕ is defined as

$$\text{Laplacian of } \phi = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}.$$

$$\text{Laplacian of } \vec{A} = \nabla^2\vec{A} = \frac{\partial^2\vec{A}}{\partial x^2} + \frac{\partial^2\vec{A}}{\partial y^2} + \frac{\partial^2\vec{A}}{\partial z^2}.$$

Directional derivative:

Consider a scalar function $\phi = \phi(x, y, z)$. Then the directional derivative ϕ in the direction of a vector $\vec{A}$ is denoted by $D_A\phi$ in the direction of $\vec{A}$ is given by

$$D_A(\phi) = \nabla\phi.\hat{n}$$

where $\hat{n}$ be the unit vector in the direction of $\vec{A}$, that is $\hat{n} = \frac{\vec{A}}{|A|}$.

Geometrical meaning of $\boldsymbol{\nabla\phi}$

If $\phi(x, y, z)$ is a scalar function then, geometrically $\nabla\phi$ represents a vector normal to the surface $\phi(x, y, z) = c$. Let $\vec{r} = x(t)i + y(t)j + z(t)k$.

$$\nabla\phi.\frac{d\vec{r}}{dt} = 0 \Rightarrow \nabla\phi \text{ is perpendicular to } \frac{d\vec{r}}{dt}.$$

Angle between the surfaces:

If $\phi_1(x, y, z)$ and $\phi_2(x, y, z)$ are the two surfaces, then the angle between their surfaces at (x_1, y_1, z_1) is

$$\cos\theta = \frac{\nabla\phi_1.\nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}.$$

Equation of tangent plane and normal line:

If $\phi(x, y, z) = c$ be the equation of a surface and $P(x_1, y_1, z_1)$ is a point on it then

a) equation of tangent plane at P is $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$.
b) Equation of normal line at P is $\frac{x-x_1}{A} = \frac{y-y_1}{B} = \frac{z-z_1}{C}$,

where $A = \left(\frac{\partial\phi}{\partial x}\right)_{(x_1,y_1,z_1)}, B = \left(\frac{\partial\phi}{\partial y}\right)_{(x_1,y_1,z_1)}, C = \left(\frac{\partial\phi}{\partial z}\right)_{(x_1,y_1,z_1)}$.

Equation of a tangent plane and normal

Consider a point A on a surface $f(x, y, z) = c$ and its position vector is denoted as r_A. Let r_P denotes the vector of any point P on the tangent plane at A. Then $AP = r_P - r_A$ and lies in the tangent plane. Also, ∇f is along the normal to the surface. Hence $AP = r_P - r_A$ is perpendicular to ∇f and hence their dot product is zero. Hence the equation of the tangent plane is

$$(r_P - r_A).\nabla f = 0$$

Consider a point A on the surface $f(x, y, z) = c$ and its position vector be r_A. Let r_P

Denotes the position vector of any point P on the normal at A. Then $AP = r_P - r_A$ and it lies along the normal at A. Also ∇f is along normal to the surface. Hence the cross product of AP and ∇f must be zero.

i.e, $(AP) \times \nabla f = 0$ i.e., $(r_P - r_A) \times \nabla f = 0$.

This gives the equation of the normal at the point A.

Theorem: If $\boldsymbol{\phi(x, y, z)}$ is a scalar function then $\boldsymbol{grad\ \phi}$ is a vector normal to the surface $\boldsymbol{\phi(x, y, z) = c}$.

Proof: Let $\vec{r}$ be the position vector of any point $P(x, y, z)$ on the surface $\phi(x, y, z) = c$. Also let

$$\vec{r} = x(t)i + y(t)j + z(t)k.$$

$$\therefore \frac{d\vec{r}}{dt} = \frac{dx}{dt}i + \frac{dy}{dt}j + \frac{dz}{dt}k$$

Is tangential to the surface at P. Also we have

$$\nabla\phi = \frac{\partial\phi}{\partial x}\,\boldsymbol{i} + \frac{\partial\phi}{\partial y}\,\boldsymbol{j} + \frac{\partial\phi}{\partial z}\,\boldsymbol{k}$$

On taking the dot product of these two vectors, we have

$$\nabla\phi\,.\frac{d\vec{r}}{dt} = \frac{\partial\phi}{\partial x}\,\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\,\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\,\frac{dz}{dt}. \qquad (1)$$

Also, let us consider $\phi(x, y, z) = c$ where $x = x(t), y = y(t), z = z(t)$ and c is a constant.

On differentiating w. r. t. t on both sides, we have $\frac{d\phi}{dt} = 0$ and using the concept of differentiation of composite functions, we have

$$\frac{\partial\phi}{\partial x}\,\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\,\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\,\frac{dz}{dt} = 0$$

or $$\nabla\phi\,.\frac{d\vec{r}}{dt} = 0. \qquad \text{(by using (1)).}$$

$\Longrightarrow \nabla\phi$ is perpendicular to $\frac{d\vec{r}}{dt}$. Since $\frac{d\vec{r}}{dt}$ is a vector tangential to the surface at P, we can conclude that $\nabla\phi$ is along the normal to the surface $\phi(x, y, z) = c$ at P.

Theorem: The directional derivative of a scalar function $\boldsymbol{\phi}$ at any point is maximum along $\boldsymbol{\nabla\phi}$.

Proof: By the definition of the directional derivative of ϕ along $\hat{n}$, we have $\nabla\phi\,.\,\hat{n}$.

By the definition of dot product, we have $\nabla\phi\,.\,\hat{n} = |\nabla\phi|\;.\;|\hat{n}|\,cos\theta$.

