Digital Signal Processing (ECE – 2222)

List of Formulae

1	DFT pair	$x(n) = \frac{1}{\mathrm{N}}\sum_{k=0}^{N-1} X(k)e^{j\frac{2\pi}{N}kn}, \quad n = 0, 1,2 \ldots\ldots N-1$ $X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}kn}, \quad k = 0, 1,2 \ldots\ldots N-1$	x(n) : Signal of length L X(k): N-point DFT of x(n) N ≥ L
2	N-point Circular convolution	$x_3(n) = x_1(n) \circledast x_2(n) = \sum_{m=0}^{N-1} x_1(m)x_2(n-m)$	$x_1(n), x_2(n)$: signals of length N or less
3	Pole-zero IIR system	$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$ $H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$	$x(n)\colon input$ $y(n)\colon output$ $X(z)$: z-transform of $x(n)$ $Y(z)$: z-transform of y(n) $H(z)$: System function K: Integer part of *(N+1)/2* N ≥ M $C = \frac{b_N}{a_N}$
4	All pole IIR system	$y(n) = \sum_{k=1}^{N} a_k y(n-k) + b_0 x(n)$ $H(z) = \frac{Y(z)}{X(z)} = \frac{b_0}{1 + \sum_{k=1}^{N} a_k z^{-k}}$	
5	All zero FIR system	$y(n) = \sum_{k=0}^{M} b_k x(n-k)$ $H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^{M} b_k z^{-k}$	
6	Cascade realization for IIR filters	$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \prod_{k=1}^{K} H_k(z)$ $H_k(z) = \frac{b_{k0} + b_{k1}z^{-1} + b_{k2}z^{-2}}{1 + a_{k1}z^{-1} + a_{k2}z^{-2}}$	

7	Parallel realization for IIR filters	$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = C + \sum_{k=1}^{K} H_k(z)$ $H_k(z) = \frac{b_{k0} + b_{k1}z^{-1}}{1 + a_{k1}z^{-1} + a_{k2}z^{-2}}$	
8	Frequency sampling realization for FIR filters	$H(z) = \frac{1}{M}(1 - z^{-M})H_p(z)$ $H_p(z) = \frac{H(0)}{1 - z^{-1}} + \sum_{k=1}^{(M-1)/2} \frac{A(k) - B(k)z^{-1}}{1 - 2cos\left(\frac{2\pi k}{M}\right)z^{-1} + z^{-2}}\ M\ odd$ $H_p(z) = \frac{H(0)}{1 - z^{-1}} + \frac{H\left(\frac{M}{2}\right)}{1 + z^{-1}} + \sum_{k=1}^{\frac{M}{2}-1} \frac{A(k) - B(k)z^{-1}}{1 - 2cos\left(\frac{2\pi k}{M}\right)z^{-1} + z^{-2}} M\ even$ $A(k) = H(k) + H(M - k)$ $B(k) = H(k)e^{-j\frac{2\pi k}{M}} + H(M - k)e^{j\frac{2\pi k}{M}}$	$H(\omega)$: Frequency response $H(k)$: $H(\omega)$ at $\omega = \omega_k = \frac{2\pi k}{M}$ M: Length of the FIR filter
9	Impulse Invariant Transformation	$z = e^{sT}$ $H(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k} \equiv H(z) = \sum_{k=1}^{N} \frac{A_k}{1 - e^{s_k T} z^{-1}}$	N: Order of the analog filter s_k: Poles of $H(s)$ z_k: Zeroes of $H(s)$ T: Sampling interval Ω: Analog filter frequency variable ω: Digital filter frequency variable
10	Bilinear Transformation	$s = \frac{2}{T}\frac{(1 - z^{-1})}{(1 + z^{-1})}$ $\Omega = \frac{2}{T} tan\left(\frac{\omega}{2}\right)$	
11	Matched z-transform	$H(s) = \frac{\prod_{k=1}^{M}(s - z_k)}{\prod_{k=1}^{N}(s - s_k)} \equiv H(z) = \frac{\prod_{k=1}^{M} 1 - e^{z_k T}z^{-1}}{\prod_{k=1}^{N} 1 - e^{s_k T}z^{-1}}$	
12	Butterworth analog low pass filter response	$\|H(\Omega)\|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}} = \frac{1}{1 + \epsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}}$ $N = \frac{log\left(\frac{1}{{\delta_2}^2} - 1\right)}{2log\left(\frac{\Omega_s}{\Omega_c}\right)}$ or N $\geq$ log(d)/ log(k) Where d=$\sqrt{(1/Ap^2) -}\ 1/\sqrt{(1/As^2) -}\ 1$ and k = $\frac{\Omega p}{\Omega s}$ Ap and As are absolute values $s_k = \Omega_c e^{j\emptyset_k}$	N: order of the filter $\|H(\Omega)\|^2$: Squared magnitude response Ω_p: Pass band edge frequency Ω_p: Stop band edge frequency

