Intensity Transformations (Histogram Processing)

Dr. Navjot Singh

Image and Video Processing

Acknowledgements

- Gonzalez, Rafael C. Digital image processing. Pearson, 4th edition, 2018.
- Jain, Anil K. Fundamentals of digital image processing. Prentice-Hall, Inc., 1989.
- Digital Image Processing course by Brian Mac Namee, Dublin Institute of Technology
- Digital Image Processing course by Christophoros Nikou, University of Ioannina

Contents

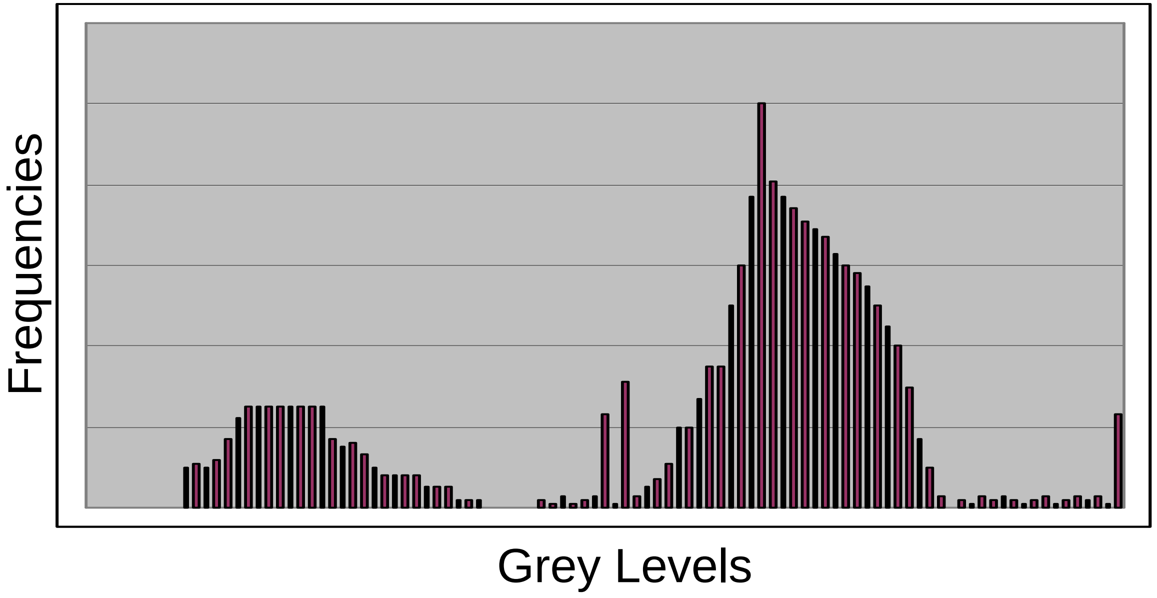
Over the next few lectures we will look at image enhancement techniques working in the spatial domain:

- Histogram processing
- Spatial filtering
- Neighbourhood operations

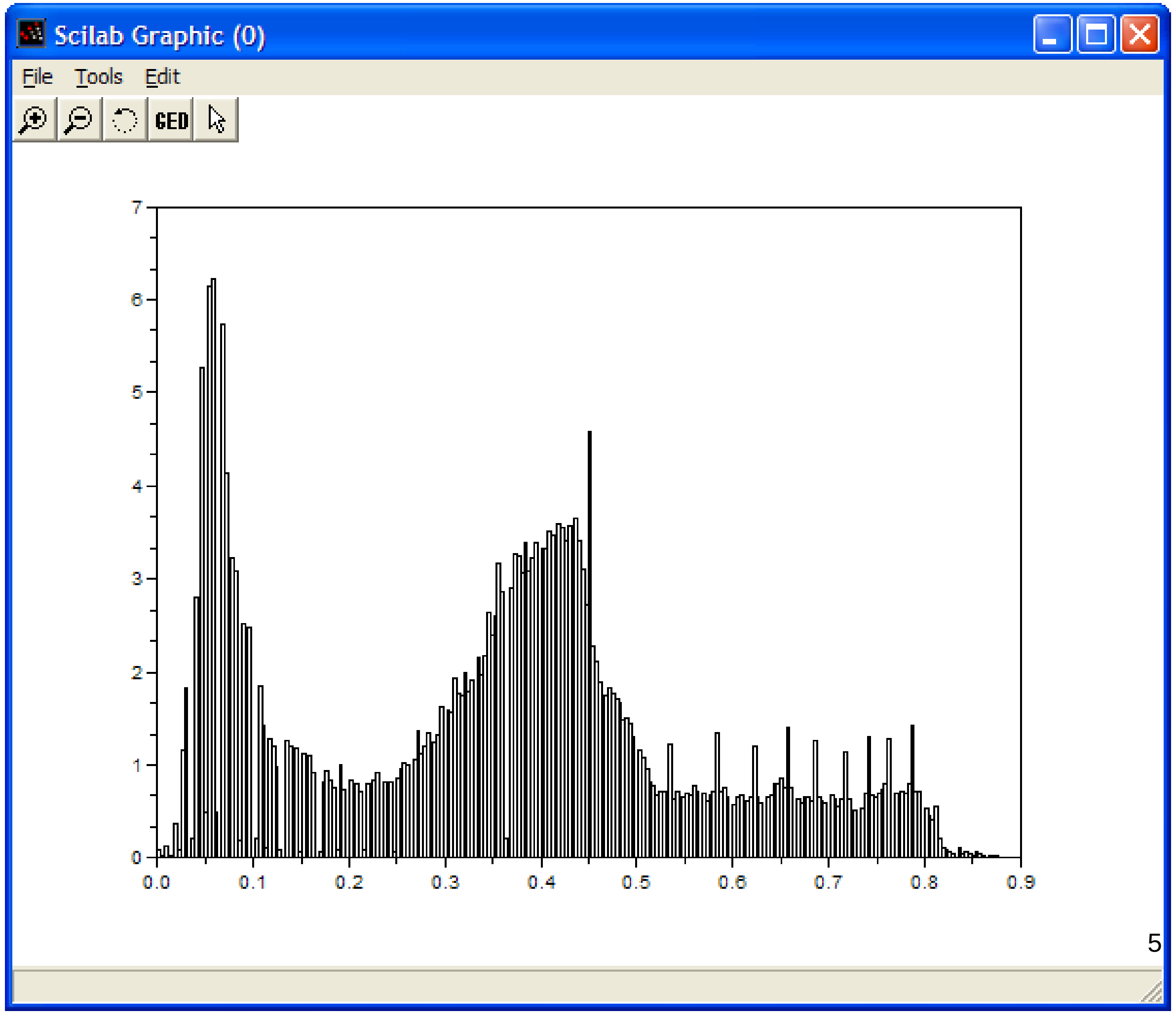
Image Histograms

The histogram of an image shows us the distribution of grey levels in the image

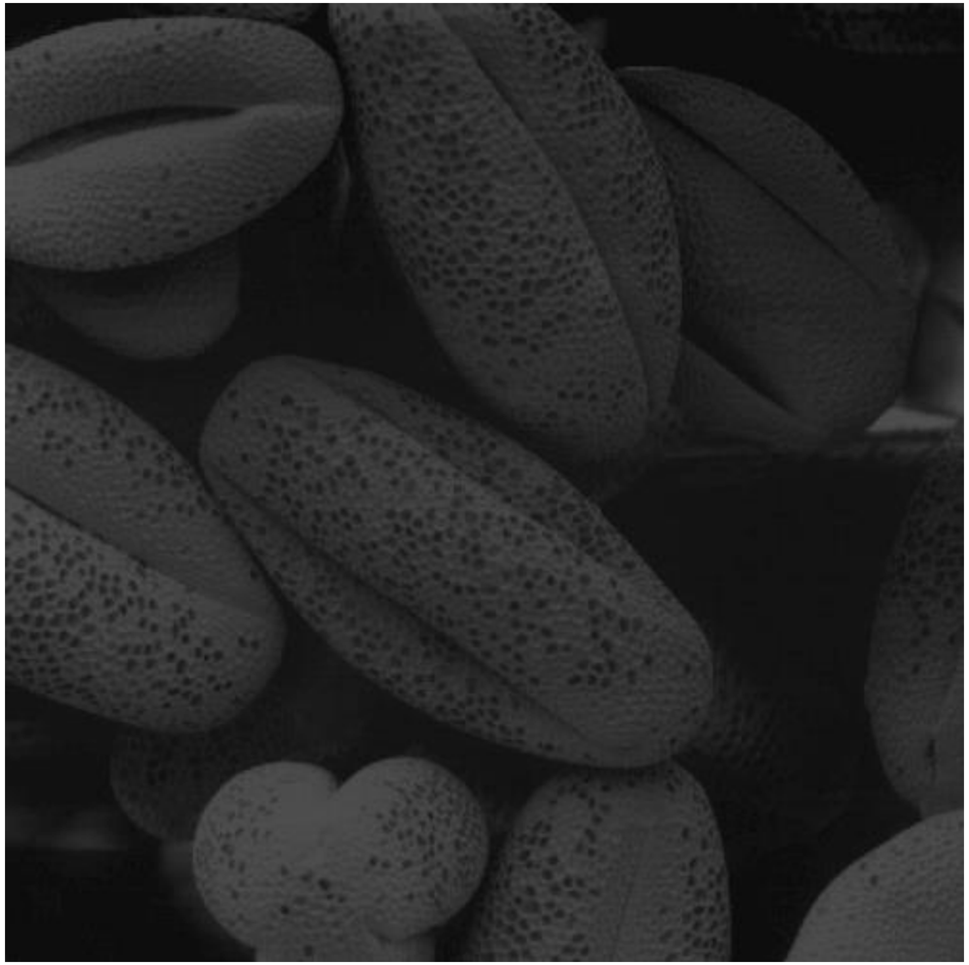
Massively useful in image processing, especially in segmentation

