Digital Image Processing

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Outline

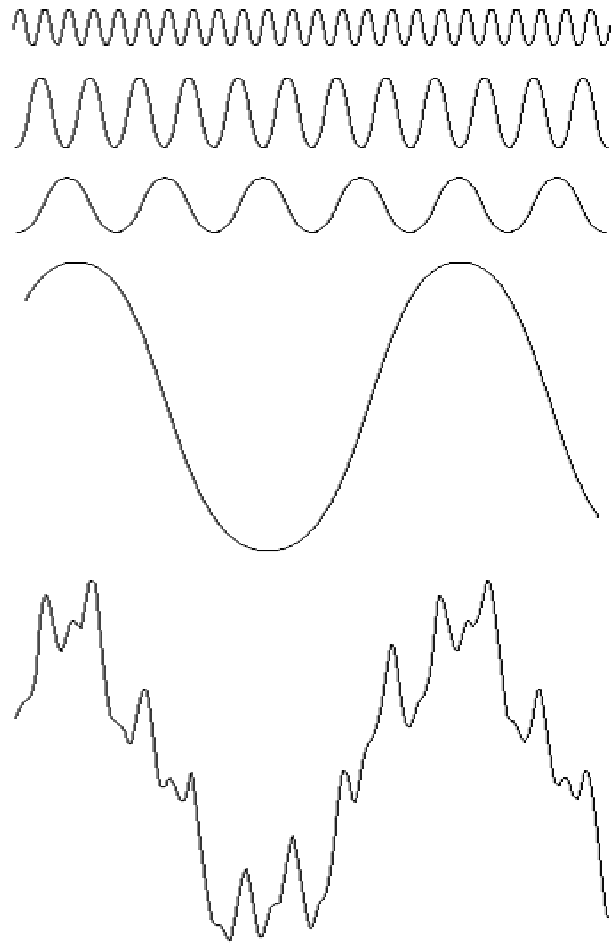
- Fourier transformation
- Discrete Fourier Transformation
- Characteristics of Fourier Transformation
- Convolution Issues

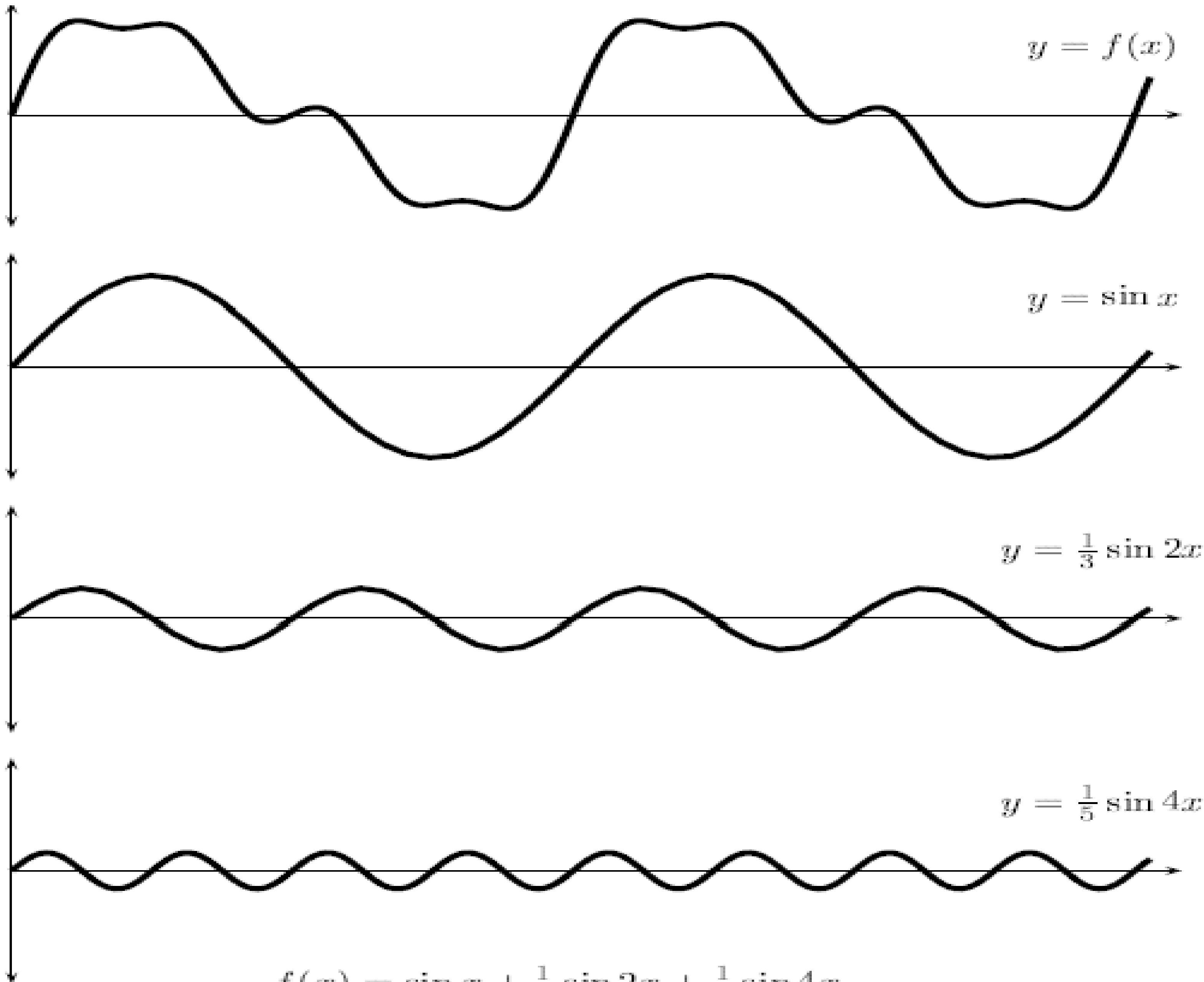
Joseph Fourier (1768-1830)

Joseph Fourier, 21 March 1768–16 May 1830. (*By permission of the Bibliothèque Municipale de Grenoble.*)

Fourier Transform: a review

- Basic ideas:
  - A periodic function can be represented by the **sum** of sines/cosines functions of different frequencies, multiplied by a different coefficient.
  - Non-periodic functions can also be represented as the **integral** of sines/cosines multiplied by weighing function.

