

Goertzel Algorithm

Dr. Sampath Kumar

Associate Professor

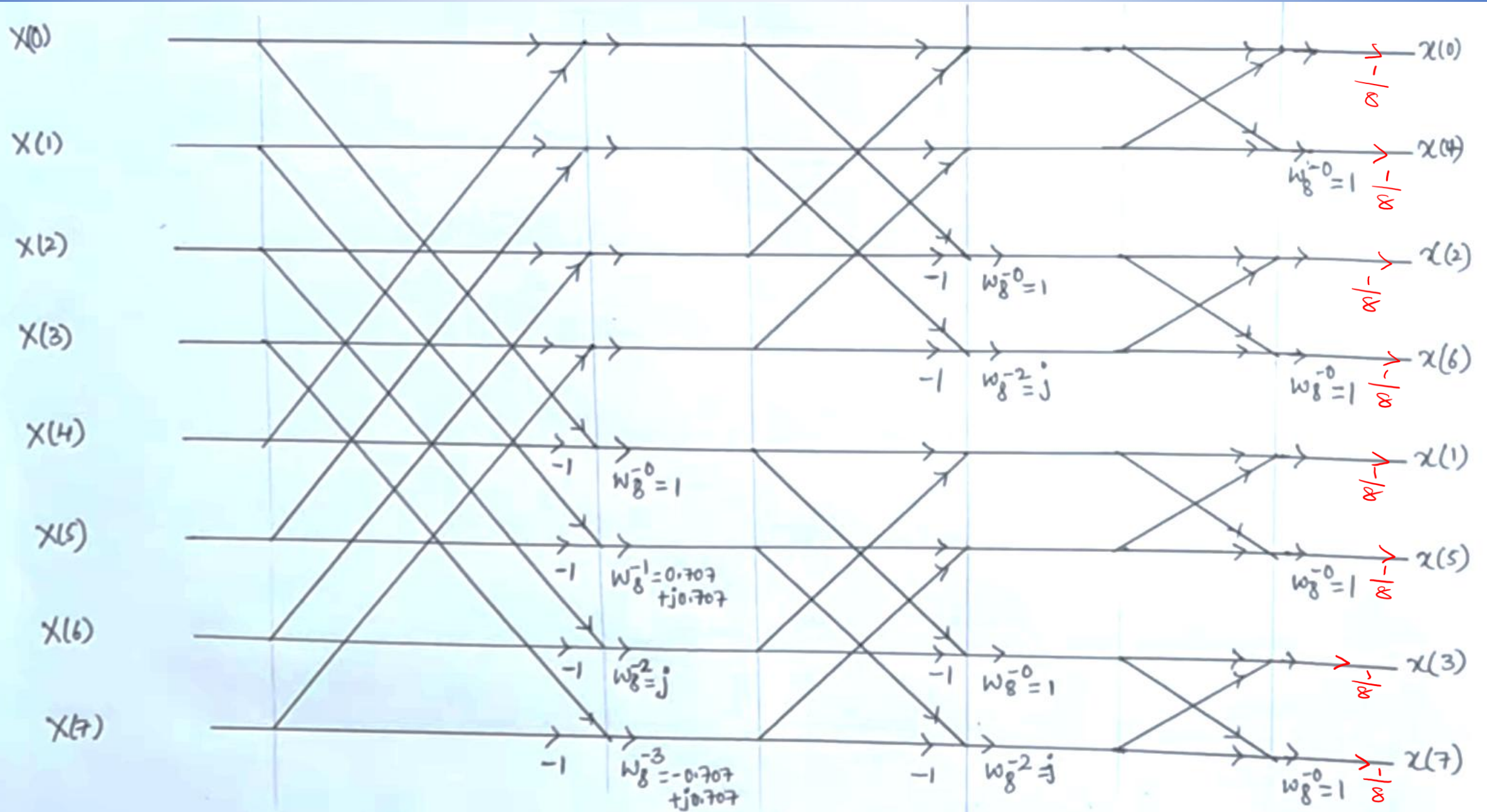
Department of ECE

MIT, Manipal

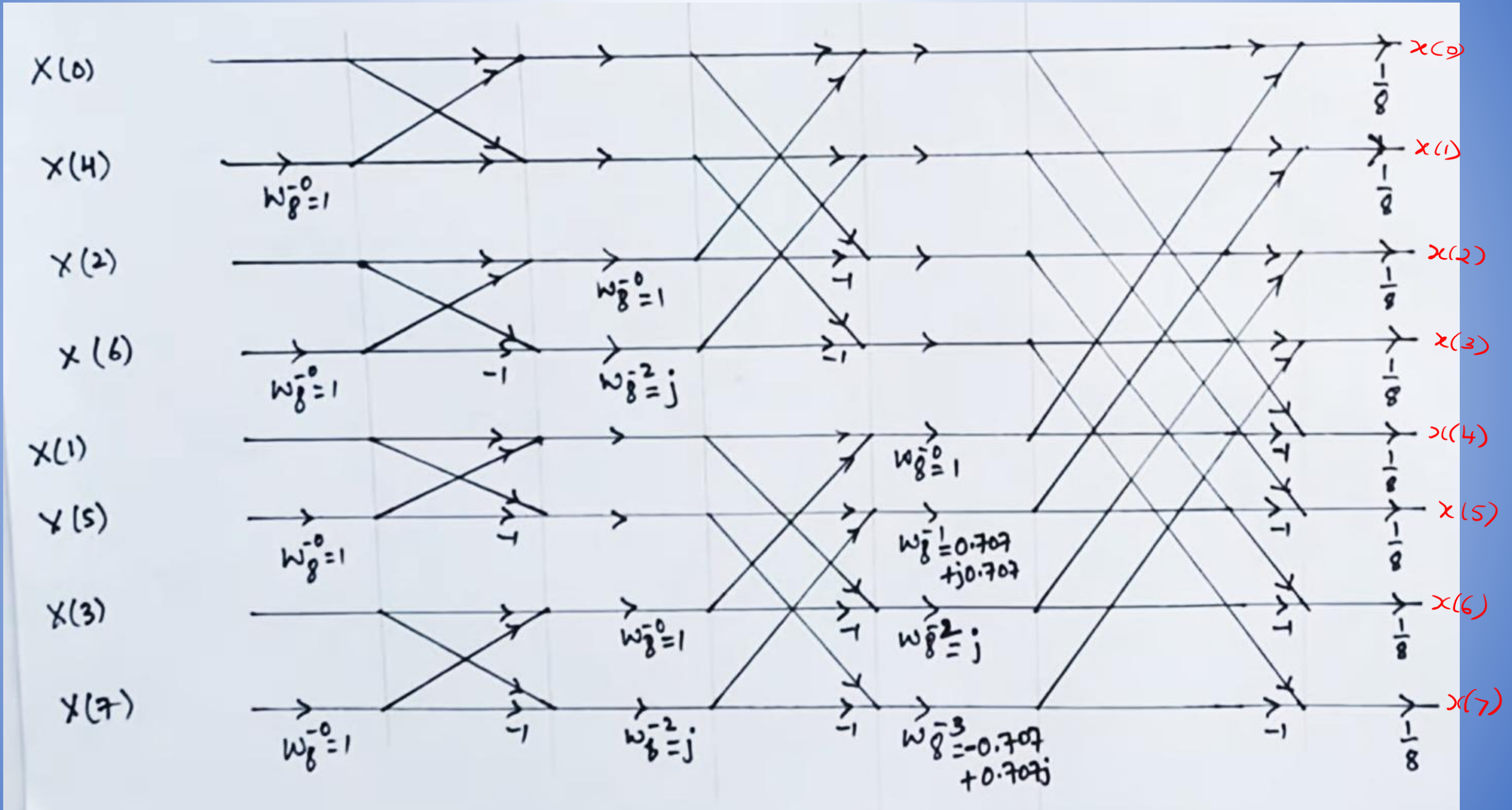
Inverse FFT

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-nk}$$

IDFT using DIT-FFT algorithm



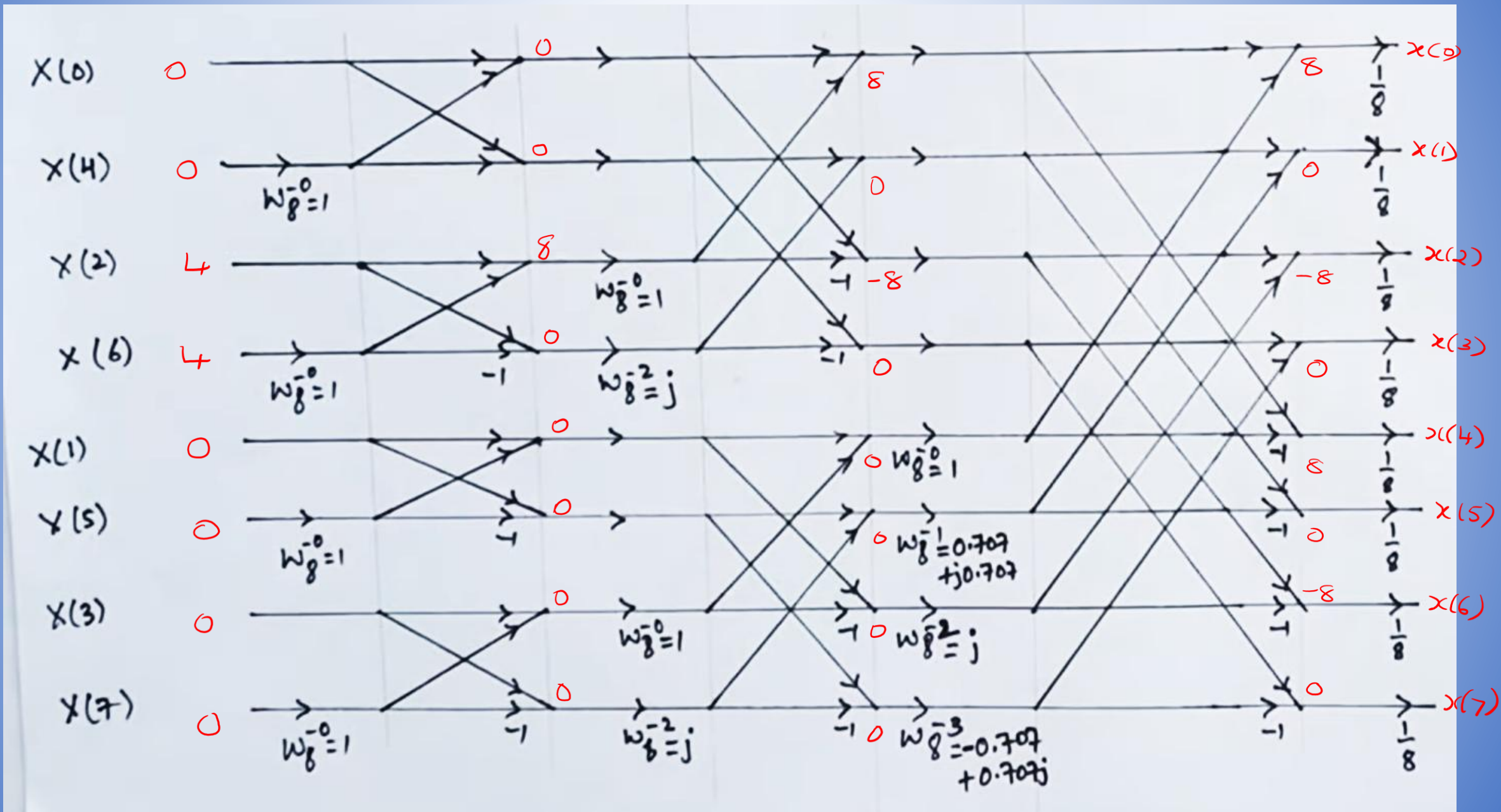
IDFT using DIF-FFT algorithm



$$X(k) = \{0, 0, 4, 0, 0, 0, 4, 0\}$$

IDFT using DIF-FFT algorithm

$$x(n) = \{1, 0, -1, 0, 1, 0, -1, 0\}$$



Linear filtering approach to computation of DFT

- Geortzel Algorithm

- Exploits the periodicity of the phase factors $\{W_N^k\}$
- DFT is computed as a linear convolution operation

We know that $W_N^{-Nk} = 1$. We multiply this to DFT expression and we get

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{nk} \cdot W_N^{-Nk} = \sum_{m=0}^{N-1} x(m) W_N^{-(N-m)k} \quad \text{--- ①}$$

Let us define a sequence $y_k(n)$ as follows:

$$y_k(n) = \sum_{m=-\infty}^{\infty} x(m) W_N^{-(n-m)k} \quad \text{--- ②}$$

The above equation looks as if $y_k(n)$ is a convolution of two sequences $x(n)$ and $W_N^{-kn} u(n)$.

