Introduction to Discrete Fourier Transform Frequency domain sampling

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Review of Fourier representation of discrete signals

$$x[n] \longleftrightarrow X(e^{j\omega})$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

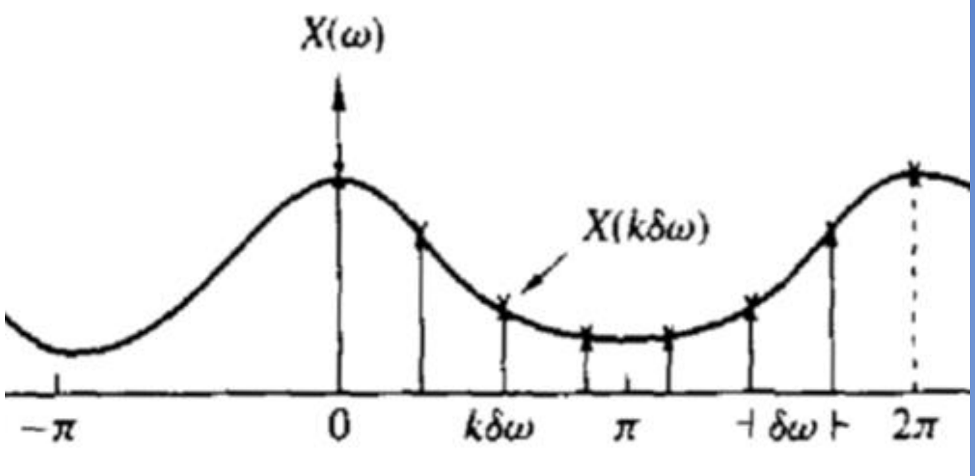
– Analysis Equation
– DTFT

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$$

– Synthesis Equation
– Inverse DTFT

- The DTFT and inverse DTFT are not symmetric. One is integration over a finite interval (2π), and the other is summation over infinite terms
- The signal, x[n] is aperiodic, and hence, the transform is a continuous function of frequency
- Not practical for (real-time) computation on a digital computer
- Go for Discrete Fourier Transform

Frequency Domain Sampling



- Consider an aperiodic signal $x(n)$ finite duration signal with FT

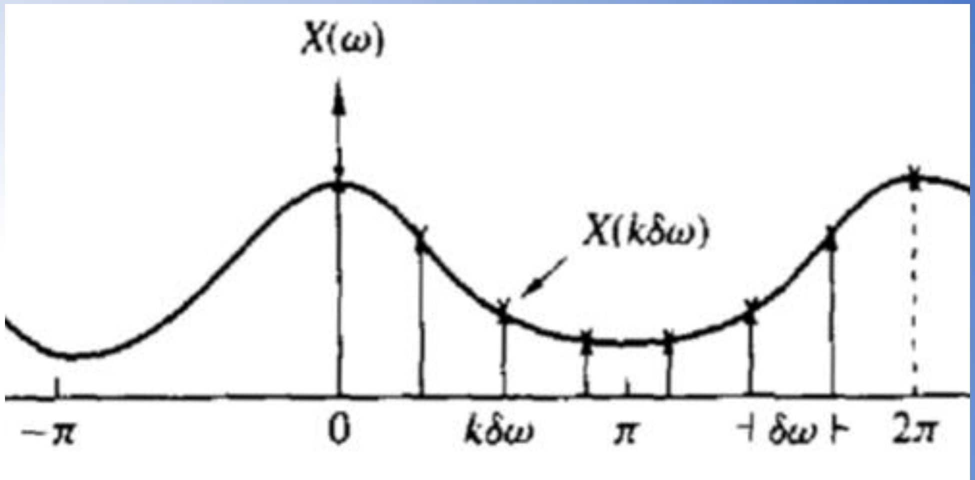
$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \qquad (1)$$

- Suppose we sample $X(\omega)$ periodically in frequency at spacing $\delta\omega$ radians between successive samples.