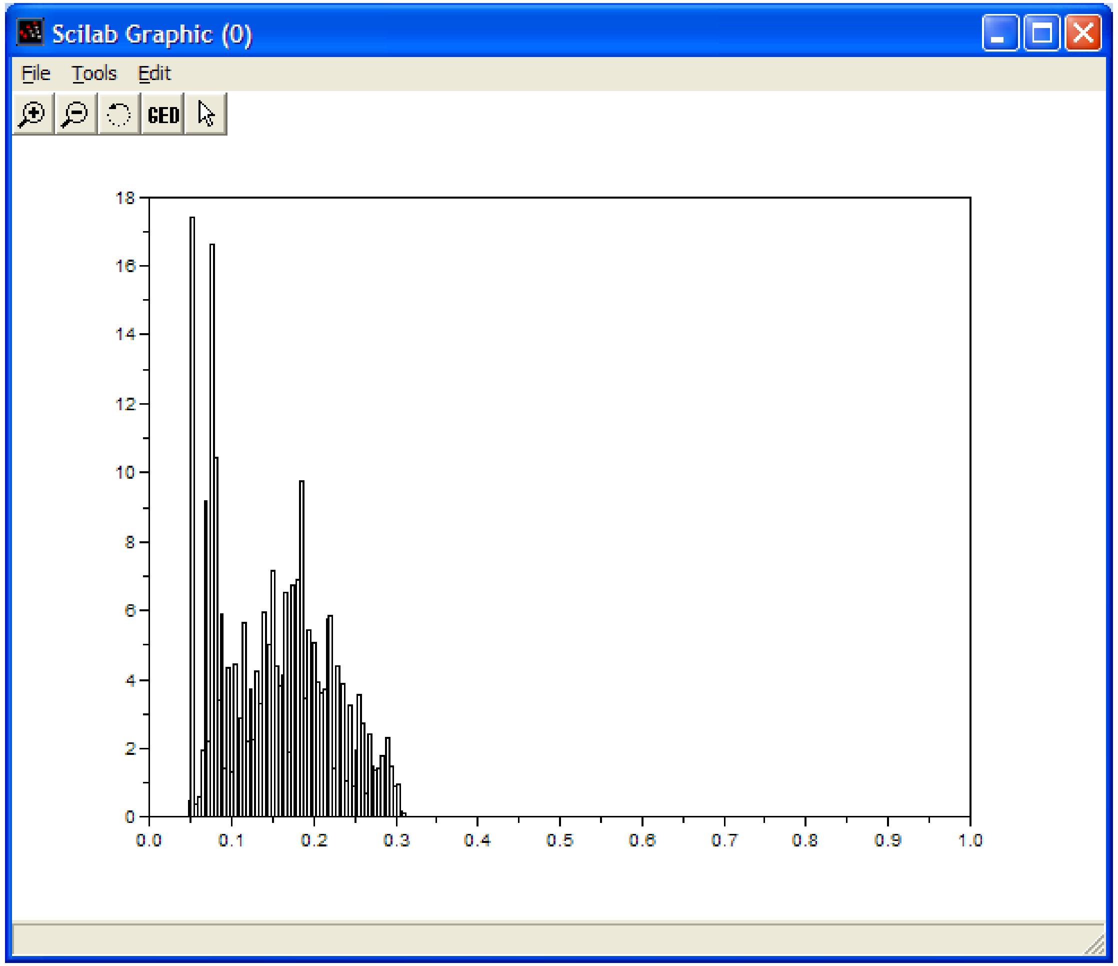


Histogram Examples

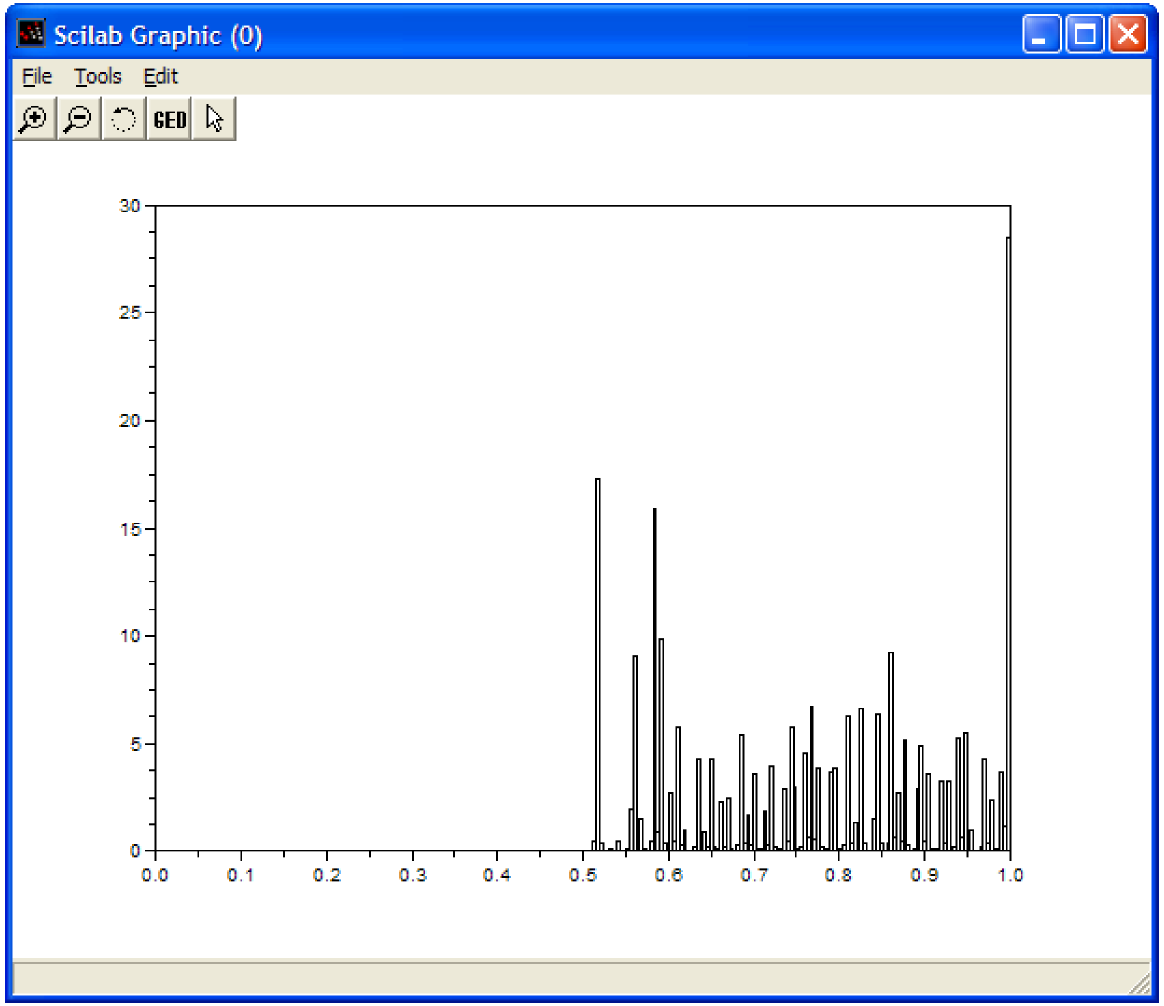


Histogram Examples (cont…): Dark Image

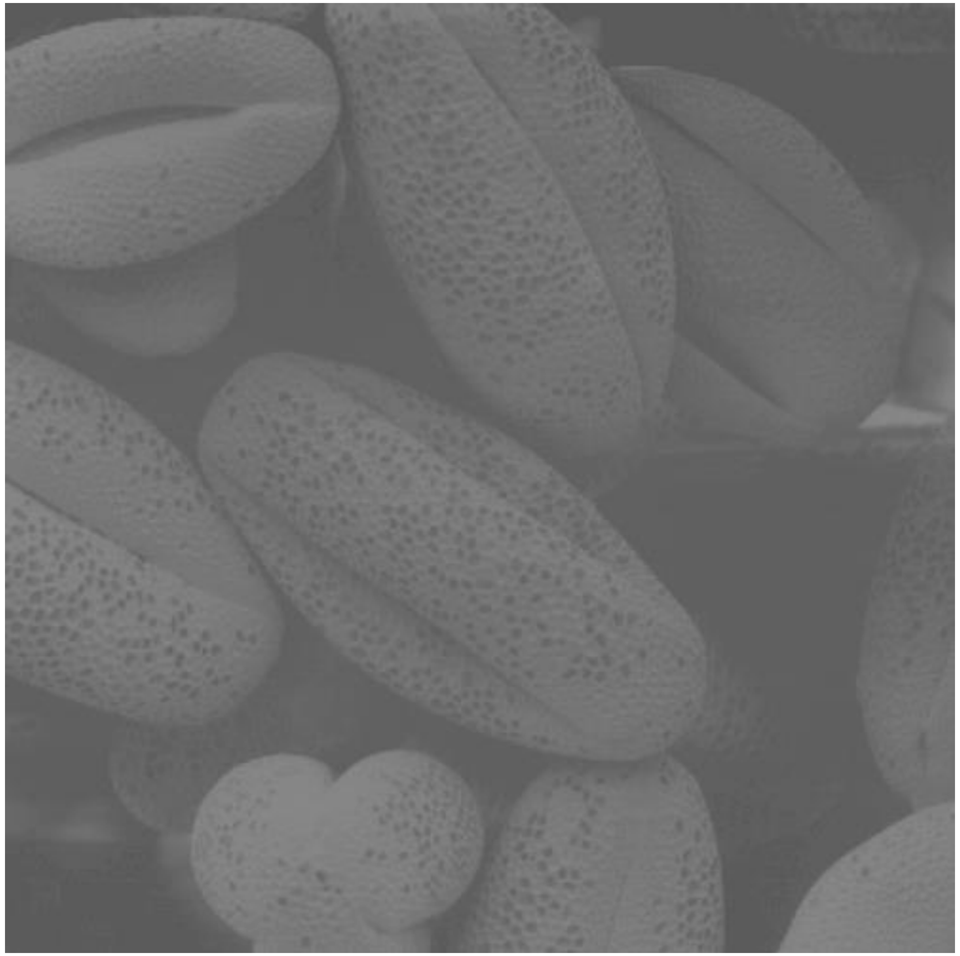


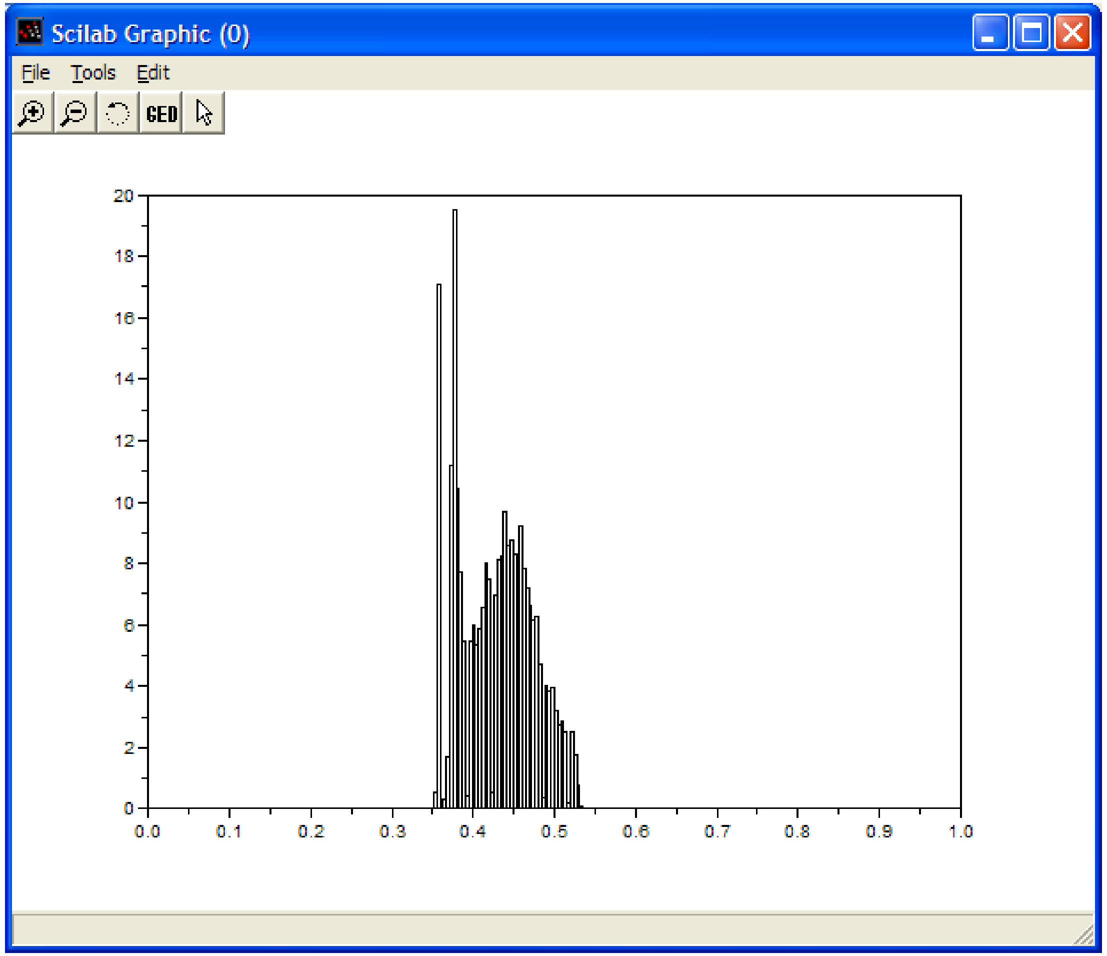


Histogram Examples (cont…): Light Image

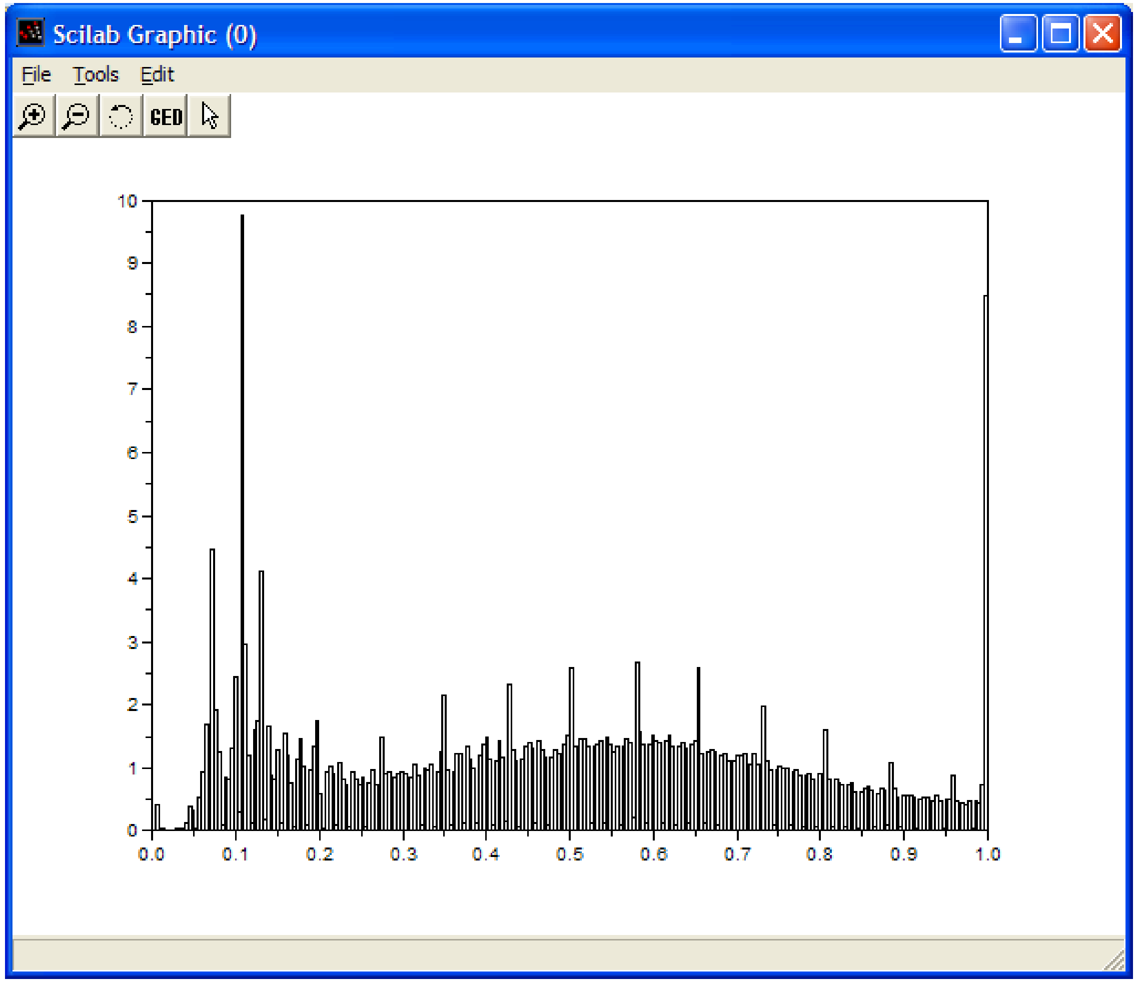


Histogram Examples (cont…): Low Contrast Image

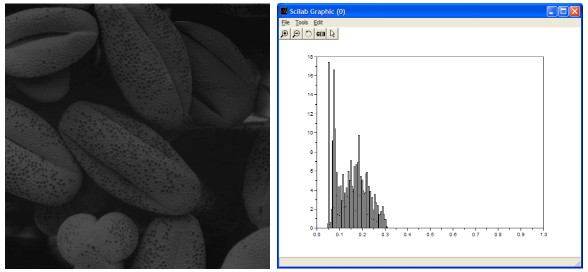




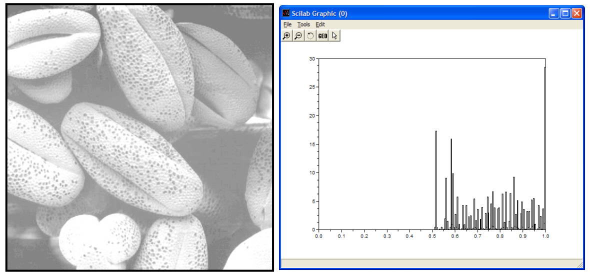
Histogram Examples (cont…): High Contrast Image



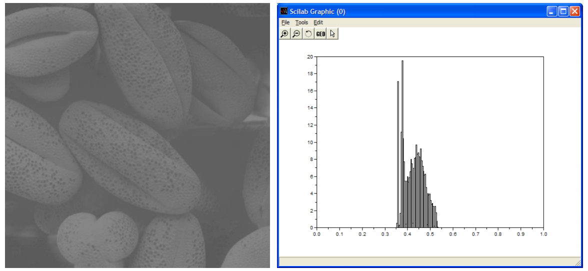
Histogram Examples (cont…)



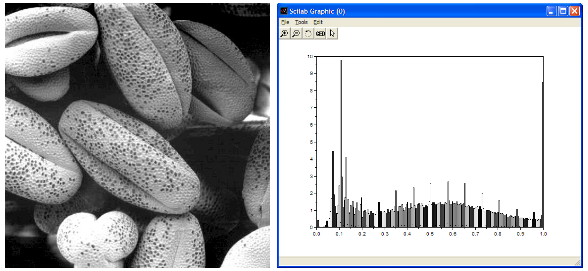
Dark Image



Light Image



Low Contrast Image



High Contrast Image

- A selection of images and their histograms
- Notice the relationships between the images and their histograms
- Note that the high contrast image has the most evenly spaced histogram

Histogram Processing

Histogram $h(r_k) = n_k$

r_k is the k^{th} intensity value

n_k is the number of pixels in the image with intensity r_k

Normalized histogram $p(r_k) = \dfrac{n_k}{MN}$

n_k: the number of pixels in the image of size $\mathrm{M} \times \mathrm{N}$ with intensity r_k

Histogram Equalisation

- Spreading out the frequencies in an image (or equalising the image) is a simple way to improve dark or washed out images.
- At first, the continuous case will be studied:
  - r is the intensity of the image in $[0, L\text{-}1]$.
  - we focus on transformations $s=T(r)$:
  - $T(r)$ is strictly monotonically increasing.
  - $T(r)$ must satisfy:

