Minimum Spanning Trees

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Design and Analysis of Algorithms

Minimum spanning trees

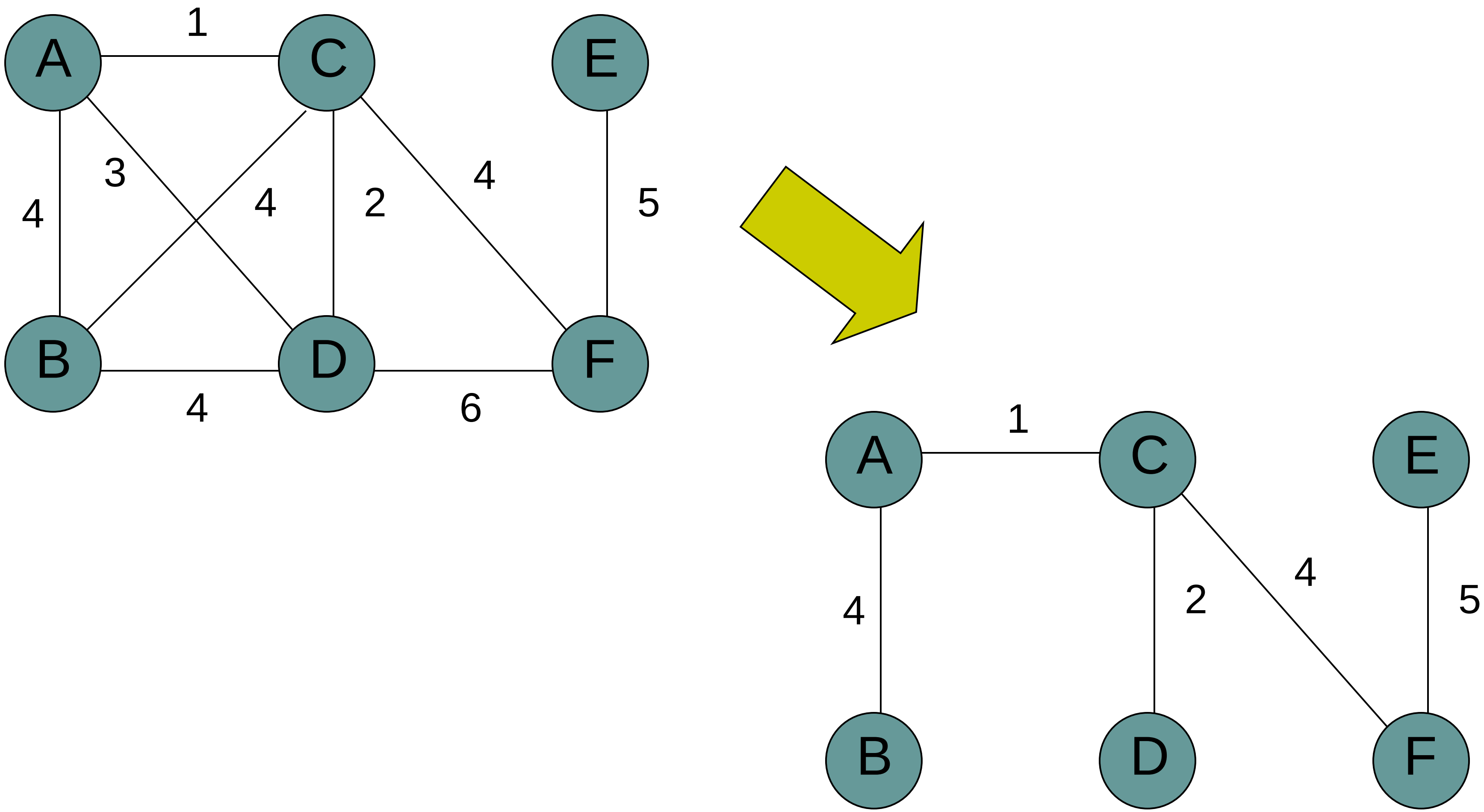
What is the lowest weight set of edges that connects all vertices of an undirected graph with positive weights

Input: An undirected, positive weight graph, G=(V,E)

Output: A tree T=(V,E') where E' $\subseteq$ E that minimizes

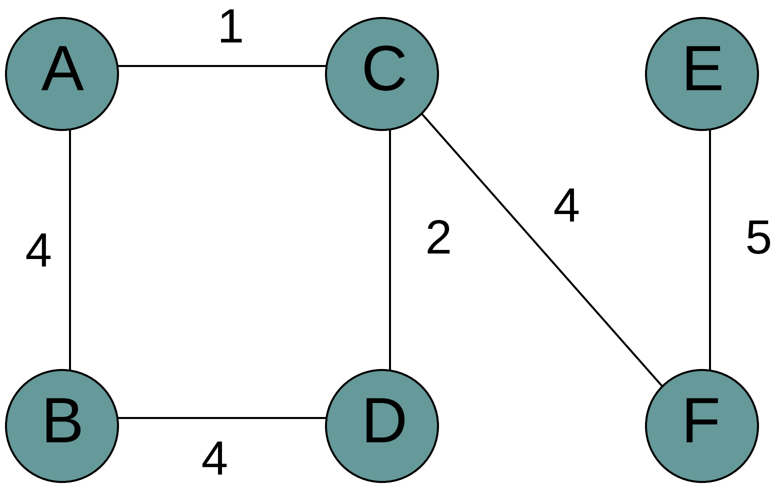
$$weight(T) = \sum_{e \in E'} w_e$$

MST example



MSTs

Can an MST have a cycle?



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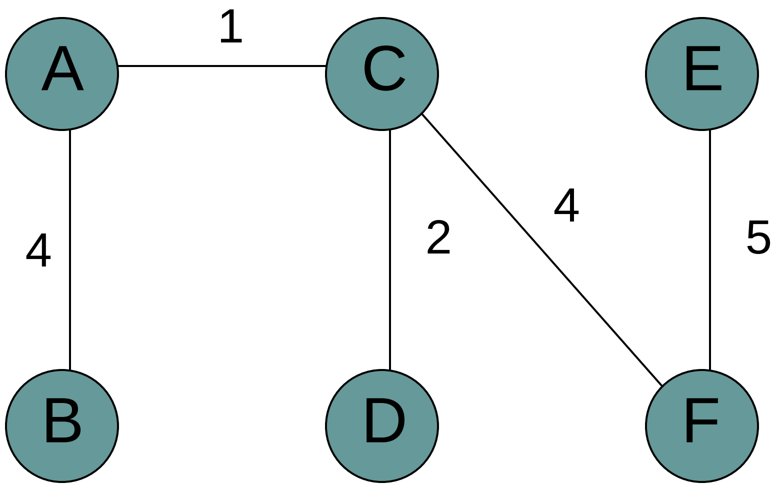


Applications

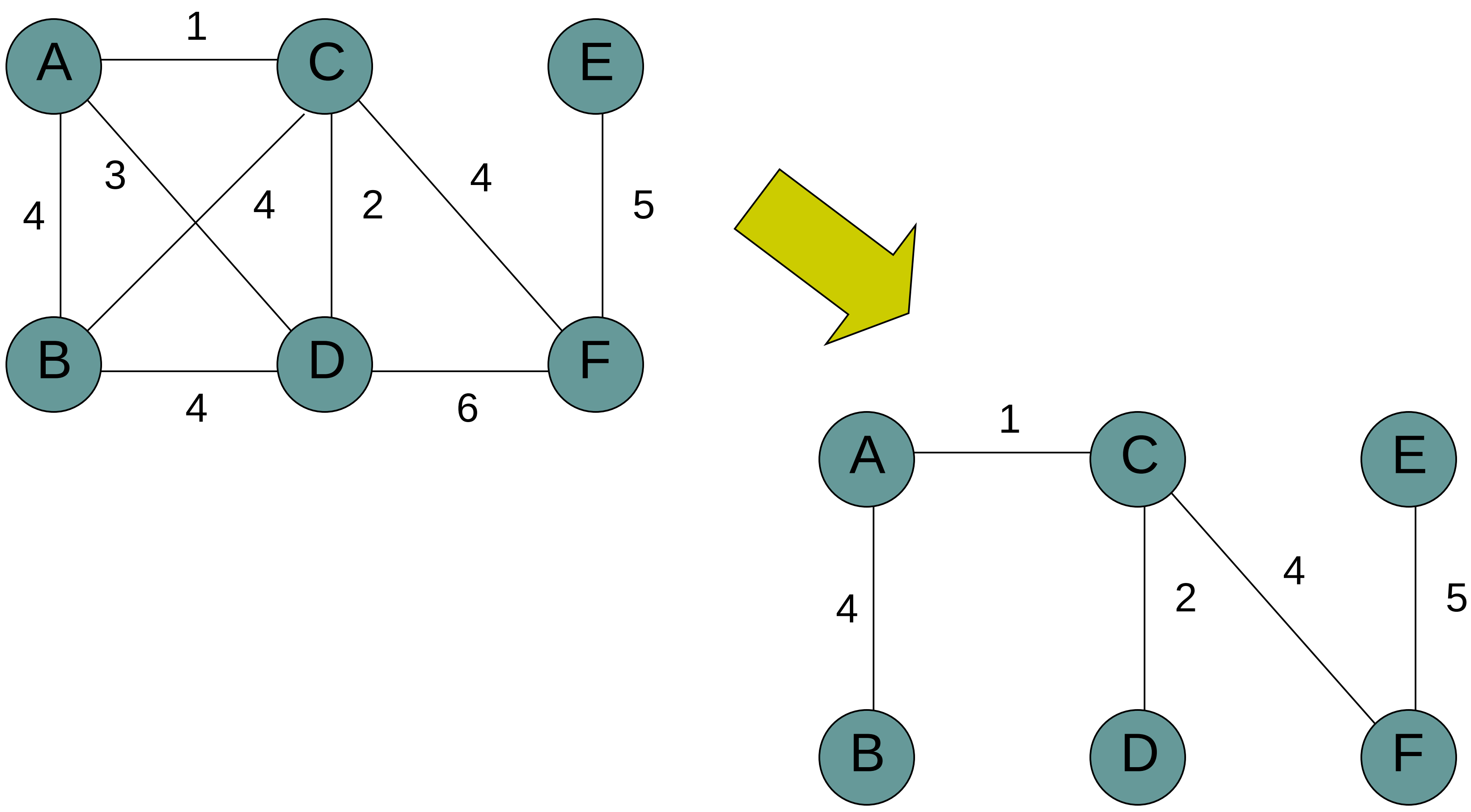
Connectivity

- Networks (e.g. communications)
- Circuit design/wiring

hub/spoke models (e.g. flights, transportation)

Traveling salesman problem?

