Principles of Communication Engineering

in

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Introduction to Signals

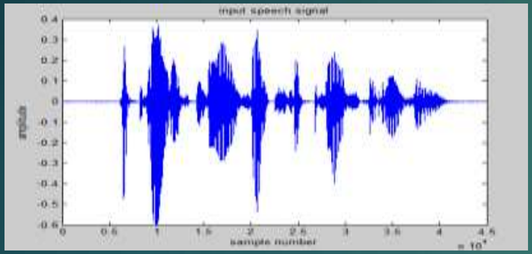
- Signal is a function of one or more variables that conveys information about some (usually physical) phenomenon.
- Signal, we mean any variable that carries or contains some kind of information that can, for example, be conveyed, displayed, or manipulated.
- Signals are represented mathematically as functions of one or more independent variables.
- The signal may depend on one or more independent variables.
- If a signal depends on only one variable, then it is known as one dimensional signal.
  - ✓ Speech signal, ac power supply, Electrocardiogram and the variation of room temperature
  - ✓ We concentrate on this type of signals.
- If a signal depends on two independent variables then the signal is known as two dimensional signal.
  - ✓ Pictures, X-ray images and sonograms.

✓ It possesses uncertainty (digital) or randomness (analog) and should have a band width.

- A Signal is defined as a physical quantity that varies with time, space or any other independent variables.

Introduction to Signals

Signals vary with **time**
Ex: Speech signal
No functional relationship to describe the signal



Signals vary with **space**
Ex: Electromagnetic field wave within a room

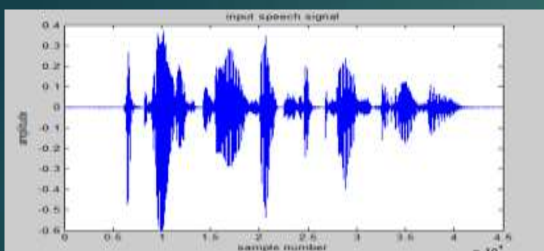
Signals vary with **dimensions**
Ex: The RGB value of an image

- ❖ Speech, we encounter, for example, in telephony, radio, and everyday life.
- ❖ Sound and music, such as that produced by the compact disc player.
- ❖ Radar signals, which are used to determine the range and bearing of distant targets.

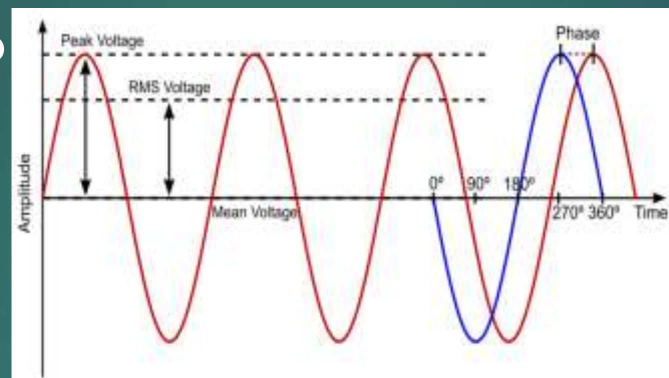
- ❖ A Signal is defined as a physical quantity that varies with time, space or any other independent variables.

Introduction to Signals

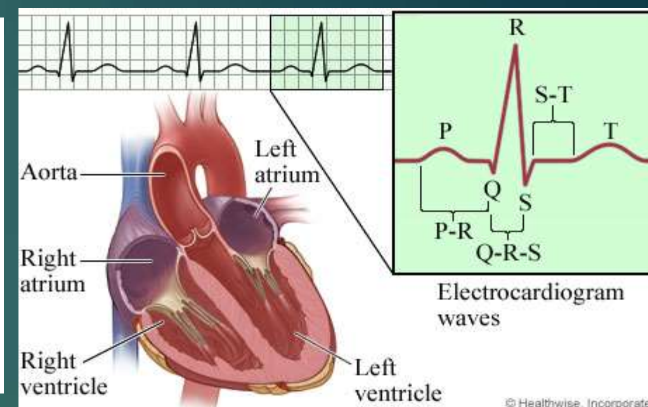
Signals can be classified based on the number of independent variables with which they are associated.
Ex: Speech signal
No functional relationship to describe the signal



Signals vary with **time**
Ex: AC power signal



Signals vary with **dimensions**
Ex: The Electrocardiogram

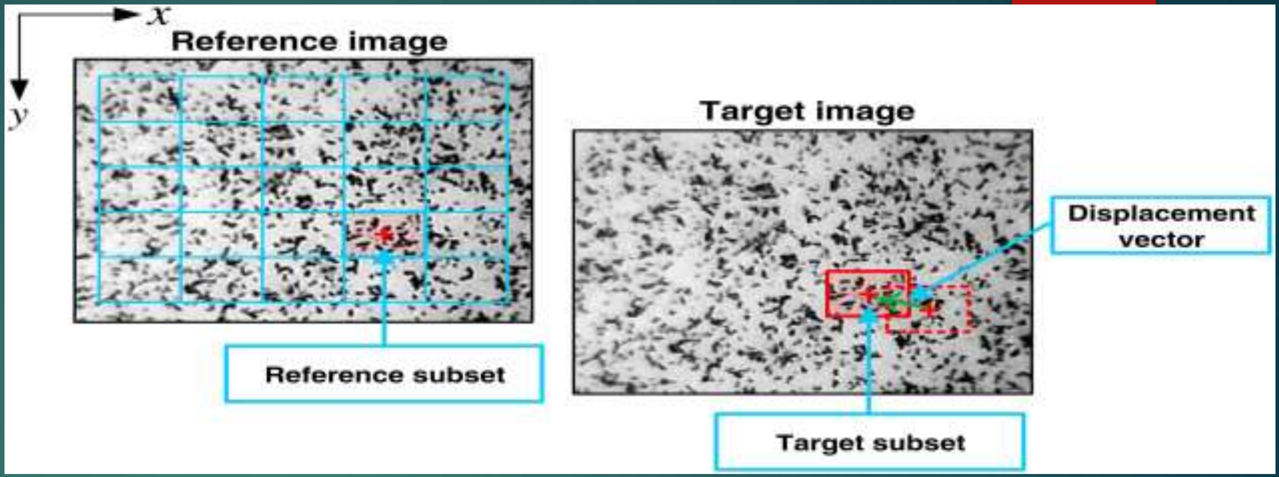


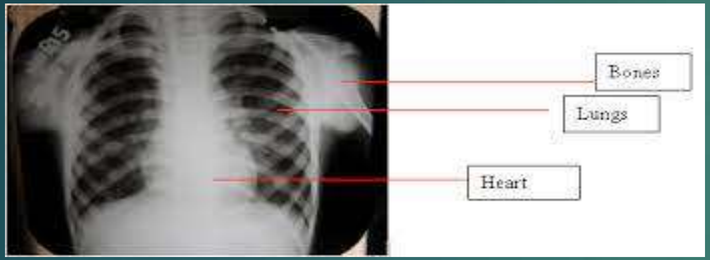
- ❖ Speech, we encounter, for example, in telephony, radio, and everyday life.
- ❖ If a signal is a function of only one variable, then it is known as one dimensional (1D) signal.
- ❖ Biomedical signals, such as electrocardiogram (Heart signals) and electroencephalogram (Brain signals).
- ❖ Sound and music, such as that produced by the compact disc player.
- ❖ Radar signals, which are used to determine the range and bearing of distant targets.

❖ A Signal is defined as a physical quantity that varies with time, space or any other independent variables.

Introduction to Signals

- ❖ Moving picture as an example of a continuous 2 dimensional signal





- ❖ Similarly, if a signal is a function of two or more variables is said to be multidimensional.
- ❖ If a signal is a function of two variables is said to be two dimensional.
    - ❖ Pictures.
    - ❖ X-ray images
    - ❖ Sonograms
- ❖ A Signal is defined as a physical quantity that varies with time, space or any other independent variables.

Introduction to Signals

- Signals may have to be transformed in order to
  - Amplify or filter out embedded information
  - Detect patterns
  - Prepare the signal to survive a transmission channel
  - Prevent interference with other signals sharing a medium
  - Undo distortions contributed by a transmission channel
  - Compensate for sensor deficiencies
  - Find information encoded in a different domain

- To do so, we also need
  - Methods to measure, characterize, model and simulate transmission channels
  - Mathematical tools that split common channels and transformations into easily manipulated building blocks.

Introduction to Signals

- **Flow of Information**

- **Measured quantity that varies with time (or position)**

- **Electrical signal received from a transducer**
  (Microphone, Thermometer, Accelerator, antenna, etc.)

- **Electrical signal that controls a process**

- **Continuous-time Signals: Voltage, Current, Temperature, Speed, …**
- **Discrete-time Signals: Daily minimum / maximum temperature, Lap intervals in races, Sampled continuous signals, …**

- **Note: Electronics (unlike optics) can only deal easily with time-dependent signals, therefore spatial signals such as images, are typically first converted into a time signal with a scanning process (TV, Fax, etc..)**

Introduction to Signals

- ❖ A Signal is a function of one or more variables that conveys information about some (usually physical) phenomenon.

