Problem set - IV

Theory: $\underline{X} : (X_1, X_2, \cdots X_n) : S \longrightarrow \mathbb{R}^n$

$\underline{X}$ is Random vector iff $X_1, X_2, \cdots X_n$ are radom variables.

C.D.f. — Joint c.d.f.

p.m.f. — Joint p.m.f.

p.d.f. — Joint p.d.f.

$$F_X(x) = P(X \le x)$$

$$F_{\underline{X}}(x_1, x_2, \cdots x_n) = P(X_1 \le x_1, X_2 \le x_2, \cdots X_n \le x_n)$$

$$\boxed{X, Y}$$

$$F_{\underline{X}}(x, y) = P(X \le x, Y \le y)$$

$$\lim_{x \to +\infty} F_{\underline{X}}(x, y) = P(Y \le y) = F_Y(y)$$

$$\lim_{y \to \infty} F_{\underline{X}}(x, y) = F_X(x)$$

$$\lim_{x \to -\infty} F_{\underline{X}}(x, y) = 0 = \lim_{y \to -\infty} F_{\underline{X}}(x, y)$$

$$\lim_{\substack{x \to \infty \\ y \to \infty}} F_{\underline{X}}(x, y) = 1$$

for each $(a_1, b_1] \times (a_2, b_2]$ in ...

$$\Delta = F_{\underline{X}}(b_1, b_2) - F_{\underline{X}}(a_1, b_2) - F_{\underline{X}}(a_2, b_1) + F_{\underline{X}}(a_1, \ldots$$

why ?

Proof :-

$$P(a_1 < X_1 \le b_1, \; a_2 < X_2 \le b_2)$$

$$= P(X \le b_1, \; a_2 < X_2 \le b_2) - P(X \le a_1, \; a_2 \le X_2 \le b\ldots$$

$$\Downarrow$$

$$P(X \le b_1, \; X_2 \le b_2) - P(X \le b_1, X_2 \le a_2) - P(X_1 \le a_1, X_2 \le b_2) + P(X_1 \le a_1, X_2 \le a_2)$$

$$\Rightarrow F_{\underline{X}}(b_1, b_2) - F_{\underline{X}}(b_1, a_2) - F_{\underline{X}}(a_1, b_2) + F_{\underline{X}}(a_1, a_2) \ge 0$$

P.m.f or P.d.f.

$$f_X(x) = P(X = x)$$

Here it becomes like this

$$f_{\underline{X}}(x_1, x_2, \ldots x_n) = P(X_1 = x_1, X_2 = x_2, \ldots X_n = x_n)$$

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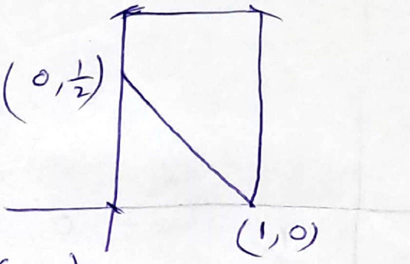
$$F(x,y) = \begin{cases} 1 & \text{if } x+2y \geq 1 \\ 0 & \text{if } x+2y < 1 \end{cases}$$



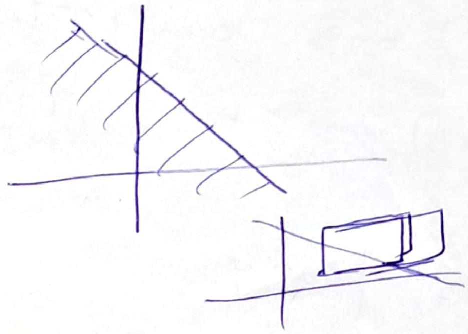
$[0,1] \times [0,1]$

$F(1,1) - F(0,1) - F(1,0) + F(0,0)$

$= 1 - 1 - 1 + 0 = -1 < 0 \Rightarrow$ Not

ii)

$$F(x,y) = \begin{cases} 0 & \text{if } x<0,\ x+y<1 \text{ or } y<0 \\ 1 & \text{o.w.} \end{cases}$$



$[\frac{1}{3}, 1] \times [\frac{1}{3}, 1]$

$F(1,1) - F(\frac{1}{3}, 1) - F(1, \frac{1}{3}) + F(\frac{1}{3}, \frac{1}{3})$ Not a c.d.f.