Where θ is the angle between $\nabla\phi$ and $\hat{n}$. Since $|\hat{n}| = 1$, we have

$$\nabla\phi\,.\,\hat{n} = |\nabla\phi|\;\; cos\theta.$$

When $\theta = 0$, $cos\theta$ has the maximum value equal to 1. If $\theta = 0$ then $\nabla\phi$ coincides with $\hat{n}$ or we can say that $\nabla\phi$ will be along $\hat{n}$.

$\therefore$ the directional derivative is maximum along $\nabla\phi$ and its maximum value is equal to $|\nabla\phi|$.

1. Find $\boldsymbol{div\,\vec{F}}$ and $\boldsymbol{curl\,\vec{F}}$ where $\vec{F} = \nabla\left(x^3 + y^3 + z^3 - 3xyz\right)$**.**

Solution: Let $\phi = x^3 + y^3 + z^3 - 3xyz$.

$$\vec{F} = grad\ \phi = \nabla\phi = \frac{\partial\phi}{\partial x}\ \boldsymbol{i} + \frac{\partial\phi}{\partial y}\ \boldsymbol{j} + \frac{\partial\phi}{\partial z}\ \boldsymbol{k}$$

$$= \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x}}(x^3 + y^3 + z^3 - 3xyz)\boldsymbol{i} + \frac{\boldsymbol{\partial}}{\boldsymbol{\partial y}}(x^3 + y^3 + z^3 - 3xyz)\boldsymbol{j}$$

$$+ \frac{\boldsymbol{\partial}}{\boldsymbol{\partial z}}(x^3 + y^3 + z^3 - 3xyz)\boldsymbol{k}$$

$$\vec{F} = (3x^2 - 3yz)i + (3y^2 - 3xz)j + (3z^2 - 3xy)k$$

$$div\vec{F} = \nabla.\ \vec{F} = \left(\frac{\partial}{\partial x}i\ + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right).\{(3x^2 - 3yz)i + (3y^2 - 3xz)j + (3z^2 - 3xy)k\}$$

$$= \frac{\partial}{\partial x}(3x^2 - 3yz)\ + \frac{\partial}{\partial y}(3y^2 - 3xz) + \frac{\partial}{\partial z}(3z^2 - 3xy)$$

Thus $div\vec{F} = 6x + 6y + 6z = 6(x + y + z)$.

Also, $curl\ \vec{F}\ \ = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \vec{f_1} & \vec{f_2} & \vec{f_3} \end{vmatrix} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3xy \end{vmatrix}$

$$= i\left\{\frac{\partial}{\partial y}(3z^2 - 3xy) - \frac{\partial}{\partial z}(3y^2 - 3xz)\right\} - j\left\{\frac{\partial}{\partial x}(3z^2 - 3xy) - \frac{\partial}{\partial z}(3x^2 - 3yz)\right\}$$

$$+k\left\{\frac{\partial}{\partial x}(3y^2 - 3xz) - \frac{\partial}{\partial y}(3x^2 - 3yz)\right\}.$$

$$= i\big(-3x - (-3x)\big) - j\big(-3y - (-3y)\big) + k\big(-3z - (-3z)\big) = 0.$$

2. Find the directional derivative of the equation $\boldsymbol{\phi = x^2yz + 4xz^2}$ at $(\mathbf{1}, -\mathbf{2}, -\mathbf{1})$ along $\mathbf{2}\boldsymbol{i} - \boldsymbol{j} - \mathbf{2}\boldsymbol{k}$.

Solution: Given $\phi = x^2yz + 4xz^2$,

We have $\nabla\phi = \left(\frac{\partial}{\partial x}\ \boldsymbol{i} + \frac{\partial}{\partial y}\ \boldsymbol{j} + \frac{\partial}{\partial z}\ \boldsymbol{k}\right)\phi = \frac{\partial\phi}{\partial x}\ \boldsymbol{i} + \frac{\partial\phi}{\partial y}\ \boldsymbol{j} + \frac{\partial\phi}{\partial z}\ \boldsymbol{k}$

$$= \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x}}(x^2yz + 4xz^2)\boldsymbol{i} + \frac{\boldsymbol{\partial}}{\boldsymbol{\partial y}}(x^2yz + 4xz^2)\boldsymbol{j} + \frac{\boldsymbol{\partial}}{\boldsymbol{\partial z}}(x^2yz + 4xz^2)\boldsymbol{k}$$

$$\Longrightarrow \nabla\phi = (2xyz + 4z^2)i + x^2z\ j + (x^2y + 8xyz)k$$

$$\therefore \nabla\phi_{(1,-2,-1)} = 8i - j + 14k.$$

The unit vector in the direction of $2i - j - 2k$ is

$$\hat{n} = \frac{2i - j - 2k}{\sqrt{4+1+4}} = \frac{2i - j - 2k}{3}.$$

Hence the required directional derivative is

$$\nabla\phi.\hat{n} = (8i - j + 14k)\,.\frac{2i - j - 2k}{3} = 8\,.\,\frac{2}{3} + (-1).\left(-\frac{1}{3}\right) + (14)\,.\left(-\frac{2}{3}\right) = -\frac{11}{3}.$$

3. Find the directional derivatives of $\boldsymbol{\phi = 4xz^3 - 3x^2y^2z}$ at $(\mathbf{2}, -\mathbf{1}, \mathbf{2})$ along

$\boldsymbol{2i - 3j - 6k}$.