$$0 \leq T(r) \leq L-1, \;\; for \;\; 0 \leq r \leq L-1$$

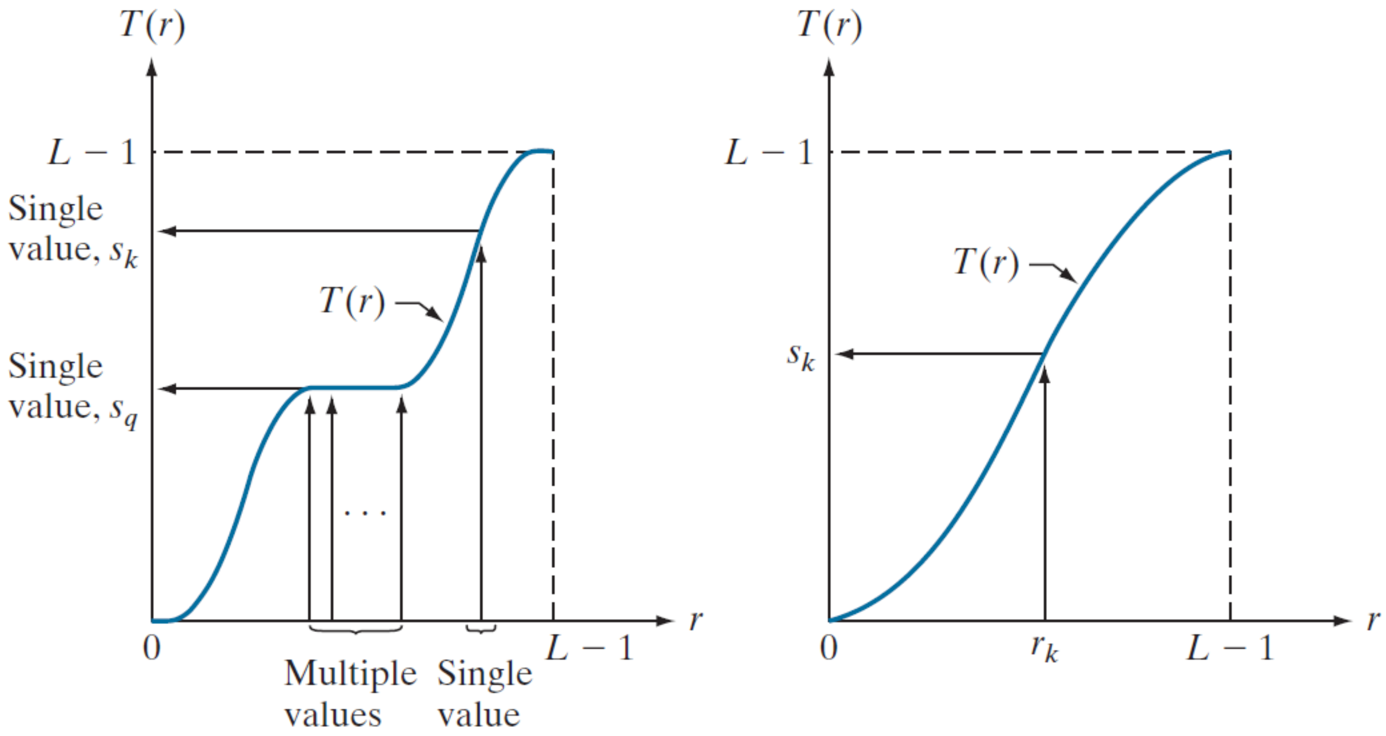
Histogram Equalisation (cont…)

- The condition for $T(r)$ to be monotonically increasing guarantees that ordering of the output intensity values will follow the ordering of the input intensity values (avoids reversal of intensities).
- The second condition ($T(r)$ in $[0,1]$) guarantees that the range of the output will be the same as the range of the input.

Histogram Equalisation (cont...)

a b

FIGURE 3.17
(a) Monotonic increasing function, showing how multiple values can map to a single value. (b) Strictly monotonic increasing function. This is a one-to-one mapping, both ways.



a) We cannot perform inverse mapping (from s to r).
b) Inverse mapping is possible.

Adapted from Gonzalez, Rafael C. Digital image processing. Pearson education India, 2009.

Histogram Equalisation (cont…)

- We can view intensities r and s as random variables and their histograms as probability density functions (pdf) $p_r(r)$ and $p_s(s)$.
- Fundamental result from probability theory:
  - If $p_r(r)$ and $T(r)$ are known and $T(r)$ is continuous and differentiable, then

$$p_s(s) = p_r(r)\frac{1}{\left|\frac{ds}{dr}\right|} = p_r(r)\left|\frac{dr}{ds}\right|$$

Histogram Equalisation (cont...)

- The pdf of the output is determined by the pdf of the input and the transformation.
- This means that we can determine the histogram of the output image.
- A transformation of particular importance in image processing is the cumulative distribution function (CDF) of a random variable:

$$s = T(r) = (L-1)\int_0^r p_r(w)\,dw$$

Histogram Equalisation (cont...)

- It satisfies the first condition as the area under the curve increases as r increases.
- It satisfies the second condition as for $r=L$-1 we have $s=L$-1.
- To find $p_s(s)$ we have to compute

$$\frac{ds}{dr} = \frac{dT(r)}{dr} = (L-1)\frac{d}{dr}\int_0^r p_r(w)\,dw = (L-1)\,p_r(r)$$

Histogram Equalisation (cont...)

Substituting this result:

$$\frac{ds}{dr} = (L-1)p_r(r)$$

to

$$p_s(s) = p_r(r)\left|\frac{dr}{ds}\right|$$

Uniform pdf

yields

$$p_s(s) = p_r(r)\left|\frac{1}{(L-1)p_r(r)}\right| = \frac{1}{L-1}, \quad 0 \le s \le L-1$$

Histogram Equalisation (cont...)

Example: Suppose that the (continuous) intensity values in an image have the PDF

$$p_r(r) = \begin{cases} \dfrac{2r}{(L-1)^2} & \text{for } 0 \le r \le L-1 \\ 0 & \text{otherwise} \end{cases}$$

Prove that the PDF of the intensities in the new image is uniform

Histogram Equalisation (cont…)

The formula for histogram equalisation in the discrete case is given

$$s_k = T(r_k) = (L-1)\sum_{j=0}^{k} p_r(r_j) = \frac{(L-1)}{MN}\sum_{j=0}^{k} n_j$$

where

- r_k: input intensity
- s_k: processed intensity
- n_j: the frequency of intensity j
- MN: the number of image pixels.

Histogram Equalisation (cont...) Example

A 3-bit 64x64 image has the following intensities:

r_k	n_k	$p_r(r_k) = n_k/MN$
$r_0 = 0$	790	0.19
$r_1 = 1$	1023	0.25
$r_2 = 2$	850	0.21
$r_3 = 3$	656	0.16
$r_4 = 4$	329	0.08
$r_5 = 5$	245	0.06
$r_6 = 6$	122	0.03
$r_7 = 7$	81	0.02

$$s_k = T(r_k) = (L-1)\sum_{j=0}^{k} p_r(r_j)$$

Applying histogram equalization:

$$s_0 = T(r_0) = 7\sum_{j=0}^{0} p_r(r_j) = 7p_r(r_0) = 1.33$$

$$s_1 = T(r_1) = 7\sum_{j=0}^{1} p_r(r_j) = 7p_r(r_0) + 7p_r(r_1) = 3.08$$

Histogram Equalisation (cont…) Example

Rounding to the nearest integer:

$$s_0 = 1.33 \to 1 \quad s_1 = 3.08 \to 3 \quad s_2 = 4.55 \to 5 \quad s_3 = 5.67 \to 6$$
$$s_4 = 6.23 \to 6 \quad s_5 = 6.65 \to 7 \quad s_6 = 6.86 \to 7 \quad s_7 = 7.00 \to 7$$

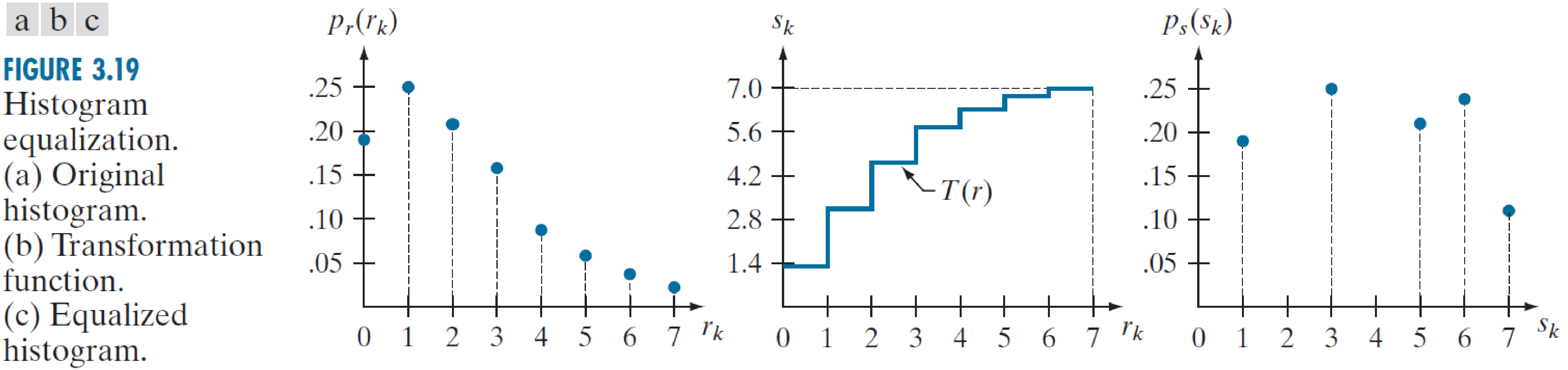
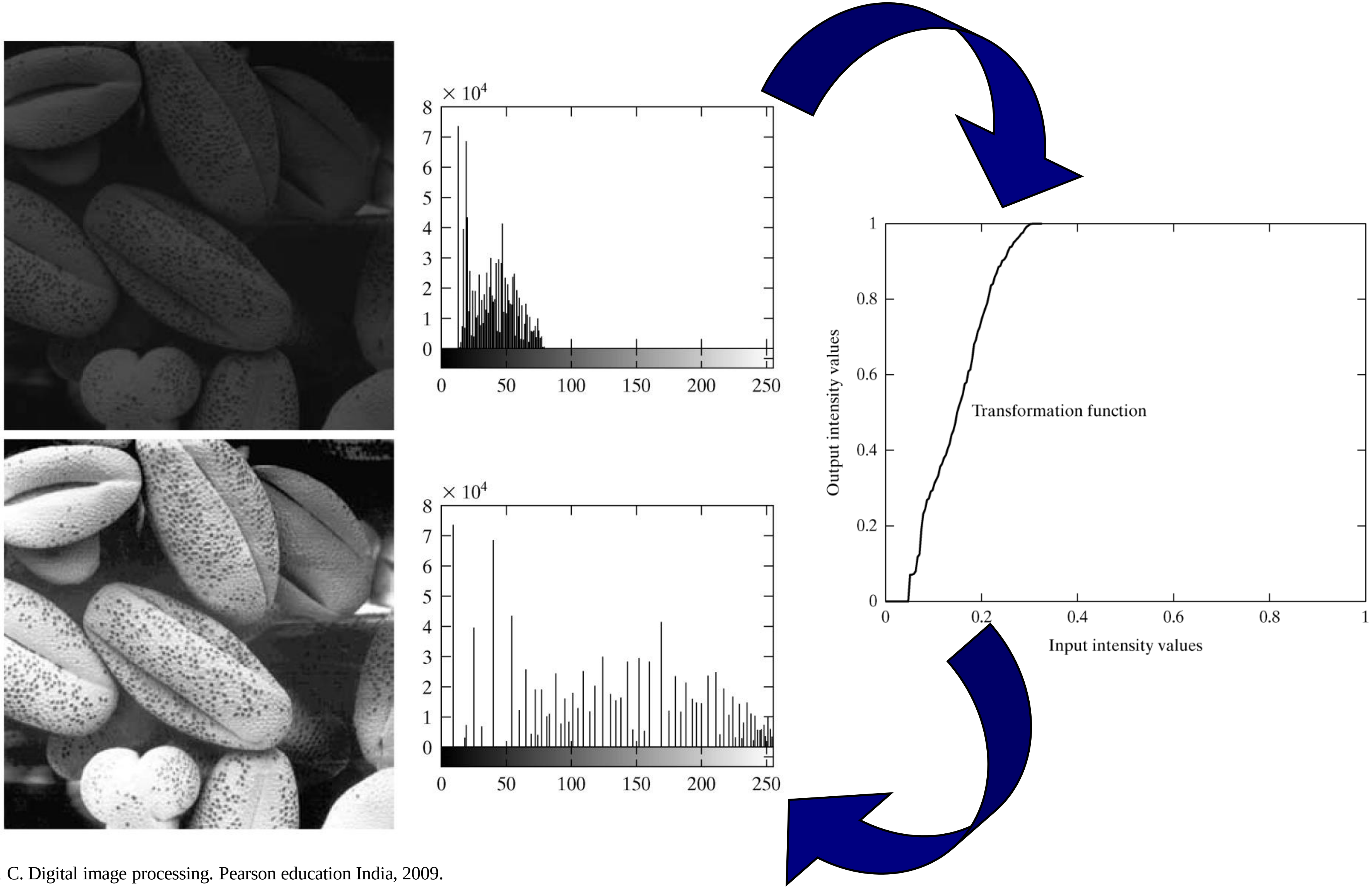


FIGURE 3.19 Histogram equalization. (a) Original histogram. (b) Transformation function. (c) Equalized histogram.