$1 - 1 - 1 + 0 = -1 < 0$

② Suppose a fair coin is tossed three times

$X = \#$ of Heads $\quad Y = \#$ absolute difference of Heads & tails

To prove: $Z = (X, Y)$ is A discrete type Random vector

$$X(\omega) = \begin{cases} 0 & \text{if } \omega \in \{TTT\} \\ 1 & \text{if } \omega \in \{TTH, HTT, THT\} \\ 2 & \text{if } \omega \in \{HHT, HTH, THH\} \\ 3 & \text{if } \omega \in \{HHH\} \\ & \text{o.w.} \end{cases}$$

$$Y(\omega) = \begin{cases} & \\ 3 & \text{if } \omega \in \{HHH, TTT\} \end{cases}$$

Y\X	0	1	2	3	P(Y=) p.m.f.
1	0	$\frac{3}{8}$	$\frac{3}{8}$	0	$6/8$
3	$1/8$	0	0	$1/8$	$2/8$
$P(X=x)$	$1/8$	$3/8$	$3/8$	$\frac{1}{8}$	$\boxed{1}$

$(0,1) \longrightarrow$ $0 \to TTT \notin 1$

$(1,1) \longrightarrow$ HTT, THT, TTH

Joint p.m.f.

$$f_Z(x,y) = \begin{cases} 0 & \text{o.w.} \\ 1/8 & \text{if } (x,y) \in \{(0,3)\ (3,3)\} \\ 3/8 & \text{if } (x,y) \in \{(1,1)(2,1)\} \end{cases}$$

(3) Let A & B be two events & two R.v. X & Y

$$X(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{o.w.} \end{cases} \qquad Y(\omega) = \begin{cases} 1 & \text{if } \omega \in B \\ 0 & \text{o.w.} \end{cases}$$

To prove: X & Y are Independent $\Leftrightarrow$ A & B are independent.

Sol^n: Joint p.m.f.

$$p(x,y) = \begin{cases} P(A^c \cap B^c) & \text{if } (x,y) = (0,0) \\ P(A \cap B^c) & \text{if } (x,y) = (1,0) \\ P(A^c \cap B) & \text{if } (x,y) = (0,1) \\ P(A \cap B) & \text{if } (x,y) = (1,1) \\ 0 & \text{o.w.} \end{cases}$$

p.m.f. of X & Y are

$$P_X(x) = \begin{cases} P(A^c) & \text{if } x=0 \\ P(A) & \text{if } x=1 \\ 0 & \text{o.w.} \end{cases}$$

$$P_Y(y) = \begin{cases} P(B^c) & \text{if } y=0 \\ P(B) & \text{if } y=1 \\ 0 & \text{o.w.} \end{cases}$$

$\Rightarrow$ Let X & Y are Independent then

$$p(x,y) = p_X(x)\, p_Y(y)$$

$$p(1,1) = p_X(1)\, p_Y(1)$$

$$P(A \cap B) = P(A)\, P(B)$$

$\Leftarrow$ Let A & B are Independent

$\Rightarrow$ A^c & B, A & B^c & A^c & B^c are independent

$\Rightarrow$ $p(x,y) = p_X(x)\, p_Y(y)$

$\Rightarrow$ X & Y are Independent

(5) X & Y be two R.V. with Joint p.d.f.

$$f(x,y) = \begin{cases} \dfrac{1+xy}{4} & \text{if } |x|<1,\ |y|<1 \\ 0 & \text{o.w.} \end{cases}$$

To prove: X & Y are not independent & X^2, Y^2 are independent

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

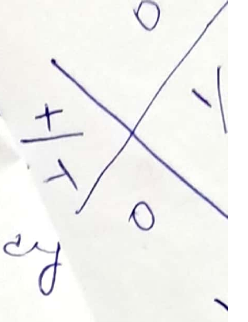
$$= \begin{cases} \int_{-1}^{1} \dfrac{1+xy}{4}\, dy & \text{if } -1<x<1 \\ 0 & \text{o.w.} \end{cases}$$

$$= \begin{cases} 1/2 & \text{if } -1<x<1 \\ 0 & \text{o.w.} \end{cases}$$

Similarly $f_Y(y) = \begin{cases} 1/2 & \text{if } -1<y<1 \\ 0 & \text{o.w.} \end{cases}$

$f(x,y) \neq f_X f_Y \Rightarrow X$ & Y are not independent.

