Stochastic Processes

Definition 0.1. *Families of Random variables which are functions of say time are known as stochastic processes.*

i.e., Stochastic process is defined as family of random variables $\{X(t), t \in T\}$. The parameter t usually represents time, so $\{X(t)\}$ represents value assumed by random variable at any time t.

Definition 0.2. *T is called index set or parameter space and is subset of $(-\infty, \infty)$.*

Definition 0.3. *If the index set is discrete, eg: $T = \{0, 1, 2, \ldots,\}$, then we have discrete-(time) parameter stochastic process. If T is continuous, eg: $T = 0 \le t \le \infty$, we call the process a continuous-(time) parameter stochastic process.*

Definition 0.4. *Values assumed by random variable $X(t)$ are called states. The set S of possible values that the random variable $X(t)$ takes for each $t \in T$ is known as state space of stochastic process and the set of possible values for the parameter t (i.e., set T) is known as parameter space.*

Further set of all possible values from the state space of stochastic process may discrete or continuous.

Definition 0.5. *If the state space is discrete, the process is referred as a chain and the states are usually identified with set of natural numbers or a subset of it.*

- *Example for discrete state space is the number of customers at service facility.*
- *Example for continuous state space is the length of the time, a customer has been waiting for the service.*

Definition 0.6. *If for $t_1 < t_2 < \ldots < t_n < t$, for all n and for all $x_1, x_2, \ldots, x_n$,*

$$P\{a \le X(t) \le b | X(t_1) = x_1, \ldots, X(t_n) = x_n\} = P\{a \le X(t) \le b | X(t_n) = x_n\},$$

then the process $\{X(t), t \in T\}$ is Markov Process.

Definition 0.7. *A discrete parameter discrete state Markov process is called* ***Markov chain***.

Example: Consider the experiment of throwing a fair die repeatedly. If Y_n denote the number of $6's$ in the first n throws, then $\{Y_n, n \geq 1\}$ is a Markov chain.

Random processes in which the occurrences of future state depends on the immediately preceding state and only on it is known as Markov chain.

A state is a condition or location of an object in the system at particular time.

Assumptions we made for Markov chains:

- Finite number of states.
- States are mutually exclusive.
- States are collectively exhaustive.
- Probability of moving from one state to other state is constant over time.

Transition probabilities:

The transition probabilities $p_{ij}(n)$ are basic entities in the study of the Markov chains. Transition probability of moving from state i to state j in n^{th} step is $p_{ij}(n) = P\{X_n = j | X_{n-1} = i\}$.

Stationary transition probabilities p_{ij} are called one step transition probabilities as they represent probability of transition from state i to state j at two successive time points or in one step.

Transition probability p_{ij} satisfy following properties:

(i) $p_{ij} \geq 0$.

(ii) $\sum\limits_{j=1}^{n} p_{ij} = 1$, for all i.

One step transition probabilities may be written in the matrix form as follows:

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{in} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \end{bmatrix}$$

This matrix is called transition probability matrix (TPM) of the Markov chain.

Examples:

1) In a certain market, only two brands of cold drinks A and B are sold. Given that man last purchased brand A, there is 80% chance that he would buy same brand in the next purchase. While if the man purchased brand B, there in 90% chance that his next purchase would br brand B. Using this information,

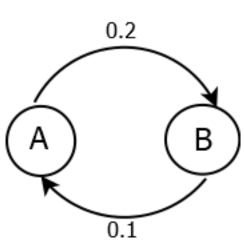
(i) Develop TPM.

(ii) Draw state transition diagram.

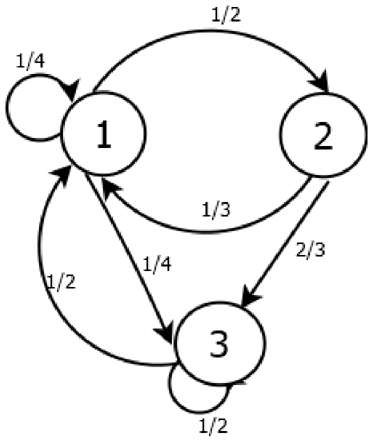
Solution:

(i) TPM is $P = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{bmatrix} \end{array}$

(ii) State transition diagram is



2) Transition diagram is



(1) Find $P\{X_4 = 3|X_3 = 2\}$

(2) Find $P\{X_3 = 1|X_2 = 1\}$

(3) If we know $P\{X_0 = 1\} = 1/3$, find $P\{X_0 = 1, X_1 = 2\}$.