Solution: Given $\phi = 4xz^3 - 3x^2y^2z$.

$$\Longrightarrow \nabla\phi = \frac{\partial\phi}{\partial x}\,\boldsymbol{i} + \frac{\partial\phi}{\partial y}\,\boldsymbol{j} + \frac{\partial\phi}{\partial z}\,\boldsymbol{k}.$$

$$\nabla\phi = (4z^3 - 6xy^2z)i - (6x^2yz)j + (12xz^2 - 3x^2y^2)k$$

$$\Longrightarrow \nabla\phi_{(2,\ -1,\ 2)} = 8i + 48j + 84k.$$

The unit vector in the direction of $2i - 3j - 6k$ is

$$\hat{n} = \frac{2i - 3j + 6k}{\sqrt{4+9+36}} = \frac{2i - 3j + 6k}{7}.$$

Hence, the required directional derivative is

$$\nabla\phi\,.\,\hat{n} = (8i + 48j + 84k)\,.\frac{2i - 3j + 6k}{7} = \frac{376}{7}.$$

4. Find the directional derivative of the function $\boldsymbol{xyz}$ along the direction of the normal to the surface $\boldsymbol{xy^2 + yz^2 + zx^2 = 3}$ at the point $(\mathbf{1}, \mathbf{1}, \mathbf{1})$. Also find the equation of the tangent plane and the normal line to this surface.

Solution: Let $\phi = xyz$ and $\psi = xy^2 + yz^2 + zx^2 - 3$, so that we have

$$\nabla\phi = \frac{\partial\phi}{\partial x}\,i + \frac{\partial\phi}{\partial y}\,j + \frac{\partial\phi}{\partial z}\,k = yz\,i + xz\,j + xy\,k$$

Hence $\qquad \nabla\phi_{(1,1,1)} = i + j + k.$

Further, $\nabla\psi = \frac{\partial\psi}{\partial x}\,i + \frac{\partial\psi}{\partial y}\,j + \frac{\partial\psi}{\partial z}\,k = (y^2 + 2xz)i + (2xy + z^2)j + (2yz + x^2)k.$

Hence $\nabla\psi_{(1,1,1)} = 3i + 3j + 3k$ is the normal to the given surface at $(1, 1, 1)$.

The unit vector along $3i + 3j + 3k$ is

$$\hat{n} = \frac{3i + 3j + 3k}{\sqrt{9+9+9}} = \frac{i + j + k}{\sqrt{3}}.$$

Hence the required directional derivative of ϕ along the normal to the given surface is

$$\nabla\phi \,.\, \hat{n} = (i + j + k)\,.\,\frac{i + j + k}{\sqrt{3}} = \sqrt{3}.$$

Also, the equation of the tangent plane is

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$$

and the equation of the normal line is

$$\frac{x - x_1}{A} = \frac{y - y_1}{B} = \frac{z - z_1}{C}$$

where $A = \frac{\partial\psi}{\partial x}, B = \frac{\partial\psi}{\partial y}, C = \frac{\partial\psi}{\partial z}$ at $(1,1,1)$.

Thus, we have the tangent plane is

$3(x - 1) + 3(y - 1) + 3(z - 1) = 0$ or $x + y + z = 3$.

The equation of the normal line is

$$\frac{x - 1}{3} = \frac{y - 1}{3} = \frac{z - 1}{3}$$

5. In which direction the directional derivative of $\boldsymbol{x^2yz^3}$ is maximum at $(\mathbf{2}, \mathbf{1}, -\mathbf{1})$ and find the magnitude of the maximum.

Solution: We know that the directional derivative is maximum along the normal vector which being $\nabla\phi$

Let $\phi = x^2yz^3$ so that we have

$$\nabla\phi = \frac{\partial\phi}{\partial x}\, i + \frac{\partial\phi}{\partial y}\, j + \frac{\partial\phi}{\partial z}\, k = 2xyz^3\, i + x^2z^3 j + 3x^2yz^2 k.$$

$\nabla\phi_{(2,1,-1)} = -4i - 4j + 12k$ which is the required direction in which the directional derivative is maximum. The magnitude of this is given by $\sqrt{16 + 16 + 144} = 4\sqrt{11}$.

6. If the directional derivative of $\boldsymbol{\phi = axy^2 + byz + cx^3z^2}$ at $(-\mathbf{1}, \mathbf{1}, \mathbf{2})$ has a maximum magnitude of 32 units in the direction parallel to $\boldsymbol{y}$ – axis. Find $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$.

Solution: Given that $\nabla\phi \,.\, \hat{n} = 32$ at $(-1, 1, 2)$.

Then $\nabla\phi = (ay^2 + 3cx^2z^2)i + (2axy + bz)j + (by + 2cx^3z)k$

$$\nabla\phi_{(-1,1,2)} = (a + 12c)i + (-2a + 2b)j + (b - 4c)k.$$

Now $\qquad \nabla\phi_{(-1,1,2)}.j = -2a + 2b = 32$ (from the given data).

$$\therefore\ -a + b = 16.$$

Also, since $\nabla\phi$ is parallel to y −axis, we must have $a + 12c = 0$ and $b - 4c = 0$

On using these $-a + b = 16, a + 12c = 0, b - 4c = 0.$

On solving $a = -12, b = 4, c = 1.$

7. In which direction the direction the directional derivative of x^2yz^3 is maximum at $(2, 1, -1)$ and find the maximum of this maximum.

Solution: We know that the directional derivative is maximum along the normal vector which being $\nabla\phi$

Let $\qquad \phi = x^2yz^3.$

Then

$$\nabla\phi = 2xyz^3i + x^2z^3j + 3x^2yz^2k$$

Further,

$$\nabla\phi_{(2,1,-1)} = -4i - 4j + 12k$$

is the required direction in which the directional derivative is maximum. The magnitude of this is $\sqrt{16 + 16 + 144} = 4\sqrt{11}.$

8. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$.