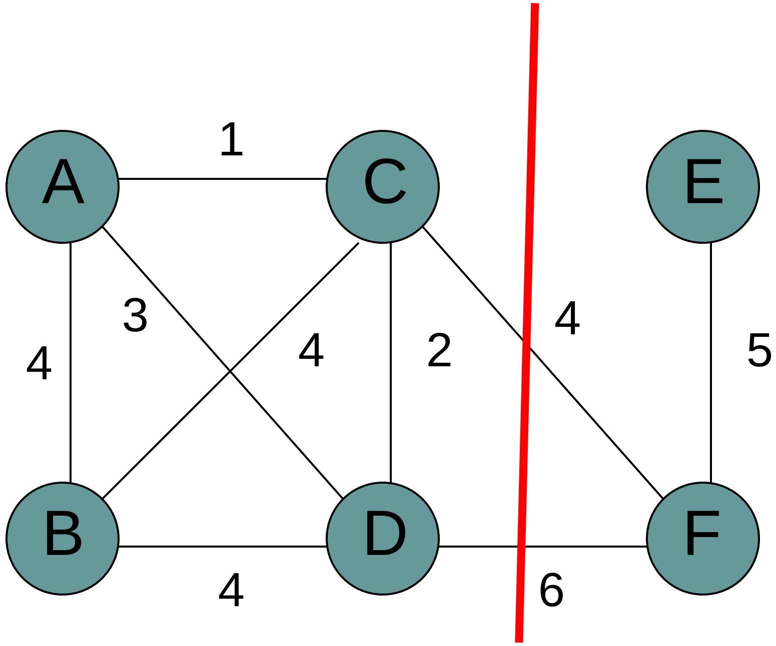
Algorithm ideas



Cuts

A cut is a partitioning of the vertices into two sets S and V-S

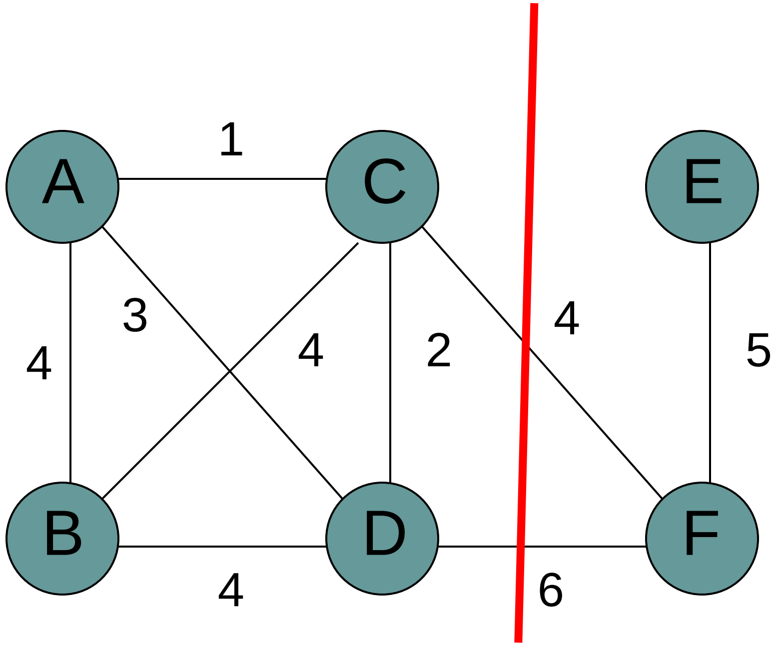
An edge "crosses" the cut if it connects a vertex u∈V and v∈V-S



Minimum cut property

Given a partition S, let edge *e* be the minimum cost edge that **crosses** the partition. *Every* minimum spanning tree contains edge *e*.

Prove this!



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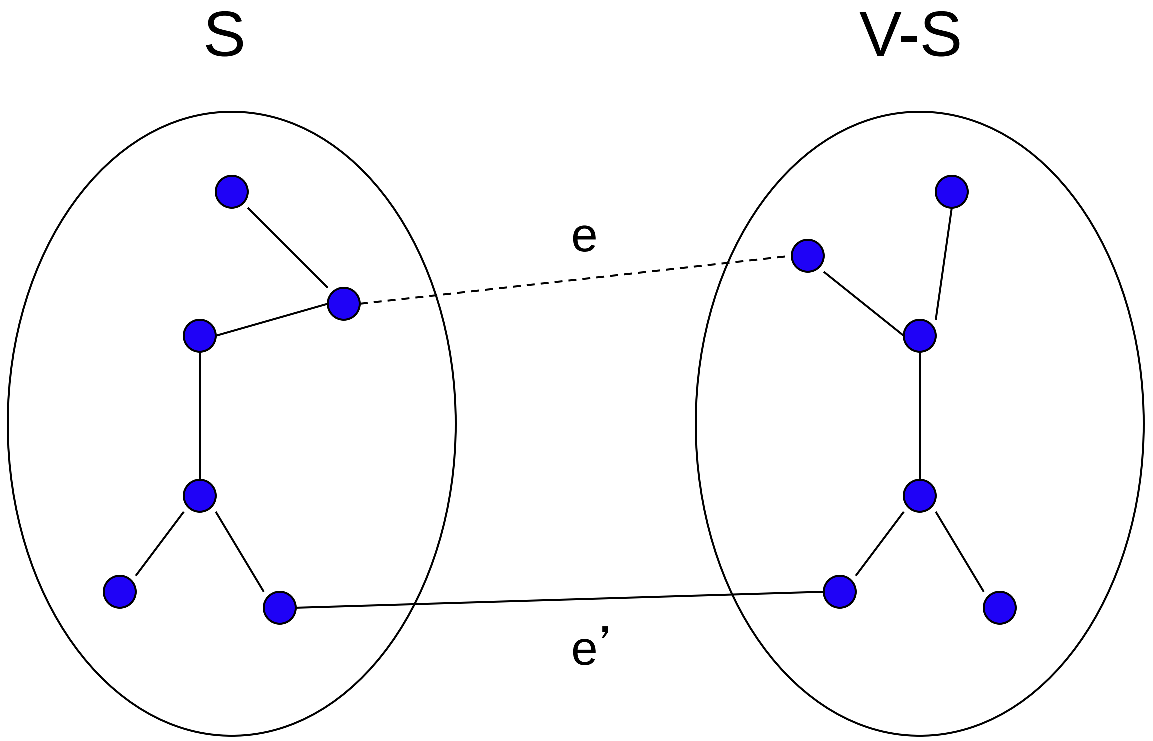

Consider an MST with edge e’ that is not the minimum edge

Minimum cut property

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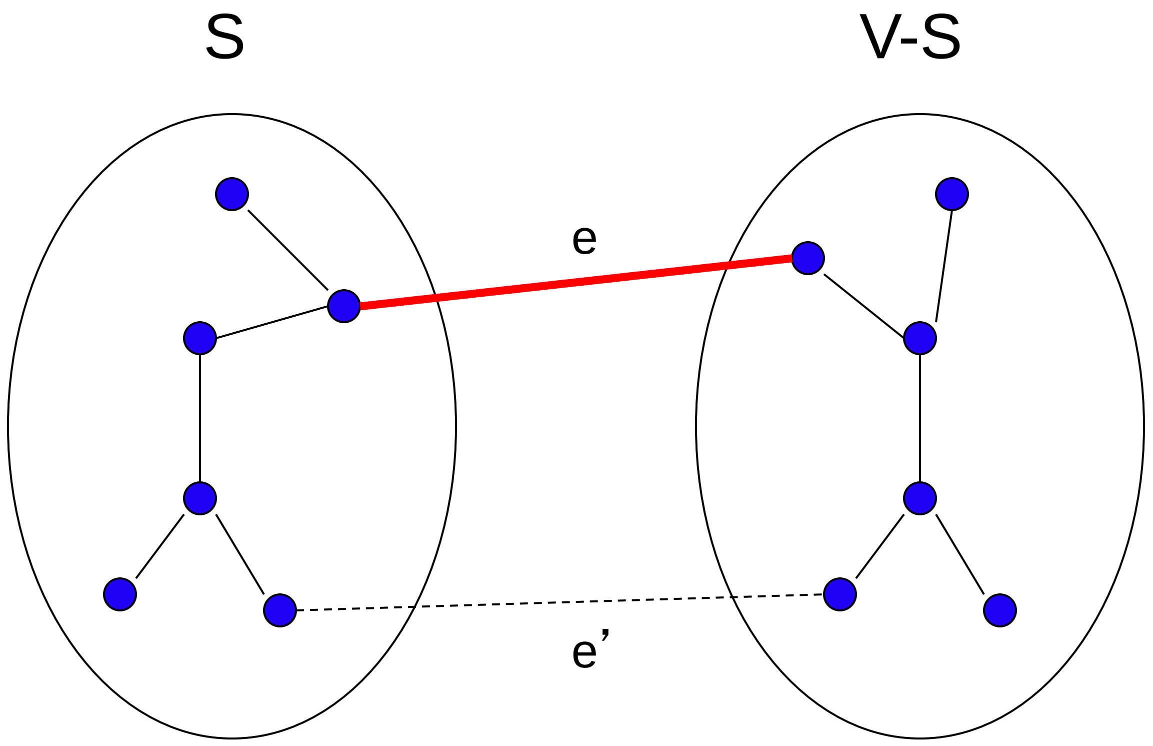
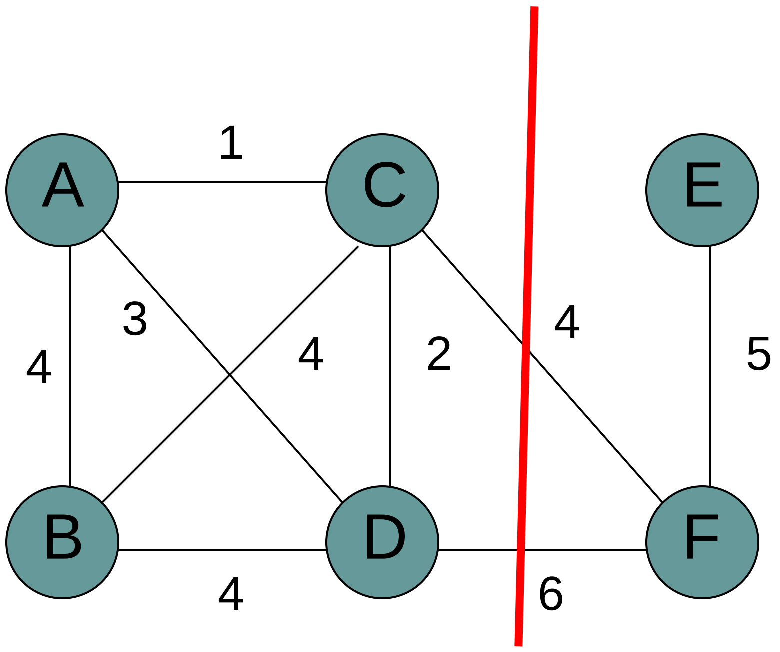


