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Section :- 'C'

Q1 $A = \{A, R, Y, N\}$ | $B = \{S, H, A\}$ | $n = 2+2+4+7$ | $C = A \cup B = \{A, R, Y, N, S, H\}$

$\boxed{|A| = 4}$ | $\boxed{|B| = 3}$ | $\boxed{n = 15}$ | $\boxed{|C| = 6}$

$D = A \times B$

$\boxed{|D| = |A| \cdot |B| = 12}$

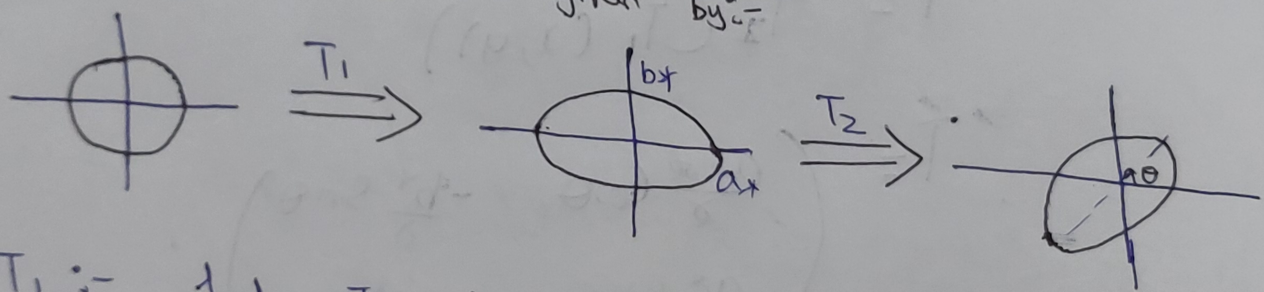
$\theta = |C| + |D| + n$

$\theta = 6 + 12 + 15$

$\boxed{\theta = 33^\circ}$

$$\left.\begin{aligned} r^* &= |C| + 1 = 7 \\ a^* &= |D| = 12 \\ b^* &= |C| = 6 \end{aligned}\right\}$$

Linear Transformation would be composition of Two Transformation given by:-



T_1 :- Let T_1 be linear transformation from $x^2 + y^2 = r_*^2$ to $\dfrac{x^2}{a_*^2} + \dfrac{y^2}{b_*^2} = 1$

$T_1 : \mathbb{R}^2 \to \mathbb{R}^2$

$T_1(1,0) = \left(\frac{a^*}{r}, 0\right)$

$T_1(0,1) = \left(0, \frac{b^*}{r}\right)$

$\therefore \boxed{T_1 = \begin{pmatrix} \frac{a^*}{r} & 0 \\ 0 & \frac{b^*}{r} \end{pmatrix}}$

$\boxed{T_1(x,y) = \left(\frac{a^* x}{r}, \frac{b^* y}{r}\right)}$

$\underline{T_2}$:- $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$

$$T_2 \begin{pmatrix} \frac{a^* x}{\gamma} \\ \frac{b^* y}{\gamma} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{\text{anticlockwise rotation matrix}} \begin{pmatrix} \frac{a^* x}{\gamma} \\ \frac{b^* y}{\gamma} \end{pmatrix}$$

$$\therefore \boxed{T_2 \left(\frac{a^* x}{\gamma}, \frac{b^* y}{\gamma}\right) = \left(\frac{a^* x}{\gamma}\cos\theta - \frac{b^* y}{\gamma}\sin\theta,\ \frac{a^* x}{\gamma}\sin\theta + \frac{b^* y}{\gamma}\cos\theta\right)}$$

$\underline{T}$:- $T : \mathbb{R}^2 \to \mathbb{R}^2$

$$T = T_2(T_1(x, y))$$

$$T = \begin{pmatrix} \frac{a^*}{\gamma}\cos\theta & -\frac{b^*}{\gamma}\sin\theta \\ \frac{a^*}{\gamma}\sin\theta & \frac{b^*}{\gamma}\cos\theta \end{pmatrix}$$

$$\therefore \boxed{T(x, y) = \left(\frac{a^* \cos\theta\, x}{\gamma} - \frac{b^* \sin\theta\, y}{\gamma},\ \frac{a^* \sin\theta\, x}{\gamma} + \frac{b^* \cos\theta\, y}{\gamma}\right)}$$

