Principles of Communication Engineering

in

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Continuous Time Fourier Transform

❖ The main drawback of Fourier series is, it is only applicable to periodic signals.

❖ There are some naturally produced signals such as ***aperiodic or nonperiodic***, which we cannot represent using Fourier Series.

❖ To overcome this shortcoming, Fourier developed a mathematical model to transform signals between time (or spatial) domain to frequency domain & vice versa, which is called ***"Fourier Transform"***

$$\widetilde{x(t)} = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} \qquad X_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \widetilde{x(t)} e^{-jn\omega_0 t}\, dt$$

✓ In the limit as $T \to \infty$, we see that $w_0 = \frac{2\pi}{T}$, becomes an infinitesimally small quantity, dw, so that

$$\frac{1}{T} \to \frac{dw}{2\pi}$$

$$\frac{X_n}{dw} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{x(t)} e^{-jwt}\, dt$$

$$x(t) = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x(t)e^{-jwt}\, dt\right) e^{jwt}\,\frac{dw}{2\pi}$$

Continuous Time Fourier Transform

❖ The inner integral, in brackets, is a function of only w and not t. Denoting the integral by $X(w)$,

$$\boldsymbol{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)\, e^{jwt} dw} \qquad \textbf{Synthesis}$$

Where $$\boldsymbol{X(w) = \int_{-\infty}^{\infty} x(t)\, e^{-jwt} dt} \qquad \textbf{Analysis}$$

These are referred to as the *Fourier transform pair*, with the function $X(w)$ referred to as the *Fourier transform* or Fourier integral of $x(t)$ as the *inverse Fourier transform* equation.

- ✓ We call $X(w)$ the Fourier transform of $x(t)$, and $x(t)$ the inverse Fourier transform of $X(w)$.
- ✓ The same information is conveyed by the statement that $x(t)$ and $X(w)$ are a Fourier transform pair. Symbolically, this statement is expressed as

$$X(w) = F[x(t)] \text{ and } x(t)=F^{-1}[X(w)]$$

Or

$$x(t) \leftrightarrow X(w)$$

$X(w)$, in general, is a complex function of the variable w. Thus, it can be written as

$$X(w)=X_R(w) + jX_I(w)$$

Continuous Time Fourier Transform

- The same information is conveyed by the statement that $x(t)$ and $X(w)$ are a Fourier transform pair. Symbolically, this statement is expressed as

$$\boldsymbol{X(w) = F[x(t)]} \textbf{ and } \boldsymbol{x(t)=F^{-1}[X(w)]}$$

Or

$$\boldsymbol{x(t) \leftrightarrow X(w)}$$

$X(w)$, in general, is a complex function of the variable w. Thus, it can be written as

$$X(w)=X_R(w) + jX_I(w)$$

$$X(w)=|X(w)|e^{j\angle X\,(w)}$$

where $|X(w)|=\sqrt{X_R(w)^2 + X_I(w)^2}$ is the magnitude and

$\angle X(w)=\tan^{-1}[\frac{X_I(w)}{X_R(w)}]$ is the angle (or phase) of $X(w)$.

- ✓ The $|X(w)|$ plotted against w is called the *magnitude spectrum of $x(t)$*, and the $\angle X(w)$ plotted against w is called the *phase spectrum*.

Properties of Continuous Time Fourier Transform

- ❖ Fourier transform possesses a number of important properties that are useful for developing conceptual insights into the transform and into the relationship between the time-domain and frequency-domain descriptions of a signal.
- ❖ They can also help to reduce the complexity of the evaluation of the Fourier transform of many signals.
- ❖ Here, we will use a shorthand notation as

$$\boldsymbol{x(t) \leftrightarrow X(w)}$$

to indicate the relationship between a time-domain signal $x(t)$ and its Fourier transform $\boldsymbol{X(w)}$

❖ Linearity

If $x_1(t) \rightarrow X_1(w)$ and $x_2(t) \rightarrow X_2(w)$

Then $\mathrm{a}\, x_1(t) + \mathrm{b}\, x_2(t) \leftrightarrow \mathrm{a}\; X_1(w) + \mathrm{b}\; X_2(w)$

$$\mathcal{F}[\mathrm{a}\, x_1(t) + \mathrm{b}\, x_2(t)] = \int_{-\infty}^{\infty} [\mathrm{a}\, x_1(t) + \mathrm{b}\, x_2(t)]\, \boldsymbol{e^{-jwt} dt}$$

$$= \mathrm{a}\; X_1(w) + \mathrm{b}\; X_2(w)$$

Properties of Continuous Time Fourier Transform

❖ Time Shifting

If $\mathrm{x}(t) \rightarrow X(w)$

Then $x(t-t_0) \leftrightarrow \boldsymbol{X}(\boldsymbol{w})e^{-jwt_0}$

$\mathcal{F}[x(t-t_0)] = \int_{-\infty}^{\infty} x(t-t_0)\, \boldsymbol{e}^{-\boldsymbol{jwt}}\boldsymbol{dt}$

A change of variables is performed by letting $\tau = (t-t_0)$, which also yields $d\tau = dt, \tau \rightarrow -\infty \; as \; t \rightarrow -\infty, and\,, \tau \rightarrow \infty \; as \; t \rightarrow \infty.$

Therefore, $\mathcal{F}[x(t-t_0)] = \int_{-\infty}^{\infty} \boldsymbol{x}(\boldsymbol{\tau})\boldsymbol{e}^{-\boldsymbol{jw}(\boldsymbol{\tau}+\boldsymbol{t_0})}\boldsymbol{d\tau}$

$$= e^{-jwt_0} \int_{-\infty}^{\infty} \boldsymbol{x}(\boldsymbol{\tau})\boldsymbol{e}^{-\boldsymbol{jw\tau}}\boldsymbol{d\tau}$$

$$= \boldsymbol{X}(\boldsymbol{w})e^{-jwt_0}$$

- ✓ **Note that this means time delay is equivalent to a linear phase shift in the frequency domain (the phase shift is proportional to frequency).**
- ✓ **We refer to a system as an all-pass filter if**

$$|X(j\omega)| = 1 \quad \angle X(j\omega) \neq 0$$

- ✓ **Phase shift is an important concept in the development of surround sound.**

Properties of Continuous Time Fourier Transform

❖ Frequency Shifting

If $\mathrm{x}(t) \rightarrow X(w)$ Then $x(t)e^{jwt_0} \leftrightarrow \boldsymbol{X(w - w_0)}$

$$\mathcal{F}[x(t)e^{jwt_0}] = \int_{-\infty}^{\infty} [x(t)e^{jwt_0}]\, \boldsymbol{e^{-jwt} dt}$$

Therefore, $\mathcal{F}[x(t)e^{jwt_0}] = \int_{-\infty}^{\infty} x(t)e^{-j(w-w_0)t} \boldsymbol{dt}$

$$= \boldsymbol{X(w - w_0)}$$

❖ Time and Frequency Scaling

If $\mathrm{x}(t) \rightarrow X(w)$

Then $\mathrm{x}(at) \leftrightarrow \frac{1}{|a|} X(\frac{w}{a})$

$$\mathcal{F}[x(at)] = \int_{-\infty}^{\infty} x(at)e^{-jwt} \boldsymbol{dt}$$

✓ A change of variables is performed by letting $\tau = at$, which also yields $d\tau = adt, \tau \rightarrow -\infty\ as\ t \rightarrow -\infty, and\ , \tau \rightarrow \infty\ as\ t \rightarrow \infty.$

Therefore, $\mathcal{F}[x(at)] = \boldsymbol{\frac{1}{a}} \int_{-\infty}^{\infty} x(\tau)e^{-j(w/a)\tau} \boldsymbol{d\tau} = \frac{1}{|a|} X(\frac{w}{a})$

Properties of Continuous Time Fourier Transform

❖ Time Reversal

If $\mathrm{x}(t) \rightarrow X(w)$

Then $x(-t) \rightarrow X(-w)$

$$\mathcal{F}[x(at)] = \frac{\mathbf{1}}{\boldsymbol{a}} \int_{-\infty}^{\infty} x(\tau) e^{-j(w/a)\tau} \boldsymbol{d\tau} = \frac{1}{|a|} X\left(\frac{w}{a}\right)$$

Substituting a= -1,

$$\mathcal{F}[x(-t)] = \frac{1}{|-1|} X\left(\frac{w}{-1}\right)$$

$$\mathcal{F}[x(-t)] = X(-w)$$

✓ If its $x(t)$ is even, then its Fourier transform, i.e,
$x(-t) = x(t)$, then $X(-w) = X(w)$

✓ If its $x(t)$ is odd, then its Fourier transform, i.e,
$x(-t) = -x(t)$, then $X(-w) = -X(w)$

✓ Time reversal is equivalent to conjugation in the frequency domain.