Using e instead of e', still connects the graph, but produces a tree with smaller weights

Minimum cut property

If the minimum cost edge that **crosses** the partition is not unique, then *some* minimum spanning tree contains edge *e*.



Kruskal's algorithm

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for all v ∈ V
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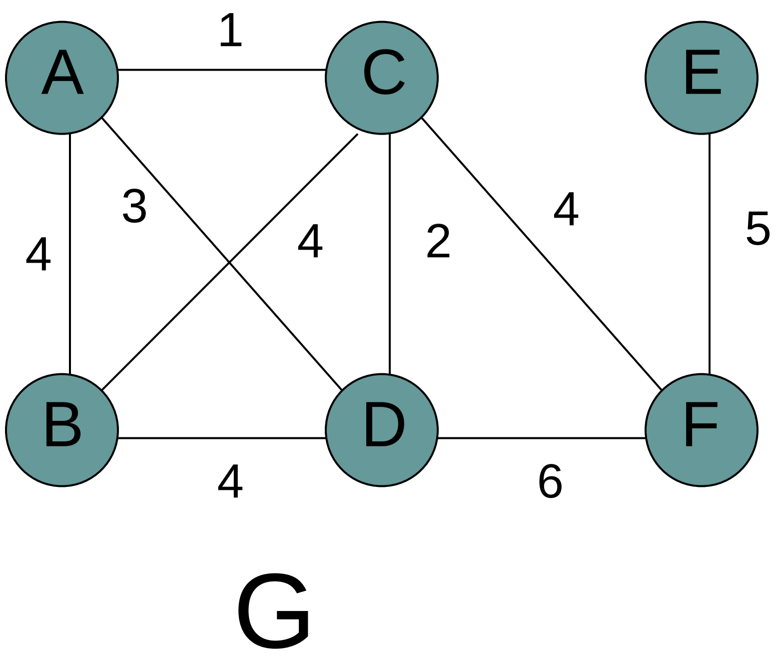
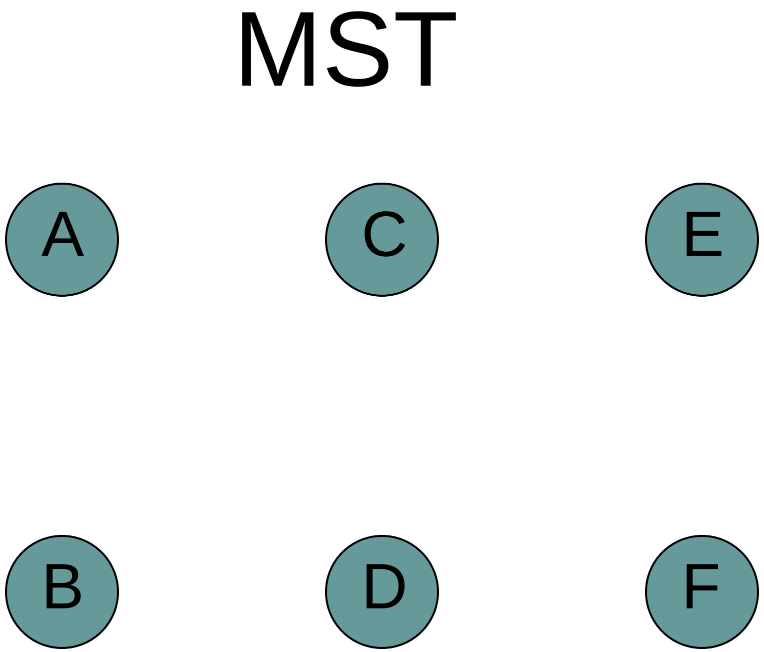
Kruskal's algorithm

Add smallest edge that connects two sets not already connected





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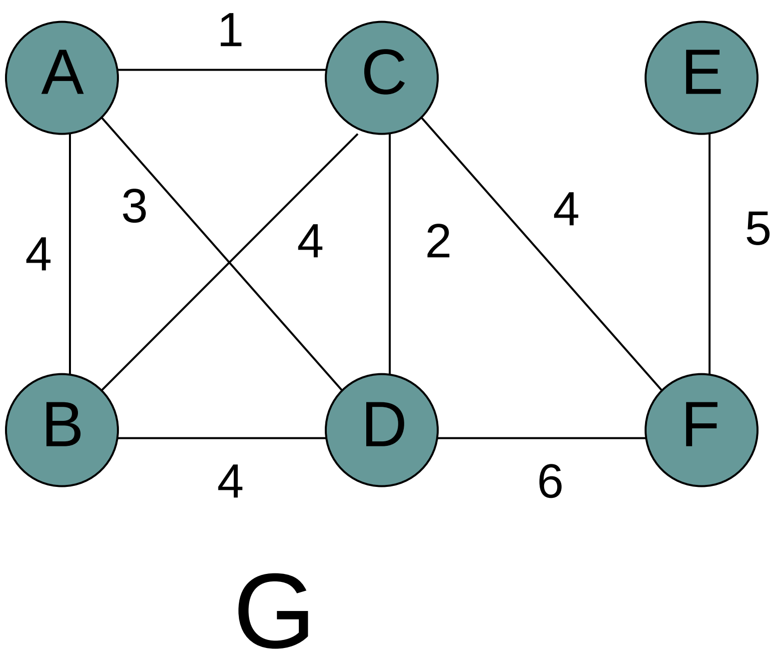

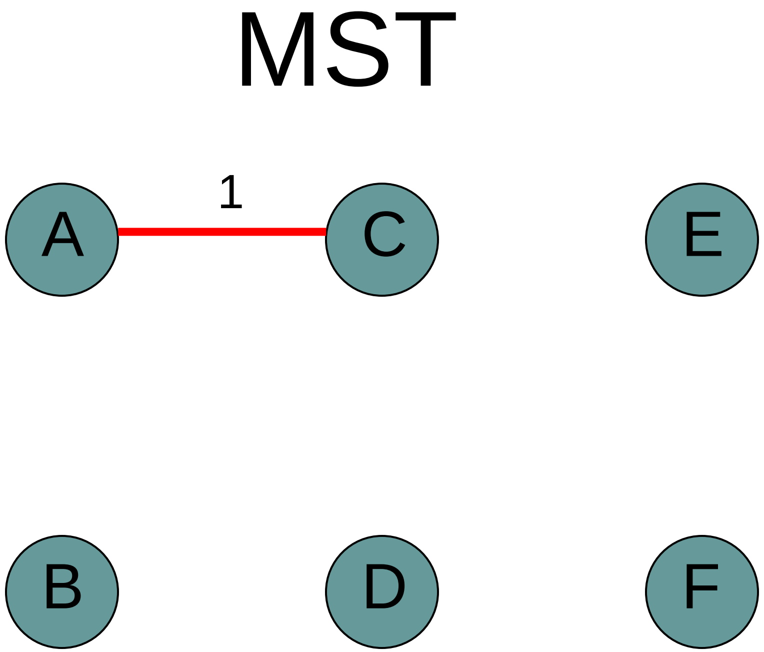

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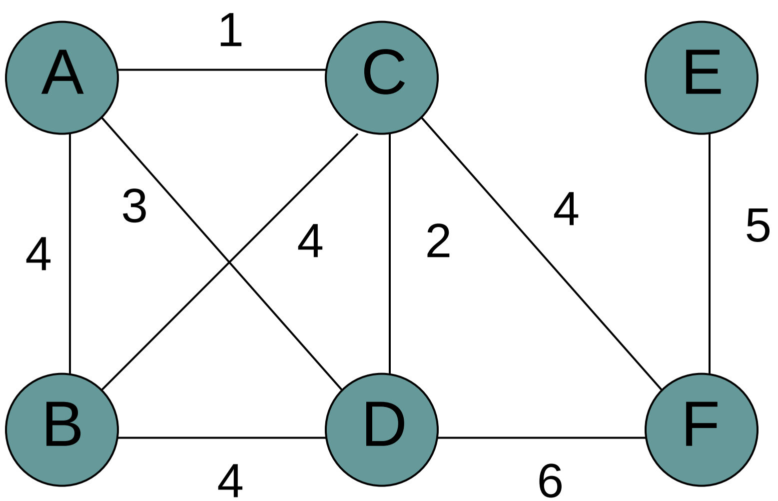
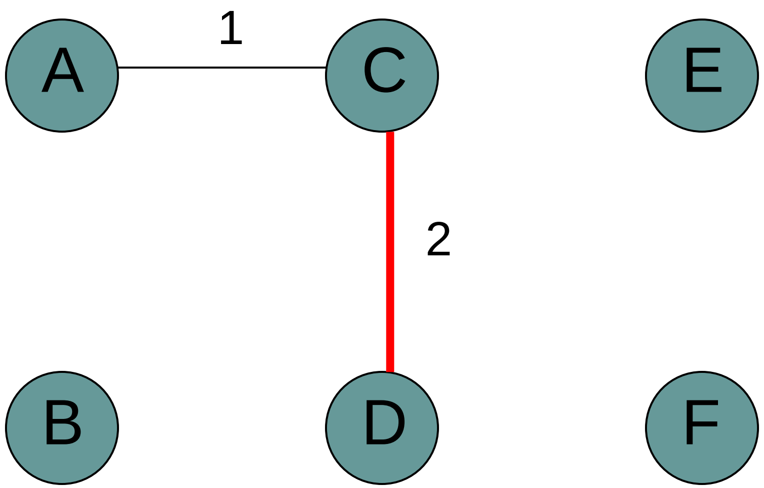