<table>
<tr><td></td><td></td><td></td><td>Ω_c: 3-dB cut-off frequency</td></tr>
<tr><td>13</td><td>Chebyshev analog low pass filter response</td><td>$|H(\Omega)|^2 = \dfrac{1}{1+\epsilon^2 {T_N}^2\left(\dfrac{\Omega}{\Omega_p}\right)}$
$T_N(x) = \begin{cases} cos(Ncos^{-1}x), & |x| \le 1 \\ cosh(Ncosh^{-1}x), & |x| > 1 \end{cases}$
$N = \dfrac{log\left[\dfrac{\sqrt{1-{\delta_2}^2}+\sqrt{1-{\delta_2}^2(1+\epsilon^2)}}{\varepsilon\delta_2}\right]}{log\left[\left(\dfrac{\Omega_s}{\Omega_p}\right)+\sqrt{\left(\dfrac{\Omega_s}{\Omega_p}\right)^2-1}\right]}$
N ≥ $\cosh^{-1}(1/d)/\cosh^{-1}(1/k)$
Where d=$\sqrt{(1-Ap^2)-}\,1/\sqrt{(1-As^2)-}\,1$ and k = $\frac{\Omega p}{\Omega s}$
$s_k = r_2cos(\emptyset_k) + jr_1sin(\emptyset_k)$
$r_1 = \Omega_p\frac{\beta^2+1}{2\beta}\,; \quad r_2 = \Omega_p\frac{\beta^2-1}{2\beta}\,; \quad \beta = \left[\frac{\sqrt{1+\epsilon^2}+1}{\varepsilon}\right]^{1/N}$</td><td>$\frac{1}{1+\epsilon^2}$: Pass band edge value of $|H(\Omega)|^2$
${\delta_2}^2$: Stop band edge value of $|H(\Omega)|^2$
s_k: Poles of H(s)
$\emptyset_k = pole\ angle = \dfrac{\pi}{2}+\dfrac{(2k+1)\pi}{2N},$
k= 0, 1, 2, ……N-1</td></tr>
<tr><td>14</td><td>Frequency Transformation for analog filters</td><td colspan="2">

