Indian Institute of Information Technology Allahabad
Univariate and Multivariate Calculus (UMC)
C1 Review Test - Tentative Marking Scheme

Program: B.Tech. 2nd Semester (IT+ECE)
Duration: **01 Hour** Full Marks: 20
Date: February 27, 2020 Time: 12:30 - 13:30 IST

Attempt all the questions. Numbers indicated on the right in [ ] are full marks of that particular question. There is no credit for a solution if the appropriate work is not shown, even if the answer is correct. All the notations are standard and same as used in the lectures. Do not write on question paper and cover pages of the answer booklet except your details.

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1. Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 - 6x^2 + 9x + 1$. Find the intervals of decrease/increase, intervals of concavity/convexity, points of local minima/local maxima, and points of inflection of f. [8]

   Solution. $f'(x) = 3(x-1)(x-3)$ and $f''(x) = 6(x-2)$.

   f is increasing on $(-\infty, 1)$ as well as on $(3, \infty)$, and f is decreasing on $(1, 3)$. [1+1+1]

   f has a local maximum at $x = 1$ and a local minimum at $x = 3$. [1+1]

   f is concave on $(-\infty, 2)$ and convex on $(2, \infty)$. [1+1]

   2 is a point of inflection for f. [1]

2. Find the Maclaurin series (if exists) of the function defined by

$$f(x) = \begin{cases} e^{-1/x^2}, & \text{for } x \neq 0 \\ 0, & \text{for } x = 0. \end{cases}$$

   Does the Maclaurin series converge to $f(x)$ for every $x \in \mathbb{R}$. [4]

   Solution. The Maclaurin series of f is given by

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots . \qquad [1]$$

   Using L'Hopital rule,

$$f'(0) = \lim_{x \to 0^+} \frac{e^{-1/x^2}}{x} = \lim_{x \to \infty} \frac{x}{e^{x^2}} = 0. \qquad [1]$$

   Similarly, $f^{(k)}(0) = 0$ for all $k = 2, 3 \ldots$. [1]

   Therefore, the Maclaurin series of f (for any $x \in \mathbb{R}$) is identically zero. Hence, it does not converge to $f(x)$ at any $x \neq 0$. [1]

3. Let $f : [0, 1] \to \mathbb{R}$ be such that $f(x) = x$ for x rational and $f(x) = 0$ for x irrational. Evaluate the upper and lower Riemann integrals of f. [8]

Solution.

Let $P = \{x_0, x_1, \ldots, x_n\}$ be any partition of $[0,1]$. Since there exists an irrational number in each sub-interval $[x_{i-1}, x_i]$, $L(P,f) = 0$, and hence $\underline{\int_{a}^{b}} f(x)dx = 0$. [1+1]

Now,

$$\begin{aligned} U(P,f) &= \sum_{i=1}^{n} x_i(x_i - x_{i-1}) && [1] \\ &= \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} x_{i-1}x_i \\ &\geq \sum_{i=1}^{n} x_i^2 - \frac{1}{2}\sum_{i=1}^{n}(x_{i-1}^2 + x_i^2) \text{ (Using AM-GM inequality)} \\ &= \frac{1}{2}\sum_{i=1}^{n}(x_{i-1}^2 - x_i^2) = \frac{1}{2}. \\ &\implies \overline{\int_{a}^{b}} f(x)dx \geq \frac{1}{2}. && [1] \end{aligned}$$

For each $n \in \mathbb{N}$, consider $P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}, \frac{n}{n} = 1\right\}$. [1]

Then,

$$\begin{aligned} U(P_n, f) = \frac{n+1}{2n} &= \frac{1}{2} + \frac{1}{2n} && [1] \\ \implies \inf\left\{U(P_n,f) : n \in \mathbb{N}\right\} &= \frac{1}{2} && [1] \\ \implies \overline{\int_{a}^{b}} f(x)dx \leq \inf\left\{U(P_n,f) : n \in \mathbb{N}\right\} &= \frac{1}{2}. && [1] \end{aligned}$$

Therefore, $\overline{\int_{a}^{b}} f(x)dx = \frac{1}{2}$.

Alternate Solution. For $n \in \mathbb{N}$, let $P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}, \frac{n}{n} = 1\right\}$. [1]

Then $||P_n|| = \max_{1 \leq i \leq n} \Delta x_i = \frac{1}{n} \to 0$ as $n \to \infty$. [1]

Moreover,

$$U(P_n, f) = \frac{n+1}{2n} = \frac{1}{2} + \frac{1}{2n} \quad \text{and} \quad L(P_n, f) = 0. \qquad [1+1]$$

By using the following result:

If (P_n) is a sequence of partitions such that $||P_n|| \to 0$ as $n \to \infty$, then

$$\underline{\int_a^b} f(x)dx = \lim_{n\to\infty} L(P_n, f) \quad \text{and} \quad \overline{\int_a^b} f(x)dx = \lim_{n\to\infty} U(P_n, f), \qquad [2]$$

we conclude that

$$\underline{\int_a^b} f(x)dx = 0 \quad \text{and} \quad \overline{\int_a^b} f(x)dx = \frac{1}{2}. \qquad [1+1]$$