G

MST



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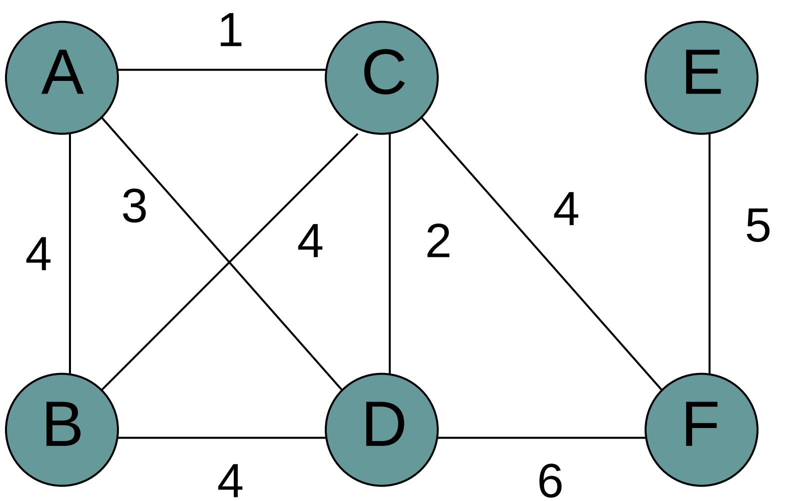

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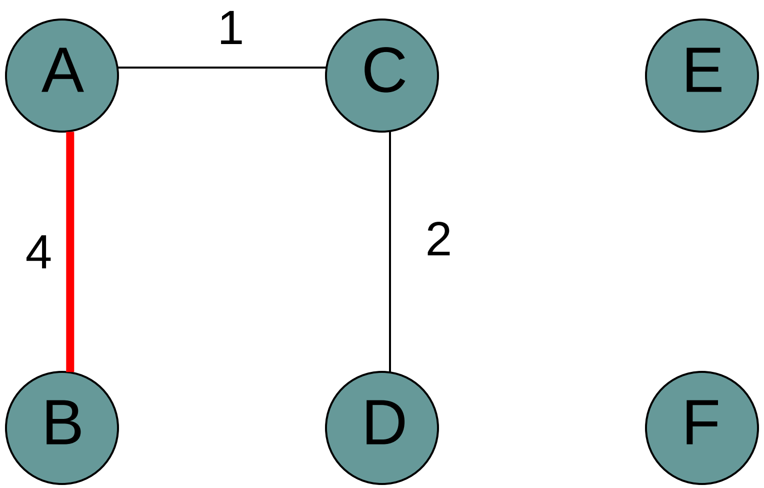

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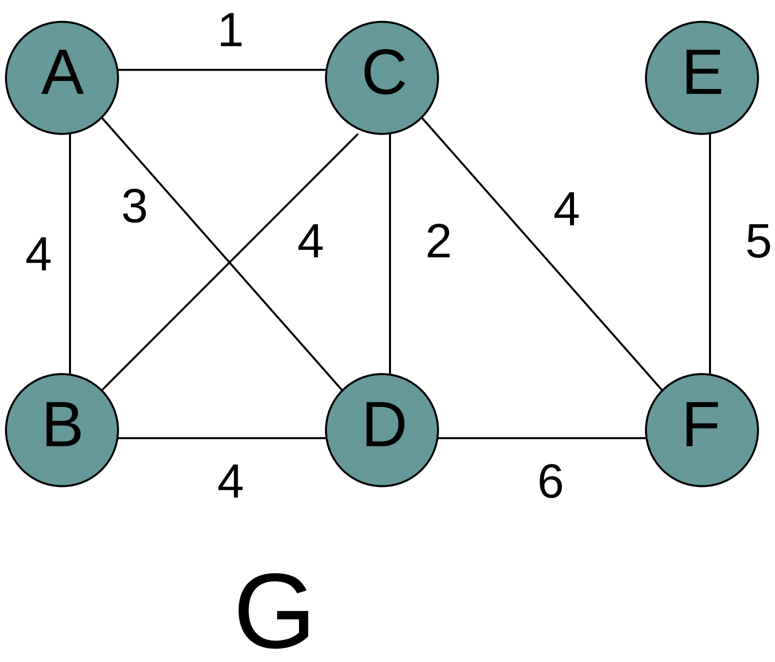

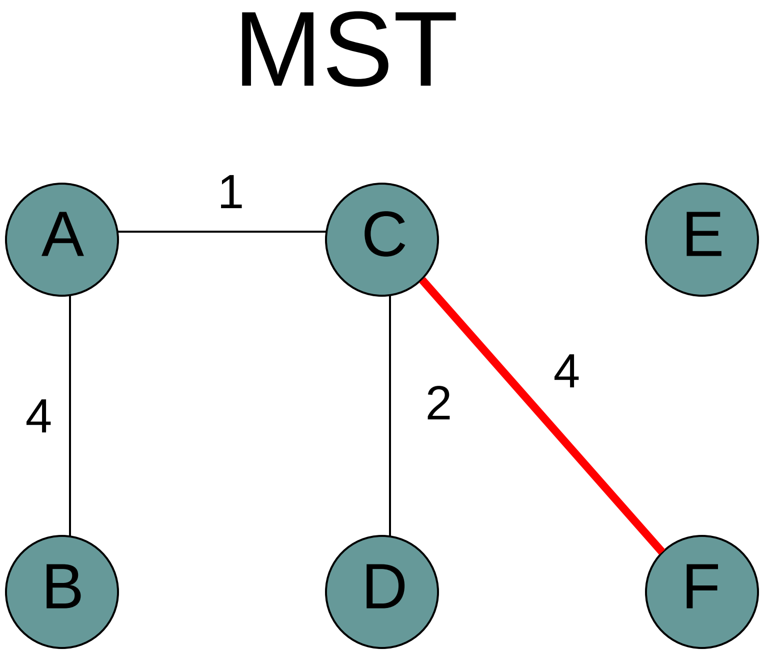

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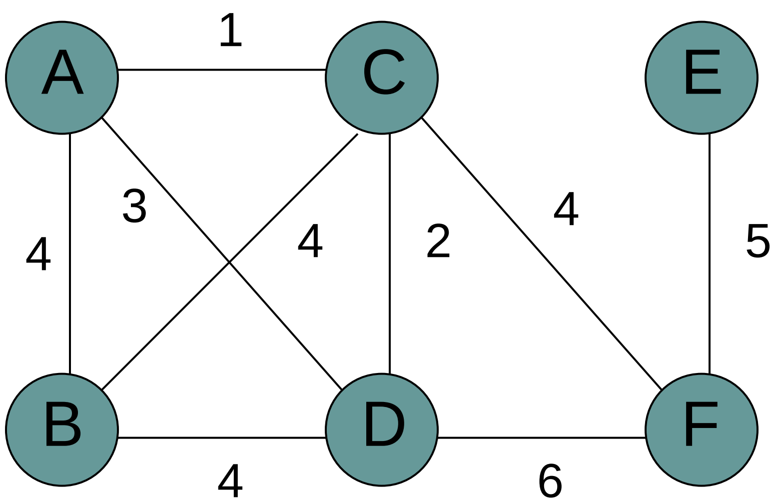

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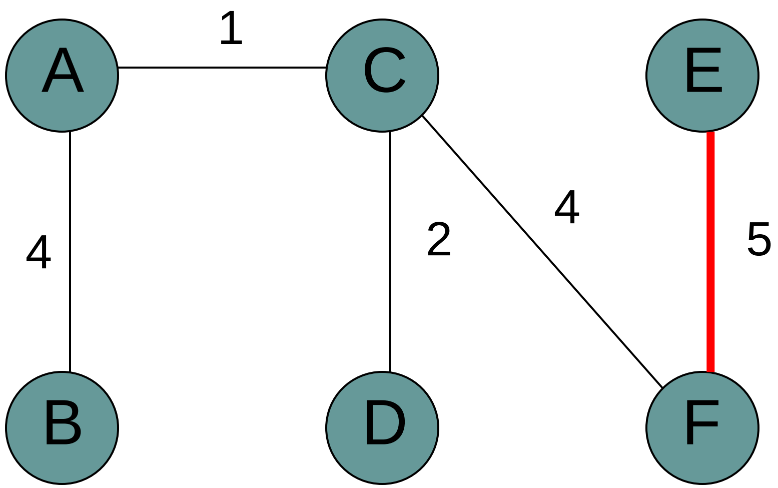

Correctness of Kruskal's

Never adds an edge that connects already connected vertices

Always adds lowest cost edge to connect two sets. By min cut property, that edge must be part of the MST

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        MAKESET(v)
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sort the edges of E by weight
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Running time of Kruskal’s

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|V| calls to MakeSet

O(|E| log |E|)

2 |E| calls to FindSet

|V| calls to Union

Running time of Kruskal’s

Disjoint set data structure

O(|E| log |E|) +

	MakeSet	FindSet |E| calls	Union |V| calls	Total
Linked lists				

Disjoint set

A B C D

Disjoint set: union

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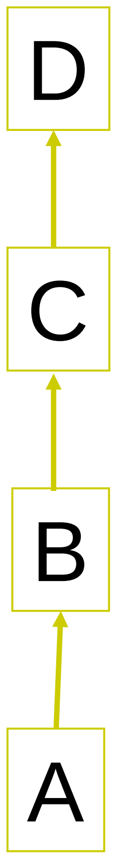
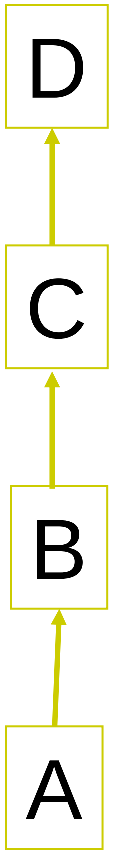


Running time?