If $x(n)$ is a finite length sequence taking values from 0 to $N-1$, then we can write

$$y_k(n) = \sum_{m=0}^{N-1} x(m) W_N^{-(n-m)k} \quad \text{--- (3)}$$

Rewriting eqn (1)

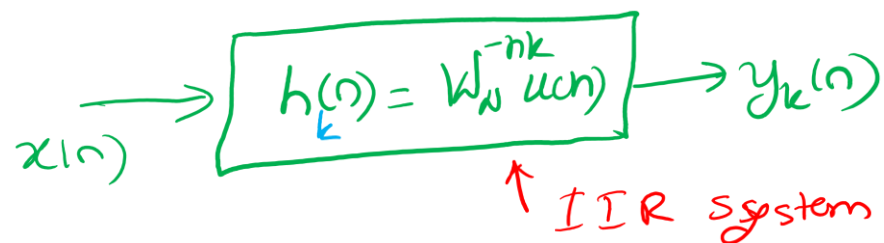
$$X(k) = \sum_{m=0}^{N-1} x(m) W_N^{-(N-m)k}$$

From equations (1) & (3) we can say that

$$X(k) = y_k(n) \Big|_{n=N} \quad \text{--- (4)}$$

We can think of $x(n)$ as an input to a linear time invariant system (filter) whose impulse response is $h_k(n) = W_N^{-nk} u(n)$

Then the output observed at time $n=N$ yields the value of the DFT at the frequency $\omega_k = \frac{2\pi k}{N}$



The system function for this filter can be defined as follows:

$$h_k(n) = W_N^{-nk} u(n) \Leftrightarrow H_k(z) = \frac{1}{1 - W_N^{-k} z^{-1}} \quad \text{--- (5)}$$

This filter has a pole on the unit circle at the frequency $\omega_k = \frac{2\pi k}{N}$

The entire DFT can be computed by passing the block of input data into a parallel bank of N single pole filters.

We can use the difference equation corresponding to the filter given by equation (5)

we have

$$H_k(z) = \frac{Y_k(z)}{X(z)} = \frac{1}{1 - W_N^{-k} z^{-1}}$$

$$Y_k(z) (1 - W_N^{-k} z^{-1}) = X(z)$$

or

$$y_k(n) - W_N^{-k} y_k(n-1) = x(n)$$

$$y_k(n) = W_N^{-k} y_k(n-1) + x(n)$$

Now we can realize the structure for equation (5) as follows:

$$H_k(z) = \frac{1}{1 - W_N^{-k} z^{-1}} \frac{(1 - W_N^k z^{-1})}{(1 - W_N^k z^{-1})} = \frac{1 - W_N^k z^{-1}}{1 - (W_N^k + W_N^{-k}) z^{-1} + z^{-2}}$$

$$H_k(z) = \frac{1 - W_N^k z^{-1}}{1 - 2 \cos\left(\frac{2\pi k}{N}\right) z^{-1} + z^{-2}} \quad \text{--- (6)}$$

$$\text{Let } H_k(z) = \frac{Y_k(z)}{X(z)} = \frac{1 - W_N^k z^{-1}}{1 - 2 \cos\left(\frac{2\pi k}{N}\right) z^{-1} + z^{-2}}$$

$$\text{Let } H_k(z) = \frac{Y_k(z)}{V_k(z)} \cdot \frac{V_k(z)}{X(z)} = (1 - W_N^k \bar{z}^{-1}) \cdot \frac{1}{1 - 2\cos\left(\frac{2\pi k}{N}\right)\bar{z}^{-1} + \bar{z}^{-2}}$$

$$\text{Let } \frac{V_k(z)}{X(z)} = \frac{1}{1 - 2\cos\left(\frac{2\pi k}{N}\right)\bar{z}^{-1} + \bar{z}^{-2}} \quad \text{--- (7)}$$

$$V_k(z) \left[1 - 2\cos\left(\frac{2\pi k}{N}\right)\bar{z}^{-1} + \bar{z}^{-2} \right] = X(z)$$

$$V_k(z) - 2\cos\left(\frac{2\pi k}{N}\right)\bar{z}^{-1} V_k(z) + V_k \bar{z}^{-2} = X(z)$$