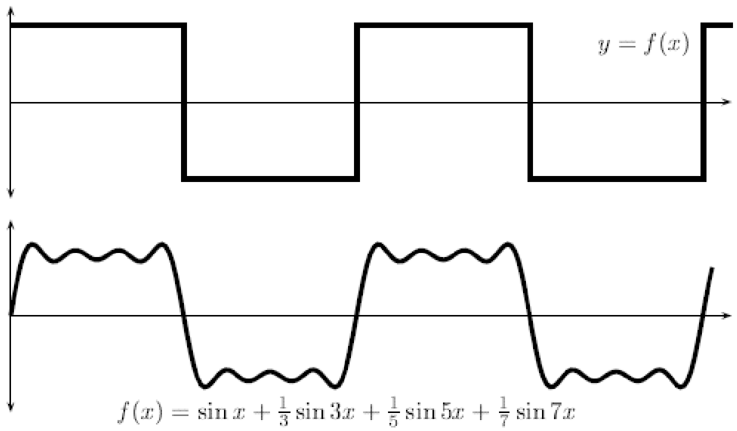




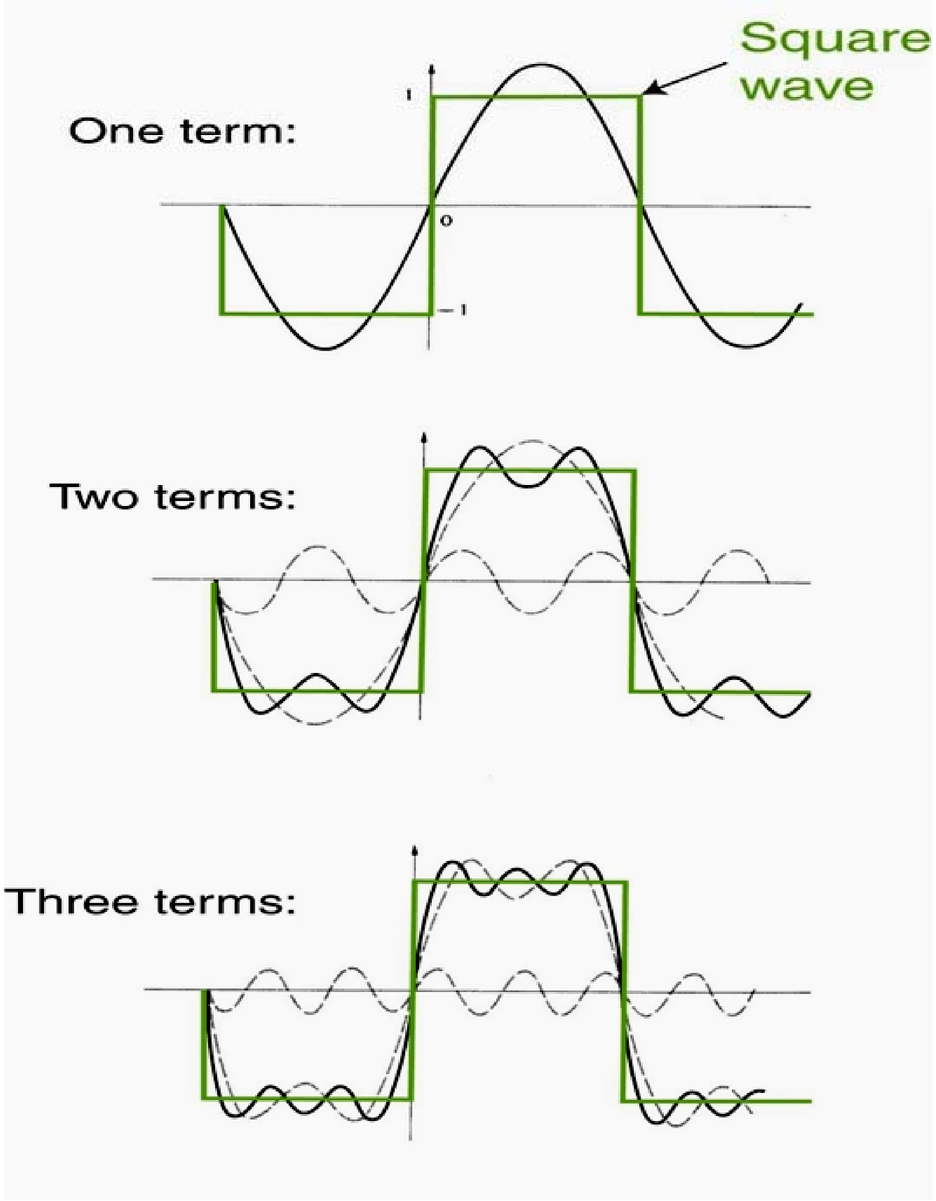
$$f(x) = \sin x + \tfrac{1}{3}\sin 2x + \tfrac{1}{5}\sin 4x$$

A function and its trigonometric decomposition

Fourier transform basis functions



A square wave and its trigonometric approximation



Approximating a square wave as the sum of sine waves.

DFT -Definition

Suppose

$$\mathbf{f} = [f_0, f_1, f_2, \ldots, f_{N-1}]$$

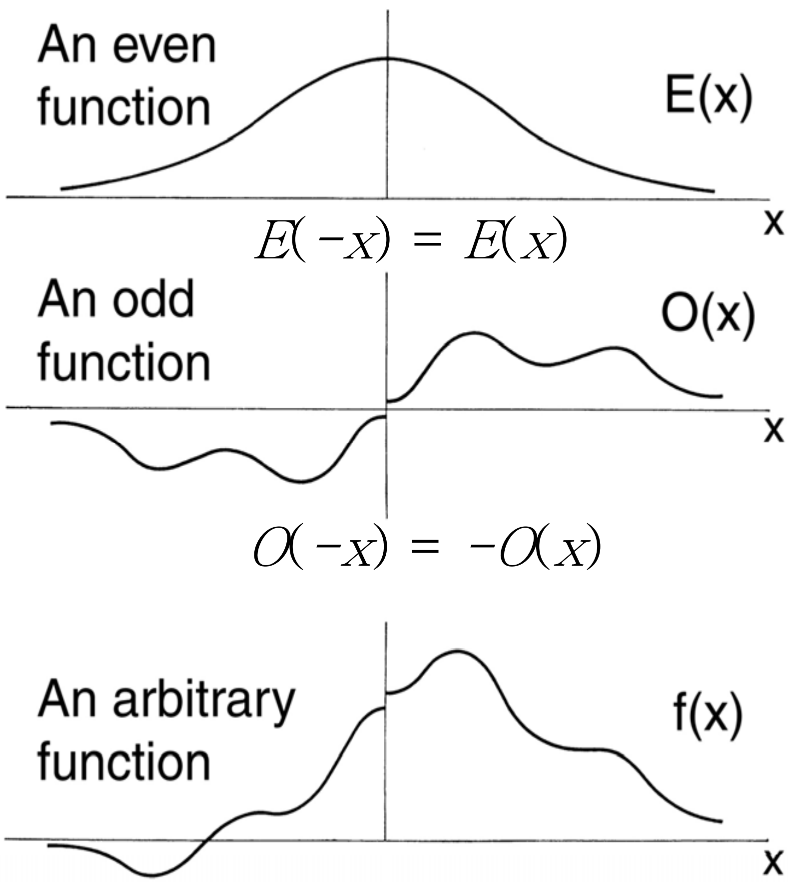
is a sequence of length N. We define its *discrete Fourier transform* to be the sequence

$$\mathbf{F} = [F_0, F_1, F_2, \ldots, F_{N-1}]$$

where

$$F_u = \frac{1}{N} \sum_{x=0}^{N-1} \exp\left[-2\pi i \frac{xu}{N}\right] f_x.$$

Any function can be written as the sum of an even and an odd function



$$E(x) \equiv [\,f(x) + f(-x)]/2$$

$$O(x) \equiv [\,f(x) - f(-x)]/2$$

$$\Downarrow$$

$$f(x) = E(x) + O(x)$$

Fourier Cosine Series

Because cos(*mt*) is an even function, we can write an even function, *f(t),* as:

$$f(t) \quad = \quad \frac{1}{\pi} \sum_{m=0}^{\infty} F_m \cos(mt)$$

where series F_m is computed as

$$\boxed{F_m \quad = \int_{-\pi}^{\pi} f(t) \cos(mt)\, dt}$$

Here we suppose *f(t)* is over the interval (–*π*,*π*).