Notice that due to discretization, the resulting histogram will rarely be perfectly flat. However, it will be extended.

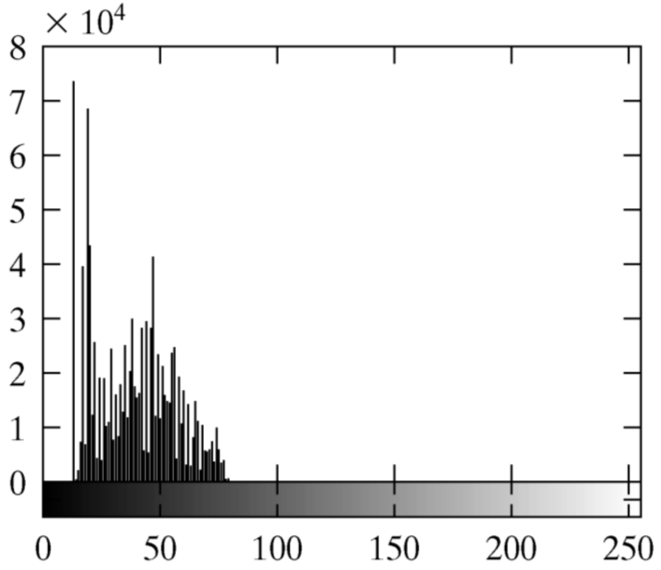
Adapted from Gonzalez, Rafael C. Digital image processing. Pearson education India, 2009.

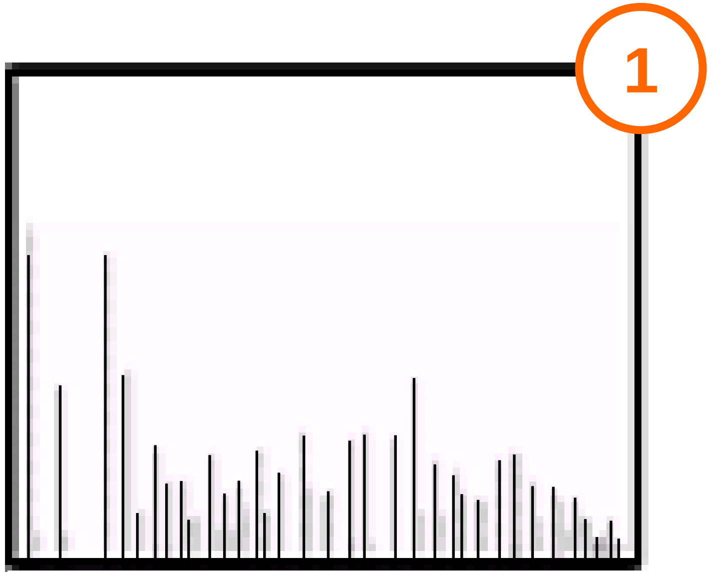
Equalisation Transformation Function

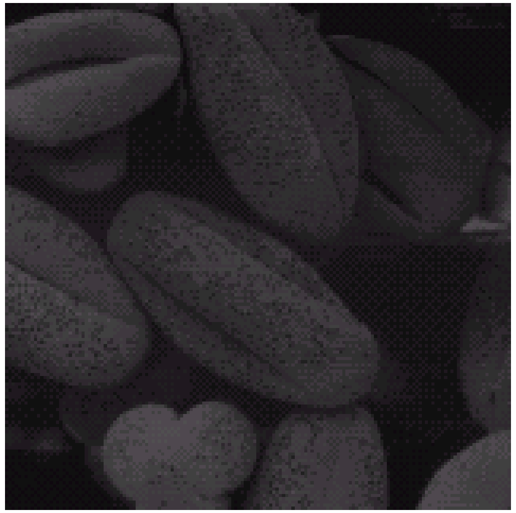


Adapted from Gonzalez, Rafael C. Digital image processing. Pearson education India, 2009.

Equalisation Examples



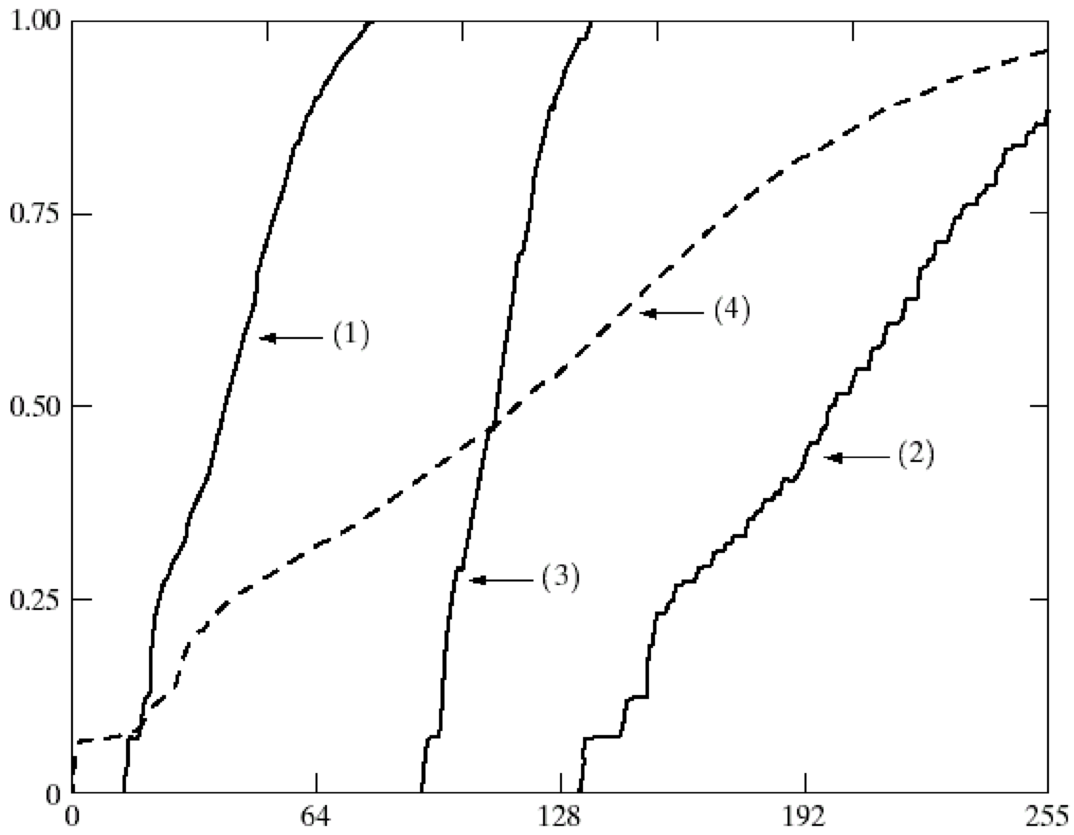




Adapted from Gonzalez, Rafael C. Digital image processing. Pearson education India, 2009.

