String Matching

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Design and Analysis of Algorithms

Strings

- Let Σ be an alphabet, e.g. Σ = (, a, b, c, …, z)
- A string is any member of Σ*, i.e. any sequence of 0 or more members of Σ
  - ‘this is a string’ $\in$ Σ*
  - ‘this is also a string’ $\in$ Σ*
  - ‘1234’ $\notin$ Σ*

String operations

- Given strings s_1 of length n and s_2 of length m
- Equality: is $s_1 = s_2$? (case sensitive or insensitive)

‘this is a string’ = ‘this is a string’

‘this is a string’ ≠ ‘this is another string’

‘this is a string’ =? ‘THIS IS A STRING’

- Running time
  - O(n) where n is length of shortest string

String operations

- Concatenate (append): create string s_1s_2

  'this is a' . ' string' $\rightarrow$ 'this is a string'

  - Running time
    - $\Theta(n+m)$

String operations

- Substitute: Exchange all occurrences of a particular character with another character

Substitute('this is a string', 'i', 'x') → 'thxs xs a strxng'

Substitute('banana', 'a', 'o') → 'bonono'

- Running time
  - Θ(n)

String operations

- Length: return the number of characters/symbols in the string

Length('this is a string') → 16

Length('this is another string') → 24

- Running time
  - O(1) or Θ(n) depending on implementation

String operations

- Prefix: Get the first j characters in the string

Prefix('this is a string', 4) → 'this'

- Running time
  - Θ(j)
- Suffix: Get the last j characters in the string

Suffix('this is a string', 6) → 'string'

- Running time
  - Θ(j)

String operations

- Substring – Get the characters between i and j inclusive

Substring('this is a string', 4, 8) → 's is '

- Running time
  - $\Theta(j - i)$
- Prefix?
  - Prefix(S, i) = Substring(S, 1, i)
- Suffix?
  - Suffix(S, i) = Substring(S, i+1, length(n))

Edit distance
(aka Levenshtein distance)

- Edit distance between two strings is the minimum number of insertions, deletions and substitutions required to transform string s_1 into string s_2

Insertion:

ABACED → ABACCED → DABACCED

Insert ‘C’ Insert ‘D’

Edit distance
(aka Levenshtein distance)

- Edit distance between two strings is the minimum number of insertions, deletions and substitutions required to transform string s_1 into string s_2

Deletion:

ABACED

Edit distance
(aka Levenshtein distance)

- Edit distance between two strings is the minimum number of insertions, deletions and substitutions required to transform string s_1 into string s_2

Deletion:

ABACED → BACED

Delete 'A'

Edit distance
(aka Levenshtein distance)

- Edit distance between two strings is the minimum number of insertions, deletions and substitutions required to transform string s_1 into string s_2

Deletion:

ABACED → BACED → BACE

Delete 'A' Delete 'D'

Edit distance
(aka Levenshtein distance)

- Edit distance between two strings is the minimum number of insertions, deletions and substitutions required to transform string s_1 into string s_2

Substitution:

ABACED → ABADED → ABADES

Sub 'D' for 'C' — Sub 'S' for 'D'

Edit distance examples

Edit(Kitten, Mitten) = 1

Operations:

Sub ‘M’ for ‘K’ Mitten

Edit distance examples

Edit(Happy, Hilly) = 3

Operations:

Sub ‘a’ for ‘i’	Hippy
Sub ‘l’ for ‘p’	Hilpy
Sub ‘l’ for ‘p’	Hilly

Edit distance examples

Edit(Banana, Car) = 5

Operations:

Delete ‘B’	anana
Delete ‘a’	nana
Delete ‘n’	naa
Sub ‘C’ for ‘n’	Caa
Sub ‘a’ for ‘r’	Car

Edit distance examples

Edit(Simple, Apple) = 3

Operations:

Delete ‘S’	imple
Sub ‘A’ for ‘i’	Ample
Sub ‘m’ for ‘p’	Apple

Is edit distance symmetric?

- that is, is $\text{Edit}(s_1, s_2) = \text{Edit}(s_2, s_1)$?

Edit(Simple, Apple) =? Edit(Apple, Simple)

- Why?
  - sub ‘i’ for ‘j’ → sub ‘j’ for ‘i’
  - delete ‘i’ → insert ‘i’
  - insert ‘i’ → delete ‘i’

Calculating edit distance

X = A B C B D A B

Y = B D C A B A

Ideas?

Calculating edit distance

X = A B C B D A ?

Y = B D C A B ?

After all of the operations, X needs to equal Y

Calculating edit distance

X = A B C B D A ?

Y = B D C A B ?

Operations: Insert

Delete

Substitute

Insert

X = A B C B D A ?

Y = B D C A B ?

Insert

X = $\boxed{\text{A B C B D A ?}}$

Edit

Y = $\boxed{\text{B D C A B}}$?

$$Edit(X,Y) = 1 + Edit(X_{1\ldots n}, Y_{1\ldots m-1})$$

Delete

X = A B C B D A ?

Y = B D C A B ?

Delete

X = $\boxed{\text{A B C B D A}}$?

Edit

Y = $\boxed{\text{B D C A B ?}}$

$$Edit(X,Y) = 1 + Edit(X_{1\ldots n-1}, Y_{1\ldots m})$$

Substition

X = A B C B D A ?

Y = B D C A B ?

Substition

$$X = \boxed{A\ B\ C\ B\ D\ A}\ ?$$

Edit

$$Y = \boxed{B\ D\ C\ A\ B}\ ?$$

$$Edit(X,Y) = 1 + Edit(X_{1\ldots n-1}, Y_{1\ldots m-1})$$

Anything else?

X = A B C B D A ?

Y = B D C A B ?

Equal

X = A B C B D A ?

Y = B D C A B ?

Equal

X = $\boxed{\text{A B C B D A}}$?

Edit

Y = $\boxed{\text{B D C A B}}$?

$$Edit(X,Y) = Edit(X_{1\ldots n-1}, Y_{1\ldots m-1})$$

Combining results

Insert: $$Edit(X,Y) = 1 + Edit(X_{1\ldots n}, Y_{1\ldots m-1})$$

Delete: $$Edit(X,Y) = 1 + Edit(X_{1\ldots n-1}, Y_{1\ldots m})$$

Substitute: $$Edit(X,Y) = 1 + Edit(X_{1\ldots n-1}, Y_{1\ldots m-1})$$

Equal: $$Edit(X,Y) = Edit(X_{1\ldots n-1}, Y_{1\ldots m-1})$$

Combining results

$$Edit(X,Y) = \min \begin{cases} 1 + Edit(X_{1\ldots n}, Y_{1\ldots m-1}) & \text{insertion} \\ 1 + Edit(X_{1\ldots n-1}, Y_{1\ldots m}) & \text{deletion} \\ Diff(x_n, y_m) + Edit(X_{1\ldots n-1}, Y_{1\ldots m-1}) & \text{equal/subs titution} \end{cases}$$

```
EDIT(X, Y)
  m ← length[X]
  n ← length[Y]
  for i ← 0 to m
          d[i, 0] ← i
  for j ← 0 to n
          d[0, j] ← j
  for i ← 1 to m
          for j ← 1 to n
                  d[i, j] = min(1 + d[i − 1, j],
                                1 + d[i, j − 1],
                                DIFF(x_i, y_j) + d[i − 1, j − 1])
  return d[m, n]
```

Running time

```
EDIT(X, Y)
m ← length[X]
n ← length[Y]
for i ← 0 to m
        d[i, 0] ← i
for j ← 0 to n
        d[0, j] ← j
for i ← 1 to m
        for j ← 1 to n
                d[i, j] = min(1 + d[i − 1, j],
                              1 + d[i, j − 1],
                              DIFF(x_i, y_j) + d[i − 1, j − 1])
return d[m, n]
```

$\Theta(nm)$

Variants

- Only include insertions and deletions
  - What does this do to substitutions?
- Include swaps, i.e. swapping two adjacent characters counts as one edit
- Weight insertion, deletion and substitution differently
- Weight **specific** character insertion, deletion and substitutions differently
- Length normalize the edit distance

String matching

- Given a pattern string P of length m and a string S of length n, find **all** locations where P occurs in S

P = ABA

S = DCABABBABABA

String matching

- Given a pattern string P of length m and a string S of length n, find **all** locations where P occurs in S

P = ABA

S = DCABABBABABA

Uses

- grep/egrep
- search
- find
- java.lang.String.contains()

Naive implementation

```
NAIVE-STRING-MATCHER(S, P)
  n ← length[S]
  m ← length[P]
  for s ← 0 to n − m
        if S[1...m] = T[s + 1...s + m]
              print "Pattern at s"
```

Is it correct?