Fourier Sine Series

Because sin(*mt*) is an odd function, we can write any odd function, *f(t),* as:

$$f(t) \;=\; \frac{1}{\pi}\sum_{m=0}^{\infty} F'_m \sin(mt)$$

where the series F'_m is computed as

$$\boxed{F'_m \;=\; \int_{-\pi}^{\pi} f(t)\sin(mt)\,dt}$$

Fourier Series

So if $f(t)$ is a general function, neither even nor odd, it can be written:

$$\boxed{f(t) = \frac{1}{\pi}\sum_{m=0}^{\infty} F_m \cos(mt) + \frac{1}{\pi}\sum_{m=0}^{\infty} F'_m \sin(mt)}$$

Even component

Odd component

where the Fourier series is

$$F_m = \int f(t)\cos(mt)\,dt \qquad F'_m = \int f(t)\sin(mt)\,dt$$

The Fourier Transform

Let $F(m)$ incorporates both cosine and sine series coefficients, with the sine series distinguished by making it the imaginary component:

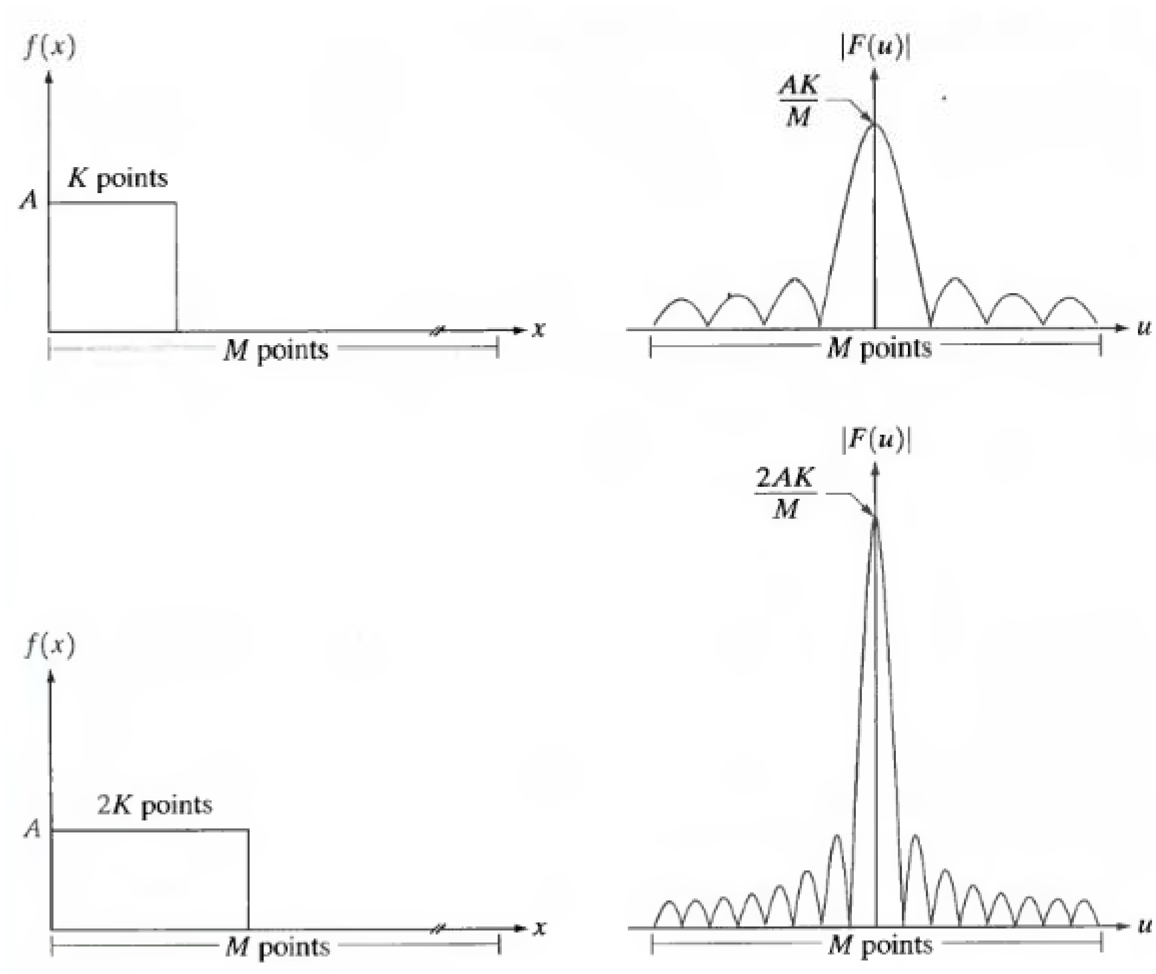
$$F(m) = F_m - jF_m^{'} = \int f(t)\cos(mt)dt - j \cdot \int f(t)\sin(mt)dt$$

Let's now allow *f(t)* range from $-\infty$ to ∞, we rewrite:

$$\boxed{\Im\{f(t)\} = F(u) = \int_{-\infty}^{\infty} f(t)\exp(-j2\pi ut)dt}$$

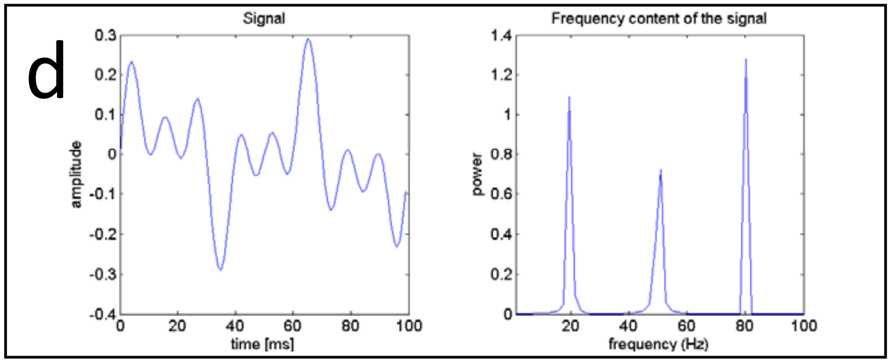
F*(*u*)** is called the **Fourier Transform** of *f(t)*. We say that *f(t)* lives in the "**time domain**," and ***F*(*u*)** lives in the "**frequency domain**." ***u is called the frequency variable.

The Fourier Transform



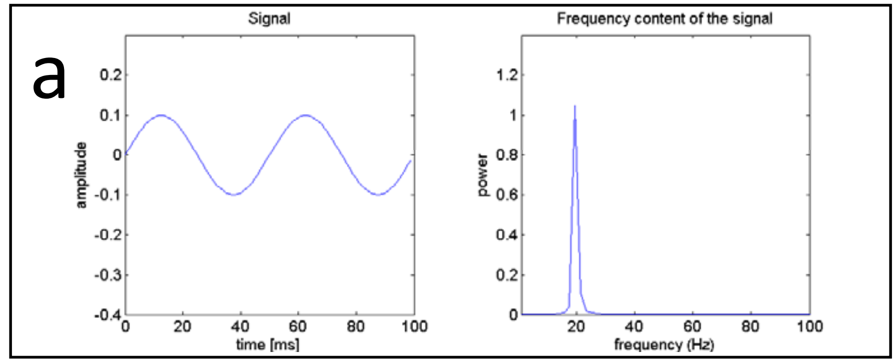
a b
c d

FIGURE 4.2 (a) A discrete function of M points, and (b) its Fourier spectrum. (c) A discrete function with twice the number of nonzero points, and (d) its Fourier spectrum.

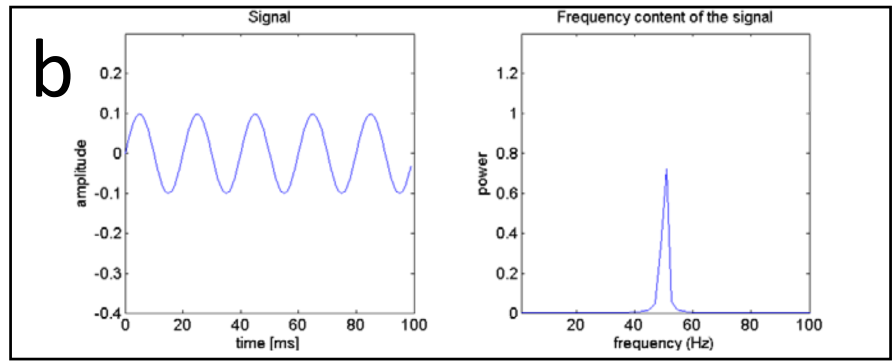


time domain frequency domain

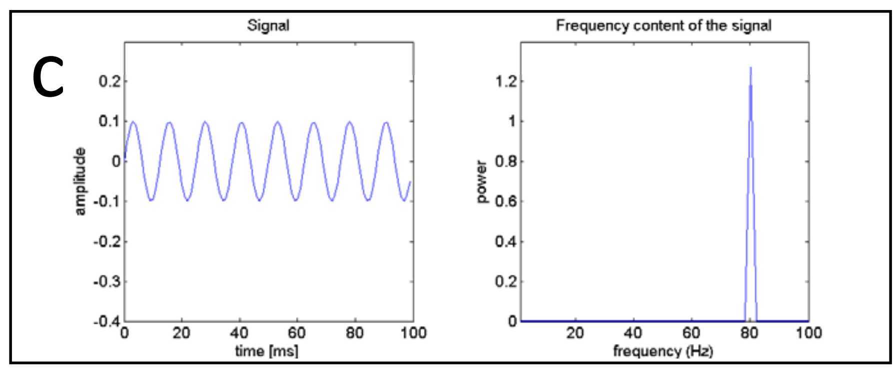
$$d = a + b + c$$



time domain frequency domain



time domain frequency domain



time domain frequency domain

The Inverse Fourier Transform

We go from *f(t)* to *F(u)* by

$$\boxed{\Im\left\{f(t)\right\} = F(u) = \int_{-\infty}^{\infty} f(t)\exp(-j2\pi ut)\,dt}$$