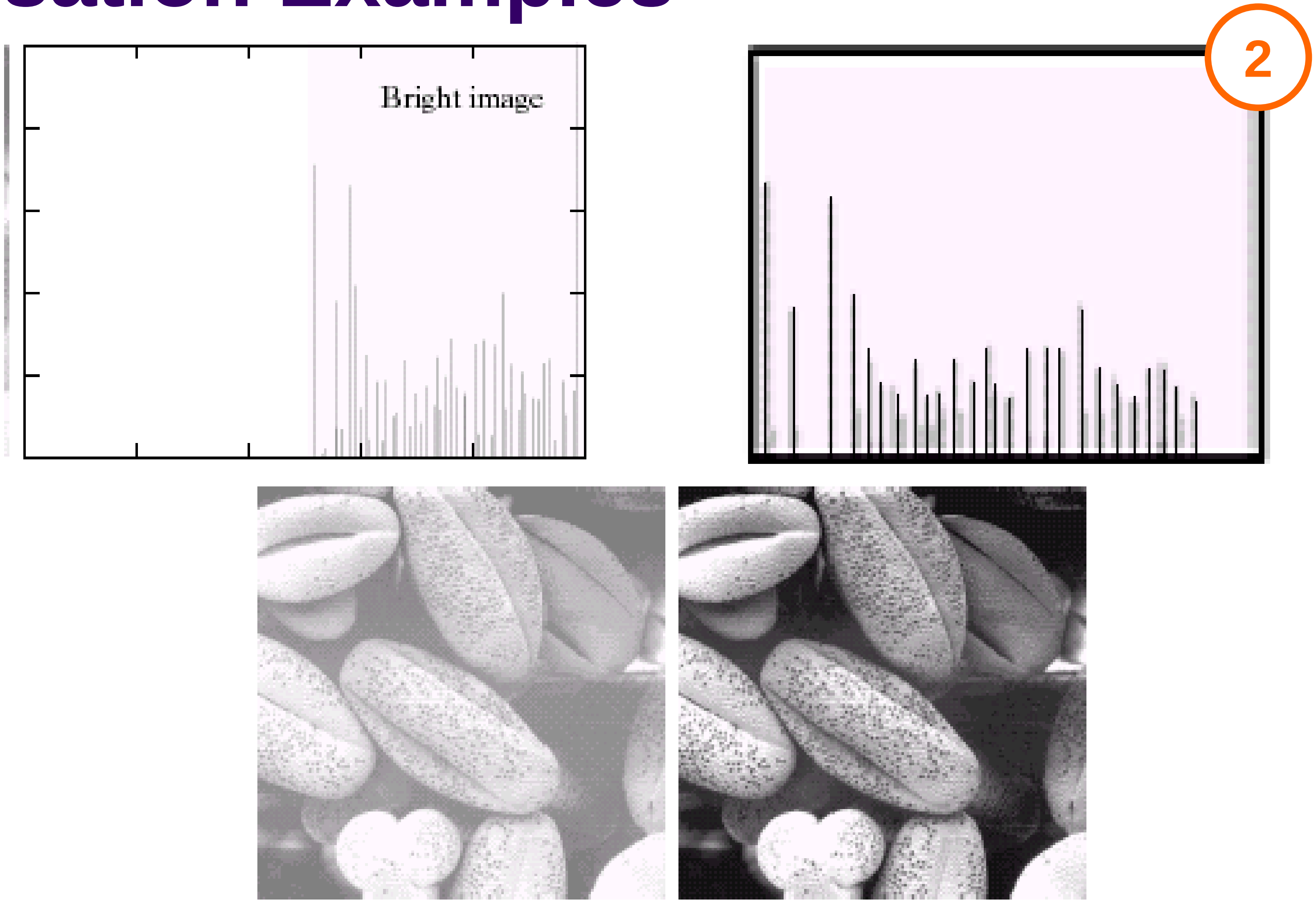
Equalisation Transformation Functions

The functions used to equalise the images in the previous example



Adapted from Gonzalez, Rafael C. Digital image processing. Pearson education India, 2009.

Equalisation Examples



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Equalisation Transformation Functions

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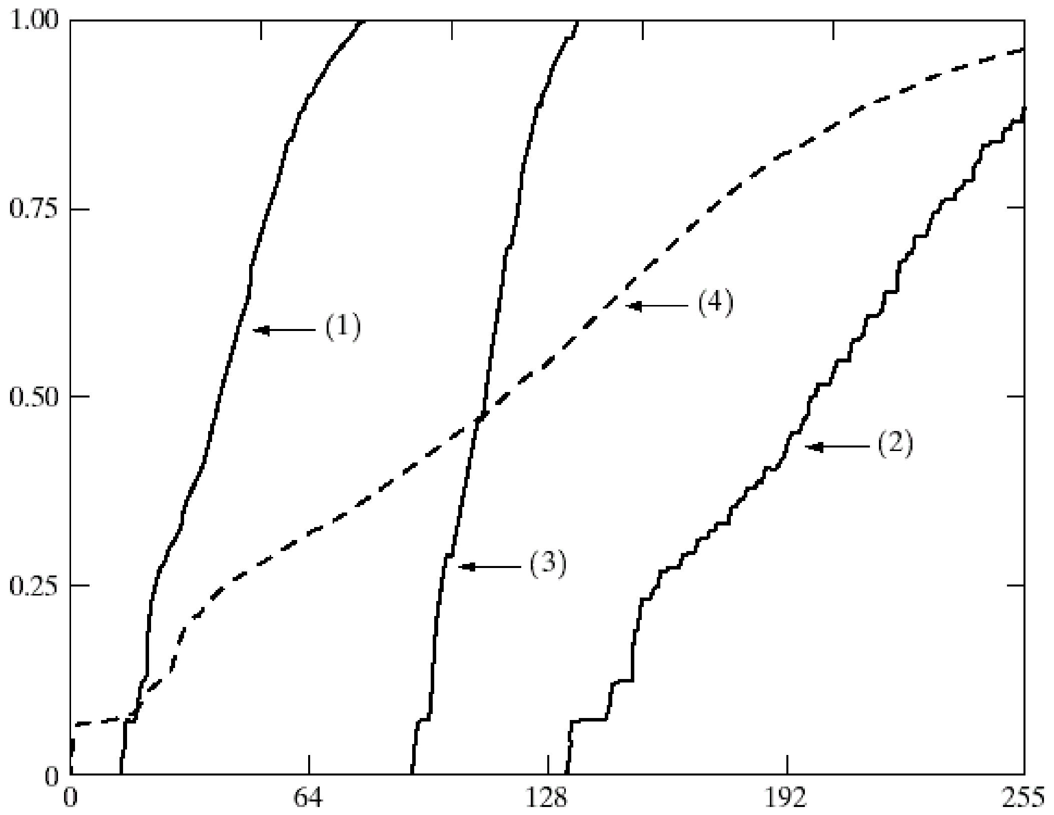
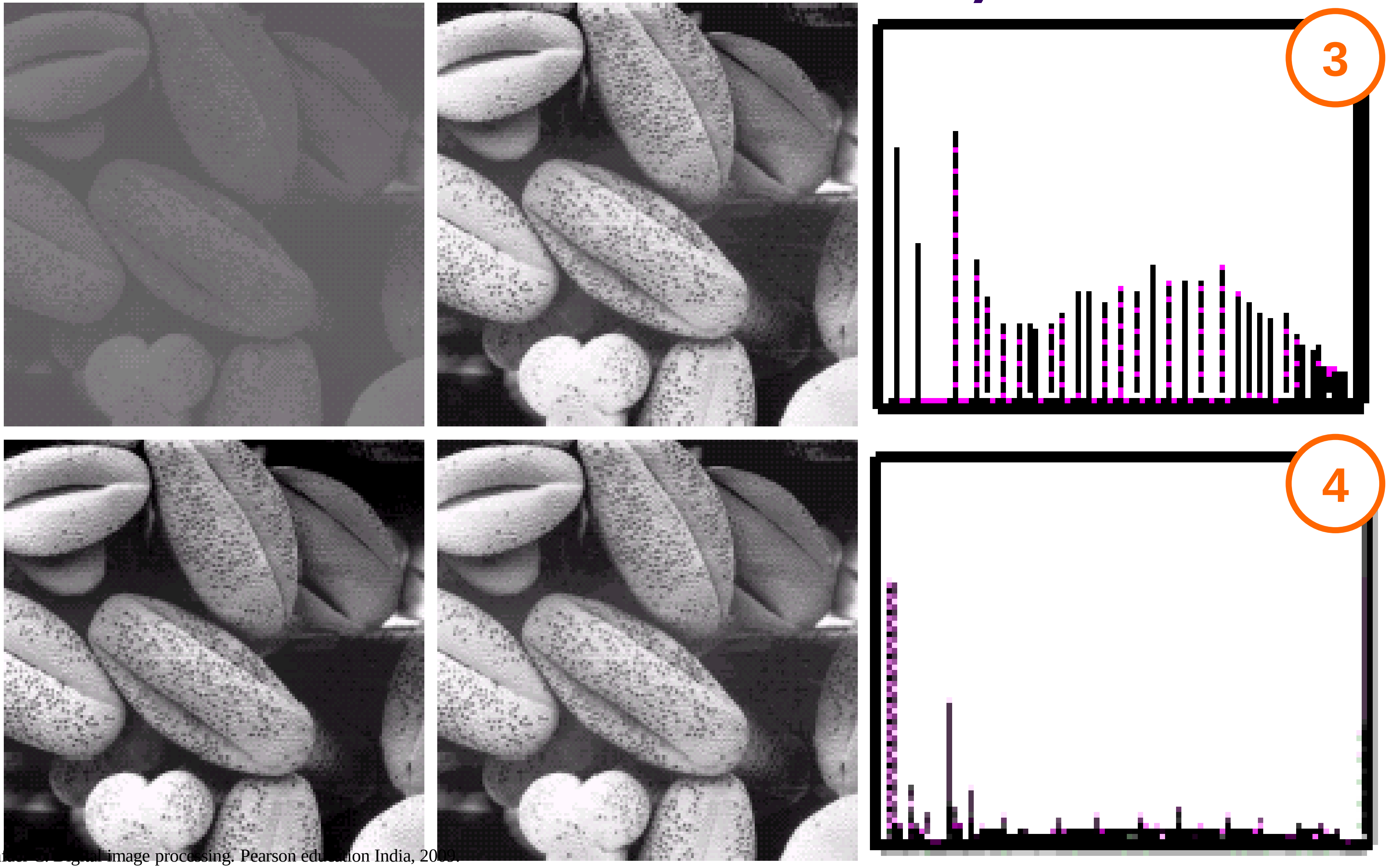


Adapted from Gonzalez, Rafael C. Digital image processing. Pearson education India, 2009.

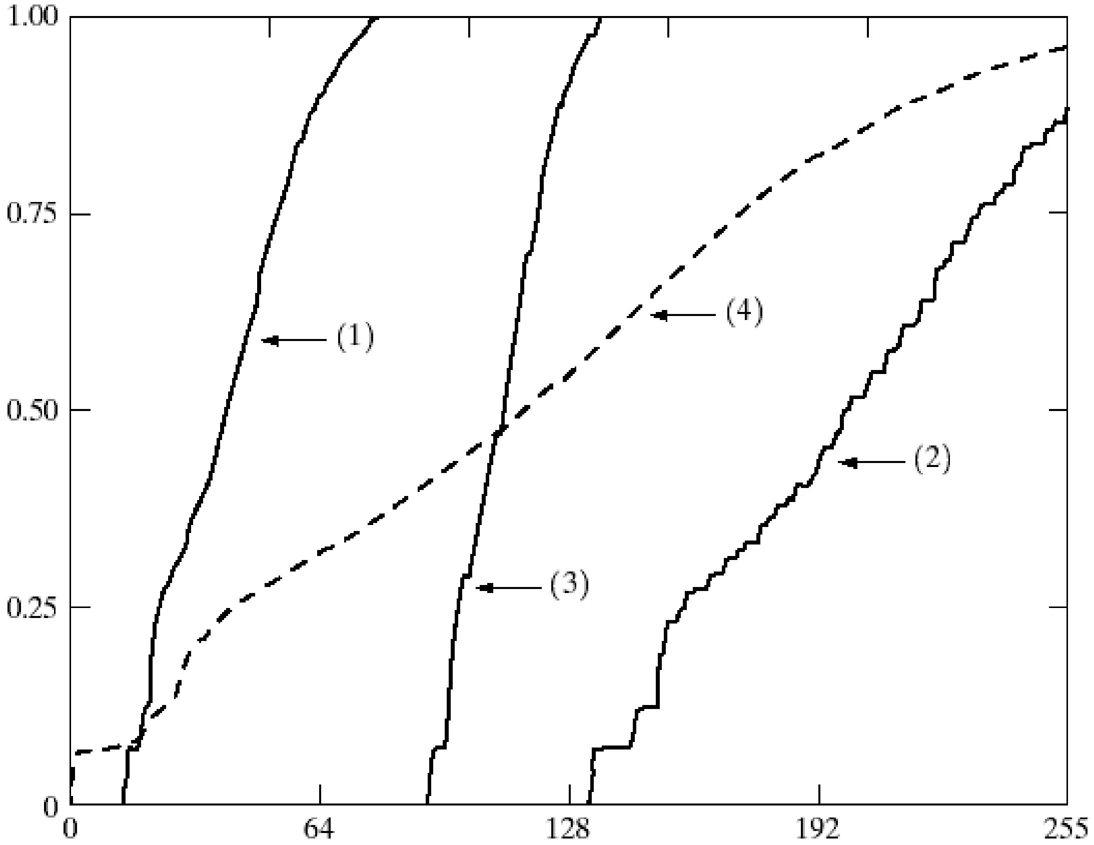
Equalisation Examples (cont…)



Adapted from Gonzalez, Rafael C. Digital image processing. Pearson education India, 2009.

