Present Value Relations

Cashflows and Assets

- **What is an "Asset"?**
  - Some examples
    - Business entity
    - Property, plant, and equipment
    - Patents, R&D
    - Stocks, bonds, options, …
    - Knowledge, goodwill, opportunities, etc.
- **From a business perspective, an asset is a sequence of cashflows**

$$Asset_t \equiv \{CF_t, CF_{t+1}, CF_{t+2}, \ldots\}$$

- An asset at every point in time is a different asset.
- It's a sequence of cash flows at different points in the future, including the present, but not the past.

- **Example:** Drug company develops a flu vaccine.
  - Strategy A: To bring to market in 1 year, invest \$1 B (billion) now and returns \$500 M (million), \$400 M and \$300 M in years 1, 2 and 3 respectively.
  - Strategy B: To bring to market in 2 years, invest \$200 M in years 0 and 1. Returns \$300 M in years 2 and 3.

- Which strategy creates more value?

- Problem. How to value/compare CF streams.

Cashflows and Assets

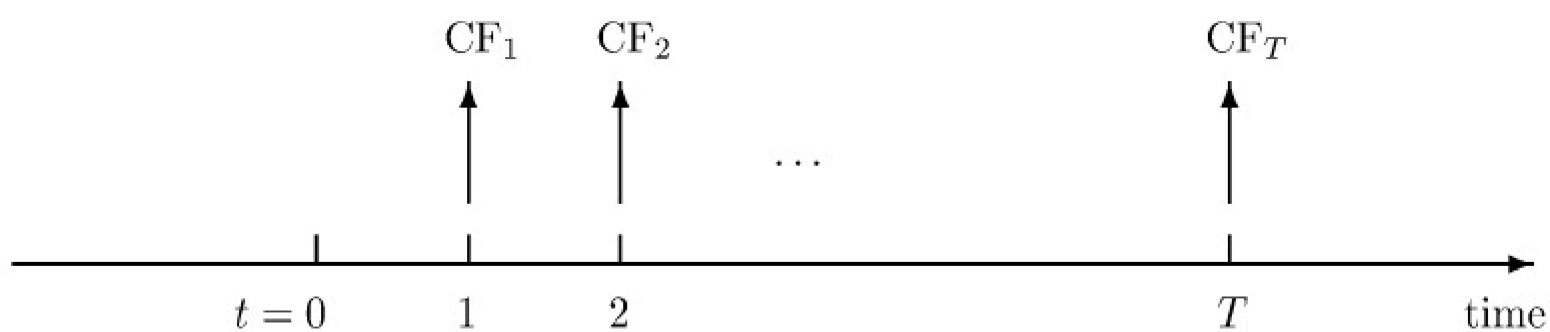
- **Valuing an asset requires valuing a sequence of cashflows**
  - Sequences of cashflows are the "basic building blocks" of finance

    $$\text{Value of Asset}_t \equiv V_t\{CF_t, CF_{t+1}, CF_{t+2}, \ldots\}$$

  - There are two distinct cases of value determination based on
    - No uncertainty about whether they're going to happen or not happen.
    - Uncertainty

- We get V_t from the market that is what we pay for it(Supply vs. Demand).

- A cash flow today is not the same as a cash flow next year.
- Draw a timeline to visualize the timing of cashflows.



The Time Value of Money

- Money you have in hand now is more valuable than money you receive in the future. This is because
  - Individuals prefer current consumption to future consumption
  - Opportunity cost of money.
  - Inflation erodes its buying power
- This is called the ***time value of money***.

- But how exactly do you compare the value of money now with the value of money in the future?
- That is where **net present value** comes in.
- Net present value: “Net” of Initial cost or investment made up front.

- Past and future cannot be combined without first converting them by multiplying a common base called ***discount factors***.
- The reason they're called discount factors is because typically they are less than 1.

- Reason why there might be some kind of discounting
  - Impatience to consume now versus later.
  - Inflation

- Cashflows at different points in time is different. The value of the collection of cashflows is just the sum of all of the values of the cashflows denominated in the same unit.

- The value of a sequence of cashflows is simply equal to today's dollars, often called *present value*.