NAIVE-STRING-MATCHER(S, P)

$n \leftarrow length[S]$

$m \leftarrow length[P]$

for $s \leftarrow 0$ **to** $n - m$

  if $S[1...m] = T[s+1...s+m]$

    print "Pattern at s"

Running time?

NAIVE-STRING-MATCHER(S, P)

$n \leftarrow length[S]$

$m \leftarrow length[P]$

for $s \leftarrow 0$ **to** $n - m$

  $\boxed{\textbf{if } S[1 \ldots m] = T[s+1 \ldots s+m]}$

    print "Pattern at s"

- What is the cost of the equality check?
  - Best case: O(1)
  - Worst case: O(m)

Running time?

```
NAIVE-STRING-MATCHER(S, P)
n ← length[S]
m ← length[P]
for s ← 0 to n − m
        if S[1...m] = T[s + 1...s + m]
                print "Pattern at s"
```

- Best case
  - Θ(n) – when the first character of the pattern does **not** occur in the string
- Worst case
  - O((n-m+1)m)

Worst case

P = AAAA

S = AAAAAAAAAAAAA

Worst case

P = AAAA

S = AAAAAAAAAAAAA

Worst case

P = AAAA

S = AAAAAAAAAAAAAA

Worst case

P = AAAA

S = AAAAAAAAAAAAA

repeated work!

Worst case

P = AAAA

S = AAAAAAAAAAAAA

Ideally, after the first match, we'd know to just check the next character to see if it is an 'A'

Patterns

- Which of these patterns will have that problem?

P = ABAB

P = ABDC

P = BAA

P = ABBCDDCAABB

Patterns

- Which of these patterns will have that problem?

  P = ABAB

  P = ABDC

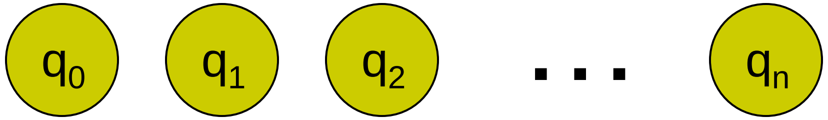
  P = BAA

  P = ABBCDDCAABB

If the pattern has a suffix that is also a prefix then we will have this problem

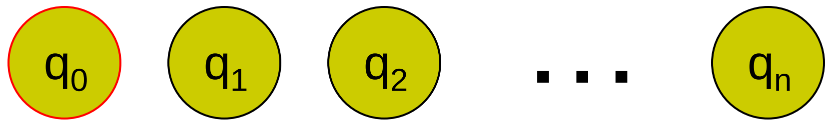
Finite State Automata (FSA)

- An FSA is defined by 5 components
  - Q is the set of states

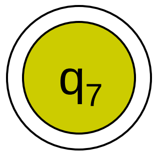


Finite State Automata (FSA)

- An FSA is defined by 5 components
  - Q is the set of states

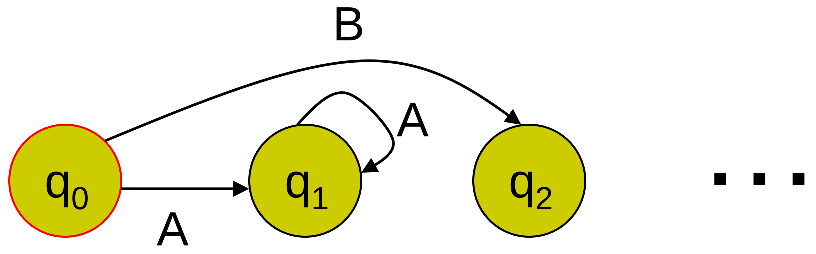


  - q_0 is the start state
  - $A \subseteq Q$, is the set of accepting states where $|A| > 0$

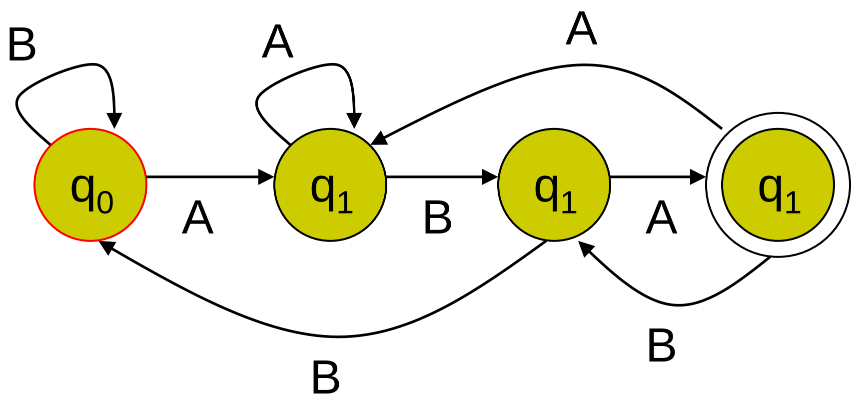


  - Σ is the alphabet (e.g. {A, B})
  - δ is the transition function from Q x Σ to Q

Q Σ	Q
q_0 A	q_1
q_0 B	q_2
q_1 A	q_1
…	



FSA operation



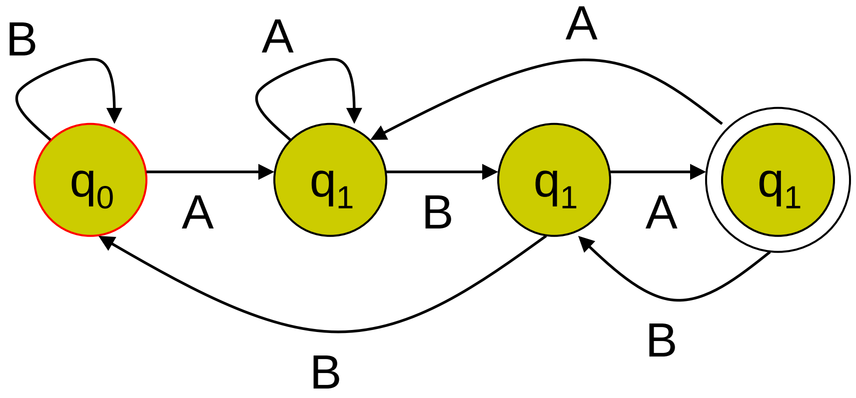
An FSA starts at state q_0 and reads the characters of the input string one at a time.

If the automaton is in state q and reads character a, then it transitions to state $\delta(q,a)$.

If the FSA reaches an accepting state ($q \in A$), then the FSA has found a match.

