Indian Institute of Information Technology Allahabad
Probability and Statistics (PAS)
C1 Review Test Tentative Marking Scheme

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1. An exciting computer game is released. Sixty percent of players complete all the levels. Thirty percent of them will then buy an advanced version of the game. Among 15 users, what is the expected number of people who will buy the advanced version? What is the probability that at least two people will buy it? [5]

   Solution: Let X be the number of people (successes), among the mentioned 15 users (trials), who will buy the advanced version of the game. Then $X \sim \text{Bin}(n, p)$, where the probability of success is

   $$\begin{aligned} p &= P(\{\text{buy advanced}\}) \\ &= P(\{\text{buy advanced} \mid \text{complete all levels}\})P(\{\text{complete all levels}\}) \\ &= (0.30)(0.60) = 0.18 \end{aligned} \qquad [2]$$

   This implies that

   $$E(X) = np = (15)(0.18) = 2.7 \qquad [1]$$

   and

   $$P(\{X \geq 2\}) = 1 - P(\{X = 0\}) - P(\{X = 1\}) = 1 - (1 - 0.18)^{15} - (15)(0.18)^1(1 - 0.18)^{14}.$$
   [2]

2. Let X be a discrete random variable with $P(\{-a < X < a\}) = 1$, $0 < a < \infty$. Show that $E(X)$ exists. [5]

   Solution: Let $b \in \mathbb{R} \setminus (-a, a)$. Then $P(\{X = b\}) = 0$ because if $P(\{X = b\}) \neq 0$, then $P(\{X \in (-a, a) \cup \{b\}\}) > 1$, which is not possible. [1]

   Let E_X and f_X be the support and the p.m.f. of X respectively. Then $E_X \subset (-a, a)$. If E_X is finite, then $E(X)$ exists. [1]

   Suppose E_X is countable, say $E_X = \{a_1, a_2, \ldots, a_n, \ldots\}$. Then $|a_n| f_X(a_n) < a f_X(a_n)$, $\forall\, n \in \mathbb{N}$. [1]

   Since the series $\sum\limits_{i=1}^{\infty} a f_X(a_i) = a$, by the comparison test, the series $\sum\limits_{i=1}^{\infty} |a_i| f_X(a_i)$ is convergent. Therefore, $E(X)$ exists. [2]

3. Let X be a random variable with probability density function

   $$f_X(x) = \begin{cases} \frac{1}{5}, & \text{if } -2 < x < 3, \\ 0, & \text{otherwise.} \end{cases}$$

   Find the cumulative distribution function of $Y = |X|$ and hence find the probability density function of Y. [7]

Solution: The cumulative distribution function of $Y = |X|$ is

$$F_Y(y) = P(\{|X| \leq y\}) = \begin{cases} 0, \text{ if } y < 0, \\ P(\{-y \leq Y \leq y\}), \text{ if } 0 \leq y < 2, \\ P(\{-2 \leq Y \leq y\}), \text{ if } 2 \leq y \leq 3, \\ 1, \text{ if } y > 3, \end{cases} \qquad [1+1]$$

$$= \begin{cases} 0, \text{ if } y < 0, \\ \frac{2y}{5}, \text{ if } 0 \leq y < 2, \\ \frac{y+2}{5}, \text{ if } 2 \leq y \leq 3, \\ 1, \text{ if } y > 3. \end{cases} \qquad [3]$$

Since F_Y is differentiable everywhere except at the points $\{0, 1, 2, 3\}$, p.d.f. of Y is

$$f_Y(y) = \begin{cases} \frac{2}{5}, \text{ if } 0 < y < 2, \\ \frac{1}{5}, \text{ if } 2 < y < 3, \\ 0, \text{ otherwise.} \end{cases} \qquad [2]$$

4. Let X be a random variable with probability density function

$$f_X(x) = \begin{cases} \frac{e^{-x}x^n}{n!}, \text{ if } x > 0, \\ 0, \text{ otherwise,} \end{cases}$$

where $n \geq 0$ is an integer. Show that $P(\{0 < X < 2(n+1)\}) > \frac{n}{n+1}$. [8]

Solution: We know that $\int\limits_0^\infty x^n e^{-x} dx = n!$. Also, [1]

$$E(X) = \int\limits_0^\infty \frac{x^{n+1}e^{-x}}{n!} dx = \frac{(n+1)!}{n!} = n+1; \qquad [1]$$

$$E(X^2) = \int\limits_0^\infty \frac{x^{n+2}e^{-x}}{n!} dx = \frac{(n+2)!}{n!} = (n+2)(n+1); \qquad [1]$$

$$Var(X) = E(X^2) - (E(X))^2 = n+1.$$

Now,

$$P(\{0 < X < 2(n+1)\}) = P(\{|X-(n+1)| < (n+1)\}) = 1 - P(\{|X-(n+1)| \geq (n+1)\}). \qquad [1]$$

By Chebyshev Inequality, $P(\{|X-(n+1)| \geq (n+1)\}) \leq \frac{1}{n+1}$. [1]
Hence, $P(\{0 < X < 2(n+1)\}) \geq 1 - \frac{1}{n+1} = \frac{n}{n+1}$.

Suppose $P(\{0 < X < 2(n+1)\}) = \frac{n}{n+1}$. This implies that $\int\limits_0^{2(n+1)} \frac{e^{-x}x^n}{n!} dx = \frac{n}{n+1}$ or $-e^{-x} \sum\limits_{k=0}^{n} \frac{x^k}{k!} \Big|_0^{2(n+1)} = \frac{n}{n+1}$. This is a contradiction since the left hand side is an irrational number. [3]

Thus, we have $P(\{0 < X < 2(n+1)\}) > \frac{n}{n+1}$ for every integer $n \geq 0$.