Disjoint set: union



O(1)

Disjoint set: find-set

Search each linked list

Disjoint set: find-set

Running time?

Disjoint set: find-set

O(n) -- n = number of things in set

Running time of Kruskal's

Disjoint set data structure

O(|E| log |E|) +

	MakeSet	FindSet |E| calls	Union |V| calls	Total
Linked lists	|V|	O(|V| |E|)	|V|	O(|V||E| + |E| log |E|) O(|V| |E|)
Linked lists + heuristics	|V|	O(|E| log |V|)	|V|	O(|E| log |V|+ |E| log |E|) O(|E| log |E|)

Prim’s algorithm

Start at some root node and build out the MST by adding the lowest weighted edge at the frontier

```
Prim(G, r)
for all v ∈ V
        key[v] ← ∞
        prev[v] ← null
key[r] ← 0
H ← MakeHeap(key)
while !Empty(H)
        u ← Extract-Min(H)
        visited[u] ← true
        for each edge (u, v) ∈ E
                if !visited[v] and w(u, v) < key(v)
                        Decrease-Key(v, w(u, v))
                        prev[v] ← u
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Prim's

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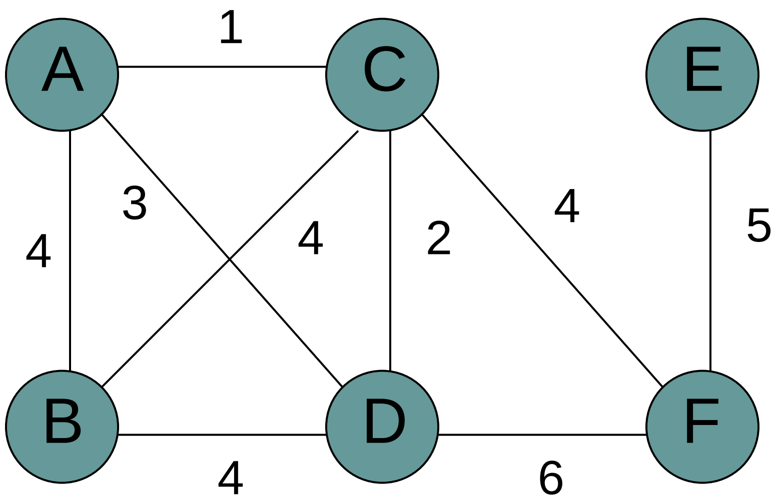

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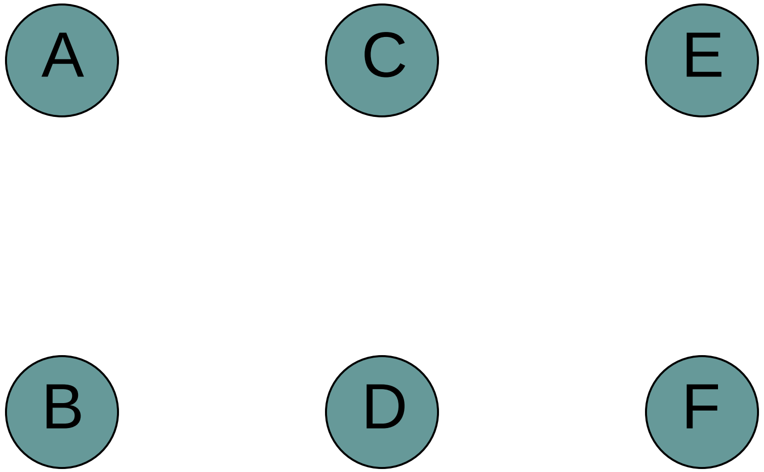