Properties of Continuous Time Fourier Transform

❖ Convolution Property

If $x_1(t) \rightarrow X_1(w)$ and $x_2(t) \rightarrow X_2(w)$

Then $x_1(t) * x_2(t) \leftrightarrow X_1(w)\, X_2(w)$

✓ The Fourier transform maps the convolution of two signals into the product of their Fourier transforms.

$$\mathcal{F}[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} [x_1(t) * x_2(t)] e^{-jwt} dt$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} [x_1(\tau) * x_2(t-\tau)] \right) e^{-jwt} dt$$

✓ Interchanging the order of integration and noting that $x_1(\tau)$ does not depend on t gives

$$\mathcal{F}[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} x_1(\tau) \left(\int_{-\infty}^{\infty} [x_2(t-\tau)] e^{-jwt} dt \right) d\tau$$

$$\mathcal{F}[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} x_1(\tau) (X_2(w) e^{-jw\tau} d\tau$$

$$= X_2(w) \int_{-\infty}^{\infty} x_1(\tau)\, e^{-jw\tau} d\tau$$

$$x_1(t) * x_2(t) = X_2(w)\, X_1(w) = X_1(w) X_2(w)$$

$$x_1(t) * x_2(t) \leftrightarrow X_1(w) X_2(w)$$

Properties of Continuous Time Fourier Transform

❖ Multiplication Property

If $x_1(t) \to X_1(w)$ and $x_2(t) \to X_2(w)$

Then $x_1(t)x_2(t) \leftrightarrow \frac{1}{2\pi}[X_1(w) * X_2(w)]$

✓ The Fourier transform maps the multiplication of two signals into the convolution of their Fourier transforms.

$$\mathcal{F}[x_1(t)x_2(t)] = \int_{-\infty}^{\infty}[x_1(t)x_2(t)]e^{-jwt}dt$$

$$=\int_{-\infty}^{\infty}(\frac{1}{2\pi}\int_{-\infty}^{\infty}X_1(\theta)e^{j\theta t}d\theta)\, x_2(t)e^{-jwt}dt$$

✓ Interchanging the order of integration and noting that$X_1(\theta)$ does not depend on t gives

$$\mathcal{F}[x_1(t)x_2(t)] == \frac{1}{2\pi}\int_{-\infty}^{\infty}X_1(\theta)(\int_{-\infty}^{\infty}[x_2(t)e^{j\theta t}]e^{-jwt}dt)d\theta$$

$$\mathcal{F}[x_1(t)x_2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}X_1(\theta)\, X_2(w-\theta)d\theta$$

$$=X_2(w)\int_{-\infty}^{\infty}x_1(\tau)\, e^{-jw\tau}d\tau$$

$$=\frac{1}{2\pi}[X_1(w) * X_2\,(w)]$$

$$x_1(t) * x_2(t) \leftrightarrow \frac{1}{2\pi}[X_1(w)X_2(w)]$$

Properties of Continuous Time Fourier Transform

- Duality Property

  If $x(t) \leftrightarrow X(w)$

  Then $\mathrm{X}(t) \leftrightarrow 2\pi x(-w)$

By definition,

$$\boldsymbol{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)\, e^{jwt} dw}$$

$$\boldsymbol{2\pi x(t) = \int_{-\infty}^{\infty} X(w)\, e^{jwt} dw}$$

✓ Replacing t with $-t$ in the above equation gives

$$\boldsymbol{2\pi x(-t) = \int_{-\infty}^{\infty} X(w)\, e^{jwt} dw}$$

✓ **Now interchanging the variables** t and w yields

$$\boldsymbol{2\pi x(-w) = \int_{-\infty}^{\infty} X(t)\, e^{-jwt} dt}$$

$$\boldsymbol{2\pi x(-w)} = \mathcal{F}[x(t)]$$

Therefore, $\mathrm{X}(t) \leftrightarrow 2\pi x(-w)$

Properties of Continuous Time Fourier Transform

❖ Parseval's Relation

If $x(t) \leftrightarrow X(w)$

Then $E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(w)^2 dw$

✓ Parseval's theorem states that the signal energies of an energy signal and its Fourier transform are equal. definition,

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t)x^*(t)dt$$

$$= \int_{-\infty}^{\infty} x(t)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \boldsymbol{X(w)}\, \boldsymbol{e^{jwt}} \boldsymbol{dw}\right)dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(w)\left(\int_{-\infty}^{\infty} x(t)\, \boldsymbol{e^{-jwt}} dt\right)\boldsymbol{dw}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(w)X(w)\boldsymbol{dw}$$

Therefore,

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(w)^2 dw$$

Summary

TABLE 3.1 Properties of the Fourier Transform

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \leftrightarrow aX(\omega) + bV(\omega)$
Right or left shift in time	$x(t - c) \leftrightarrow X(\omega)e^{-j\omega c}$
Time scaling	$x(at) \leftrightarrow \frac{1}{a}X\left(\frac{\omega}{a}\right)\ a > 0$
Time reversal	$x(-t) \leftrightarrow X(-\omega) = \overline{X(\omega)}$
Multiplication by a power of t	$t^n x(t) \leftrightarrow j^n \frac{d^n}{d\omega^n} X(\omega)\ n = 1, 2, \ldots$
Multiplication by a complex exponential	$x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0)\quad \omega_0$ real
Multiplication by $\sin(\omega_0 t)$	$x(t)\sin(\omega_0 t) \leftrightarrow \frac{j}{2}[X(\omega + \omega_0) - X(\omega - \omega_0)]$
Multiplication by $\cos(\omega_0 t)$	$x(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}[X(\omega + \omega_0) + X(\omega - \omega_0)]$
Differentiation in the time domain	$\frac{d^n}{dt^n}x(t) \leftrightarrow (j\omega)^n X(\omega)\ n = 1, 2, \ldots$
Integration in the time domain	$\int_{-\infty}^{t} x(\lambda)\, d\lambda \leftrightarrow \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$
Convolution in the time domain	$x(t) * v(t) \leftrightarrow X(\omega)V(\omega)$
Multiplication in the time domain	$x(t)v(t) \leftrightarrow \frac{1}{2\pi}X(\omega) * V(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} x(t)v(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{X(\omega)}V(\omega)\, d\omega$
Special case of Parseval's theorem	$\int_{-\infty}^{\infty} x^2(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2\, d\omega$
Duality	$X(t) \leftrightarrow 2\pi x(-\omega)$

Discrete-Time Fourier Transform (DTFT)

❖ ***DTFT***

✓ The discrete-time aperiodic signal is treated in the same way as the continuous-time case, i.e., as an extension of the DTFS to the case of periodic signal as $N \rightarrow \infty$

✓ Consequently, the frequency axis is a **continuum.**

✓ The synthesis equation is now an integral, but still restricted to $w \in [-\pi, \pi]$.

❖ **Fourier Transform Representation of Aperiodic Discrete-Time Signals**

✓ The discrete-time Fourier Series (DTFS) representation of a periodic signal $x(n)$ (with period N and frequency $\omega_0 = \frac{2\pi}{N}$) can be written as

$$x(n) = \sum_{k=<N>} X_k e^{jk\omega_0 n}$$

Substituting $X_k = \frac{1}{N}\sum_{m=<N>} x(m)e^{-jk\omega_0 m}$ in the DTFS definition, we obtain

$$x(n) = \sum_{k=<N>}\left(\frac{1}{N}\sum_{m=<N>} x(m)e^{-jk\omega_0 m}\right)e^{jk\omega_0 n}$$

or

$$x(n) = \sum_{k=<N>}\left(\sum_{m=<N>} x(m)e^{-jk\omega_0 m}\right)e^{jk\omega_0 n}\,\frac{\omega_0}{2\pi}$$

Discrete-Time Fourier Transform (DTFT)

❖ ***FT Representation of Aperiodic Discrete-Time Signals***

✓ Since the inner and outer summation is over any arbitrary range of m of width N

$$x(n) = \sum_{k=k0}^{k0+N-1}\left(\sum_{m=-N/2}^{\frac{N}{2}-1} x(m)e^{-jk\omega_0 m}\right) e^{jk\omega_0 n}\,\frac{\omega_0}{2\pi} \qquad \text{for } N \text{ even}$$

$$x(n) = \sum_{k=k0}^{k0+N-1}\left(\sum_{m=-\frac{N-1}{2}}^{\frac{N-1}{2}} x(m)e^{-jk\omega_0 m}\right) e^{jk\omega_0 n}\,\frac{\omega_0}{2\pi} \qquad \text{for } N \text{ odd}$$