- Implicit assumptions/requirements for net present value calculations
  - Cashflows are known (magnitudes, signs, timing) so there's no uncertainty
  - Discount rates are known
  - No frictions in currency conversions

Future Value

- **What determines the growth of $1 over T years?**
- $1 today should be worth more than $1 in the future because
  - more people want money today than money tomorrow (Supply and demand)
  - opportunity cost of capital **r**

- This r is often called the ***interest rate, growth rate, or cost of capital***.

- When we have $1 in year 1 and we want to figure out what is it worth in today's dollars, that's a discount.
- When we're going the other way, if I have $1 today and I want to know how much is that dollar worth a year from now, it should be greater than 1.

- Future values of \$1 today

  If the interest rate is r,

$$\$1 \text{ in Year } 0 = \$1 \times (1+r) \text{ in Year } 1$$

$$\$1 \text{ in Year } 0 = \$1 \times (1+r)^2 \text{ in Year } 2$$

$$\vdots$$

$$\$1 \text{ in Year } 0 = \$1 \times (1+r)^T \text{ in Year } T$$

Present Value

- What is the present value of $1 received in T year from now?
  - Present values of $1 received T years from now

If the interest rate is r,

$\$1/(1+r)$ in Year $0 = \$1$ in Year 1

$\$1/(1+r)^2$ in Year $0 = \$1$ in Year 2

$\vdots$

$\$1/(1+r)^T$ in Year $0 = \$1$ in Year T

- Example. Bank pays an annual interest of 4% on 2-year CDs and you deposit $10,000. What is your balance two years later?

  FV = 10, 000 × (1+0.04)² = $10, 816

- Example. (A) $10 M in 5 years or (B) $15 M in 15 years. Which is better if *r = 5%?*

  - $PV_A = \frac{10}{(1.05)^5} = 7.84$
  - $PV_B = \frac{15}{(1.05)^{15}} = 7.22$

- We now have an explicit expression for V_0:

$$V_0 = CF_0 + \frac{CF_1}{(1+r)} + \frac{CF_2}{(1+r)^2} + \dots$$

- Using this expression, any cash flow can be valued!
  - All capital budgeting and corporate finance reduces to this expression
  - However, we require many assumptions (perfect markets; assume that interest rates are constant)
  - Risk should be incorporated into r. Where r is ‘opportunity cost of capital’ or ‘required rate of return’.
  - The discount rate for the investment equals the rate of return that could be earned on an investment in the financial markets with similar risk.
  - A project creates value only if it generates a higher return than similar investments in the financial market (E(r)>CoC).

Example

- You have $\$1$ today and the interest rate is 5%.
- How much will you have in …

1 year … $\$1 \times 1.05 = \$1.05$

2 years … $\$1 \times 1.05 \times 1.05 = \$1.103$

3 years … $\$1 \times 1.05 \times 1.05 \times 1.05 = \$1.158$

T years … $\$1 \times 1.05 \times 1.05 \times \ldots \times 1.05 = \$1.05T$

- These cashflows are equivalent to each other. They all have the same value.
- $\$1$ today is equivalent to $\$(1+r)^T$ in T years
- $\$1$ in T years is equivalent to $\$1 / (1+r)^T$ today

Example

- Your firm spends $800,000 annually for electricity at its Boston headquarters. A sales representative from Johnson Controls wants to sell you a new computer-controlled lighting system that will reduce electrical bills by roughly $90,000 in each of the next three years. If the system costs $230,000, fully installed, should you go ahead with the investment?

- Lighting system

Year	0	1	2	3
Cashflow	-230,000	90,000	90,000	90,000
			$270,000	

- Assume the cost savings are known with certainty and the interest rate is 4%.