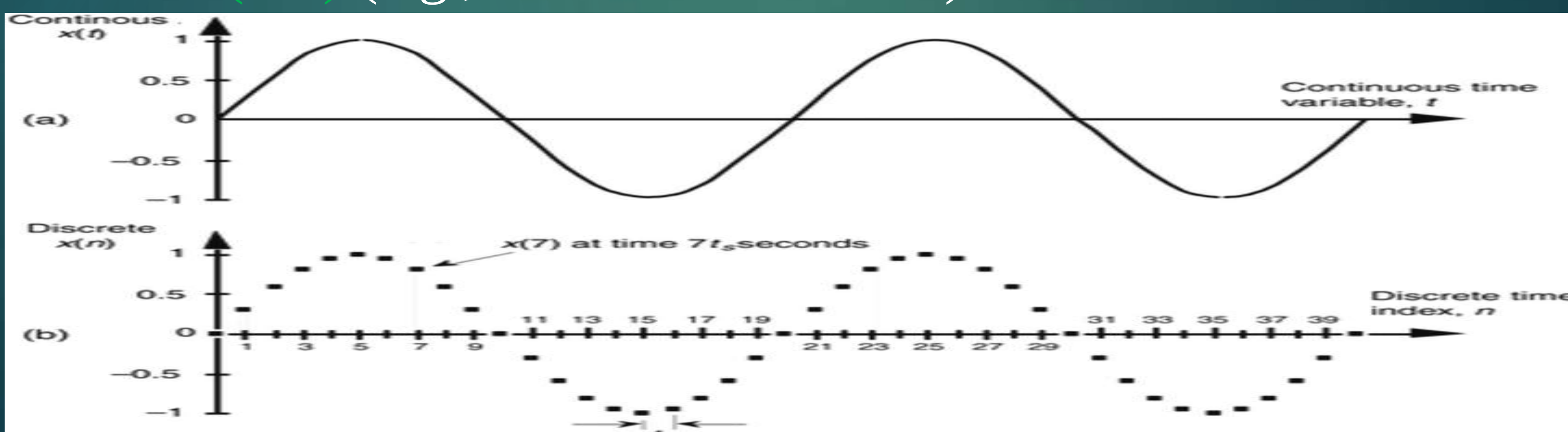
- ❖ For a function *f*, in the expression *f*(t_1, t_2,…, t_n), each of the $\{t_k\}$ is called an independent variable, while the function value itself is referred to as a dependent variable.

- ❖ Some examples of signals include:

✓ A voltage or current in an electronic circuit
✓ The position, velocity, or acceleration of an object
✓ A force or torque in a mechanical system
✓ A flow rate of a liquid or gas in a chemical process
✓ A digital image, digital video, or digital audio
✓ A stock market index

Classification of Signals

- ❑ Number of independent variables (i.e., dimensionality):
  - ✓ A signal with one independent variable is said to be one dimensional (e.g., audio signal)
  - ✓ A signal with more than one independent variable is said to be multi-dimensional (e.g., image)
- ❑ Continuous or discrete independent variables:
  - ✓ A signal with continuous independent variables is said to be continuous time (CT) (e.g., voltage waveform)
  - ✓ A signal with discrete independent variable is said to be discrete time (DT) (e.g., stock market index)

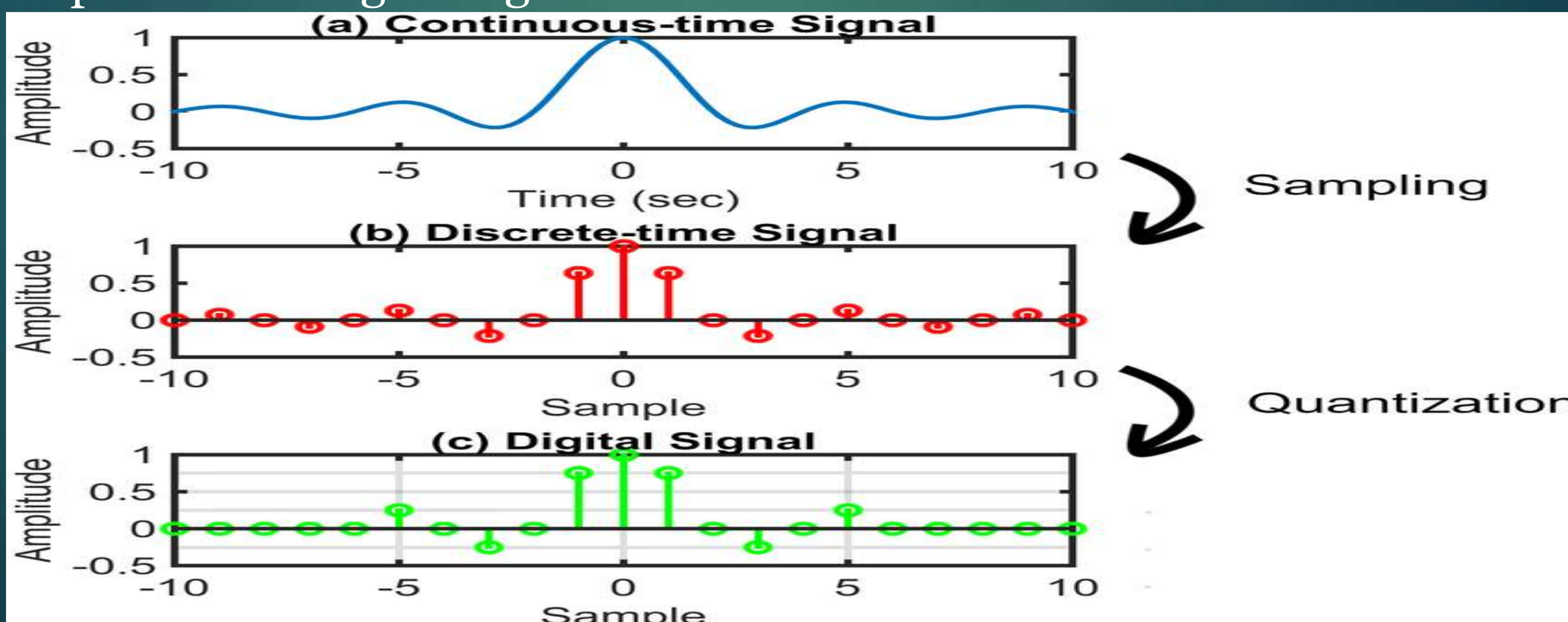


Classification of Signals

- Continuous-time Signals: The signals that are defined fr every instant of time are known as continuous-time signals. They are denoted by $x(t)$.

- Discrete-time Signals: The signals that are defined at discrete-instant of time are known as discrete-time signals. The discrete-time signals are continuous in amplitude and discrete-in time. They are denoted by $x(n)$.

- Digital Signals: The signals that are discrete-in-time and quantized in amplitude are digital signals.



Dr. Ramesh K Bhukya
Signal Processing

Elementary Continuous Time Signals

- **Unit step function:** The step function is an important signal used for analysis of many systems. If a step function has unity magnitude then it is called unit step function and denoted by u(t).

  The unit step function $\begin{cases} u(t) = 1 \quad for \quad t \geq 0 \\ u(t) = 0 \quad for \quad t < 0 \end{cases}$

- **Unit ramp function:** The unit ramp function can be defined as

$$\begin{cases} r(t) = t \quad for \quad t \geq 0 \\ r(t) = 0 \quad for \quad t < 0 \end{cases}$$

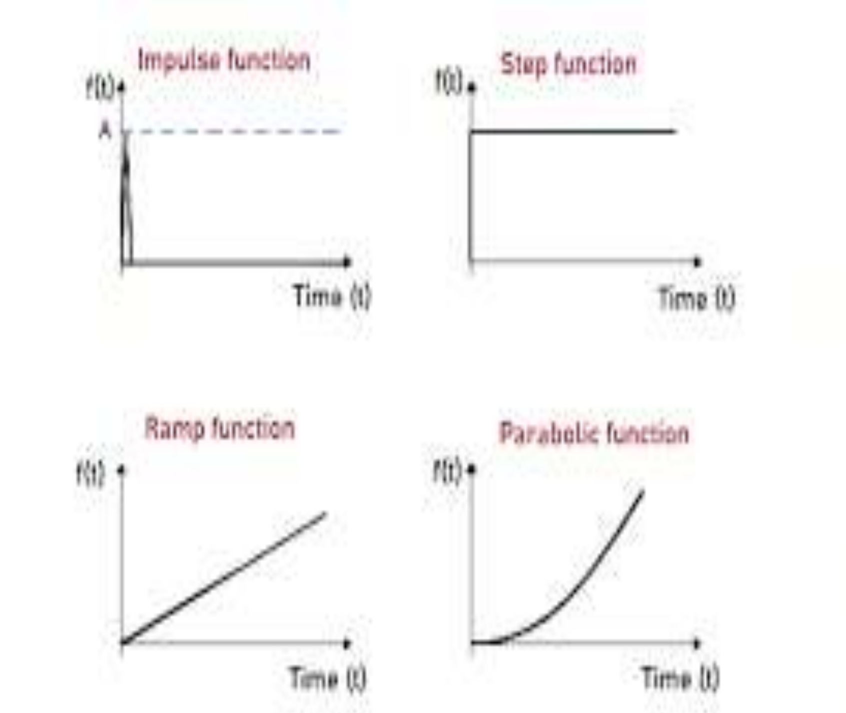
- **Unit parabolic function:** The unit parabolic function is given by

$$\begin{cases} p(t) = \frac{t^2}{2} \quad for\, t \geq 0 \\ p(t) = 0 \quad for\, t < 0 \end{cases}$$

- **Impulse function:** The impulse function occupies an important place in signal analysis.

