IIR filter design
Bilinear Transformation

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Bilinear transformation method for IIR filter design

This method overcomes the limitation of the previous method.

Transforms the $j\Omega$ axis into unit ⊙ only once.

Avoids aliasing of frequency components

All points in LHP are mapped inside unit ⊙
" RHP " outside "

Consider an analog filter with system function

$$H(s) = \frac{b}{s+a} = \frac{Y(s)}{X(s)}$$

$$Y(s)(s+a) = X(s)\,b.$$

This system can also be characterized by the differential equation,

$$\frac{dy}{dt} + a\,y(t) = b\,x(t) \qquad \text{——————} \quad (1)$$

Integrate the derivative and approximate the integral using trapezoidal formula,

$$y(t) = \int_{t_0}^{t} y'(\tau)\, d\tau + y(t_0) \qquad \text{——— (2)}$$

Approximate this integral by trapezoidal formula, at $t = nT$ and $t_0 = nT - T$

$$\int_{t_0}^{t} y'(\tau)\, d\tau = \frac{t - t_0}{2}\left[y'(t_0) + y'(t) \right]$$

Substitute for t and t_0,

$$\int_{t_0}^{t} y'(\tau)\, d\tau = \frac{nT - nT + T}{2}\left[y'(nT) + y'(nT - T) \right]$$

Substitute in equation (2),

$$y(nT) = \frac{T}{2}\left[y'(nT) + y'(nT - T) \right] + y(nT - T) \qquad \text{——— (3)}$$

Evaluate the differential eqn in ① at $t = nT$,

$$① \Rightarrow \frac{dy(t)}{dt} + a y(t) = b x(t)$$

$$y'(nT) + a y(nT) = b x(nT)$$

$$y'(nT) = -a y(nT) + b x(nT)$$

Substitute this in eqn ③,

$$③ \Rightarrow y(nT) = \frac{T}{2}\left[y'(nT) + y'(nT-T)\right] + y(nT-T)$$

$$= \frac{T}{2}\left[-a y(nT) + b x(nT) - a y(nT-T) + b x(nT-T)\right] + y(nT-T)$$

Take $y(nT) \equiv y(n)$ and $x(nT) \equiv x(n)$,
and $nT - T = n-1$,

$$y(n) = \frac{T}{2}\left[-a y(n) + b x(n) - a y(n-1) + b x(n-1)\right] + y(n-1)$$

Grouping all output & input terms,

$$\left(1+\frac{aT}{2}\right)y(n)-\left(1-\frac{aT}{2}\right)y(n-1)=\frac{bT}{2}\left[x(n)+x(n-1)\right]$$

Thus we obtain a difference equation

Taking z-transform,

$$\left(1+\frac{aT}{2}\right)Y(z)-\left(1-\frac{aT}{2}\right)z^{-1}Y(z)=\frac{bT}{2}\left[X(z)+z^{-1}X(z)\right]$$

$$H(z)=\frac{Y(z)}{X(z)}=\frac{\frac{bT}{2}\left[1+z^{-1}\right]}{1+\frac{aT}{2}-z^{-1}+\frac{aT}{2}z^{-1}}$$

$$=\frac{\frac{bT}{2}\left[1+z^{-1}\right]}{\left(1-z^{-1}\right)+\frac{aT}{2}\left(1+z^{-1}\right)}$$

$$\Rightarrow H(z) = \frac{b}{\frac{2}{T}\frac{(1-z^{-1})}{(1+z^{-1})} + \frac{2}{T}\cdot\frac{aT}{2}\frac{(1+z^{-1})}{(1+z^{-1})}}$$

$$= \frac{b}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + a}$$

We know, $H(s) = \frac{b}{s+a}$

Therefore the mapping from s-plane to z-plane is

$$\boxed{s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

This is called bilinear transformation

This transformation holds in general for N^{th} order differential equation too.

Characteristics

$$S = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$= \frac{2}{T}\left(\frac{z-1}{z+1}\right)$$

$$= \frac{2}{T}\,\frac{re^{jw}-1}{re^{jw}+1}$$

$$= \frac{2}{T}\left[\frac{r\cos w + jr\sin w - 1}{r\cos w + jr\sin w + 1}\right]$$

$$= \frac{2}{T}\left[\frac{r\cos w - 1 + jr\sin w}{r\cos w + 1 + jr\sin w}\right]\left[\frac{r\cos w + 1 - jr\sin w}{r\cos w + 1 - jr\sin w}\right]$$

