Introduction to Risk and Return

Asset Returns

- Asset returns over a given period are often uncertain.
- **Expected return** = best forecast at beginning of period

$$E(r_t) = \frac{E(P_t - P_{t-1}) + E(D_t)}{P_{t-1}}$$

- **Realized return**

$$r_t = \frac{P_t - P_{t-1} + D_t}{P_{t-1}} = \frac{P_t - P_{t-1}}{P_{t-1}} + \frac{D_t}{P_{t-1}}, \qquad \text{(Cap gain+Div. Yield)}$$

- **Risk premium**, or expected excess return

  Risk premium= $E(r_t) - r_f$

Where,

- P_{t-1} is the price at period t−1
- P_t is the price at the end of period
- D_t is the dividend at the end of period
- r_f is the risk free rate

- **Discrete returns (simple returns or holding period return)** are the most commonly used, and represent periodic (e.g. daily, weekly, monthly, etc.) price movements.
- The realized return is called the **holding-period return.**

- Example: You invested in 1 share of Apple (AAPL) for \$95 and sold a year later for \$200. The company did not pay any dividend during that period. What will be the cash return on this investment?

- Rate of Return = (\$200 + 0 - \$95) ÷ 95 = 110.53%

	Stock Price		Cash distribution (dividend)	Return	
	Beginning of the Year (April 9, 2015)	End of the Year (April 8, 2016)		Cash	Rate
Company	A	B	C	D=C+B-A	E=D/A
Duke Energy (DUK)	77.23	79.63	3.30	5.70	7.4%
Emerson Electric (EMR)	58.40	53.84	1.90	-2.66	-24.6%
Sears Holdings (SHLD)	43.24	14.45	-	-28.79	-266.6%
Walmart (WMT)	80.29	68	2.00	-10.29	-212.8%

Multiple-Period Realized Return

- The time series of past realized returns do not explicitly provide investors' original assessments of the probabilities of those returns; we observe only dates and associated holding period returns.
- There are two methods to determine the average return to this realized returns : the arithmetic mean and geometric mean.
  - **Arithmetic Average:**

    If there are n observations

    $$r_a = (r_1 + r_2 + \cdots + r_n) / n$$

    - Not equivalent per-period return because it neglects compounding
    - Useful for forecasting the return next period

- **Geometric Average**

  If there are n observations

  $$r_g = [(1 + r_1)(1 + r_2) \times \cdots \times (1 + r_n)]^{1/n} - 1$$

  - Gives the equivalent per-period return
  - It takes into account the compounding that occurs from period to period

- “What was the average of the yearly rates of return?”
  - The arithmetic average rate of return answers the question
- “What was the growth rate of your investment?”
  - The geometric average rate of return answers the question

- **Appropriate Average Calculation**

What annual rate of return can we expect for next year?	The arithmetic average rate of return calculated using annual rates of return.
What annual rate of return can we expect over a multiyear horizon?	The geometric average rate of return calculated over a similar past period.

- Example: Compute the arithmetic and geometric average for the following stock.

Year	Annual Rate of Return	Value of the stock
0		\$25
1	40%	\$35
2	-50%	\$17.50

- Arithmetic Average=(40+(-50)) ÷ 2 = **-5%**
- Geometric Average = $[(1+R_1) \times (1+R_2)]^{1/2} - 1$

  $= [(1.4) \times (1+(-.5))]^{1/2} - 1$

  = **-16.33%**

- Internal rate of return, IRR
  - Return if one can re-invest cash-flows at this rate
  - “Dollar-weighted average”
  - The IRR in the rate that makes:

initial price = present value of future net profits

$$P(0) = \sum_{t=1}^{\infty} \frac{C(t)}{(1 + IRR)^t}$$

Expected rate of return

- NPV and other valuation techniques need cost of capital.
- The cost of capital is simply the return expected by those who provide capital for the business.
  - Opportunity cost
  - Required rate of return
  - Risk-adjusted discount rate
  - Determined by “the market”
- **Expected return** is what the investor expects to earn from an investment in the future.
- Since financial resources are finite, the expected rate of return will be higher for riskier assets than for safer assets.
- Expected rate of return compensates for time-value and risk.
- Expected rate of return = Riskless rate + Risk premium

- Return on an asset is a *random variable, characterized by*
  - *all possible outcomes, and*
  - *probability of each outcome (state)*
- **Example**. The S&P 500 index and the stock of MassAir, a regional airline company, has the following returns:

State	1	2	3
Probability	0.20	0.60	0.20
Return on S& P 500(%)	-5	10	20
Return on MassAir (%)	-10	10	40

- The variation in asset returns can be substantial.

Example

State of the economy	Probability of the state of the economy	End selling price of the stock	Beginning price of the stock	Cash return	Percentage Rate of return	Product=rate of return X Prob. of State of economy
A	B	C	D	E=C-D	F=E/D	G=B X F
Recession	20%	$9,000	$10,000	-$1,000	-10%	-2.0%
Moderate Growth	30%	$11,200	$10,000	$1,200	12%	3.6%
Strong Growth	50%	$12,200	$10,000	$2,200	22%	11.0%
Sum	100%				**Expected returns**	**12.6%**

- The probabilities assigned to the three economic conditions have to be determined subjectively, which requires management to have thorough understanding of both the investment cash flows and the general economy.

Continuously Compounded Returns

- **Log returns** are often used in academic research and financial modelling.
- Log returns are calculated as the difference between the log of two prices.

$$R_t = ln(P_t) - ln\ (P_{t-1}) = ln\left(\frac{P_t}{P_{t-1}}\right)$$

This can be written as, $R_t = ln\left(\frac{P_t}{P_{t-1}}\right) = ln(\,1 + r_t)$

$where, r_t$, is the simple growth rate in prices between $t-1$ and $t\left(r_t = \frac{P_t - P_{t-1}}{P_{t-1}}\right)$

$or,\ \exp(\,R_t) = 1 + r_t = \frac{P_t}{P_{t-1}}$

or, $\quad P_t = \exp(\,R_t)P_{t-1}$

so R_t is the continuously compounded growth rate in prices between $t\ -\ 1$ and t.