Year	0	1	2	3
Cashflow	-230,000	90,000	90,000	90,000
Discount factor		1.04	$(1.04)^2$	$(1.04)^3$
Present value (PV)	-230,000	86,538	83,210	80,010

Net Present value (NPV)

$$= -230{,}000 + \frac{90{,}000}{1.04} + \frac{90{,}000}{(1.04)^2} + \frac{90{,}000}{(1.04)^3}$$

- $\text{NPV} = -230{,}000 + \underbrace{86{,}538 + 83{,}210 + 80{,}010}_{\$249{,}758} = \$19{,}758$ Go ahead

- Lighting system, cont.
- Electricity prices can fluctuate, so you're not sure how much the firm will save by investing in the lighting system. Your best guess is that the firm will save $90,000 in each of the next three years, but the savings could be higher or lower. Risk is comparable to an investment in utility stocks, which have an expected rate of return of 7%.

$$\text{NPV} = -230{,}000 + \frac{90{,}000}{1.07} + \frac{90{,}000}{(1.07)^2} + \frac{90{,}000}{(1.07)^3} = \$6{,}188$$

- Go ahead. The project is now less valuable, but it still creates value since NPV > 0.

- **Example:** Drug company develops a flu vaccine.
  - Strategy A: To bring to market in 1 year, invest \$1,000 M (million) now and returns \$500 M (million), \$400 M and \$300 M in years 1, 2 and 3 respectively.
  - Strategy B: To bring to market in 2 years, invest \$200 M in years 0 and 1. Returns \$300 M in years 2 and 3.
- Assume that r = 5%
- Strategy A:

Time	0	1	2	3
Cashflow	-1,000	500	400	300
PV	-1,000	476.2	362.8	259.2
			Total PV	98.2

- Strategy B:

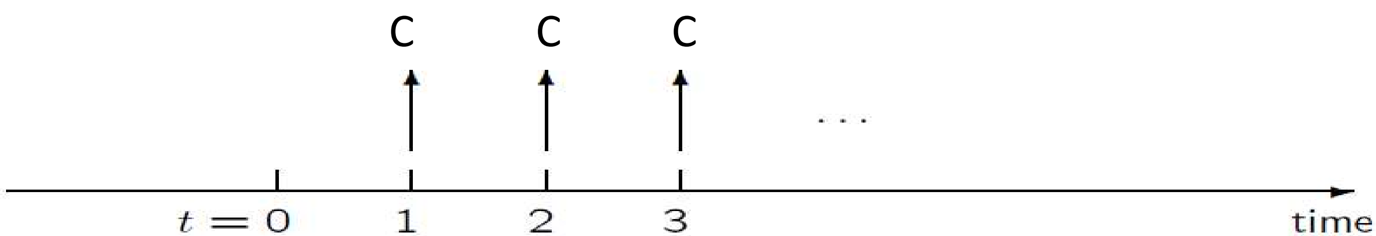
Time	0	1	2	3
Cashflow	-200	-200	300	300
PV	-200	-190.5	272.1	259.2
			Total PV	140.8

- Firm should choose strategy B, and its value would increase by \$140.8 M

Perpetuity

Perpetuity is a situation where a stream of cash flow payments continues indefinitely or is an annuity with infinite maturity. Perpetuity pays constant cashflow C forever.

– How can we value it?



$$PV = \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \dots \qquad (1)$$

Multipling $(1+r)$ on both sides,

$$(1+r) \times PV = C + \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \dots \qquad (2)$$

By subtracting (1) from (2), $r \times PV = C \quad , or \qquad PV = \frac{C}{r}$

Perpetuity with growth

Cashflow grows with rate ***g*** forever.

$$PV = \frac{C}{(1+r)} + \frac{C(1+g)}{(1+r)^2} + \frac{C(1+g)^2}{(1+r)^3} + \dots \qquad (3)$$

Multipling $\frac{(1+r)}{(1+g)}$ on both sides,

$$\frac{(1+r)}{(1+g)} \times PV = \frac{C}{(1+g)} + \frac{C}{(1+r)} + \frac{C(1+g)}{(1+r)^2} + \frac{C(1+g)^2}{(1+r)^3} + \dots \qquad (4)$$

Subtracting (3) from (4), $\left[\frac{(1+r)}{(1+g)} - 1\right] \times PV = \frac{C}{(1+g)}$

$$PV = \frac{C}{r\text{-}g},$$

where, $r > g$ (the amount that the cash is growing can never exceed the discount rate)

- If r equals g, we have an infinite number of C over 1 plus r. And C over 1 plus r is a finite constant. The sum is infinite. So at some point, that's going to exceed total world GDP, and then beyond it.
- If r is equal to g, the cash that you're presumably going to be paying to somebody is actually increasing at the exact same rate that the discount rate is growing.
- So there's no way to sustain that forever. The growth rates cannot persist forever. The amount that the cash is growing can never exceed the discount rate.