$a^* = 12$; $\gamma^* = 7 = \gamma$

$b^* = 6$;

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$$\boxed{T(x,y) = \left(\frac{12\cos 33^\circ x}{7} - \frac{6\sin 33^\circ y}{7}, \frac{12\sin 33^\circ x}{7} + \frac{6\cos 33^\circ y}{7}\right)}$$

Q2

$K = |A| = 4;$

Eigen values of T :-

Transition matrix of T with respect to standard bases of $\mathbb{R}^2$ is given by:-

$$T = \begin{pmatrix} \frac{a^* \cos\theta}{r} & \frac{-b^* \sin\theta}{r} \\ \frac{a^* \sin\theta}{r} & \frac{b^* \cos\theta}{r} \end{pmatrix}$$

$|T - \lambda I| = 0$ Let λ_1, λ_2 be eigen values of T :-

$\therefore$ $Tr(T) = \frac{\cos\theta}{r}(a^* + b^*) = \lambda_1 + \lambda_2$

$|T| = \frac{a^* b^*}{r^2} = \lambda_1 \lambda_2$

$\therefore$ $\lambda_1 + \lambda_2 = \frac{18\cos\theta}{7}$

$\lambda_1 \lambda_2 = \frac{12 \times 6}{49} = \frac{72}{49}$

Equation with roots λ_1, λ_2

$$\boxed{\lambda^2 - \frac{18\cos\theta}{7}\lambda + \frac{72}{49} = 0}$$

Let $a = \frac{12}{7}$; $b = \frac{6}{7}$

$\therefore \quad \lambda^2 - (a+b)\cos\theta\,\lambda + ab = 0 \rightarrow \lambda_1, \lambda_2$

Case I :- Real roots

$$(a+b)^2\cos^2\theta - 4ab \geq 0$$

$$\left(\cos\theta - \frac{2\sqrt{ab}}{a+b}\right)\left(\cos\theta + \frac{2\sqrt{ab}}{a+b}\right) \geq 0$$

Real roots (eigen values) exist only if

$$\cos\theta \notin \left(-\frac{2\sqrt{ab}}{a+b}, \frac{2\sqrt{ab}}{a+b}\right)$$

$\sqrt{ab} = \frac{6\sqrt{2}}{7}$; $a+b = \frac{18}{7}$

$$\cos\theta \notin \left(\frac{-\cancel{12}^{2}\sqrt{2}}{\cancel{18}_{3}}, \frac{12\sqrt{2}}{18}\right)$$

$$\cos\theta \notin \left(-\frac{2\sqrt{2}}{3}, \frac{2\sqrt{2}}{3}\right) \Rightarrow \cos\theta \notin (-0.94, 0.94)$$

$\theta = 33^\circ$

$\therefore \cos 33^\circ = 0.83$

$\therefore$ $\boxed{\text{No real eign values exist}}$

Eign vectors :- No ~~real~~ eign vectors exist in real field

Case II :- Complex roots :-

Let roots be of form $re^{i\alpha}$ & $re^{-i\alpha}$

$\therefore r(e^{i\alpha} + e^{-i\alpha}) = (a+b)\cos\theta$

$\therefore r(2\cos\alpha) = (a+b)\cos\theta$

$r = \left(\frac{a+b}{2}\right)\frac{\cos\theta}{\cos\alpha}$

$\therefore r^2 = ab \quad \therefore \boxed{r = \sqrt{ab}}$

$\therefore \boxed{\cos\alpha = \frac{a+b}{2\sqrt{ab}}\cos\theta}$

$$\alpha = \cos^{-1}\left(\frac{(a+b)\cos\theta}{2\sqrt{ab}}\right)$$

$a+b = \frac{18}{7}$

$\sqrt{ab} = \frac{6\sqrt{2}}{7}$

$$\boxed{\alpha = \cos^{-1}\left(\frac{18\cos 33^\circ}{12\sqrt{2}}\right) = \cos^{-1}\left(\frac{3\cos 33^\circ}{2\sqrt{2}}\right)}$$