This derivation is not there for the exam

- Since $X(\omega)$ is periodic with period 2π, therefore only samples in the fundamental frequency range are required.
- Let we take *N* equidistant samples ➔ $\delta\omega = \dfrac{2\pi}{N}$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \qquad (1)$$



- Now calculate Eq. (1) at $\omega_k = \dfrac{2\pi k}{N}$, i.e.,

$$X\left(\frac{2\pi k}{N}\right) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\frac{2\pi kn}{N}} \qquad (2)$$

- Summation in Eq. (2) can be written as

$$X\left(\frac{2\pi k}{N}\right) = \cdots + \sum_{n=-N}^{-1} x(n) e^{-j\frac{2\pi kn}{N}} + \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi kn}{N}}$$

$$+ \sum_{n=N}^{2N-1} x(n) e^{-j\frac{2\pi kn}{N}} + \cdots$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n) e^{-j\frac{2\pi kn}{N}} \qquad (3)$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n) e^{-j\frac{2\pi kn}{N}} \qquad (3)$$

- If we change the index in the inner summation from $n \rightarrow n - lN$ & interchanging the order of summation, we have,

$$X\left(\frac{2\pi k}{N}\right) = \sum_{n=0}^{N-1} \left[\sum_{l=-\infty}^{\infty} x(n-lN)\right] e^{-j\frac{2\pi kn}{N}} \qquad (4)$$

For $k = 0, 1, \ldots, N-1$

- Here, signal

$$x_p(n) = x(n-lN)$$

is a periodic sequence with fundamental period N

$$X\left(\frac{2\pi k}{N}\right) = \sum_{n=0}^{N-1} x_p(n)\, e^{-j\frac{2\pi kn}{N}}$$

For $k = 0, 1, \ldots, N-1$

Discrete Fourier Transform (DFT)

$$X\left(\frac{2\pi k}{N}\right)=\sum_{n=0}^{N-1} x_p(n)\, e^{-j\frac{2\pi kn}{N}} \qquad (4)$$

- The $x_p(n)$ can be expresses using FS as,

$$x_p(n)=\sum_{k=0}^{N-1} c_k\, e^{j\frac{2\pi kn}{N}} \quad n=0,1,\cdots,N-1 \qquad (5)$$

where Fourier coefficient c_k is given as,

$$c_k=\frac{1}{N}\sum_{n=0}^{N-1} x_p(n)\, e^{-j\frac{2\pi kn}{N}} \quad k=0,1,\cdots,N-1 \qquad (6)$$

By comparing Eq. (4) & (6) we observe,

$$c_k=\frac{1}{N}X\left(\frac{2\pi k}{N}\right) \qquad (7)$$

Substitute Eq. (7) in (5)

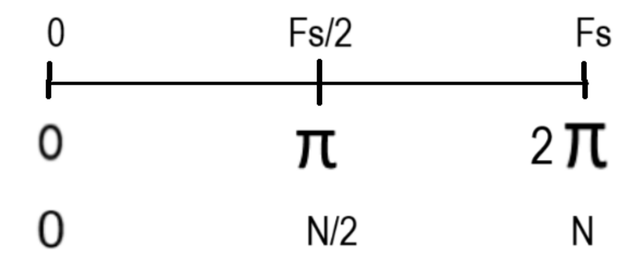
$$x_p(n)=\frac{1}{N}\sum_{k=0}^{N-1} X\left(\frac{2\pi k}{N}\right)e^{j\frac{2\pi kn}{N}} \quad n=0,1,\cdots,N-1 \qquad (8)$$

Inverse Discrete Fourier Transform (IDFT)

Discrete Fourier Transform (DFT)

$$X\left(\frac{2\pi k}{N}\right)=\sum_{n=0}^{N-1} x_p(n)\, e^{-j\frac{2\pi kn}{N}}$$

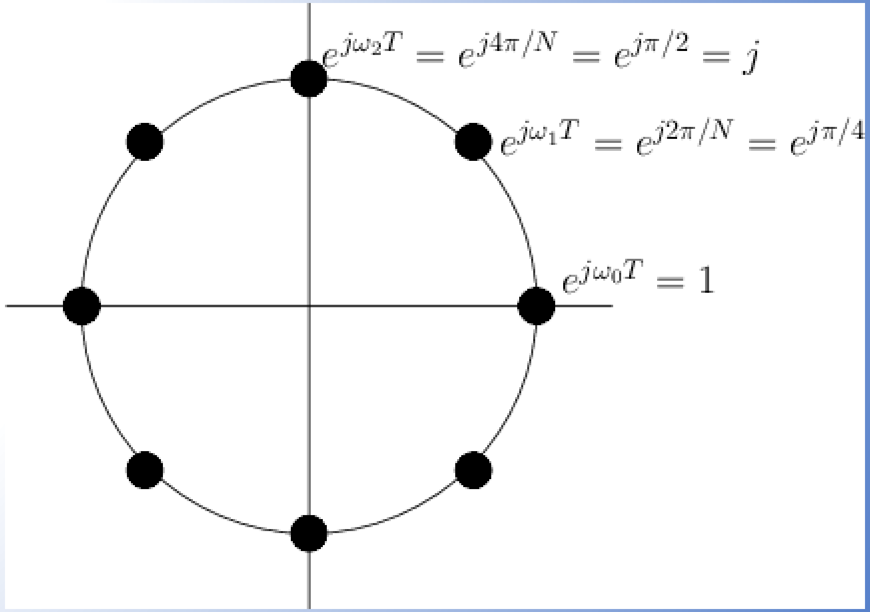
For $k = 0,\ 1,\ \ldots,\ N-1$



Inverse Discrete Fourier Transform (IDFT)

