Power Spectrum Estimation

The estimation of spectral characteristics of signals (more specifically random signals) is an important ~~issue~~ aspect in signal processing.

First, let us consider finite energy signal $x_a(t)$.

Then energy $E = \int_{-\infty}^{\infty} |x_a(t)|^2 \, dt < \infty$. Fourier transform exists and is $X_a(F) = \int_{-\infty}^{\infty} x_a(t) e^{-j 2\pi F t} dt$

From Parseval relation, $E = \int_{F=-\infty}^{\infty} |X_a(F)|^2 dF$

The quantity $|X_a(F)|^2 = S_{xx}(F) \equiv$ Energy spectrum density (ESD) --- (1)

This quantity represent distribution of signal energy as a function of frequency. Total energy in any frequency band can be obtained by integrating ESD.

If $R_{xx}(\tau)$ is autocorrelation function i.e

$R_{xx}(\tau) = \int_{t=-\infty}^{\infty} x_a^*(t)\, x_a(t+\tau)\, dt$, then it follows

that $S_{xx}(F) = FT[R_{xx}(\tau)] = \int_{\tau=-\infty}^{\infty} R_{xx}(\tau) e^{-j 2\pi F\tau} d\tau$.

--- (2)

Signal $x_a(t)$ is bandlimited to B and sampled at rate $F_s \geq 2B$ to get sampled data $x(n)$.

With normalised frequency $f = \frac{F}{F_s}$,

$$X(f) = \sum_{n=-\infty}^{\infty} x(n) e^{-j 2\pi f n}$$

From sampling theory with $f = \frac{F}{F_s}$, we know that

$$X(f) = F_s \cdot \sum_{k=-\infty}^{\infty} X_a(F - kF_s)$$

With no aliasing, $X(f) = F_s \cdot X_a(F) \; ; \quad |F| \leq F_s/2$

$$\therefore \quad S_{xx}(f) = |X(f)|^2 = F_s^2 \, |X_a(f)|^2 \quad \text{---} (3)$$

If $r_{xx}(m)$ is autocorrelation, then

$$S_{xx}(f) = \sum_{m=-\infty}^{\infty} r_{xx}(m) \, e^{-j2\pi f m} \quad \text{---} (4)$$

similarly for random signals (Power signals) the power spectrum density (PSD) $= \Gamma_{xx}(F)$ is

$$\Gamma_{xx}(F) = |X_a(F)|^2 = \int_{-\infty}^{\infty} \gamma_{xx}(\tau) \, e^{-j2\pi F\tau} \, d\tau \quad \text{where} \quad \text{---} (5)$$

$\gamma_{xx}(\tau)$ is the autocorrelation function.

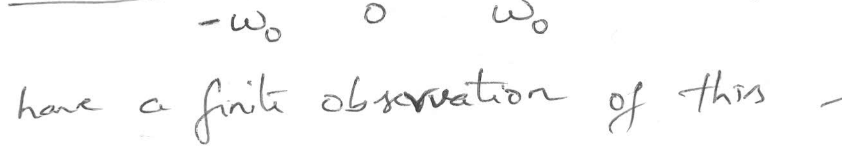
In all these definitions it is assumed that the signal is of infinite duration. But in practice we need to estimate ESD or PSD from the observation of signal over a finite time interval. The finite record length of the data sequence is a major limitation on the quality of the spectrum. The effects are i) Spectral leakage ii) Resolution problem (Resolution reduces). This is illustrated below.

consider $x(n) = \cos(\omega_0 n)$

Then ~~$X(\omega)$ = DTFT of $x(n)$~~ = Spectrum of $x(n)$ i.e

$$\text{DTFT}[x(n)] = X(\omega) = \frac{1}{2}\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$$

Plot of this is



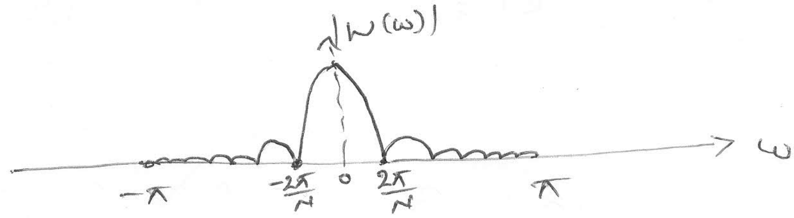
When we have a finite observation of this signal i.e

$\tilde{x}(n) = x(n) \cdot w(n)$ where $w(n)$ is rectangular window. If we take finite data length $= N$, then

$$w(n) = 1, \quad 0 \le n \le N-1$$
$$= 0 \quad \text{elsewhere.}$$

DTFT of $w(n) = W(\omega) = \frac{\sin(\omega N/2)}{\sin(\omega/2)} e^{-j\omega(N-1)/2}$