Fourier Transform

Given *F*(*u*), *f(t)* can be obtained by the inverse Fourier transform

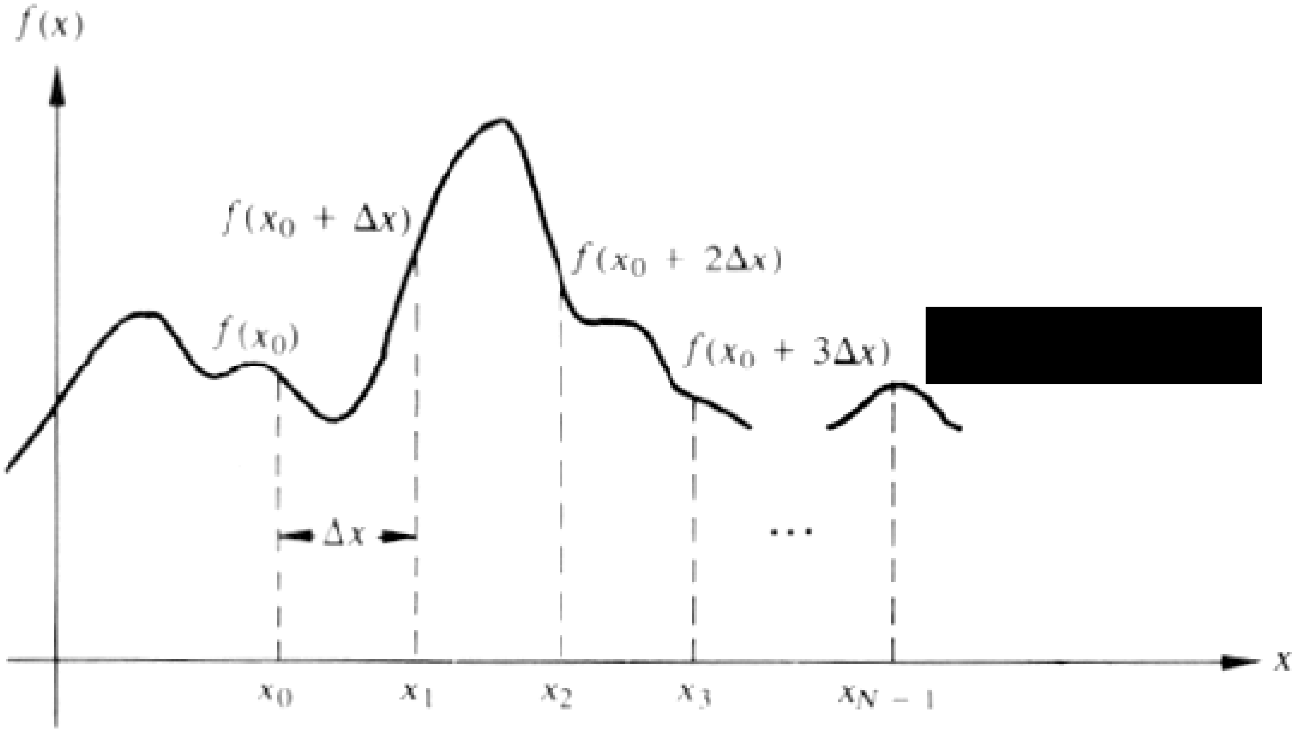
$$\boxed{\Im^{-1}\left\{F(u)\right\} = f(t) = \int_{-\infty}^{\infty} F(u)\exp(j2\pi ut)\,du}$$

Inverse Fourier Transform

Discrete Fourier Transform (DFT)

- A continuous function f(x) is discretized as:

$$\{f(x_0), f(x_0+\Delta x), f(x_0+2\Delta x), \ldots, f(x_0+(M-1)\Delta x)\}$$



Discrete Fourier Transform (DFT)

Let x denote the discrete values (x=0,1,2,…,M-1), i.e.

$$f(x) = f(x_0 + x\Delta x)$$

then

$$\{f(x_0), f(x_0+\Delta x), f(x_0+2\Delta x), \ldots, f(x_0+(M-1)\Delta x)\}$$

$$\Downarrow$$

$$\{f(0), f(1), f(2), \ldots, f(M-1)\}$$

Discrete Fourier Transform (DFT)

- The discrete Fourier transform pair that applies to sampled functions is given by:

$$\boxed{F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)\exp(-j2\pi ux/M)}$$ *u=0,1,2,…,M-1*

and

$$\boxed{f(x) = \sum_{u=0}^{M-1} F(u)\exp(j2\pi ux/M)}$$ *x=0,1,2,…,M-1*

2-D Discrete Fourier Transform

- In 2-D case, the DFT pair is:

$$\boxed{F(u,v)=\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}f(x,y)\exp(-j2\pi(ux/M+vy/N))}$$

u=0,1,2,…,M-1 and v=0,1,2,…,N-1

and:

$$\boxed{f(x,y)=\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}F(u,v)\exp(j2\pi(ux/M+vy/N))}$$

x=0,1,2,…,M-1 and y=0,1,2,…,N-1

Polar Coordinate Representation of FT

- The Fourier transform of a real function is generally complex and we use polar coordinates:

$$F(u,v) = R(u,v) + j \cdot I(u,v)$$

Polar coordinate

$$F(u,v) = \left|F(u,v)\right| \exp(j\phi(u,v))$$

Magnitude: $$\left|F(u,v)\right| = [R^2(u,v) + I^2(u,v)]^{1/2}$$

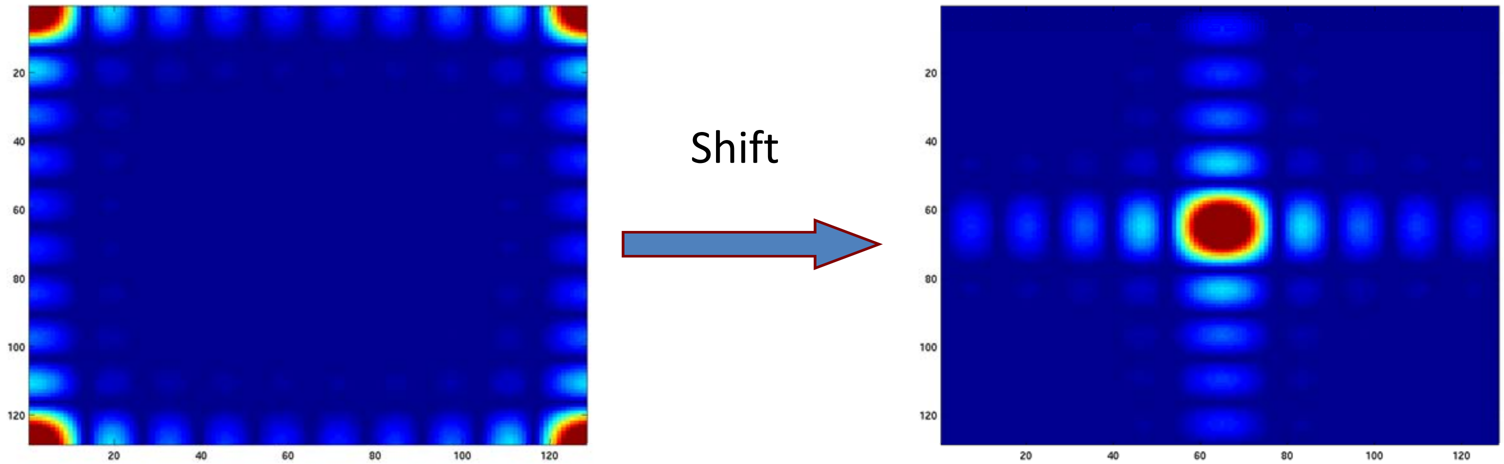
Phase: $$\phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$$