Fs = sampling frequency

$$x_p(n)=\frac{1}{N}\sum_{k=0}^{N-1} X\left(\frac{2\pi k}{N}\right)e^{j\frac{2\pi kn}{N}} \quad n=0,1,\cdots,N-1$$



$$\text{DFT:} \quad X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \qquad k = 0, 1, 2, \ldots, N-1$$

$$\text{IDFT:} \quad x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi kn/N}, \qquad n = 0, 1, 2, \ldots, N-1$$

Properties of the DFT

N-point DFT pair $x(n)$ and $X(k)$ is $x(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X(k)$

Periodicity. If $x(n)$ and X(k) are an N-point DFT pair, then

$$x(n+N) = x(n) \qquad \text{for all } n$$

$$X(k+N) = X(k) \qquad \text{for all } k$$

Linearity. If

$$x_1(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_1(k)$$

and

$$x_2(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_2(k)$$

then for any real-valued or complex-valued constants a_1 and a_2,

$$a_1 x_1(n) + a_2 x_2(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} a_1 X_1(k) + a_2 X_2(k)$$

TABLE 7.1 Symmetry Properties of the DFT

N-Point Sequence $x(n)$, $0 \le n \le N-1$	N-Point DFT
$x(n)$	$X(k)$
$x^*(n)$	$X^*(N-k)$
$x^*(N-n)$	$X^*(k)$
$x_R(n)$	$X_{ce}(k) = \frac{1}{2}[X(k) + X^*(N-k)]$
$jX_I(n)$	$X_{co}(k) = \frac{1}{2}[X(k) - X^*(N-k)]$
$x_{ce}(n) = \frac{1}{2}[x(n) + x^*(N-n)]$	$X_R(k)$
$x_{co}(n) = \frac{1}{2}[x(n) - x^*(N-n)]$	$jX_I(k)$
Real Signals	
Any real signal	$X(k) = X^*(N-k)$
$x(n)$	$X_R(k) = X_R(N-k)$
	$X_I(k) = -X_I(N-k)$
	$\lvert X(k)\rvert = \lvert X(N-k)\rvert$
	$\angle X(k) = -\angle X(N-k)$
$x_{ce}(n) = \frac{1}{2}[x(n) + x(N-n)]$	$X_R(k)$
$x_{co}(n) = \frac{1}{2}[x(n) - x(N-n)]$	$jX_I(k)$

Next slide contains the proof

Example: Symmetry property

Any real signal $X(k) = X^*(N-k)$

$$X^*(k) = \left[\sum_{n=0}^{N-1} x(n)\ e^{-j2\pi kn/N}\right]^*$$

$$= \left[\sum_{n=0}^{N-1} x(n)\ e^{j2\pi kn/N} . 1\right] \qquad \{x^*(n) = x(n) \text{ for real sequence}\}$$

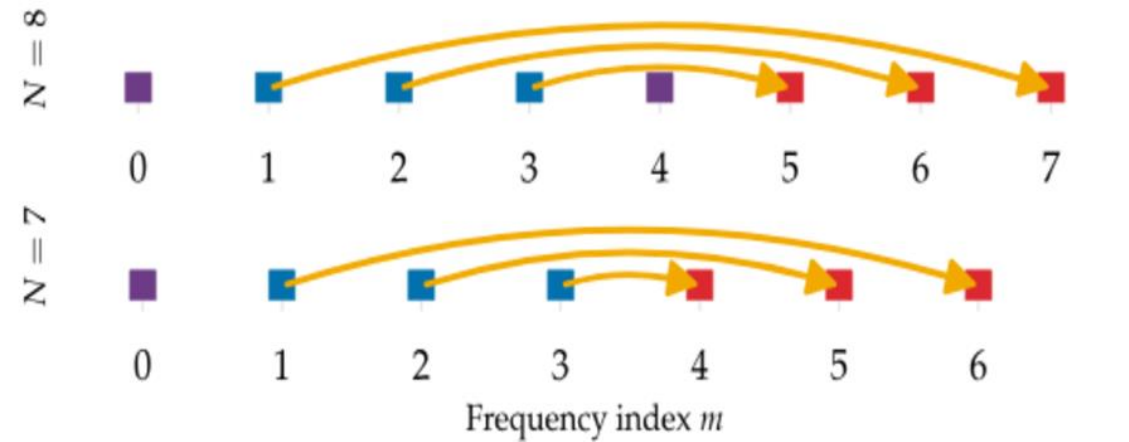
$$= \left[\sum_{n=0}^{N-1} x(n)\ e^{j2\pi kn/N} . e^{-j2\pi n}\right] \qquad \{\because e^{-j2\pi n} = 1\}$$

