Principles of Communication Engineering

Indian Institute of Information Technology Allahabad (IIITA), Prayagraj-211015

Dr. Ramesh Kumar Bhukya, *Ph.D. (IIT Guwahati)*,
Assistant Professor,
Department of Electronics and Communication Engineering,
Indian Institute of Information Technology Allahabad,
Deoghat, Jhalwa, Prayagraj-211015, Uttar Pradesh, India.
Email ID: rkbhukya@iiita.ac.in

Fourier Series

Fourier Series

- The Fourier series is a mathematical tool that allows the representation of any periodic signal as the sum of harmonically related sinusoids.

  Any periodic signal, i.e., one for which $x(t) = x(t+T)$, can be expressed by a Fourier series provided that
  - ✓ If it is discontinuous, there are finite number of discontinuities in the period T
  - ✓ It has a finite average value over the period T
  - ✓ It has a finite number of positive and negative maxima in the period T

- When these *Dirichlet conditions* are satisfied, the Fourier series exist.
  - ✓ The Fourier series is of two types
    - Trigonometric Fourier series
    - Exponential Fourier series

- **Trigonometric Fourier Series**

  The Trigonometric Fourier Series is expressed as

  $x(t) = a_0 + \sum_{n=1}^{\infty}[a_n cos(nw_0t) + b_n sin(nw_0t)]$

$$a_0 = \frac{1}{T}\int_0^T x(t)dt$$

$$a_n = \frac{2}{T}\int_0^T x(t)cos(nw_0t)dt$$

$$b_n = \frac{2}{T}\int_0^T x(t)sin(nw_0t)dt$$

Fourier Series

❖ Polar Form Representation of the Fourier Series

Case I

$$a_n = c_n \cos(\theta_n) \text{ and } b_n = -c_n \sin(\theta_n)$$

$$c_0 = a_0 \quad and \quad c_n = \sqrt{a_n{}^2 + b_n{}^2} \; for\; n \geq 1$$

$$\theta_n = \tan^{-1}\frac{-bn}{a_n}$$

Substituting $a_n = c_n \cos(\theta_n)$, $b_n = -c_n \sin(\theta_n)$

$$x(t) = a_0 + \sum_{n=1}^{\infty}[c_n \cos(\theta_n)\, cos(nw_0 t) - c_n \sin(\theta_n)\, sin(nw_0 t)]$$

$$= a_0 + \sum_{n=1}^{\infty} c_n cos(nw_0 t + \theta_n)$$

$$x(t) = c_0 + \sum_{n=1}^{\infty} c_n cos(nw_0 t + \theta_n)$$

Case II $a_n = c_n \sin(\emptyset_n)$, $b_n = c_n cos(\emptyset_n)$

$$c_0 = a_0 \quad and \quad c_n = \sqrt{a_n{}^2 + bn^2} \quad for\; n \geq 1$$

$$\emptyset_n = \tan^{-1}\frac{a_n}{b_n}$$

Fourier Series

❖ **Polar Form Representation of the Fourier Series**

Case II

$$a_n = c_n \sin(\emptyset_n) , b_n = c_n cos(\emptyset_n)$$

$$c_0 = a_0 \quad and \quad c_n = \sqrt{a_n{}^2 + bn^2} \quad for\; n \geq 1$$

$$\emptyset_n = \tan^{-1}\frac{a_n}{b_n}$$

Substituting

$$a_n = c_n \sin(\emptyset_n) , b_n = c_n cos(\emptyset_n)$$

$$x(t) = a_0 + \sum_{n=1}^{\infty}[c_n \sin(\emptyset_n)\, cos(nw_0 t) + c_n cos(\emptyset_n)\, sin(nw_0 t)]$$

$$= a_0 + \sum_{n=1}^{\infty} c_n sin(nw_0 t + \emptyset_n)$$

$$x(t) = c_0 + \sum_{n=1}^{\infty} c_n sin(nw_0 t + \emptyset_n)$$

Fourier Series

❖ **Evaluation of Fourier Series Coefficients**

$$\int_0^T sin(mw_0t)dt = 0\,, \qquad for\ all\ \ m$$

$$\int_0^T cos(nw_0t)dt = 0\ \ , \qquad for\ all\ n \neq 0$$

✓ The average value of a sinusoid over m and n complete cycles in the period T is zero. The following three cross product terms are also zero for the stated relationships of m and n

$$\int_0^T sin(mw_0t)cos(nw_0t)dt = 0\,, \qquad for\ all\ \ m\,,n$$

$$\int_0^T sin(mw_0t)sin(nw_0t)dt = \begin{cases} 0, m \neq n \\ \frac{T}{2}, m = n \end{cases}$$

$$\int_0^T cos(mw_0t)cos(nw_0t)dt = \begin{cases} 0, m \neq n \\ \frac{T}{2}, m = n \end{cases}$$

Fourier Series

❖ Evaluation of Fourier Series Coefficients

Case I *Proof*

$$a_0 = \frac{1}{T}\int_0^T x(t)dt$$

$$x(t) = a_0 + [a_1\cos(w_0t) + b_1\sin(w_0t)] + [a_2\cos(2w_0t) + b_2\sin(2w_0t) + \ \dots + [a_n cos(nw_0t) + b_n sin(nw_0t)] \dots$$

$$\int_0^T x(t)dt = \int_0^T a_0\,dt + \left[\int_0^T a_1\cos(w_0t)\,dt\right.$$

Fourier Series

❖ **Evaluation of Fourier Series Coefficients**

Case II *Proof*

$$a_n = \frac{2}{T}\int_0^T x(t)cos(nw_0t)dt$$

$$x(t) \\ = a_0 + [a_1\cos(w_0t) + b_1\sin(w_0t)] + [a_2\cos(2w_0t) + b_2\sin(2w_0t) + \ \ldots \\ + [a_ncos(nw_0t) + bnsin(nw_0t)]$$

$$\int_0^T x(t)\cos(nw_0t)\,dt = \int_0^T a_0\cos(nw_0t)\,dt + \left[\int_0^T a_1\cos(w_0t)\cos(nw_0t)\,dt\right.$$