Equalisation Transformation Functions

The functions used to equalise the images in the previous examples



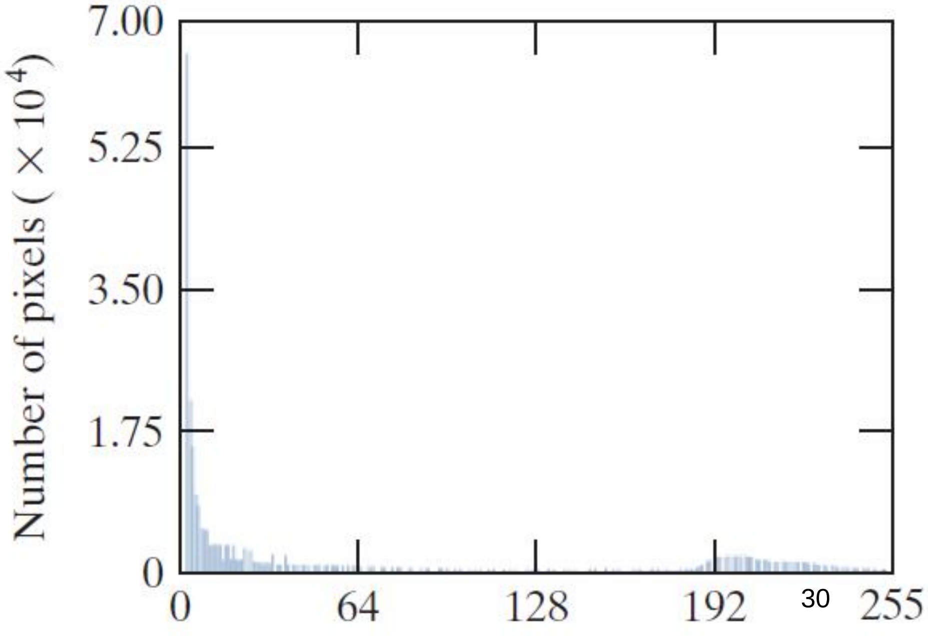
Adapted from Gonzalez, Rafael C. Digital image processing. Pearson education India, 2009.

Histogram Specification or Matching

- Histogram equalization does not always provide the desirable results.

a b

FIGURE 3.23
(a) An image, and
(b) its histogram.



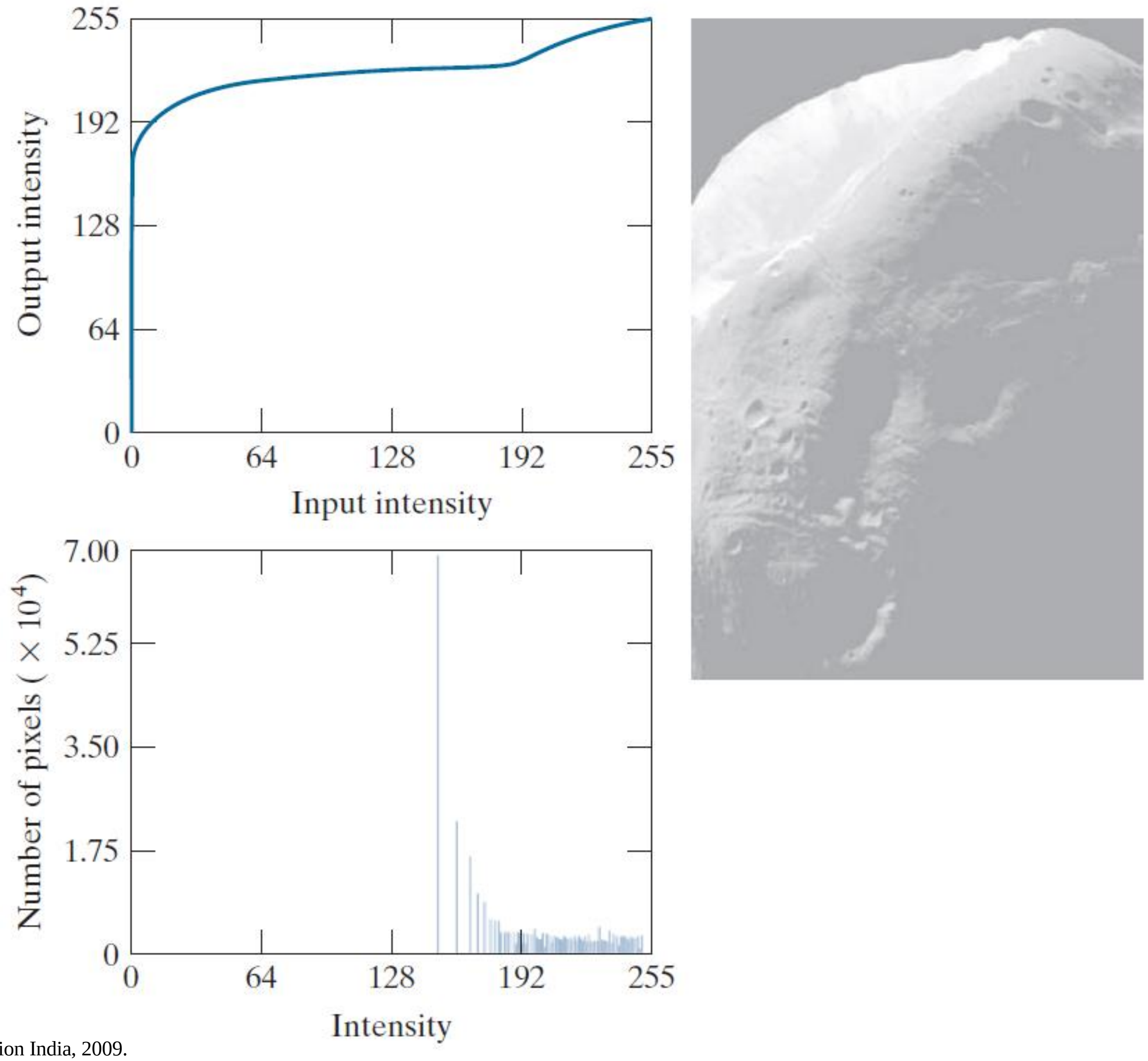
- Image of Phobos (Mars moon) and its histogram.
- Many values near zero in the initial histogram

Adapted from Gonzalez, Rafael C. Digital image processing. Pearson education India, 2009.

Histogram Specification (cont…)

a b
c

FIGURE 3.24
(a) Histogram equalization transformation obtained using the histogram in Fig. 3.23(b).
(b) Histogram equalized image.
(c) Histogram of equalized image.



Adapted from Gonzalez, Rafael C. Digital image processing. Pearson education India, 2009.

Histogram specification (cont.)

- In these cases, it is more useful to specify the final histogram.
- Problem statement:
  - Given $p_r(r)$ from the image and the target histogram $p_z(z)$, estimate the transformation $z=T(r)$.
- The solution exploits histogram equalization.

Histogram specification (cont…)

- Equalize the initial histogram of the image:

$$s = T(r) = (L-1)\int_0^r p_r(w)\,dw \qquad G(z) = T(r)$$

- Equalize the target histogram:

$$s = G(z) = (L-1)\int_0^r p_z(w)\,dw$$

- Obtain the inverse transform: $z = G^{-1}(s) = G^{-1}(T(r))$

  In practice, for every value of r in the image:
- get its equalized transformation $s=T(r)$.
- perform the inverse mapping $z=G^{-1}(s)$, where $s=G(z)$ is the equalized target histogram.

Histogram specification (cont…)

Assuming continuous intensity values, suppose that an image has the intensity PDF

$$p_r(r) = \begin{cases} \dfrac{2r}{(L-1)^2}, & \text{for } 0 \le r \le L-1 \\ 0\,, & \text{otherwise} \end{cases}$$

Find the transformation function that will produce an image whose intensity PDF is

$$p_z(z) = \begin{cases} \dfrac{3z^2}{(L-1)^3}, & \text{for } 0 \le z \le (L-1) \\ 0, & \text{otherwise} \end{cases}$$

Histogram specification (cont…)

Find the histogram equalization transformation for the input image

$$s = T(r) = (L-1)\int_0^r p_r(w)dw = (L-1)\int_0^r \frac{2w}{(L-1)^2}dw = \frac{r^2}{L-1}$$

Find the histogram equalization transformation for the specified histogram

$$G(z) = (L-1)\int_0^z p_z(t)dt = (L-1)\int_0^z \frac{3t^2}{(L-1)^3}dt = \frac{z^3}{(L-1)^2} = s$$

The transformation function

$$z = [(L-1)^2 s]^{1/3} = \left[(L-1)^2 \frac{r^2}{L-1}\right]^{1/3} = [(L-1)r^2]^{1/3}$$

Histogram specification (cont…)

The discrete case:

- Equalize the initial histogram of the image:

$$s_k = T(r_k) = (L-1)\sum_{j=0}^{k} p_r(r_j) = \frac{(L-1)}{MN}\sum_{j=0}^{k} n_j$$

$$G(z) = T(r)$$

- Equalize the target histogram:

$$s_k = G(z_q) = (L-1)\sum_{i=0}^{q} p_z(r_i)$$

- Obtain the inverse transform: $z_q = G^{-1}(s_k) = G^{-1}(T(r_k))$

Histogram Specification (cont...) Example

Consider again the 3-bit 64x64 image:

r_k	n_k	$p_r(r_k) = n_k/MN$
$r_0 = 0$	790	0.19
$r_1 = 1$	1023	0.25
$r_2 = 2$	850	0.21
$r_3 = 3$	656	0.16
$r_4 = 4$	329	0.08
$r_5 = 5$	245	0.06
$r_6 = 6$	122	0.03
$r_7 = 7$	81	0.02

It is desired to transform this histogram to:

$$p_z(z_0) = 0.00 \quad p_z(z_1) = 0.00 \quad p_z(z_2) = 0.00 \quad p_z(z_3) = 0.15$$
$$p_z(z_4) = 0.20 \quad p_z(z_5) = 0.30 \quad p_z(z_6) = 0.20 \quad p_z(z_7) = 0.15$$

with $z_0 = 0, z_1 = 1, z_2 = 2, z_3 = 3, z_4 = 4, z_5 = 5, z_6 = 6, z_7 = 7.$

Histogram Specification (cont...) Example

The first step is to equalize the input (as before):

$$s_0 = 1,\ s_1 = 3,\ s_2 = 5,\ s_3 = 6,\ s_4 = 6,\ s_5 = 7,\ s_6 = 7,\ s_7 = 7$$

The next step is to equalize the output:

$$G(z_0) = 0 \quad G(z_1) = 0 \quad G(z_2) = 0 \quad G(z_3) = 1$$
$$G(z_4) = 2 \quad G(z_5) = 5 \quad G(z_6) = 6 \quad G(z_7) = 7$$

Notice that $G(z)$ is not strictly monotonic. We must resolve this ambiguity by choosing, e.g. the smallest value for the inverse mapping.