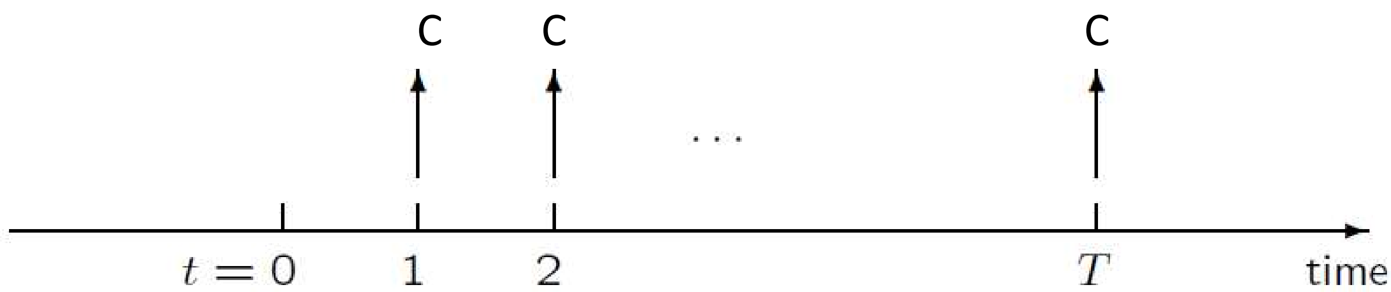
- Firms in the S&P 500 are expected to pay, collectively, \$20 in dividends next year. If growth is constant, what should the level of the index be if dividends are expected to grow 5% annually? 6% annually? Assume r = 8%.
- Growing perpetuity

$$g = 5\%, \quad PV = \frac{20}{1.08} + \frac{20(1.05)}{(1.08)^2} + \cdots = \frac{20}{0.08 - 0.05} = \$667$$

$$g = 6\%, \quad PV = \frac{20}{1.08} + \frac{20(1.06)}{(1.08)^2} + \cdots = \frac{20}{0.08 - 0.06} = \$1000$$

Annuity

- An annuity is a security that pays a fixed amount every year for a finite number of years. Annuity pays constant cashflow C for T periods.



Today is *t = 0 and cash flow starts at t = 1.*

$$PV = \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \ldots + \frac{C}{(1+r)^T} \qquad (5)$$

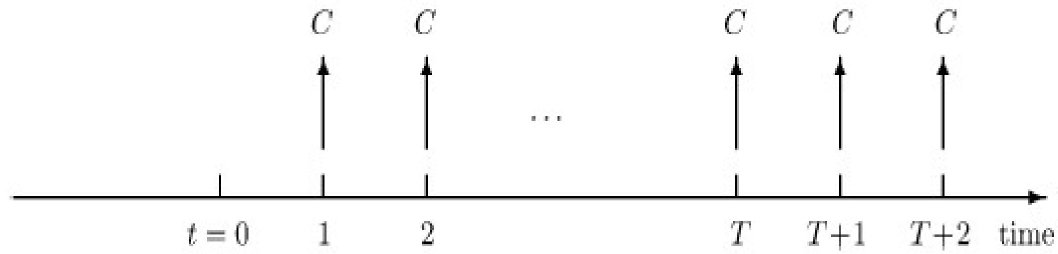
Multiplying $(1+r)$ on both sides,

$$(1+r) \times PV = C + \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \ldots + \frac{C}{(1+r)^{T-1}} \qquad (6)$$

Subtracting (5) from (6),

$$r \times PV = C - \frac{C}{(1+r)^T}, \quad \text{or,} \quad PV = \frac{C}{r}\left[1 - \frac{1}{(1+r)^T}\right]$$