FSA operation

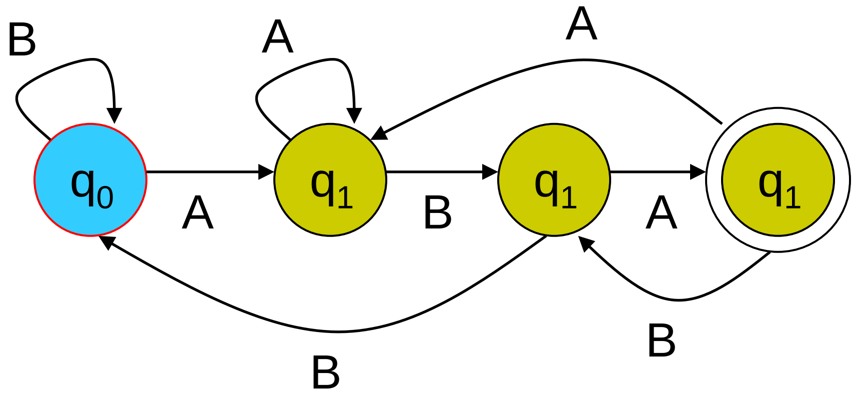
P = ABA



What pattern does this represent?

FSA operation

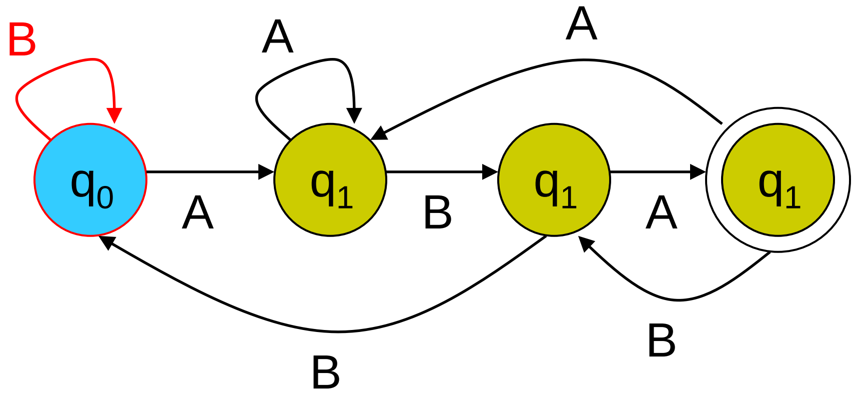
P = ABA



S = BABABBABABA

FSA operation

P = ABA



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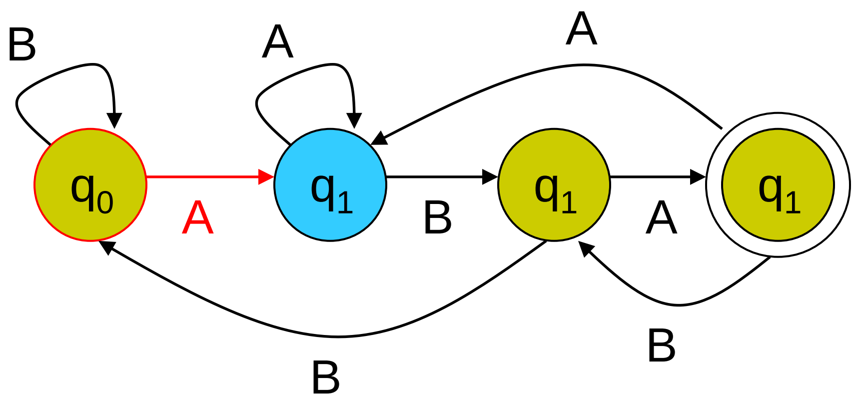
FSA operation

P = ABA



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FSA operation

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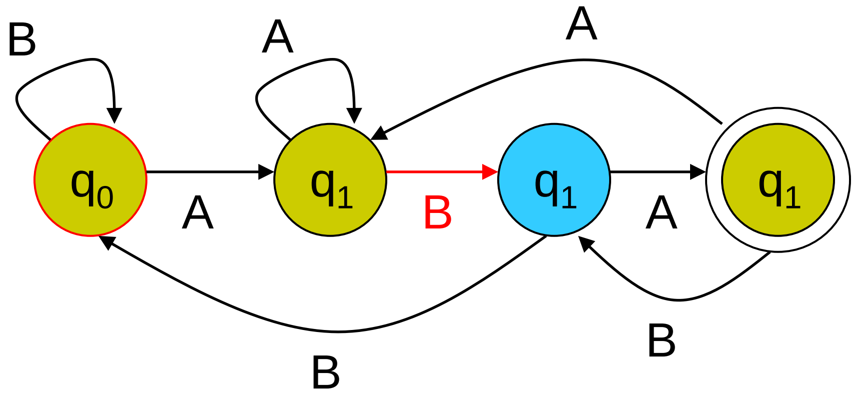

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FSA operation

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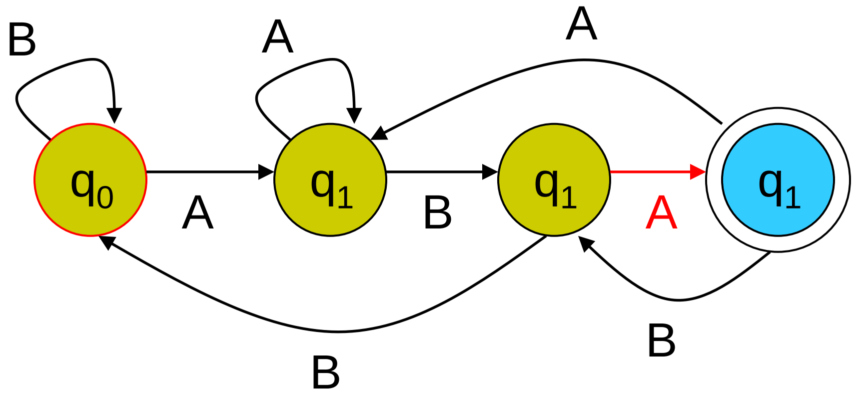

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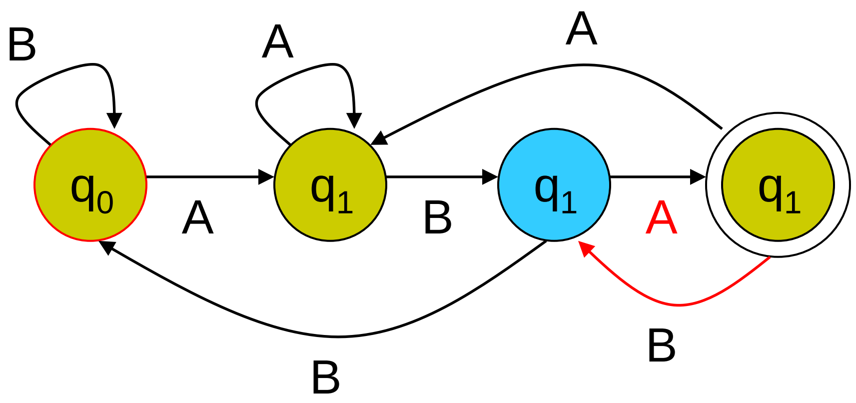
FSA operation

P = ABA



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FSA operation

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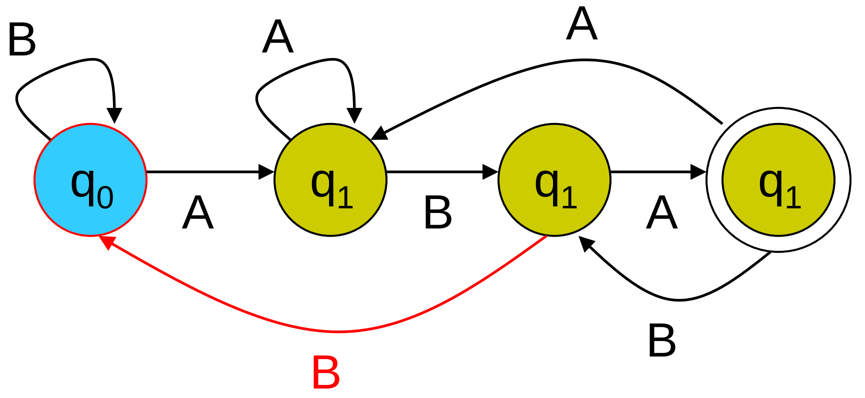