  It is defined as $\int_{-\infty}^{\infty} \delta(t)dt = 1$

  And $\delta(t)$=0 for $t \neq 0$



Typical test signals for the design and analysis of control processes

Elementary Continuous Time Signals

Relationship Between Standard Test Signals

- Impulse
- Step
- Ramp
- Parabolic

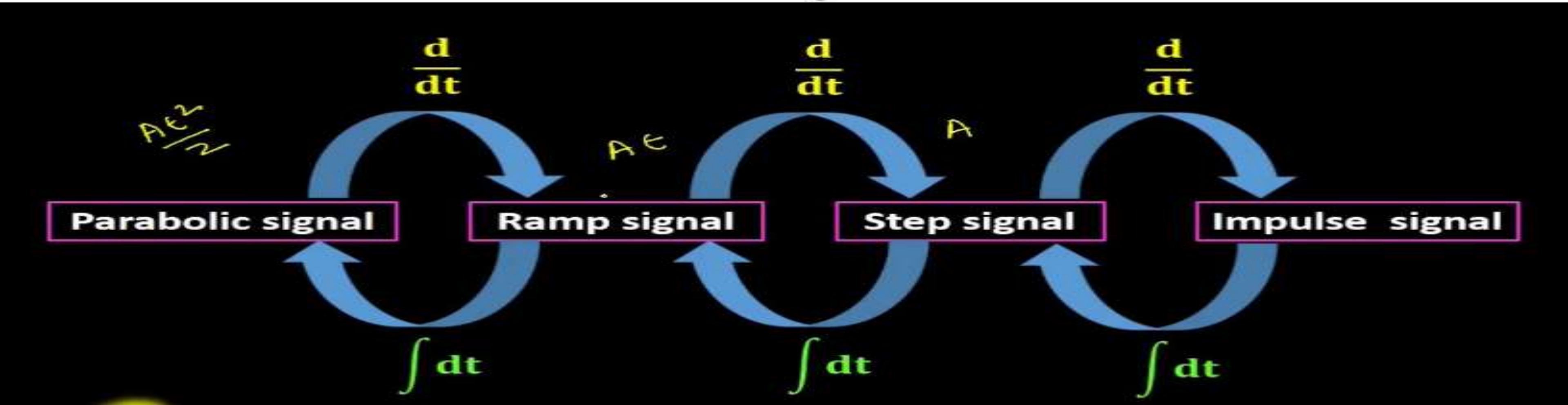
$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases}$$

$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$p(t) = \begin{cases} \dfrac{At^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$\int$ $\quad \frac{d}{dt}$



Classification of Signals

- ❑ A **continuous-valued CT** signal is said to be **analog** (e.g., voltage waveform)
- ❑ A **discrete-valued DT** signal is said to be **digital** (e.g., digital audio)
  - ✓ Graphical Representation of signals (CT and DT Signal)
  - ✓ Functional Representation of signals (CT and DT Signal)
  - ✓ Tabular Representation of signals (CT and DT Signal)
  - ✓ Sequence Representation of signals (CT and DT Signal)

Signal Characteristics and Model

- ❖Operations on the time dependence of a signal
  - ✓Time Shift
  - ✓Time Scaling
  - ✓Time Reversal
  - ✓Combinations

  - ✓Amplitude scaling
  - ✓Amplitude shifting

Signals

- A CT signal is called a **function**.
- A DT signal is called a **sequence**.

- Although, strictly speaking, a sequence is a special case of a function (where the domain of the function is the integers), we will use the term function exclusively to mean a function that is not a sequence.

- The n^{th} element of a sequence *x* is denoted as either *x(n) or* x_n.

- Strictly speaking, an expression like "*f(t)*" means the **value** of the function *f* evaluated at the point *t*.

- Unfortunately, engineers often use an expression like *"f(t)"* to refer to the **function** *f* (rather than the value of *f* evaluated at the point *t*), and this sloppy notation can lead to problems (e.g., ambiguity) in some situations.

- In contexts where sloppy notation may lead to problems, one should be careful to clearly distinguish between a function and its value.

Properties of Signals

- Even Signals
  - ✓ A function x is said to be **even** if it satisfies

    $$x(t) = x(-t) \quad \textit{for all} \quad t.$$

  - ✓ A sequence x is said to be **even** if it satisfies

    $$x(n) = x(-n) \quad \textit{for all} \quad n.$$

- ❖ Geometrically, the graph of an **even signal** is **symmetric** about the origin.

- Odd signals
  - ✓ A function x is said to be **odd** if it satisfies

    $$x(t) = -x(-t) \quad \textit{for all} \quad t.$$

  - ✓ A sequence x is said to be **odd** if it satisfies

    $$x(n) = -x(-n) \quad \textit{for all} \quad n.$$

- ❖ Geometrically, the graph of an odd signal is **anti-symmetric** about the origin.

- Periodic Signals
  - ✓ A function x is said to be **periodic** with **period** T (or **T-periodic**) if, for some strictly-positive real constant T, the following condition holds:

    $$x(t) = x(t+T) \quad \textit{for all} \quad t.$$

Properties of Signals

❖ A T-periodic function x is said to have **frequency** $\frac{1}{T}$ and **angular frequency** $\frac{2\pi}{T}$.

❑ Periodic Signals

✓ A sequence x is said to be **periodic** with **period** N (or **N-periodic**) if, for some strictly-positive real constant N, the following condition holds:

$$x(n)=x(n+N) \quad \text{for all } n.$$

❖ An N-periodic sequence x is said to have **frequency** $\frac{1}{N}$ and **angular frequency** $\frac{2\pi}{N}$.

❖ A function/sequence that is not periodic is said to be **aperiodic.**

Periodic and Aperiodic Signals

- A periodic signal has a waveform that repeats over and over, with the time between the repeats defined as the period of the signal.
  - ✓ The continuous-time signal $x(t)$ is periodic if and only if

$$\boldsymbol{x(t) = x(t+T)}, \quad \textbf{for all } \boldsymbol{t}$$

  Where T is positive and is the period of the signal.

  - ✓ A periodic signal $x(t)$ remains unchanged when time-shifted by one period.
- A signal $x(t)$ that is not periodic will be referred to as an ***aperiodic*** signal

- **A discrete-time signal** $x(n)$ is periodic if and only if

$$x(n)= x(n+N), \quad \textbf{for all } \boldsymbol{n}$$

where N is a positive integer and is the period of the signal measured in terms of number of sample spacing (samples/cycle).

- If there is no value of N that satisfies $x(n)= x(n+N)$, the signal is called ***aperiodic or nonperiodic.***

- ***<u>Note:</u> It should be noted that no physical signals are periodic since they all begin at some time and/or cease to exist at some later time.***

Energy and Power Signals

- The energy of a continuous-time and a discrete-time signal is defined as follows:
- **Continuous-time signal**

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt$$

- **Discrete-time signal**

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} |x(t)|^2 \, dt$$

energy and power of CT signal

- **Or, it may be defined as**

$$P_x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} |x(t)|^2 \, dt$$

- The power of a continuous-time and a discrete-time signal is defined as follows:
- **Continuous-time signal**

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

- **Discrete-time signal**

energy and power of DT signal

$$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$x(n) = x(n+N), \quad \text{for all } n$$

- ***<u>Note:</u> All practical signals have finite energies and are therefore energy signals. It is impossible to generate a true power signal in practice because such a signal has infinite duration and infinite energy.***
- ***All finite periodic signals are power signals; however, not all power signals are periodic   Example: Unit Step Signal***

Even and Odd Signals

- A signal $\boldsymbol{x(t)}$ is referred to as an ***even*** signal if it is identical with its reflection about the origin.
  - ✓ In continuous time a signal is even if $\boldsymbol{x(t)} = \boldsymbol{x(-t)}$
  - ✓ A signal is referred to as odd signal if $\boldsymbol{x(t)} = \boldsymbol{-x(-t)}$

- A real-valued discrete-time signal $\boldsymbol{x(n)}$ is referred to as an ***even*** signal if it is identical with its reflection about the origin, that is,

$$\boldsymbol{x(n)} = \boldsymbol{x(-n)}, \quad \text{for all } \boldsymbol{n}$$

  - ✓ An even signal has the same value at the instants n and $-n$ for all values of n.
  - ➢ **An even signal is symmetrical about the vertical axis.**

- A real-valued discrete-time signal is said to be an odd signal if

$$\boldsymbol{x(n)} = \boldsymbol{-x(-n)}, \quad \text{for all } \boldsymbol{n}$$

  - ✓ The value of an odd signal at the instant n is the negative of its value at the instant $-n$.
  - ➢ **An odd signal is anti-symmetric about the vertical axis. An odd signal must necessarily be "0" at $\boldsymbol{n = 0}$.**

Conjugate-symmetric and Conjugate-anti-symmetric Signals

- ❖ A complex discrete-time signal $\boldsymbol{x(n)}$ is said to be conjugate-symmetric if

$$\boldsymbol{x(n)} = \boldsymbol{x}^*(-\boldsymbol{n}), \quad \text{for all } \boldsymbol{n}$$

  - ✓ A real conjugate-symmetric signal is called an **even signal.**
- ❖ A complex discrete-time signal is said to be conjugate-anti-symmetric if

$$\boldsymbol{x(n)} = -\boldsymbol{x}^*(-\boldsymbol{n}), \quad \text{for all } \boldsymbol{n}$$

  - ✓ A real conjugate-anti-symmetric signal is called an odd signal.
  - ✓ For a conjugate-anti-symmetric signal $\boldsymbol{x(n)}$, the sample value at $\boldsymbol{n=0}$ must be purely imaginary.