Fourier Transform: shift

- It is common to multiply input image by $(-1)^{x+y}$ prior to computing the FT. This shift the center of the FT to (M/2,N/2).

$$\Im\{f(x,y)\} = F(u,v)$$

$$\Im\left\{f(x,y)(-1)^{x+y}\right\} = F(u-M/2, v-N/2)$$



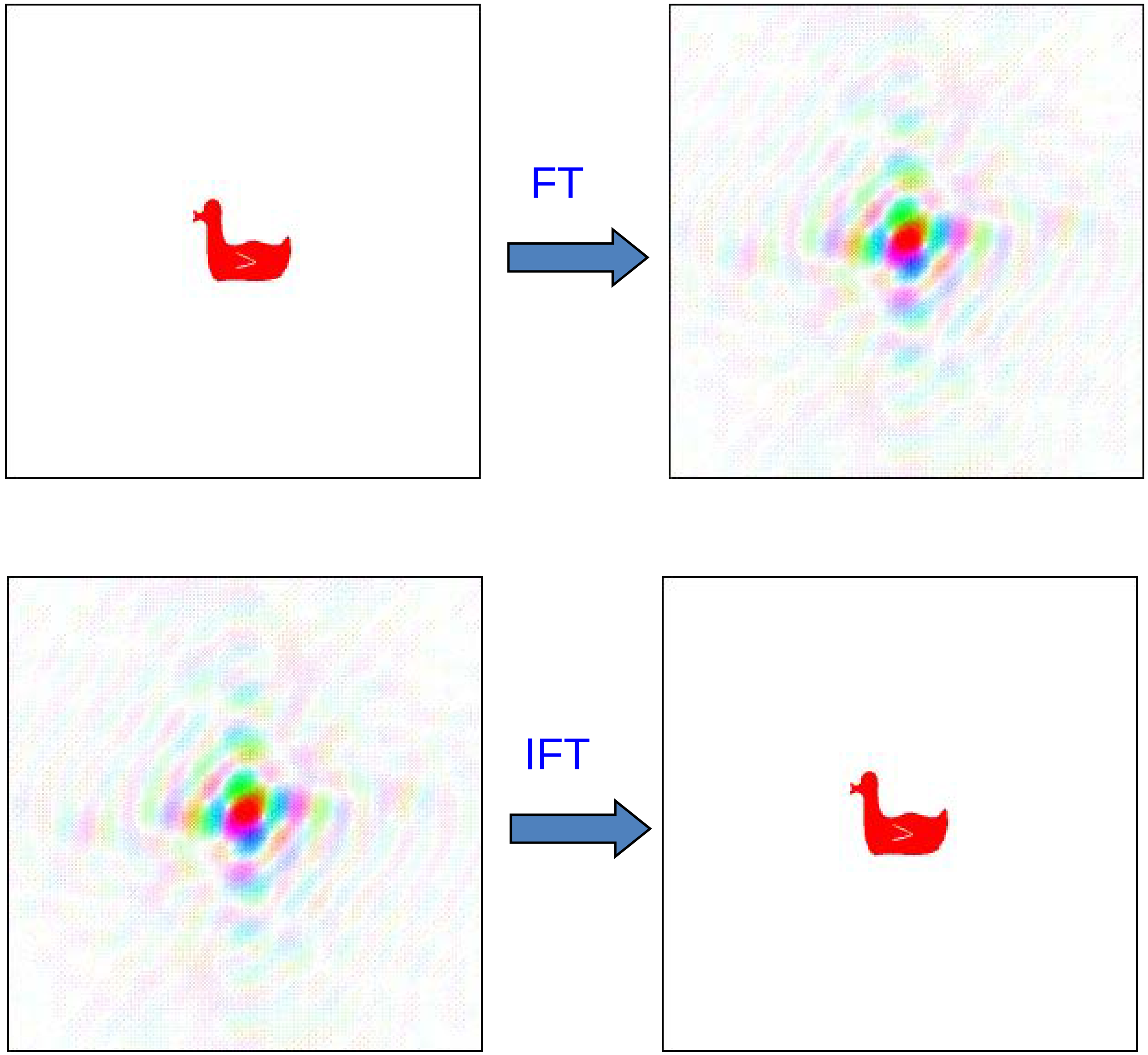
Symmetry of FT

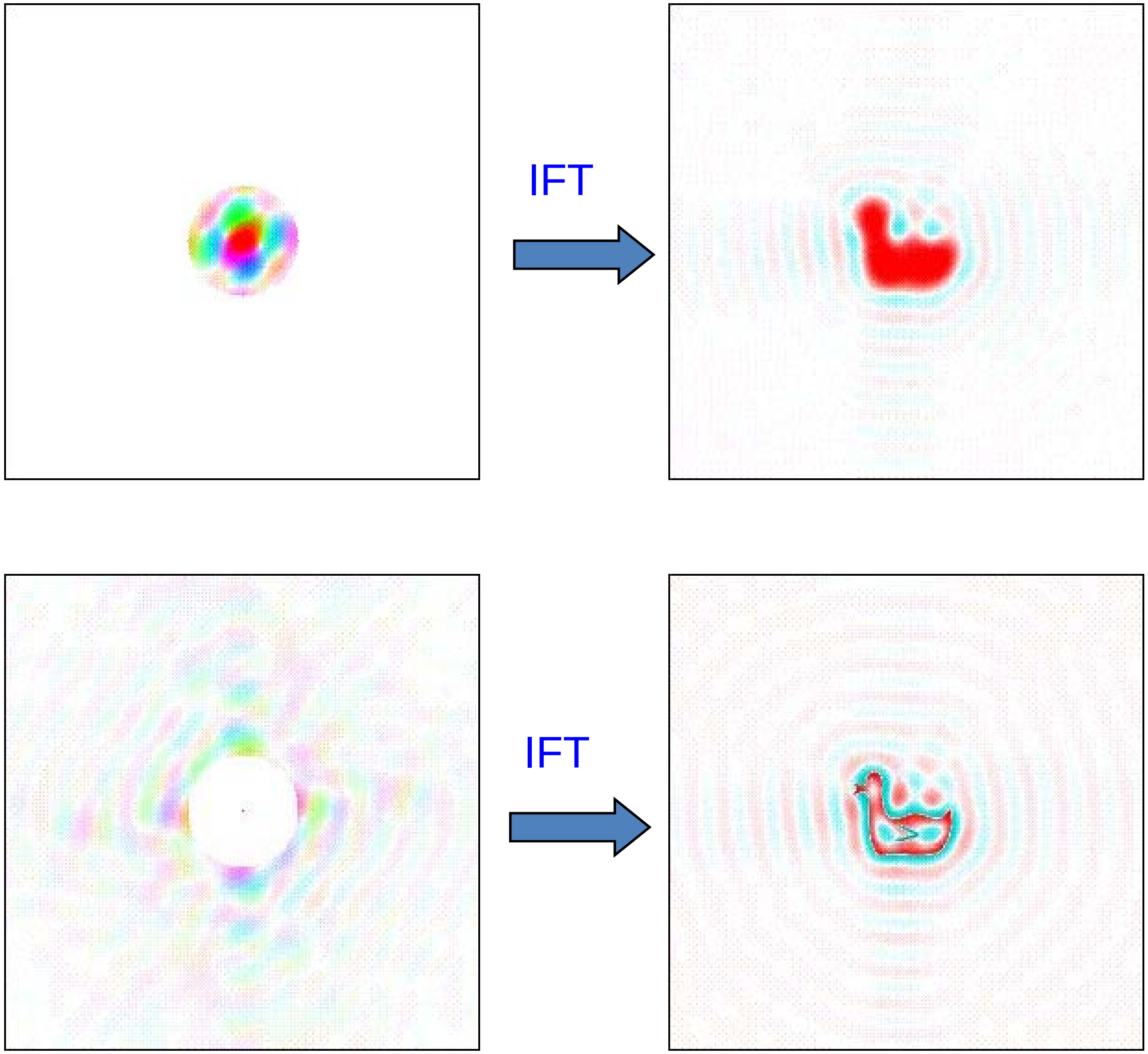
- For real image f(x,y), FT is conjugate symmetric:

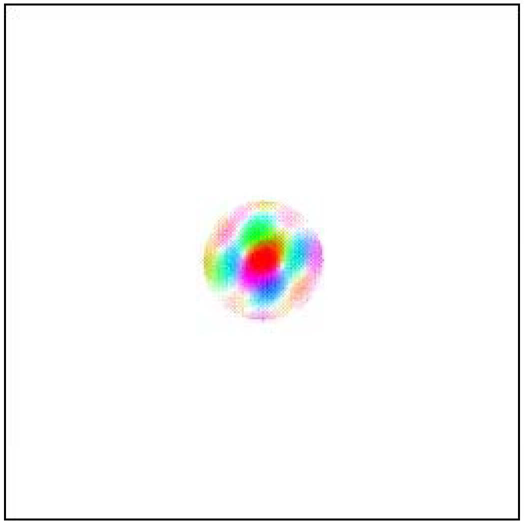
$$F(u,v) = F^*(-u,-v)$$

- The magnitude of FT is symmetric:

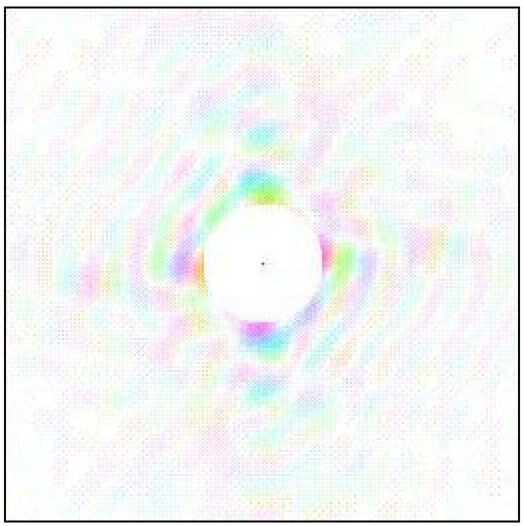
$$\left|F(u,v)\right| = \left|F(-u,-v)\right|$$


FT
IFT


IFT
IFT



The central part of FT, i.e. the low frequency components are responsible for the general gray-level appearance of an image.



The high frequency components of FT are responsible for the detail information of an image.

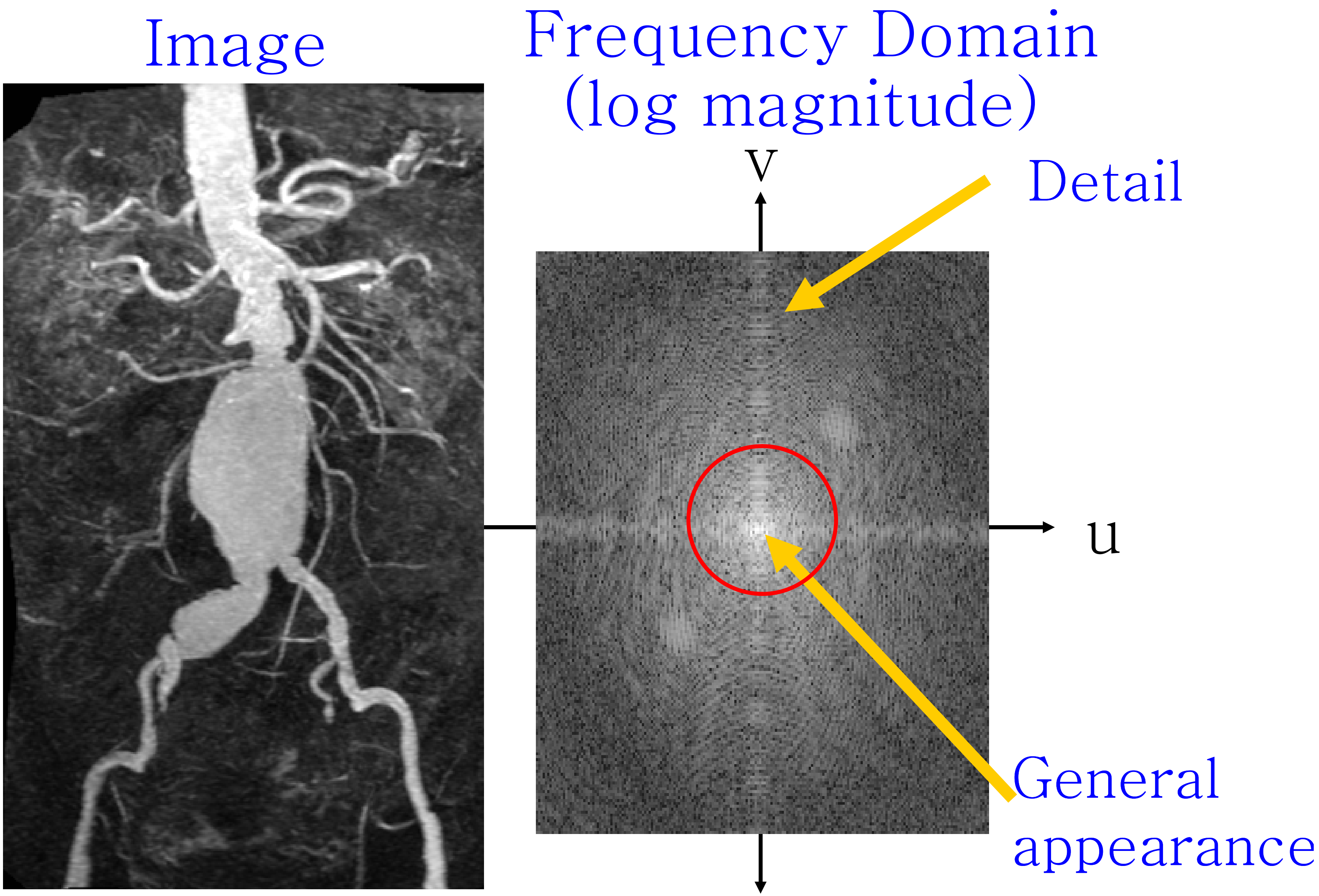
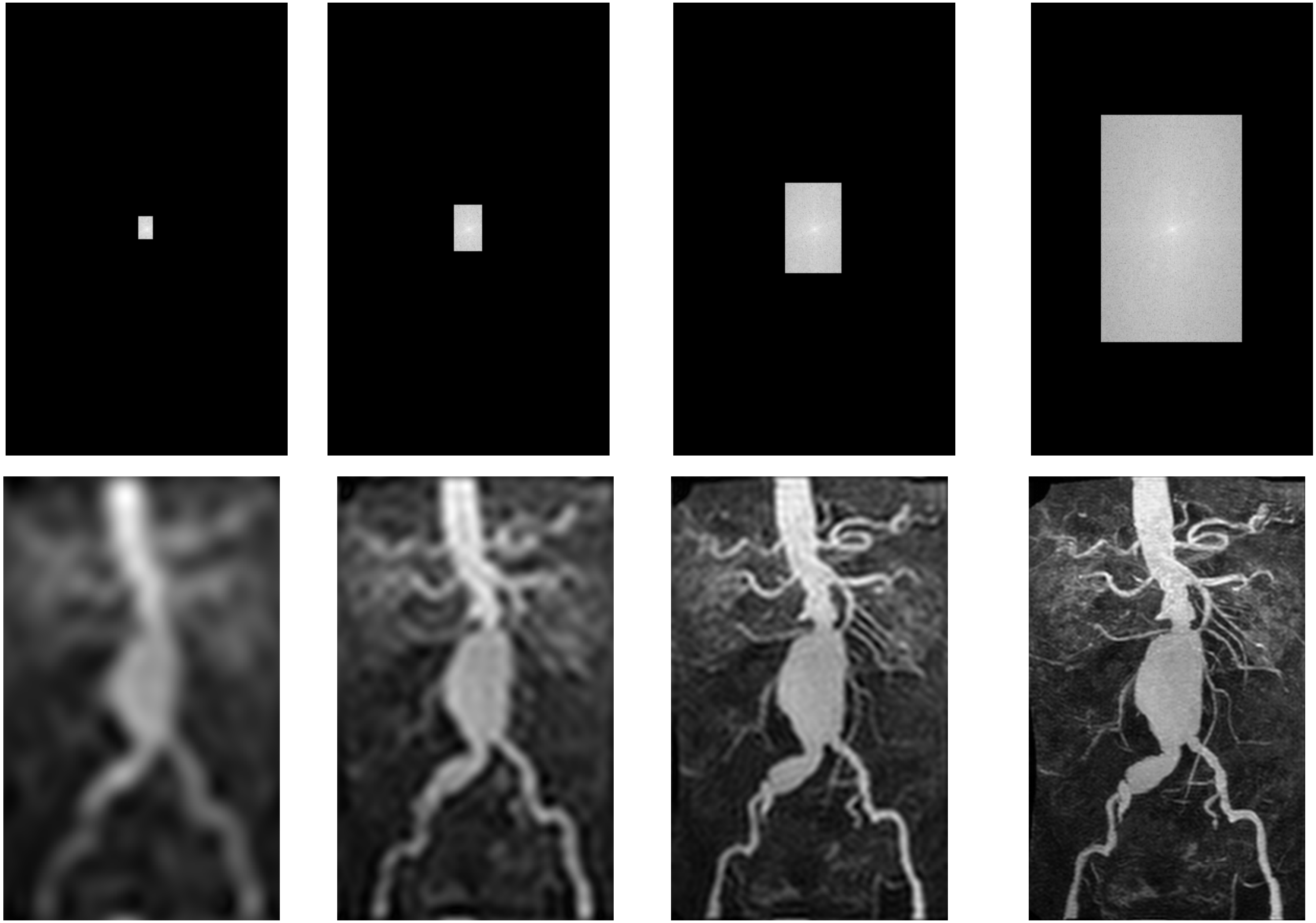