Fourier Series

❖ Evaluation of Fourier Series Coefficients

Case III *Proof*

$$b_n = \frac{2}{T}\int_0^T x(t) sin(nw_0 t)dt$$

$$x(t) = a_0 + [a_1 \cos(w_0 t) + b_1 \sin(w_0 t)] + [a_2 \cos(2w_0 t) + b_2 \sin(2w_0 t) + \ \ldots + [a_n cos(nw_0 t) + bnsin(nw_0 t)] \ldots$$

$$\int_0^T x(t)\sin(nw_0 t)\,dt = \int_0^T a_0 \sin(nw_0 t)\,dt + \left[\int_0^T b_1 \cos(w_0 t)\sin(nw_0 t)\,dt\right.$$

Fourier Series

- **Symmetry Conditions**
  - ✓ **$x(t)$ is an even signal, i.e., $x(t) = x(-t)$**

$$a_0 = \frac{1}{T}\left(\int_0^{\frac{T}{2}} x(t)\,dt + \int_0^{\frac{T}{2}} x(t)\,dt\right)$$

$$x(t) = 2 + t^2 + t^4$$

$$a_0 = \frac{2}{T}\int_0^{T/2} x(t)dt$$

$$a_n = \frac{4}{T}\int_0^{T/2} x(t)cos(nw_0t)dt \quad \text{and} \quad b_n = 0$$

Proof

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)dt$$

$$a_0 = \frac{1}{T}\left(\int_{-\frac{T}{2}}^{0} x(t)dt + \int_0^{\frac{T}{2}} x(t)dt\right)$$

$$a_0 = \frac{1}{T}\left(\int_0^{\frac{T}{2}} x(-t)dt + \int_0^{\frac{T}{2}} x(t)dt\right)$$

Signal is even so :

$$a_0 = \frac{1}{T}\left(\int_0^{\frac{T}{2}} x(t)dt + \int_0^{\frac{T}{2}} x(t)dt\right)$$

$$a_0 = \frac{2}{T}\int_0^{T/2} x(t)dt$$

$$a_0 = \frac{2}{T}\left(\int_0^{\frac{T}{2}} x(t)\,dt\right)$$

Fourier Series

- **Symmetry Conditions**
  - ✓ $\boldsymbol{x(t)}$ **is an even signal, i.e.,** $\boldsymbol{x(t) = x(-t)}$

$$x(t) = 2 + t^2 + t^4$$

$$a_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \cos(n\omega_0 t)\, dt$$

$$a_n = \frac{2}{T} \left(\int_{-\frac{T}{2}}^{0} x(t) \cos(n\omega_0 t)\, dt + \int_{0}^{\frac{T}{2}} x(t) \cos(n\omega_0 t)\, dt \right)$$

$$= \frac{2}{T} \left(\int_{0}^{\frac{T}{2}} x(-t) \cos(-n\omega_0 t)\, dt + \int_{0}^{\frac{T}{2}} x(t) \cos(n\omega_0 t)\, dt \right)$$

$$= \frac{2}{T} \left(\int_{0}^{\frac{T}{2}} x(t) \cos(n\omega_0 t)\, dt + \int_{0}^{\frac{T}{2}} x(t) \cos(n\omega_0 t)\, dt \right)$$

$$a_n = \frac{4}{T} \int_{0}^{\frac{T}{2}} x(t) \cos(n\omega_0 t)\, dt$$

Fourier Series

❖ **Symmetry Conditions**

✓ **$x(t)$ is an even signal, i.e., $x(t) = x(-t)$**

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\,sin(n\omega_0 t)\,dt$$

$$b_n = \frac{2}{T}\left(\int_{-\frac{T}{2}}^{0} x(t)\,sin(n\omega_0 t)\,dt + \int_{0}^{\frac{T}{2}} x(t)\,sin(n\omega_0 t)\,dt\right)$$

$$b_n = \frac{2}{T}\left(\int_{0}^{\frac{T}{2}} x(-t)\,sin(-n\omega_0 t)\,dt + \int_{0}^{\frac{T}{2}} x(t)\,sin(n\omega_0 t)\,dt\right)$$

$$b_n = \frac{2}{T}\left(-\int_{0}^{\frac{T}{2}} x(t)\,sin(n\omega_0 t)\,dt + \int_{0}^{\frac{T}{2}} x(t)\,sin(n\omega_0 t)\,dt\right)$$

$$b_n = 0$$

Fourier Series

❖ **Symmetry Conditions**

✓ **$x(t)$ is an odd signal, i.e., $x(t) = -x(-t)$**

$$x(t) = t + t^3 + t^5$$

$$a_0 = 0 \qquad a_n = 0$$

$$b_n = \frac{4}{T}\int_0^{\frac{T}{2}} x(t)\, sin(n\omega_0 t)\, dt$$

Proof

$$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, dt \qquad a_0 = \frac{1}{T}\left(\int_{-\frac{T}{2}}^{0} x(t)\, dt + \int_0^{\frac{T}{2}} x(t)\, dt\right)$$

$$a_0 = \frac{1}{T}\left(\int_0^{\frac{T}{2}} x(-t)\, dt + \int_0^{\frac{T}{2}} x(t)\, dt\right) \qquad a_0 = \frac{1}{T}\left(-\int_0^{\frac{T}{2}} x(t)\, dt + \int_0^{\frac{T}{2}} x(t)\, dt\right)$$

Fourier Series

❖ **Symmetry Conditions**

✓ **$x(t)$ is an odd signal, i.e., $x(t) = -x(-t)$**

$$x(t) = t + t^3 + t^5$$

$$a_0 = 0 \qquad a_n = 0$$

$$b_n = \frac{4}{T}\int_0^{\frac{T}{2}} x(t)\, sin(n\omega_0 t)\, dt$$

Proof

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\cos(n\omega_0 t)\, dt \qquad = \frac{2}{T}\left(\int_{-\frac{T}{2}}^{0} x(t)\cos(n\omega_0 t)\, dt + \int_0^{\frac{T}{2}} x(t)\cos(n\omega_0 t)\, dt\right)$$

$$= \frac{2}{T}\left(\int_0^{\frac{T}{2}} x(-t)\cos(-n\omega_0 t)\, dt + \int_0^{\frac{T}{2}} x(t)\cos(n\omega_0 t)\, dt\right)$$

$$= \frac{2}{T}\left(-\int_0^{\frac{T}{2}} x(t)\cos(n\omega_0 t)\, dt + \int_0^{\frac{T}{2}} x(t)\cos(n\omega_0 t)\, dt\right)=0$$