Solution: The angle between the surfaces is defined to be equal to the angle between their normal and we know that $\nabla\phi$ is a vector normal to the surface. Let the equation of the given two surfaces are

$$\phi_1 = x^2 + y^2 + z^2 - 9 \text{ and } \phi_2 = x^2 + y^2 - z - 3.$$

We have

$$\nabla\phi = \frac{\partial\phi}{\partial x}\, i + \frac{\partial\phi}{\partial y}\, j + \frac{\partial\phi}{\partial z}\, k$$

Hence, $\nabla\phi_1 = 2xi + 2yj + 2zk$ and $\nabla\phi_2 = 2xi + 2yj - k.$

$\therefore \nabla\phi_{1(2,-1,2)} = 4i - 2j + 4k$ and $\nabla\phi_{2(2,-1,2)} = 4i - 2j - k$.

If θ is the angle between these two normal, we have

$$\cos\theta = \frac{\nabla\phi_1.\nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|} = \frac{16+4-4}{\sqrt{16+4+16}\sqrt{16+4+1}} = \frac{8}{3\sqrt{21}}.$$

$$\Rightarrow \theta = Cos^{-1}\left(\frac{8}{3\sqrt{21}}\right).$$

9. Find a unit vector normal to the surface $xy^3z^2 = 4$ at the point $(-1, 1, 2)$.

Solution: Consider $(x, y, z) = xy^3z^2 - 4$.

Then $\nabla f = (y^3z^2)i + (3xy^2z^2)j + (2xy^3z)k$.

Therefore $\nabla f\ at\ (-1,1,2) = 4i - 12j - 4k$.

A unit normal to the surface is $= \frac{\nabla f}{|\nabla f|} = \frac{4i-12j-4k}{\sqrt{4^2+(-12)^2+(-4)^2}} = \frac{4i-12j-4k}{\sqrt{176}}$.

10. Find the equation of the tangent plane to the surface $z = x^2 - y^2$ at the point $(2, -1, 6)$.

Solution: Let P be any point on the tangent plane at $A(2, -1,6)$ and let the position vector of P and A be r_P and r_A.

i.e, $r_P = ix + iy + kz$ and $r_A = 2i - j + 6k$.

Hence $AP = r_P - r_A = i(x-2) + j(y+1) + k(z-6)$.

Also ∇f at $(2, -1,6)$ is $4i + 2j - k$.

Hence the equation of the tangent plane is $AP.\nabla f = 0$.

i.e, $\{i(x-2) + j(y+1) + k(z-6)\}.\{4i + 2j - k\} = 0$

i.e., $4x + 2y - z = 0$.

11. Find the equation of the normal line to the surface $xyz = 4$ at the point $(1, 2, 2)$.

Solution: Here $r_P = xi + yj + zk$ and $r_A = i + 2j + 2k$.

Hence $AP = r_P - r_A = i(x-1) + j(y-2) + k(z-2)$.

Also $[\nabla f]_{(1,2,2)} = 4i + 2j + 2k$.

The equation of the normal line is represented by $(r_P - r_A) \times \nabla f = 0$.

i.e., $\begin{vmatrix} i & j & k \\ x-1 & y-2 & z-2 \\ 4 & 2 & 2 \end{vmatrix} = 0$

i.e.,

$$i\{2(y-2)-2(z-2)\}+j\{4(z-2)-2(x-1)\}+k\{2(x-1)-4(y-2)\}=0.$$

This equation is satisfied provided the coefficients of i, j, k are zero simultaneously.

i.e., $2(y-2)-2(z-2)=0, 4(z-2)-2(x-1)=0$ and $2(x-1)-4(y-2)=0$.

Solving, we get the equation of the normal line as

$$\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-2}{1}.$$

12. If $\vec{F}=\nabla(xy^3z^2)$ find $div\,\vec{F}$ and $curl\,\vec{F}$ at the point $(1,-1,1)$.

Solution: Let $\phi = xy^3z^2$.

$$\therefore \vec{F}=\nabla\phi=\frac{\partial\phi}{\partial x}\,i+\frac{\partial\phi}{\partial y}\,j+\frac{\partial\phi}{\partial z}\,k=y^3z^2i+3xy^2z^2j+2xy^3z\,k.$$

Hence $div\,\vec{F}=\nabla\,.\,\vec{F}=\left(\frac{\partial}{\partial x}\,i+\frac{\partial}{\partial y}\,j+\frac{\partial}{\partial z}\,k\right).(y^3z^2i+3xy^2z^2j+2xy^3z\,k)$

$$=\frac{\partial}{\partial x}(y^3z^2)+\frac{\partial}{\partial y}(3xy^2z^2)+\frac{\partial}{\partial z}(2xy^3z)$$

$$=0+6xyz^2+2xy^3=2xy(3z^2+2y^2).$$

Finally, $div\,\vec{F}$ at $(1,-1,1)=-8$.

Also, $curl\,\vec{F}=\nabla\times\vec{F}=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3z^2 & 3xy^2z^2 & 2xy^3z \end{vmatrix}$

$$=i(6xy^2z-6xy^2z)-j(2y^3z-2y^3z)+k(3y^2z^2-3y^2z^2)$$

$$\therefore curl\,\vec{F}=0.$$

13. If $\vec{F}=(3x^2y-z)i+(xz^3+y^4)j-2x^3z^2k$ find $grad(div\,\vec{F})$ at $(2,-1,0)$.

Solution: $div\,\vec{F}=\nabla\,.\vec{F}=\left(\frac{\partial}{\partial x}\,i+\frac{\partial}{\partial y}\,j+\frac{\partial}{\partial z}\,k\right).\left((3x^2y-z)i+(xz^3+y^4)j-2x^3z^2k\right)$

$$=\frac{\partial}{\partial x}(3x^2y-z)+\frac{\partial}{\partial y}(xz^3+y^4)+\frac{\partial}{\partial z}(-2x^3z^2)$$

$\nabla\,.\vec{F}=6xy+4y^3-4x^3z=\phi\ (say).$

Now $grad\left(div\,\vec{F}\right) = grad\ \phi = \nabla\,.\ \phi$.

