FIR filter design
Linear Phase FIR Filter Characteristics

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Linear Phase FIR Filters

$h(n)$ - length M, $0 \dots M-1$

$x(n) \rightarrow \boxed{} \rightarrow y(n)$

$$H(z) = \sum_{n=0}^{M-1} h(n) z^{-n}$$

$$= h(0) + h(1)z^{-1} + \dots h(M-1) z^{-(M-1)}$$

$$H(\omega) = |H(\omega)| e^{j\phi(\omega)}$$

Group delay, $\tau_g = -\dfrac{d(\phi(\omega))}{d\omega}$

$$h(n) = \pm h(M-1-n)$$

$= +h(M-1-n)$ - symmetric ✓

$= -h(M-1-n)$ - antisymm. ✓

Linear phase property delays the input signal but preserves the signal shape with no distortion. Watch demo @ https://youtu.be/zCdV9IUCSy8

$$H(\omega) = H_r(\omega) \cdot e^{-j\omega\alpha}, \quad \alpha = \frac{M-1}{2} \quad - \text{symm}$$

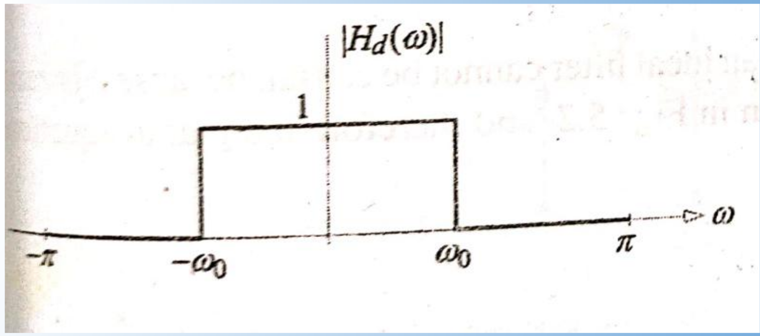
$$H(\omega) = H_r(\omega) \cdot e^{-j(\omega\alpha - \pi/2)}$$

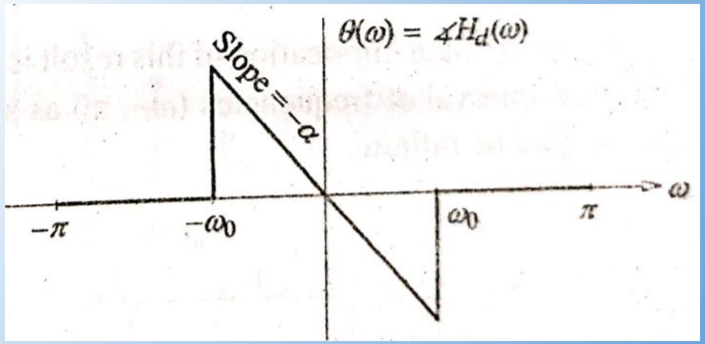
$\Rightarrow$ antisymm.

$$\tau_g = -\frac{d\phi(\omega)}{d\omega}$$

$$= \alpha = \frac{M-1}{2}$$

$$H_d(\omega) = \begin{cases} e^{-j\omega\alpha}, & |\omega| < \omega_0 \\ 0, & \omega_0 < |\omega| < \pi \end{cases}$$





Linear phase filters have same group delay

4 Cases of linearphase FIR:

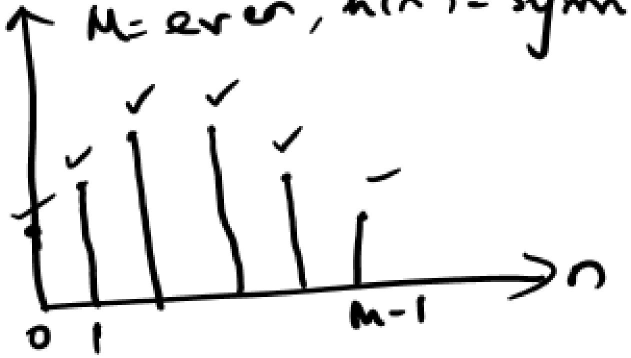
1) M = even → symm., antisymm.

