Indian Institute of Information Technology Allahabad
Discrete Mathematical Structures
Solution of C1 Review test

Program: B.Tech. 2^{nd} Semester (IT+IB)
Duration: **60+ 10 minutes** Full Marks: 16
Date: May 22, 2022 Time:: 5:00 PM - 6:10 PM

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1. Let us define sets A and B as follows:

$$A = \{\textbf{Your first name}\} \text{ and } B = \{\textbf{ Your last name}\}$$

In the absense of last name, take $B = \{l, a, s, t, n, m, e\}$
For example: if your name is **Peter Massopust**, then $A = \{p, e, t, r\}$ and $B = \{m, a, s, o, p, u, t\}$.

Now, $C := A \cup B = \{p, e, t, r, m, a, s, o, u\}$ and $D := A \cap B = \{p, t\}$. Then

(I) Construct two distinct partial order relations (name these relations R_1 and R_2 respectively) on C. [4]
Solution: Let us define a relation on C as follows:
$R_1 := \{(p, p), (e, e), (t, t), (r, r), (m, m), (a, a), (s, s), (o, o), (u, u)\}$.
It is simple to check that R_1 is reflexive, anti-symmetric and transitive. [2]

Now, we may define another relation on C as follows:
$R_2 = \{(p, p), (e, e), (t, t), (r, r), (m, m), (a, a), (s, s), (o, o), (u, u), (p, e), (e, t), (p, t)\}$
It is plain to see that R_2 is reflexive, anti-symmetric and transitive. [2]

(II) Find all maximal and minimal elements of the constructed partial ordered sets (C, R_1) and (C, R_2). [4]
Solution: In (C, R_1) : the set of maximal elements= $\{p, e, t, r, m, a, s, o, u\}$, [1]
and the set of minimal elements= $\{p, e, t, r, m, a, s, o, u\}$. [1]

In (C, R_2) : the set of maximal elements= $\{t, r, m, a, s, o, u\}$, [1]
and the set of minimal elements= $\{p, r, m, a, s, o, u\}$. [1]

(III) Find the supremum and infimum (if they exist) of the constructed partial ordered sets (C, R_1) and (C, R_2). [2]
Solution: Since there are more than one maximal elements, the partial ordered sets (C, R_1) and (C, R_2) have no supremum. [1]
Since there are more than one minimal elements, the partial ordered sets (C, R_1) and (C, R_2) have no infimum. [1]

(IV) Determine whether the following sets are finite, countably infinite (countable) or uncountable: [3]

(a) $X =$ the collection of all functions from C to D.
Solution: Finite (countable) because $|X| = |D|^{|C|} = 2^9$. [1]

(b) $Y =$ the collection of all functions from C to $\mathbb{N}$, where $\mathbb{N}$ denotes the set of natural numbers.
Solution: Countably infinite because

$$|Y| = |\mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}|.$$

[1]

(c) $Z =$ the collection of all functions from $\mathbb{N}$ to C.
Solution: Uncountable because $|Z| \geq 2^{|\mathbb{N}|} = |P(\mathbb{N})|$. [1]

2. Let $n \in \mathbb{N}$ and suppose we are given real numbers $a_1 \geq a_2 \geq \ldots \geq a_n \geq 0$. Show that Arithmetic mean (AM) $= \frac{a_1+a_2+\ldots a_n}{n} \geq (a_1 a_2 \ldots a_n)^{\frac{1}{n}} =$ GM (Geometric mean). [3]
Solution:

The above statement, we will prove by mathematical induction. For $n = 2$, we have

$$\frac{a_1 + a_2}{2} \geqslant (a_1 a_2)^{1/2},$$

which can be obtained by following lines: [1]

$$a_1 a_2 \leq \frac{(a_1 + a_2)^2}{4} \text{ iff } (a_1 + a_2)^2 \geq 4a_1 a_2 \text{ iff } (a_1 + a_2)^2 - 4a_1 a_2 \geq 0 \text{ iff } (a_1 - a_2)^2 \geq 0.$$

Suppose the statement is true for $n - 1$ such numbers, that is,

$$\frac{b_1 + b_2 + \ldots b_{n-1}}{n-1} \geq (b_1 b_2 \ldots b_{n-1})^{\frac{1}{n-1}},$$

where $b_1 \geq b_2 \geq \ldots \geq b_{n-1} \geq 0$. [1]
Given that $a_1 \geq a_2 \geq a_3 \ldots \geq a_n$. Since $G = (a_1 a_2 \cdots a_n)^{1/n}$, we have $a_n \leqslant G_1 \leqslant a_1$. We claim that $a_1 + a_n \geq \frac{a_1 a_n}{G} + G$. Now,

$$\begin{aligned} a_1 + a_n - G - \frac{a_1 a_n}{G} &= \frac{a_1}{G}(G - a_n) + (a_n - G) \\ &= \frac{1}{G}(a_1 - G)(G - a_n) \geq 0, \end{aligned}$$

the previous claim follows. Using the inductive hypothesis, we have

$$\frac{a_2 + a_3 + \cdots + a_{n-1} + \frac{a_1 a_n}{G}}{n-1} \geq (G^n/G)^{1/n-1}.$$

This yields

$$a_2 + a_3 + \cdots + a_{n-1} + \frac{a_1 a_n}{G} \geq (n-1)G.$$

Now, the above can be written as

$$\frac{a_2 + a_3 + \cdots + a_{n-1} + \frac{a_1 a_2}{G} + G}{n} \geq G.$$

Using $a_1 + a_n \geq \frac{a_1 a_n}{G} + G$, we obtain

$$\frac{a_1 + a_2 + \cdots + a_{n-1} + a_n}{n} \geq G,$$

completing the proof. [1]