- Several benefits of using *log returns*, both theoretic and algorithmic.
  - Log-normality: if we assume that prices are distributed log normally, then $\ln(1 + r_t)$ is conveniently normally distributed.
  - Approximate raw-log equality: when returns are very small, the following approximation ensures they are close in value to raw returns: $\ln(1 + r_t) \approx r_t$
  - Time-additivity: the compound return over *T* periods is merely the difference in log between initial and final periods.

$$\sum_t \ln(1 + r_t) = \ln(1 + r_1) + \cdots + \ln(1 + r_T) = \ln(p_T) - \ln(p_0)$$

- *log returns* assume continuous compounding.
- Log returns *aggregate across time*, while discrete returns *aggregate across assets.*

APR and EAR

- **Annual percentage rate (APR)**
  - APR is the yearly rate of interest that an individual must pay on a loan, or that they receive on a deposit account.
- Annual percentage rate (APR) = Quoted rate

  = interest per period * number of periods per year

- **Effective Annual Rate (EAR):** If interest is compounded m times a year and quoted rate is r:

$$\text{Effective annual rate (EAR)} = [(1 + r/m)]^m - 1$$

- *The r/m is per-period rate for each period.*
- What happens as m gets big?

  In the limit as m → ∞, interest is 'continuously compounded'

  EAR = e^r – 1

Example

- An internet company, e-Money, is offering a money market account with an APR of 5.25%. What is the effective annual interest rate offered by e-Money if the compounding interval is

  (a) annual

  (b) monthly

  (c) daily

  (d) continuously?

- We know that, EAR $= [(1+r/m)]^m - 1$
  and for continuously compounding, EAR = $e^r - 1$

- The effective annual interest rate for an APR of 5.25% is
  (a) 5.2500% with annual compounding (m=1)
  (b) 5.3782% with monthly compounding (m=12)
  (c) 5.3899% with daily compounding (m=365)
  (d) 5.3903% with continuous compounding ($m \to \infty$)

- Example: Which loan is cheapest?
  - 15%, compounded daily
  - 15.5%, compounded quarterly
  - 16%, compounded annually

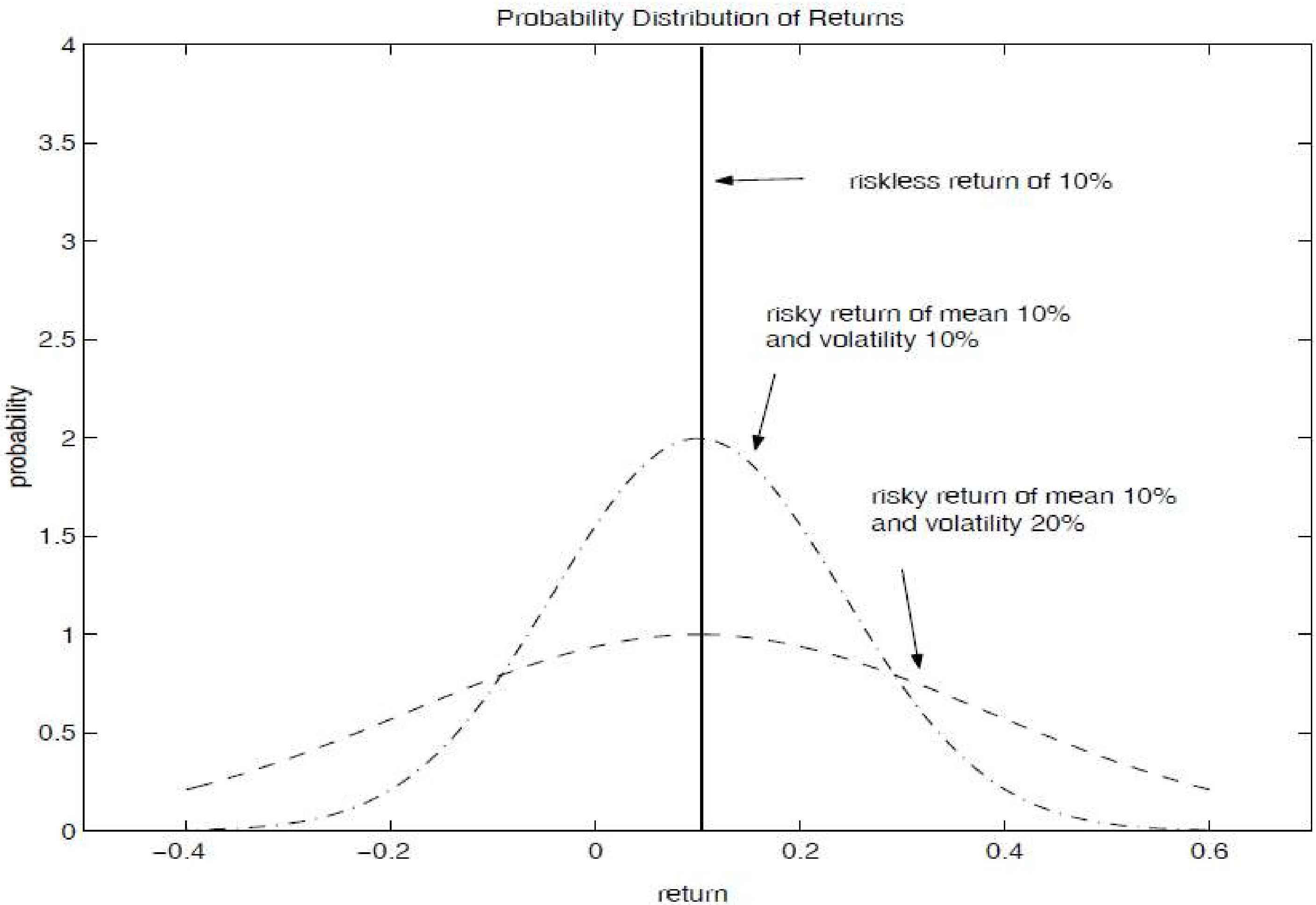
Introduce risk into the valuation process

- Questions:
  1. How do we define and measure risk?
  2. How to estimate the required rate of return for a given level of risk?
  3. How are risks of different assets related to each other?
  4. How do you translate this risk measure into a risk premium?
- Related questions:
  – How risky are stocks and what have their returns been historically?
  – Is the stock market “efficient”?
  – How can we gauge the performance of portfolio managers?

