Indian Institute of Information Technology Allahabad
Univariate and Multivariate Calculus (UMC)
Quiz 1 Tentative Marking Scheme

Program: B.Tech. 2^{nd} Semester (IT+ECE)
Duration: **45 Minutes** Full Marks: 15
Date: January 20, 2020 Time: 17:15 - 18:00 IST

Attempt all the questions. Numbers indicated on the right in [ ] are full marks of that particular problem. There is no credit for a solution if the appropriate work is not shown, even if the answer is correct. All the notations are standard and same as used in the lectures. Do not write on question paper and cover pages of the answer booklet except your details. This question paper has one page.

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1. Let S be a nonempty subset of $\mathbb{R}$ and $c \in \mathbb{R}$. Define the additive translate $c + S$ of S as follows:

$$c + S = \{c + x : x \in S\}.$$

If S is bounded, show that

$$\sup(c + S) = c + \sup S. \quad [5]$$

Solution. Let $\alpha = \sup S$, then $s \leq \alpha$, for all $s \in S$.
$\Rightarrow c + s \leq c + \alpha$, for all $s \in S$.
$\Rightarrow c + \alpha = c + \sup S$ is an upper bound of $c + S$. [1]
$\Rightarrow \sup(c + S) \leq c + \sup S$. [1]
Let β be any upper bound of $c + S$, i.e., $c + s \leq \beta$ for all $s \in S$.
$\Rightarrow s \leq \beta - c$ for all $s \in S$, i.e., $\beta - c$ is an upper bound of S. [1]
$\Rightarrow \alpha \leq \beta - c$ ($\because \alpha = \sup S$). [1]
$\Rightarrow c + \alpha = c + \sup S \leq \beta$.
Therefore, $\sup(c + S) = c + \sup S$. [1]

2. Let $x_n \to x$. Suppose that (x_n) has infinitely many positive and negative terms. Show that $x = 0$. [5]

Solution. Let $x > 0$. For $\epsilon = \frac{x}{2}$, there exists $n_0 \in \mathbb{N}$ such that $|x_n - x| < \frac{x}{2}$, for all $n \geq n_0$. [2]

$\Rightarrow x_n > \frac{x}{2} > 0$ for all $n \geq n_0$. [2]

Thus, there are only finitely many negative terms, which leads to a contradiction. [1]

The case $x < 0$ can be done in a similar way.

Alternative Method. Let (x_{n_k}) and (x_{m_k}) be the subsequences of (x_n) formed by the infinite negative and positive terms, respectively. [1]

Then, $x_{n_k} \to x$ and $x_{m_k} \to x$ [1+1]

As $x_{n_k} < 0$ we have $x \leq 0$. Similarly, as $x_{m_k} > 0$, we have $x \geq 0$. [1+1]

Therefore, $x = 0$.

3. Let $\alpha, \beta \in \mathbb{R}$ such that $0 < \alpha, \beta < 1$. Discuss the convergence/divergence of the sequence

$$x_n = \big((1-\alpha)^n + (2-\beta)^n\big)^{\frac{1}{n}}. \qquad [5]$$

Solution. Note that $0 < 1 - \alpha < 1$ and $1 < 2 - \beta < 2$

$\Rightarrow (1-\alpha) < (2-\beta) \Rightarrow (1-\alpha)^n \leq (2-\beta)^n$. [1]

$$\begin{aligned}
\Rightarrow 2 - \beta &= \big((2-\beta)^n\big)^{1/n} \\
&\leq x_n && [1] \\
&\leq \big((2-\beta)^n + (2-\beta)^n\big)^{1/n} && [1] \\
&= 2^{1/n}\big((2-\beta)^n\big)^{1/n} \\
&= 2^{1/n}(2-\beta).
\end{aligned}$$

Since, $2^{1/n} \to 1$, by sandwich theorem, $x_n \to 2 - \beta$. [1+1]