$$\boxed{r = \sqrt{ab} = \frac{6\sqrt{2}}{7}}$$

Eigen values of $T = \lambda_1, \lambda_2$

$$= \frac{6\sqrt{2}}{7} e^{i\cos^{-1}\left(\frac{3\cos 33^\circ}{2\sqrt{2}}\right)}, \frac{6\sqrt{2}}{7} e^{-i\cos^{-1}\left(\frac{3\cos 33^\circ}{2\sqrt{2}}\right)}$$

Eigen values of $(T)^k = \lambda_1^k, \lambda_2^k$.

$$= \left(\frac{6\sqrt{2}}{7}\right)^k e^{ki\cos^{-1}\left(\frac{3\cos 33^\circ}{2\sqrt{2}}\right)}, \left(\frac{6\sqrt{2}}{7}\right)^k e^{-ki\cos^{-1}\left(\frac{3\cos 33^\circ}{2\sqrt{2}}\right)}$$

for $k=4$:-

$$= \boxed{\left(\frac{6\sqrt{2}}{7}\right)^4 e^{4i\cos^{-1}\left(\frac{3\cos 33^\circ}{2\sqrt{2}}\right)}, \left(\frac{6\sqrt{2}}{7}\right)^4 e^{-4i\cos^{-1}\left(\frac{3\cos 33^\circ}{2\sqrt{2}}\right)}}$$

Eigen vector of T = Eigen vector of T^k

Proof:- $TX = \lambda X$

$$\left.\begin{aligned} T^2X &= T(T(x)) = \lambda^2 x \\ &\vdots \\ T^kX &= \lambda^k X \end{aligned}\right\} \therefore X \text{ vector remains same}$$

Eigen Vectors in this Case :-

$$T x = \lambda x$$

$$(T - \lambda I)\, x = 0$$

Sub Case 1 :-

$x_2, x_1 \in \mathbb{C}$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} a\cos\theta - \lambda & -b\sin\theta \\ a\sin\theta & b\cos\theta - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$\boxed{\text{Rank of } T < 2}$

$$(a\cos\theta - \lambda)\, x_1 - b x_2 \sin\theta = 0$$

Let $x_2 = K$

$$x_1 = \frac{bK\sin\theta}{a\cos\theta - \lambda} = \frac{6K/7 \sin\theta}{\frac{12\cos\theta}{7} - \frac{6\sqrt{2}}{7} e^{\pm i\alpha}}$$

$$x_1 = \frac{K \sin\theta}{2\cos\theta - \sqrt{2}\, e^{\pm i\cos^{-1}\left(\frac{3\cos 3}{2\sqrt{2}}\right)}}$$

Eigen vectors :- $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$X = L \cdot S \left(\begin{bmatrix} \dfrac{\sin 33^\circ}{2\cos 33^\circ - \sqrt{2}\, e^{\pm i \cos^{-1}\left(\frac{3\cos 33^\circ}{2\sqrt{2}}\right)}} \\ 1 \end{bmatrix} \right)$$

$$\boxed{\text{Eigen vector} = \begin{bmatrix} \dfrac{\tan 33^\circ}{2 - \sqrt{2}\sec 33^\circ\, e^{\pm i \cos^{-1}\left(\frac{3\cos 33^\circ}{2\sqrt{2}}\right)}} \\ 1 \end{bmatrix}}$$

Thus eigen vector exist only for $\boxed{\text{complex field}}$ & Not for $\boxed{\text{real field.}}$

Conclusion :-

$\lambda_1, \lambda_2 \rightarrow$ Real $\rightarrow$ X (No eigen vector exist)

$\lambda_1, \lambda_2 \rightarrow$ Complex $\rightarrow$ (Eigen value exist under complex field as calculated above)

$(re^{\pm i\alpha})$

X X X