Defining Risk

- Risk, refers to the likelihood that we will receive a return on an investment that is different from the return we expected to make.
- Thus, risk includes not only the bad outcomes, i.e, returns that are lower than expected, but also good outcomes, i.e., returns that are higher than expected.
- Example. Moments of return distribution. Consider three assets:

State	Mean	Standard Deviation
r_0 (%)	10.0	0.0
r_1 (%)	10.0	10.0
r_2 (%)	10.0	20.0



- Between Asset 0 and 1, which one would you choose?
- Between Asset 1 and 2, which one would you choose?
- Investors care about expected return and risk.

Risk vs. Uncertainty

- Risk is the possibility of loss. Suppose you buy HDFC at Rs 675. Your risk is that the price may come down after you buy the stock.
- Uncertainty does not always involve losses. Suppose you are asked whether the Reserve Bank of India will cut interest rates. You cannot answer the question based on RBI's past decisions. The reason is that interest rate changes are a function of several variables, many of which are uncertain.
- Risk is measurable. Based on the historical price movement, you may conclude that HDFC will decline by not more than 20% in one month. Uncertainty is subjective and cannot be measured.

Risk and return

- The spread of the actual returns around the expected return is measured by the variance or standard deviation of the distribution.
- The greater the deviation of the actual returns from expected returns, the greater the variance.
- A common measure of equity risk is standard deviation of returns . The standard deviation is the square root of the variance.
- We use the standard deviation simply because that's in the same units, whereas the variance is in units squared per year.

Statistics review

Let random variable $= x$

- <u>Population parameters</u>

Mean $= \mu \equiv E[x_i]$; Variance $= \sigma^2 \equiv E[(x_i - \mu)^2]$

Standard deviation $= \sigma = \sqrt{E[(x_i - \mu)^2]}$

- Sample estimator of N observations:

Sample mean $= \hat{\mu} \equiv \frac{1}{N}\sum_{i=1}^{N} x_i$

Sample variance $= \hat{\sigma}^2 \equiv \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \hat{\mu})^2$

Sample standard deviation $= \hat{\sigma}$

- The mean μ, variance σ^2 and standard deviation σ are the theoretical or population values of the underlying securities.
- The historical estimates are given by the sample mean $\hat{\mu}$, the sample variance $\hat{\sigma}^2$, the sample standard deviation $\hat{\sigma}$.
- Expected return is the best forecast at beginning of period.
- Realised return = dividend yield + capital gain.
- Risk premium is the expected excess return.

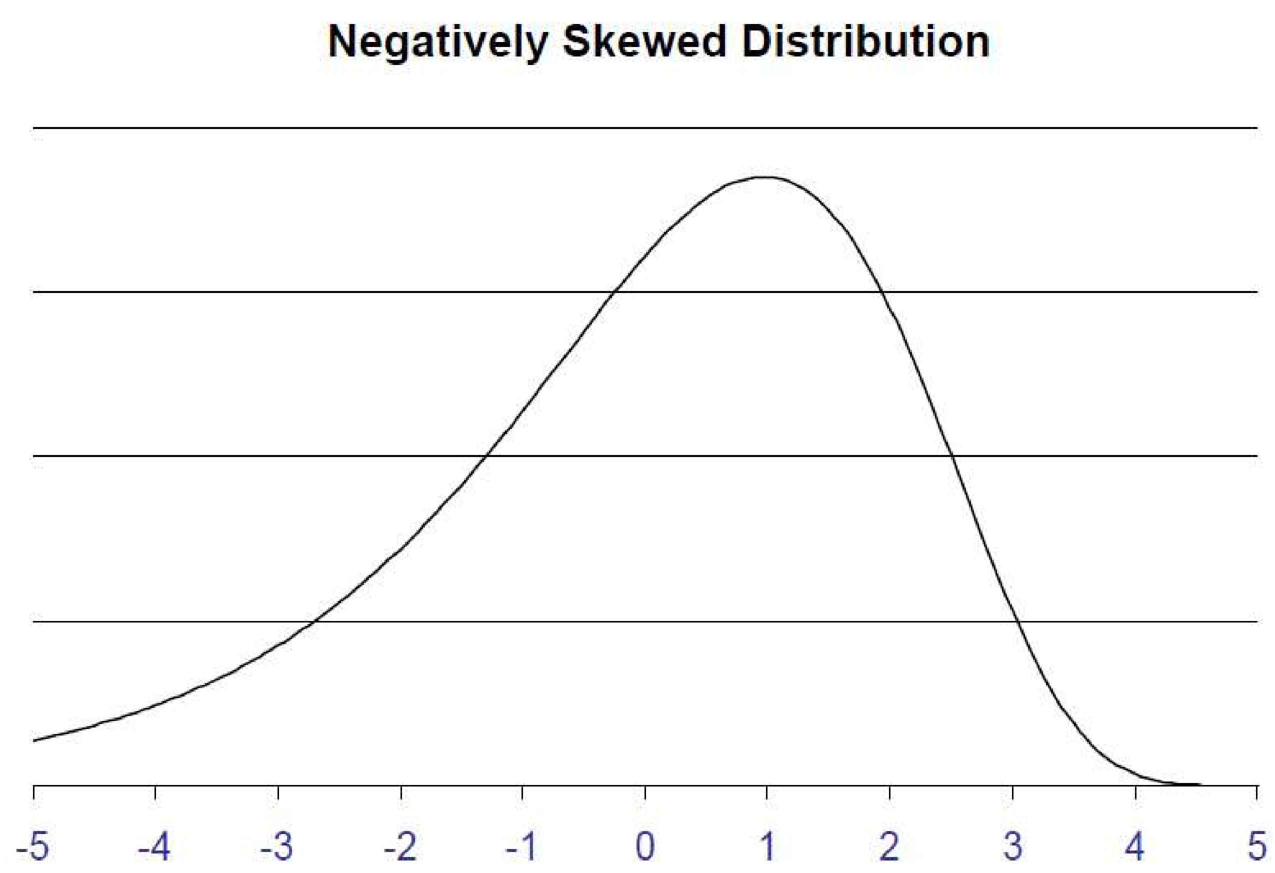
- Standard Deviation (Volatility)
  - Variance = σ
  - Often represented in mathematical notation as σ, or referred to as volatility
  - An investment with higher σ is viewed as a higher risk investment
  - Measures the dispersion of returns
- Scaling Volatility
  - Volatility scales with the square root of time
  - We can normally assume 252 trading days in a given year, and 21 trading days in a given month

$$\sigma_{annual} = \sigma_{daily} * \sqrt{252}$$
$$\sigma_{monthly} = \sigma_{daily} * \sqrt{21}$$