Hence $grad\left(div\,\vec{F}\right) = \left(\frac{\partial}{\partial x}\,\boldsymbol{i} + \frac{\partial}{\partial y}\,\boldsymbol{j} + \frac{\partial}{\partial z}\,\boldsymbol{k}\right).(6xy + 4y^3 - 4x^3z)$.

$$grad\left(div\,\vec{F}\right) = (6y - 12x^2z)i + (6x + 12y^2)j + (-4x^3)k.$$

Finally, $grad\left(div\,\vec{F}\right)_{(2,-1,0)} = -6i + 24j - 32k$.

14. Find $\boldsymbol{curl\left(curl\,\vec{A}\right)}$ given that $\boldsymbol{\vec{A} = xy\,i + y^2z\,j + yz^2k}$.

Solution: $Curl\,\vec{A} = \nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & y^2z & yz^2 \end{vmatrix}$

$= i(z^2 - y^2) - j(0 - 0) + k(0 - x) = (z^2 - y^2)i - xk.$

Now $curl\left(curl\,\vec{A}\right) = \nabla \times \left(\nabla \times \vec{A}\right)$

$$= \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z^2 - y^2 & 0 & -x \end{vmatrix} = i(0 - 0) - j(-1 - 2z) + k(0 + 2y)$$

$$\therefore curl\left(curl\,\vec{A}\right) = (1 + 2z)j + 2yk.$$

15. If $\boldsymbol{\vec{r} = xi + yj + zk}$ and $\boldsymbol{r = |\vec{r}|}$ then prove that $\boldsymbol{\nabla(r^n) = nr^{n-2}\,\vec{r}}$.

Solution: $r = \overrightarrow{|r|} = \sqrt{x^2 + y^2 + z^2} \implies r^2 = x^2 + y^2 + z^2$.

Now on differentiating the above partially w. r. t. $x, y, and\ z$ we have

$$2r\frac{\partial r}{\partial x} = 2x, 2r\frac{\partial r}{\partial y} = 2y, 2r\frac{\partial r}{\partial z} = 2z.$$

Or

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}.$$

Hence $\nabla(r^n) = \left(\sum \frac{\partial}{\partial x} i\right)(r^n)$

$$= \sum nr^{n-1}\frac{\partial r}{\partial x}\ i = \sum nr^{n-1}\left(\frac{x}{r}\right)\ i$$

$$= \sum nr^{n-2}x\ i = nr^{n-2}\sum x\ i = nr^{n-2}\,\vec{r}.$$

Thus $\nabla(r^n) = grad(r^n) = nr^{n-2}\,\vec{r}$.

16. If $\vec{r} = xi + yj + zk$ and $r = |\vec{r}|$ then prove that $\nabla\,.\,(r^n\,\vec{r}) = (n+3)r^n$.

Solution: We have $r^n\vec{r} = r^n \sum x\, i = \sum(r^n x)\, i$.

$$\therefore \nabla\,.\,(r^n\vec{r}) = \left(\sum \frac{\partial}{\partial x}\, i\right)\,.\sum(r^n x)\, i$$

$$= \sum \frac{\partial}{\partial x}(r^n x) = \sum\left(r^n + nr^{n-1}\frac{\partial r}{\partial x}\; x\right)$$

$$= \sum\left(r^n + nr^{n-1}\frac{x}{r}\; x\right) = \sum(r^n + nr^{n-2}\; x^2)$$

On expanding the summation, we obtain

$$\nabla\,.\,(r^n\vec{r}) = (r^n + nr^{n-2}x^2) + (r^n + nr^{n-2}y^2) + (r^n + nr^{n-2}z^2)$$

$$= 3r^n + nr^{n-2}\,(x^2+y^2+z^2) = 3r^n + nr^{n-2}r^2 = 3r^n + nr^n.$$

$$\therefore \nabla\,.\,(r^n\vec{r}) = div(r^n.\vec{r}) = (n+3)r^n.$$

17. If $\vec{r} = xi + yj + zk$ and $r = |\vec{r}|$ prove that $\nabla \times (r^n\vec{r}) = 0$.

Solution: We have $r^n\vec{r} = r^n\sum x\, i = \sum(r^n x)\, i = r^n x\, i + r^n y\, j + r^n z\, k$.

$$\therefore \nabla \times (r^n\vec{r}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$$= \sum i\left(\frac{\partial}{\partial y}(r^n z) - \frac{\partial}{\partial z}(r^n y)\right) = \sum i\left(nr^{n-1}\frac{\partial r}{\partial y}z - nr^{n-1}\,\frac{\partial r}{\partial z}y\right)$$

$$= \sum i\left(nr^{n-1}\frac{y}{r}z - nr^{n-1}\,\frac{z}{r}y\right) = \sum i(nr^{n-2}yz - nr^{n-2}\,yz) = \vec{0}.$$

Thus $\nabla \times (r^n\vec{r}) = curl\,(r^n\vec{r}) = 0$.