✓ The fundamental period $N \to \infty$. The inner summation covers an infinite range. The outer summation approaches an integral in $w = kw_0$ that covers a range of

$$k_0 \le k \le k_0 + N - 1$$
$$k_0 \le k < k_0 + N$$
$$k_0 \le \frac{\omega}{\omega_0} < k_0 + N$$
$$k_0\omega_0 \le \omega < k_0\omega_0 + N\omega_0$$
$$k_0\omega_0 \le \omega < k_0\omega_0 + 2\pi$$

✓ Therefore, $\quad x(n) = \frac{1}{2\pi}\int\left(\sum_{m=-\infty}^{\infty} x(m)e^{-j\omega m}\right)e^{j\omega n}\,d\omega$

✓ Denoting the summation by $X(e^{j\omega})$, $\quad x(n) = \frac{1}{2\pi}\int X(e^{j\omega})e^{j\omega n}\,d\omega$

Where $\quad X(e^{j\omega}) = \sum_{m=-\infty}^{\infty} x(m)e^{-j\omega m}$

Discrete-Time Fourier Transform (DTFT)

❖ ***FT Representation of Aperiodic Discrete-Time Signals***

✓ The $x(n)$ usually referred to as the ***synthesis equation,*** because it synthesizes an arbitrary signal from its complex exponential components.

$$x(n) = \frac{1}{2\pi}\int\left(\sum_{m=-\infty}^{\infty} x(m)e^{-j\omega m}\right)e^{j\omega n}\,d\omega$$

✓ Denoting the summation by $X\left(e^{j\omega}\right)$, $\quad \boldsymbol{x(n) = \frac{1}{2\pi}\int X(e^{j\omega})e^{j\omega n}\,d\omega}$

✓ On the other hand, $X(e^{j\omega})$ is referred to as the ***analysis equation,*** because it analyses how much of each complex exponential signal is present in the original signal.

$$\boldsymbol{X(e^{j\omega}) = \sum_{m=-\infty}^{\infty} x(m)e^{-j\omega m}}$$

✓ We call $X(e^{j\omega})$ the discrete-time Fourier transform (DTFT) of $x(n)$, and $x(n)$ the inverse discrete-time Fourier transform (IDTFT) of $X(e^{j\omega})$.

✓ This nomenclature can be represented as,

$X\left(e^{j\omega}\right) = DTFT[x(n)] = \mathcal{F}[x(n)] \quad and \;\; x(n) = IDTFT\left[X\left(e^{j\omega}\right)\right] = \mathcal{F}^{-1}\left[X\left(e^{j\omega}\right)\right]$

✓ DT Fourier Transform pair.

$$x(n) \leftrightarrow X\left(e^{j\omega}\right)$$

Discrete-Time Fourier Transform (DTFT)

❖ ***FT Representation of Aperiodic Discrete-Time Signals***

✓ The Fourier transform $X(e^{j\omega})$ is a complex function of the real variable ω and can be written in rectangular form as .

$$\boldsymbol{X(e^{j\omega}) = X_R(e^{j\omega}) + jXI(e^{j\omega})}$$

where $X_R(e^{j\omega})$ and $XI(e^{j\omega})$ are, respectively, the real and imaginary parts of $X(e^{j\omega})$

$$X_R(e^{j\omega}) = \frac{1}{2}[X(e^{j\omega}) + X^*(e^{j\omega})]$$

$$X_I(e^{j\omega}) = \frac{1}{2j}[X(e^{j\omega}) - X^*(e^{j\omega})]$$

where $X^*(e^{j\omega})$ denotes the complex conjugate of $X(e^{j\omega})$.

✓ The Fourier transform $X(e^{j\omega})$ can be alternatively be expressed in the polar form

$$X(e^{j\omega}) = |X(e^{j\omega})|e^{j\theta(\omega)}$$

where $\theta(\omega) = \angle X(e^{j\omega})$

✓ The relation between the rectangular and polar forms of $X(e^{j\omega})$ follows as

$$X_R(e^{j\omega}) = |X(e^{j\omega})|\cos(\theta(\omega))$$

$$X_I(e^{j\omega}) = |X(e^{j\omega})|\sin(\theta(\omega))$$

Discrete-Time Fourier Transform (DTFT)

❖ ***FT Representation of Aperiodic Discrete-Time Signals***

$$\left|X\left(e^{j\omega}\right)\right| = \sqrt{(X_R{}^2\left(e^{j\omega}\right) + X_I{}^2\left(e^{j\omega}\right))}$$

$$\theta(\omega) = \angle X\left(e^{j\omega}\right) = \tan^{-1}\left[\frac{X_I\left(e^{j\omega}\right)}{X_R\left(e^{j\omega}\right)}\right]$$

✓ Thus, for a real signal, it follows that

$$\left|\boldsymbol{X}\left(\boldsymbol{e}^{-j\omega}\right)\right| = \sqrt{(\boldsymbol{X}_R{}^2\left(\boldsymbol{e}^{-j\omega}\right) + \boldsymbol{X}_I{}^2\left(\boldsymbol{e}^{-j\omega}\right))}$$

$$\left|\boldsymbol{X}\left(\boldsymbol{e}^{-j\omega}\right)\right| = \sqrt{(\boldsymbol{X}_R{}^2\left(\boldsymbol{e}^{j\omega}\right) + \boldsymbol{X}_I{}^2\left(\boldsymbol{e}^{j\omega}\right))}$$

$$\left|\boldsymbol{X}\left(\boldsymbol{e}^{-j\omega}\right)\right| = \left|\boldsymbol{X}\left(\boldsymbol{e}^{j\omega}\right)\right|$$

✓ The magnitude spectrum $\left|\boldsymbol{X}\left(\boldsymbol{e}^{-j\omega}\right)\right|$ is an even function of w. Likewise, for a real signal, we note from that as

$$\angle X\left(e^{-j\omega}\right) = \tan^{-1}\left[\frac{X_I\left(e^{-j\omega}\right)}{X_R\left(e^{-j\omega}\right)}\right] = \tan^{-1}\left[-\frac{X_I\left(e^{j\omega}\right)}{X_R\left(e^{j\omega}\right)}\right] = -\tan^{-1}\left[\frac{X_I\left(e^{j\omega}\right)}{X_R\left(e^{j\omega}\right)}\right] = -\angle X\left(e^{j\omega}\right)$$

$$\angle X\left(e^{-j\omega}\right) = -\angle X\left(e^{j\omega}\right)$$

The phase spectrum $\angle X\left(e^{j\omega}\right)$ is an odd function of w.

Discrete-Time Fourier Transform (DTFT)

❖ ***Periodicity of DTFT***

✓ The DTFT is a periodic function in w with a period 2π. That is

$$X\left(e^{j(\omega+2\pi)}\right) = X\left(e^{j\omega}\right)$$

Proof: By definition,

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

✓ For any integer k, we have

$$X\left(e^{j(\omega+2\pi k)}\right) = \sum_{n=-\infty}^{\infty} x(n)e^{-j(\omega+2\pi k)n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi kn}\ e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = X\left(e^{j\omega}\right)$$

✓ We have used the fact that $e^{-j2\pi kn} = 1$. Hence $X\left(e^{j\omega}\right)$ is periodic with period 2π.

✓ However, this property is a consequence of the fact that frequency range for any discrete-time signal is unique over the frequency interval of $(-\pi, \pi)$ or $(0, 2\pi)$, and any frequency outside this interval is equivalent to a frequency within this interval.

Discrete-Time Fourier Transform (DTFT)

❖ ***Convergence of DTFT***

✓ An infinite series may or may not converge. The Fourier transform $X\left(e^{j\omega}\right)$ of $x(n)$ is said to exist if the series converges in some sense.

$$X_{K(e^{j\omega})} = \sum_{n=-K}^{K} x(n)e^{-j\omega n}$$

✓ The partial sum of the weighted complex exponentials. Then, for uniform convergence of $X\left(e^{j\omega}\right)$,

$$\lim_{K\to\infty} X_{(e^{j\omega})} - X_{K(e^{j\omega})}=0$$

$$\lim_{K\to\infty} X_{K(e^{j\omega})} = X_{(e^{j\omega})}$$

✓ Uniform convergence is guaranteed if x(n) is absolutely summable. Indeed, if

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

$$|X(e^{j\omega}| = |\textstyle\sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}| = \leq \textstyle\sum_{n=-\infty}^{\infty} x(n)||e^{-j\omega n}|$$

$$|X(e^{j\omega})| \leq \textstyle\sum_{n=-\infty}\infty |x(n)| < \infty$$

✓ Some sequences are not absolutely summable, but they are square summable,

$$\textstyle\sum_{n=-\infty}^{\infty} |x(n)|^2 \leq (\textstyle\sum_{n=-\infty}^{\infty} |x(n)|)^2$$