Plot of $|W(\omega)|$ is as follows



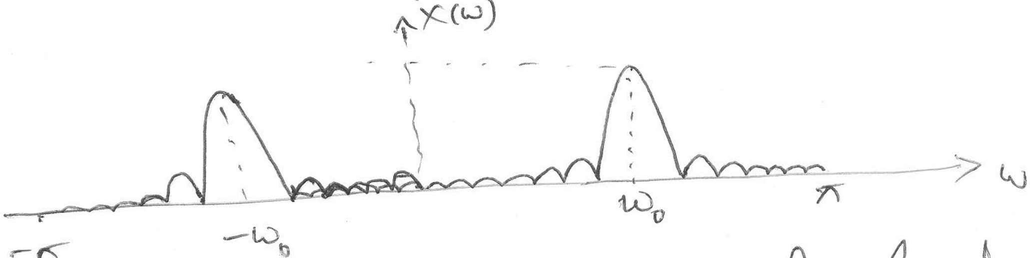
Now from DTFT property

$\tilde{X}(\omega) = X(\omega) * W(\omega)$

(Multiplication in time $\equiv$ convolution in Fourier domain)

$\therefore \tilde{X}(\omega) = \frac{1}{2}\left[W(\omega-\omega_0) + W(\omega+\omega_0)\right]$

This spectrum is plotted below



We note that the spectrum $X(\omega)$ is localised to single frequency whereas spectrum $\tilde{X}(\omega)$ is not; instead it is spread out ~~to~~ over whole frequency range. This means that power is "leaked out" into the entire frequency range. This is "spectral leakage" and due to windowing. ~~Leakage~~ Tapering the window will reduce the leakage.

The second effect of windowing (to get finite length data) that it reduces spectral resolution.

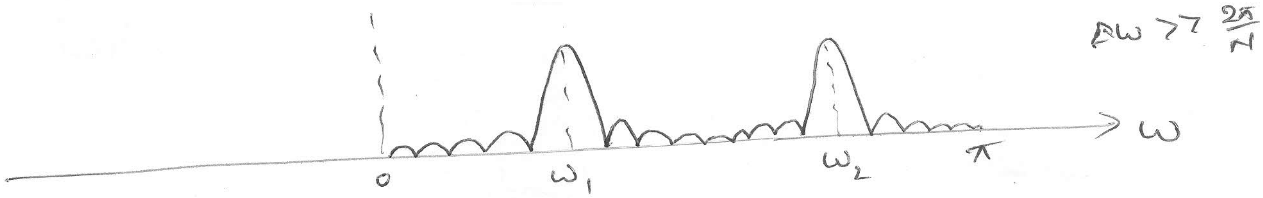
Let $x(n) = \cos(\omega_1 n) + \cos\omega_2(n)$. The difference b/w successive frequency is $\Delta\omega = (\omega_2 - \omega_1)$. Minimum $\Delta\omega$ that can be resolved in spectral resolution.

As before let $\bar{x}(n) = x(n) \cdot w(n)$

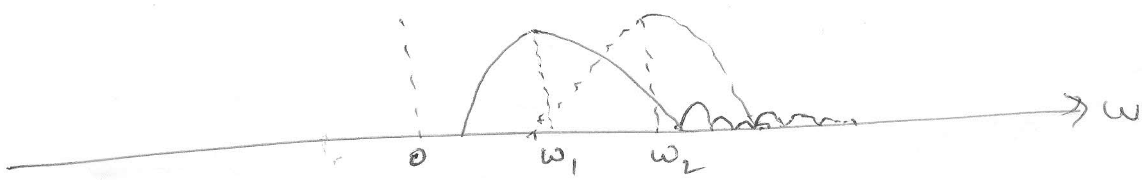
Then $\tilde{X}(\omega) = \frac{1}{2}\left[W(\omega-\omega_1) + W(\omega-\omega_2) + W(\omega+\omega_1) + W(\omega+\omega_2)\right]$

Let window length $= N$.

First let us take ~~to~~ $\Delta\omega \geq \frac{2\pi}{N}$. The plot of one-sided spectrum (0 to π) is



If $\Delta\omega < \frac{2\pi}{N}$, two peaks overlap. Hence not resolved



Thus to get better resolution (small $\Delta\omega$) we need to have large N. Tapering the window will also result in poor resolution.

Note:

In practice DFT is used to get the spectrum. M point DFT is considered where $M \geq N$. Note that taking $M > N$ ~~does~~ does not improve the resolution, but we are able to get the spectrum at more interpolated points. The resolution is decided by data length N, not DFT length M.