18. If $\vec{r} = xi + yj + zk \quad and\; r = \vec{r}$ **then prove that** $\nabla^2(r^n) = n\,(n+1)r^{n-2}$

Solution: $\nabla^2(r^n) = \sum \frac{\partial^2}{\partial x^2}(r^n) = \sum \frac{\partial}{\partial x}\frac{\partial}{\partial x}(r^n)$

$$=\sum \frac{\partial}{\partial x}\left(nr^{n-1}\frac{\partial r}{\partial x}\right) = \sum \frac{\partial}{\partial x}\left(nr^{n-1}\left(\frac{x}{r}\right)\right)$$

$$= \sum \frac{\partial}{\partial x}(nr^{n-2}x) = n\sum\left(r^{n-2} + (n-2)r^{n-3}\left(\frac{x}{r}\right)x\right)$$

$$= n\sum(r^{n-2} + (n-2)r^{n-4}x^2)$$

On expanding the summation we get

$$= n\{(r^{n-2} + (n-2)r^{n-4}x^2) + (r^{n-2} + (n-2)r^{n-4}y^2) + (r^{n-2} + (n-2)r^{n-4}z^2)\}$$

$$= n\{3r^{n-2} + (n-2)r^{n-4}(x^2 + y^2 + z^2)\}$$

$$= n\{3r^{n-2} + (n-2)r^{n-4}(r^2)\}$$

$$= n\{3r^{n-2} + (n-2)r^{n-2}\}$$

$$\therefore \nabla^2(r^n) = n(n+1)r^{n-2}$$

19. Show that $\vec{F} = (2xy^2 + yz)i + (2x^2y + xz + 2z^2y)j + (2zy^2 + xy)k$ is a conservative force field. Find its scalar potential

Solution: We have to show that $curl\,\overrightarrow{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (2xy^2 + yz) & (2x^2y + xz + 2z^2y) & (2zy^2 + xy) \end{vmatrix}$$

$$= \quad i(4yz + x - x - 4yz) - j(y - y) + k(4xy + z - 4xy - z) = 0$$

$\Longrightarrow \vec{F}$ is a conservative

Now we have find $\emptyset\ such\ that\ \nabla\emptyset = \vec{F}$

$$\frac{\partial\phi}{\partial x}\ i + \frac{\partial\phi}{\partial y}\ j + \frac{\partial\phi}{\partial z}\ k = (2xy^2 + yz)i + (2x^2y + xz + 2z^2y)j + (2zy^2 + xy)k$$

$$\Rightarrow \frac{\partial\phi}{\partial x} = 2xy^2 + yz \Longrightarrow \emptyset = \int(2xy^2 + yz)dx + f_1(y,z) \Longrightarrow \emptyset = x^2y^2 + xyz + f_1(y,z)$$

$$\Rightarrow \frac{\partial\phi}{\partial y} = (2x^2y + xz + 2z^2y) \Longrightarrow \emptyset = \int(2x^2y + xz + 2z^2y)dy + f_2(x,z)$$

$$\Longrightarrow \emptyset = x^2y^2 + xyz + z^2y^2 + f_2(x,z)$$

$$\Rightarrow \frac{\partial\phi}{\partial z} = (2zy^2 + xy) \Longrightarrow \emptyset = \int(2zy^2 + xy)dz + f_3(x,y)$$

$$\Longrightarrow \emptyset = z^2y^2 + xyz + f_3(x,y)$$

Let us choose $f_1(y,z) = z^2y^2, f_2(x,z) = 0, f_3(x,y) = x^2y^2$

Thus $\emptyset = x^2y^2 + xyz + z^2y^2$ be the required scalar potential.

20. Find the value of the constant 'a' such that $\vec{F} = (axy - z^3)i + (\,a - 2)x^2j\ +$ $(1 - a)xz^2k$ is irrotational and hence find a scalar function such that $\nabla\emptyset = \vec{F}$.

Solution: We have to show that $curl\vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (axy - z^3) & (a-2)x^2 & (1-a)xz^2 \end{vmatrix}$$

$$= \quad i(0-0) - j\{(1-a)z^2 + 3z^2\} + k\{(a-2)2x - ax\} = \vec{0}$$

$$(a-4)z^2 + (a-4)xk = \vec{0}$$

The above equation is identically satisfied when a = 4

Now consider $\nabla\emptyset = \left(\vec{F}\right)_{a=4}$

$$\frac{\partial\phi}{\partial x}\, i + \frac{\partial\phi}{\partial y}\, j + \frac{\partial\phi}{\partial z}\, k = (4xy - z^3)i + 2x^2 j - 3xz^2 k$$

$$\Rightarrow \frac{\partial\phi}{\partial x} = 4xy - z^3 \Longrightarrow \emptyset = \int (4xy - z^3)dx \;\; + f_1(y,z) \;\Longrightarrow \emptyset = 2x^2y - x\,z^3 + f_1(y,z)$$

$$\Rightarrow \frac{\partial\phi}{\partial y} = \boldsymbol{2x^2} \Longrightarrow \emptyset = \int \boldsymbol{2x^2}dy \;\; + f_2(x,z) \;\Longrightarrow \emptyset = 2x^2y + f_2(x,z)$$

$$\Rightarrow \frac{\partial\phi}{\partial z} = -3xz^2 \Longrightarrow \emptyset = \int -3xz^2\; dz + f_3(x,y) \;\Longrightarrow \emptyset = -xz^3 + f_3(x,y)$$

Let us choose$f_1(y,z) = 0, f_2(x,z) = -xz^3, f_3(x,y) = 2x^2y$

Thus $\emptyset = 2x^2y - xz^3$ be the required scalar potential

21. **If** $\vec{F} = (x+y+az)i + (\,bx + 2y - z)j \; + (x + cy + 2z)k$ **find a, b, c such that** $curl\vec{F} = 0$ **and then find** $\emptyset$ **such that** $\nabla\emptyset = \vec{F}$