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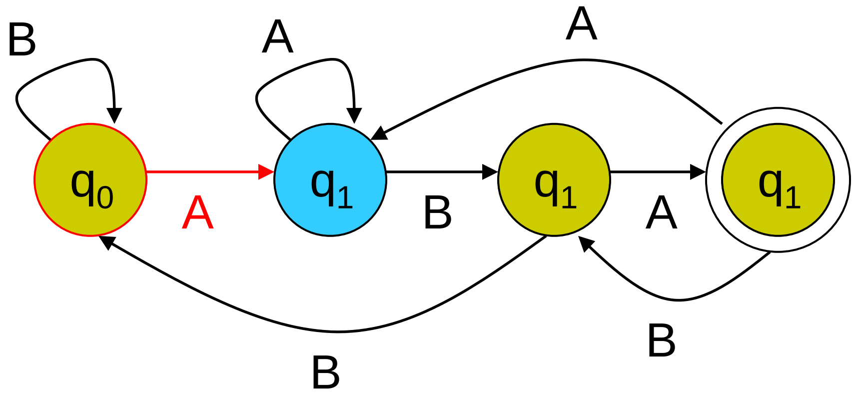

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FSA operation

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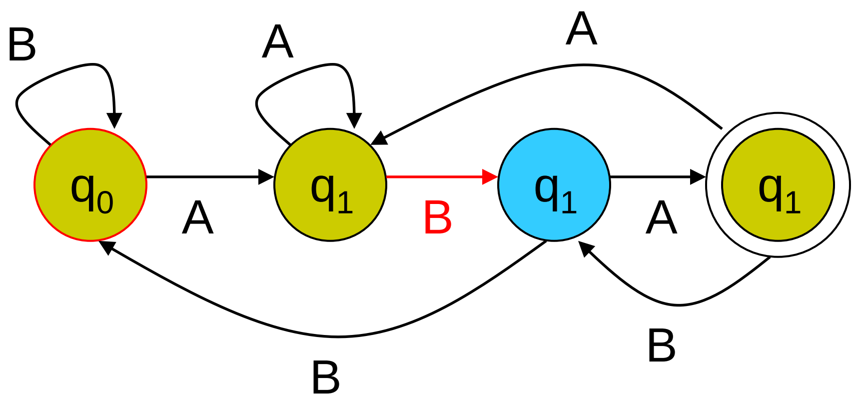
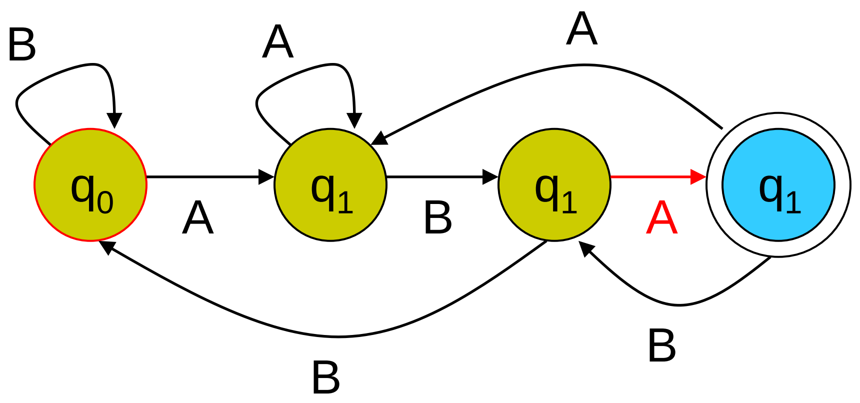


S = BABABBABABA

FSA operation

P = ABA



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FSA operation

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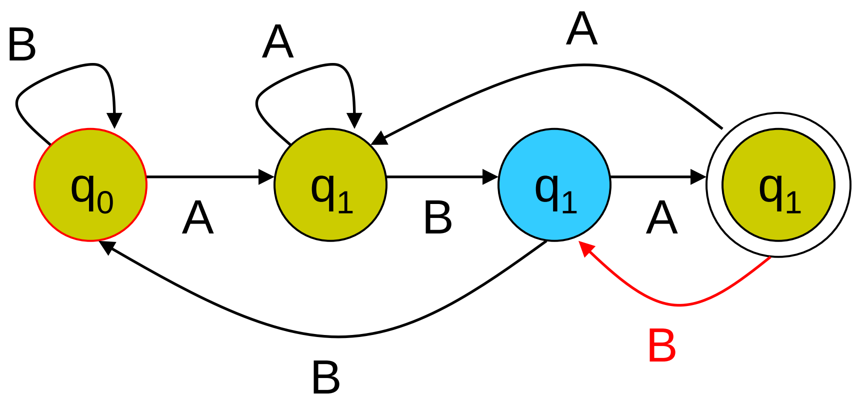
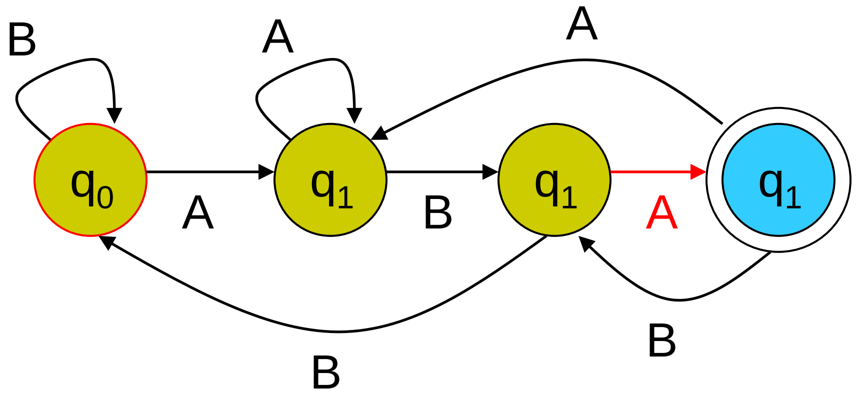


S = BABABBABABA

FSA operation

P = ABA



S = BABABBABABA

Suffix function

- The *suffix function* σ(x,y) is the length of the longest suffix of x that is a prefix of y

$$\sigma(x,y) = \max_i \left(x_{m-i+1\ldots m} = y_{1\ldots i} \right)$$

σ(abcdab, ababcd) = ?

Suffix function

- The *suffix function* σ(x,y) is the length of the longest suffix of x that is a prefix of y

$$\sigma(x, y) = \max_i (x_{m-i+1 \ldots m} = y_{1 \ldots i})$$

σ(abcdab, ababcd) = 2

Suffix function

- The *suffix function* σ(x,y) is the index of the longest suffix of x that is a prefix of y

$$\sigma(x, y) = \max_i (x_{m-i+1\ldots m} = y_{1\ldots i})$$

σ(daabac, abacac) = ?

Suffix function

- The *suffix function* σ(x,y) is the length of the longest suffix of x that is a prefix of y

$$\sigma(x, y) = \max_i (x_{m-i+1 \ldots m} = y_{1 \ldots i})$$

σ(daabac, abacac) = 4

Suffix function

- The *suffix function* σ(x,y) is the length of the longest suffix of x that is a prefix of y

$$\sigma(x, y) = \max_i (x_{m-i+1 \ldots m} = y_{1 \ldots i})$$

σ(dabb, abacd) = ?

Suffix function

- The *suffix function* σ(x,y) is the length of the longest suffix of x that is a prefix of y

$$\sigma(x, y) = \max_i (x_{m-i+1 \ldots m} = y_{1 \ldots i})$$

σ(dabb, abacd) = 0

Building a string matching automata

- Given a pattern $P = p_1, p_2, \ldots, p_m$, we'd like to build an FSA that recognizes P in strings

P = ababaca

Ideas?