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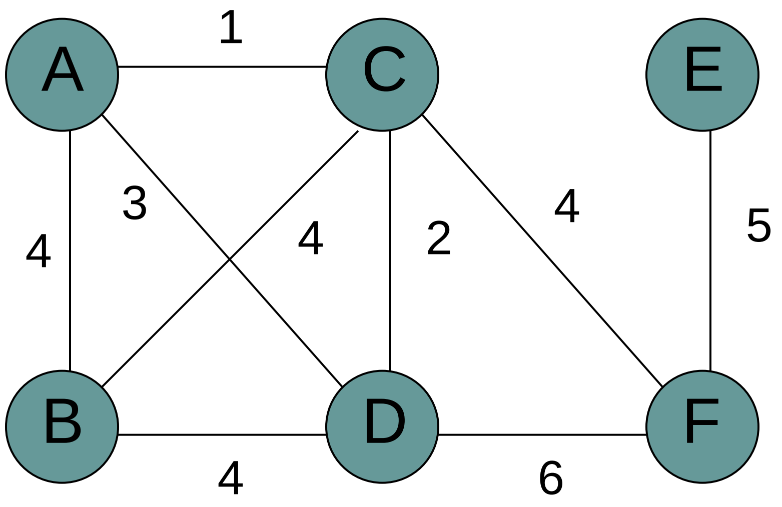

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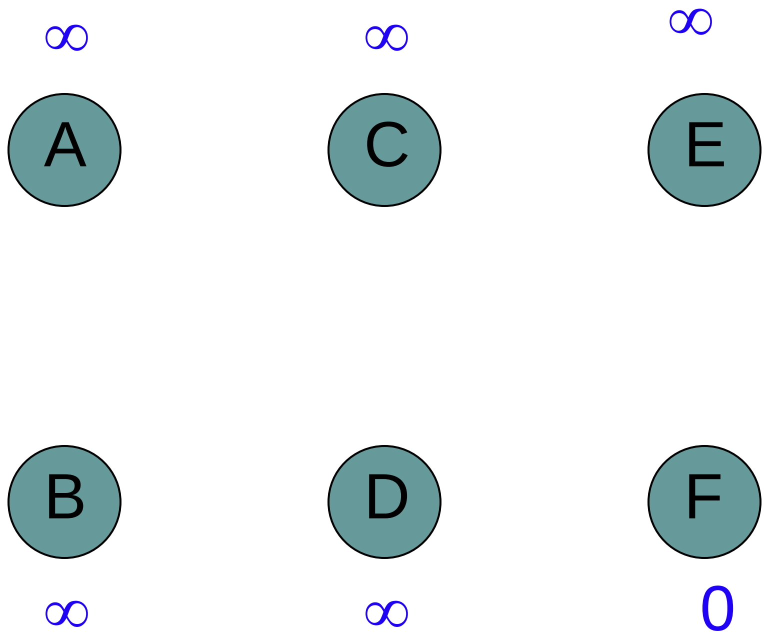

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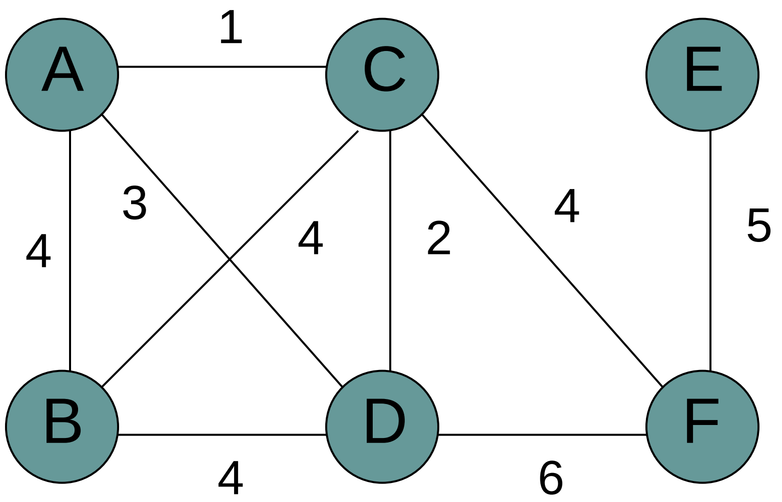

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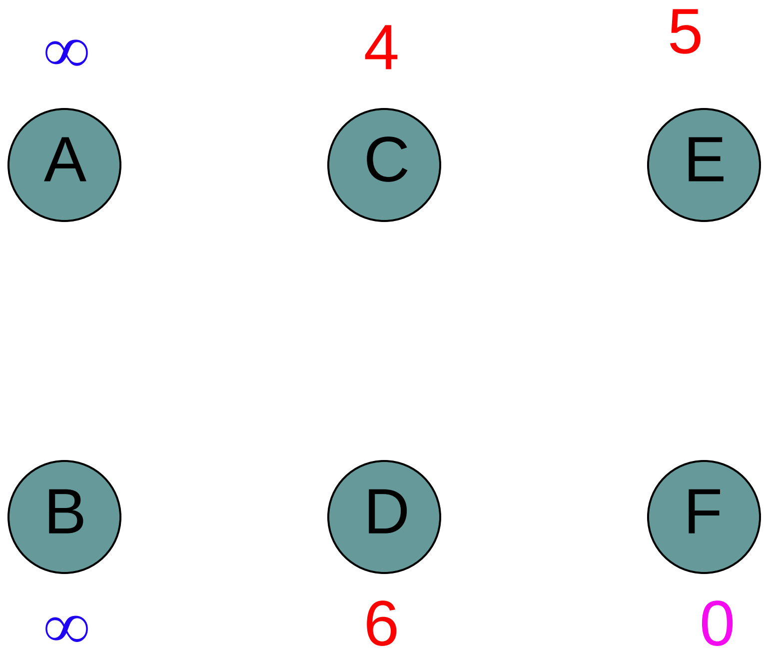

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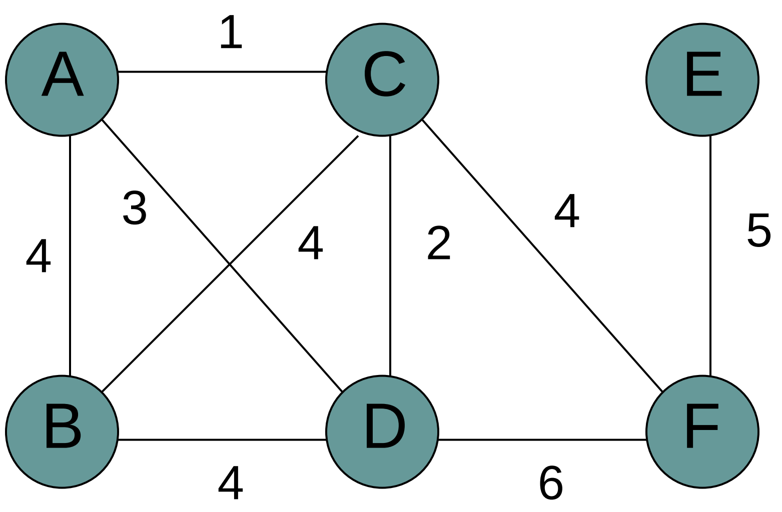

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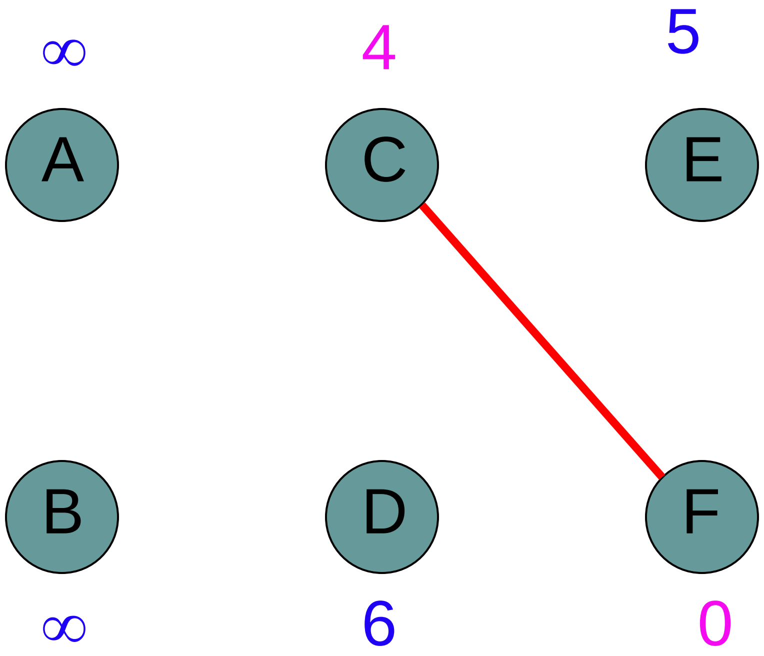

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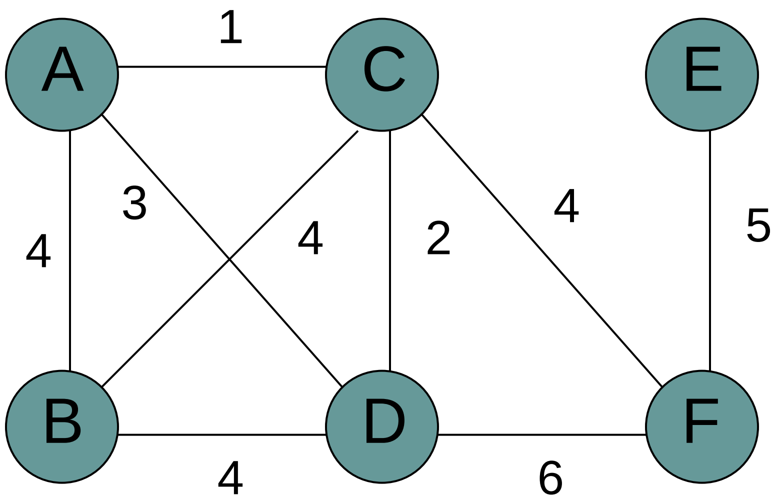

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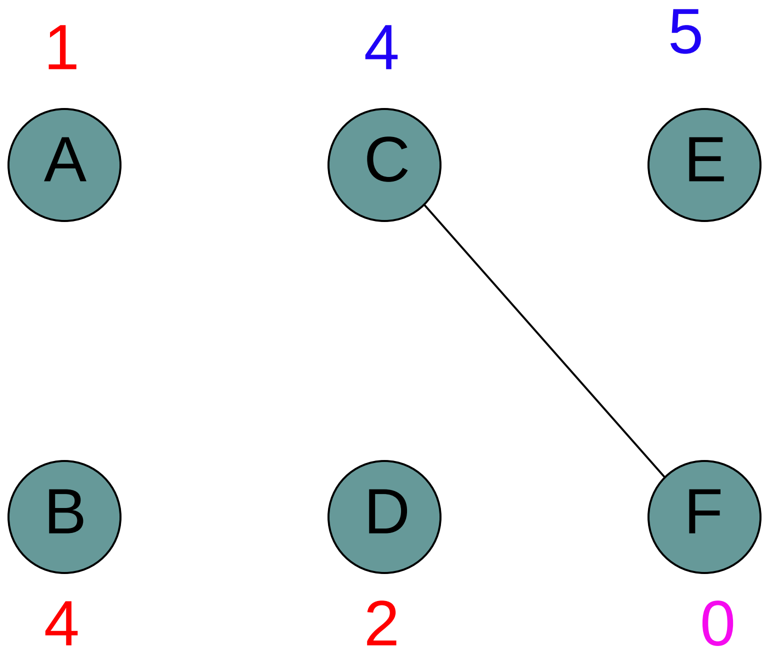

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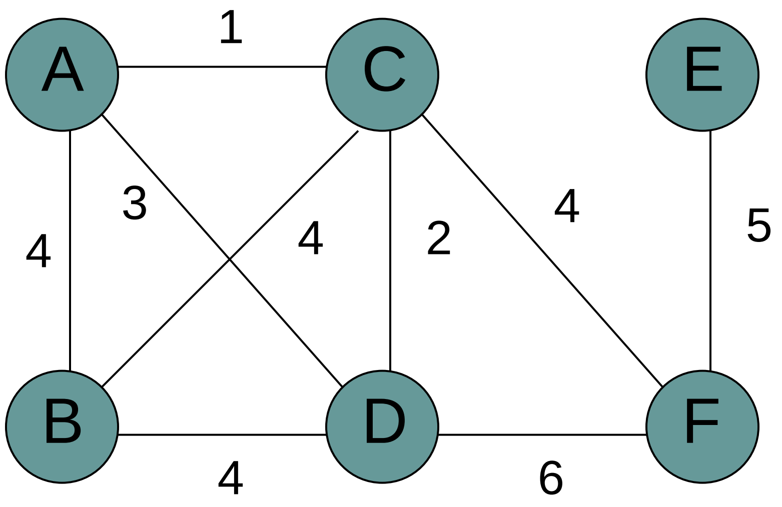

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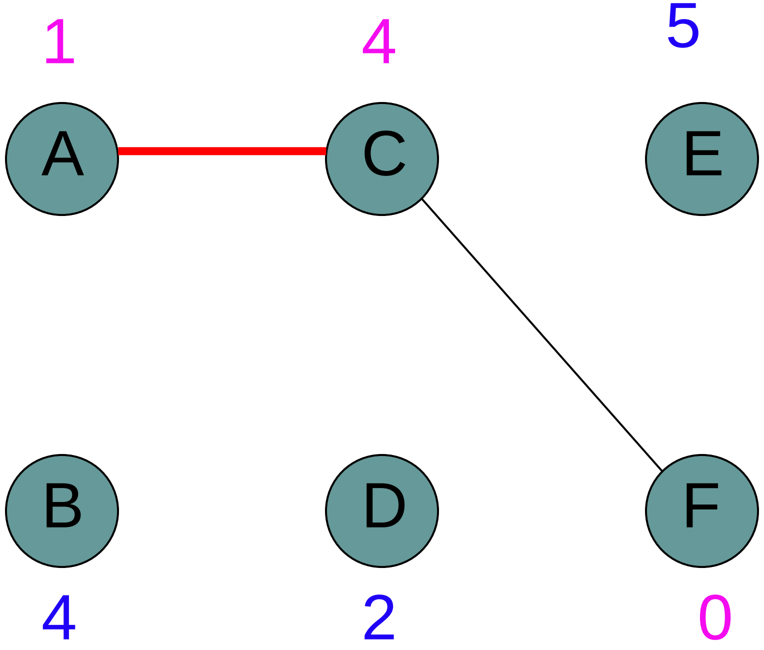

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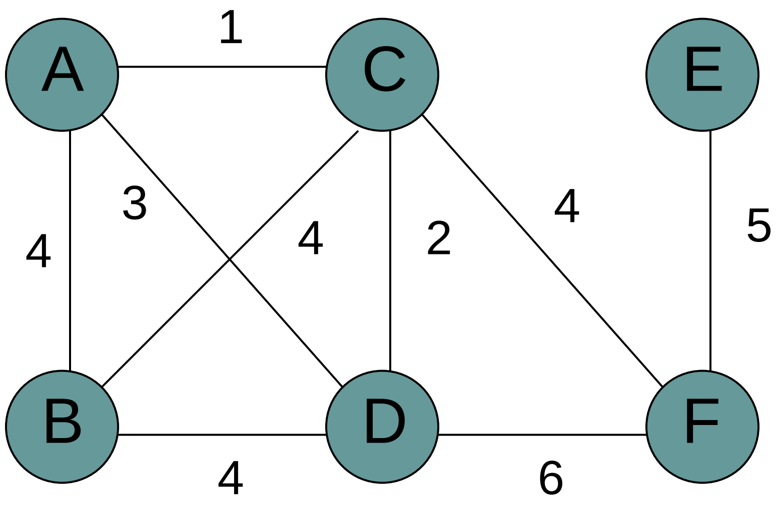

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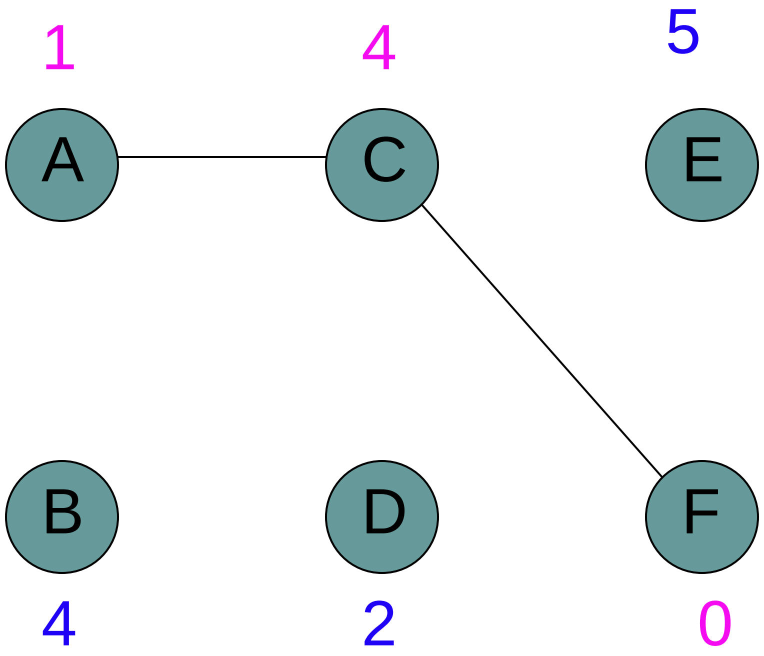

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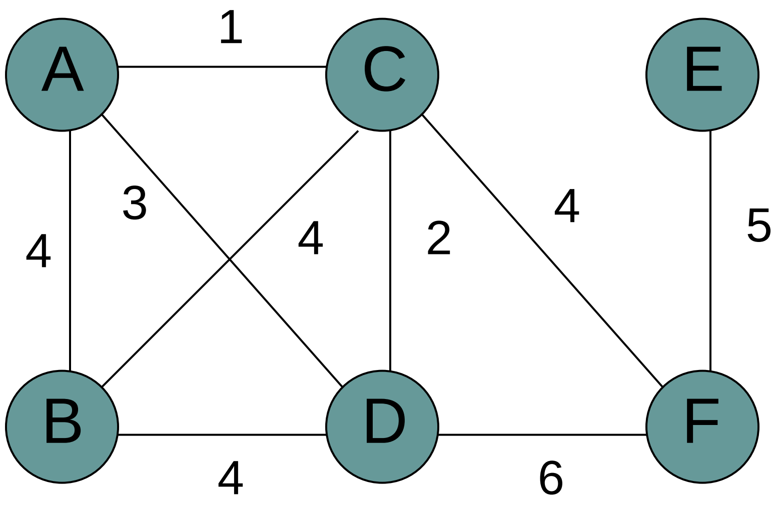