Solution: We have to show that $curl\vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+y+az) & (\,bx+2y-z) & (x+cy+2z) \end{vmatrix}$$

$$= \quad i(c+1) - j(1-a) + (b-1) = \vec{0}$$

The above equation is identically satisfied when a = 1, b =1, c = -1

Now consider $\nabla\emptyset = \vec{F}$ when a = 1, b =1, c = -1

$$\frac{\partial\phi}{\partial x}\, i + \frac{\partial\phi}{\partial y}\, j + \frac{\partial\phi}{\partial z}\, k = (x+y+z)i + (\,x + 2y - z)j \; + (x - y + 2z)k$$

$\Rightarrow \frac{\partial\phi}{\partial x} = (x+y+z) \Longrightarrow \emptyset = \int (x+y+z)dx \;\; + f_1(y,z) \;\Longrightarrow \emptyset = \frac{x^2}{2} + xy + \; x\,z +$

$f_1(y,z)$

$\Rightarrow \frac{\partial\phi}{\partial y} = (\, x+2y-z) \Longrightarrow \emptyset = \int (\, x+2y-z)dy \;\;\; + f_2(x,z) \;\Longrightarrow \emptyset = \frac{x^2}{2} +$

$y^2 - yz + f_2(x,z)$

$$\Rightarrow \frac{\partial\phi}{\partial z} = (x-y+2z) \Longrightarrow \emptyset$$

$$= \int (x-y+2z)dz + f_3(x,y) \;\Longrightarrow \emptyset = xz - yz + z^2 + f_3(x,y)$$

Let us choose $f_1(y,z) = y^2 - yz + z^2, f_2(x,z) = \frac{x^2}{2} + xz + z^2,\; f_3(x,y) = \frac{x^2}{2} + xy + y^2$

Thus $\emptyset = \frac{x^2}{2} + xz + xy + y^2 - yz + z^2$ be the required scalar potential

22. Prove that $div\big(curl\vec{A}\big) = 0,\; ie\; \nabla.\big(\nabla \times \vec{A}\big) = 0$

Proof: Let $\vec{A} = a_1 i \; + a_2 j + \;\; a_3 k$ be a vector point function of x, y, z

$$curl\vec{A} = \nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix}$$

Now $div(curl\vec{A}) = \nabla.(\nabla \times \vec{A})$

$$= \sum \frac{\partial}{\partial X} i.\sum i \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right) = \sum \left(\frac{\partial^2 a_3}{\partial x \partial y} - \frac{\partial^2 a_2}{\partial x \partial z}\right)$$

On expanding we get,

$$\left(\frac{\partial^2 a_3}{\partial x \partial y} - \frac{\partial^2 a_2}{\partial x \partial z}\right) + \left(\frac{\partial^2 a_1}{\partial y \partial z} - \frac{\partial^2 a_3}{\partial y \partial x}\right) + \left(\frac{\partial^2 2}{\partial z \partial x} - \frac{\partial^2 a_1}{\partial z \partial y}\right) = 0$$

Thus, $div\big(curl\vec{A}\big) = 0$, for any vector function $\vec{A}$

23. Prove that $curl\big(curl\vec{A}\big) = grad\big(div\vec{A}\big) - \nabla^2\vec{A}$

ie., $\nabla \times \big(\nabla \times \vec{A}\big) = \nabla\big(\nabla \cdot \vec{A}\big) - \nabla^2\vec{A}$

Proof : Let $\vec{A} = a_1 i \; + a_2 j + \;\; a_3 k$ be a vector point function of x, y, z

$$curl\vec{A} = \nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix} = \sum i \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z} \right)$$

Now $curl\left(curl\vec{A}\right) = \nabla \times \left(\nabla \times \vec{A}\right)$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right) & \left(\frac{\partial a_1}{\partial z} - \frac{\partial a_3}{\partial x}\right) & \left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y}\right) \end{vmatrix}$$

$$= \sum i \left\{ \frac{\partial}{\partial y} \left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y} \right) - \frac{\partial}{\partial z} \left(\frac{\partial a_1}{\partial z} - \frac{\partial a_3}{\partial x} \right) \right\}$$

$= \sum i \left(\frac{\partial^2 a_2}{\partial x \partial y} + \frac{\partial^2 a_3}{\partial z \partial x} \right) - \sum i \left(\frac{\partial^2 a_1}{\partial y^2} + \frac{\partial^2 a_1}{\partial z^2} \right)$, by rearranging

(In order to get ∇^2 in the second term and observing that we do not have the second order partial derivative w.r.t x we must think of adding and subtracting the same.)

Adding and subtracting $\sum i \frac{\partial^2 a_1}{\partial x^2}$ we get

$$= \sum i \left(\frac{\partial^2 a_1}{\partial x^2} + \frac{\partial^2 a_2}{\partial x \partial y} + \frac{\partial^2 a_3}{\partial z \partial x} \right) - \sum i \left(\frac{\partial^2 a_1}{\partial x^2} + \frac{\partial^2 a_1}{\partial y^2} + \frac{\partial^2 a_1}{\partial z^2} \right)$$

$$= \sum i \frac{\partial}{\partial x} \left(\frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z} \right) - \sum \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) a_1 \, i$$

$$= \sum i \frac{\partial}{\partial x} \left(div\vec{A}\right) - \nabla^2 \sum a_1 \, i = grad\left(div\vec{A}\right) - \nabla^2 \vec{A}$$

Thus $curl\left(curl\vec{A}\right) = grad\left(div\vec{A}\right) - \nabla^2 \vec{A}$