X^2 & Y^2 are Independent



$$P(X^2 \le u, Y^2 \le v) = P(-u^{1/2} < X \le u^{1/2}, -v^{1/2} \ldots)$$

$$= \int_{-v^{1/2}}^{v^{1/2}} \int_{-u^{1/2}}^{u^{1/2}} \frac{1+xy}{4} \, dx \, dy$$

$$= u^{1/2} v^{1/2}$$

for $0 \le u \le 1$
$0 \le v \le 1$

$$P(X^2 \le u) = P(-u^{1/2} \le X \le u^{1/2})$$

$$= \int_{-u^{1/2}}^{u^{1/2}} \frac{1}{2} \, dx = u^{1/2}$$

$$F_{X^2}(x) = \begin{cases} u^{1/2} & 0 \le u \le 1 \\ 0 & \text{o.w.} \end{cases}$$

$$F_{Y^2}(y) = \begin{cases} v^{1/2} & 0 \le v \le 1 \\ 0 & \text{o.w.} \end{cases}$$

$$\Rightarrow F_{\underline{Z}}(x, y) = F_{X^2}(x) F_{Y^2}(y)$$

(6) Toss a fair coin 3 times

X = # of Heads on first toss

Y = # of Heads on last two tosses

Z = # of Heads on first two tosses

a) Find Joint p.m.f. for X & Y. Compute $\text{cov}(X, Y)$

b) Find Joint p.m.f. for X & Z Compute $\text{cov}(X, Z)$

Sol^n:

$$X(\omega) = \begin{cases} 0 & \text{if } \omega \in \{THT, TTH, THH, TTT\} \\ 1 & \text{if } \omega \in \{HTT, HHT, HTH, HHH\} \end{cases}$$

$$Y(\omega) = \begin{cases} 0 & \text{if } \omega \in \{TTT, HTT\} \\ 1 & \text{if } \omega \in \{TTH, THT, HTH, HHT\} \\ 2 & \text{if } \omega \in \{HHH, THH\} \end{cases}$$

Y\X	0	1	
0	1/8	1/8	2/8
1	2/8	2/8	4/8
2	1/8	1/8	2/8
	4/8	4/8	

Joint p.m.f. $Z = (x, y)$

$$p_Z(x,y) = \begin{cases} \frac{1}{8} & \text{if } (x,y) \in \{(0,0)\ (1,0)\ (0,2)\ (1,2)\} \\ \frac{2}{8} & \text{if } (x,y) \in \{(0,1)(1,1)\} \\ 0 & \text{o.w.} \end{cases}$$

$$p_X(x) = \begin{cases} 1/2 & \text{if } x=0 \\ \frac{1}{2} & \text{if } x=1 \\ 0 & \text{o.w.} \end{cases}$$

$$p_Y(y) = \begin{cases} 2/8 & \text{if } Y=0,2 \\ \frac{1}{2} & \text{if } Y=1 \\ 0 & \text{o.w.} \end{cases}$$

$p_Z = p_X\, p_Y \Rightarrow X \,\&\, Y$ are Ind.
$\Rightarrow \text{cov}(X,Y) = 0$

(b) $$z(w) = \begin{cases} 0 & \text{if } w \in \{TTT, TTH\} \\ 1 & \text{if } w \in \{HTH, THH, TH\cancel{T}, HTT\} \\ 2 & \text{if } w \in \{HHH, HHT\} \end{cases}$$

Z\X	0	1	Z ↓
0	2/8	0	2/8
1	2/8	2/8	4/8
2	0	2/8	2/8
X →	4/8	4/8	

$E(XZ) =$

$E(X) = 4/8$

$E(\boxed{Z}) = 1$

$$X Z(\omega) = \begin{cases} 0 & \text{if } \omega \in \{THT, TTH, THH\} \\ 1 & \text{if } \omega \in \{HTT, HTH\} \\ 2 & \text{if } \omega \in \{HHH, HHT\} \end{cases}$$

$$E(XZ) = 0 \times \frac{1}{8} + 1 \times \frac{2}{8} + 2 \times \frac{2}{8} = \frac{6}{8}$$

$$\text{Cov}(X, Z) = \frac{6}{8} - \frac{4}{8} = \frac{1}{4}$$

(7) Roll a dice $(n = 1, 2, --6)$
Two events S_1 & S_2 are defined as folo

$$S_1 = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{Otherw} \end{cases}$$

$$S_2 = \begin{cases} 1 & \text{if } n \text{ is prime} \\ 0 & \text{o.w.} \end{cases}$$

find Joint p.m.f. $P(S_1, S_2)$
Also covariance & correlation cofficient. between S_1 & S_2