Building a string matching automata

$P = ababaca$

- $Q = q_1, q_2, \ldots, q_m$ corresponding to each symbol, plus a q_0 starting state
- the set of accepting states, $A = \{q_m\}$
- vocab Σ all symbols in P, plus one more representing all symbols not in P
- The transition function for $q \in Q$ and $a \in \Sigma$ is defined as:
  - $\delta(q, a) = \sigma(p_{1\ldots q}a, P)$

Transition function

P = ababaca

- $\delta(q, a) = \sigma(p_{1\ldots q}a, P)$

state	a	b	c	P
q_0	?			a
q_1				b
q_2				a
q_3				b
q_4				a
q_5				c
q_6				a
q_7				

σ(a, ababaca)

Transition function

P = ababaca

- $\delta(q, a) = \sigma(p_{1\ldots q}a, P)$

state	a	b	c	P
q_0	1	?		a
q_1				b
q_2				a
q_3				b
q_4				a
q_5				c
q_6				a
q_7				

σ(b, ababaca)

Transition function

P = ababaca

- $\delta(q, a) = \sigma(p_{1\ldots q}a, P)$

state	a	b	c	P
q_0	1	0	?	a
q_1				b
q_2				a
q_3				b
q_4				a
q_5				c
q_6				a
q_7				

σ(b, ababaca)

Transition function

P = ababaca

- $\delta(q, a) = \sigma(p_{1\ldots q}a, P)$

state	a	b	c	P
q_0	1	0	0	a
q_1				b
q_2				a
q_3				b
q_4				a
q_5				c
q_6				a
q_7				

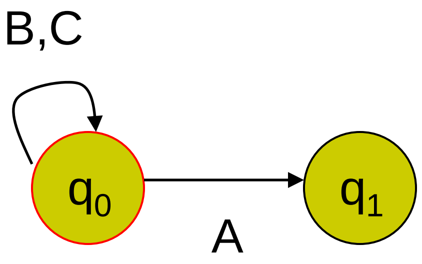
σ(b, ababaca)

Transition function

P = ababaca

- $\delta(q, a) = \sigma(p_{1\ldots q}a, P)$

state	a	b	c	P
q_0	1	0	0	a
q_1				b
q_2				a
q_3				b
q_4				a
q_5				c
q_6				a
q_7				



Transition function

P = ababaca

- $\delta(q, a) = \sigma(p_{1\ldots q}a, P)$

state	a	b	c	P
q_0	1	0	0	a
q_1	1	2	0	b
q_2	3	0	0	a
q_3	?			b
q_4				a
q_5				c
q_6				a
q_7				

We've seen 'aba' so far

σ(abaa, ababaca)

Transition function

P = ababaca

- $\delta(q, a) = \sigma(p_{1\ldots q}a, P)$

state	a	b	c	P
q_0	1	0	0	a
q_1	1	2	0	b
q_2	3	0	0	a
q_3	1			b
q_4				a
q_5				c
q_6				a
q_7				

We've seen 'aba' so far

σ(abaa, ababaca)

Transition function

P = ababaca

- $\delta(q, a) = \sigma(p_{1\ldots q}a, P)$

state	a	b	c	P
q_0	1	0	0	a
q_1	1	2	0	b
q_2	3	0	0	a
q_3	1	4	0	b
q_4	5	0	0	a
q_5	1	?		c
q_6				a
q_7				

We've seen 'ababa' so far

Transition function

P = ababaca

- $\delta(q, a) = \sigma(p_{1\ldots q}a, P)$

state	a	b	c	P
q_0	1	0	0	a
q_1	1	2	0	b
q_2	3	0	0	a
q_3	1	4	0	b
q_4	5	0	0	a
q_5	1	?		c
q_6				a
q_7				

We've seen 'ababa' so far

$\sigma(ababab, ababaca)$

Transition function

P = ababaca

- $\delta(q, a) = \sigma(p_{1\ldots q}a, P)$

state	a	b	c	P
q_0	1	0	0	a
q_1	1	2	0	b
q_2	3	0	0	a
q_3	1	4	0	b
q_4	5	0	0	a
q_5	1	4		c
q_6				a
q_7				

We've seen 'ababa' so far

σ(ababab, ababaca)

Transition function

P = ababaca

- $\delta(q, a) = \sigma(p_{1\ldots q}a, P)$

state	a	b	c	P
q_0	1	0	0	a
q_1	1	2	0	b
q_2	3	0	0	a
q_3	1	4	0	b
q_4	5	0	0	a
q_5	1	4	6	c
q_6	7	0	0	a
q_7	1	2	0	

Matching runtime

- Once we've built the FSA, what is the runtime?
  - $\Theta(n)$ - Each symbol causes a state transition and we only visit each character once
- What is the cost to build the FSA?
  - How many entries in the table?
    - $m|\Sigma|$ - Best case: $\Omega(m|\Sigma|)$
  - How long does it take to calculate the suffix function at each entry?
    - Naïve: $O(m^2)$
  - Overall naïve: $O(m^3|\Sigma|)$
  - Overall fast implementation $O(m|\Sigma|)$

Rabin-Karp algorithm

- Use a function T that computes a numerical representation of P
- Calculate T for all m symbol sequences of S and compare

P = ABA

S = BABABBABABA

Rabin-Karp algorithm

- Use a function T that computes a numerical representation of P
- Calculate T for all m symbol sequences of S and compare

P = ABA

Hash P

T(P)

S = BABABBABABA

Rabin-Karp algorithm

- Use a function T that computes a numerical representation of P
- Calculate T for all m symbol sequences of S and compare

P = ABA

S = $\underbrace{\text{BAB}}\text{ABBABABA}$

T(BAB)

=

T(P)

Hash m symbol sequences and compare

Rabin-Karp algorithm

- Use a function T that computes a numerical representation of P
- Calculate T for all m symbol sequences of S and compare

P = ABA

match

$$S = B\underbrace{ABA}_{T(ABA)}BBABABA$$

T(ABA)

=

T(P)

Hash m symbol sequences and compare

Rabin-Karp algorithm

- Use a function T that computes a numerical representation of P
- Calculate T for all m symbol sequences of S and compare

P = ABA

S = BA$\underbrace{\text{BAB}}$BABABA

T(BAB)

=

T(P)

Hash m symbol sequences and compare

Rabin-Karp algorithm

- Use a function T that computes a numerical representation of P
- Calculate T for all m symbol sequences of S and compare

P = ABA

Hash m symbol sequences and compare

S = BABABBABABA

$\underbrace{\text{BAB}}$ . . .

T(BAB)

=

T(P)

Rabin-Karp algorithm

P = ABA

For this to be useful/efficient, what needs to be true about T?

$$S = BAB\underbrace{ABB}_{T(BAB) = T(P)}ABABA \quad \ldots$$

Rabin-Karp algorithm

P = ABA

$$S = \text{BA}\underbrace{\text{BAB}}_{\substack{T(BAB) \\ = \\ T(P)}}\text{BABABA} \quad \ldots$$

For this to be useful/efficient, what needs to be true about T?

- Given $T(s_{i \ldots i+m-1})$ we must be able to efficiently calculate $T(s_{i+1 \ldots i+m})$

Calculating the hash function

- For simplicity, assume $\Sigma = (0, 1, 2, \ldots, 9)$. (in general we can use a base larger than 10).
- A string can then be viewed as a decimal number
- How do we efficiently calculate the numerical representation of a string?

T('9847261') = ?

Horner's rule

$$T(p_{1\ldots m}) = p_m + 10(p_{m-1} + 10(p_{m-2} + \ldots + 10(p_2 + 10p_1)))$$

9847261

9 * 10 = 90

(90 + 8)*10 = 980

(980 + 4)*10 = 9840

(9840 + 7)*10 = 98470

… = 9847621

Horner’s rule

$$T(p_{1\ldots m}) = p_m + 10(p_{m-1} + 10(p_{m-2} + \ldots + 10(p_2 + 10p_1)))$$

9847261

9 * 10 = 90

(90 + 8)*10 = 980

(980 + 4)*10 = 9840

(9840 + 7)*10 = 98470

… = 9847621

Running time?

