Properties of DFT contd...

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Circular time shift of a sequence

$$x((n-l))_N \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X(k)e^{-j2\pi kl/N}$$

$$\text{DFT}\{x((n-l))_N\} = \sum_{n=0}^{N-1} x((n-l))_N e^{-j2\pi kn/N}$$

$$= \sum_{n=0}^{l-1} x((n-l))_N e^{-j2\pi kn/N}$$

$$+ \boxed{\sum_{n=l}^{N-1} x(n-l)e^{-j\pi kn/N}}$$

But $x((n-l))_N = x(N-l+n)$.

$$\sum_{n=0}^{l-1} x((n-l))_N e^{-j2\pi kn/N} = \sum_{n=0}^{l-1} x(N-l+n)e^{-j2\pi kn/N}$$

$$= \sum_{m=N-l}^{N-1} x(m)e^{-j2\pi k(m+l)/N}$$

Put N – l + n = m

Put n - l = m

$$\sum_{n=l}^{N-1} x(n-l)e^{-j2\pi kn/N} = \sum_{m=0}^{N-1-l} x(m)e^{-j2\pi k(m+l)/N}$$

Therefore, $\text{DFT}\{x((n-l))\} = \sum_{m=0}^{N-1} x(m)e^{-j2\pi k(m+l)/N}$

$$= X(k)e^{-j2\pi kl/N}$$

Circular frequency shift.

$$x(n)e^{j2\pi ln/N} \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X((k-l))_N$$

This is the dual to the circular time-shifting property and its proof is similar

Complex-conjugate properties.

$$x^*(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X^*((-k))_N = X^*(N-k)$$

Proof is an exercise

The IDFT of $X^*(k)$ is

$$\frac{1}{N}\sum_{k=0}^{N-1} X^*(k)e^{j2\pi kn/N} = \left[\frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi k(N-n)/N}\right]$$

Therefore,

$$x^*((-n))_N = x^*(N-n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X^*(k)$$

Circular correlation.

$$\tilde{r}_{xy}(l) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} \tilde{R}_{xy}(k) = X(k)Y^*(k)$$

where $\tilde{r}_{xy}(l)$ is the (unnormalized) circular crosscorrelation sequence, defined as

$$\tilde{r}xy(l) = \sum_{n=0}^{N-1} x(n)y^*((n-l))N$$

circular autocorrelation of $x(n)$,

$$\tilde{r}_{xx}(l) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} \tilde{R}_{xx}(k) = |X(k)|^2$$

Multiplication of two sequences.

$$x_1(n)x_2(n) \overset{\mathrm{DFT}}{\underset{N}{\longleftrightarrow}} \frac{1}{N} X_1(k) \; \textcircled{N} \; X_2(k)$$

Its proof follows simply by interchanging the roles of time and frequency in the expression for the circular convolution of two sequences.

Parseval's Theorem.

$$\sum_{n=0}^{N-1} x(n) y^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) Y^*(k)$$

Proof The property follows immediately from the circular correlation property

$$\sum_{n=0}^{N-1} x(n) y^*(n) = \tilde{r}_{xy}(0)$$

and

$$\tilde{r}_{xy}(l) = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{R}_{xy}(k) e^{j2\pi kl/N}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) Y^*(k) e^{j2\pi kl/N}$$

Evaluate IDFT at $l = 0$

special case where $y(n) = x(n)$

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

TABLE 7.2 Properties of the DFT

Property	Time Domain	Frequency Domain
Notation	$x(n), y(n)$	$X(k), Y(k)$
Periodicity	$x(n) = x(n+N)$	$X(k) = X(k+N)$
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(k) + a_2X_2(k)$
Time reversal	$x(N-n)$	$X(N-k)$
Circular time shift	$x((n-l))_N$	$X(k)e^{-j2\pi kl/N}$
Circular frequency shift	$x(n)e^{j2\pi ln/N}$	$X((k-l))_N$
Complex conjugate	$x^*(n)$	$X^*(N-k)$
Circular convolution	$x_1(n) \textcircled{N} x_2(n)$	$X_1(k)X_2(k)$
Circular correlation	$x(n) \textcircled{N} y^*(-n)$	$X(k)Y^*(k)$
Multiplication of two sequences	$x_1(n)x_2(n)$	$\frac{1}{N}X_1(k) \textcircled{N} X_2(k)$
Parseval's theorem	$\sum_{n=0}^{N-1} x(n)y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$

The DFT as a Linear Transformation

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \qquad k = 0, 1, \ldots, N-1$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}, \qquad n = 0, 1, \ldots, N-1$$

$W_N = e^{-j2\pi/N}$ which is an Nth root of unity.

We note that the computation of each point of the DFT can be accomplished by N complex multiplications and $(N-1)$ complex additions. Hence the N-point DFT values can be computed in a total of N^2 complex multiplications and $N(N-1)$ complex additions.

DFT matrix

$$\mathbf{x}_N = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}, \qquad \mathbf{X}_N = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}$$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \qquad k = 0, 1, \ldots, N-1$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}, \qquad n = 0, 1, \ldots, N-1$$

$$W_N = e^{-j2\pi/N}$$

$$\mathbf{W}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

$$\mathbf{x}_N = \mathbf{W}_N^{-1} \mathbf{X}_N$$

$$\mathbf{X}_N = \mathbf{W}_N \mathbf{x}_N$$

$$\begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

$$\mathbf{x}_N = \frac{1}{N} \mathbf{W}_N^* \mathbf{X}_N$$

where $\mathbf{W}_N^*$ denotes the complex conjugate of the matrix $\mathbf{W}_N$

$$\mathbf{W}_N^{-1} = \frac{1}{N} \mathbf{W}_N^*$$

$$\mathbf{W}_N \mathbf{W}_N^* = N \mathbf{I}_N$$

orthogonal (unitary) matrix.

Compute the DFT of the four-point sequence

$$x(n) = (0 \quad 1 \quad 2 \quad 3)$$

$W_N^{kn} \Rightarrow$ Twiddle factor

e.g. $W_4^1 = \left(e^{-j\frac{2\pi}{4}}\right)^1 = \underline{-j}$

$W_8^1 = \left(e^{-j\frac{2\pi}{8}}\right)^1 = \underline{\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}}$

Solution. The first step is to determine the matrix $\mathbf{W}_4$. By exploiting the periodicity property of $\mathbf{W}_4$ and the symmetry property $W_N^{k+N/2} = -W_N^k$

$$\mathbf{W}_4 = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^0 & W_4^2 \\ 1 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$\mathbf{X}_4 = \mathbf{W}_4\mathbf{x}_4 = \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

The IDFT of $\mathbf{X}_4$ may be determined by conjugating the elements in $\mathbf{W}_4$ to obtain $\mathbf{W}_4^*$


Thank you