Other Statistics

- **Median**
  - 50th percentile: prob (x < median) = 0.50

- **Skewness**
  - Is the distribution symmetric?
  - Negative: big losses are more likely than big gains
  - Positive: big gains are more likely than big losses


Negatively Skewed Distribution
-5
-4
-3
-2
-1
0
1
2
3
4
5

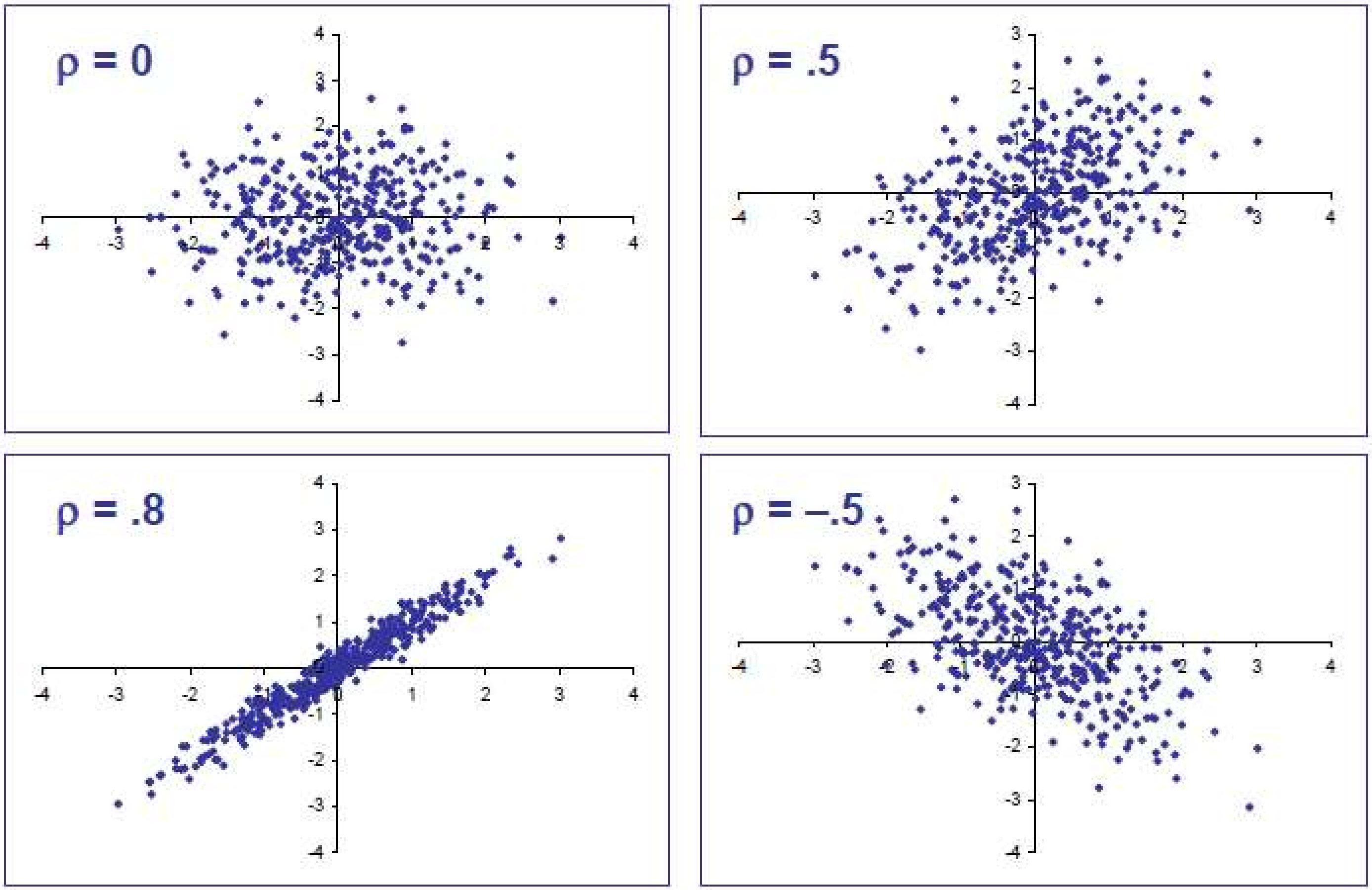
- **Two random variables**

How do x and y covary? Do they typically move in the same direction opposite each other?