  - ✓ Any complex discrete-time signal $\boldsymbol{x(n)}$ can be expressed as a sum of its conjugate-symmetric part $x_{cs}(n)$ and its conjugate-anti-symmetric part $x_{ca}(n)$

$$x(n) = x_{cs}(n) + x_{ca}(n)$$

Where

$$x_{cs}(n) = \frac{\boldsymbol{x(n)} + \boldsymbol{x}^*(-\boldsymbol{n})}{2}$$

And

$$x_{ca}(n) = \frac{\boldsymbol{x(n)} - \boldsymbol{x}^*(-\boldsymbol{n})}{2}$$

Bounded, Absolutely Summable, and Square-summable Signals

- A discrete-time signal $\boldsymbol{x(n)}$ is said to be ***bounded*** if each of its samples is of magnitude less than or equal to a finite positive number B_x, that is,

$$|\boldsymbol{x(n)}| \leq B_x < \infty$$

- A Unit step signal is a bounded signal with a bound B_x=1.
  - ✓ A signal $\boldsymbol{x(n)}$ is said to be *absolutely summable* if

$$\sum_{n=-\infty}^{\infty} |\boldsymbol{x(n)}| < \infty$$

  - ✓ A signal $\boldsymbol{x(n)}$ is said to be *square-summable* if

$$\sum_{n=-\infty}^{\infty} |\boldsymbol{x(n)}|^2 < \infty$$

  - ✓ A *square-summable* signal has finite energy and is an energy signal if it also has zero power.

Causal, Anti-causal, and Non-causal Signals

❖ A signal that does not start before $t = 0$ is a causal signal.
In other words, $x(t)$ is a causal signal if
$$x(t) = 0, \qquad t < 0$$

❖ A signal that starts before $t = 0$ is a *non-causal* signal. The signal is *anti-causal* if
$$x(t) = 0, \qquad t \geq 0$$

❖ Any signal that does not contain any singularities (a delta function or its derivative) at $t = 0$ can be written as the sum of a causal part $x^+(t)$ and anti-causal part $x^-(t)$, i.e., $x(t) = x^+(t) + x^-(t)$

❖ Example, the exponential $x(t) = e^{-at}$ can be written as
$$x(t) = e^{-at}u(t) + e^{-at}u(-t)$$
where the first term represents causal part of $x(t)$ and the second term represents the anti-causal part of $x(t)$.

Note: Multiplying the signal by the unit step ensures that the resulting signal is causal.

Systems

- A **system** is an entity that processes one or more input signals in order to produce one or more output signals

- Classification of systems
  - Number of inputs:
    - A system with **one** input is said to be **single input (SI).**
    - A system with **more than one** input is said to be **multiple input (MI).**
  - Number of outputs:
    - A system with **one** output is said to be **single output (SO).**
    - A system with **more than one** output is said to be multiple output (MO).
  - Types of signals processed:
    - A system can be classified in terms of the **types of signals** that it processes.
    - Consequently, terms such as the following (which describe signals) can also be used to describe systems:
      - One-dimensional and multi-dimensional
      - Continuous-time and discrete-time
      - Analog and digital

Continuous-Time and Discrete-Time Systems

- System: Any physical device which performs certain operation on the input signal and produces an output signal

<u>Example:</u> Amplifiers, Filters

- ✓ Every physical system is broadly characterized by its ability to accept an input such as voltage, current, force, pressure, displacement and to produce an output in response to this input.

- ✓ In brief, a system can be viewed as a process that results in transforming input signals into output signals.

Input $x(t)$ → Continuous-Time System → Output $y(t)$

Input $x(n)$ → Discrete-Time System → Output $y(n)$

- Impulse response: The response of the system to an applied impulses is called "Impulse response" of the system.
  It is represented with $h(t)$ or $h(n)$.
- <u>System is a device or combination of devices, which can operate on signals and produces corresponding response. Input to a system is called as excitation and output from it is called as response.</u>

Continuous-Time and Discrete-Time Systems

❖ **Linear and Non-linear Systems:**

A system is said to be linear when it satisfies superposition and homogenate principles.

Consider two systems with inputs as $x_1(t)$, $x_2(t)$, and outputs as $y_1(t), y_2(t)$ respectively. Then, according to the superposition and homogenate principles,

$y_3(t)=ay_1(t)+by_2(t)$ and $y_3{}^{!}(t)=ax_1(t)+bx_2(t)$

If $y_3(t)=y_3{}^{!}(t)$, then the system is linear system otherwise Non-linear system.

homogeneity: x(t) -> y(t) => ax(t) -> ay(t)

superposition: x1(t) -> y1(t), x2(t) -> y2(t) => x1(t) + x2(t) -> y1(t) + y2(t)

Continuous-Time and Discrete-Time Systems

- Systems are classified into the following categories:
  - ✓ Time-variant and Time-invariant Systems
  - ✓ Linear and Non-linear Systems
  - ✓ Static and Dynamic Systems
  - ✓ Causal and Non-causal Systems
  - ✓ Stable and Unstable Systems
  - ✓ Systems With and Without memory
  - ✓ Invertible and Non-invertible Systems
- **Time-variant and Time-invariant Systems**

  If the behavior of the system is independent of time at which input is applied to the system, then it is called "Time-invariant system".

  or

  The time shift in the input results the same corresponding time shift in the output then it is called "Time-invariant system" otherwise it is called "Time variant system". $y(t, t_1) = y(t - t_1)$

  $y(t, T) = y(t - T) \longrightarrow$ Time Invariant System

  $y(t, T) \neq y(t - T) \longrightarrow$ Time Variant System

Convolution of Two Signals

- **Derivation of Convolution Integral:**

Linear Time Invariant System (LTI System):

A continuous or discrete time system is said to be linear time invariant system if it satisfies both "Linearity and Time Invariant" Properties.

- ✓ **Impulse response:** The response of the system to an applied impulses is called "Impulse response" of the system.

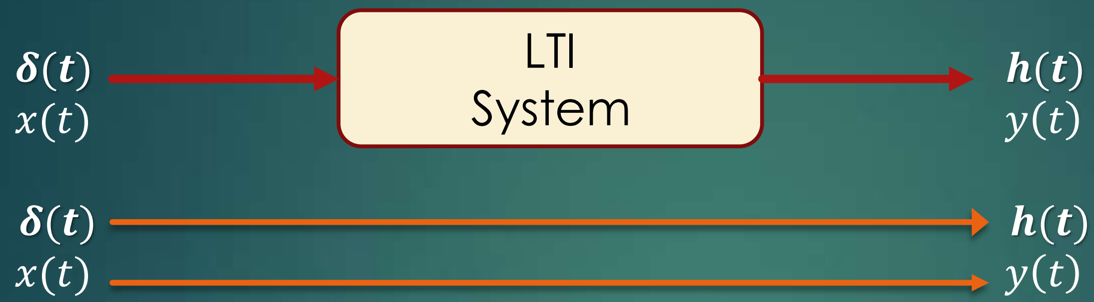
Continuous time system: $\boldsymbol{h(t)}$
Discrete time system: $\boldsymbol{h(n)}$

Consider an LTI system: $y(t) = T\{x(t)\}$

- ✓ The impulse response completely characterizes the behavior of any LTI system.
- ✓ Given the impulse response, we determine the output due to an arbitrary input signal by expressing the input as a weighted superposition of time-shifted impulses.
- ✓ For linearity and time invariance, the output signal must be a weighted superposition of time-shifted impulse responses. This weighted superposition is termed the *convolution sum* for discrete-time systems and the *convolution integral* for continuous-time systems.
- ✓ Note: A fundamental problem in system analysis is ***<u>determining the response to some specified input.</u>***