$$= \left[\sum_{n=0}^{N-1} x(n)\ e^{j2\pi kn/N} . e^{-j2\pi nN/N}\right]$$

$$= \left[\sum_{n=0}^{N-1} x(n)\ e^{-j2\pi(N-k)n/N}\right]$$

$$X^*(k) = X(N-k)$$

Take conjugate on both sides to prove $X(k) = X^*(N-k)$



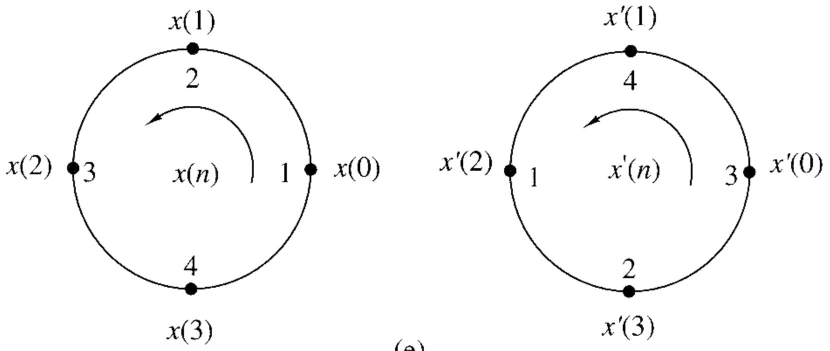
Circular shift of a sequence:

In general, the circular shift of the sequence can be represented as the index modulo N. Thus we can write

$$x'(n) = x(n-k, \text{modulo } N)$$
$$\equiv x((n-k))_N$$

For example, if $k = 2$ and $N = 4$, we have

$$x'(n) = x((n-2))_4$$



(e)

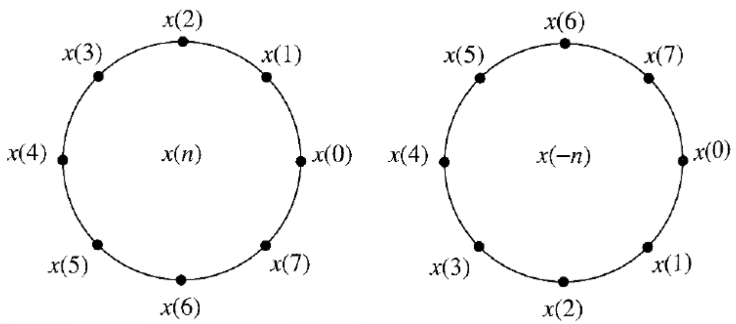
which implies that

$$x'(0) = x((-2))_4 = x(2)$$
$$x'(1) = x((-1))_4 = x(3)$$
$$x'(2) = x((0))_4 = x(0)$$
$$x'(3) = x((1))_4 = x(1)$$

Hence $x'(n)$ is simply $x(n)$ shifted circularly by two units in time, where the counter-clockwise direction has been arbitrarily selected as the positive direction.

Next property: **Time reversal of a sequence**

$$x((-n))_N = x(N-n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X((-k))_N = X(N-k)$$



$$\text{DFT}\{x(N-n)\} = \sum_{n=0}^{N-1} x(N-n)e^{-j2\pi kn/N}$$

change the index from n to $m = N - n$, then

$$\text{DFT}\{x(N-n)\} = \sum_{m=0}^{N-1} x(m)e^{-j2\pi k(N-m)/N}$$

$$= \sum_{m=0}^{N-1} x(m)e^{j2\pi km/N}$$

$$= \sum_{m=0}^{N-1} x(m)e^{-j2\pi m(N-k)/N} = X(N-k)$$


Thank you