2) M = odd → symm, antisymm.

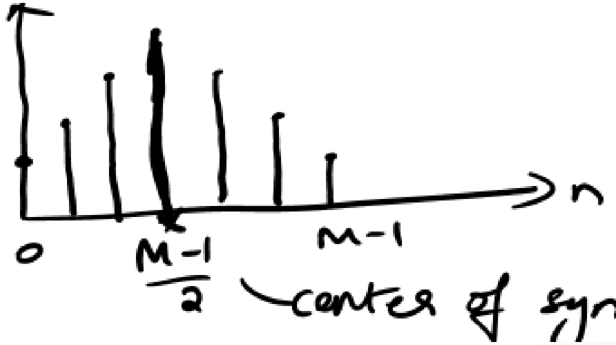
$$h(n) = +h(M-1-n)$$

$$h(n) = -h(M-1-n)$$

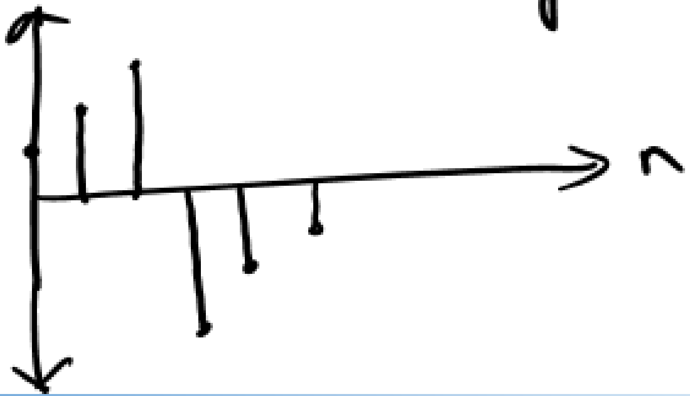
M = even, h(n) = symm



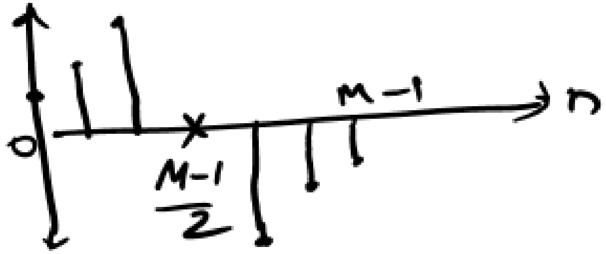
M = odd, h(n) = symm



M = even, antisymm



M = odd, h(n) - antisymm



Case 1 : M = even

$$h(n) = \pm h(M-1-n)$$

$$H(z) = \sum_{n=0}^{M-1} h(n) z^{-n}$$

$$= \sum_{n=0}^{\frac{M}{2}-1} h(n) z^{-n} + \sum_{n=\frac{M}{2}}^{M-1} h(n) z^{-n}$$

$$= \sum_{n=0}^{\frac{M}{2}-1} h(n) z^{-n} \pm \sum_{n=\frac{M}{2}}^{M-1} h(\underline{M-1-n}) z^{-n}$$

substitute $\underline{\underline{k}} = M-1-n$, $n = \frac{M}{2} \Rightarrow k = M-1-\frac{M}{2} = \frac{M}{2}-1$

