Analog filter design
Chebyshev Filter

Dr. Sampath Kumar

Associate Professor

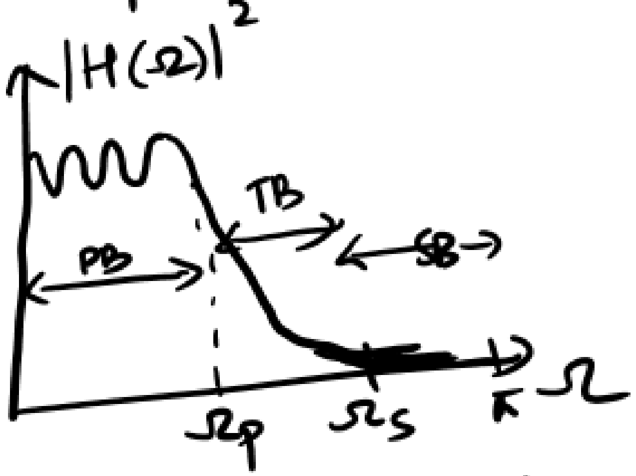
Department of ECE

MIT, Manipal

Chebyshev filter design

Type I Chebyshev

- all pole filters

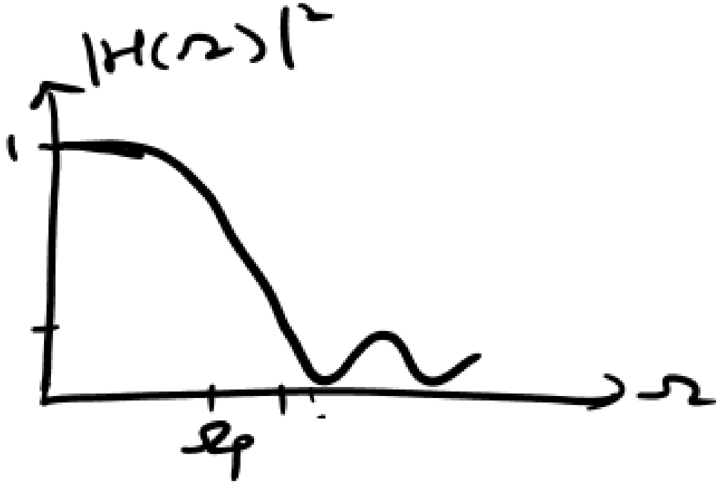


- equiripple behaviour in passband
- monotonic behaviour in stopband

✓

Type II Chebyshev

- both poles & zeros



- monotonic - PB
- equiripples - SB

X

Chebyshev Type I filter:

$$|H(\Omega)|^2 = \frac{1}{1+\varepsilon^2 T_N^2\left(\frac{\Omega}{\Omega_p}\right)}$$

$$A_p = \frac{1}{\sqrt{1+\varepsilon^2}} = 1-\delta_p$$

$T_N(x)$ – Chebyshev polynomial.

$$T_N(x) = \cos(N\cos^{-1}x), \quad |x| \le 1$$
$$= \cosh(N\cosh^{-1}x), \quad |x| > 1$$

$\cos(a+b) = \cos a \cos b - \sin a \sin b$

$T_N(x), T_{N-1}(x), T_{N+1}(x)$

$x = \cos\theta$

$$T_N(x) = \cos(N\cos^{-1}x) = \cos(N\cos^{-1}\cos\theta) = \cos(N\theta)$$

$$T_{N+1}(x) = \cos\left[(N+1)\cos^{-1}x\right] = \cos\left[(N+1)\theta\right]$$
$$= \cos N\theta\cos\theta - \sin N\theta\sin\theta.$$

$T_N(x) = \cos(N\theta)$

$T_{N+1}(x) = \cos[(N+1)\theta] = \cos N\theta \cos\theta - \sin N\theta \sin\theta$.

$T_{N-1}(x) = \cos[(N-1)\theta] = \cos N\theta \cos\theta + \sin N\theta \sin\theta$.

$T_{N+1}(x) + T_{N-1}(x) = 2\cos N\theta \cos\theta$.

$= 2T_N(x)\cdot x$

$T_{N+1}(x) = 2T_N(x)\cdot x - T_{N-1}(x)$

when $N=0$, $T_0(x) = \cos(0\cdot\cos^{-1}x) = 1$

$N=1$, $T_1(x) = \cos(1\cdot\cos^{-1}x) = x$

$N=2$, $T_2(x) = 2T_1(x)\cdot x - T_0(x) = 2x^2 - 1$

$N=3$, $T_3(x) = 2T_2(x)\cdot x - T_1(x) =$

$= 2x(2x^2-1) - x = 4x^3 - 3x$.