$$a_0 = 0$$

Fourier Series

❖ **Symmetry Conditions**

✓ **$x(t)$ is an odd signal, i.e., $x(t) = -x(-t)$**

$$x(t) = t + t^3 + t^5$$

$$a_0 = 0 \qquad a_n = 0$$

$$b_n = \frac{4}{T}\int_0^{\frac{T}{2}} x(t)\,sin(n\omega_0 t)\,dt$$

Proof

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\,sin(n\omega_0 t)\,dt \qquad = \frac{2}{T}\left(\int_{-\frac{T}{2}}^{0} x(t)\,sin(n\omega_0 t)\,dt + \int_0^{\frac{T}{2}} x(t)\,sin(n\omega_0 t)\,dt\right)$$

$$= \frac{2}{T}\left(\int_0^{\frac{T}{2}} x(-t)\,sin(-n\omega_0 t)\,dt + \int_0^{\frac{T}{2}} x(t)\,sin(n\omega_0 t)\,dt\right)$$

$$b_n = \frac{2}{T}\left(-\int_0^{\frac{T}{2}} x(t)\,sin(n\omega_0 t)\,dt + \int_0^{\frac{T}{2}} x(t)\,sin(n\omega_0 t)\,dt\right)$$

$$b_n = \frac{4}{T}\int_0^{\frac{T}{2}} x(t)\,sin(n\omega_0 t)\,dt$$

Fourier Series

❖ **Symmetry Conditions**

✓ $\boldsymbol{x(t)}$ **is Half-wave symmetry, i.e.,** $x(t) = -x\left(t \pm \frac{T}{2}\right)$

$$a_n = b_n = 0 \quad For\ n\ is\ even$$

$$a_n = \frac{4}{T}\int_0^{\frac{T}{2}} x(t)\cos(n\omega_0 t)\,dt \quad For\ n\ is\ odd$$

$$b_n = \frac{4}{T}\int_0^{\frac{T}{2}} x(t)\,sin(n\omega_0 t)\,dt \quad For\ n\ is\ odd$$

Proof

$$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\,dt \qquad = \frac{1}{T}\left(\int_{-\frac{T}{2}}^{0} x(t)\,dt + \int_0^{\frac{T}{2}} x(t)\,dt\right)$$

$$= \frac{1}{T}\left(\int_0^{\frac{T}{2}} x(t - T/2)\,dt + \int_0^{\frac{T}{2}} x(t)\,dt\right)$$

$$a_0 = \frac{1}{T}\left(-\int_0^{\frac{T}{2}} x(t)\,dt + \int_0^{\frac{T}{2}} x(t)\,dt\right) \qquad a_0 = 0$$

Fourier Series

❖ **Symmetry Conditions**

✓ **$x(t)$ is Half-wave symmetry, i.e.,** $x(t) = -x\left(t \pm \frac{T}{2}\right)$

$$a_n = b_n = 0 \quad For\ n\ is\ even$$

$$a_n = \frac{4}{T}\int_0^{\frac{T}{2}} x(t)\cos(n\omega_0 t)\,dt \quad For\ n\ is\ odd$$

$$b_n = \frac{4}{T}\int_0^{\frac{T}{2}} x(t)\,sin(n\omega_0 t)\,dt \quad For\ n\ is\ odd$$

Proof

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\cos(n\omega_0 t)\,dt \quad = \frac{2}{T}\left(\int_{-\frac{T}{2}}^{0} x(t)\cos(n\omega_0 t)\,dt + \int_0^{\frac{T}{2}} x(t)\cos(n\omega_0 t)\,dt\right)$$

$$= \frac{2}{T}\left(\int_0^{\frac{T}{2}} x(t - T/2)\cos\left(n\omega_0(t - \frac{T}{2})\right)dt + \int_0^{\frac{T}{2}} x(t)\cos(n\omega_0 t)\,dt\right)$$

$$= \frac{2}{T}\left(-\int_0^{\frac{T}{2}} x(t)\cos\left(n\omega_0(t - \frac{T}{2})\right)dt + \int_0^{\frac{T}{2}} x(t)\cos(n\omega_0 t)\,dt\right)$$

$$\cos\left(n\omega_0(t - \frac{T}{2})\right) = \cos\left(n\omega_0 t - n\omega_0\frac{T}{2}\right) \quad = \cos\left(n\omega_0 t - n\frac{2\pi}{T}\frac{T}{2}\right) \quad = \cos(n\omega_0 t - n\pi)$$

Fourier Series

- **Symmetry Conditions**
  - ✓ **$x(t)$ is Half-wave symmetry, i.e.,** $x(t) = -x\left(t \pm \frac{T}{2}\right)$

$$a_n = b_n = 0 \quad \text{For n is even}$$

$$a_n = \frac{4}{T}\int_0^{\frac{T}{2}} x(t)\cos(n\omega_0 t)\,dt \quad \text{For n is odd}$$

$$b_n = \frac{4}{T}\int_0^{\frac{T}{2}} x(t)\,sin(n\omega_0 t)\,dt \quad \text{For n is odd}$$

Proof

$$a_n = \frac{2}{T}\left(-\int_0^{\frac{T}{2}} x(t)\cos(n\omega_0 t)\,dt + \int_0^{\frac{T}{2}} x(t)\cos(n\omega_0 t)\,dt\right) \quad \text{n even}$$

$$a_n = \frac{2}{T}\left(\int_0^{\frac{T}{2}} x(t)\cos(n\omega_0 t)\,dt + \int_0^{\frac{T}{2}} x(t)\cos(n\omega_0 t)\,dt\right) \quad \text{n odd}$$

$$a_n = 0, \qquad n\ even$$

$$a_n = \frac{4}{T}\int_0^{\frac{T}{2}} x(t)\cos(n\omega_0 t)\,dt, \qquad n\ odd$$

Fourier Series

- **Symmetry Conditions**
  - ✓ **$x(t)$ is Half-wave symmetry, i.e.,** $x(t) = -x\left(t \pm \frac{T}{2}\right)$