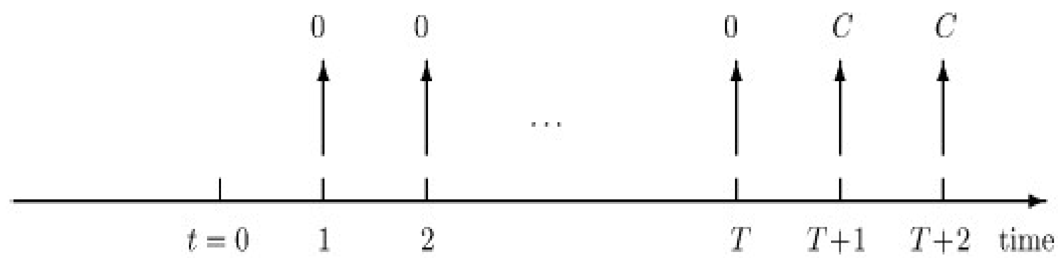
$$PV = \frac{C}{r} - \frac{C}{r}\frac{1}{(1+r)^T}$$



T-Period Annuity =

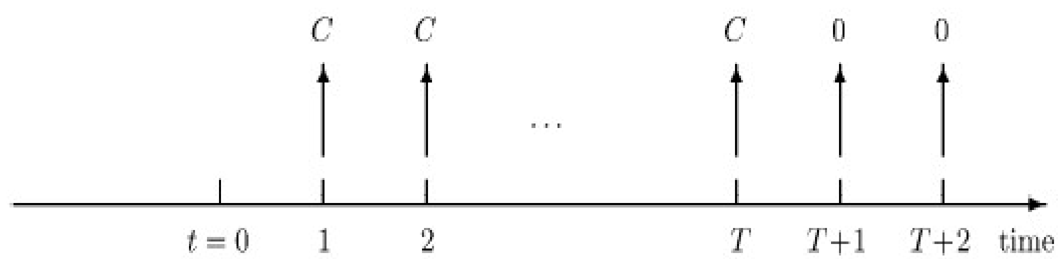
Perpetuity -

Date T Perpetuity



FV (Annuity) =

PV (Annuity) × $(1+r)^T$



- Example 1. An insurance company sells an annuity of \$10,000 per year for 20 years. Suppose *r = 5%. What should the company* sell it for?

$$PV = \frac{10{,}000}{0.05} \times \left(1 - \frac{1}{(1.05)^{20}}\right) = 10{,}000 \times 12.46 = \$124{,}622.1$$

- Example 2. You just moved to a new city and, after seeing the affordable prices, decide to buy a home. If you borrow \$800,000, what is your monthly mortgage payment? The interest rate on a 30-year fixed-rate mortgage is 5.7% (or 0.475% monthly, 5.7% / 12).

$$PV = 800{,}000 = C \times \left(\frac{1}{0.00475} - \frac{1}{0.00475(1+0.00475)^{360}}\right) = C \times 172.295$$

$$C = 800{,}000 / 172.295 = \$4{,}643.20$$

- Example 3: You just won the lottery and it pays \$100,000 a year for 20 years. Are you a millionaire? Suppose that r = 10%.

$$PV = 100{,}000 \times \frac{1}{0.10}\left(1 - \frac{1}{(1.10)^{20}}\right) = 100{,}000 \times 8.8514 = \$851{,}356$$

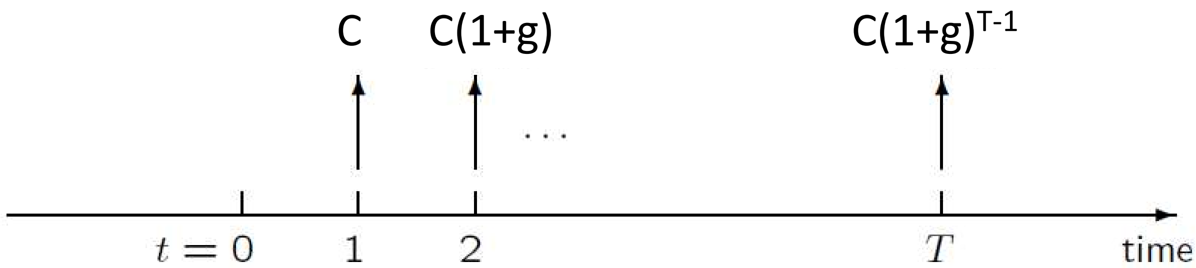
  - What if the payments last for 50 years?