✓ For such sequences, we can impose a mean-square convergence condition:

$$\lim_{K\to\infty} \int_{-\pi}^{\pi} |X\,(e^{j\omega})\text{-}\, X_{K(e^{j\omega})}|^2 \mathrm{dw} = 0$$

Discrete-Time Fourier Transform (DTFT)

❖ ***Gibbs Phenomenon***

✓ Consider a finite energy signals of Fourier transform $X(e^{j\omega})$ of $x(n)$ is

$$X(e^{j\omega}) = \begin{cases} 1 & |\omega] \leq d\omega_c \\ 0 & \omega_c < |\omega| \leq \pi \end{cases}$$

✓ The inverse DTFT of $X(e^{j\omega})$ is given by

$$x(n) = \frac{1}{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega = \frac{1}{2\pi} \int_{w_c}^{w_c} e^{j\omega n} \; d\omega$$

$$= \frac{1}{2\pi} \frac{e^{jwn}}{jn} \Big|_{-\omega_c}^{\omega_c} = \frac{\sin(\omega_c n)}{n\pi} n \neq 0$$

✓ For $n = 0$, the inverse Fourier transform expression reduces to

$$x(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X\,(e^{j\omega})\, d\omega = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} 1\, d\omega = \frac{\omega_c}{\pi}$$

Hence $$X(e^{j\omega}) = \begin{cases} \dfrac{\omega_c}{\pi} & n = 0 \\ \dfrac{\sin(\omega_c n)}{n\pi} & n \neq 0 \end{cases}$$

$$x(n) = \frac{\sin(\omega_c n)}{n\pi} \quad -\infty < n < \infty \quad || \quad \sum_{-\infty}^{\infty} x(n) e^{-j\omega n} = \sum_{-\infty}^{\infty} \frac{\sin(\omega_c n)}{n\pi} e^{-j\omega n}$$

$$X_k(e^{j\omega}) = \sum_{n=-K}^{K} \frac{\sin(\omega_c n)}{n\pi} e^{-j\omega n}$$

Discrete-Time Fourier Transform (DTFT)

- ***Properties of Discrete-time Fourier Transform***
- Fourier transform possesses a number of important properties that are useful for developing conceptual insights into the transform and into the relationship between the time-domain and frequency-domain representations of a signal.

- **Linearity**
- ✓ The relationship between a discrete-time sequence $x(n)$ and its Fourier transform $X(e^{j\omega})$

$$x(n) \leftrightarrow X(e^{j\omega})$$

If $\quad x_1(n) \leftrightarrow X_1(e^{j\omega}) \quad$ and $\quad x_2(n) \leftrightarrow X_2(e^{j\omega})$

then $\quad ax_1(n) + bx_2(n) \leftrightarrow aX_1(e^{j\omega}) + bX_2(e^{j\omega})$

Proof

$$\mathcal{F}[ax_1(n) + bx_2(n)] = \sum_{n=-\infty}^{\infty} [ax_1(n) + bx_2(n)]e^{-j\omega n}$$

$$= a\sum_{n=-\infty}^{\infty} x_1(n)e^{-j\omega n} + b\sum_{n=-\infty}^{\infty} x_2(n)e^{-j\omega n}$$

$$= aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

Discrete-Time Fourier Transform (DTFT)

❖ ***Properties of Discrete-time Fourier Transform***

- The relationship between the time-domain and frequency-domain representations of a signal.

❖ **Time Shifting**

✓ The relationship between a discrete-time sequence $x(n)$ and its Fourier transform $X(e^{j\omega})$

If $\quad x(n) \leftrightarrow X(e^{j\omega})$

then $\quad x(n-n_0) \leftrightarrow X(e^{j\omega})\, e^{-j\omega n_0}$

Proof The Fourier transform of $x(n-n_0)$ ***is***

$$\mathcal{F}[x(n-n_0)] = \sum_{n=-\infty}^{\infty} [x(n-n_0)]e^{-j\omega n}$$

Let $(n-n_0) = m$

$$\mathcal{F}[x(n-n_0)] = e^{-j\omega n_0} \sum_{m=-\infty}^{\infty} [x(m)]e^{-j\omega m} = X(e^{j\omega})\, e^{-j\omega n_0}$$

✓ When a signal is shifted in time, the magnitude of its DTFT remains unaltered.

$$\mathcal{F}[x(n)] = X(e^{j\omega}) = |X(e^{j\omega})|\, e^{j\angle X(e^{j\omega})}$$

Then $\quad \mathcal{F}[x(n-n_0)] = X_1(e^{j\omega})\, e^{-j\omega n_0} = |X(e^{j\omega})|\, e^{j\angle X(e^{j\omega}-\omega n_0)}$

Discrete-Time Fourier Transform (DTFT)

❖ ***Properties of Discrete-time Fourier Transform***

- The relationship between the time-domain and frequency-domain representations of a signal.

❖ **Frequency Shifting**

✓ The relationship between a discrete-time sequence $x(n)$ and its Fourier transform $X(e^{j\omega})$

If $\quad x(n) \leftrightarrow X(e^{j\omega})$

then $\quad x(n)e^{jn\omega_0} \leftrightarrow X(e^{j(\omega-\omega_0)})$

Proof The Fourier transform of $x(n)e^{jn\omega_0}$ is

$$\mathcal{F}[x(n)e^{jn\omega_0}] = \sum_{n=-\infty}^{\infty} x(n)e^{jn\omega_0}e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)e^{-j(\omega-\omega_0)n}$$

$$= X(e^{j(\omega-\omega_0)})$$

✓ Hence, a frequency shift corresponds to multiplication in time domain by a complex sinusoid whose frequency is equal to the frequency shift.

Discrete-Time Fourier Transform (DTFT)

❖ ***Properties of Discrete-time Fourier Transform***

- The relationship between the time-domain and frequency-domain representations of a signal.

❖ **Time Reversal**

✓ The relationship between a discrete-time sequence $x(n)$ and its Fourier transform $X(e^{j\omega})$

If $$x(n) \leftrightarrow X(e^{j\omega})$$

then $$y(n) = x(-n) \longleftrightarrow Y(e^{j\omega}) = X(e^{-j\omega})$$

Proof By definition,

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y(n) e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x(-n) e^{-j\omega n}$$

✓ Substituting $m = -n$ into the equation, we obtain

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(m) e^{j\omega m} = \sum_{n=-\infty}^{\infty} x(m) e^{-j(-\omega)n} = X(e^{-j\omega})$$

✓ If $x(n)$ is even, then its Fourier transform is also even

$$x(n) = x(-n) \leftrightarrow X(e^{j\omega}) = X(e^{-j\omega})$$

✓ If $x(n)$ is odd, then so is its Fourier transform, that is

$$x(n) = -x(-n) \leftrightarrow X(e^{j\omega}) = -X(e^{-j\omega})$$

Discrete-Time Fourier Transform (DTFT)

❖ ***Properties of Discrete-time Fourier Transform***

- The relationship between the time-domain and frequency-domain representations of a signal.

❖ **Time Expansion**

✓ Let m be a positive integer and define the signal

$$x_{(m)}(n) = x\left(\frac{n}{m}\right) \quad \text{if } n \text{ is a multiple of } m$$
$$= 0 \quad \text{if } n \text{ is not a multiple of } m$$

$$y(n) = x_{(m)}(n) \leftrightarrow Y(e^{j\omega}) = X(e^{jm\omega})$$

Proof By definition,

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y(n)e^{-j\omega m} = \sum_{n=-\infty}^{\infty} x_{(m)}(n)e^{-j(\omega)n} = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{m}\right)e^{-j(\omega)n}$$

✓ A change of variables is performed by letting $r = \frac{m}{n}$, which also yields $r = -\infty$ as $n = -\infty$, and r $= \infty$ as $n = \infty$. Therefore,

$$y(e^{j\omega}) = \sum_{r=-\infty}^{\infty} x(r)e^{-j\omega m\tau} = X(e^{jm\omega})$$

✓ The signal spread out and slowed down in time by taking $m > 1$, its Fourier transform is compressed.

Discrete-Time Fourier Transform (DTFT)

❖ ***Properties of Discrete-time Fourier Transform***

- The relationship between the time-domain and frequency-domain representations of a signal.

❖ **Differentiation in Time Domain**

✓ The discrete-time parallel to the differentiation property of the continuous-time Fourier transform involves the use of the first-difference operation.

If $\quad x(n) \leftrightarrow X(e^{j\omega})$

Then

$$y(n) = x(n) - x(n-1) \leftrightarrow Y(e^{j\omega}) = (1 - e^{-j\omega})X(e^{j\omega})$$

Proof Given that $\quad x(n) = X(e^{j\omega})$

✓ Using the time-shifting property, we get

$$x(n-1) \leftrightarrow X(e^{j\omega})e^{-j\omega}$$

✓ Now, using the linearity property, we get

$$x(n) - x(n-1) \leftrightarrow X(e^{j\omega}) - X(e^{j\omega})e^{-j\omega}$$

$$x(n) - x(n-1) \leftrightarrow (1 - e^{-j\omega})X(e^{j\omega})$$

✓ A common use of this property is in situations where evaluation of the Fourier transform is easier for the first difference than for the original sequence.

