Indian Institute of Information Technology, Allahabad
C2 Review Test
Probability and Statistics (PAS)
B. Tech. (3rd Semester)

Date: October 18, 2022 (12:15 PM - 01:15 PM) Total Marks: 27

Important Instructions

1. Answer all questions. Writing on question paper is not allowed.
2. Attempt all the parts of question 4 at the same place. Parts done separately will not be graded.
3. Number the pages of your answer booklet. On the first page of your answer booklet, make a table (as shown below) to indicate the page number in which respective questions have been answered. If you did not attempt a particular question, write down NA.

Question No.	1	2	3	4	5	6	7	8	9
Page No.									

4. Use of any electronic gadgets is not allowed.

---

1. If X and Y are independent Poisson variables such that

$$P(X=1)=P(X=2) \text{ and } P(Y=2)=P(Y=3).$$

Find the variance of $X-2Y$. [6]

Solution: The p.m.f of a Poisson distribution $P(\lambda)$

$$f(x)=\begin{cases} e^{-\lambda}\frac{\lambda^x}{x!}, \text{ if } x\in\{0,1,2,\ldots \\ 0, \text{ otherwise}\end{cases}$$ [1]

Now, suppose $X\sim P(\lambda_1)$ and $X\sim P(\lambda_2)$. Since $P(X=1)=P(X=2)$, $\lambda_1=2$. [1]

Since $P(Y=2)=P(Y=3)$, $\lambda_2=3$. [1]

Since X and Y are independent, $Cov(X,Y)=0$. [1]

So $Var(X-2Y)=Var(X)+4Var(Y)=14$. [2]

2. A random variable X follows exponential distribution with the expected value 0.5. Find the expected value of X^2. [2]

Solution: For an exponential distribution, the probability density function will be given by $f(x)=\lambda e^{-\lambda x}$, where mean $=\frac{1}{\lambda}$ and variance $=\frac{1}{\lambda^2}$.

Given, mean $=0.5\Rightarrow$ variance $=0.25$. [1]

Mean $=E(X)$, $Var(X)=E(X^2)-E(X)^2$

$\Rightarrow E(X^2)=0.25+0.25=0.50$. [1]

3. Let X and Y be two discrete random variables with the probability mass functions

$$f_X(x)=\begin{cases} p_1, \text{ if } x=a_1 \\ 1-p_1, \text{ if } x=a_2 \\ 0, \text{ otherwise}\end{cases} \quad \text{and } f_Y(y)=\begin{cases} q_1, \text{ if } y=b_1 \\ 1-q_1, \text{ if } y=b_2 \\ 0, \text{ otherwise}\end{cases}$$

respectively. Show that X and Y are independent if and only if the correlation coefficient between X and Y is zero. [5]

Solution: Suppose X and Y are independent. Then $E(XY) = E(X)E(Y) \Rightarrow Cov(X,Y) = 0$. Thus the correlation coefficient between X and Y is zero. [1]

Conversely, suppose the correlation coefficient between X and Y is zero. So, $E(XY) = E(X)E(Y)$.

It is easy to see that $P(X = x, Y = y) = 0$, if $(x,y) \notin \{a_1, a_2\} \times \{b_1, b_2\}$. [1]

Suppose, $P(X = a_1, Y = b_1) = p$. Then

$$\begin{aligned} P(X = a_1, Y = b_2) &= P(X = a_1) - P(X = a_1, Y = b_1) = p_1 - p \\ P(X = a_2, Y = b_1) &= P(Y = b_1) - P(X = a_1, Y = b_1) = q_1 - p \\ P(X = a_2, Y = b_2) &= 1 - P(\{X = a_1\} \cup \{Y = b_1\}) = 1 - p_1 - q_1 + p \end{aligned}$$

Since $E(XY) = E(X)E(Y)$,

$$\begin{aligned} a_1 b_1 p + a_1 b_2 (p_1 - p) + a_2 b_1 (q_1 - p) + a_2 b_2 (1 - p_1 - q_1 + p) \\ = (a_1 p_1 + a_2(1 - p_1))(b_1 q_1 + b_2(1 - q_1)) \end{aligned}$$

Hence, $p = p_1 q_1$. [2]

Thus, $P(X = x, Y = y) = P(X = x)P(Y = y)$, for every $(x,y) \in \mathbb{R}^2$. Hence, X and Y are independent. [1]

4. Let (X, Y) be a random vector with joint probability density function f defined by $f(x,y) = \frac{1}{2}$ inside the square with corners at the points $(0,1), (1,0), (-1,0), (0,-1)$ in the $(x,y)-$ plane, and $f(x,y) = 0$ otherwise. [14]

   (a) Find marginal probability density functions of X and Y.

   (b) Are X and Y independent?

   (c) Are X and Y uncorrelated?

   (d) Find the conditional probability density function of X given $Y = y$, for $-1 < y < 0$ and the conditional probability density function of Y given $X = x$, for $0 < x < 1$.

   Solution:

(a) The marginal probability density function of X is

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f(x,y)dy \\
&= \begin{cases} \int_{-(1+x)}^{(1+x)} \frac{1}{2}dy, & \text{if } -1<x<0 \\ \int_{-(1-x)}^{(1-x)} \frac{1}{2}dy, & \text{if } 0\le x<1 \\ 0, & \text{otherwise} \end{cases} && [1+1+1+1] \\
&= \begin{cases} 1+x, & \text{if } -1<x<0 \\ 1-x, & \text{if } 0\le x<1 \\ 0, & \text{otherwise} \end{cases} && [1]
\end{aligned}
$$

Similarly, the marginal probability density function of Y is

$$
\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f(x,y)dx \\
&= \begin{cases} \int_{-(1+y)}^{(1+y)} \frac{1}{2}dx, & \text{if } -1<y<0 \\ \int_{-(1-y)}^{(1-y)} \frac{1}{2}dx, & \text{if } 0\le y<1 \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} 1+y, & \text{if } -1<y<0 \\ 1-y, & \text{if } 0\le y<1 \\ 0, & \text{otherwise} \end{cases} && [1]
\end{aligned}
$$

(b) Since $f(x,y) \neq f_X(x)f_Y(y)$, X and Y are not independent. [1]

(c) $E(XY) = \int_{-1}^{0}\int_{-(1+x)}^{(1+x)} \frac{xy}{2}dydx + \int_{0}^{1}\int_{-(1-x)}^{(1-x)} \frac{xy}{2}dydx = 0.$ [1]

$E(Y) = \int_{-1}^{0}\int_{-(1+x)}^{(1+x)} \frac{y}{2}dydx + \int_{0}^{1}\int_{-(1-x)}^{(1-x)} \frac{y}{2}dydx = 0.$ [1]

Thus, $Cov(X,Y) = 0$, Hence, they are uncorrelated. [1]

(d) The conditional probability density function of X given $Y=y$ is

$$
\begin{aligned}
f_{X|Y}(x|y) &= \frac{f(x,y)}{f_Y(y)} \\
&= \begin{cases} \frac{1}{2(1+y)}, & \text{if } -(1+y)<x<(1+y) \\ 0, & \text{otherwise} \end{cases} && [1+1]
\end{aligned}
$$

The conditional probability density function of Y given $X = x$ is

$$
\begin{aligned}
f_{Y|X}(y|x) &= \frac{f(x,y)}{f_X(x)} \\
&= \begin{cases} \frac{1}{2(1-x)}, \text{ if } -(1-x) < y < (1-x) \\ 0, \text{ otherwise} \end{cases}
\end{aligned}
\qquad [1+1]
$$