(4) If we know $P\{X_0 = 1\} = 1/3$, find $P\{X_0 = 1, X_1 = 2, X_2 = 3\}$.

(5) Find TPM.

Solution: (1) $P\{X_4 = 3|X_3 = 2\} = p_{23} = 2/3$.

(2) $P\{X_3 = 1|X_2 = 1\} = p_{11} = 1/4$.

(3) $P\{X_0 = 1, X_1 = 2\} = P(X_0 = 1)P(X_1 = 2|X_0 = 1) = 1/3.p_{12} = 1/3.1/2 = 1/6$.

(4) $P\{X_0 = 1, X_1 = 2, X_2 = 3\} = P(X_0 = 1)P(X_1 = 2|X_0 = 1)P(X_2 = 3|X_1 = 2)$

$$= 1/3.p_{12}p_{23} = 1/9.$$

(5) TPM is $P = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{ccc} 1/4 & 1/2 & 1/4 \\ 1/3 & 0 & 2/3 \\ 1/2 & 0 & 1/2 \end{array} \right] \end{array}.$

3) The factory has 2 machines and one repair crew. Assume that probability of any one machine breaking down on a given day is α. Assume that if the repair crew is working on a machine, the probability that will complete the repair in a day is β. For a simplicity, ignore the probability of repair completion or breakdown taking place except at the end of the day. Let X_n be the number of machines in operation at the end of the n^{th} day. Assume the behavior of X_n can be modeled as a Markov chain.

Solution: Probability of machine breakdown=α.

Probability of machine got repaired by a crew in a day=β.

Probability of non completion of repair by crew in a day=$1-\beta$.

Probability of non breakdown of machine =$1-\alpha$.

Let $X_n = \{0,1,2\}$.

TPM is

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 1-\beta & \beta & 0 \\ \alpha(1-\beta) & (1-\alpha)(1-\beta)+\alpha\beta & \beta(1-\alpha) \\ \alpha^2 & 2\alpha(1-\alpha) & (1-\alpha)^2 \end{bmatrix} \end{array}.$$

4) An Urn initially contains 5 black balls and 5 white balls. The following experiment is repeated indefinitely. A ball is drawn from the Urn, if the ball is white, it is put back in the urn otherwise it is left out. Let X_n be the number of black balls remaining in the urn after n draws from the urn. Find Transition probability matrix.

Solution: $S = \{0,1,2,3,4,5\}$.

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1/6 & 5/6 & 0 & 0 & 0 & 0 \\ 0 & 2/7 & 5/7 & 0 & 0 & 0 \\ 0 & 0 & 3/8 & 5/8 & 0 & 0 \\ 0 & 0 & 0 & 4/9 & 5/9 & 0 \\ 0 & 0 & 0 & 0 & 5/10 & 5/10 \end{bmatrix} \end{array}$$

Higher order Probabilities

Chapman-Kolmogorov equation

We have so far considered unit step transition probabilities, the probability of X_{n+1} given X_n.
One step transition probability from state i to state j is $p_{ij}^{(1)} = p\{X_{n+1} = j|X_n = i\}$.
2 step transition probability is $p_{ij}^{(2)} = p\{X_{n+2} = j|X_n = i\}$.
m step transition probability is $p_{ij}^{(m)} = p\{X_{n+m} = j|X_n = i\}$, $i, j \in S, n \geq 0$.
In the matrix form we denote

- $P^{(1)} = [P_{ij}]$.
- $P^{(2)} = P^{(1)}P^{(1)}$.
- $P^{(m+1)} = P^{(m)}P^{(1)}$.
- $P^{(m+n)} = P^{(m)}P^{(n)}$.

In order to compute unconditional probabilities, we need to define initial state probability distribution. A Markov chain is fully specified once the transition probability matrix and the initial state distribution have been defined.

The initial state distribution is a probability distribution of state at initial time 0. i.e., distribution of X_0 given by $P(X_0 = i) = \alpha_i, \forall i \in S$.