Discrete-Time Fourier Transform (DTFT)

❖ ***Properties of Discrete-time Fourier Transform***

- The relationship between the time-domain and frequency-domain representations of a signal.

❖ **Differentiation in Frequency Domain**

If $x(n) \leftrightarrow X(e^{j\omega})$

Then $-jnx(n) \leftrightarrow \dfrac{dX(e^{j\omega})}{d\omega}$ ***or*** $nx(n) \leftrightarrow j\dfrac{dX(e^{j\omega})}{d\omega}$

Proof By definition,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

✓ Differentiating both sides with respect to w, we obtain

$$\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{\infty} ([-jnx(n)])e - j\omega n$$

$$\frac{dX(e^{j\omega})}{d\omega} = F[-jnx(n)]$$

✓ Therefore, Multiplying both sides by j, we get

$$-jnx(n) \leftrightarrow \frac{dX(e^{j\omega})}{d\omega}$$

$$nx(n) \leftrightarrow j\frac{dX(e^{j\omega})}{d\omega}$$

Discrete-Time Fourier Transform (DTFT)

- ***Properties of Discrete-time Fourier Transform***
- The relationship between the time-domain and frequency-domain representations of a signal.
- **Convolution Property**
- ✓ The relationship between a discrete-time sequence $x(n)$ and its Fourier transform $X(e^{j\omega})$

$$x(n) \leftrightarrow X(e^{j\omega})$$

If $\quad x_1(n) \leftrightarrow X_1(e^{j\omega}) \quad$ and $\quad x_2(n) \leftrightarrow X_2(e^{j\omega})$

then $\quad x_1(n) * x_2(n) \leftrightarrow X_1(e^{j\omega}) * X_2(e^{j\omega})$

- ✓ The Fourier transform maps the convolution of two signals into the product of their Fourier transforms.

Proof The Fourier transform of $x_1(n) * x_2(n)$ is

$$F[x_l(n) * x_2(n)] = \sum_{n=-\infty}^{\infty} [x_1(n) * x_2(n)] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} x_1(m) x_2(n-m)\right) e^{-j\omega n}$$

- ✓ By interchanging the order of summation,

$$F[x,(n) * x_2(n)] = \sum_{m=-\infty}^{\infty} x_1(m) \left(\sum_{n=-\infty}^{\infty} x_2(n-m) e^{-j\omega n}\right)$$

Discrete-Time Fourier Transform (DTFT)

❖ **Convolution Property**

✓ The relationship between a discrete-time sequence $x(n)$ and its Fourier transform $X(e^{j\omega})$

If $\quad x_1(n) \leftrightarrow X_1(e^{j\omega}) \quad$ and $\quad x_2(n) \leftrightarrow X_2(e^{j\omega})$

then

$$x_1(n) * x_2(n) \leftrightarrow X_1(e^{j\omega}) * X_2(e^{j\omega})$$

Proof The Fourier transform of $x_1(n) * x_2(n)$ is

$$F[x,(n) * x_2(n)] = \sum_{m=-\infty}^{\infty} x_1(m)\left(\sum_{n=-\infty}^{\infty} x_2(n-m)e^{-j\omega n}\right)$$

✓ Applying the time-shifting property, the bracketed term is $X_2(e^{j\omega})e^{-j\omega m}$. Substituting this into this equation yields

$$F[x,(n) * x_2(n)] = \sum_{m=-\infty}^{\infty} x_1(m)\left(X_2(e^{j\omega})e^{-j\omega m}\right)$$

$$= X_2(e^{j\omega}) \sum_{m=-\infty}^{\infty} x_1(m)\left(e^{-j\omega m}\right)$$

✓ Therefore,

$$F[x_l(n) * x_2(n)] \leftrightarrow X_2(e^{j\omega}) * X_1(e^{j\omega}) \leftrightarrow X_1(e^{j\omega}) * X_2(e^{j\omega})$$

$$x_1(n) * x_2(n) \leftrightarrow X_1(e^{j\omega}) * X_2(e^{j\omega})$$

Discrete-Time Fourier Transform (DTFT)

❖ **Accumulation Property**

✓ The relationship between a discrete-time sequence $x(n)$ and its Fourier transform $X(e^{j\omega})$

$$x(n) \leftrightarrow X(e^{j\omega})$$

then

$$\sum_{k=-\infty}^{\infty} x(k) \leftrightarrow \frac{1}{1-e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{m=-\infty}^{\infty} \delta(\omega - 2\pi m)$$

Proof Convolving a signal $x(n)$ with a unit step function $u(n)$, we obtain

$$x(n) * u(n) = \sum_{k=-\infty}^{\infty} x(k) u(n-k)$$

✓ Since $u(n-k) = \begin{cases} 1 & n-k \geq 0 \quad \rightarrow k \leq n \\ 0 & n-k < 0 \quad \rightarrow k > n \end{cases}$

$$x(n) * u(n) = \sum_{k=-\infty}^{\infty} x(k)$$

✓ Now we can prove the accumulation property of the Fourier transform.

$$\sum_{k=-\infty}^{\infty} x(k) = x(n) * u(n) \qquad F[\sum_{k=-\infty}^{\infty} x(k)] = F[x(n) * u(n)]$$

Discrete-Time Fourier Transform (DTFT)

❖ **Accumulation Property**

$$\sum_{k=-\infty}^{\infty} x(k) = x(n) * u(n) \qquad F[\sum_{k=-\infty}^{\infty} x(k)] = F[x(n) * u(n)]$$

✓ Using the convolution property, we obtain

$$F\left[\sum_{k=-\infty}^{\infty} x(k)\right] = X(e^{j\omega})U(e^{j\omega})$$

$$= X(e^{j\omega})\left(\frac{1}{1-e^{-j\omega}} + \pi \sum_{m=-\infty}^{\infty} \delta(\omega - 2\pi m)\right)$$

$$= X(e^{j\omega})\left(\frac{1}{1-e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j\omega}) \sum_{m=-\infty}^{\infty} \delta(\omega - 2\pi m)\right)$$

$$= X(e^{j\omega})\left(\frac{1}{1-e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{m=-\infty}^{\infty} \delta(\omega - 2\pi m)\right)$$

✓ Therefore,

$$F\left[\sum_{k=-\infty}^{\infty} x(k)\right] \leftrightarrow X(e^{j\omega})\left(\frac{1}{1-e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{m=-\infty}^{\infty} \delta(\omega - 2\pi m)\right)$$

Discrete-Time Fourier Transform (DTFT)

❖ Multiplication (Modulation or Windowing) Property

✓ The relationship between a discrete-time sequence $x(n)$ and its Fourier transform $X(e^{j\omega})$

If $\quad x_1(n) \leftrightarrow X_1(e^{j\omega}) \quad$ and $\quad x_2(n) \leftrightarrow X_2(e^{j\omega})$

then

$$x_1(n)x_2(n) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})\,d\theta$$

Proof The Fourier transform of $x_1(n)x_2(n)$ is given by

$$F[x_1(n)x_2(n)] = \sum_{n=-\infty}^{\infty}[x_1(n)x_2(n)]$$

$$= \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})\,d\theta$$

$$= \sum_{n=-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\theta n)})\,d\theta\, x_2(n)\right]e^{-j\omega n}$$

Discrete-Time Fourier Transform (DTFT)

❖ Multiplication (Modulation or Windowing) Property

Proof The Fourier transform of $x_1(n)x_2(n)$ is given by

$$F[x_1(n)x_2(n)] = \sum_{n=-\infty}^{\infty}[x_1(n)x_2(n)] \qquad = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})\,d\theta$$

$$= \sum_{n=-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\theta n)})\,d\theta\, x_2(n)\right]e^{-j\omega n}$$

✓ Applying the frequency-shifting property, the bracketed term is $X_2(e^{j(\omega-\theta)})$. Substituting this into the equation yields

$$= \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})\left(\sum_{n=-\infty}^{\infty}[x_2(n)e^{j\theta n}]e^{-j\omega n}\right)d\theta \qquad = \frac{1}{2\pi}\int_{2\pi} x_1(e^{j\theta})x_2(e^{j(\omega-\theta)})\,d\theta$$

✓ Therefore, $\quad x_1(n)x_2(n) \leftrightarrow \frac{1}{2\pi}[X_1(e^{j\omega}) \circledast X_2(e^{j\omega})]$

Discrete-Time Fourier Transform (DTFT)