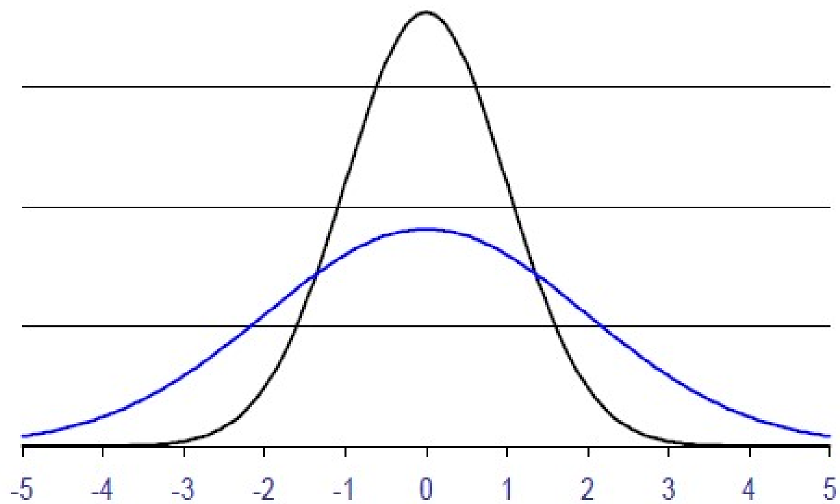
- **Covariance:** $\sigma_{x,y} = E[(x - \mu_x)(y - \mu_y)]$
  - If$\sigma_{x,y} > 0$, then x and y tend to move in the same direction
  - If$\sigma_{x,y} < 0$, then x and y tend to move in opposite directions

- **Correlation** : $\rho_{xy} = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$; $\quad -1 \leq \rho_{xy} \leq 1$

Examples of Correlation Between Two Random Variables-



- **Value-at-Risk (VaR)**
  - The VaR is the measure of loss associated with extreme negative returns.
  - How bad can things get over the next day (or week)?
  - Also quantile of a distribution. The quantile, q, of a distribution is the value below which lies q % of the possible values of that distribution.
  - 1st or 5th percentile: prob $(x < VaR) = 0.01$ or 0.05
  - For 1% VaR, ‘We are 99% certain that we won’t lose more than $Y in the next 24 hours’



- **Normal Distribution**
  - Bell-shaped, symmetric
  - A model of randomness
  - Central Limit Theorem
  - $x \sim N(\mu, \sigma^2)$
    x is normally distributed with mean μ and variance σ^2
  - 'Standard normal' mean 0 and variance 1 [or $N(0,1)$]

- **Confidence Intervals**
  If R is normally distributed, then …
  - 68% of observations fall within +/–1.00std. deviations from mean
  - 90%of observations fall within +/–1.65std. deviations from mean
  - 95%of observations fall within +/–1.96std. deviations from mean
  - 99%of observations fall within +/–2.58std. deviations from mean

- **Estimating the mean**
  - Given a sample $x_1, x_2, \ldots, x_N$
  - Don't know μ, σ^2

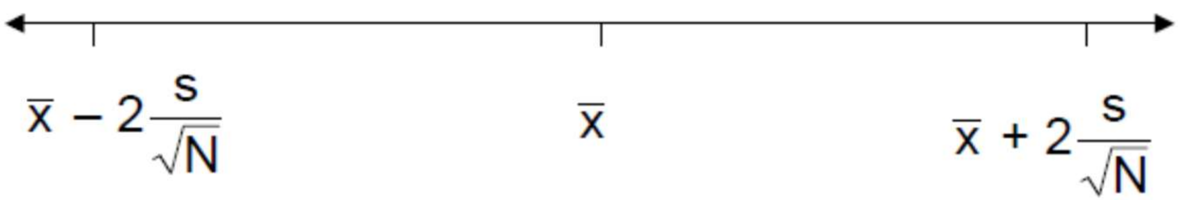
    $\Rightarrow$ estimate μ by sample average $\bar{x}$

    estimate σ^2 by sample variance s^2

- **How precise is $\boldsymbol{\bar{x}}$?**

  std dev $(\bar{x}) \approx s/\sqrt{N}$

  95% confidence interval for μ



- **Example:** From 1946 – 2001, the average return on the U.S. stock market was 0.63% monthly above the Tbill rate, and the standard deviation of monthly returns was 4.25%. Using these data, how precisely can we estimate the risk premium?

- Sample: $\bar{x}$ = 0.63%, s = 4.25%, N =56 years= 672 months

  Std error=std dev $(\bar{x})$ = 4.25 / $\sqrt{672}$ = 0.164%

  95% confidence interval

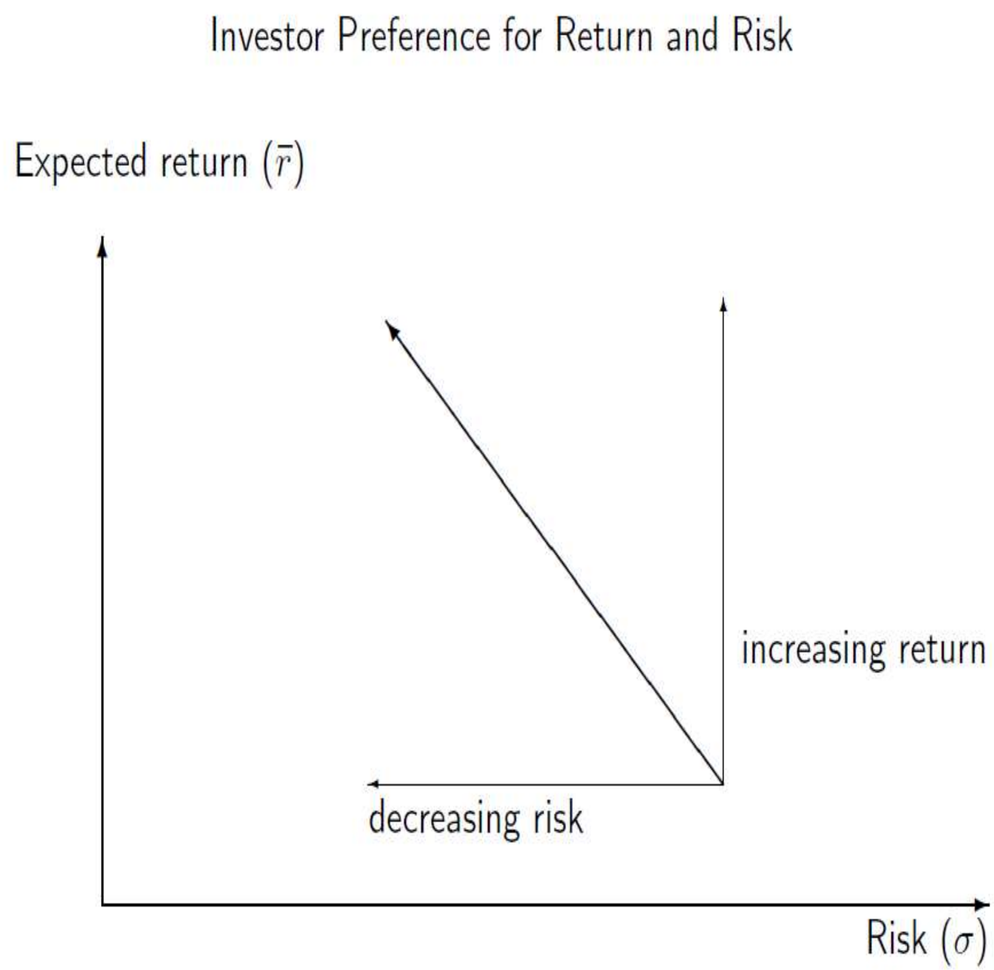
  Lower bound = 0.63 – 2 × 0.164 = 0.30%