Properties of Cheb polynomials :

1) $|T_N(x)| \le 1$ for all $|x| \le 1$

2) $|T_N(x)| > 1$ for all $|x| > 1$

3) N = odd, $T_N(x)$ odd fn of x.

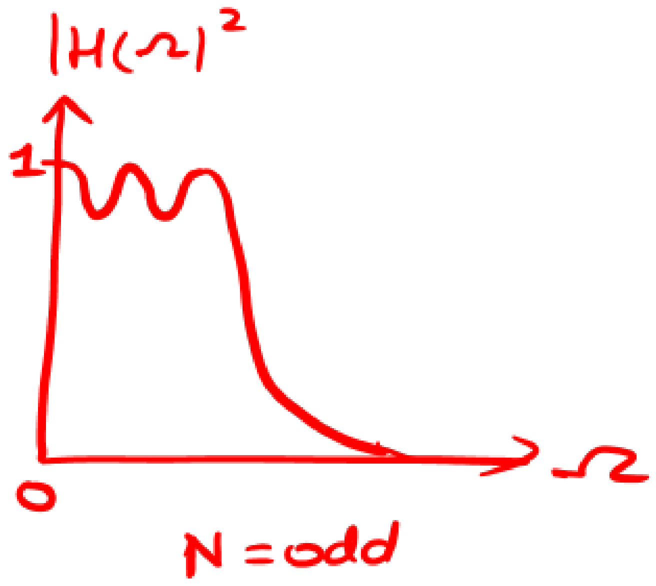
N = even, $T_N(x)$ even "

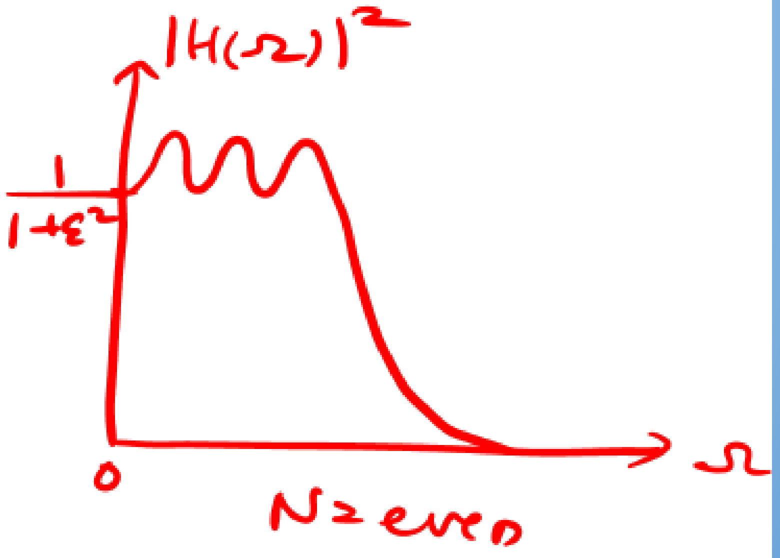
4) $T_N(1) = 1$, for all N

5) roots of $T_N(x)$ ± 1, $-1 \le x \le 1$

6) N = odd, $T_N(0) = 0 \Rightarrow |H(0)|^2 = 1$

N = even, $T_N(0) = \pm 1 \Rightarrow |H(0)|^2 = \frac{1}{1+\epsilon^2}$





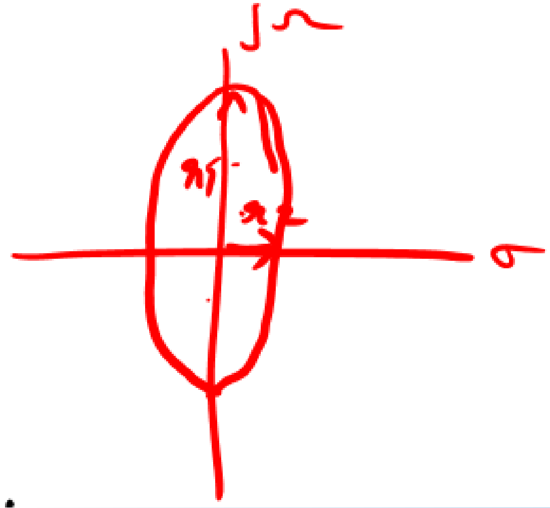
Location of poles of Type I cheb filter:

$$|H(\Omega)|^2 = \frac{1}{1+\varepsilon^2 T_N^2\left(\frac{j\Omega}{j\Omega_p}\right)} = \frac{1}{1+\varepsilon^2 T_N^2\left(\frac{s}{j\Omega_p}\right)} = H(s)\cdot H(-s)$$

$$\text{dmr} = 0, \quad 1+\varepsilon^2 T_N^2\left(\frac{s}{j\Omega_p}\right) = 0$$

