Indian Institute of Information Technology Allahabad
Univariate and Multivariate Calculus Test
Tentative Marking Scheme

Program: B.Tech. 2^{nd} Semester (IT+ECE)
Duration: **01 Hour** Full Marks: 21
Date: May 22, 2021 Time: 11:00 - 12:00 IST

Attempt all questions.

1. Negate the following sentence:

   "In every city in India there exists at least one person who can speak minimum three languages and who can read and write two languages". [2]

   Solution. The negation of the sentence is:

   "There exists a city in India in which every person can speak maximum two languages or can read or write more than or less than two languages". [2]

2. Let $A \subset \mathbb{R}$ such that $\sup A = \inf A$. What can you say about A. Provide justifications for your answer. [2]

   Solution. We claim that A is a singleton. Let $x, y \in A$ such that $x \neq y$. Without loss of generality, we can assume $x < y$. Then $\inf A \leq x < y \leq \sup A$. This is a contradiction since $\sup A = \inf A$. [2]

3. Let $x_1 = \frac{5}{2}$. For $n \geq 1$, define $x_{n+1} = \frac{1}{7}\left(x_n^3 + 6\right)$. Discuss the convergence/divergence of this sequence, and find its limits in case it is convergent. [5]

   Solution. Given $x_1 = \frac{5}{2} \Rightarrow x_n > 0$.

   Now, $x_2 = \frac{173}{56} \Rightarrow x_2 > x_1$.

   $$x_{n+1} - x_n = \frac{x_n^3 - x_{n-1}^3}{7} = \frac{(x_n - x_{n-1})(x_n^2 + x_{n-1}^2 + x_n x_{n-1})}{7} > 0. \qquad [2]$$

   By induction, (x_n) is an increasing sequence.

   Let (x_n) be bounded above. Then (x_n) will converge to its supremum. Let ℓ be the limit of (x_n).

   Since $x_n \geq \frac{5}{2}$ for all n, we have $\ell \geq \frac{5}{2}$. [1]

   It follows that,

   $$7\ell = \ell^3 + 6 \Rightarrow \ell = 1, 2, -3 < \frac{5}{2}, \qquad [1]$$

   a contradiction. Hence, the sequence is not bounded above and divergent. [1]

4. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function such that $f(x+y) = f(x) + f(y)$, for all $x, y \in \mathbb{R}$. If f is continuous at 0, show that f is continuous everywhere. [3]

   Solution. We first note that $f(0) = 0$, $f(-x) = -f(x)$ and $f(x-y) = f(x) - f(y)$. [1]

Let $x \in \mathbb{R}$ and $x_n \to x$. Then $x_n - x \to 0$. [1]

Since f is continuous at 0, $f(x_n) - f(x) = f(x_n - x) \to f(0) = 0$. Hence, $f(x_n) \to f(x)$. Therefore, f is continuous at x. [1]

5. Find an interval $[a, b]$ in which the equation

$$e^{-x} = 4 - x^3$$

has a solution. [3]

Solution. Set $f(x) = e^{-x} - 4 + x^3$.

$f(0) = -3 < 0$ and $f(2) = e^{-2} + 4 > 0$ [1+1]

The function f is sum of continuous function, hence continuous. Thus, by IVP there exists a point c in $[0, 2]$ such that $f(c) = 0$. [1]

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(0) > 0$ and $f(a) < a^2$ for some $0 < a \in \mathbb{R}$. Show that there exists a point $b \in (0, a)$ such that $f(b) = b^2$. [3]

Solution. Let $g(x) = f(x) - x^2$. Then g is continuous. [1]

$g(0) = f(0) > 0$ and $g(a) = f(a) - a^2 < 0$. [1]

By IVP there exists a real number b in $(0, a)$ such that $g(b) = 0$, that is, $f(b) = b^2$. [1]

7. Let $f : (1, 4) \longrightarrow (0, 3)$ be a continuous function such that $f([2, 5/2]) \subseteq [2, 5/2]$. Show that there exists a $c \in [2, 5/2]$ such that $f(c) = c$. [3]

Solution. Let $g(x) = f(x) - x$. Since f is continuous, g is also continuous. [1]

Also $g(2) \geq 0$ and $g(5/2) \leq 0$. [1]

By IVP, there exists a $c \in [2, 5/2]$ such that $g(c) = f(c) - c = 0 \Rightarrow f(c) = c$. [1]