Θ(m)

Calculating the hash on the string

- Given $T(s_{i \ldots i+m-1})$ how can we efficiently calculate $T(s_{i+1 \ldots i+m})$?

m = 4

$$9\underbrace{6380}_{T(s_{i \ldots i+m-1})}1572348267$$

$$T(s_{i+1 \ldots i+m}) = 10(T(s_{i \ldots i+m-1}) - 10^{m-1} s_i) + s_{i+m}$$

Calculating the hash on the string

- Given $T(s_{i\ldots i+m-1})$ how can we efficiently calculate $T(s_{i+1\ldots i+m})$?

m = 4

801

$$\underbrace{9638015}_{T(s_{i\ldots i+m-1})}72348267$$

subtract highest order digit

$$T(s_{i+1\ldots i+m}) = 10(T(s_{i\ldots i+m-1}) \boxed{-10^{m-1}s_i}) + s_{i+m}$$

Calculating the hash on the string

- Given $T(s_{i\ldots i+m-1})$ how can we efficiently calculate $T(s_{i+1\ldots i+m})$?

m = 4

8010

$$963\boxed{801}572348267$$

$$\underbrace{3801}_{T(s_{i\ldots i+m-1})}$$

shift digits up

$$T(s_{i+1\ldots i+m}) = \boxed{10}(T(s_{i\ldots i+m-1}) - 10^{m-1}s_i) + s_{i+m}$$

Calculating the hash on the string

- Given $T(s_{i\ldots i+m-1})$ how can we efficiently calculate $T(s_{i+1\ldots i+m})$?

m = 4

8015

963$\boxed{8015}$72348267

$\underbrace{3801}_{T(s_{i\ldots i+m-1})}$

add in the lowest digit

$$T(s_{i+1\ldots i+m}) = 10(T(s_{i\ldots i+m-1}) - 10^{m-1}s_i) \boxed{+ s_{i+m}}$$

Calculating the hash on the string

- Given T(si…i+m-1) how can we efficiently calculate T(si+1…i+m)?

m = 4

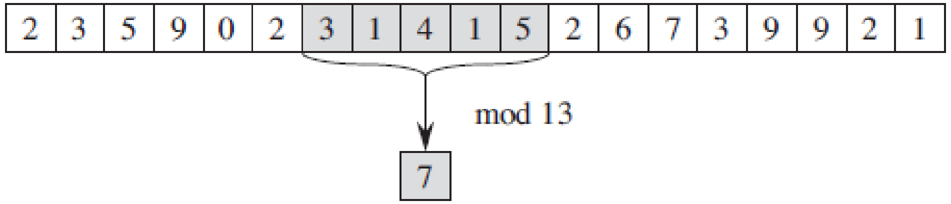
Running time?

$$\underbrace{9\,6\,3\,8\,0}_{} 1572348267$$

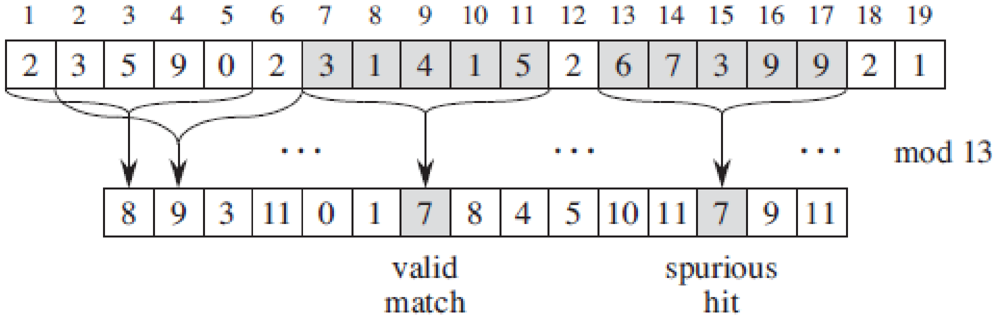
$9\underbrace{6380}_{T(s_{i\ldots i+m-1})}1572348267$

- $\Theta(m)$ for $s_{1\ldots m}$
- O(1) for the rest

$$T(s_{i+1\ldots i+m}) = 10(T(s_{i\ldots i+m-1}) - 10^{m-1}s_i) + s_{i+m}$$

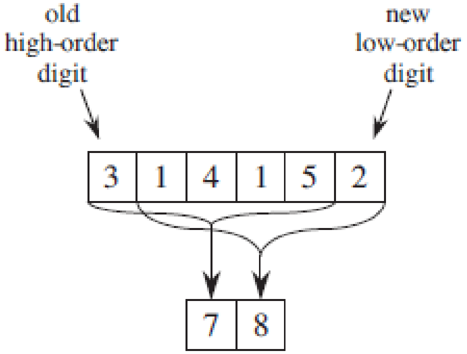


(a)



(b)

Figure 32.5 The Rabin-Karp algorithm. Each character is a decimal digit, and we compute values modulo 13. **(a)** A text string. A window of length 5 is shown shaded. The numerical value of the shaded number, computed modulo 13, yields the value 7. **(b)** The same text string with values computed modulo 13 for each possible position of a length-5 window. Assuming the pattern $P = 31415$, we look for windows whose value modulo 13 is 7, since $31415 \equiv 7 \pmod{13}$. The algorithm finds two such windows, shown shaded in the figure. The first, beginning at text position 7, is indeed an occurrence of the pattern, while the second, beginning at text position 13, is a spurious hit. **(c)** How to compute the value for a window in constant time, given the value for the previous window. The first window has value 31415. Dropping the high-order digit 3, shifting left (multiplying by 10), and then adding in the low-order digit 2 gives us the new value 14152. Because all computations are performed modulo 13, the value for the first window is 7, and the value for the new window is 8.



$$
\begin{aligned}
14152 &\equiv (31415 - \underbrace{3}_{\text{old high-order digit}} \cdot 10000) \cdot \underbrace{10}_{\text{shift}} + \underbrace{2}_{\text{new low-order digit}} \pmod{13} \\
&\equiv (7 - 3 \cdot 3) \cdot 10 + 2 \pmod{13} \\
&\equiv 8 \pmod{13}
\end{aligned}
$$

(c)

Adapted from Cormen, T.H., Leiserson, C.E., Rivest, R.L. and Stein, C., Introduction to algorithms. MIT press, 2009

Rabin-Karp algorithm

```
RABIN-KARP-MATCHER(T, P, d, q)
n = T.length
m = P.length
h = d^(m-1) mod q
p = 0
t_0 = 0
for i = 1 to m                          // preprocessing
    p = (dp + P[i]) mod q
    t_0 = (d t_0 + T[i]) mod q
for s = 0 to n - m                      // matching
    if p == t_s
        if P[1..m] == T[s+1..s+m]
            print "Pattern occurs with shift" s
    if s < n - m
        t_{s+1} = (d(t_s - T[s+1]h) + T[s+m+1]) mod q
```

Algorithm so far…

- Is it correct?
  - Each string has a unique numerical value and we compare that with each value in the string
- Running time
  - Preprocessing:
    - $\Theta(m)$
  - Matching
    - $\Theta(n-m+1)$

Is there any problem with this analysis?

Algorithm so far…

- Is it correct?
  - Each string has a unique numerical value and we compare that with each value in the string
- Running time
  - Preprocessing:
    - $\Theta(m)$
  - Matching
    - $\Theta(n-m+1)$

How long does the check $T(P) = T(s_{i\ldots i+m-1})$ take?