Histogram Specification (cont...) Example

Perform inverse mapping: find the smallest value of z_q that is closest to s_k:

$s_k = T(r_i)$	$G(z_q)$
$s_0 = 1$	$G(z_0) = 0$
$s_1 = 3$	$G(z_1) = 0$
$s_2 = 5$	$G(z_2) = 0$
$s_3 = 6$	$G(z_3) = 1$
$s_4 = 6$	$G(z_4) = 2$
$s_5 = 7$	$G(z_5) = 5$
$s_6 = 7$	$G(z_6) = 6$
$s_7 = 7$	$G(z_7) = 7$

$$s_k \rightarrow z_q$$

$$1 \rightarrow 3$$

$$3 \rightarrow 4$$

$$5 \rightarrow 5$$

$$6 \rightarrow 6$$

$$7 \rightarrow 7$$

e.g. every pixel with value $s_0=1$ in the histogram-equalized image would have a value of 3 (z_3) in the histogram-specified image.

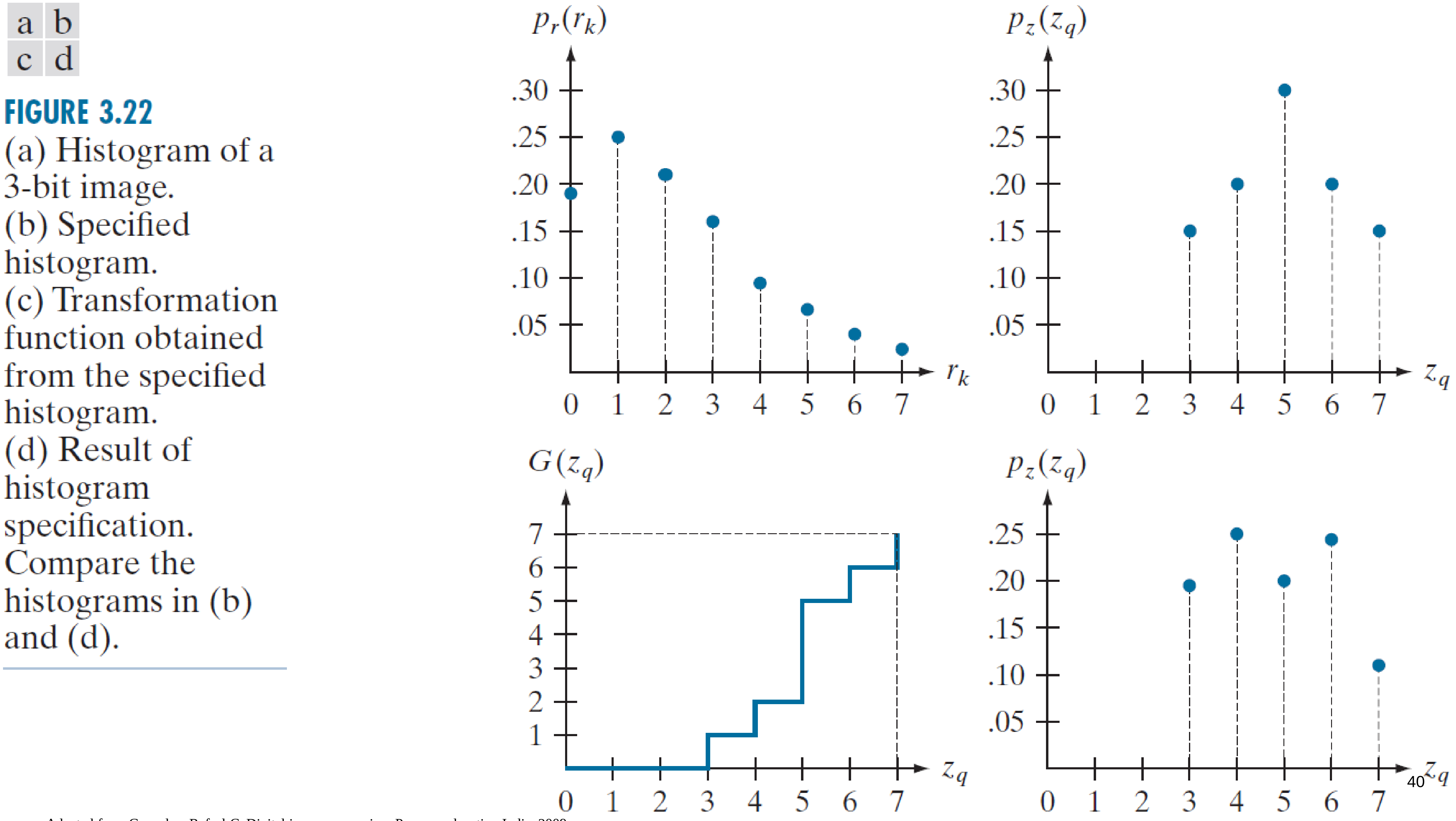
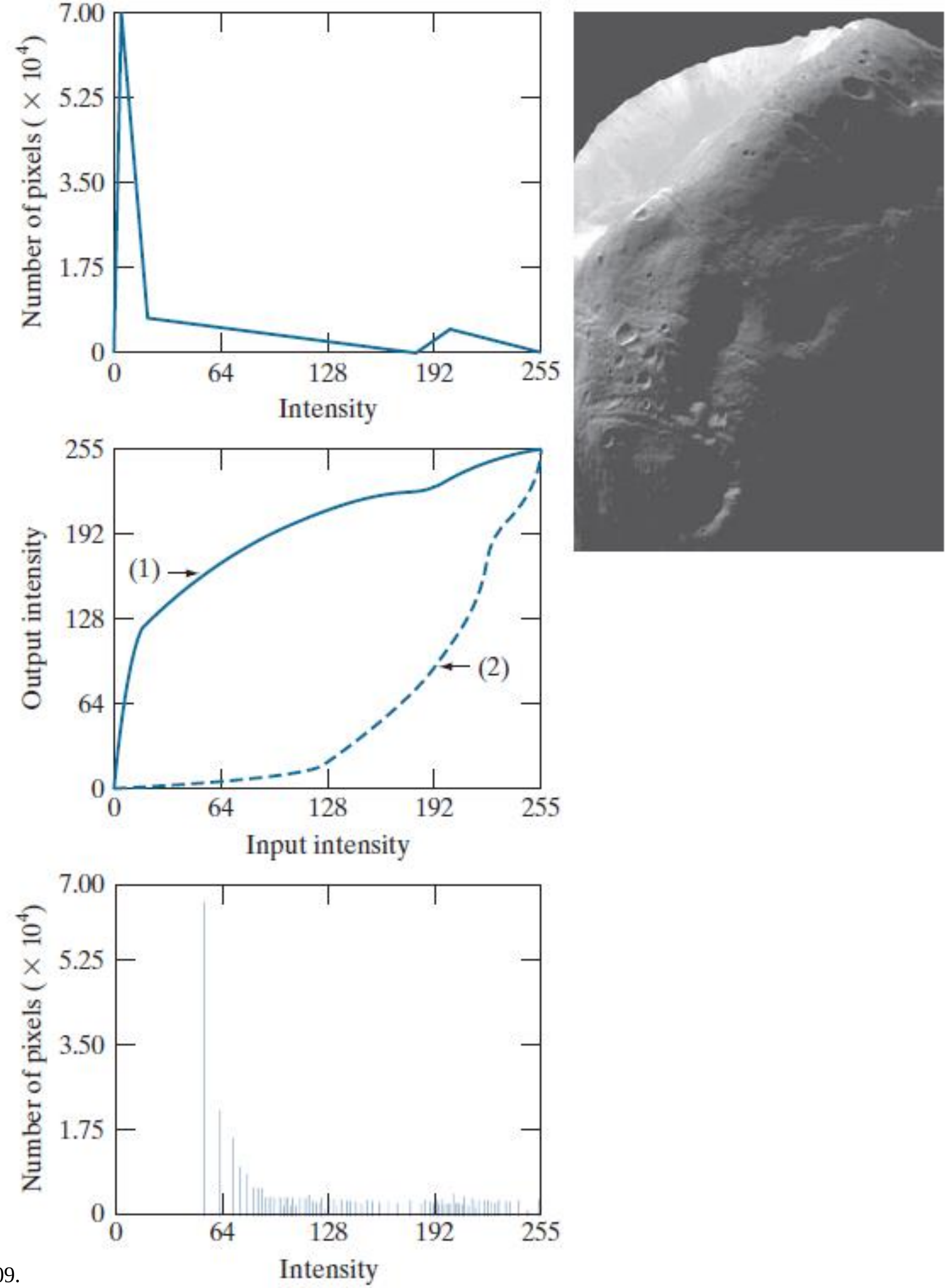


FIGURE 3.22
(a) Histogram of a 3-bit image.
(b) Specified histogram.
(c) Transformation function obtained from the specified histogram.
(d) Result of histogram specification. Compare the histograms in (b) and (d).

Adapted from Gonzalez, Rafael C. Digital image processing. Pearson education India, 2009.

a c
b
d

FIGURE 3.25 Histogram specification. (a) Specified histogram. (b) Transformation $G(z_q)$, labeled (1), and $G^{-1}(s_k)$, labeled (2). (c) Result of histogram specification. (d) Histogram of image (c).



Adapted from Gonzalez, Rafael C. Digital image processing. Pearson education India, 2009.

Summary

We have looked at:

- Different kinds of image enhancement
- Histograms
- Histogram equalisation
- Histogram specification

Next time we will start to look at spatial filtering and neighbourhood operations