Proof

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\sin(n\omega_0 t)\,dt \quad = \frac{2}{T}\left(\int_{-\frac{T}{2}}^{0} x(t)\sin(n\omega_0 t)\,dt + \int_{0}^{\frac{T}{2}} x(t)\sin(n\omega_0 t)\,dt\right)$$

$$= \frac{2}{T}\left(\int_{0}^{\frac{T}{2}} x(t-T/2)\sin(n\omega_0(t-T/2)\,dt + \int_{0}^{\frac{T}{2}} x(t)\sin(n\omega_0 t)\,dt\right)$$

$$b_n = \frac{2}{T}\left(-\int_{0}^{\frac{T}{2}} x(t)\sin\left(n\omega_0(t-\frac{T}{2})\right)dt + \int_{0}^{\frac{T}{2}} x(t)\sin(n\omega_0 t)\,dt\right)$$

$$\sin\left(n\omega_0(t-\frac{T}{2})\right) = \sin\left(n\omega_0 t - n\omega_0\frac{T}{2})\right) = \sin\left(n\omega_0 t - n\frac{2\pi}{T}\frac{T}{2}\right) = \sin(n\omega_0 t - n\pi)$$

$$b_n = \frac{2}{T}\left(-\int_{0}^{\frac{T}{2}} x(t)\sin(n\omega_0 t)\,dt + \int_{0}^{\frac{T}{2}} x(t)\sin(n\omega_0 t)\,dt\right) \qquad b_n = 0, \qquad n\ even$$

$$b_n = \frac{2}{T}\left(\int_{0}^{\frac{T}{2}} x(t)\sin(n\omega_0 t)\,dt + \int_{0}^{\frac{T}{2}} x(t)\sin(n\omega_0 t)\,dt\right) \qquad b_n = \frac{4}{T}\int_{0}^{\frac{T}{2}} x(t)\sin(n\omega_0 t)\,dt \quad n\ odd$$

Fourier Series

- **Symmetry Conditions**
  - ✓ ***x(t)* is Quarter-wave symmetry**
  - ✓ It has either odd or even symmetry and
  - ✓ It has half-wave symmetry

$$a_0 = 0$$

$$a_n = \frac{8}{T}\int_0^{\frac{T}{2}} x(t)\cos(n\omega_0 t)\,dt$$

$$b_n = \frac{8}{T}\int_0^{\frac{T}{2}} x(t)\,sin(n\omega_0 t)\,dt$$

Name of symmetry	Condition	Property	a_0	a_n	b_n
Even	$x(t) = x(-t)$	Cosine term only	$\frac{4}{T}\int_0^{\frac{T}{2}} x(t)\,dt$	$\frac{4}{T}\int_0^{\frac{T}{2}} x(t)\cos(n\omega_0 t)\,dt$	0
Odd	$x(t) = -x(-t)$	Sine term only	0	0	$\frac{4}{T}\int_0^{\frac{T}{2}} x(t)\sin(n\omega_0 t)\,dt$
Half-wave	$x(t) = -x\left(t \pm \frac{T}{2}\right)$	Odd n only	0	$\frac{4}{T}\int_0^{\frac{T}{2}} x(t)\cos(n\omega_0 t)\,dt$	$\frac{4}{T}\int_0^{\frac{T}{2}} x(t)\sin(n\omega_0 t)\,dt$

Fourier Series

❖ **Exponential Fourier Series**

✓ The exponential Fourier series is expressed as

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} \qquad X_n = \frac{1}{T}\int_0^T x(t) e^{-jn\omega_0 t}\, dt$$

✓ **Relationship Between Trigonometric and Exponential Fourier Series**

$$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)]$$

$$a_0 + \sum_{n=1}^{\infty} \left[a_n \frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2} + b_n \frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j}\right]$$

$$a_0 + \sum_{n=1}^{\infty} \left[a_n \frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2} - jb_n \frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2}\right]$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left[\left(\frac{a_n - jb_n}{2}\right) e^{jn\omega_0 t} + \left(\frac{a_n + jb_n}{2}\right) e^{-jn\omega_0 t}\right]$$

Fourier Series

- **Exponential Fourier Series**
  - ✓ **Relationship Between Trigonometric and Exponential Fourier Series**

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left[\left(\frac{a_n - jb_n}{2}\right)e^{jn\omega_0 t} + \left(\frac{a_n + jb_n}{2}\right)e^{-jn\omega_0 t}\right]$$

$$X_n = \frac{a_n - jb_n}{2} \qquad X_n = \frac{1}{2}\left(\frac{2}{T}\int_0^T x(t)\cos(n\omega_0 t)\,dt - j\frac{2}{T}\int_0^T x(t)\sin(n\omega_0 t)\,dt\right)$$

$$X_n = \frac{1}{2} * \frac{2}{T}\int_0^T x(t)(\cos(n\omega_0 t) - j\sin(n\omega_0 t))\,dt \qquad X_n = \frac{1}{T}\int_0^T x(t)e^{-jnw_0 t}\,dt$$

$$X_0 = \frac{1}{T}\int_0^T x(t)\,dt = a_0 \qquad X_n = \frac{a_n - jb_n}{2} \qquad X_{-n} = \frac{a_{-n} - jb_{-n}}{2}$$

Fourier Series

- **Exponential Fourier Series**
  - ✓ **Relationship Between Trigonometric and Exponential Fourier Series**

$$a_n = \frac{2}{T}\int_0^T x(t)\cos(n\omega_0 t)\,dt$$

$$a_{-n} = \frac{2}{T}\int_0^T x(t)\cos(-n\omega_0 t)\,dt$$

$$a_{-n} = \frac{2}{T}\int_0^T x(t)\cos(n\omega_0 t)\,dt = a_n$$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin(n\omega_0 t)\,dt$$

$$b_{-n} = \frac{2}{T}\int_0^T x(t)\sin(-n\omega_0 t)\,dt$$

$$b_{-n} = -\frac{2}{T}\int_0^T x(t)\sin(n\omega_0 t)\,dt = -b_n$$

$$X_{-n} = \frac{a_n + jb_n}{2}$$

$$x(t) = X_0 + \sum_{n=1}^{\infty}\left[X_n e^{jn\omega_0 t} + X_{-n}\, e^{-jn\omega_0 t}\right]$$

$$x(t) = \sum_{n=-\infty}^{\infty}\left[X_n e^{jn\omega_0 t}\right]$$

$$a_0 = X_0$$

$$a_n = 2\,\mathrm{Re}\{X_n\} \qquad a_n = X_n + X_{-n}$$

$$b_n = -2\,\mathrm{Im}\{X_n\} \qquad b_n = j(X_n - X_{-n})$$

Fourier Series

- **Exponential Fourier Series**
  - **Relationship Between Trigonometric and Exponential Fourier Series**
  - *The magnitude spectrum is* $|X_n| = |X_{-n}| = \frac{\sqrt{a_n^2 + b_n^2}}{2} = \frac{c_n}{2}$
  - *The magnitude spectrum is an even function* $c_n = \begin{cases} 2|X_n|, n \geq 1 \\ X_0, n = 0 \end{cases}$
  - *The phase spectrum is*

$$\angle X_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right) = \theta_n$$

$$\angle X_{-n} = \tan^{-1}\left(\frac{-b_{-n}}{a_{-n}}\right)$$

$$\tan^{-1}\left(\frac{b_n}{a_n}\right) = -\tan^{-1}\left(-\frac{b_n}{a_n}\right) = -\theta_n = -\angle x_n$$

The phase spectrum is an odd function.