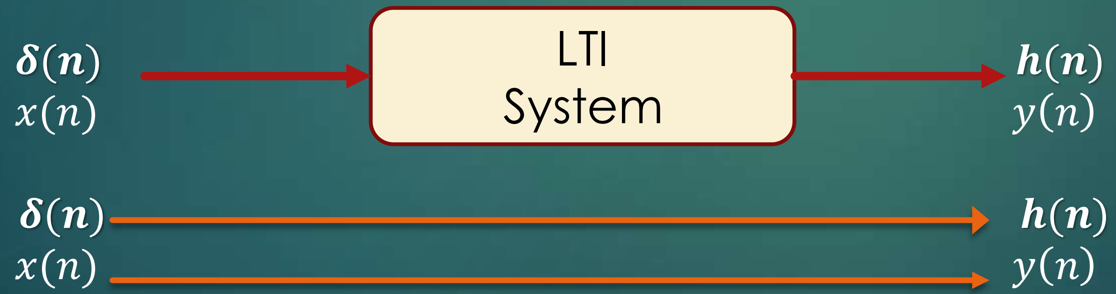
Convolution of Two Signals

- **Unit Impulse Response:**
  The impulse response is defined as the output of an LTI system due to a unit impulse signal input applied at time $t = 0$ or $n = 0$



  Where $\boldsymbol{\delta(t)}$ is the unit impulse function and $\boldsymbol{h(t)}$ is the unit impulse response of a **continuous-time LTI system (CTS).**

- **Similarly for a discrete-time system (DTS),**
  The impulse response is defined as the output of an LTI system due to a unit impulse signal input applied at time $t = 0$ or $n = 0$



  Where $\boldsymbol{\delta(n)}$ is the unit impulse function and $\boldsymbol{h(n)}$ is the unit impulse response or unit sample response of a **discrete-time LTI system (DTS).**

Convolution of Two Signals

❖ **Convolution Integral:**

A continuous-time LTI system with input $x(t)$. Using the superposition and time-invariant property of LTI systems, we can express output $y(t)$ as a linear combination of the responses of the system to shifted impulse signals.

$$\boldsymbol{\delta(t)} \longrightarrow \boldsymbol{h(t)}$$

$$x(t) \longrightarrow y(t)$$

$$\boldsymbol{\delta(t-\tau)} \longrightarrow \boldsymbol{h(t-\tau)}$$

$$x(\boldsymbol{\tau})\ \boldsymbol{\delta(t-\tau)} \longrightarrow x(\boldsymbol{\tau})\ \boldsymbol{h(t-\tau)}$$

Where $\boldsymbol{\delta(t)}$ is the unit impulse function and $\boldsymbol{h(t)}$ is the unit impulse response of a **continuous-time LTI system (CTS).**

$$x(t) = \int_{-\infty}^{\infty} x(\boldsymbol{\tau})\ \boldsymbol{\delta(t-\tau)}d\tau \longrightarrow y(t) = \int_{-\infty}^{\infty} x(\boldsymbol{\tau})\ \mathrm{h}(\boldsymbol{t-\tau})d\tau$$

Thus, we obtain the system output $y(t)$ to an arbitrary input $x(t)$ in terms of the unit impulse response $\boldsymbol{h(t)}$,

$$y(t) = \int_{-\infty}^{\infty} x(\boldsymbol{\tau})\ \mathrm{h}(\boldsymbol{t-\tau})d\tau$$

The integral relationship is called the *convolution integral* of signals $x(t)$ and $\boldsymbol{h(t)}$. This operation is represented symbolically as

$$y(t)=x(t)*\boldsymbol{h(t)}$$

Convolution of Two Signals

❖ **Commutative property:**

A continuous-time LTI system with input $x(t)$. Using the superposition and time-invariant property of LTI systems, we can express output $y(t)$ as a linear combination of the responses of the system to shifted impulse signals.

$\boldsymbol{\delta(t)} \longrightarrow \boldsymbol{h(t)}$

$x(t) \longrightarrow y(t)$

$\boldsymbol{\delta(t-\tau)} \longrightarrow \boldsymbol{h(t-\tau)}$

$x(\boldsymbol{\tau})\ \boldsymbol{\delta(t-\tau)} \longrightarrow x(\boldsymbol{\tau})\ \boldsymbol{h(t-\tau)}$

Where $\boldsymbol{\delta(t)}$ is the unit impulse function and $\boldsymbol{h(t)}$ is the unit impulse response of a **continuous-time LTI system (CTS).**

$$x(t) * \boldsymbol{h(t)} = \boldsymbol{h(t)} * x(t)$$

$$x(t) * \boldsymbol{h(t)} = \int_{-\infty}^{\infty} x(\boldsymbol{\tau})\ \boldsymbol{\delta(t-\tau)} d\tau$$

If we perform a change of variables by letting $(\boldsymbol{t-\tau}) = \alpha$, then $\boldsymbol{\tau}=(t-\alpha)$, $d\alpha = -d\tau$, $\alpha \rightarrow \infty$ as $\boldsymbol{\tau} \rightarrow \infty$, and $\alpha \rightarrow -\infty$ as $\boldsymbol{\tau} \rightarrow \infty$. Therefore,

$$x(t) * \boldsymbol{h(t)} = -\int_{-\infty}^{\infty} x(\mathbf{t-\alpha})\ \boldsymbol{h}(\alpha) d\alpha = \int_{-\infty}^{\infty} \boldsymbol{h}(\alpha) x(\mathbf{t-\alpha}) d\alpha = \boldsymbol{h(t)} * x(t)$$

$$y(t) = x(t) * \boldsymbol{h(t)} = \boldsymbol{h(t)} * x(t)$$

<u>**Note:**</u> The role of the input signal and impulse response are interchangeable. It is often used to simplify the evaluation or interpretation of the convolution integral.

Convolution of Two Signals

- **Associative property:**

The continuous convolution is associative, i.e.,

$$x(t) * [\boldsymbol{h_1(t)} * \boldsymbol{h_2(t)}] = [x(t) * \boldsymbol{h_1}] * \boldsymbol{h_2(t)}$$

By definition

$$x(t) * [\boldsymbol{h_1(t)} * \boldsymbol{h_2(t)}] = \boldsymbol{x(t)} * [\int_{-\infty}^{\infty} \boldsymbol{h_1(\tau) h_2(t-\tau) d\tau}]$$
$$= \int_{-\infty}^{\infty} \boldsymbol{x(\alpha)} \left[\int_{-\infty}^{\infty} \boldsymbol{h_1(\tau) h_2(t-\alpha-\tau) d\tau}\right] \boldsymbol{d\alpha}$$

A change of variables is performed by letting $\boldsymbol{\tau} = \beta - \alpha$, which also yields $d\tau = d\beta$, $\beta \rightarrow \infty$ as $\boldsymbol{\tau} \rightarrow \infty$, and $\beta \rightarrow -\infty$ as $\boldsymbol{\tau} - \infty$. Therefore,

$$x(t) * \boldsymbol{h(t)} = -\int_{-\infty}^{\infty} x(\mathbf{t-\alpha})\, \boldsymbol{h}(\alpha) d\alpha = \int_{-\infty}^{\infty} \boldsymbol{h}(\alpha) x(\mathbf{t-\alpha}) d\alpha = \boldsymbol{h(t)} * x(t)$$

$$x(t) * [\boldsymbol{h_1(t)} * \boldsymbol{h_2(t)}] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \boldsymbol{x(\alpha)\, h_1(\beta-\alpha) h_2(t-\beta) d\beta d\alpha}$$

Interchanging the order of integration gives the desired result

$$x(t) * [\boldsymbol{h_1(t)} * \boldsymbol{h_2(t)}] = \int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} \boldsymbol{x(\alpha)\, h_1(\beta-\alpha) d\alpha}] \boldsymbol{h_2(t-\beta) d\beta}$$
$$= \int_{-\infty}^{\infty} [x(\beta) * h_1(\beta)] h_2\,(t - \boldsymbol{\beta}) d\boldsymbol{\beta} = [x(t) * h_1(t)] * h_2(t)$$

Note: There is no need to indicate which convolution is to be performed first.

$$x(t) * \boldsymbol{h_1(t)} * \boldsymbol{h_2(t)} \equiv x(t) * \boldsymbol{h_2(t)} * \boldsymbol{h_1(t)} \equiv \boldsymbol{h_1(t)} * \boldsymbol{h_2(t)} * x(t)$$

Convolution of Two Signals

❖ **Distributive property:**

The convolution operation possesses the distributive property,, **i.e.,**

$$x(t) * [\boldsymbol{h_1(t)} + \boldsymbol{h_2(t)}]=[x(t) * \boldsymbol{h_1}] + x(t) * \boldsymbol{h_2(t)}$$

By definition

$$x(t) * [\boldsymbol{h_1(t)} + \boldsymbol{h_2(t)}]= \int_{-\infty}^{\infty} \boldsymbol{x(\tau)[h_1(t-\tau) + h_2\,(t-\tau)]d\tau}$$

$$= \int_{-\infty}^{\infty} \boldsymbol{x(\tau)h_1(t-\tau)} \int_{-\infty}^{\infty} \boldsymbol{x(\tau)\, h_2(t-\tau)]d\tau}$$

$$=[\boldsymbol{x(t) * h_1(t)}] + [\boldsymbol{x(t) * h_2(t)}]$$

❖ **Shift property:**

According to the shift property,

If $y(t)=x(t)*\boldsymbol{h(t)}$, then $x(t) * h(t-t_0) = x(t-t_0) * h(t)=y(t-t_0)$