$$z = re^{jw}$$

$$S = \sigma + j\Omega$$

$$e^{jw} = \cos w + j\sin w$$

$$s = \frac{2}{T}\frac{\left[r^2\cos^2\omega + r\cos\omega - jr^2\cos\omega\sin\omega - r\cos\omega - 1 - jr\sin\omega + jr^2\cos\omega\sin\omega + jr\sin\omega - j^2r^2\sin^2\omega \right]}{(r\cos\omega+1)^2 - (jr\sin\omega)^2}$$

$$= \frac{2}{T} \times \frac{r^2 + 2jr\sin\omega - 1}{r^2 + 2r\cos\omega + 1}$$

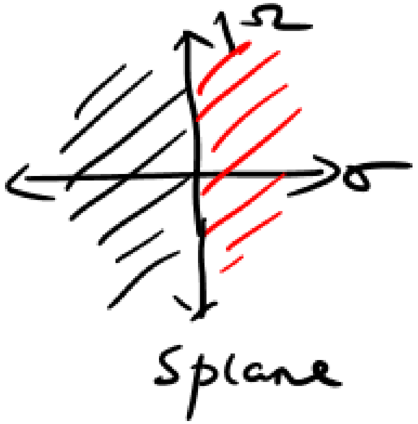
$$s = \left(\frac{2}{T}\left[\frac{r^2-1}{r^2+2r\cos\omega+1}\right) + j\frac{2}{T}\frac{2r\sin\omega}{r^2+2r\cos\omega+1}\right)$$

$$s = \sigma + j\Omega$$

$r<1,\ \sigma<0$

$r>1,\ \sigma>0$

$\boxed{r=1,\ \sigma=0}$



$$\Omega = \frac{2}{T} \times \frac{\sin\omega}{1+\cos\omega}$$

$$= \frac{2}{T} \cdot \frac{2\sin\omega/2 \cos\omega/2}{2\cos^2\omega/2}$$

$$\boxed{\Omega = \frac{2}{T}\tan\frac{\omega}{2}}$$

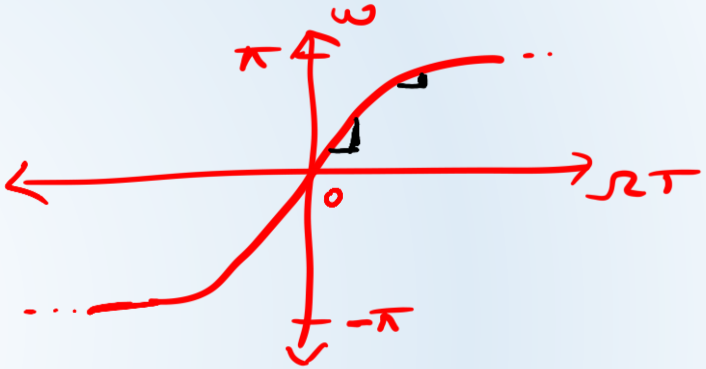
$$\Rightarrow \omega = 2\tan^{-1}\left(\frac{\Omega T}{2}\right)$$

$\Omega = \pm\infty$, $\omega = \pm\pi$

one to one mapping

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = 2\cos^2\theta - 1$



freq compression
warping

nonlinear

Q. Analog filter, $H_a(s) = \dfrac{s+0.1}{(s+0.1)^2+16}$

Digital filter, using bilinear transf.
↓ resonant freq $\omega_r = \pi/2$.

Soln: $H(s) = \dfrac{s+0.1}{(s+0.1)^2-(j4)^2} = \dfrac{s+0.1}{(s+0.1-j4)(s+0.1+j4)}$

Poles, $s_k = -0.1 \pm j4$

$(s = \sigma + j\Omega)$

$\Omega_r = 4$

$\omega_r = \pi/2$

$\Omega_r = \dfrac{2}{T}\tan\dfrac{\omega_r}{2}$

$4 = \dfrac{2}{T}\tan\dfrac{\pi/2}{2} \Rightarrow T = 1/2$

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = 4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$= \frac{s+0.1}{(s+0.1)^2+16}\Bigg|_{s=}$$

$$= \frac{4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+0.1}{\left(4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+0.1\right)^2+16}$$

$$H(z) = \frac{0.128 + 0.006z^{-1} - 0.122z^{-2}}{1+0.995z^{-2}}$$

$$= 0.128\frac{(z^2+0.046z-0.953)}{z^2+0.995}$$

Poles @ $0.987e^{\pm j\pi/2}$

Zeros @ $-1, 0.953$

3) Matched z-transform

Direct mapping of poles & zeros

$$H(s) = \frac{\prod^{M} (s - z_k)}{\prod_{k=1}^{N} (s - p_k)} \xrightarrow{\text{map}} H(z) = \frac{\prod_{k=1}^{M} \left(1 - e^{z_k T} z^{-1}\right)}{\prod_{k=1}^{N} \left(1 - e^{p_k T} z^{-1}\right)}$$


Thank you