$S_2 \backslash S_1$	0	1	
0	1/6	2/6	3/6
1	2/6	1/6	3/6
	3/6	3/6	

↳ Joint p.m.f

$E(S_1, S_2) = 1/6$

$E(S_1) = 3/6$

$E(S_2) = 3/6$

we can write

$E(S_1^2) = 3/6 \quad E(S_2^2) = 3/6$

$$\text{Cov}(S_1 S_2) = \frac{1}{6} - \frac{9}{36} = -\frac{1}{12}$$

$$\text{Var}(S_1) = \frac{3}{6} - \frac{9}{36} = \frac{9}{36} \Rightarrow \sqrt{\text{var} S_1} = 1/2$$

$$\sqrt{\text{Var}(S_2)} = \frac{1}{2}$$

$$\Rightarrow \rho(S_1, S_2) = \frac{\text{Cov}(S_1, S_2)}{\sqrt{\text{Var}(S_1)\text{Var}S_2}} = \frac{-\frac{1}{12}}{1/4} = -1/3$$

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Joint p.m.f. of X & Y is

$$p(x,y) = \begin{cases} c(2x+y) & \text{if } (x,y) \in \{0,1,2\} \times \{0,1,2,3\} \\ 0 & \text{o.w.} \end{cases}$$

a) find c

b) $P(X=2, Y=1)$ & $P(X \geq 1, Y \leq 2)$

c) find Marginal p.m.f. of X & Y

d) Are X & Y are independent

Sol^n: $c = 1/42$

$P(X=2, Y=1) = 5/42$

$P(X \geq 1, Y \leq 2) = P(X=1, Y \leq 2) + P(X=2, Y \leq 2)$

$= P(X=1, Y=0) + P(X=1, Y=1) + P(X=1, Y=2)$
$+ P(X=2, Y=0) + P(X=2, Y=1) + P(X=2, Y=2)$

$= 4/7$

c)

$$p_X(x) = \begin{cases} p(0,0)+p(0,1)+p(0,2)+p(0,3) & \text{if } x=0 \\ p(1,0)+p(1,1)+p(1,2)+p(1,3) & \text{if } x=1 \\ & \text{if } x=2 \\ 0 & \text{o.w.} \end{cases}$$

similarly we can find p_Y

product $\cancel{=} \Rightarrow$ Not Ind

(9) $Cov(aX+b, Y) = a\, cov(X,Y)$

$var(aX+b) = a^2 var(X)$

$$\rho(aX+b, Y) = \frac{cov(aX+b, Y)}{\sqrt{var(aX+b), Y}} = \frac{a\, cov(X,Y)}{a\sqrt{varX \times varY}}$$

$= \rho(X,Y)$

$X \& Y$

$T = \frac{5X}{9} - \frac{160}{9}$ $\quad S = \frac{5Y}{9} - \frac{160}{9}$

$\text{cov}(X,Y) = 4$ $\quad \& \quad \rho(X,Y) = 0.8$

$\Rightarrow \rho(T,S) = 0.8$

~~cov(T, S) =~~

formula: $\text{cov}(a_1X + b_1, a_2Y + b_2) = a_1a_2 \,\text{cov}(X,Y)$

$\Rightarrow \text{cov}(T,S) = \frac{25}{81} \cos(X,Y) = \frac{100}{81}$

~~10 $X = \begin{cases} 4 & \text{if drawn card is Heart} \end{cases}$~~

~~14~~ ~~14~~ ~~14~~

~~13~~

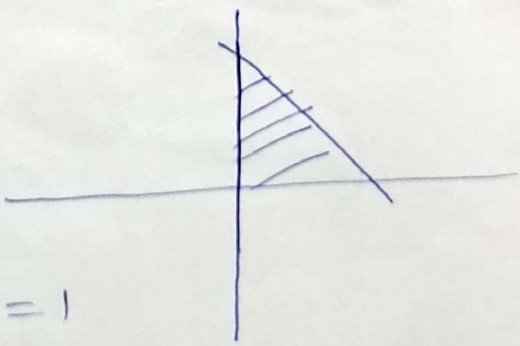
→ 21 Let X, Y be Random vector with Joint p.d.f.