  Upper bound = 0.63 + 2 × 0.164 = 0.96%

  Annual rate (× 12): 3.6% < μ < 11.5%

Investor preferences for return and risk

- Key assumptions on investor preferences
  1. Higher mean in return is preferred:
  2. Higher standard deviation (SD) in return is disliked:
  3. Investors care only about mean and SD (or variance).
- Under 1-3, standard deviation (SD) gives a measure of risk.



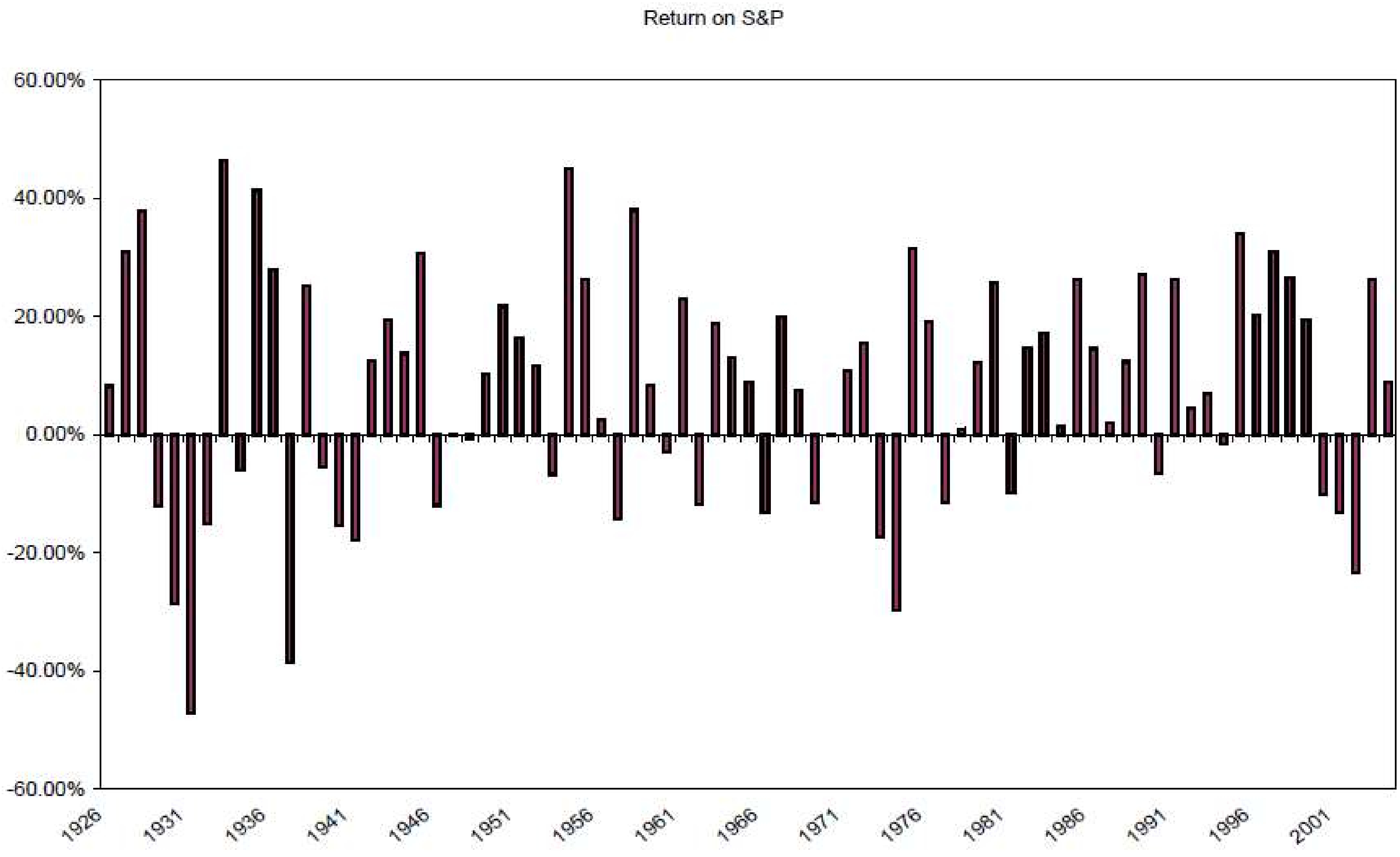
Empirical Properties of Stock Returns

- **Time-series behaviour**
  - How risky are stocks?
  - How risky is the overall stock market?
  - Can we predict stock returns?
  - How does volatility change over time?
- **Cross-sectional behaviour**
  - What types of stocks have the highest returns?
  - What types of stocks are riskiest?
  - Can we predict which stocks will do well and which won't?