$n = M-1 \Rightarrow k = M-1-(M-1) = 0$

$$\therefore H(z) = \sum_{n=0}^{\frac{M}{2}-1} h(n) z^{-n} \pm \sum_{k=0}^{\frac{M}{2}-1} h(k) z^{-(M-1-k)}$$

$$k \to n \Rightarrow H(z) = \sum_{n=0}^{\frac{M}{2}-1} h(n) z^{-n} \pm \sum_{n=0}^{\frac{M}{2}-1} h(n) z^{-(M-1-n)}$$

$$H(z) = \sum_{n=0}^{\frac{M}{2}-1} h(n)\left[z^{-n} \pm z^{-(M-1-n)}\right]$$

$$= z^{-\left(\frac{M-1}{2}\right)} \sum_{n=0}^{\frac{M}{2}-1} h(n)\left[z^{\left(\frac{M-1}{2}-n\right)} \pm z^{-\left(M-1-n-\frac{M-1}{2}\right)}\right]$$

$$= z^{-\left(\frac{M-1}{2}\right)} \sum_{n=0}^{\frac{M}{2}-1} h(n)\left[z^{\left(\frac{M-1}{2}-n\right)} \pm z^{-\left(\frac{M-1}{2}-n\right)}\right] \quad \text{—— } (1)$$

$$H(e^{j\omega}) \rightarrow H(z)\big|_{z=e^{j\omega}}$$

(a) M = even, symm. imp response:

$$(1) \Rightarrow H(e^{j\omega}) = e^{-j\omega\left(\frac{M-1}{2}\right)} \sum_{n=0}^{\frac{M}{2}-1} h(n)\Big[\underbrace{e^{j\omega\left(\frac{M-1}{2}-n\right)} + e^{-j\omega\left(\frac{M-1}{2}-n\right)}}_{e^{j\theta}+e^{-j\theta} = 2\cos\theta}\Big]$$

$$= \underbrace{e^{-j\omega\left(\frac{M-1}{2}\right)}}_{\text{phase, } \phi(\omega)} \underbrace{\sum_{n=0}^{\frac{M}{2}-1} h(n)\cdot 2\cos\omega\left(\frac{M-1}{2}-n\right)}_{H_r(\omega)}$$

$$H(e^{j\omega}) = H_r(\omega) \cdot e^{-j\omega\left(\frac{M-1}{2}\right)}$$

Magn resp, $H_r(\omega) = \sum_{n=0}^{M/2-1} h(n) \left[2\cos\omega\left(\frac{M-1}{2}-n\right)\right]$

Phase resp, $\phi(\omega) = -\omega\left(\frac{M-1}{2}\right)$ if $H_r(\omega) > 0$

$= -\omega\left(\frac{M-1}{2}\right)+\pi$ if $H_r(\omega) < 0$.

(b) M = even, ~~antisymm~~.

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{M-1}{2}\right)} \sum_{n=0}^{M/2-1} h(n) \left[\underbrace{e^{j\omega\left(\frac{M-1}{2}-n\right)} - e^{-j\omega\left(\frac{M-1}{2}-n\right)}}\right]$$

$$= e^{-j\omega\left(\frac{M-1}{2}\right)} \sum_{n=0}^{\frac{M}{2}-1} h(n) \left[2j \sin\omega\left(\frac{M-1}{2}-n\right)\right]$$

$j = e^{j\pi/2}$

$$= e^{-j\omega\left(\frac{M-1}{2}\right)} \sum_{n=0}^{\frac{M}{2}-1} h(n) \cdot 2 \cdot e^{j\frac{\pi}{2}} \cdot \sin\omega\left(\frac{M-1}{2}-n\right)$$

$$= \underbrace{e^{-j\omega\left(\frac{M-1}{2}\right)+j\frac{\pi}{2}}}_{\phi(\omega)} \cdot \underbrace{\sum_{n=0}^{\frac{M}{2}-1} h(n) \cdot 2 \cdot \sin\omega\left(\frac{M-1}{2}-n\right)}_{H_r(\omega)}$$