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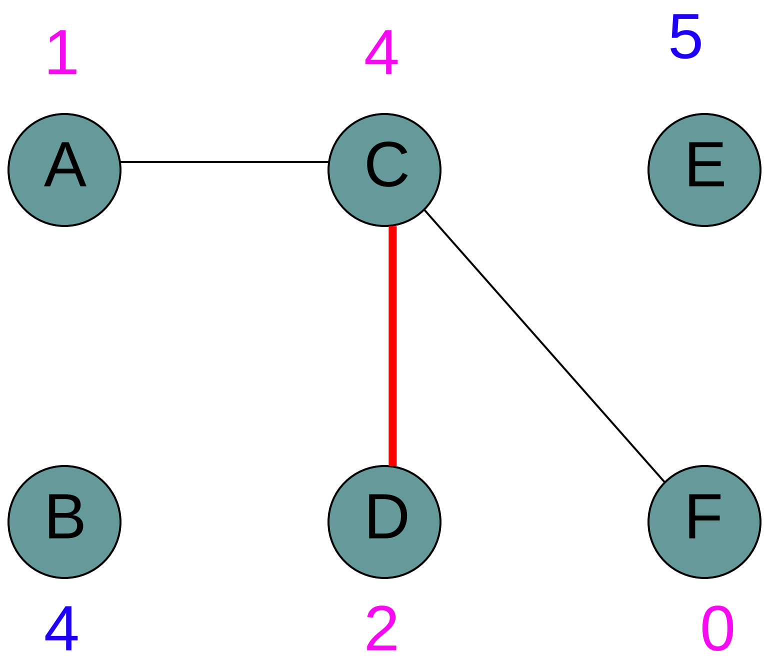

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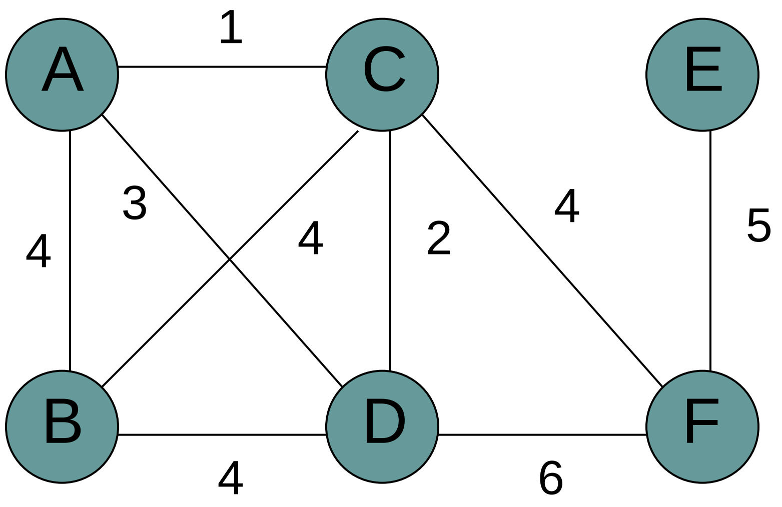

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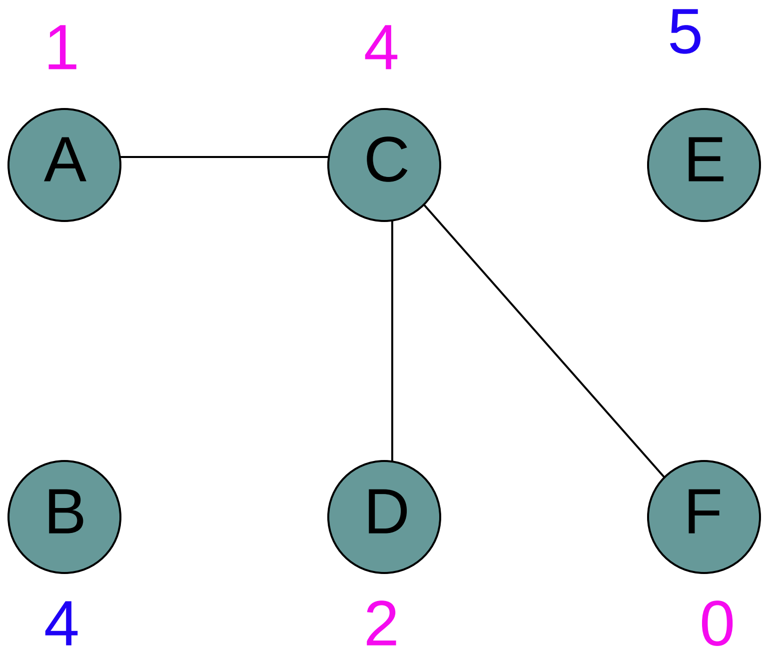

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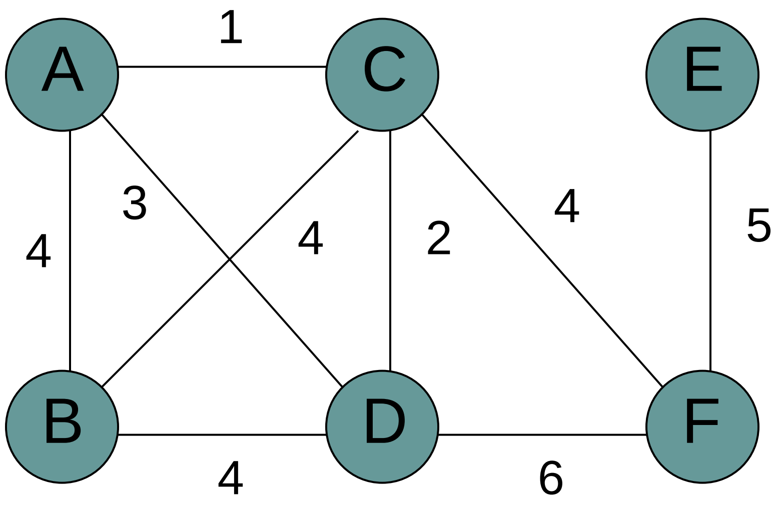

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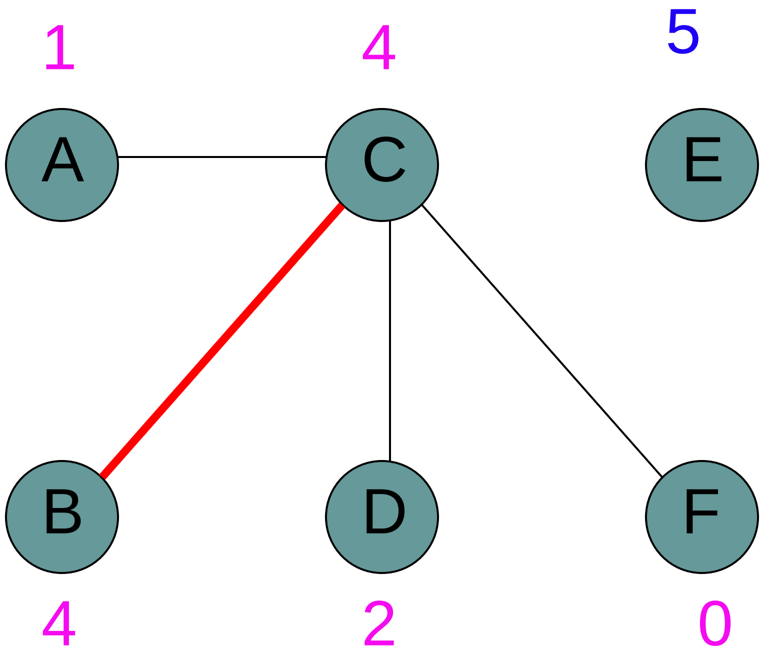

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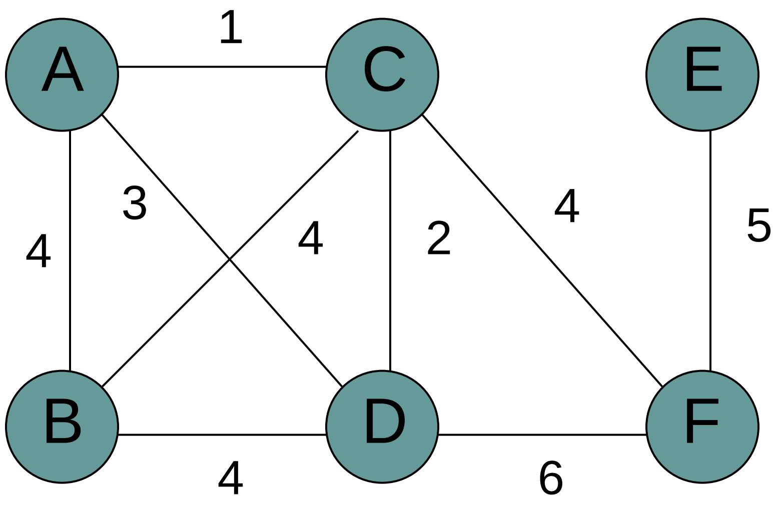

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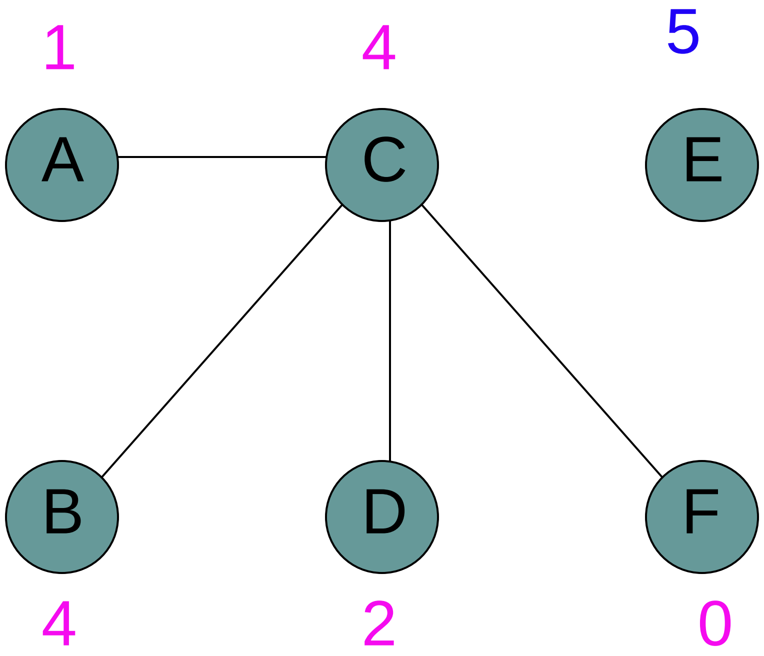

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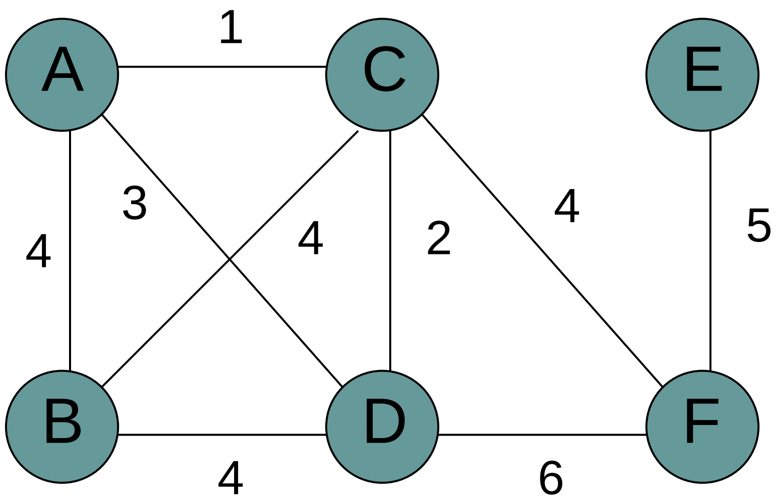

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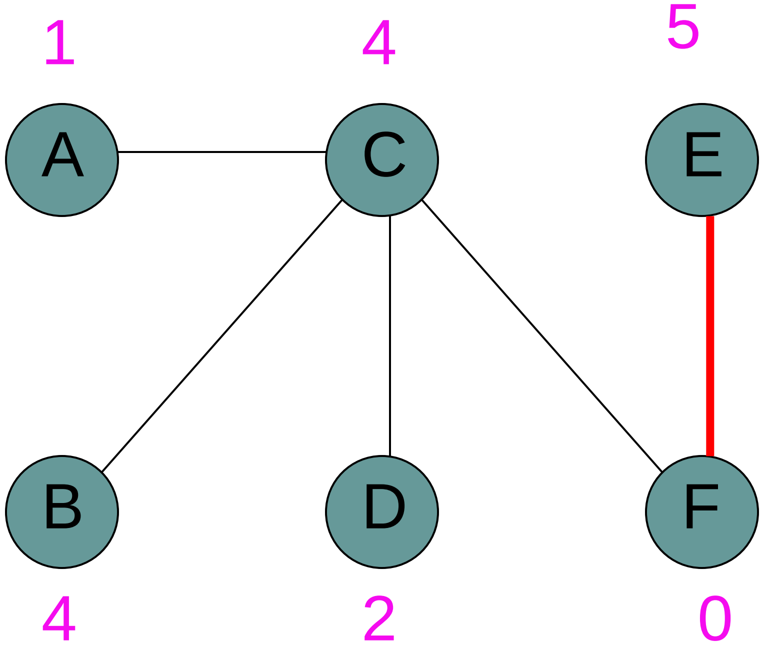

Correctness of Prim's?

Can we use the min-cut property?

- Given a partion S, let edge *e* be the minimum cost edge that **crosses** the partition. *Every* minimum spanning tree contains edge *e*.

Let S be the set of vertices visited so far

The only time we add a new edge is if it's the lowest weight edge from S to V-S

Running time of Prim’s