24. **Prove that** $div\left(\vec{A} \times \vec{B}\right) = \left(\vec{B} \cdot curl\vec{A}\right) - \left(\vec{A} \cdot curl\vec{B}\right)$

ie., $\nabla \cdot \left(\vec{A} \times \vec{B}\right) = \vec{B} \cdot \left(\nabla \times \vec{A}\right) - \vec{A} \cdot \left(\nabla \times \vec{B}\right)$

Proof: Let $\vec{A} = a_1 i + a_2 j + a_3 k$ and $\vec{B} = b_1 i + b_2 j + b_3 k$, be two vector point functions of x, y, z

$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \sum i (a_2 b_3 - a_3 b_2)$$

Now $div\left(\vec{A} \times \vec{B}\right) = \nabla \cdot \left(\vec{A} \times \vec{B}\right)$

$$= \left(\sum \frac{\partial}{\partial x} i\right) \cdot \sum i(a_2b_3 - a_3b_2)$$

$$= \sum \frac{\partial}{\partial x}(a_2b_3 - a_3b_2)$$

$$= \sum \left(a_2 \frac{\partial b_3}{\partial x} + b_3 \frac{\partial a_2}{\partial x} - a_3 \frac{\partial b_2}{\partial x} - b_2 \frac{\partial a_3}{\partial x}\right)$$

On expanding we get

$$\left(a_2 \frac{\partial b_3}{\partial x} + b_3 \frac{\partial a_2}{\partial x} - a_3 \frac{\partial b_2}{\partial x} - b_2 \frac{\partial a_3}{\partial x}\right) + \left(a_3 \frac{\partial b_1}{\partial y} + b_1 \frac{\partial a_3}{\partial y} - a_1 \frac{\partial b_3}{\partial y} - b_3 \frac{\partial a_1}{\partial y}\right)$$
$$+ \left(a_1 \frac{\partial b_2}{\partial z} + b_2 \frac{\partial a_1}{\partial z} - a_2 \frac{\partial b_1}{\partial z} - b_1 \frac{\partial a_2}{\partial z}\right)$$

ie., $\quad = \sum b_1 \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right) - \sum a_1 \left(\frac{\partial b_3}{\partial y} - \frac{\partial b_2}{\partial z}\right)$

$$= \left(\sum b_1 i\right) . \sum \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right) i - \left(\sum a_1 i\right) . \sum \left(\frac{\partial b_3}{\partial y} - \frac{\partial b_2}{\partial z}\right) i$$

$(\therefore \sum A_1 B_1 = \sum A_1\, i \,. \sum B_1\, i)$

$$= \left(\sum b_1 i\right). \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix} - \left(\sum a_1 i\right). \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$$

Thus $div(\vec{A} \times \vec{B}) = (\vec{B} \cdot curl\vec{A}) - (\vec{A} \cdot curl\vec{B})$

25. If $\vec{r} = xi + yj + zk$ and $\vec{r} = |\vec{r}|$, then prove that

(a) $\nabla \cdot (r^n \vec{r}) = (n+3)r^n$ **(b)** $\nabla \times (r^n \vec{r}) = 0$

Solution: (a) We have $\nabla \cdot (\emptyset \vec{A}) = \emptyset(\nabla \cdot \vec{A}) + \nabla\emptyset \cdot \vec{A}$

$\therefore \nabla \cdot (r^n \vec{r}) = r^n(\nabla \cdot \vec{r}) + \nabla r^n . \vec{r}$ ------------------------(1)

Now $\nabla \cdot \vec{r} = \left(\sum \frac{\partial}{\partial x} i\right) . (\sum x\, i) = 1 + 1 + 1 = 3$

Also $\therefore \nabla r^n = n r^{n-2} \vec{r}$

Using these in the R.H.S of (1) we get,

$\nabla \cdot (r^n \vec{r}) = 3r^n + nr^n = (3+n)r^n$

Thus $\nabla \cdot (r^n \vec{r}) = (n+3)r^n$

(b) We have $\nabla\times\left(\emptyset\,\vec{A}\right)=\emptyset\left(\nabla\times\vec{A}\right)+\nabla\emptyset\times\vec{A}$

$\therefore\ \nabla\times(r^n\,\vec{r})=r^n(\nabla\times\vec{r})+\nabla r^n\times\vec{r}$ ----------(2)

$$\nabla\times\vec{r}=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}=\vec{0}$$

Also $\nabla r^n\times\vec{r}=\vec{0}(\ nr^{n-2}\vec{r})\times\vec{r}=\ nr^{n-2}(\vec{r}\times\vec{r})=0$

Thus from (2), $\nabla\times(r^n\,\vec{r})=0$

26. **Prove that** $curl(grad\emptyset)=\vec{0}$ ie., $\nabla\times(\nabla\emptyset)=\vec{0}$

Proof: Let $\emptyset$ be a scalar point function of *x, y, z.*

$$grad\emptyset=\nabla\emptyset=\frac{\partial\emptyset}{\partial x}i+\frac{\partial\emptyset}{\partial y}j+\frac{\partial\emptyset}{\partial z}k$$

$$curl(grad\emptyset)=\nabla\times(\nabla\emptyset)=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\emptyset}{\partial x} & \frac{\partial\emptyset}{\partial y} & \frac{\partial\emptyset}{\partial z} \end{vmatrix}$$

i.e., $=\ \sum\frac{\partial}{\partial x}\left\{\frac{\partial}{\partial y}\left(\frac{\partial\emptyset}{\partial z}\right)-\frac{\partial}{\partial z}\left(\frac{\partial\emptyset}{\partial y}\right)\right\}i=\sum\left(\frac{\partial^2\emptyset}{\partial y\partial z}-\frac{\partial^2\emptyset}{\partial z\partial y}\right)i=\vec{0}$

Thus $curl(grad\emptyset)=\vec{0}$, for any scalar function $\emptyset$.