Indian Institute of Information Technology Allahabad
Discrete Mathematical Structures
Tentative Marking Scheme of C2 Review

Program: B.Tech. 2nd Semester (IT+IB)
Duration: **60 minutes** Full Marks: 20
Date: July 4, 2022 Time:: 5:15 PM - 6:15 PM

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1. Determine whether the following statements are true or false. In either case, give a proper justification (proof or counterexample). [15]

   (a) Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, and $G_3 = (V_3, E_3)$ be graphs, where $V_1 = \{1, 2, \ldots, 6\}$, $E_1 = \{12, 23, 26, 34, 45\}$;
   $V_2 = \{a_1, a_2, \ldots, a_6\}$, $E_2 = \{a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_3a_6\}$;
   $V_3 = \{v_1, v_2, \ldots, v_6\}$, $E_3 = \{v_1v_2, v_2v_3, v_3v_4, v_3v_5, v_5v_6\}$.
   Then $G_1 \ncong G_2$ and $G_1 \cong G_3$. [1.5+1.5]
   Solution:$G_1 \ncong G_2$ **is true while** $G_1 \cong G_3$ **is false**.
   Note that, there is only one vertex in each G_1, G_2 and G_3 with degree 3. These are respectively 2, a_3 and v_3 with $e(2) = 4$, $e(a_3) = 3$ and $e(v_3) = 3$. If $G_1 \cong G_2$ (or $G_1 \cong G_3$) then then there is an isomorphism $f : G_1 \to G_2$ such that $f(2) = a_3$ (respectively $g : G_1 \to G_3$ such that $f(2) = v_3$). But, the maps do not preserve the eccentricity of 2, a_3 and v_3.

   (b) Let $G_4 = (V_4, E_4)$ be a graph, where $V_4 = \{1, 2, \ldots, 8\}$
   and $E_4 = \{13, 14, 16, 24, 25, 26, 27, 35, 37, 48, 56, 58, 78\}$. Then G_4 is planar. [2]
   Solution: False
   If we use contract the edges 16, 37 and 48 then the resulting graph is K_5. Now by kuratowski's theorem, G_4 is non-planar. [2]
   [2]

   (c) The number of simple graphs with n vertices is equal to $2^{n(n-1)/2}$. [2]
   Solution: True.
   The maximum number of edges possible in a simple graph with n vertices is $\binom{n}{2} = n(n-1)/2$ Therefore, the number of simple graphs with n vertices is equal to $2^{n(n-1)/2}$.

   (d) Let $(G, *)$ be a finite group. If G has no nontrivial subgroups then G is of prime order. [2]
   Solution: True.
   On the contrary, let us assume that G is of composite order. Without loss of generality, $|G| = pq$ where p and q are primes. Let a be any non identity element of G then possible order of a are p, q, pq.
   Case 1: If order of a is p or q then $H =< a >$ is non trivial subgroup of G, which is not possible.
   Case 2: If order of a is pq then G becomes a cyclic group and for any divisor of order of a cyclic group there exists a subgroup which implies that G has non

trivial subgroup. It leads us to a contradiction.
Hence, G is of prime order.

(e) Let S_{10} be the group of permutations on $\{1,2,3,4,5,6,7,8,9,10\}$ and let $g=(123)(4567)\in S_{10}$. Then the order of g is 7. [2]
Solution: False.
Since $o(g)=o((123)(4567))=l.c.m(o(123),o(4567))=l.c.m(3,4)=12.$

(f) If every nontrivial subgroup of a group G is cyclic then G is abelian. [2]
Solution: False.
Let $G=S_3$, the symmetric group of all permutations on $\{1,2,3\}$. Then we have the following nontrivial subgroups of S_3 :

$$H_1=\{Id,(12)\},\ H_2=\{Id,(13)\},\ H_3=\{Id,(23)\},\ H_4=\{Id,(123),(132)\}.$$

All the mentioned subgroups are cyclic because

$$H_1=<(12)>,\ H_2=<(13)>,\ H_3=<(23)>,\ H_4=<(123)>.$$

Now, since

$$(12)(13)=(132)\neq(123)=(13)(12),$$

the group S_3 is not abelian. This proves the claim.

(g) If G is a group of even order, then G has an element $a\neq e$ satisfying $a^2=e$, where e denotes the identity of the group G. [2]
Solution: True.
Given that $|G|=2k$ from some $k\in\mathbb{N}$. If there exists an element a such that $a^2=e$. Then we are done. Otherwise, suppose for the sake of contradiction that

$$a^2\neq e,\quad \text{for all } a\in G\backslash\{e\}.$$

That is, no member of $G\backslash\{e\}$ is the inverse of itself. Here $G\backslash\{e\}$ has odd number of elements. This is a contradiction.

2. Determine the following for the above graph G_4 given in $Q1$ (b) : [5]

(a) Radius $rad(G_4)$
(b) Diameter $diam(G_4)$
(c) Center $C(G_4)$
(d) Girth $g(G_4)$
(e) Clique number $\omega(G_4)$

Solution:

$d(1,2)=2,\ d(1,3)=1,\ d(1,4)=1,\ d(1,5)=2,\ d(1,6)=1,\ d(1,7)=2,\ d(1,8)=2;$
$d(2,3)=2,\ d(2,4)=1,\ d(2,5)=1,\ d(2,6)=1,\ d(2,7)=1,\ d(2,8)=2;$
$d(3,4)=2,\ d(3,5)=1,\ d(3,6)=2,\ d(3,7)=1,\ d(3,8)=2;$
$d(4,5)=2,\ d(4,6)=2,\ d(4,7)=2,\ d(4,8)=1;$
$d(5,6)=1,\ d(5,7)=2,\ d(5,8)=1;$

$d(6,7)=2$, $d(6,8)=2$;
$d(7,8)=1$. [1.5]

$e(1)=e(2)=3(3)=e(4)=3(5)=e(6)=e(7)=e(8)=2$. [.5]

$rad(G_4)=min\{e(i)\,|\,1\leq i\leq 8\}=2$. [.5]

$diam(G_4)=max\{e(i)\,|\,1\leq i\leq 8\}=2$. [.5]

Center $C(G_4)=\{6,7,8\}$.

Girth $g(G_4)=3$. Consider the cycle $C(2-5-6)$. [.5+.5]

Clique number $\omega(G_4)=3$. Consider the cycle $K_3=C(2-5-6)$. [.5+.5]