$$PV = 100{,}000 \times \frac{1}{0.10}\left(1 - \frac{1}{(1.10)^{50}}\right) = 100{,}000 \times 9.915 = \$991{,}481$$

  - How about forever – a perpetuity?

$$PV = \frac{100{,}000}{0.10} = \$1{,}000{,}000$$

Annuity with Growth



$$PV = \frac{C}{(1+r)} + \frac{C(1+g)}{(1+r)^2} + \ldots + \frac{C(1+g)^{T\text{-}1}}{(1+r)^T} \qquad (7)$$

Multipling $\frac{(1+r)}{(1+g)}$ on both sides,

$$\frac{(1+r)}{(1+g)} \times PV = \frac{C}{(1+g)} + \frac{C}{(1+r)} + \cdots \frac{C(1+g)^{T-2}}{(1+r)^{T-1}} \qquad (8)$$

Subtracting (7) from (8), $\left[\frac{(1+r)}{(1+g)} - 1\right] \times PV = \frac{C}{(1+g)} - \frac{C(1+g)^{T-1}}{(1+r)^T}$

$$PV = \frac{C}{r\text{-}g}\left[1 - \left(\frac{1+g}{1+r}\right)^T\right], \quad r \neq g$$

- Example 3 . Saving for retirement - Suppose that you are now 30 and would like \$2 million at age 65 for your retirement. You would like to save each year an amount that grows by 5% each year. How much should you start saving now, assuming that r=8%?

$$FV = PV \times (1+r)^T$$

Given, $FV = 2{,}000{,}000$

We know PV with annuity, $PV = \dfrac{C}{r\text{-}g}\left[1-\left(\dfrac{1+g}{1+r}\right)^T\right],$

Given, $r = 0.08,\ g = 0.05,\ T = 35,$

$$\frac{FV}{(1+r)^T} = \frac{C}{r\text{-}g}\left[1-\left(\frac{1+g}{1+r}\right)^T\right]$$

$$C = \frac{2{,}000{,}000}{(1.08)^{35} \times 20.8976} = \$6{,}472.96$$

Compounding

- On many investments or loans, interest is credited or charged more often than once a year.
- **Examples**
  - Bank accounts – daily
  - Mortgages and leases – monthly
  - Bonds – semiannually
- Annual percentage rate (APR)
  - APR is the yearly rate of interest that an individual must pay on a loan, or that they receive on a deposit account.
  - Annual percentage rate (APR) = Quoted rate

    = interest per period * number of periods per year

- Effective annual rate (EAR)
  - If interest is compounded k times a year and let *r denote APR* the quoted annual percentage rate, then

$$\text{Effective annual rate (EAR)} = [1+(r/k)]^k - 1$$

- *The r/k is per-period rate for each period.*
- What happens as k gets big?

  In the limit as $k \rightarrow \infty$, interest is 'continuously compounded'

  EAR ≈ e^r – 1
- 'e' is the base of the natural logarithm ≈ 2.7182818

- **Example.** Bank of America's one-year CD offers 5% APR, with semi-annual compounding. If you invest \$10,000, how much money do you have at the end of one year? What is the actual annual rate of interest you earn?

- Quoted APR of $r_{APR} = 5\%$ is not the effective annual rate.
  It is only used to compute the 6-month interest rate as follows:
  $$(5\%/2) = 2.5\%.$$
  The actual annual rate, the effective annual rate (EAR), is
  $$r_{EAR} = (1 + 0.025)^2 - 1 = 5.0625\%.$$

  Investing \$10,000, at the end of one year you have:
  $$10,000(1 + 0.050625) = 10,506.25.$$

- Discounting rule
  - Bonds
    - Make semiannual payments, interest compounded semiannually Discount semiannual cashflows by APR / 2
  - Mortgages
    - Make monthly payments, interest compounded monthly Discount monthly cashflows by APR / 12

- Always use the EAR when compounding and discounting
- Due to interest compounding, the EAR is higher than the APR whenever the compounding frequency is higher than once a year.