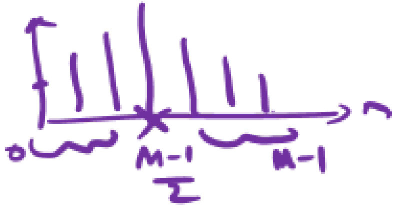
$$= e^{\underbrace{-j\omega\left(\frac{M-1}{2}\right)+j\frac{\pi}{2}}_{\phi(\omega)}} \cdot \underbrace{\sum_{n=0}^{\frac{M}{2}-1} h(n)\cdot 2\cdot \sin\omega\left(\frac{M-1}{2}-n\right)}_{H_r(\omega)}$$

$$H_r(\omega) = \sum_{n=0}^{M/2-1} h(n)\cdot 2 \sin\omega\left(\frac{M-1}{2}-n\right)$$

$$\phi(\omega) = \frac{\pi}{2}-\omega\left(\frac{M-1}{2}\right) \qquad \text{if } H_r(\omega)>0$$

$$= \frac{\pi}{2}-\omega\left(\frac{M-1}{2}\right)+\pi = \frac{3\pi}{2}-\omega\left(\frac{M-1}{2}\right) \qquad \text{if } H_r(\omega)<0$$

Case 2 : M = odd



$$H(z) = \sum_{n=0}^{M-1} h(n)\, z^{-n}$$

$$= \sum_{n=0}^{\frac{M-1}{2}-1} h(n)\, z^{-n} + h\left(\frac{M-1}{2}\right) z^{-\left(\frac{M-1}{2}\right)} + \sum_{n=\frac{M-1}{2}+1}^{M-1} h(n)\, z^{-n}$$

$$= h\left(\frac{M-1}{2}\right) z^{-\left(\frac{M-1}{2}\right)} + \sum_{n=0}^{\frac{M-3}{2}} h(n)\, z^{-n} + \sum_{n=\frac{M+1}{2}}^{M-1} h(n)\, z^{-n} \qquad \boxed{h(n) = \pm h(M-1-n)}$$

$$= h\left(\frac{M-1}{2}\right) z^{-\left(\frac{M-1}{2}\right)} + \sum_{n=0}^{\frac{M-3}{2}} h(n)\, z^{-n} \pm \sum_{n=\frac{M+1}{2}}^{M-1} h(\underline{M-1-n})\, z^{-n}$$

Subs. $k = M-1-n$; $n = \frac{M+1}{2} \Rightarrow k = M-1-\left(\frac{M+1}{2}\right) = \frac{M-3}{2}$

$n = M-1 \Rightarrow k = M-1-(M-1) = 0$

$$H(z) = h\left(\frac{M-1}{2}\right) z^{-\left(\frac{M-1}{2}\right)} + \sum_{n=0}^{\frac{M-3}{2}} h(n)\, z^{-n} \pm \sum_{\cancel{k}\,n=0}^{\frac{M-3}{2}} h(\cancel{k}\,n)\, z^{-(M-1-\cancel{k}\,n)}$$

$$H(z) = h\left(\frac{M-1}{2}\right) z^{-\left(\frac{M-1}{2}\right)} + \sum_{n=0}^{\frac{M-3}{2}} h(n)\left[z^{-n} \pm z^{-(M-1-n)}\right]$$

$$= z^{-\left(\frac{M-1}{2}\right)}\left[h\left(\frac{M-1}{2}\right) + \sum_{n=0}^{\frac{M-3}{2}} h(n)\left\{z^{\left(\frac{M-1}{2}-n\right)} \pm z^{-\left(\frac{M-1}{2}-n\right)}\right\}\right] \quad ②$$

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{M-1}{2}\right)}\left[h\left(\frac{M-1}{2}\right) + \sum_{n=0}^{\frac{M-3}{2}} h(n)\left\{\underbrace{e^{j\omega\left(\frac{M-1}{2}-n\right)} \pm e^{-j\omega\left(\frac{M-1}{2}-n\right)}}\right\}\right]$$

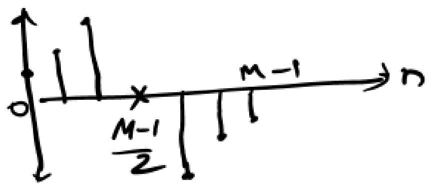
(a) M = odd, symm imp. resp:

$$H_R(\omega) = h\left(\frac{M-1}{2}\right) + \sum_{n=0}^{\frac{M-3}{2}} h(n) \cdot 2 \cdot \cos\omega\left(\frac{M-1}{2}-n\right)$$

$$\phi(\omega) = -\omega\left(\frac{M-1}{2}\right) \qquad \text{if } H_R(\omega) > 0$$

$$= -\omega\left(\frac{M-1}{2}\right) + \pi \qquad \text{if } H_R(\omega) < 0$$

M=odd, $h(n)$ -antisymm



(b) M = odd, antisymm.