Image
Frequency Domain
(log magnitude)
v
Detail
u
General
appearance



Frequency Domain Filtering

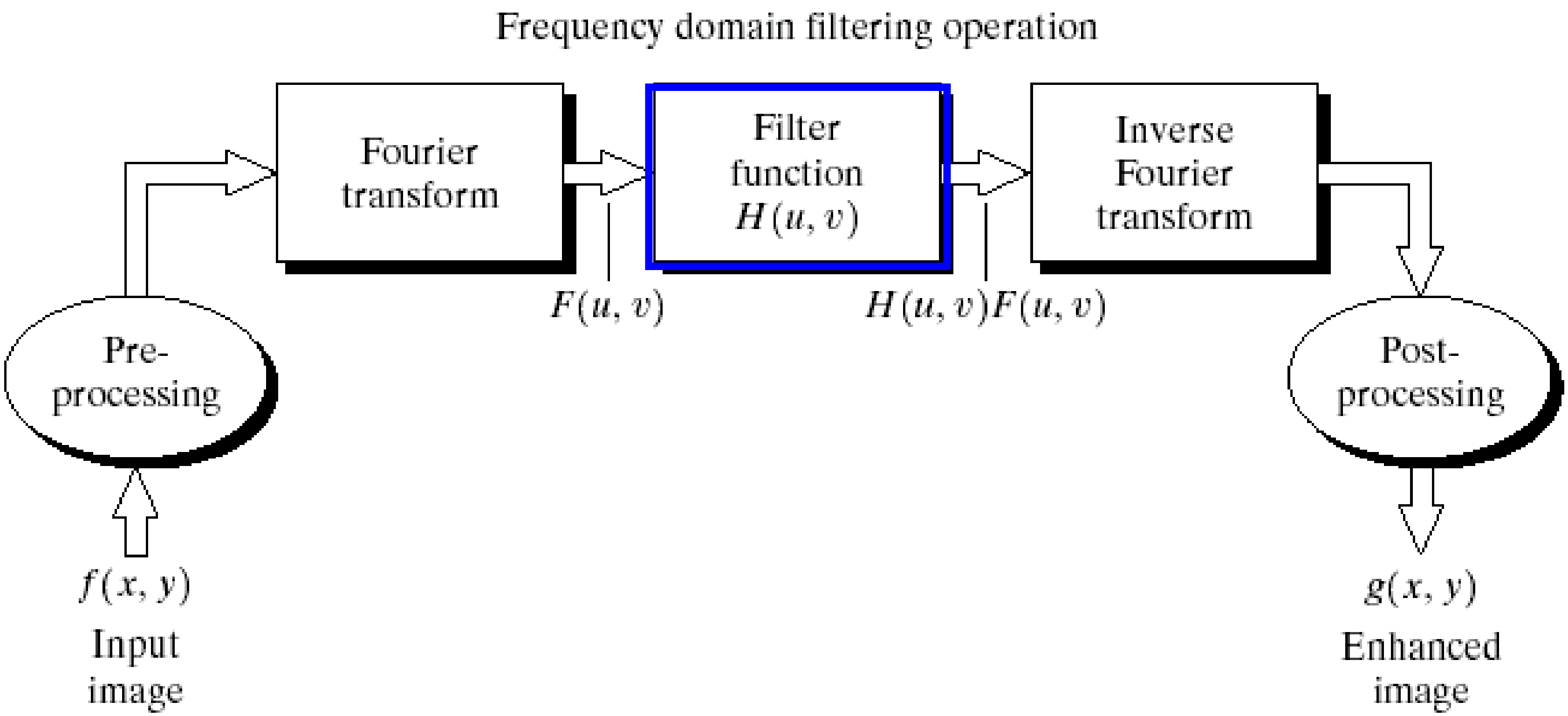


FIGURE 4.5 Basic steps for filtering in the frequency domain.

Frequency Domain Filtering

- Edges and sharp transitions (e.g., noise) in an image contribute significantly to high-frequency content of FT.
- Low frequency contents in the FT are responsible to the general appearance of the image over smooth areas.
- Blurring (smoothing) is achieved by attenuating range of high frequency components of FT.