And $x(t-t_1)* h(t-t_2)=y(t-t_1-t_2)$

Proof By definition

$$x(t) * \boldsymbol{h(t)} = \int_{-\infty}^{\infty} x(\boldsymbol{\tau})\ \boldsymbol{h(t-\tau)}d\tau = \mathbf{y(t)}$$

$$x(t) * \boldsymbol{h(t} - t_0) = \int_{-\infty}^{\infty} x(\boldsymbol{\tau})\ \boldsymbol{h(t} - t_0 - \boldsymbol{\tau})d\tau = \mathbf{y(t} - t_0)$$

Convolution of Two Signals

❖ **Convolution with an Impulse:**

The convolution of a signal $x(t)$ with a unit impulse function results in the signal $x(t)$ itself

$$x(t) * \delta(t) = x(t)$$

By definition

$$x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\boldsymbol{\tau})\delta(\boldsymbol{t} - \boldsymbol{\tau})d\tau$$

From the sampling property of the impulse function, we have

$$x(\boldsymbol{\tau})\delta(\boldsymbol{t} - \boldsymbol{\tau}) = x(t)\delta(\boldsymbol{t} - \boldsymbol{\tau})$$

And therefore,

$$x(\boldsymbol{\tau})\delta(\boldsymbol{t}) = \int_{-\infty}^{\infty} x(t)\delta(\boldsymbol{t} - \boldsymbol{\tau})d\tau = x(t)\int_{-\infty}^{\infty} \delta(\boldsymbol{t} - \boldsymbol{\tau})d\tau = x(t)$$

❖ **Width property:**

The width of the nonzero extent (the interval of time between the first and last nonzero values) of the continuous convolution of two functions equals the sum of the widths of the nonzero extents of the two functions.

✓ The durations (widths) of $x(t)$ and $h(t)$ are finite, given by W_x and W_h, respectively, then the duration (width) of $x(t)*h(t)$ is $W_x + W_h$.

Convolution of Two Signals

❖ **Differentiation Property:**

According to the differentiation property,

If $y(t)$=$x(t)$*$\boldsymbol{h(t)}$, then $\left(\frac{d}{dt}x(t)\right) * h(t) = x(t) * \left(\frac{d}{dt}h(t)\right) = \frac{d}{dt}y(t)$

By definition

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\boldsymbol{\tau})\boldsymbol{h}(\boldsymbol{t}-\boldsymbol{\tau})d\tau$$

By differentiating both sides of the above equation w.r.t, we get

$\frac{d}{dt}y(t) = \int_{-\infty}^{\infty} x(\boldsymbol{\tau})(\boldsymbol{\frac{d}{dt}h(t-\tau)})d\tau = x(t) * \left(\frac{d}{dt}h(t)\right)$

❖ **Time-scaling property:**

According to the time-scaling property,

If $y(t) = x(t) * h(t)$, then $x(at) * h(at) = \frac{1}{|a|}y(at)$

This property states that if both $x(t)$ and $h(t)$ are time-scaled by $\boldsymbol{a}$, their convolution is also time-scaled by $\boldsymbol{a}$ and multiplied by $1/|\boldsymbol{a}|$.

Convolution of Two Signals

❖ **Time-scaling property:**

According to the time-scaling property,

If $y(t) = x(t) * h(t)$, then $x(at) * h(at) = \frac{1}{|a|} y(at)$

Proof Case I: a>0. By definition

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\boldsymbol{\tau})\ h(t-\tau) d\tau = y(t)$$

$$x(at) * h(at) = \int_{-\infty}^{\infty} x(\boldsymbol{a\tau})\ h(a(t-\tau)) d\tau = \int_{-\infty}^{\infty} x(\boldsymbol{a\tau})\ h(at-a\tau)) d\tau$$

A change of variables is performed by letting $a\tau = \alpha$, which also yields $d\tau = \left(\frac{1}{a}\right) d\alpha$, $\alpha \to \infty$ as $\tau \to \infty$, and $\alpha \to -\infty$ as $\tau \to -\infty$. Therefore,

$$x(at) * h(at) = \frac{1}{a} \int_{-\infty}^{\infty} x(\alpha)\ h(at-\alpha)) d\alpha = \frac{1}{a} y(at)$$

Convolution of Two Signals

❖ **Time-scaling property:**

Case II: a<0. By definition

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\boldsymbol{\tau})\ h(t-\tau)d\tau = y(t)$$

$$x(-at) * h(-at) = \int_{-\infty}^{\infty} x(-\boldsymbol{a\tau})\ h(-a(t-\tau))d\tau = \int_{-\infty}^{\infty} x(-\boldsymbol{a\tau})\ h(-at+a\tau))d\tau$$

A change of variables is performed by letting -$a\tau = \alpha$, which also yields $d\tau = -\left(\frac{1}{a}\right) d\alpha$, $\alpha \rightarrow \infty$ as $\tau \rightarrow -\infty$, and $\alpha \rightarrow -\infty$ as $\tau \rightarrow \infty$. Therefore,

$$x(at) * h(at) = \frac{1}{a} \int_{-\infty}^{\infty} x(\alpha)\ h(at-\alpha))d\alpha = \frac{1}{a} y(at)$$

From the above two cases it is evident that

$$x(at) * h(at) = \frac{1}{|a|} y(at)$$

Convolution of Two Signals

- ❖ **Discrete-Time LTI Systems: The Convolution Sum**
  - ❖ Convolution Sum

    Linear systems are governed by the superposition principle.

    Let the responses of the system to two inputs $x_1(n)$ and $x_2(n)$ be $y_1(n)$ And $y_2(n)$, respectively.

    The system is linear if the response to the input $x(n) = a_1x_1(n)+a_2x_2(n)$ is equal to $y(n) = a_1y_1(n)+a_2y_2(n)$

    By using multiplication property: $x(n) * \delta(n-k) = x(k) * \delta(n-k)$
    The multiplication of a signal by a time-shifted impulse results in a time-shifted impulse with the amplitude given by the value of the signal at that time the impulse occurs.

    This property is used to express $x(n)$ as weighted sum of time-shifted impulses:

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

Convolution of Two Signals

❖ Convolution Sum

For a linear system, knowing the system response to impulse $\delta(n)$, the system response to any arbitrary input could be obtained by summing the system to various impulse components. Let $h(n)$ be the system response to impulse input $\delta(n)$. $x(n) \rightarrow y(n)$

To indicate the input and the corresponding response of the system. Using the superposition and time-invariant property of LTI systems, we can express output $y(n)$ as a linear combination of the responses of the system to shifted impulse signals.

$$\delta(n) \rightarrow h(n)$$
$$\delta(n-k) \rightarrow h(n-k)$$
$$x(k)\delta(n-k) \rightarrow x(k)h(n-k)$$

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k) \rightarrow y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

Thus, we have obtained the system output $y(n)$ to an arbitrary input x(n)
In terms of the unit impulse response $h(n)$
$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$ is called the *convolution sum* of sequences $x(n)$ and $h(n)$

$$y(n) = x(n) * h(n)$$

Properties of Convolution Sum

- Commutative property

The convolution operation possesses the commutative property, that is,

$$x(n) * h(n) \; = \; h(n) * x(n)$$

The role of the input signal and the impulse response is interchangeable. It is often used to simplify the evaluation or interpretation of the convolution sum

$$y(n) = x(n) * h(n)$$

Proof: By definition,

$$x(n) * h(n) \; = \; \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

A change of variable is performed by letting $n - k \; = \; m$,
Which also yields $k \; = \; n - m$, $m = -\infty$ as $n = \; -\infty$, and $m = \infty$ as $n = \infty$.
Therefore,

$$x(n) * h(n) \; = \; \sum_{-\infty}^{\infty} x(n-m)h(m) = \sum_{-\infty}^{\infty} h(m)x(n-m) = \; h(n) * x(n)$$

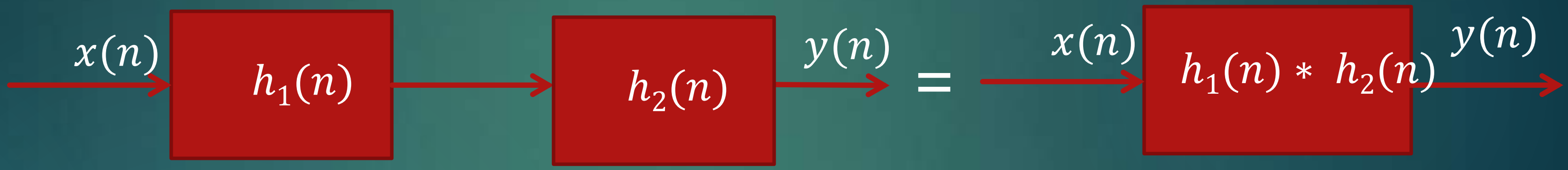
Properties of Convolution Sum

❖ Associative property

The convolution sum possesses the associative property that is:

$$x(n) * [h_1(n) * h_2(n)] = [x(n) * h_1(n)] * h_2(n)$$

This implies that a cascade combination of LTI systems can be replaced by a single system whose impulse response is the convolution of the individual impulse response