$$H_R(\omega) = \underbrace{h\left(\frac{M-1}{2}\right)}_{=0} + \sum_{n=0}^{\frac{M-3}{2}} h(n) \cdot 2 \cdot \sin\omega\left(\frac{M-1}{2}-n\right)$$

(b) $M = \text{odd}$, antisymm

$$H_r(\omega) = \sum_{n=0}^{\frac{M-3}{2}} h(n) \cdot 2 \sin \omega \left(\frac{M-1}{2} - n \right)$$

$$\phi(\omega) = \frac{\pi}{2} - \omega \left(\frac{M-1}{2} \right) \quad \text{if } H_r(\omega) > 0$$

$$= \frac{3\pi}{2} - \omega \left(\frac{M-1}{2} \right) \quad \text{if } H_r(\omega) < 0.$$

Linear phase FIR filter frequency response	i) Symmetric impulse response, odd length $$H(\omega) = e^{-j\omega\left(\frac{M-1}{2}\right)}\left[2\sum_{n=0}^{\frac{M-3}{2}} h(n)cos\left\{\omega\left(\frac{M-1}{2}-n\right)\right\}\right]$$ ii) Symmetric impulse response, even length $$H(\omega) = e^{-j\omega\left(\frac{M-1}{2}\right)}\left[2\sum_{n=0}^{\frac{M}{2}-1} h(n)cos\left\{\omega\left(\frac{M-1}{2}-n\right)\right\}\right]$$ iii) Anti-symmetric impulse response, odd length $$H(\omega) = je^{-j\omega\left(\frac{M-1}{2}\right)}\left[2\sum_{n=0}^{\frac{M-3}{2}} h(n)sin\left\{\omega\left(\frac{M-1}{2}-n\right)\right\}\right]$$ iv) Anti-symmetric impulse response, even length $$H(\omega) = je^{-j\omega\left(\frac{M-1}{2}\right)}\left[2\sum_{n=0}^{\frac{M}{2}-1} h(n)sin\left\{\omega\left(\frac{M-1}{2}-n\right)\right\}\right]$$

Choice of filters:

1) Antisymm, odd : $H_R(\omega) = 0$ when $\omega = 0$ & $\omega = \pi$

LPF ✗, HPF ✗

2) Antisymm, even : $H_R(\omega) = 0$ when $\omega = 0$

LPF ✗

3) Symmetric imp resp is used for design of LPF, HPF, BPF, BSF.

Zero location symmetry of linear phase FIR:

$$H(z) = \sum_{k=0}^{M-1} h(k) z^{-k}$$

$$= h(0) + h(1) z^{-1} + \dots + h(M-2) z^{-(M-2)} + h(M-1) z^{-(M-1)}$$

Linear phase:
$$= h(0) + h(1) z^{-1} + \dots \pm h(1) z^{-(M-2)} \pm h(0) z^{-(M-1)}$$

$$= h(0)\left[1 \pm z^{-(M-1)}\right] + h(1)\left[z^{-1} \pm z^{-(M-2)}\right] + \dots \quad (1)$$

$$H(z^{-1}) = h(0)\left[1 \pm z^{(M-1)}\right] + h(1)\left[z^{1} \pm z^{+(M-2)}\right] + \dots$$