Convolution in Time Domain

$g(x,y) = h(x,y) \otimes f(x,y)$

$$g(x,y) = \sum_{x'=0}^{M-1} \sum_{y'=0}^{M-1} h(x',y') f(x-x', y-y')$$

$$\equiv f(x,y) * h(x,y)$$

- $f(x,y)$ is the input image
- $g(x,y)$ is the filtered
- $h(x,y)$: impulse response

Convolution Theorem

$$G(u,v)=F(u,v)\cdot H(u,v)$$

$$\Downarrow$$

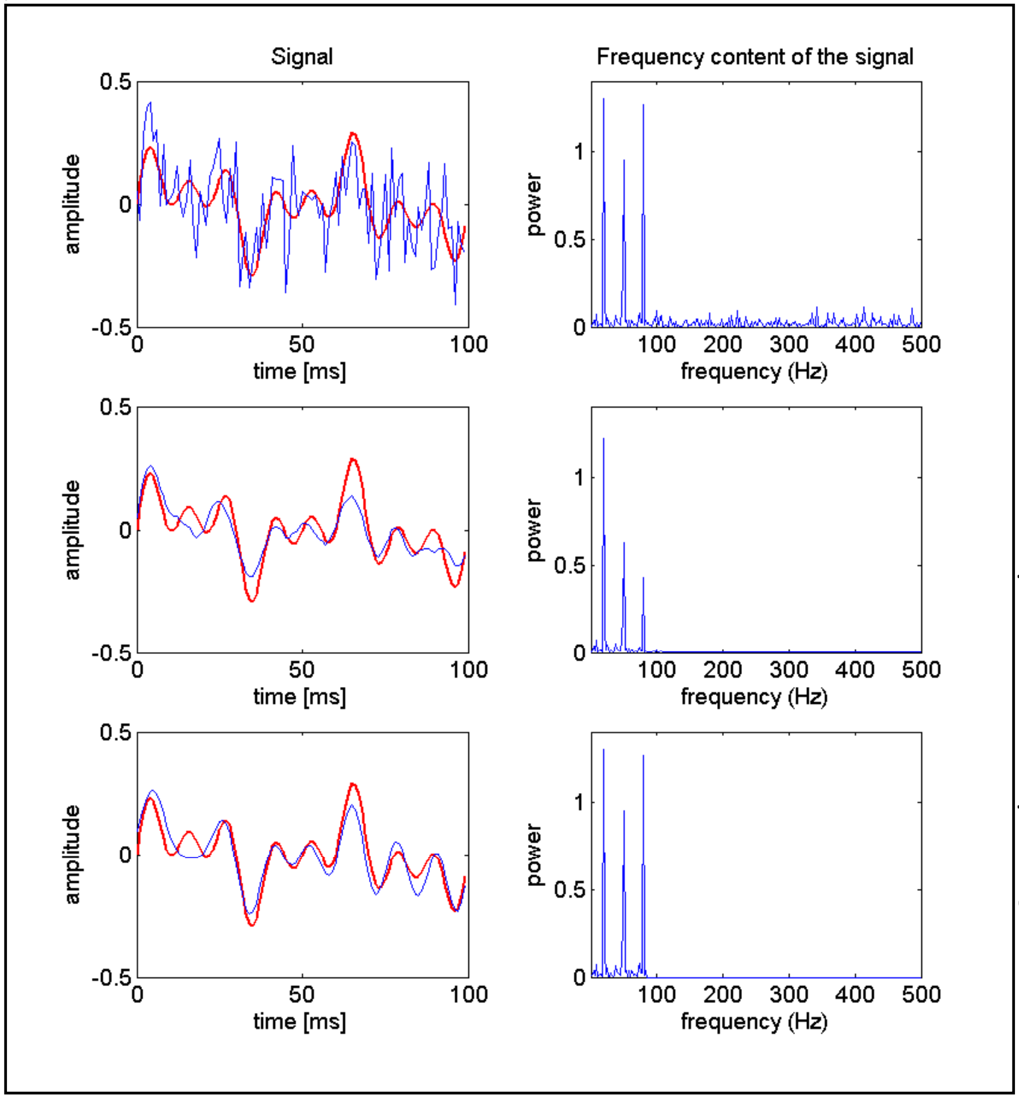
$$g(x,y)=h(x,y)\otimes f(x,y)$$

Multiplication in Frequency Domain

$\Updownarrow$

Convolution in Time Domain

- Filtering in Frequency Domain with H(u,v) is equivalent to filtering in Spatial Domain with f(x,y).



blue line = sum of 3 sinusoids (20, 50, and 80 Hz) + random noise

red line = sum of 3 sinusoids without noise

blue line = sum of 3 sinusoids after filtering in time domain

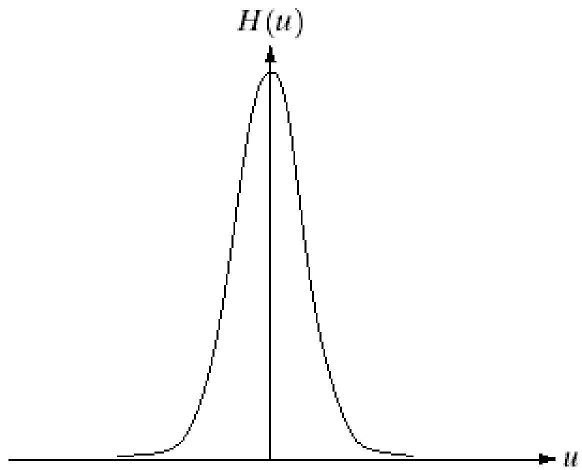
1x average [1 1 1 1 1] / 5

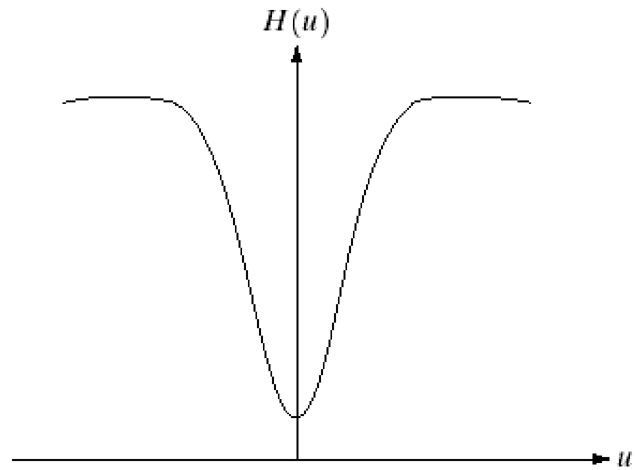
blue line = sum of 3 sinusoids after filtering in frequency domain

cut-off 90 Hz

Examples of Filters

Frequency domain

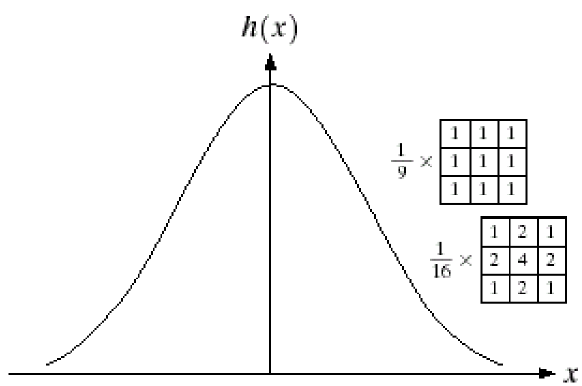


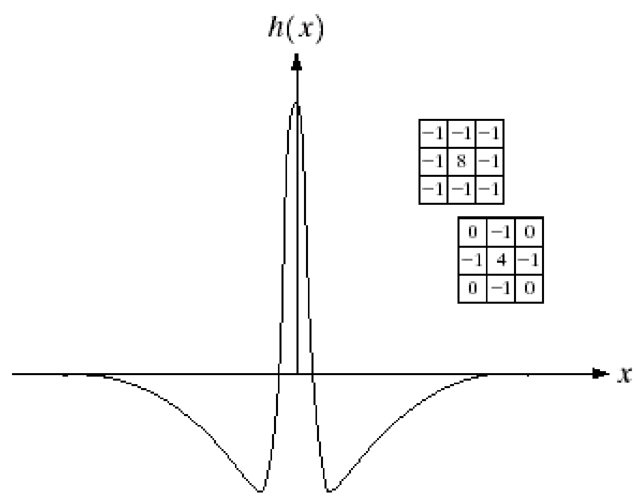


Gaussian lowpass filter

Gaussian highpass filter

Spatial domain





Separability