Prototype Low pass filter has band edge frequency $\boldsymbol{\Omega_p}$

<table>
<tr><th>Type of Transformation</th><th>Transformation</th><th>Band edge frequency of new filter</th></tr>
<tr><td>Low pass</td><td>$s \longrightarrow \dfrac{\Omega_p}{\Omega_{pn}}s$</td><td>$\Omega_{pn}$</td></tr>
<tr><td>High pass</td><td>$s \longrightarrow \dfrac{\Omega_p\Omega_{pn}}{s}$</td><td>$\Omega_{pn}$</td></tr>
<tr><td>Band pass</td><td>$s \longrightarrow \Omega_p\dfrac{s^2+\Omega_l\Omega_u}{s(\Omega_u-\Omega_l)}$</td><td>$\Omega_l\ ,\ \Omega_u$</td></tr>
<tr><td>Band stop</td><td>$s \longrightarrow \Omega_p\dfrac{s(\Omega_u-\Omega_l)}{s^2+\Omega_l\Omega_u}$</td><td>$\Omega_l\ ,\ \Omega_u$</td></tr>
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<table>
<tr><td rowspan="6">15</td><td rowspan="6">Frequency Transformation for digital filters</td><td colspan="3">***Prototype Low pass filter has band edge frequency*** ω_p</td></tr>
<tr><td>**Type of Transformation**</td><td>**Transformation**</td><td>**Parameters**</td></tr>
<tr><td>Low pass</td><td>$z^{-1} \longrightarrow \frac{z^{-1} - a}{1 - az^{-1}}$</td><td>$\omega_{pn} = band\ edge\ frequency\ of\ new\ filter$
$a = \frac{sin\left[\left(\omega_p - \omega_{pn}\right)/2\right]}{sin\left[\left(\omega_p + \omega_{pn}\right)/2\right]}$</td></tr>
<tr><td>High pass</td><td>$z^{-1} \longrightarrow \frac{z^{-1} + a}{1 + az^{-1}}$</td><td>$\omega_{pn} = band\ edge\ frequency\ of\ new\ filter$
$a = \frac{cos\left[\left(\omega_p - \omega_{pn}\right)/2\right]}{cos\left[\left(\omega_p + \omega_{pn}\right)/2\right]}$</td></tr>
<tr><td>Band pass</td><td>$z^{-1} \longrightarrow -\frac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}$</td><td>$\omega_l = lower\ band\ edge\ frequency$
$\omega_u = upper\ band\ edge\ frequency$
$a_1 = \frac{-2\alpha K}{(K+1)}$
$a_2 = \frac{(K-1)}{(K+1)}$
$\alpha = \frac{cos[(\omega_u + \omega_l)/2]}{cos[(\omega_u - \omega_l)/2]}$
$K = cot\left(\frac{\omega_u - \omega_l}{2}\right) tan\left(\frac{\omega_p}{2}\right)$</td></tr>
<tr><td>Band stop</td><td>$z^{-1} \longrightarrow \frac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}$</td><td>$\omega_l = lower\ band\ edge\ frequency$
$\omega_u = upper\ band\ edge\ frequency$
$a_1 = \frac{-2\alpha}{(K+1)}$
$a_2 = \frac{(1-K)}{(1+K)}$
$\alpha = \frac{cos[(\omega_u + \omega_l)/2]}{cos[(\omega_u - \omega_l)/2]}$
$K = tan\left(\frac{\omega_u - \omega_l}{2}\right) tan\left(\frac{\omega_p}{2}\right)$</td></tr>
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<tr>
<td>16</td>
<td>Linear phase FIR filter frequency response</td>
<td>i) Symmetric impulse response, odd length

$$H(\omega) = e^{-j\omega\left(\frac{M-1}{2}\right)}\left[2\sum_{n=0}^{\frac{M-3}{2}} h(n)cos\left\{\omega\left(\frac{M-1}{2}-n\right)\right\}\right]$$

ii) Symmetric impulse response, even length

$$H(\omega) = e^{-j\omega\left(\frac{M-1}{2}\right)}\left[2\sum_{n=0}^{\frac{M}{2}-1} h(n)cos\left\{\omega\left(\frac{M-1}{2}-n\right)\right\}\right]$$

iii) Anti-symmetric impulse response, odd length

$$H(\omega) = je^{-j\omega\left(\frac{M-1}{2}\right)}\left[2\sum_{n=0}^{\frac{M-3}{2}} h(n)sin\left\{\omega\left(\frac{M-1}{2}-n\right)\right\}\right]$$

iv) Anti-symmetric impulse response, even length

$$H(\omega) = je^{-j\omega\left(\frac{M-1}{2}\right)}\left[2\sum_{n=0}^{\frac{M}{2}-1} h(n)sin\left\{\omega\left(\frac{M-1}{2}-n\right)\right\}\right]$$
</td>
<td>M: length of the filter

$H(\omega)$: Frequency response</td>
</tr>
<tr>
<td>17</td>
<td>Linear phase FIR filter design using window functions</td>
<td colspan="2">