$$s_k = \sigma_k + j\Omega_k.$$

Eqn of ellipse : $\dfrac{\sigma_k^2}{r_2^2} + \dfrac{\Omega_k^2}{r_1^2} = 1$



$$r_1 = \Omega_p\left(\frac{\beta^2+1}{2\beta}\right)$$

$$r_2 = \Omega_p\left(\frac{\beta^2-1}{2\beta}\right)$$

$$\beta = \left[\frac{\sqrt{1+\varepsilon^2}+1}{\varepsilon}\right]^{1/N}$$

(x_k, y_k)

$$x_k = r_2 \cos\phi_k$$

$$y_k = r_1 \sin\phi_k.$$

$$\phi_k = \frac{\pi}{2} + (2k+1)\frac{\pi}{2N}$$

$$x_k + jy_k = r_2\cos\phi_k + j\, r_1 \sin\phi_k$$

Order of Type 1 Chebyshev filter.

$$|H(\Omega)|^2 = \frac{1}{1+\varepsilon^2 T_N^2\left(\frac{\Omega}{\Omega_p}\right)}$$

$A_p^2 = \frac{1}{1+\varepsilon^2}$

At $\Omega = \Omega_s$, $|H(\Omega_s)|^2 = A_s^2 = \dfrac{1}{1+\varepsilon^2 T_N^2\left(\frac{\Omega_s}{\Omega_p}\right)}$

$$\Rightarrow 1+\varepsilon^2 T_N^2\left(\frac{\Omega_s}{\Omega_p}\right) = \frac{1}{A_s^2}$$

$$T_N^2\left(\frac{\Omega_s}{\Omega_p}\right) = \frac{\frac{1}{A_s^2}-1}{\varepsilon^2} = \frac{\frac{1}{A_s^2}-1}{\frac{1}{A_p^2}-1} = \frac{1}{d^2}$$

$$d = \sqrt{\frac{1/A_p^2 - 1}{1/A_s^2 - 1}}$$

$$k = \frac{\Omega_p}{\Omega_s}$$

$$A_p^2 = \frac{1}{1+\varepsilon^2}$$

$$T_N^2\left(\frac{1}{k}\right) = \frac{1}{d^2}$$

$$T_N\left(\frac{1}{k}\right) = \frac{1}{d}$$

$|x|>1$, $\cosh\left(N \cosh^{-1}(1/k)\right) = \frac{1}{d}$

$$\boxed{N = \frac{\cosh^{-1}(1/d)}{\cosh^{-1}(1/k)}} \quad \text{—— (1)}$$

$$N = \frac{\cosh^{-1}(1/d)}{\cosh^{-1}(1/k)}$$

$$\delta_s = \frac{1}{\sqrt{1+\delta^2}}$$

$$\frac{1}{d} = \sqrt{\frac{\frac{1}{A_s^2}-1}{\frac{1}{A_p^2}-1}} = \frac{\sqrt{\frac{1}{\delta_s^2}-1}}{\varepsilon} = \frac{\delta}{\varepsilon}$$

$$\boxed{N = \frac{\cosh^{-1}(\delta/\varepsilon)}{\cosh^{-1}(\Omega_s/\Omega_p)}} \quad \text{—} \ (2)$$

Q. Design a digital Chebyshev filter that satisfies the constraints

$$0.707 \le |H(e^{j\omega})| \le 1 \quad , \quad 0 \le \omega \le 0.2\pi = \omega_p$$

$= A_p$

$$|H(e^{j\omega})| \le 0.1 = A_s \quad , \quad 0.5\pi \le \omega \le \pi$$

$= \omega_s$

Use bilinear transf and assume $T = 1\,\text{sec}$.

Soln: Given, A_p, A_s, ω_p, ω_s

$$\Omega_p = \frac{2}{T} \tan \frac{\omega_p}{2} = 2 \tan \frac{0.2\pi}{2} = 2 \tan 0.1\pi = 0.649$$

$$\Omega_s = \frac{2}{T} \tan \frac{\omega_s}{2} = 2 \tan \frac{0.5\pi}{2} = 2$$

Order of filter, $N = \dfrac{\cosh^{-1}(1/d)}{\cosh^{-1}(1/k)} = \dfrac{\cosh^{-1}(1/0.1)}{\cosh^{-1}(1/0.32)} = 1.65 \approx \underline{\underline{2}}$

$$d = \sqrt{\frac{\frac{1}{A_p^2} - 1}{\frac{1}{A_s^2} - 1}} = \sqrt{\frac{\frac{1}{0.707^2} - 1}{\frac{1}{0.1^2} - 1}} = 0.1 \quad , \quad k = \frac{\Omega_p}{\Omega_s} = \frac{0.649}{2} = 0.32$$