Fourier Series

- **DIRICHLET CONDITIONS**
  - Over any period, $x(t)$ must be *absolutely* integrable, i.e.,

$$a_0 = X_0 \qquad a_n = 2\,\mathrm{Re}\{X_n\}\frac{2}{T_0} \qquad b_n = 0$$

$$\int_0^T |x(t)|\,dt < \infty$$

  - This guarantees that each coefficient X_n will be finite since

$$|X_n| = \frac{1}{T}\left|\int_0^T x(t)e^{-jn\omega_0 t}\,dt\right| \le \frac{1}{T}\int_0^T \left|x(t)e^{-jn\omega_0 t}\right| dt = \frac{1}{T}\int_0^T |x(t)|\,dt$$

$$\int_0^T |x(t)|\,dt < \infty \qquad |X_n| < \infty$$

  - In any finite interval of time, $x(t)$ is of bounded variation, i.e., there are no more than a finite number of maxima and minima during any single period of the signal.
  - In any finite interval of time, there are only a finite number of discontinuities. Further, each of these discontinuities is finite.

$$x(t_0) = \frac{1}{2}\left[x(t_0^+) + x(t_0^-)\right]$$

Discrete-Time Fourier Series (DTFS)

Discrete-Time Fourier Series (DTFS)

- A discrete-time signal $x(n)$ is periodic with period N if

$$x(n) = x(n+N)$$

- The set of all discrete-time complex exponential signals that are periodic with period N is given by

$$\phi_k(n) = e^{jk\omega_0 n} = e^{jk(2\pi/N)n}$$

- The discrete-time periodic signal $x(n)$ can be represented by a summation of complex exponential $\phi_k(n)$ of the form

$$x(n) = \sum_k X_k \phi_k(n) = \sum_k X_k e^{jk\omega_0 n} = \sum_k X_k e^{jk(2\pi/N)n}$$

- The discrete-time exponentials whose frequencies are separated by 2π (or integer multiples of 2π) are identical.

$$\phi_0(n) = e^{j0(2\pi/N)n} = \phi_N(n) = e^{jN(2\pi/N)n}$$

$$\phi_1(n) = e^{j(2\pi/N)n} = \phi_{N+1}(n) = e^{j(N+1)(2\pi/N)n}$$

Discrete-Time Fourier Series (DTFS)

- In general

$$\phi_k(n) = \phi_{k+N}(n)$$

- The Fourier series of a periodic signal $x(n)$ consists of only N harmonics and can be expressed as

$$x(n) = \sum_{k=k_0}^{k_0+N-1} X_k e^{jk\omega_0 n}$$

- Where $k_0$is arbitrary. Since k_0 is arbitrary, we can use the notation

$$x(n) = \sum_{k=N} X_k e^{jk\omega_0 n}$$

- **Evaluation of DTFS Coefficients**
- To determine the Fourier series coefficients X_k, we replace the summation variable k by m on the right side and multiply both sides by $e^{-j\omega_0 kn}$

$$x(n)e^{-j\omega_0 kn} = \sum_{m=N} X_m e^{j\omega_0 (m-k)n}$$

Discrete-Time Fourier Series (DTFS)

- Then, we sum over the values of n in $[0, N-1]$ to get

$$\sum_{n=0}^{N-1} x(n)e^{-j\omega_0 kn} = \sum_{n=0}^{N-1} \sum_{m=(N)} X_m e^{j\omega_0 (m-k)m}$$

- By interchanging the order of summation, we can write

$$\sum_{n=0}^{N-1} x(n)e^{-j\omega_0 kn} = \sum_{m=(N)} X_m \sum_{n=0}^{N-1} e^{j\omega_0 (m-k)m}$$

- We know that

$$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}$$

- For $\alpha = 1$, we have

$$\sum_{n=0}^{N-1} \alpha^n = N$$

Discrete-Time Fourier Series (DTFS)

- If $m-k$ is not an integer of N (i.e., $(m-k)$ ≠rN for r=0, ±1, ±2, ...), we can let α = $X_m e^{j\omega_0(m-k)n}$

$$\sum_{n=0}^{N-1} e^{j\omega_0(m-k)n} = \frac{1-e^{j\omega_0(m-k)N}}{1-e^{jw_0(m-k)}} = \frac{1-e^{j2\pi/N(m-k)N}}{1-e^{j2\pi/N(m-k)}} = 0$$

- If $m-k$ is an integer multiple of N,

$$\sum_{n=0}^{N-1} e^{j\omega_0(m-k)n} = N$$

- Combining above equations, we write

$$\sum_{n=0}^{N-1} e^{j\omega_0(m-k)n} = N\delta(m-k-rN)$$

- Where $\delta(m-k-rN)$ is the unit sample occurring at $m=k+rN$. We yields

$$\sum_{n=0}^{N-1} x(n)e^{-j\omega_0 kn} = \sum_{m=(N)} X_m N\delta(m-k-rN)$$

Discrete-Time Fourier Series (DTFS)

- The nonzero value in the sum corresponds to $m = k$, and the right hand side equation evaluates to NX_k

$$X_k = \frac{1}{N}\sum_{n=0}^{N-1} x\,(n)e^{-j\omega_0 kn}$$

- Because each of the terms in the summation is periodic with N, the summation can be taken over any N successive values of n.