Proof: By definition,

$$x(n) * [h_1(n) * h_2(n)] = x(n) * \sum_{k=-\infty}^{\infty} h_1(k)h_2(n-k)$$

$$= x(n) * h(n) = \sum_{m=-\infty}^{\infty} x(m)h(n-m)$$

$$= \sum_{m=-\infty}^{\infty} \mathrm{x}(m) \left[\sum_{k=-\infty}^{\infty} h_1(k)h_2(n-m-k)\right]$$

$$= \sum_{m=-\infty}^{\infty} \mathrm{x}(m) \left[\sum_{k=-\infty}^{\infty} h_1(k)h_2(n-(m+k))\right]$$

Properties of Convolution Sum

- Associative property

  The convolution sum possesses the associative property that is:

  Proof: By definition,

$$x(n) * [h_1(n) * h_2(n)] = x(n) * \sum_{k=-\infty}^{\infty} h_1(k)h_2(n-k)$$

$$= x(n) * h(n) = \sum_{m=-\infty}^{\infty} x(m)h(n-m)$$

$$= \sum_{m=-\infty}^{\infty} \mathrm{x}(m) \left[\sum_{k=-\infty}^{\infty} h_1(k)h_2(n-m-k)\right]$$

$$= \sum_{m=-\infty}^{\infty} \mathrm{x}(m) \left[\sum_{k=-\infty}^{\infty} h_1(k)h_2(n-(m+k))\right]$$

  Change of variable $r = m + k$ that is $k = r - m$. Which also yields that, $r = -\infty$ as $k = -\infty$, and $r = \infty$ as $k = \infty$. Therefore,

$$x(n) * [h_1(n) * h_2(n)] = \sum_{m=-\infty}^{\infty} \mathrm{x}(m) \left[\sum_{k=-\infty}^{\infty} h_1(r-m)h_2(n-r))\right]$$

  Interchanging order of summation gives desired result:

$$x(n) * [h_1(n) * h_2(n)] = \sum_{r=-\infty}^{\infty} \left[\sum_{m=-\infty}^{\infty} \mathrm{x}(m) h_1(r-m)h_2(n-r))\right]$$

$$x(n) * [h_1(n) * h_2(n)] = [x(n) * h_1(n)] * h_2(n)$$

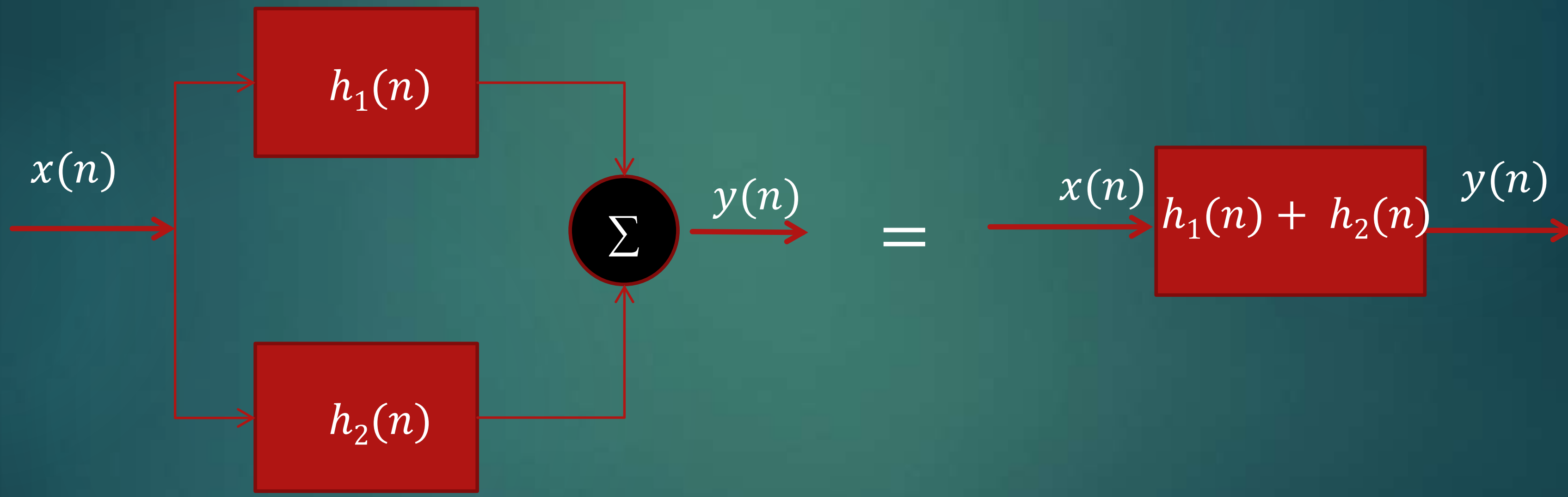
Properties of Convolution Sum

❖ **Distributive property**

The convolution operation possesses the distributive property that is:

$$x(n) * [h_1(n) + h_2(n)] = [x(n) * h_1(n)] + [x(n) * h_2(n)]$$

This implies that a cascade combination of LTI systems can be replaced by a single system whose impulse response is the convolution of the individual impulse response



Proof: By definition, $x(n) * [h_1(n) + h_2(n)] = \sum_{k=-\infty}^{\infty} \mathrm{x}(k)\ [h_1(n-k) + h_2\ (n-k)]$

$$= \sum_{k=-\infty}^{\infty} \mathrm{x}(k)\ h_1(n-k) + \sum_{k=-\infty}^{\infty} \mathrm{x}(k)\ h_2(n-k)$$

$$= [x(n) * h_1(n)] + [x(n) * h_2(n)]$$

Properties of Convolution Sum

❖ **Shift property**

According to shift property

If $x(n) * h(n) = y(n)$

then $x(n) * h(n-n_0) = x(n-n_0) * h(n) = y(n-n_0)$

And $x(n-n_1) * h(n-n_2) = y(n-n_1-n_2)$

Proof: By definition,

$x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = y(n)$

$x(n) * h(n-n_0) = \sum_{k=-\infty}^{\infty} x(k)h(n-n_0-k) = y(n-n_0)$

❖ **Impulse convolution**

Convolution of a signal $x(n)$ with a unit impulse signal results in the signal $x(n)$ itself.

$$x(n) * \delta(n) = x(n)$$

$$x(n) * \delta(n) = \mathrm{E} x(k)\delta(n-k)$$

$$x(n)\delta(n-k) = x(k)\delta(n-k)$$

$$x(n) * \delta(n) = \sum_{k=-\infty}^{\infty} x(n)\delta(n-k) = x(n)\sum_{k=-\infty}^{\infty} \delta(n-k) = x(n)$$

Properties of Convolution Sum

❖ **width property**

$$N_y = N_x + N_h - 1$$

Proof: By definition,

$$N_x = RE_x - LE_x + 1 \quad ; \quad N_h = RE_h - LE_h + 1$$

$$RE_y = RE_x + RE_h \ \ and \ LE_y = LE_x + LE_h$$

$$y(n) = x(n) * h(n)$$

$$N_y = RE_y - LE_y + 1$$

$$= RE_x + RE_h - LE_x - LE_h + 1$$

$$= (RE_x - LE_x + 1) + (RE_h - LE_h + 1) - 1$$

$$N_y = N_x + N_h - 1$$

❖ **Sum property**

$$S_y = \sum_{k=-\infty}^{\infty} y(n), \quad S_x = \sum_{n=-\infty}^{\infty} x(n)\,, S_h = \sum_{n=-\infty}^{\infty} h(n)$$

Then $\sum_{k=-\infty}^{\infty} y(n) = \sum_{n=-\infty}^{\infty} x(n) \sum_{n=-\infty}^{\infty} h(n)$

Or equivalently, $S_y = S_x S_h$

$$S_y = \sum_{k=-\infty}^{\infty} y(n) = \sum_{n=-\infty}^{\infty}[x(n) * h(n)] = \sum_{n=-\infty}^{\infty}[\sum_{k=-\infty}^{\infty} x(k)h(n-k)]$$

Interchanging the order of the summation,

$$S_y = \sum_{k=-\infty}^{\infty} x(k) \sum_{n=-\infty}^{\infty} h(n-k) = \sum_{k=-\infty}^{\infty} x(k) \sum_{m=-\infty}^{\infty} h(m) = S_x S_h$$

Relations Between LTI System Properties and Impulse Response

- **LTI Systems With and Without Memory**

A DTS is said to be memory less if and only if the impulse response

$$h(n) = K\delta(n)$$

Where $K = h(0),\ \ is\ a\ constant$

Proof

$$y(n) = h(n) * x(n)$$
$$= \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

$$= \cdots + h(-2)x(n+2) + h(-1)x(n+1) + h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + \cdots$$

$$h(n) = K\delta(n)$$

Where $K = h(0)$

$$y(n) = Kx(n)$$

If a DTS has an impulse response $h(n)$ that is not identically zero for $n \neq 0$, then the system has memory.