Taking Inverse Z.T.

$$v_k(n) - 2\cos\left(\frac{2\pi k}{N}\right)v_k(n-1) + v_k(n-2) = x(n) \quad \text{--- (8)}$$

$$v_k(n) = x(n) + 2\cos\left(\frac{2\pi k}{N}\right)v_k(n-1) - v_k(n-2) \quad \text{--- (9)}$$

Now Let

$$\frac{y_k(z)}{\sqrt{k}(z)} = (1 - W_n^k z^{-1}) \quad \text{--- (10)}$$

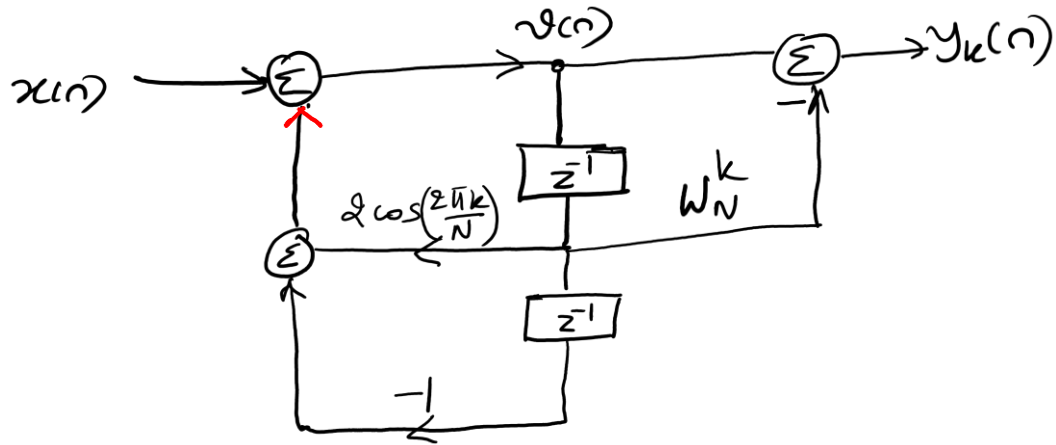
$$Y_k(z) = V_k(z) - W_N^k z^{-1} V_k(z) \quad \text{--- (11)}$$

Taking inverse Z.T. we get

$$y_k(n) = v_k(n) - W_N^k v_k(n-1) \quad \text{--- (12)}$$

$$v_k(n) = x(n) + 2\cos\left(\frac{2\pi k}{N}\right)v_k(n-1) - v_k(n-2) \quad \text{--- (9)}$$

Let us now realize a system using eqns (9) and (12) as follows:



Goertzel algorithm is attractive when the DFT to be computed at a relatively smaller number M of values where $M \leq \log_2 N$

Direct form II realization for computing k^{th} DFT point

If $x(n) = \{2, 0, 2, 0\}$
find $X(2)$

Solⁿ $X(k) = y_k(n) \Big|_{n=N}$

$k=2, N=4$

$X(2) = y_k(n) \Big|_{n=4}$

and $y_k(n) - W_N^{-k} y_k(n-1) = x(n)$

$y_2(n) - W_4^{-2} y_2(n-1) = x(n)$

where $W_4^{-2} = e^{j\frac{2\pi}{4} \cdot -2} = e^{j\pi} = -1$

$y_2(n) + y_2(n-1) = x(n)$

$X(2) = y_2(4)$

$$y_2(n) + y_2(n-1) = x(n)$$

$$\begin{aligned} n &= 0, 1, 2, 3 \\ y_2(0) &= x(0) - y_2(-1) \\ &= 2 - 0 = 2 \end{aligned}$$

$$\begin{aligned} y_2(1) &= x(1) - y_2(0) \\ &= 0 - 2 = -2 \end{aligned}$$

$$\begin{aligned} y_2(2) &= x(2) - y_2(1) \\ &= 2 - (-2) = 4 \end{aligned}$$

$$\begin{aligned} y_2(3) &= x(3) - y_2(2) \\ &= 0 - 4 = -4 \end{aligned}$$

$$\begin{aligned} y_2(4) &= x(4) - y_2(3) = 0 - (-4) \\ &= 4 \end{aligned}$$

$$x[2] = y_2[4] = 4$$

*Thank
you*