❖ **Conjugation and Conjugate Property**

✓ If $x(n) = X(e^{j\omega})$

✓ Then $x^*(n) = X^*(e^{-j\omega})$

Proof The Fourier transform of $x^*(n)$ is given by

$$F[x^*(n)] = \sum_{n=-\infty}^{\infty} x^*(n) e^{-j\omega n} = \left[\sum_{n=-\infty}^{\infty} x(n) e^{j\omega n}\right]^*$$

$$= \left[\sum_{n=-\infty}^{\infty} x(n) e^{-j(-\omega)n}\right]^*$$

$$F[x^*(n)] = \left[X(e^{-j\omega})\right]^* = X^*(e^{-j\omega})$$

✓ **Case I** If is real, that is, if $x^*(n) = x(n)$ Then $F[x^*(n)] = F[x(n)]$

✓ Then $X^*(e^{-j\omega}) = X(e^{j\omega})$ $X(e^{-j\omega}) = X^*(e^{j\omega})$ (conjugate symmetric)

✓ The DTFT of a real signal is *conjugate symmetric*

Discrete-Time Fourier Transform (DTFT)

❖ **Conjugation and Conjugate Property**

✓ If $x(n) = X(e^{j\omega})$

✓ Then $x^*(n) = X^*(e^{-j\omega})$

✓ **Case I** If $x(n)$ is real and even, that is, if $x(n) = x^*(n) = x(-n)$

$$F[x^*(n)] = F[x(n)] = F[x(-n)]$$

$$X^*(e^{-j\omega}) = X(e^{j\omega}) == X(e^{-j\omega})$$

✓ Then $X(e^{j\omega}) = X^*(e^{j\omega}) = X(e^{-j\omega})$ (real and even)

✓ The DTFT of a real signal $X(e^{j\omega})$ is *real and even*

Discrete-Time Fourier Transform (DTFT)

- ❖ **Parseval's Relation**
- ✓ Let $x(n)$ be an energy signal and if $x(n) \leftrightarrow X(e^{j\omega})$
- ✓ Then $E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int \left|X(e^{j\omega})\right|^2 \mathrm{d}\omega$

Proof: Consider the left-hand side equation, we have

$$\mathrm{E}x = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-\infty}^{\infty} x(n)x^*(n) = \sum_{n=-\infty}^{\infty} x(n)\left(\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,\mathrm{d}\omega\right)$$

$$= \sum_{n=-\infty}^{\infty} x(n)\left(\frac{1}{2\pi}\int_{2\pi} X^*(e^{j\omega})e^{-j\omega n}\,\mathrm{d}\omega\right)$$

$$= \frac{1}{2\pi}\int_{2\pi} X^*(e^{jw})\left(\sum_{n=-\infty}^{\infty} x(n)e^{-j\omega t}\right)\mathrm{d}\omega$$

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{2\pi} X^*(e^{j\omega})X(e^{j\omega})\,\mathrm{d}\omega = \frac{1}{2\pi}\int_{2\pi} \left|X(e^{jw})\right|^2 \mathrm{d}\omega$$

Discrete-Time Fourier Transform (DTFT)

❖ Fourier Transform of Periodic Signals

✓ The signal $x(n)$ has the DTFT representation

$$x(n) = \sum_{k=\langle N \rangle} X_k e^{jk\omega_0 n}$$

***Proof*:** Taking Fourier Transform on both sides, we get

$$F[x(n)] = F\left[\sum_{\langle N \rangle} X_k e^{jk\omega_0 n}\right]$$

$$X(e^{j\omega}) = \sum_{K=\langle N \rangle} X_k F\left[e^{jk\omega_0 n}\right]$$

Using $e^{jw_0 n} \leftrightarrow \sum_{m=-\infty}^{m=\infty} 2\pi\delta(w - w_0 - 2\pi m)$

$$F\left[e^{jk\omega_0 n}\right] = 2\pi\delta(\omega - k\omega_0), \qquad 0 \le \omega \le 2\pi$$

$$X(e^{j\omega}) = \sum_{k=\langle N \rangle} X_k 2\pi\delta(\omega - k\omega_0), \qquad 0 \le \omega \le 2\pi$$

$$X(e^{j\omega}) = 2\pi \sum_{k=\langle N \rangle} X_k \delta(\omega - k\omega_0) \quad, 0 \le \omega \le 2\pi$$

Since the DTFT is periodic with period 2π. $Nw_0 = 2\pi$. Thus, $X(e^{j\omega})$ can be compactly

$$X(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} X_k \delta(\omega - k\omega_0)$$

Discrete-Time Fourier Transform (DTFT)

❖ **Signal Transmission Through Linear Time-invariant Systems**

✓ The signal $x(n)$ and $y(n)$ are the input and output of a linear time-invariant systems (LTI) system with impulse response $h(n)$, then

$$y(n) = x(n) * h(n)$$

✓ Application of the time convolution property yields

$$F[y(n)] = F[x(n)] * F[h(n)]$$

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

$$|y(e^{j\omega})|e^{j\angle y(e^{j\omega})} = \left[X(e^{j\omega})e^{j\angle xe^{j\omega}}\right]\left(|H(e^{j\omega})|e^{j\angle H(e^{j\omega})}\right)$$

$$|y(e^{j\omega})| = |x(e^{j\omega})||H(e^{j\omega})|$$

$$\angle Y(e^{j\omega}) = \angle X(e^{j\omega}) + \angle H(e^{j\omega})$$

❖ **Response to Complex Exponentials**

$$y(n) = x(n) * h(n) = \sum_{m=-\infty}^{\infty} h(m)x(n-m)$$

$$y(n) = \sum_{m=-\infty}^{\infty} h(m)e^{j\omega_0(n-m)} \qquad y(n) = \sum_{m=-\infty}^{\infty} h(m)e^{-j\omega_0 m}e^{j\omega_0 n}$$

$$y(n) = H(e^{j\omega_0})e^{j\omega_0 n} \qquad H(e^{j\omega_0}) = \sum_{m=-\infty}^{\infty} h(m)e^{-j\omega_0 m}$$

$$H(e^{j\omega_0}) = |H(e^{j\omega_0})|e^{j\angle H(e^{j\omega_0})} = |H(e^{j\omega_0})|e^{j\theta(\omega_0)}$$

Discrete-Time Fourier Transform (DTFT)

❖ **Response to Sinusoidal Signal**

✓ The signal $x(n)$ be the input to the linear time-invariant systems (LTI) system be

$$x(n) = A\cos(\omega_0 n), \qquad -\infty < n < \infty$$

$$x(n) = \frac{A}{2}e^{j\omega_0 n} + \frac{A}{2}e^{-j\omega_0 n}$$

✓ Due to the linearity, the response $y(n)$ to the input $x(n)$ is given by

$$y(n) = \frac{A}{2}H\left(e^{j\omega_0}\right)e^{j\omega_0 n} + \frac{A}{2}H\left(e^{-j\omega_0}\right)e^{j\omega_0 n}$$

$$y(n) = A\left|H\left(e^{-j\omega_0}\right)\right|\cos[\left(\omega_0 n + H\left(e^{j\omega_0}\right)\right)]$$

$$x(n) = \sum_{i=1}^{L} A_i\cos(\omega_i n + \phi_i), \ -\infty < n < \infty$$

✓ The input to the system consists of an arbitrary linear combination of sinusoidal signals of the form

$$y(n) = \sum_{i=1}^{L} A_i\left|H\left(e^{j\omega_i}\right)\right|\cos[\omega_i n + \phi_i + \theta(\omega_i)], \qquad -\infty < n < \infty$$

❖ **Response to Causal Exponential Sequence**

$$x(n) = e^{j\omega_0 n}u(n)$$

$$y(n) = h(n) * x(n) = \sum_{m}^{\infty} h(m)x(n-m)$$

Discrete-Time Fourier Transform (DTFT)

❖ **Response to Causal Exponential Sequence**

$$x(n) = e^{j\omega_0 n}u(n)$$

$$y(n) = h(n) * x(n) = \sum_{m}^{\infty} h(m)x(n-m)$$

$$\sum_{m=0}^{\infty} h(m)e^{j\omega_0(n-m)}u(n-m) = \sum_{m=0}^{n} h(m)e^{-j\omega_0 m}e^{j\,\omega_0 n}$$

$$\left(\sum_{m=0}^{\infty} h(m)e^{-j\omega_0 m}\right)e^{j\omega_0 n} + \left(\sum_{m=n+1}^{\infty} h(m)e^{-j\omega_0 m}\right)e^{j\omega_0 n} - \left(\sum_{m=n+1}^{\infty} h(m)e^{-j\omega_0 m}\right)e^{j\omega_0 n}$$

$$y(n) = H\left(e^{j\omega_0}\right)e^{j\omega_0 n} - \left(\sum_{m}^{\infty} h(m)e^{-j\omega_0 m}\right)e^{j\omega_0 n}, \qquad n > 0$$

$$ytr(n) = -\left(\sum_{m=n+1}^{\infty} h(m)e^{-j\omega_0 m}\right)e^{j\omega_0 n}$$

$$|ytr(n)| = \left|\sum_{m=n+1}^{\infty} h(m)e^{-j\omega_0(m-n)}\right| \leq \sum_{m=n+1}^{\infty} |h(m)| \leq \sum_{m=0}^{\infty} |h(m)|$$