Modular arithmetics

- The run time assumptions we made were assuming arithmetic operations were constant time, which is not true for large numbers
- To keep the numbers small, we'll use modular arithmetics, i.e. all operations are performed mod q
  - a+b = (a+b) mod q
  - a*b = (a*b) mod q
  - …

Modular arithmetics

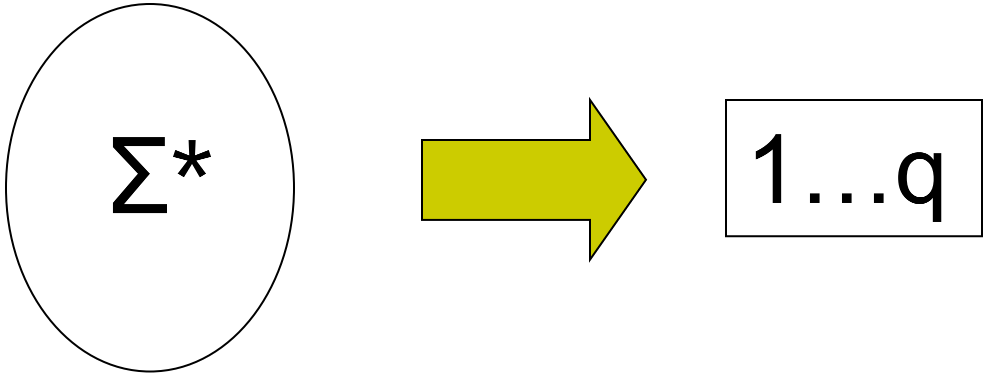
- If T(A) = T(B), then T(A) mod q = T(B) mod q
  - In general, we can apply mods as many times as we want and we will not effect the result
- What is the downside to this modular approach?
  - Spurious hits: if T(A) mod q = T(B) mod q that does **not** necessarily mean that T(A) = T(B)
  - If we find a hit, we must check that the actual string matches the pattern

Runtime

- Preprocessing
  - $\Theta(m)$
- Running time
  - Best case:
    - $\Theta(n-m+1)$ – No matches and no spurious hits
  - Worst case
    - $\Theta((n-m+1)m)$

Average case running time

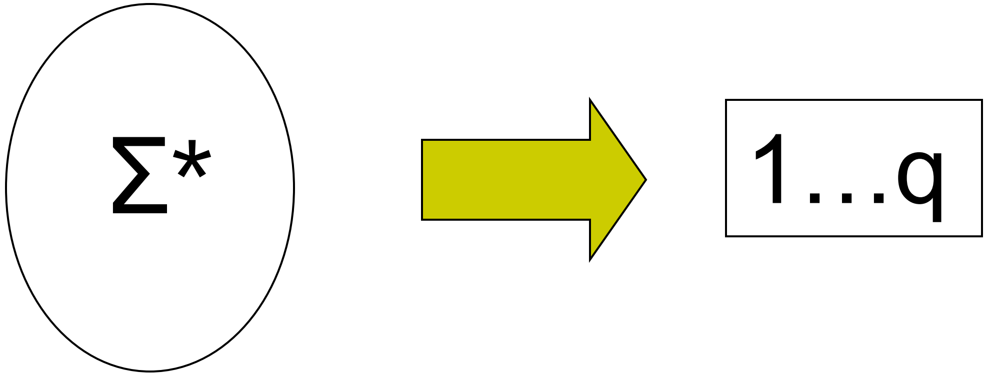
- Assume v valid matches in the string
- What is the probability of a spurious hit?
  - As with hashing, assume a uniform mapping onto values of q:



  - What is the probability under this assumption?

Average case running time

- Assume v valid matches in the string
- What is the probability of a spurious hit?
  - As with hashing, assume a uniform mapping onto values of q:



  - What is the probability under this assumption? 1/q

Average case running time

- How many spurious hits?
  - n/q
- Average case running time:

$$\underbrace{O(n-m+1)}_{\text{iterate over the positions}} + \underbrace{O(m(v+n/q))}_{\text{checking matches and spurious hits}}$$

Knuth-Morris-Pratt Algorithm

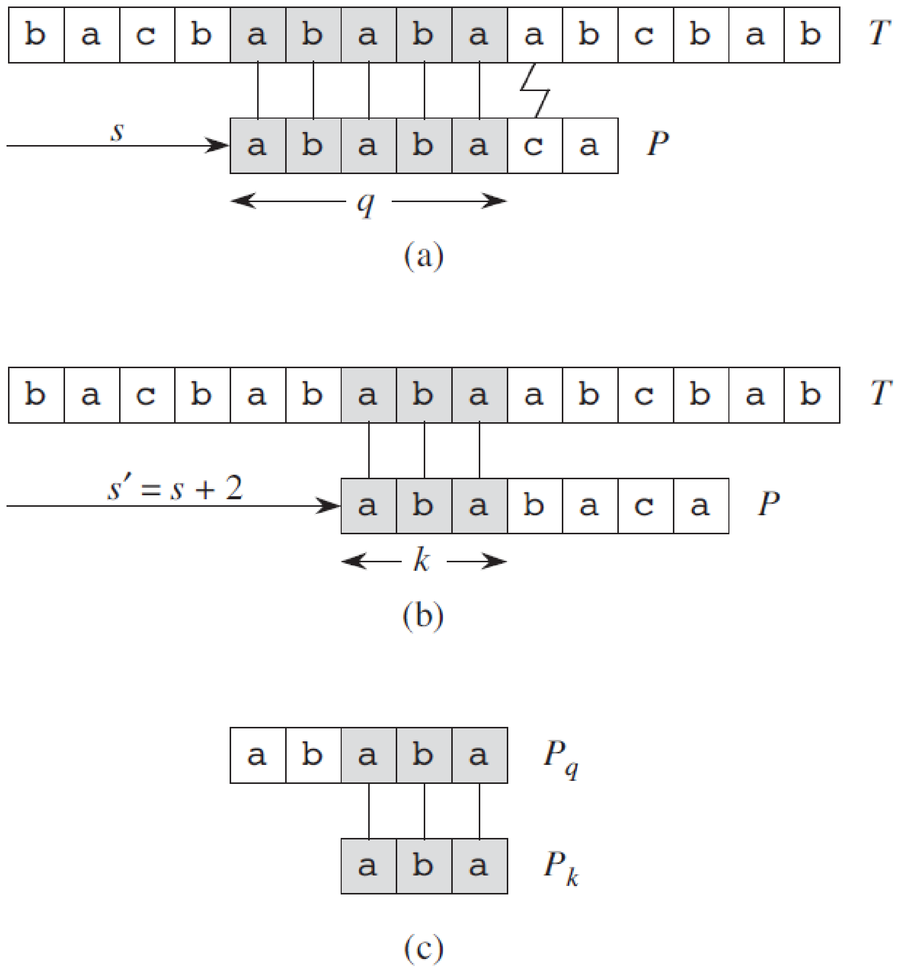


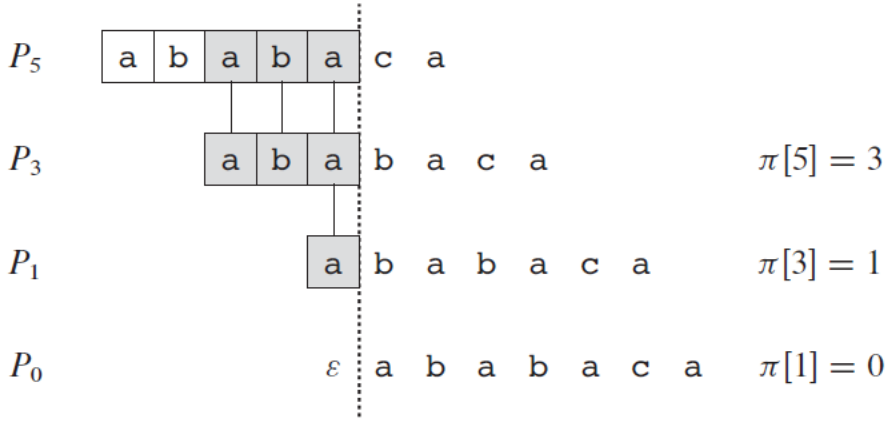
Figure 32.10 The prefix function π. **(a)** The pattern P = ababaca aligns with a text T so that the first $q = 5$ characters match. Matching characters, shown shaded, are connected by vertical lines. **(b)** Using only our knowledge of the 5 matched characters, we can deduce that a shift of $s + 1$ is invalid, but that a shift of $s' = s+2$ is consistent with everything we know about the text and therefore is potentially valid. **(c)** We can precompute useful information for such deductions by comparing the pattern with itself. Here, we see that the longest prefix of P that is also a proper suffix of P_5 is P_3. We represent this precomputed information in the array π, so that $\pi[5] = 3$. Given that q characters have matched successfully at shift s, the next potentially valid shift is at $s' = s + (q - \pi[q])$ as shown in part (b).