$$x(n) = \sum_{k=N} X_k e^{jk\omega_0 n}$$

and

$$X_k = \frac{1}{N}\sum_{n=N} x\,(n)e^{-jk\omega_0 n}$$

- **Magnitude and Phase Spectrum of Discrete-Time Periodic Signals (Fourier Spectra)**
- In general, the Fourier coefficients X_k, are complex, and they can be represented in the polar form as

$$X_k = |X_k|e^{j\angle X_k}$$

Discrete-Time Fourier Series (DTFS)

- The spectrum of the discrete-time periodic signal, in contrast, is band limited and has at most N components
- The DTFS coefficients X_k are periodic with period N, i.e.,

$$X_{k+N} = X_k$$

- ***Proof***

$$X_k = \frac{1}{N} \sum_{n=N} x(n) e^{-jk\omega_0 n}$$

$$X_{k+N} = \frac{1}{N} \sum_{n=(N)} x(n) e - j(k+N)\omega_0 n$$

$$= \frac{1}{N} \sum_{n=(N)} x(n)\ e^{-jk\omega_0 n}\ e^{-j(2\pi/N)Nn}$$

$$= \frac{1}{N} \sum_{n=(N)} x(n)\ e^{-jk\omega_0 n}\ e^{-j(2\pi)n}$$

$$= \frac{1}{N} \sum_{n=(N)} x(n)\ e^{-jk\omega_0 n}$$

$$X_{k+N} = X_k$$

Discrete-Time Fourier Series (DTFS)

❖ **Properties of DTFS**

✓ Similarities between the properties of discrete-time and continuous-time Fourier series.

✓ To indicate the relationship between a periodic signal and its Fourier series coefficients.

❖ ***Linearity***

✓ If x(n) and y(n) denote two periodic signals with period N, and

$$x(n) \longleftrightarrow X_k \qquad\qquad y(n) \longleftrightarrow Y_k$$

Then

$$z(n) = ax(n) + by(n) \leftrightarrow Z_k = aX_k + bY_k$$

❖ ***Proof*** The Fourier series coefficients of z(n) is given by

$$Z_k = \frac{1}{N}\sum_{n=(N)} z\,(n)e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=(N)} [ax(n) + by(n)]e^{-jk\omega_0 n}$$

$$a\frac{1}{N}\sum_{n=(N)} x\,(n)e^{-jk\omega_0 n} + b\frac{1}{N}\sum_{n=(N)} y(n)e^{-jk\omega_0 n}$$

$$\boldsymbol{Z_k = aX_k + bY_k}$$

Discrete-Time Fourier Series (DTFS)

- ***Time Shifting***
- ✓ When a time shift is applied to a periodic signal x(n), the period N of the signal is preserved. If

$$x(n) \leftrightarrow X_k$$

Then $$y(n) = x(n - n_0) \leftrightarrow Y_k = Xke^{-jk\omega_0 n_0}$$

- ✓ When a signal is shifted in time, the magnitudes of its Fourier series coefficients remain unaltered. That is, $|Y_k| = |X_k|$.

- ***Proof*** By definition,

$$Y_k = \frac{1}{N} \sum_{n=(N)} y(n)e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=0}^{N-1} y(n)e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=0}^{N-1} x(n - n_0)e^{-jk\omega_0 n}$$

A change of variables is performed by letting $m = (n - n_0)$, which also yields ($m \rightarrow - - n_0$) as ($n \rightarrow 0$), and ($m \rightarrow (N - 1 - n_0)$) as ($n \rightarrow (N - 1)$). Therefore,

$$Y_k = \frac{1}{N} \sum_{m=-n_0}^{N-1-n_0} x(m)e^{-jk\omega_0 (m+n_0)}$$

Discrete-Time Fourier Series (DTFS)

❖ ***Time Shifting***

Proof

$$Y_k = \frac{1}{N}\sum_{m=-n_0}^{N-1-n_0} x\,(m)e^{-jk\omega_0(m)}e^{-jk\omega_0(n_0)}$$

$$Y_k = X_k e^{-jk\omega_0 n_0} \qquad |Y_k| = |X_k|$$

❖ ***Frequency Shifting***

✓ If

$$x(n) \leftrightarrow X_k$$

✓ Then

$$y(n) = e^{jM\omega_0 n}x(n) \leftrightarrow Y_k = X_{k-M}$$

Proof By definition,

$$y_k = \frac{1}{N}\sum_{n=N} y(n)e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=0}^{N-1} y(n)e^{-jk\omega_0 n}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} e^{jM\omega_0 n}x(n)e^{-jk\omega_0 n} \qquad = \frac{1}{N}\sum_{n=0}^{N-1} x\,(n)e^{-j(k-M)\omega_0 n} = X_{k-M}$$

✓ A frequency shift corresponds to multiplication in time domain by a complex sinusoid whose frequency is equal to the time shift.

Discrete-Time Fourier Series (DTFS)

❖ ***Time Reversal***

✓ If $$x(n) \leftrightarrow X_k$$

✓ Then $$y(n) = x(-n) \leftrightarrow Y_k = X_{\_\,k}$$

Proof

$$Y_k = \frac{1}{N}\sum_{n=N} y(n)e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=0}^{N\cdot 1} y(n)e^{-jk\omega_0 n}$$

$$= \frac{1}{N}\sum_{n=0}^{N\cdot 1} x(-n)e^{-jk\omega_0 n} \qquad = \frac{1}{N}\sum_{m=-(N-1)}^{N\cdot 1} x(m)e^{-j(-k)\omega_0 m} = X_{-k}$$

✓ An interesting consequence of the time-reversal property is that if x(n) is even then its Fourier series coefficients are also even, i.e., if $x(-n) = x(n)$ then
$$X_{-k} = X_k$$

✓ Similarly, if x(n) is odd, then its Fourier series coefficients, i.e., if $x(-n) = -x(n)$ then
$$X_{-k} = -X_k$$

✓ The time reversal applied to a discrete-time signal results in a time reversal of the corresponding sequence of Fourier series coefficients.

Discrete-Time Fourier Series (DTFS)

❖ ***Time Scaling***

✓ If $$x(n) \longleftrightarrow X_k$$

✓ Then $$y(n) = x_{\left(m\right)}(n) \leftrightarrow Y_k = \frac{1}{m}\mathrm{X_k}$$

✓ The Fourier series coefficients $Y_k = \frac{1}{m}\mathrm{X_k}$ are also periodic with period mN.