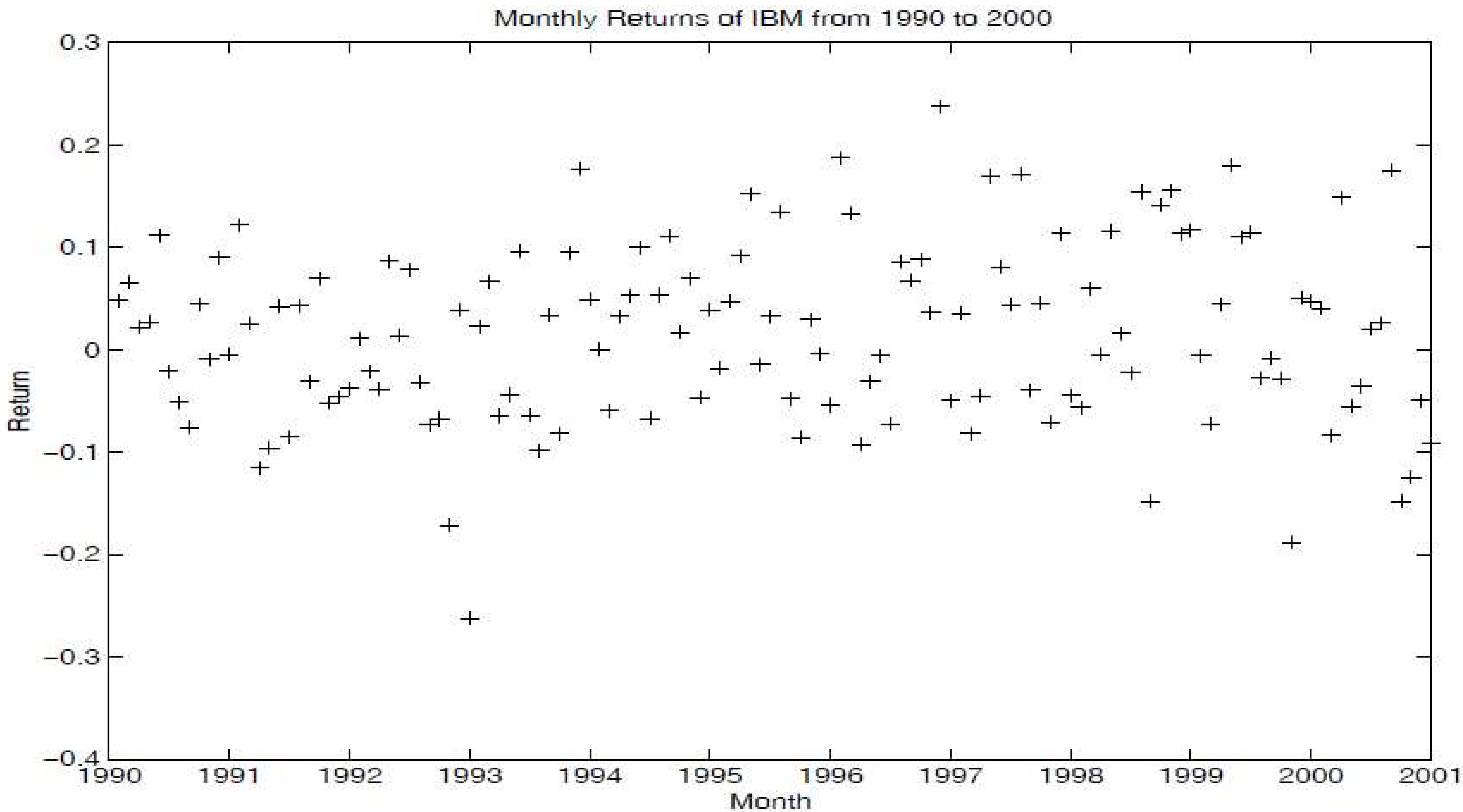
- **What properties should stock prices have in “Efficient” markets?**
  - Stock market prices are random, unpredictable
  - Prices should react quickly and correctly to news
  - Investors cannot earn abnormal, risk-adjusted returns (or at least it shouldn’t be easy)

Historical Return and Risk

- Annual Returns - S&P 500 Index (1926 – 2004)



- Monthly Returns - IBM (1990 – 2000)