Linear and Non-linear Phase

- ✓ Consider a discrete-time LTI system with impulse response $h(n)$ and frequency response $H(e^{jw})$.
- ✓ A signal $x(n)$ with Fourier transform $\mathrm{X}(e^{jw})$ be applied to the input of the system.
- ✓ A signal $y(n)$ with Fourier transform $\mathrm{Y}(e^{jw})$ denote the output of the system.
  - In distortion less transmission, the input $x(n)$ and the output $y(n)$ satisfy the condition

$$\boldsymbol{y(n) = Gx(n - n_d)}$$

    - *Where the constant G accounts for a change in amplitude and the constant* $\mathbf{n_d}$ *accounts for a delay in transmission.*

$$\boldsymbol{Y(e^{j\omega}) = \mathrm{GX}(e^{j\omega})e^{-j\omega n_d}}$$

$$\boldsymbol{\frac{Y(e^{j\omega})}{X(e^{j\omega})} = H(e^{j\omega}) = Ge^{-j\omega n_d}}$$

- ❑ **The magnitude response $|H(e^{j\omega})|$ must be a constant. $|H(e^{j\omega})| = G$**
- ❑ **The phase response $\angle H(e^{j\omega})$ must be a linear function of w with slope -n_d and intercept zero.**

$$\angle H(e^{j\omega}) = -\mathrm{w}n_d$$

Phase Delay and Group Delay

❖**Phase delay:** The time delay experienced by a single-frequency signal (i.e., a sinusoidal signal) when the signal passes through a system is referred to as the system phase delay and is defined as

$$\tau_p(w) = -\frac{\angle H(e^{j\omega})}{w} = -\frac{\theta(w)}{w}$$

✓Assume that the input signal is a single frequency signal

$$x(n) = A\cos(w_o n + \theta) = \frac{A}{2} e^{j(w_o n+\theta)} + \frac{A}{2} e^{-j(w_o n+\theta)}$$

✓The system frequency response is

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{\angle H(e^{j\omega})}$$

✓Then, the system output signal is given by

$$y(n) = \frac{A}{2}|H(e^{j\omega_0})| e^{j(\omega_0 n+\theta+\angle H(e^{j\omega_0}))} + \frac{A}{2}|H(e^{-j\omega_0})| e^{-j(\omega_0 n+\theta-\angle H(e^{-j\omega_0}))}$$

✓Since

$$|H(e^{j\omega_0})| = |H(e^{-j\omega_0})| \text{ and } \angle H(e^{j\omega_0}) = -\angle H(e^{-j\omega_0})$$

✓Therefore, we get

$$y(n) = \frac{A}{2}|H(e^{j\omega_0})| e^{j(\omega_0 n+\theta+\angle H(e^{j\omega_0}))} + \frac{A}{2}|H(e^{j\omega_0})| e^{-j(\omega_0 n+\theta+\angle H(e^{J\omega_0}))}$$

Phase Delay and Group Delay

$$y(t) = A\left|H\left(e^{j\omega_0}\right)\right|cos\left(\omega_0 n + \angle H\left(e^{j\omega_0}\right) + \theta\right)$$

$$y(t) = A\left|H\left(e^{j\omega_0}\right)\right|cos\left(\omega_0\left(n + \frac{\angle H\left(e^{j\omega_0}\right)}{\omega_0}\right) + \theta\right)$$

$$y(n) = A\left|H\left(e^{j\omega_0}\right)\right| cos\left(\omega_0\left(n - \tau_p(\omega_0)\right) + \theta\right)$$

✓ **Where**

$$\tau_p(\omega)\Big|_{\omega=\omega_0} = \tau_p(\omega_0) = -\frac{\angle H\left(e^{j\omega_0}\right)}{\omega_0} = -\frac{\theta(\omega_0)}{\omega_0}$$

❖ **Group delay:** When the input signal contains many sinusoidal components with different frequencies that are not harmonically related, each component will go through different phase delays when processed by a frequency-selective LTI system, and the signal delay is determined using a different parameter called the group delay

$$\tau_g(\omega) = -\frac{d\angle H\left(e^{j\omega}\right)}{d\omega} = -\frac{d\theta(\omega)}{d\omega}$$

Phase Delay and Group Delay

- ✓ The group delay by using a single-frequency modulating and carrier signals with zero phase for simplicity.
- ✓ The input signal (Double-side band suppressed carrier i.e., DSB-SC modulated signal) is given by

$$S(n) = A\, cos(w_m n)\, cos(w_c n)$$

- ✓ The cosine-product trigonometric identity to rewrite the input signal as

$$S(n) = \frac{A}{2}\cos[(w_c + w_m)n] + \frac{A}{2}\cos[(w_c - w_m)n] = \frac{A}{2}\cos(w_1 n) + \frac{A}{2}\cos(w_2 n)$$

Where, $w_1 = w_c + w_m$

and $w_2 = w_c - w_m$

- ✓ The system output signal is given by

$$y(n) = \frac{A}{2}\left|H\left(e^{jw_1}\right)\right|\cos\left[w_1 n + \angle H\left(e^{jw_1}\right)\right] + \frac{A}{2}\left|H\left(e^{jw_2}\right)\right|\cos\left[w_2 n + \angle H\left(e^{jw_2}\right)\right]$$

$$y(n) = \frac{A}{2}\cos[(w_1 n + \theta(w_1)] + \frac{A}{2}\cos[(w_2 n + \theta(w_2)]$$

- Where $\theta(w_1)$ and $\theta(w_2)$ are the phase shifts produced by the system frequencies w_1 and w_2, respectively.

Phase Delay and Group Delay

- The equivalently, we can express $y(n)$ as

$$y(n) = \frac{A}{2}\cos\left(\omega_c n + \omega_m n + \frac{\theta(\omega_1)+\theta(\omega_2)}{2} + \frac{\theta(\omega_1)-\theta(\omega_2)}{2}\right)$$

$$+\frac{A}{2}\cos\left(\omega_c n - \omega_m n + \frac{\theta(\omega_1)+\theta(\omega_2)}{2} - \frac{\theta(\omega_1)-\theta(\omega_2)}{2}\right)$$

$$= \frac{A}{2}\cos\left(\left[\omega_e n + \frac{\theta(\omega_1)+\theta(\omega_2)}{2}\right] + \left[\omega_m n + \frac{\theta(\omega_1)-\theta(\omega_2)}{2}\right]\right)$$

$$+\frac{A}{2}\cos\left(\left[\omega_c n + \frac{\theta(\omega_1)+\theta(\omega_2)}{2}\right] - \left[\omega_m n + \frac{\theta(\omega_1)-\theta(\omega_2)}{2}\right]\right)$$

$$y(n) = \mathrm{A}\cos\left(w_c n + \frac{\theta(w_1)+\theta(w_2)}{2}\right)\cos\left(w_m n + \frac{\theta(w_1)-\theta(w_2)}{2}\right)$$

$$y(n) = A\cos\left(\omega_c\left[n + \frac{\theta(\omega_1)+\theta(\omega_2)}{2\omega_c}\right]\right)\cos\left(\omega_m\left[n + \frac{\theta(\omega_1)-\theta(\omega_2)}{2\omega_m}\right]\right)$$

$$y(n) = A\cos(\omega_c[n - \tau_p(\omega_c)])\cos(\omega_m[n - \tau_g(\omega_c)])$$

Phase Delay and Group Delay

- comparing the output signal $y(n)$ with the input signal $s(n)$, we make following observations:

❑ The carrier component at frequency w_c in $y(n)$ lags its counterpart in $s(n)$ by $\frac{\theta(\omega_1)+\theta(\omega_2)}{2\omega_c}$, which represents a time delay

$$\tau_p(\omega_c) = -\frac{\theta(\omega_1)+\theta(\omega_2)}{2\omega_c} = -\frac{\theta(\omega_1)+\theta(\omega_2)}{\omega_1+\omega_2}$$

❑ The modulating signal component at frequency ω_m in $y(n)$ lags its counterpart in $s(n)$ by $\frac{\theta(\omega_1)-\theta(\omega_2)}{2\omega_c}$, which represents a time delay

$$\tau_g(\omega_c) = -\frac{\theta(\omega_1)-\theta(\omega_2)}{2\omega_m} = -\frac{\theta(\omega_1)-\theta(\omega_2)}{\omega_1-\omega_2}$$