Window functions for FIR filter design

<table>
<tr><th>Name of the window</th><th>Window function 0≤n≤M-1</th><th>Main lobe width</th><th>Peak side lobe (dB)</th><th>Normalized transition width#</th><th>Stop band attenuation (dB)</th></tr>
<tr><td>Rectangular</td><td>1</td><td>4π/M</td><td>-13</td><td>0.9/(M-1)</td><td>21</td></tr>
<tr><td>Hanning</td><td>$0.5-0.5cos\left(\frac{2\pi n}{M-1}\right)$</td><td>8π/M</td><td>-32</td><td>3.1/(M-1)</td><td>44</td></tr>
<tr><td>Hamming</td><td>$0.54-0.46cos\left(\frac{2\pi n}{M-1}\right)$</td><td>8π/M</td><td>-43</td><td>3.3/(M-1)</td><td>53</td></tr>
<tr><td>Blackman</td><td>$0.42-0.5cos\left(\frac{2\pi n}{M-1}\right)+0.08cos\left(\frac{4\pi n}{M-1}\right)$</td><td>12π/M</td><td>-58</td><td>5.5/(M-1)</td><td>75</td></tr>
<tr><td>Keiser*</td><td>$\frac{I_0\left[\beta\sqrt{\left(\frac{M-1}{2}\right)^2-\left(n-\frac{M-1}{2}\right)^2}\right]}{I_0\left[\beta\left(\frac{M-1}{2}\right)\right]}$</td><td></td><td></td><td></td><td>> 70</td></tr>
</table>

*Keiser window parameters can be controlled by β. **I_0**[.] is modified Bessel function.
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<tr><td></td><td></td><td colspan="2">#Transition width is normalized to 2π or equivalently to sampling frequency F_s</td></tr>
<tr><td>18</td><td>Frequency sampling design of FIR filter</td><td>$h(n) = \frac{1}{M}\left\{G(0) + 2\sum_{k=1}^{\frac{(M-1)}{2}} G(k)cos\frac{2\pi k}{M}\left(n+\frac{1}{2}\right)\right\},$
$M\ odd$
$h(n) = \frac{1}{M}\left\{G(0) + 2\sum_{k=1}^{\frac{M}{2}-1} G(k)cos\frac{2\pi k}{M}\left(n+\frac{1}{2}\right)\right\},$
$M\ even$
$h(n) = \frac{1}{M}\left[H(0) + 2\sum_{k=1}^{\frac{M-1}{2}} Re\left\{H(k)e^{\frac{j2\pi nk}{M}}\right\}\right]$ M odd
$h(n) = \frac{1}{M}\left[H(0) + 2\sum_{k=1}^{\frac{M}{2}-1} Re\left\{H(k)e^{\frac{j2\pi nk}{M}}\right\}\right]$ M even
$H_d(\omega) = H_r(\omega)e^{-j\omega(M-1)/2}$
Where ω=$\frac{2\pi k}{M}$
$H(k) = H_r(k)e^{\frac{-j2\pi k}{M}\frac{(M-1)}{2}}$</td><td>M: length of the filter
$H_d(\omega)$: Desired frequency respons
$G(k) = (-1)^k H_r\left(\frac{2\pi k}{M}\right)$</td></tr>
<tr><td>19</td><td>Non-parametric power spectrum estimators</td><td><table>
<tr><td>Estimate</td><td>Quality factor</td><td>Computational requirement</td></tr>
<tr><td>Bartlett</td><td>$Q_B = 1.1N\Delta f$</td><td>$\frac{N}{2}\log_2\frac{0.9}{\Delta f}$</td></tr>
<tr><td>Welch (50% overlap)</td><td>$Q_W = 1.39N\Delta f$</td><td>$N\log_2\frac{5.12}{\Delta f}$</td></tr>
<tr><td>Blackman-Tukey</td><td>$Q_{BT} = 2.34N\Delta f$</td><td>$N\log_2\frac{1.28}{\Delta f}$</td></tr>
</table></td><td><i>Quality factor</i>
$Q = \frac{\{E[estimate]\}^2}{var[estimate]}$
Δf: <i>Frequency resolution</i>
<i>N: Data frame length</i></td></tr>
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