Now we compute unconditional probabilities. The probability of state j at particular time n can be computed as $p(X_n = j) = \sum_{i \in S} p\{X_n = j|X_0 = i\}p\{X_0 = i\} = \sum p_{ij}^{n}\alpha_i$.

Probability of chain realization can be computed as follows:

$$p\{X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n\} = p\{X_0 = i_0\}p\{X_1 = i_1|X_0 = i_0\}p\{X_2 = i_2|X_1 = i_1\}$$
$$\ldots p\{X_n = i_n|X_{n-1} = i_{n-1}\}.$$

Examples

1) Let $\{X_n, n = 0, 1, 2, 3, \dots\}$ be a Markov chain with state space $\{0, 1, 2\}$ and the initial probability vector $p(0) = (1/4, 1/2, 1/4)$. One step transition probability matrix P is given by

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 1/4 & 3/4 & 0 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/4 & 3/4 \end{bmatrix} \end{array}$$

. Find, (i) $p(X_0 = 0, X_1 = 1, X_2 = 1)$, (ii)$p(X_2 = 1)$,
(iii) $p(X_7 = 0|X_5 = 0)$.

Solution: Given that $p(X_0 = 0) = 1/4, p(X_0 = 1) = 1/2$ and $p(X_0 = 2) = 1/4$.

(i)

$$\begin{aligned} p(X_0 = 0, X_1 = 1, X_2 = 1) &= p(X_0 = 0)p(X_1 = 1|X_0 = 0)p(X_2 = 1|X_1 = 1) \\ &= 1/4.3/4.1/3 = 1/16. \end{aligned}$$

(ii) $$P^2 = P.P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 5/16 & 7/16 & 1/4 \\ 7/36 & 4/9 & 13/36 \\ 1/12 & 13/48 & 31/48 \end{bmatrix} \end{array}.$$

$$\begin{aligned} p(X_2 = 1) &= \sum_{i \in S} p(X_0 = i)p(X_2 = 1|X_0 = i) \\ &= p(X_0 = 0)p(X_2 = 1|X_0 = 0) + p(X_0 = 1)p(X_2 = 1|X_0 = 1) + p(X_0 = 2)p(X_2 = 1|X_0 = 2) \\ &= 1/4.p_{01}^2 + 1/2.p_{11}^2 + 1/3.p_{21}^2 \\ &= 1/4.7/16 + 1/2.4/9 + 1/3.13/48 = 0.3993. \end{aligned}$$

(iii) $p(X_7 = 0|X_5 = 0) = p_{00}^2 = 5/16$.

2) Let $\{X_n, n = 0, 1, 2, 3, \dots\}$ be a Markov chain with state space $\{0, 1, 2\}$ and the initial probability distribution $p(X_0 = i) = 1/3, i = 0, 1, 2$. One step transition probability matrix P is given by

$$P = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{bmatrix} \end{array}$$

. Evaluate the followings: (i) $p(X_1 = 1|X_0 = 2)$, (ii)$p(X_2 = 2|X_1 = 1)$, (iii)$p(X_2 = 2, X_1 = 1|X_0 = 2)$, (iv) $p(X_2 = 2, X_1 = 1, X_0 = 2)$, (v)$p(X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2)$.

Solution: (i) $p(X_1 = 1|X_0 = 2) = P_{21} = 3/4$.

(ii)$p(X_2 = 2|X_1 = 1) = P_{12} = 1/4$.

(iii)$p(X_2 = 2, X_1 = 1|X_0 = 2) = \dfrac{P(X_2 = 2, X_1 = 1, X_0 = 2)}{P(X_0 = 2)} = \dfrac{P(X_0 = 2)P(X_1 = 1|X_0 = 2)P(X_2 = 2|X_1 = 1)}{P(X_0 = 2)}$
$= P_{21}P_{12} = 3/4.1/4 = 3/16.$

(iv)$p(X_2 = 2, X_1 = 1, X_0 = 2) = P(X_0 = 2)P(X_1 = 1|X_0 = 2)P(X_2 = 2|X_1 = 1) = 1/3.P_{21}P_{12}$
$= 1/3.3/4.1/4 = 1/16.$

(v)$p(X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2) = 3/64.$