$$f(x,y) = \begin{cases} cx+1 & \text{if } x,y \geq 0 \ \& \ x+y<1 \\ 0 & \text{o.w.} \end{cases}$$

a) find the value of c

b) find the Marginal p.d.f. of $X \& Y$

c) $P(Y < 2X^2)$



a) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) = 1$

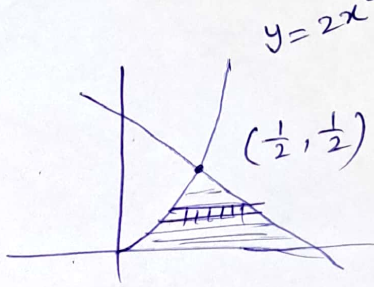
$\Rightarrow \int_0^1 \int_0^{1-x} (cx+1)\, dy\, dx = 1$

$\Rightarrow c = 3$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$= \begin{cases} \int_0^{1-x} (3x+1)\, dy & \text{if } 0 \le x < 1 \\ 0 & \text{o.w.} \end{cases}$$

$$= \begin{cases} (3x+1)(1-x) & \text{if } 0 \le x \le 1 \\ 0 & \text{o.w.} \end{cases}$$



c) $P(Y < 2x^2)$

$$= \cancel{\iint} \quad \iint_{A} f(x,y)\, dA = \frac{53}{96}$$

AND

(19) Probability distribution for no. of eggs in a clutch is $P(\lambda)$ & probability that each each egg will hatch is p. show by calculation that probability distribution of no. of chicks that hatch is $P(\lambda p)$.

$X =$ no. of eggs $\sim P(\lambda)$

$Y =$ no. of chicks Hatching

~~$P(Y=y)$~~ Probability of k chicks Hatching from a clutch of size n is ${}^{n}C_{k}\, p^k (1-p)^{n-k}$

Probability of clutch of size n is $\frac{e^{-\lambda} \lambda^n}{n!}$

Probability of K chicks hatching from a clutch is

$$Pr(K \text{ chicks}) = \sum_{n=K}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!} \, {}^nC_K \, p^K q^{n-K}$$

$$= \frac{e^{-\lambda p}(\lambda p)^K}{K!}$$

$$\Rightarrow Y \sim P(\lambda p)$$

Sol^n (12) Let X denote the time until the miner reaches safety, and let Y denote the door he initially chooses. Now,

$$E[X] = E[E[X|Y]]$$

$$= \sum_y E[X|Y=y]\, P(Y=y)$$

$$= E[X|Y=1]\, P(Y=1) + E[X|Y=2]\, P(Y=2) + E[X|Y=3]\, P(Y=3).$$

$$= \frac{1}{3}\Big(\underbrace{E[X|Y=1]}_{2} + \underbrace{E[X|Y=2]}_{3+E[X]} + \underbrace{E[X|Y=3]}_{5+E[X]}\Big)$$

$$\Rightarrow \boxed{E[X] = 10}$$

To understand why $E[X|Y=2] = 3+E[X]$:

If miner chooses the second door, then he spends three hours in the tunnel and then returns to the mine. But once he returned to the mine the problem is as before, and hence his expected additional time under safety is just $E[X]$. Hence $E[X|Y=2] = 3+E[X]$.

(14) At a party N men throw their hats into the center of a room. The hats are mixed up and each man randomly select one. Let X be the # of men those get their own hat. Find $E(X)$ & $Var(X)$.

Solⁿ:- $X = \#$ of men who got own hat

For $i = 1, 2, \ldots, N$

$$X_i = \begin{cases} 1 & \text{if } i^{th} \text{ man get their own hat} \\ 0 & \text{o.w.} \end{cases}$$

Then $X = X_1 + X_2 + \cdots + X_N$

For each i,

$$E(X_i) = 1 \cdot (P(X_i = 1) + 0 \times P(X_i = 0)$$

$$= \sum_{x=0}^{1} x \, P(X_i = x) = \frac{1}{N} = E(X_i^2)$$

$$E(X) = E(X_1 + \cdots + X_n)$$

$$= E(X_1) + \cdots + E(X_n)$$

$$= N \times \frac{1}{N} = 1.$$

$$Var(X) = E(X^2) - E(X)^2 = 1.$$

$$E(X^2) = E((X_1 + X_2 + \cdots + X_N)^2) = E(X_1^2) + \cdots + E(X_N^2) + 2(E(X_1 X_2) + \cdots + \cdots)$$

$$E(X_i X_j) = \sum_{(x,y) \in \{(0,0), (1,0), (0,1), (1,1)\}} xy \, P(X_i = x, X_j = y) = P(X_i = 1, X_j = 1) = \frac{1}{N(N-1)}.$$