- If interest is paid *k* times per year then the future value after *n* years is:

$$FV = PV\left(1+\frac{r}{k}\right)^{k.n}$$

- **Example:**

  Car loan—If you borrow $10,000, how much would you owe in a year computed with daily compounding, at the rate of 6.75% per year ?

  Daily interest rate = 6.75 / 365 = 0.0185%

  Day 1:Balance = 10,000.00 ×1.000185 = 10,001.85

  Day 2:Balance = 10,001.85 ×1.000185 = 10,003.70

  ……

  Day 365:Balance = 10,696.26 ×1.000185= 10,698.24

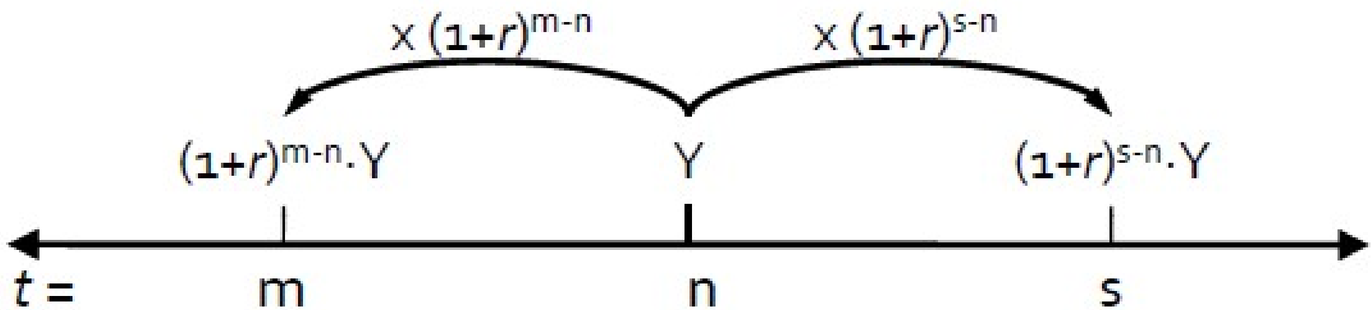
  EAR = 6.982% > 6.750%

- EAR can be much different than the stated annual percentage rate (APR).
- Example. Suppose r_{APR} = 5%:

k	Value of \$1 in a year	r_{EAR}
1	1.050000	5.0000%
2	1.050625	5.0625%
12	1.051162	5.1162%
365	1.051268	5.1267%
8,760	1.051271	5.1271%
$\vdots$	$\vdots$	$\vdots$
∞	$e^{0.05} = 1.051271$	5.1271%

Compounding / Discounting

- We can…
  - move money forward in time by **compounding.**
  - move money backward in time by **discounting.**



- Only relative time matters
- Multiplying by $(1+r)^{m-n}$ = *dividing by* $(1+r)^{n-m}$.

Inflation

Inflation: Change in real purchasing power of $1 over time

- Different from time-value of money. The purchasing power of money can vary over time, irrespective of the time value of money.
- Time value of money simply says that people are impatient and they prefer money now to money later.
- But inflation is a comment about the purchasing power of that dollar now versus later.
- Nominal vs. Real Rates
  - Nominal interest rates - typical market rates.
  - Real interest rates - interest rates adjusted for inflation.

If π is denoted as annual inflation rate, then

$$(\text{Real CF})_t = \frac{(\text{Nominal CF})_t}{(1+\pi)^t}$$

$$(1+r_{real}) = \frac{(1+r_{nominal})}{(1+\pi)}$$

$$or,\ r_{real} = \frac{(1+r_{nominal})}{(1+\pi)} - 1$$

$$r_{real} \approx r_{nominal} - \pi$$

- Example. \$1.00 invested at a 6% interest rate grows to \$1.06 next year. If inflation is 4% per year, then the real value is \$1.06/1.04 = 1.019. The real return is 1.9%.

$$(1+r_{real}) = \frac{(1+r_{nominal})}{(1+\pi)}$$

- For NPV Calculations, treat inflation consistently
  - Discount real cashflows using real interest rates
  - Discount nominal cashflows using nominal interest rates
    - Nominal cashflows ⇒ expressed in actual-dollar cashflows
    - Real cashflows ⇒ expressed in constant purchasing power
    - Nominal rate ⇒ actual prevailing interest rate
    - Real rate ⇒interest rate adjusted for inflation

Example

This year you earned $100,000. You expect your earnings to grow 2% annually, in real terms, for the remaining 20 years of your career. Interest rates are currently 5% and inflation is 2%. What is the present value of your income?