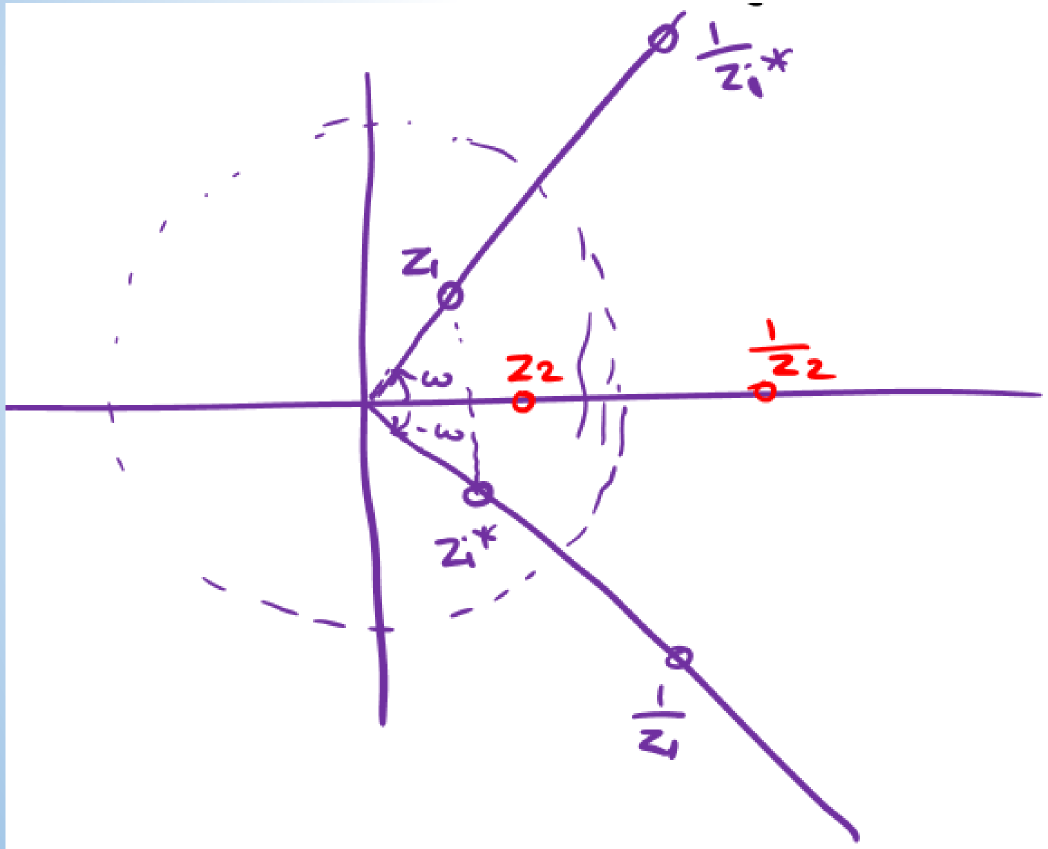
$$= z^{M-1} \cdot \left\{ h(0)\left[z^{-(M-1)} \pm z^{0}\right] + h(1)\left[z^{-(M-2)} \pm z^{-1}\right] + \dots \right\}$$

$$\boxed{z^{-(M-1)} H(z^{-1}) = \pm H(z)}$$

Roots of $H(z)$ & $H(z^{-1})$ are identical.

$H(z) \downarrow z_i$; $H(z^{-1}) \downarrow \frac{1}{z_i}$; $\downarrow$ reciprocal pairs

- If z_i is a zero of $H(z)$, $\frac{1}{z_i}$ is also a zero (reciprocal pair)
- If $h(n)$ is real, roots – complex conjugate pairs $\left(z_i^*, \frac{1}{z_i^*}\right)$