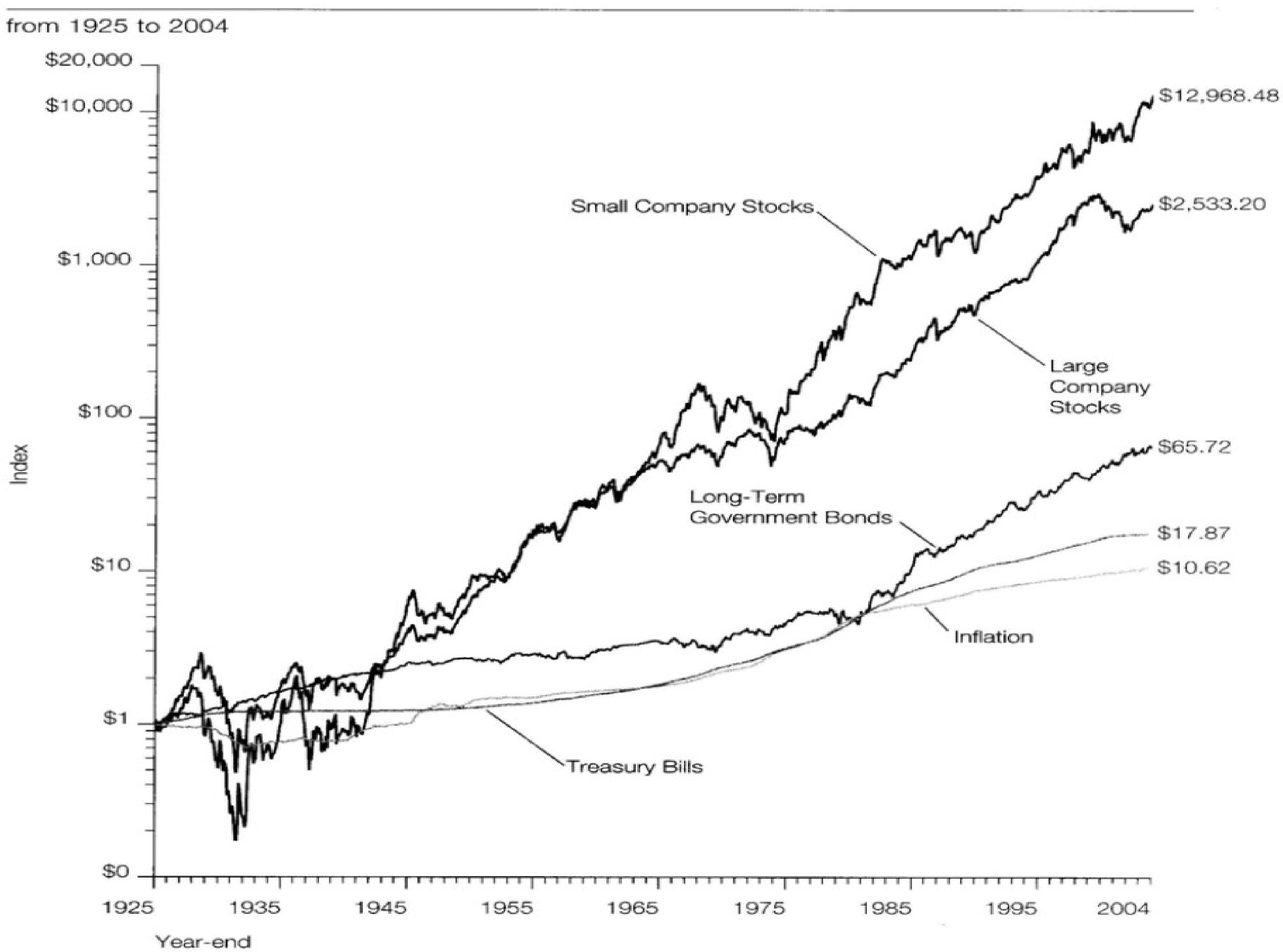


- Average Annual Total Returns from 1926 to 2005 (USA)

Asset	Mean (%)	StD (%)
T-bills	3.8	3.1
Long term T-bonds	5.8	9.2
Long term corp. bonds	6.2	8.5
Large stocks	12.3	20.2
Small stocks	17.4	32.9
Inflation	3.1	4.3

Return indices of investments in the U.S. capital markets

Wealth Indices of Investments in the U.S. Capital Markets
Year-End 1925 = $1.00



Statistical review

- Consider two random variables: *x and y*

State	1	2	3	...	n
Probability	p_1	p_2	p_3	...	p_n
Value of x	x_1	x_2	x_3	...	x_n
Value of y	y_1	y_2	y_3	...	y_n

where $\sum_{i=1}^{n} p_i = 1$

1. Mean: The expected or forecasted value of a random outcome

$$E[x] = \bar{x} = \sum_{j=1}^{n} p_j \, x_j$$

2. Variance: A measure of how much the realized outcome is likely to differ from the expected outcome.

$$Var[x] = \sigma_x^2 = E[(x - \bar{x})^2] = \sum_{j=1}^{n} p_j . (x_j - \bar{x})^2$$

- Standard Deviation (Volatility):

$$SD[x] = \sigma_x = \sqrt{\mathrm{var}[x]}$$

3. Skewness: A measure of asymmetry in distribution.

$$\mu_3[x] = \sqrt[3]{E[(x - \bar{x})^3]/\sigma_x}$$

4. Kurtosis: A measure of fatness in tail distribution.

$$\mu_4[x] = \sqrt[4]{E[(x - \bar{x})^4]/\sigma_x}$$

- **Covariance and Correlation**

1. Covariance: A measure of how much two random outcomes "vary together".

$$Cov[x, y] = \sigma_{xy} = E[(x - \bar{x})(y - \bar{y})] = \sum_{j=1}^{n} p_j \,.\, (x_j - \bar{x})(y_j - \bar{y})$$

2. Correlation: A standardized measure of co-variation.

$$Cor[x, y] = \rho_{xy} = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$$

- Note:
  a) *ρ_{xy} must lie between -1 and 1.*
  b) The two random outcomes are
     - *Perfectly positively correlated if ρ_{xy} =+1*
     - *Perfectly negatively correlated if $\rho_{xy} = -1$*
     - *Uncorrelated if $\rho_{xy} = 0$.*
  c) If one outcome is certain, then $\rho_{xy} = 0$.

Example

- The S&P 500 index and the stock of MassAir, a regional airline company, give the following returns:

State	1	2	3
Probability	0.20	0.60	0.20
Return on S& P 500(%)	-5	10	20
Return on MassAir (%)	-10	10	40

- Suppose that random variables *x and y are the returns on S&P* 500 index and MassAir, respectively.

1. Expected Value:

$$\bar{x} = (0.2)(-0.05) + (0.6)(0.10) + (0.2)(0.20) = 0.09$$

$$\bar{y} = 0.12$$

2. Variance:

$$\sigma_x^2 = (0.2)(-0.05-0.09)^2 + (0.6)(0.10-0.09)^2 + (0.2)(0.20-0.09)^2$$

$$= 0.0064$$

$$\sigma_y^2 = 0.0256$$

3. Standard Deviation (SD):

$$\sigma_{\mathrm{x}} = \sqrt{0.0064} = 8.00\%$$

$$\sigma_{\mathrm{y}} = 16.00\%$$

With mean and SD:

$\bar{x} = 0.09, \sigma_x = 0.08,$

$\bar{y} = 0.12, \sigma_y = 0.16.$

4. Covariance:

$$\sigma_{xy} = (0.2)(.0.05.0.09)(.0.10.0.12) + (0.6)(0.10.0.09)(0.10.0.12) + (0.2)(0.20.0.09)(0.40.0.12) = 0.0122.$$

5. Correlation:

$$\rho_{xy} = \frac{0.0122}{(0.08)(0.16)} = 0.953125.$$

Computation Rules

Let a and b be two constants

$i)$ $E[ax] = aE[x]$

$ii)$ $E[ax + by] = aE[x] + bE[y]$

$iii)$ $E[xy] = E[x].E[y] + Cov[x, y]$

$iv)$ $Var[ax] = a^2Var[x] = a^2\sigma_x^2$

$v)$ $Var[ax + by] = a^2\sigma_x^2 + b^2\sigma_y^2 + 2ab\sigma_{xy}$

$vi)$ $Cov[x + y, z] = Cov[x, z] + Cov[y, z]$

$vii)$ $Cov[ax, by] = abCov[x, y] = ab\sigma_{xy}$