Proof The Fourier series coefficients of $y(n) = x_{\left(m\right)}(n)$ are given by

$$Y_k = \frac{1}{m}\frac{1}{N}\sum_{n=0}^{m(N-1)} y(n)e^{-jk\left(\frac{\omega}{m}\right)n}$$

$$Y_k = \frac{1}{m}\frac{1}{N}\sum_{n=0}^{m(N-1)} x_{\left(m\right)}(n)e^{-jk\left(\frac{\omega}{m}\right)n}$$

$$Y_k = \frac{1}{m}\frac{1}{N}\sum_{n=0}^{m(N-1)} x(n/m)e^{-jk\left(\frac{\omega}{m}\right)n}$$

✓ A change of variables is performed by letting $r = \frac{n}{m}$, which also yields $r = 0$ as $n = 0$ and $r = N - 1$ as $n = m(N - 1)$. Therefore,

$$Y_k = \frac{1}{m}\frac{1}{N}\sum_{r=0}^{(N-1)} x(r)e^{-jk\omega r} = \frac{1}{m}X_k$$

Discrete-Time Fourier Series (DTFS)

❖ ***Periodic Convolution***

- ✓ If $\quad x(n) \longleftrightarrow X_k \qquad y(n) \longleftrightarrow Y_k$
- ✓ Then $\quad z(n) = x(n) \circledast y(n) = \sum_{r=<n>} x(r)y(n-r) \leftrightarrow Zk = NX_kY_k$

Proof

For periodic signals with the same period, a special form of convolution, known as periodic convolution, is defined as

$$Z_k = \frac{1}{N}\sum_{n=<N>} z(n)e^{-jk\omega n} = \frac{1}{N}\sum_{n=<N>}(\sum_{r=<N>} x(r)y(n-r))\ e^{-jk\omega n}$$

$$Z_k = \sum_{r=<N>} x(r)(\frac{1}{N}\sum_{n=<N>} y(n-r)\ \ e^{-jk\omega n})$$

✓ From the time shifting property, i.e., if $\quad y(n) \leftrightarrow Yk$

then $\quad y(n-r) \leftrightarrow Yke^{-jr\omega n}$

We have $\quad Z_k = N\,\frac{1}{N}\sum_{r=<N>} x(r)e^{-jr\omega n}Y_k = NX_kY_k$

✓ **The convolution in time transform to multiplication of the frequency domain representations.**

Discrete-Time Fourier Series (DTFS)

❖ ***Multiplication***

✓ If $x(n)$ and $y(n)$ denote two periodic signals with period N, and

$$x(n) \leftrightarrow X_k \qquad y(n) \leftrightarrow Y_k$$

✓ Then $z(n) = x(n)y(n) \ \leftrightarrow\leftrightarrow Z_k = \sum_{r=<N>} X_r Y_{k\ \_\ r}$

Proof

Consider the signal $z(n)$,

$$Z(n) = \mathrm{x(n)y}(n) = \sum_{r=<N>} \mathrm{X_r} e^{jr\omega n} \sum_{m=<N>} Y_m \ e^{jm\omega n}$$

$$z(n) = \sum_{r=<N>} \mathrm{X_r} \sum_{m=<N>} Y_m \ e^{j(m+r)\omega n}$$

✓ A change of variables is performed by letting $k = m + r$, which also yields $m = (k - r), (k \rightarrow \mathrm{r})$ as $(\mathrm{m} \rightarrow 0)$, and $[k \rightarrow (r + N - 1)]$ as $[m \rightarrow (N - 1)]$.

Therefore,

$$z(n) = \sum_{k\ <N>} \left(\sum_{r=<N>} \mathrm{X_r\ Y}_{k\ \_\ r}\right) e^{jk\omega n} = \sum_{k<N>} Z_k \ e^{jk\omega n}$$

Thus, $$Z_k = \sum_{r=<N>} X_r Y_{k\ \_\ r}$$

Discrete-Time Fourier Series (DTFS)

❖ ***First Difference***

✓ If $x(n)$ and $y(n)$ denote two periodic signals with period N, and

$$x(n) \longleftrightarrow X_k \qquad\qquad y(n) \longleftrightarrow Y_k$$

✓ Then $\quad y(n) = x(n) - x(n-1) \leftrightarrow Yk = \left(1 - e^{-jk\omega}\right)X_k$

Proof

✓ Given $\quad x(n) \longleftrightarrow X_k$

✓ Using the time-shifting property, we get

$$x(n-1) \leftrightarrow Xke^{-jk\omega}$$

✓ Now, using the linearity property, we get

$$x(n) - x(n-1) \leftrightarrow Xk - Xke^{-jk\omega}$$

$$x(n) - x(n-1) \leftrightarrow Xk(1 - e^{-jk\omega})$$

❖ ***Running Sum or Accumulation***

✓ If $x(n)$ and $y(n)$ denote two periodic signals with period N, and

$$x(n) \longleftrightarrow X_k \qquad\qquad y(n) \longleftrightarrow Y_k$$

✓ Then $\quad y(n) = \sum_{-\infty}^{n} x(k) \leftrightarrow Y_k = \left(\frac{1}{1-e^{-jk\omega}}\right) X_k \;, k \neq 0$

Discrete-Time Fourier Series (DTFS)

❖ ***Running Sum or Accumulation***

Proof

✓ Consider the running sum

$$y(n) = \sum_{k=-\infty}^{n} x(k)$$

$$y(n) = x(n) + \sum_{k=-\infty}^{n-1} x(k)$$

$$y\,(n) = x(n) + y(n-1)$$

$$y\,(n) - y(n-1) = x(n)$$

$$Y_k - Y_k e^{-jk\omega} = X_k$$

$$Y_k = \left(\frac{1}{1-e^{-jk\omega}}\right) X_k$$

✓ The discrete-time Fourier series coefficient Y_k of the running sum $y(n) = \sum_{k=-\infty}^{n} x(k)$ is finite-valued and periodic only if X_0=o.