Relations Between LTI System Properties and Impulse Response

❖ **LTI Systems With and Without Memory**

A DTS is said to be memory less if and only if the impulse response

$$h(n) = K\delta(n)$$

Where $K = h(0),\ \textit{is a constant}$

Proof

$$y(n) = h(n) * x(n)$$
$$= \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

$$= \cdots + h(-2)x(n+2) + h(-1)x(n+1) + h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + \cdots$$

$$h(n) = K\delta(n)$$

Where $K = h(0)$

$$y(n) = Kx(n)$$

If a DTS has an impulse response $h(n)$ that is not identically zero for $n \neq 0$, then the system has memory.

Relations Between LTI System Properties and Impulse Response

❖ **Causal for LTI Systems**

A DTS is said to be causal if and only if the impulse response

$$h(n) = 0, \qquad n < 0$$

Proof

$$y(n) = h(n) * x(n)$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

$$= \left[\sum_{k=-\infty}^{-1} h(k)x(n-k)\right] + \left[\sum_{k=0}^{\infty} h(k)x(n-k)\right]$$

For a DTS LTI system,

$h(n) = 0 \; for \; n < 0$

And the discrete convolution takes the new form $y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$

k = 0 ?

?

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

Relations Between LTI System Properties and Impulse Response

❖Stability for LTI Systems

A DTS is said to be stable if and only if the impulse response is absolutely summable, that is

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

Proof

$$\boldsymbol{y(n) = h(n) * x(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)}$$

$$\boldsymbol{|y(n)| = |\sum_{k=-\infty}^{\infty} h(k)x(n-k)|}$$

$$\boldsymbol{|y(n)| \leq \sum_{k=-\infty}^{\infty} |h(k)||x(n-k)|}$$

$|x(n)| \leq Bx <\infty$ then $|x(n-k)| \leq Bx$

$$\boldsymbol{|y(n)|} \leq \mathrm{Bx} \sum_{\mathrm{k}=-\infty}^{\infty} \boldsymbol{|h(k)|}$$

$$\boldsymbol{|y(n)|} \leq \mathrm{BxBy}$$

?

$$\sum_{n=-\infty}^{\infty} |h(k)| = Bx < \infty$$

Then $\boldsymbol{y(n)}$ is bounded in magnitude and, hence, the system is stable.

Relations Between LTI System Properties and Impulse Response

❖ Invertibility for LTI Systems

A DTS is said to be invertible with impulse response $h(n)$ if and only if an inverse system exists that, when connected in series with the original system, produces an output equal to the input to the first system.

$$x(n) * [h(n) * h_I(n)] = x(n)$$

This requirement implies that

$$[h(n) * h_I(n)] = \delta(n)$$

❖ Unit Step Response of LTI Systems

$$s(n) = \sum_{k=-\infty}^{n} h(k)$$

Proof $s(n) = h(n) * u(n) = \sum_{k=-\infty}^{\infty} h(k)u(n-k)$

$u(n-k) = \begin{cases} 1 & k \leq n \\ 0 & k > n \end{cases}$ and $s(n) = \sum_{k=-\infty}^{n} h(k)$

$$s(n) = h(n) + \sum_{k=-\infty}^{n-1} h(k)$$

$$s(n) = h(n) + s(n-1)$$

$$h(n) = s(n) - s(n-1)$$

Correlation of Signals

❖ Cross-correlation Sequence of Discrete-time Energy Signals

$$R_{xy}(m) = \sum_{n=-\infty}^{\infty} x(n)y^*(n-m) = \sum_{n=-\infty}^{\infty} x(n+m)y^*(n)$$

$$R_{xy}(m) = \sum_{n=-\infty}^{\infty} x(n)y(n-m) = \sum_{n=-\infty}^{\infty} x(n+m)y(n)$$

$$R_{yx}(m) = \sum_{n=-\infty}^{\infty} y(n)x\,(n-m) = \sum_{n=-\infty}^{\infty} y(n+m)x\,(n)$$

$$R_{xy}(m) = R_{yx}(-m)$$

$$R_{xy}(m) = x(m) * y(-m)$$

Proof

$$R_{xy}(m) = x(m) * y(-m) = \sum_{n=-\infty}^{\infty} x(n)y\big(-(m-n)\big) = \sum_{n=-\infty}^{\infty} x\,(n)y(n-m)$$

Correlation of Signals

❖Cross-correlation Sequence of Power Signals

$$R_{xy}(m) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x\,(n) y^*(n-m) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x\,(n+m) y^*(n)$$

$$R_{xy}(m) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x\,(n) y(n-m) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x\,(n+m) y(n)$$

$$R_{xy}(m) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x\,(n) y(n-m)$$

$$R_{xy}(m) = \frac{1}{2N+1} \sum_{n=-N}^{N} x\,(n) y(n-m)$$

Correlation of Signals

❖ Autocorrelation Sequence of Discrete-time Signals

$$R_{xx}(m) = \sum_{n=-\infty}^{\infty} x(n)x(n-m) = \sum_{n=-\infty}^{\infty} x(n+m)x(n)$$

$$R_{xx}(m) = x(m) * x(-m)$$

Proof

$$R_{xx}(m) = x(m) * x(-m) = \sum_{n=-\infty}^{\infty} x(n)x\big(-(m-n)\big) = \sum_{n=-\infty}^{\infty} x\,(n)x(n-m)$$

$$R_{xx}(m) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x\,(n)x(n-m) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x\,(n+m)x(n)$$

$$R_{xx}(m) = \frac{1}{2N+1} \sum_{n=-N}^{N} x\,(n)x(n-m)$$

Correlation of Signals

❖Properties of Cross-correlation and Autocorrelation Sequences

$$R_{xy}(m) = R_{yx}(-m)$$

$$R_{xx}(m) = R_{xx}(-m)$$

Proof $$R_{xx}(m) = \sum_{n=-\infty}^{\infty} x(n)x(n-m)$$

Case I: The autocorrelation sequence is an even function

$$R_{xx}(m) = R_{xx}(-m)$$

$$R_{xy}(m) = R_{yx}(-m)$$

Proof: By definition, $$R_{xx}(m) = \sum_{n=-\infty}^{\infty} x(n)x(n-m)$$

Let, $n - m = k$

$$R_{xx}(m) = \sum_{n=-\infty}^{\infty} x(k+m)x(k)$$
$$= \sum_{n=-\infty}^{\infty} x(k)x(k+m)$$
$$= \sum_{n=-\infty}^{\infty} x(k)x\big(k-(-m)\big)$$
$$= R_{xx}(-m)$$

Correlation of Signals

❖ Properties of Cross-correlation and Autocorrelation Sequences

Relation to signal energy and signal power is given by the following cases:

✓ If $x(n)$ is an energy signal

$$R_{xx}(0) = E_x = \sum_{n=-\infty}^{\infty} x^2(n)$$

Proof: $$R_{xx}(m) = \sum_{n=-\infty}^{\infty} x(n)x(n-m)$$

at $m = 0$,

$$R_{xx}(0) = \sum_{n=-\infty}^{\infty} x(n)x(n) = \sum_{n=-\infty}^{\infty} x^2(n) = E_x$$

✓ If x(n) is a power signal

Proof: By definition,

$$R_{xx}(0) = P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x^2(n)$$

$$R_{xx}(m) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(n)x(n-m)$$

at $m = 0$

$$R_{xx}(0) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(n)x(n)$$

$$R_{xx}(0) = P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x^2(n) = P_x$$

Correlation of Signals

❖Properties of Cross-correlation and Autocorrelation Sequences

The cross-correlation sequence satisfies the condition

$$\left|R_{xy}(m)\right| \le \sqrt{R_{xx}(0)R_{yy}(0)} = \sqrt{E_x E_y}$$

and the autocorrelation sequence satisfies this condition

$$|R_{xx}(m)| \le R_{xx}(0) = E_x$$

Proof: By definition, Consider a linear combination of two finite-energy discrete-time signals

$$a\,x(n) + b\,y(n-m)$$

The energy of this signal is

$$\sum_{n=-\infty}^{\infty}[a\,x(n) + b\,y(n-m)\,]^2 = a^2\sum_{n=-\infty}^{\infty} x^2(n)\,\mathrm{dt} + b^2\sum_{n=-\infty}^{\infty} y^2(n-m)$$

$$+\,2ab\,\sum_{n=-\infty}^{\infty} x(n)y(n-m)$$

$$= a^2\,R_{xx}(0) + b^2\,R_{yy}(0) + 2ab\,R_{xy}(m) \ge 0$$

Now, assuming $b \ne 0$,

$$(\frac{a}{b})^2\,R_{xx}(0) + 2(\frac{a}{b})\,R_{xy}(m) + R_{yy}(0) \ge 0$$

$$4[R^2{}_{xy}(m) - R_{xx}(0)\,R_{yy}(0)\,] \le 0$$

$$\left|R_{xy}(m)\right| \le \sqrt{R_{xx}(0)R_{yy}(0)} = \sqrt{E_x E_y}$$

For autocorrelation function where $y(n) = x(n)$,

$$|R_{xx}(m)| \le R_{xx}(0) = E_x$$