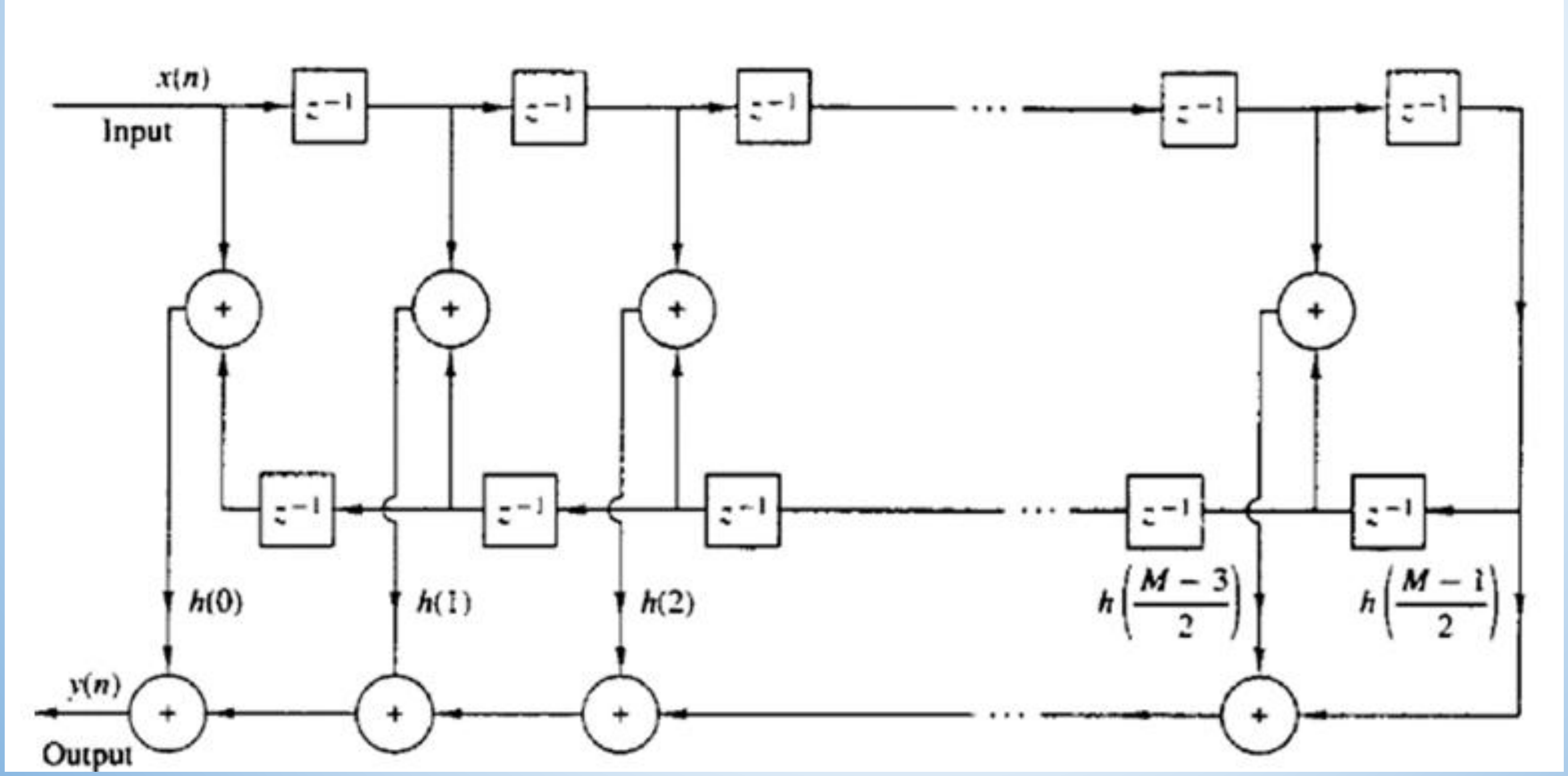
$$z_i = r_i e^{j\phi_i}$$

$$\frac{1}{z_i} = \frac{1}{r_i} e^{-j\phi_i}$$

HW: Draw an efficient tapped delay line structure for linear phase FIR with M = odd & symm imp resp.

Soln:

$$H(z) = h\left(\frac{M-1}{2}\right) z^{-\left(\frac{M-1}{2}\right)} + \sum_{n=0}^{\frac{M-3}{2}} h(n)\left[z^{-n} + z^{-(M-1-n)}\right]$$




Thank you