Real Interest Rate = (1.05 / 1.02) –1 = 2.94%

Real Cashflows

Year	1	2	...	20
Cashflow	102,000	104,040	...	148,595
Discount factor	1.0294	1.0294^{2}		1.0294^{20}
Present value (PV)	99,086	98,180		83,219

- **Present Value = $1,818,674**

Currencies

- How do we discount cashflows in foreign currencies?
- Discounting rule
  - Discount each currency at its own interest rate: discount $’s at the U.S. interest rate, £’s at the U.K. interest rate, ….
  - This gives PV of each cashflow stream in its own currency.
  - Convert to domestic currency at the current exchange rate.

- Example : Your firm just signed a contract to deliver 2,000 batteries in each of the next 2 years to a customer in Japan, at a per unit price of ¥800. It also signed a contract to deliver 1,500 in each of the next 2 years to a customer in Britain, at a per unit price of £6.2. Payment is certain and occurs at the end of the year.
- The British interest rate is $r^{£} = 5\%$ and the Japanese interest rate is $r^{¥} = 3.5\%$. The exchange rates are $s^{¥/\$} = 118$ and $s^{\$/£} = 1.6$.
- What is the value of each contract?

Japan

- $CF_t = 2{,}000 \times 800 = ¥1{,}600{,}000$
- $\text{PV contract} = \frac{1{,}600{,}000}{1.035} + \frac{1{,}600{,}000}{1.035^2} = ¥3{,}039{,}511$
- $\text{PV contract} = 3{,}039{,}511 \times (1 / 118_{¥/\$}) = \$25{,}759$

Britain

- $CF_t = 1{,}500 \times 6.2 = £9{,}300$
- $\text{PV contract} = \frac{9{,}300}{1.05} + \frac{9{,}300}{1.05^2} = £17{,}293$
- $\text{PV contract} = 17{,}293 \times 1.6_{\$/£} = \$27{,}668$

Example. Mortgage calculation

- Pay 20% down payment, and borrow the rest from the bank using the property as collateral.
- Pay a fixed monthly payment for the life of the mortgage.
- Have the option to prepay the mortgage anytime before the maturity date of the mortgage.

Suppose that you bought a house for $500,000 with $100,000 down payment and financed the rest with a thirty-year fixed rate mortgage at 8.5% APR compounded monthly.

The monthly payment M is determined by

$$400{,}000 = \sum_{t=1}^{360} \frac{M}{[1+(0.085/12)]^{t}}$$

$$= \frac{M}{(0.085/12)}\left\{1-\frac{1}{[1+(0.085/12)]^{360}}\right\}$$

$$= \frac{M}{(0.085/12)} \times (0.9212)$$

$$M = \$3{,}075.65$$

Effective annual interest rate (EAR):

$$[1+(0.085/12)]^{12} - 1 = 1.08839 - 1 = 8.839\%.$$

- The monthly payments are as follows:

t (month)	Principal	Interest	Sum	Remaining P.
1	242.37	2833.33	3075.7	399,757.63
2	244.08	2831.62	3075.7	399,513.55
3	245.81	2829.89	3075.7	399,267.74
⋮	⋮	⋮	⋮	⋮
120	561.29	2514.42	3075.7	354,415.49
121	565.26	2510.44	3075.7	353,850.23
⋮	⋮	⋮	⋮	⋮
240	1309.27	1766.43	3075.7	248,068.95
241	1318.54	1757.16	3075.7	246,750.41
⋮	⋮	⋮	⋮	⋮
359	3032.60	43.10	3075.7	3,054.07
360	3054.07	21.63	3075.7	0.00

- Total monthly payment is the same for each month.
- The percentage of principal payment increases over time.
- The percentage of interest payment decreases over time.