Discrete-Time Fourier Series (DTFS)

❖ ***Conjugation and Conjugate Symmetry***

✓ If x(n) and y(n) denote two periodic signals with period N, and

$$x(n) \longleftrightarrow X_k \qquad\qquad y(n) \longleftrightarrow Y_k$$

✓ Then $$y(n) = x^*(n) \leftrightarrow Y_k = X^*{}_{-k}$$

Proof

$$Y_{k=} \frac{1}{N} \sum_{n=<N>} y(n)e^{-jkw_0 n} = \frac{1}{N} \sum_{n=<N>} x^n(n)e^{-jkw_0 n}$$

$$= \left(\frac{1}{N} \sum_{n=<N>} x(n)e^{jkw_0 n} \right)^* = \left(\frac{1}{N} \sum_{n=<N>} x(n)e^{-j(-k)w_0 n} \right)^*$$

$$= (X_{-k})^* = X_{-k}{}^*$$

✓ **Case I** If $x(n)$ is real, i.e., if

$$x^*(n) = x(n)$$

✓ Then

$$X^*{}_{-k} = X_k$$

✓ Therefore, $$X_{-k} = X^*{}_k$$

✓ That is, if $x(t)$ is real and even, then so are its Fourier series coefficients.

Discrete-Time Fourier Series (DTFS)

❖ ***Conjugation and Conjugate Symmetry***

✓ **Case II** If $x(n)$ is real and odd, then its Fourier series coefficients are purely imaginary and odd.

$$X_{-k} = X^*{}_k = -X_k$$

✓ **Case III** Even and odd decomposition of real signals: If x(n) is real and

$$x(n) \leftrightarrow X_k$$

Then $$x_e(n) = \varepsilon\{x(n)\} \leftrightarrow Re\{X_k\}$$

1.

✓ The even part of a signal $x(n)$ is defined as

$$x_e(n) = \frac{1}{2}[x(n) + x(-n)]$$

Proof

✓ Using the linearity property, we get

$$x_e(n) \leftrightarrow \frac{1}{2} 2\mathrm{Re}\{X_k\}$$

$$x_e(n) \leftrightarrow \mathrm{Re}\{X_k\}$$

$$x_e(n) \leftrightarrow \frac{1}{2}[X_k + X_{-k}]$$

And finally, we get $$x_e(t) \leftrightarrow \frac{1}{2}[X_k + X^*{}_k]$$

Discrete-Time Fourier Series (DTFS)

❖ ***Conjugation and Conjugate Symmetry***

✓ **Case III** Even and odd decomposition of real signals: If x(n) is real and

$$x(n) \leftrightarrow X_k$$

Then $$x_o(n) = O\{x(n)\} \leftrightarrow jIm\{X_k\}$$

2.

✓ The odd part of a signal $x(n)$ is defined as

$$x_o(n) \leftrightarrow \frac{1}{2}[x(n) - x(-n)]$$

Proof

✓ Using the linearity property, we get

$$x_o(n) \leftrightarrow \frac{1}{2}[X_k - X_{-k}]$$

$$x_o(n) \leftrightarrow \frac{1}{2}[X_k - X^*{}_k]$$

And finally, we get $$x_o(n) \leftrightarrow \frac{1}{2}2\mathrm{jIm}\{X_k\}$$

Discrete-Time Fourier Series (DTFS)

❖ ***Parseval's Relation***

✓ If $x(n)$ is periodic signal with the same period N, and

$x(n) \leftrightarrow X_k$ Then $\frac{1}{N}\sum_{n=\langle N\rangle}|x(n)|^2 = \sum_{n=\langle N\rangle}|X_k|^2$

Proof

✓ Consider the LHS of the equation, we have

$$\frac{1}{N}\sum_{n=\langle N\rangle}|x(n)|^2 = \frac{1}{N}\sum_{n=\langle N\rangle} x(n)x^*(n)$$

$$= \frac{1}{N}\sum_{n=\langle N\rangle} x(n)\left(\sum_{n=\langle N\rangle} X_k e^{jkw_o n}\right)^*$$

$$= \frac{1}{N}\sum_{n=\langle N\rangle} x(n)\left(\sum_{n=\langle N\rangle} X^*{}_k e^{-jkw_o n}\right)$$

$$= \sum_{n=\langle N\rangle} X^*{}_k\left(\frac{1}{N}\sum_{n=\langle N\rangle} x(n)e^{-jkw_o n}\right)$$

$$= \sum_{n=\langle N\rangle} X^*{}_k X_k$$

$$= \sum_{n=\langle N\rangle} |X_k|^2$$

Discrete-Time Fourier Series (DTFS)

❖ ***Systems with Periodic Inputs***

✓ The response of an LTI system to a sinusoidal input leads to a characterization of system behavior that is termed the *frequency response of the system*.

✓ The impulse response of a system be h(n) and the input be x(n)=$e^{iw_0 n}$, then the convolution integral gives the output as

$$y(n) = h(n) * x(n)$$

$$= \sum_{m=-\infty}^{\infty} h(m)x(n-m) = \sum_{m=-\infty}^{\infty} h(m)e^{jw_o(n-m)}$$

$$= \sum_{m=-\infty}^{\infty} h(m)e^{-jw_o m}\, e^{jw_o n}$$

$$y(n) = H(e^{jw_o})e^{jw_o n}$$

$$where \rightarrow H(e^{jw_o}) = \sum_{m=-\infty}^{\infty} h(m)e^{-jw_o m}$$

✓ In polar form $H\left(e^{jw_o}\right) = \left|H(e^{jw_o})\right| e^{\angle H(e^{jw_o})}$

✓ Phase response $y(n) = \left|H\left(e^{jw_o}\right)\right| e^{j\left(w_o t + \angle H\left(e^{jw_o}\right)\right)}$

Discrete-Time Fourier Series (DTFS)

❖ ***Systems with Periodic Inputs***

✓ In polar form $H(e^{jw_o}) = |H(e^{jw_o})|e^{\angle H(e^{jw_o})}$

✓ Phase response $y(n) = |H(e^{jw_o})|e^{j\left(w_o t+\angle H(e^{jw_o})\right)}$

✓ By representing arbitrary signals as weighted superposition's of Eigen functions, we transform the operation of convolution to multiplication.

$$x(n) = \sum_{k=\langle N\rangle} X_k e^{jkw_o n}$$

✓ Then, the output of the system is given by

$$y(n) = \sum_{k=\langle N\rangle} X_k H(e^{jkw_o})e^{jkw_o n} = \sum_{k=\langle N\rangle} Y_k e^{jkw_o n}$$

Where, $w_o = \frac{2\pi}{N}$ and $Y_k = X_K H(e^{jkw_o})$