Coming back to our analysis of PSD estimation: we observe that finite record length of random data distorts the calculated PSD ~~from~~ using DFT or DFT of autocorrelation (due to windowing)

Why can't we have ~~to~~ infinite data length?

One problem is storage. Other problem is out signals of interest are ~~not~~ non stationary process.

If signal is stationary process we could have taken large N and hence get better quality spectrum. But this is not possible with non-stationary random signals. Hence in power spectrum estimation, our goal is to select as short data record as possible that still allows us to resolve the spectral characteristics with good quality (good resolution, low variance of the estimate etc)

There are different techniques for estimating PSD.
They are broadly classified into

1. Non parametric methods where measured data is directly used, no assumptions are made about how data are generated.
2. Parametric methods where some assumption on data generation are used and a model for the data generator is first obtained.

Estimation of autocorrelation & Power spectrum of random signals : "Periodogram"

We consider a single realization of the random process. Let $x_a(t)$ be such random signal. It is sampled at $F_s \geq 2B$ where B is the highest frequency contained in the spectrum of $x_a(t)$. This results in N-length random signal $x(n)$; $0 \leq n \leq N-1$.

PSD can be obtained from autocorrelation function

The time averaged autocorrelation sequence of $x(n)$ is ($x(n)$ has length N, m is "lag")

$$r'_{xx}(m) = \frac{1}{N-m} \sum_{n=0}^{N-m-1} x^*(n)\, x(n+m) \; ; \quad 0 \leq m \leq N-1 \qquad \text{---- (6)}$$

Random process can be best characterised by mean (expected value) and variance. It can be proved that expected value of $r'_{xx}(m)$ given by (6) gives true autocorrelation (not ~~ever~~ biased) and

$\lim_{N \to \infty} \mathrm{var}(r'_{xx}(m)) = 0$. Such estimate is referred to as unbiased and consistant estimate.

However the $\mathrm{var}[r'_{xx}(m)]$ is poor. i.e. it is large for large lag or when m reaches $N-1$. This results in poor estimate of PSD. Alternate to this is to use biased autocorrelation given by

$$r_{xx}(m) = \frac{1}{N} \sum_{n=0}^{N-m-1} x^*(n)\, x(n+m) \quad ; \quad 0 \le m \le N-1 \quad \text{--- (7)}$$

(This has a bias $|m|\, \gamma_{xx}(m)/N$ where $\gamma_{xx}(m)$ is true autocorrelation). It can be still shown that $\cancel{\lim_{N\to\infty} \mathrm{var}[r_{xx}(m)]}$ $\lim_{N \to \infty} \mathrm{var}[r_{xx}(m)] = 0$

or $r_{xx}(m)$ is also consistant.

Now PSD is Fourier transform of $r_{xx}(m)$

$$\therefore \ PSD = P_{xx}(f) = \sum_{m=-(N-1)}^{(N-1)} r_{xx}(m)\, e^{-j 2\pi f m} \quad \text{---- (8)}$$

(Note that ACF length is twice that of the signal and it is even symmetric)

If we substitute for $r_{xx}(m)$ in (8) from (7), we will get PSD as

$$P_{xx}(f) = \frac{1}{N} \left| \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi f n} \right|^2 = \frac{1}{N} |X(f)|^2 \quad \text{--- (9)}$$

where $X(f)$ is DTFT of $x(n)$.

This representation for PSD is referred to as "Periodogram".

Note that this is only an estimate of PSD but not exactly the true PSD of original signal.
It can be proved that

$$\lim_{N \to \infty} \mathrm{var}[\text{Periodogram}] = \Gamma_{xx}^{2}(f)$$

where $\Gamma_{xx}(f)$ is True Power spectrum density. Since $\mathrm{var}[P_{xx}(f)] \neq 0$, the periodogram estimate of PSD is not consistant. This is another poor quality indication of Periodogram method of PSD estimation apart from spectral leakage and poor resolution problem. These are the main drawbacks of Periodogram.

Other methods are developed to improve the quality. They are classified as nonparametric and parametric methods

<u>Non-parametric methods</u>:

<u>Principle</u>: These methods make no assumption about how the data were generated. N-length data record is used to first get periodogram. Then processing is done further to reduce the variance and hence to improve the quality of PSD estimate. N is selected to get the required frequency resolution Δf.
(Note that the frequency resolution is, at best, equal to spectral width of rectangular window of length N. This is equal to $\frac{1}{N}$ (as $\Delta\omega = \frac{2\pi}{N}$, first zero crossing of $W(\omega)$ of rectangular window $w(n)$))

The three methods are

1. ~~Bartlett~~ Bartlett method
2. Welch method
3. Blackman & Tukey method