```
PRIM(G, r)
for all v ∈ V
        key[v] ← ∞
        prev[v] ← null
key[r] ← 0
H ← MAKEHEAP(key)
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                if !visited[v] and w(u, v) < key(v)
                        DECREASE-KEY(v, w(u, v))
                        prev[v] ← u
```

Running time of Prim's

```
PRIM(G, r)
for all v ∈ V
        key[v] ← ∞
        prev[v] ← null
key[r] ← 0
H ← MAKEHEAP(key)
while !Empty(H)
        u ← EXTRACT-MIN(H)
        visited[u] ← true
        for each edge (u, v) ∈ E
                if !visited[v] and w(u, v) < key(v)
                        DECREASE-KEY(v, w(u, v))
                        prev[v] ← u
```

$\Theta(|V|)$

1 call to MakeHeap

|V| calls to Extract-Min

|E| calls to Decrease-Key

Running time of Prim's

	1 MakeHeap	$\lvert V\rvert$ ExtractMin	$\lvert E\rvert$ DecreaseKey	Total
Array	$\theta(\lvert V\rvert)$	$O(\lvert V\rvert^2)$	$O(\lvert E\rvert)$	$O(\lvert V\rvert^2)$
Bin heap	$\theta(\lvert V\rvert)$	$O(\lvert V\rvert \log \lvert V\rvert)$	$O(\lvert E\rvert \log \lvert V\rvert)$	$O((\lvert V\rvert+\lvert E\rvert) \log \lvert V\rvert)$ $O(\lvert E\rvert \log \lvert V\rvert)$
Fib heap	$\theta(\lvert V\rvert)$	$O(\lvert V\rvert \log \lvert V\rvert)$	$O(\lvert E\rvert)$	**$O(\lvert V\rvert \log \lvert V\rvert + \lvert E\rvert)$**

Kruskal's: $O(\lvert E\rvert \log \lvert E\rvert)$

Acknowledgements

- Cormen, T.H., Leiserson, C.E., Rivest, R.L. and Stein, C., Introduction to algorithms. MIT press, 2009
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