❑ The approximate phase response $\theta(w)$ in the vicinity of $w = w_c$ by the two-term Taylor expansion

$$\theta(\omega) = \theta(\omega_c) + \left.\frac{d\theta(\omega)}{d\omega}\right|_{\omega=\omega_c} (\omega-\omega_c)$$

❑ Evaluating $\theta(\omega_1)$ and $\theta(\omega_2)$, we obtain

$$\theta(\omega_1) = \theta(\omega_c) + \left.\frac{d\theta(\omega)}{d\omega}\right|_{\omega=\omega_c} (\omega_1-\omega_c)$$

and $$\theta(\omega_2) = \theta(\omega_c) + \left.\frac{d\theta(\omega)}{d\omega}\right|_{\omega=\omega_c} (\omega_2-\omega_c)$$

Properties of Discrete-time Fourier Transform

- Evaluating $\theta(\omega_1)$ and $\theta(\omega_2)$, we obtain

$$\theta(\omega_1) = \theta(\omega_c) + \left.\frac{d\theta(\omega)}{d\omega}\right|_{\omega=\omega_c} (\omega_1 - \omega_c)$$

and $$\theta(\omega_2) = \theta(\omega_c) + \left.\frac{d\theta(\omega)}{d\omega}\right|_{\omega=\omega_c} (\omega_2 - \omega_c)$$

- Now, from above equations, we obtain

$$\tau_p(\omega_c) = -\frac{\theta(\omega_c)}{\omega_c}$$

- Similarly, $$\tau_g(\omega_c) = -\left.\frac{d\theta(\omega)}{d\omega}\right|_{\omega=\omega_c} = -\left.\frac{d\angle H(\omega)}{d\omega}\right|_{\omega=\omega_c}$$

Energy Spectral Density

- ✓ Parseval's theorem relates the total signal energy in a signal $x(n)$ to its Fourier transform through

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X\left(e^{j\omega}\right)\right|^2 d\omega = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left|X\left(e^{j\omega}\right)\right|^2 df$$

- ✓ Hence, energy spectral density of the signal x(n) and is denoted by $\Psi_x\left(e^{j\omega}\right)$ as

$$\Psi_x\left(e^{j\omega}\right) = \left|X\left(e^{j\omega}\right)\right|^2$$

Properties of Discrete-time Fourier Transform

❖ **Relationship between Input and Output Energy Spectral Densities of an LTI System**

- ✓ Consider an LTI system with frequency response $H(e^{j\omega})$, input signal $x(n)$ output signal y(n).
- ✓ If $x(n)$ and y(n) are energy signals, then their energy spectral densities are $\Psi_x(e^{j\omega}) = |X(e^{j\omega})|^2$ and $\Psi_x(e^{j\omega}) = |Y(e^{j\omega})|^2$, respectively. Since we know that

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

It follows that

$$|Y(e^{j\omega})|^2 = |H(e^{j\omega})X(e^{j\omega})|^2$$
$$|Y(e^{j\omega})|^2 = |H(e^{j\omega})|^2|X(e^{j\omega})|^2$$
$$\Psi_y(e^{j\omega}) = |H(e^{j\omega})|^2\Psi_x(e^{j\omega})$$

❖ **Relation of ESD to Autocorrelation**

- ✓ The autocorrelation function $R_{xx}(\tau)$ of a real energy signal is defined as

$$R_{xx}(m) = \sum_{n=-\infty}^{\infty} x(n)x(n-m)$$
$$R_{xx}(m) = x(m) * x(-m)$$

Properties of Discrete-time Fourier Transform

- **Relation of ESD to Autocorrelation**
  - ✓ The autocorrelation function $R_{xx}(\tau)$ of a real energy signal is defined as

$$R_{xx}(m) = \sum_{n=-\infty}^{\infty} x(n)x(n-m)$$

$$R_{xx}(m) = x(m) * x(-m)$$

  - ✓ Taking the Fourier transform of this equation, we have

$$\mathcal{F}[R_{xz}(m)] = X(e^{j\omega})X(e^{-j\omega}) = X(e^{j\omega})X^*(e^{j\omega})$$

$$\mathcal{F}[R_{xx}(m)] = \left|X(e^{j\omega})\right|^2$$

$$\mathcal{F}[R_{xx}(m)] = \Psi_x(e^{j\omega})$$

$$R_{xx}(m) \leftrightarrow \Psi_x(e^{j\omega})$$

- **Power Spectral Density**
  - ✓ Power spectral density (PSD) has the same relation to power signals as ESD has to energy signals.

$$x_N(n) = \begin{cases} x(n) & -N \leq n \leq N \\ 0 & \text{otherwise} \end{cases}$$

  - ✓ Using Parseval's theorem, we have

$$E_{x_N} = \sum_{n=-\infty}^{\infty} |x_N(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X_N(e^{j\omega})\right|^2 dw$$

Properties of Discrete-time Fourier Transform

❖ Power Spectral Density

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$$E_{x_N} = \sum_{n=-\infty}^{\infty} |x_N(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X_N(e^{j\omega})\right|^2 dw$$

- ✓ Substituting the value of $x_N(n)$, we obtain

$$\sum_{n=-\infty}^{N} | x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X_N(e^{j\omega})\right|^2 d\omega$$

$$\lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=-N}^{N} |x(n)|^2 = \lim_{N\to\infty} \frac{1}{2N+1}\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X_N(e^{j\omega})\right|^2 d\omega$$

$$\lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=-N}^{N} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lim_{N\to\infty} \frac{\left|X_N(e^{j\omega})\right|^2}{2N+1} d\omega$$

$$P_x = \lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=-N}^{N} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lim_{N\to\infty} \frac{\left|X_N(e^{j\omega})\right|^2}{2N+1} d\omega$$

Properties of Discrete-time Fourier Transform

❖ **Power Spectral Density**

✓ Substituting the value of $x_N(n)$, we obtain

$$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \lim_{N\to\infty} \frac{\left|X_N(e^{j\omega})\right|^2}{2N+1} d\omega$$

✓ The left-hand side of this represents the average power P_x of the signal $x(n)$. Therefore, $P_x = \frac{1}{2\pi}\int_{-\pi}^{\pi} G_x(e^{j\omega})d\omega$

Where $G_x(e^{j\omega}) = \lim_{N\to\infty} \frac{\left|X_N(e^{j\omega})\right|^2}{2N+1}$

❖ **Relationship between Input and Output Power Spectral Densities of an LTI System**

✓ Consider an LTI System with frequency response $H(e^{jw})$, input $x_N(n)$, and output $y_N(n)$.

$$Y_N(e^{jw}) = H(e^{jw})X_N(e^{jw})$$

It follows as

$$\left|Y_N(e^{j\omega})\right|^2 = \left|H(e^{jw})X_N(e^{j\omega})\right|^2$$

$$\left|Y_N(e^{j\omega})\right|^2 = \left|H(e^{j\omega})\right|^2\left|X_N(e^{j\omega})\right|^2$$

$$\lim_{N\to\infty} \frac{\left|Y_N(e^{j\omega})\right|^2}{2N+1} = \left|H(e^{j\omega})\right|^2 \lim_{N\to\infty} \frac{\left|X_N(e^{j\omega})\right|^2}{2N+1}$$

$$G_y(e^{j\omega}) = \left|H(e^{j\omega})\right|^2 G_x(e^{j\omega})$$

Properties of Discrete-time Fourier Transform

❖ Relation of PSD to Autocorrelation

✓ The autocorrelation function $R_{xx}(m)$ of a power signal is defined as

$$R_{xx}(m) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(n)x(n-m)$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-\infty}^{\infty} x_N(n)x_N(n-m)$$

$$R_{xx}(m) = \lim_{N\to\infty} \frac{1}{2N+1} [x_N(m) * x_N(-m)]$$

✓ Taking the Fourier transform of the equation, we obtain

$$\mathcal{F}[R_{xx}(m)] = \lim_{N\to\infty} \frac{1}{2N+1} X_N(e^{j\omega})X_N(e^{-j\omega})$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} X_N(c^{j\omega})X_N^*(e^{j\omega})$$

$$\mathcal{F}[R_{xx}(m)] = \lim_{N\to\infty} \frac{1}{2N+1} \left|X_N(e^{j\omega})\right|^2$$

$$\mathcal{F}[R_{xx}(m)] = G_x(e^{j\omega})$$

$$R_{xx}(m) \leftrightarrow G_x(e^{j\omega})$$

- Thus, the autocorrelation function $R_{xx}(m)$ and PSD makes a Fourier transform pair.

Thank you so much for your Kind Attention