Adapted from Cormen, T.H., Leiserson, C.E., Rivest, R.L. and Stein, C., Introduction to algorithms. MIT press, 2009

Knuth-Morris-Pratt Algorithm

i	1	2	3	4	5	6	7
$P[i]$	a	b	a	b	a	c	a
$\pi[i]$	0	0	1	2	3	0	1

(a)



(b)

Figure 32.11 An illustration of Lemma 32.5 for the pattern $P =$ ababaca and $q = 5$. **(a)** The π function for the given pattern. Since $\pi[5] = 3$, $\pi[3] = 1$, and $\pi[1] = 0$, by iterating π we obtain $\pi^*[5] = \{3, 1, 0\}$. **(b)** We slide the template containing the pattern P to the right and note when some prefix P_k of P matches up with some proper suffix of P_5; we get matches when $k = 3$, 1, and 0. In the figure, the first row gives P, and the dotted vertical line is drawn just after P_5. Successive rows show all the shifts of P that cause some prefix P_k of P to match some suffix of P_5. Successfully matched characters are shown shaded. Vertical lines connect aligned matching characters. Thus, $\{k : k < 5 \text{ and } P_k \sqsupset P_5\} = \{3, 1, 0\}$. Lemma 32.5 claims that $\pi^*[q] = \{k : k < q \text{ and } P_k \sqsupset P_q\}$ for all q.

Adapted from Cormen, T.H., Leiserson, C.E., Rivest, R.L. and Stein, C., Introduction to algorithms. MIT press, 2009

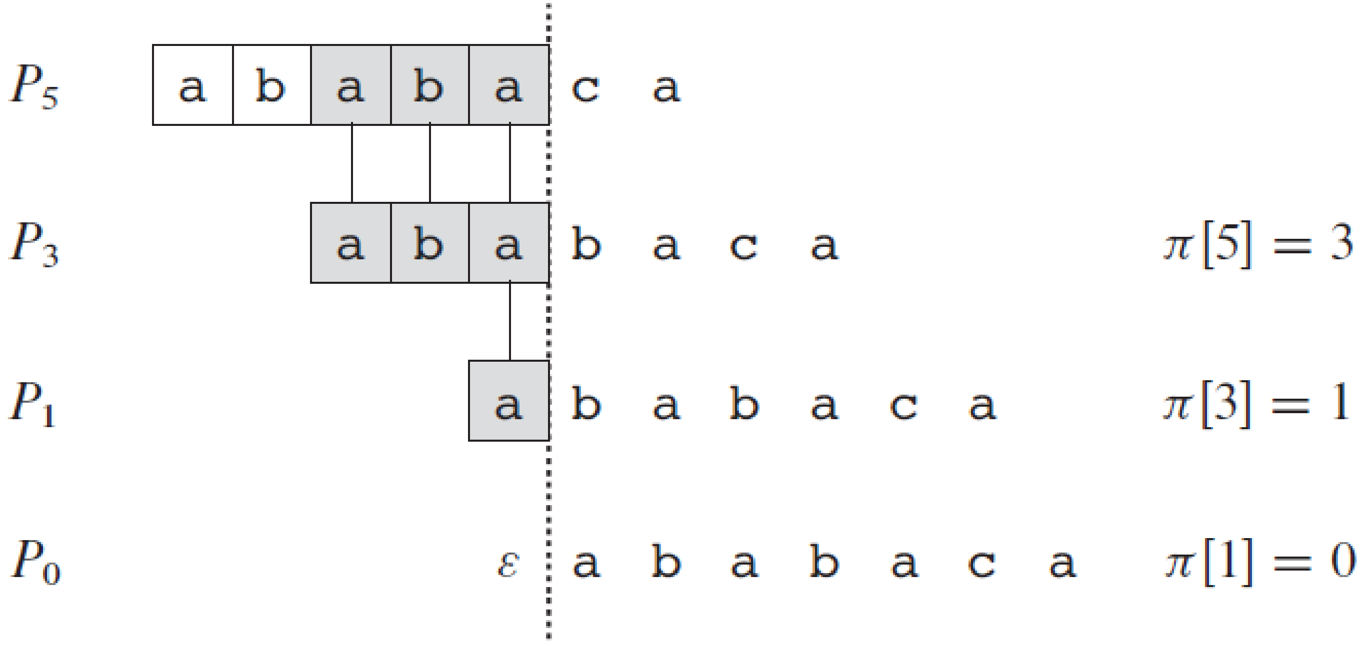
Knuth-Morris-Pratt Algorithm

Lemma 32.5 (Prefix-function iteration lemma)

Let P be a pattern of length m with prefix function π. Then, for $q = 1, 2, \ldots, m$, we have $\pi^*[q] = \{k : k < q \text{ and } P_k \sqsupset P_q\}$.

i	1	2	3	4	5	6	7
$P[i]$	a	b	a	b	a	c	a
$\pi[i]$	0	0	1	2	3	0	1

(a)



(b)

Adapted from Cormen, T.H., Leiserson, C.E., Rivest, R.L. and Stein, C., Introduction to algorithms. MIT press, 2009

Knuth-Morris-Pratt Algorithm

```
KMP-MATCHER(T, P)
n = T.length
m = P.length
π = COMPUTE-PREFIX-FUNCTION(P)
q = 0                                      // number of characters matched
for i = 1 to n                             // scan the text from left to right
    while q > 0 and P[q + 1] ≠ T[i]
        q = π[q]                           // next character does not match
    if P[q + 1] == T[i]
        q = q + 1                          // next character matches
    if q == m                              // is all of P matched?
        print "Pattern occurs with shift" i − m
        q = π[q]                           // look for the next match
```

```
COMPUTE-PREFIX-FUNCTION(P)
m = P.length
let π[1..m] be a new array
π[1] = 0
k = 0
for q = 2 to m
    while k > 0 and P[k + 1] ≠ P[q]
        k = π[k]
    if P[k + 1] == P[q]
        k = k + 1
    π[q] = k
return π
```


Adapted from Cormen, T.H., Leiserson, C.E., Rivest, R.L. and Stein, C., Introduction to algorithms. MIT press, 2009

Matching running times

Algorithm	Preprocessing time	Matching time
Naïve	0	$O((n\text{-}m+1)m)$
FSA	$\Theta(m\|\Sigma\|)$	$\Theta(n)$
Rabin-Karp	$\Theta(m)$	$O(n\text{-}m+1)m)$
Knuth-Morris-Pratt	$\Theta(m)$	$\Theta(n)$

Acknowledgements

- Cormen, T.H., Leiserson, C.E., Rivest, R.L. and Stein, C., Introduction to algorithms. MIT press, 2009
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- Prof. David Plaisted, The University of North Carolina at Chapel Hill