$$x_k = r_2 \cos \phi_k$$

$$y_k = r_1 \sin \phi_k$$

$$r_1 = \Omega_p \left(\frac{\beta^2 + 1}{2\beta} \right)$$

$$r_2 = \Omega_p \left(\frac{\beta^2 - 1}{2\beta} \right)$$

$$\beta = \left[\frac{1 + \sqrt{1 + \varepsilon^2}}{\varepsilon} \right]^{1/N}$$

$$\varepsilon = \sqrt{\frac{1}{A_p^2} - 1}$$

$N = 2$

Poles : r_1, r_2, ϕ_k

$$\varepsilon = \sqrt{\frac{1}{A_p^2} - 1} = \sqrt{\frac{1}{0.707^2} - 1} = 1$$

$$\beta = \left[\frac{1 + \sqrt{1 + 1^2}}{1} \right]^{1/2} = 1.553$$

$$r_1 = 0.649 \left(\frac{1.553^2 + 1}{2 \times 1.553} \right) = 0.7139$$

$$r_2 = 0.649 \left(\frac{1.553^2 - 1}{2 \times 1.553} \right) = 0.295$$

$$\phi_k = \frac{\pi}{2} + (2k+1)\frac{\pi}{2N} \quad , \; k = 0, 1$$

$$\phi_0 = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$$

$$\phi_1 = \frac{\pi}{2} + \frac{3\pi}{4} = \frac{5\pi}{4} = -\frac{3\pi}{4}$$

$$\phi_k = \pm \frac{3\pi}{4}$$

$$x_k + j y_k = r_2 \cos\phi_k + j\, r_1 \sin\phi_k$$

$$= 0.2949 \cos\left(\frac{3\pi}{4}\right) + j\, 0.7139 \sin\left(\frac{3\pi}{4}\right)$$

$$= -0.209 \pm j\, 0.5048$$

To find the s/m fn:

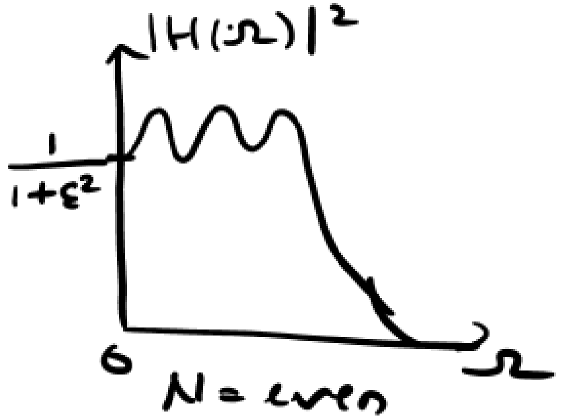
$$H(s) = \frac{\text{Constant} = C}{[s-(-0.209+j0.5048)][s-(-0.209-j0.5048)]}$$

$$= \frac{C}{(\underbrace{s+0.209}_{a} - j\underbrace{0.5048}_{b})(\underbrace{s+0.209}_{a} + j\underbrace{0.5048}_{b})}$$

$(a+b)(a-b) = a^2 - b^2$

$$= \frac{C}{(s+0.209)^2 - (j0.5048)^2}$$

$$H(s) = \frac{C}{s^2 + 0.418s + 0.299}$$



At $\Omega = 0$, $|H(0)| = A_p = \frac{1}{\sqrt{1+\varepsilon^2}}$ if N = even

$$H(s)\Big|_{s=0} = 0.707$$

$$\frac{C}{0.299} = 0.707 \Rightarrow C = 0.211$$

$$H(s) = \frac{0.211}{s^2 + 0.418s + 0.299}$$

$$H(z) = \frac{0.211}{4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 0.418 \times 2\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.299}$$

$$H(z) = \frac{0.211\,(1+z^{-1})^2}{4(1-z^{-1})^2 + 0.836(1-z^{-1})(1+z^{-1}) + 0.299\,(1+z^{-1})^2}$$

$$= \frac{0.211 + 0.422\,z^{-1} + 0.211\,z^{-2}}{5.135 - 7.402\,z^{-1} + 3.463\,z^{-2}} \quad //$$

$$= \frac{5.135\,(0.04 + 0.08\,z^{-1} + 0.04\,z^{-2})}{5.135\,(1 - 1.44\,z^{-1} + 0.67\,z^{-2})}$$

$$= \frac{0.04 + 0.08\,z^{-1} + 0.04\,z^{-2}}{1 - 1.44\,z^{-1} + 0.67\,z^{-2}} \quad //$$

HW

- Solve the above problem using impulse invariance method


Thank
you