$$E\left(X \mid Y=\tfrac{1}{2}\right) = \int_{-\infty}^{\infty} x f_{X|Y}\left(x \mid y=\tfrac{1}{2}\right) dx$$

$$= \int_0^{1/2} x \frac{2x}{(\frac{1}{2})^2} dx = 8 \int_0^{1/2} x^2 dx$$

$$= \frac{8}{3} \times \frac{1}{8} = \frac{1}{3}$$

$$E\left(X^2 \mid Y=\tfrac{1}{2}\right) = 8 \int_0^{1/2} x^2 \times x \, dx$$

$$= 8 \left(\frac{x^4}{4}\right)_0^{1/2} = \frac{8}{4} \times \frac{1}{16} = \frac{1}{8}$$

Thus $Var\left(X \mid Y=\tfrac{1}{2}\right) = \frac{1}{8} - \left(\frac{1}{3}\right)^2 = \frac{1}{8} - \frac{1}{9}$

$$= \frac{1}{72}$$

$$P\left(0 < Y < \tfrac{1}{3}\right) = \int_0^{1/3} f_Y(y) dy = \int_0^{1/3} 4y^3 dy$$

$$= 4 \left(\frac{y^4}{4}\right)_0^{1/3} = \frac{1}{81}$$

$$P\left(\tfrac{1}{3} < Y < \tfrac{2}{3} \mid X = \tfrac{1}{2}\right) = \int_{1/2}^{2/3} f_{Y|X=\frac{1}{2}}(y) dy = \int_{1/2}^{2/3} \frac{2y \, dy}{1-(\frac{1}{2})^2} \quad \left(\because x < y \ \& \ \tfrac{1}{3} > \tfrac{1}{2}\right)$$

$$= \frac{4 \times 2}{3} \left(\frac{y^2}{2}\right)_{1/2}^{2/3} = \frac{4}{3}\left(\frac{4}{9} - \frac{1}{4}\right)$$

$$= \frac{4}{3}\left(\frac{16-9}{36}\right) = \frac{4}{3} \times \frac{7}{36} = \frac{7}{27}$$

4) For $i = 1, 2, \ldots, N$

$$X_i = \begin{cases} 1 & \text{if } i\text{th man get their own hat} \\ 0 & \text{o.w.} \end{cases}$$

Then $X = X_1 + X_2 + \cdots + X_N$

For each i, $E(X_i) = \sum_{x=0}^{1} x \, P(X_i = x) = 0 \cdot P(X_i = 0) + 1 \cdot P(X_i = 1)$

$$= \frac{1}{N}$$

Thus $E(X) = E(X_1) + \cdots + E(X_N) = N \cdot \frac{1}{N} = 1$

For $i \neq j$

$$E(X_i X_j) = \sum_{(x,y) \in \{(0,0),(1,0),(0,1),(1,1)\}} xy \, P(X_i = x, X_j = y)$$

$$= P(X_i = 1, X_j = 1) = \frac{1}{N(N-1)}$$

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Thus for $i \neq j$,

$$Cov(X_i, X_j) = E(X_i X_j) - E(X_i)\,E(X_j)$$
$$= \frac{1}{N(N-1)} - \frac{1}{N}\cdot\frac{1}{N} = \frac{1}{N}\left(\frac{1}{N(N-1)}\right)$$

Also $\forall\, i = 1, 2, \ldots, N$

$$Var(X_i) = E(X_i^2) - (E(X_i))^2$$
$$= 1^2 P(X_i = 1) - \frac{1}{N^2} = \frac{1}{N} - \frac{1}{N^2} = \frac{\cancel{N}(N-1)}{N^2}$$

Hence

$$Var(X) = \sum_{i=1}^{N} Var(X_i) + 2 \sum_{1 \le i < j \le N} Cov(X_i, X_j)$$
$$= \cancel{N}\cdot\left(\frac{(N-1)}{N^2}\right) + 2\binom{N}{2}\frac{1}{N}\left(\frac{1}{N(N-1)}\right)$$
$$= \frac{N-1}{N} + \cancel{N(N-1)}\,\frac{1}{N}\left(\frac{1}{\cancel{N(N-1)}}\right)$$
$$= \frac{N-1}{N} + \frac{1}{N} = \frac{N-1+1}{N} = \frac{N}{N} = 1$$

Thus $\boxed{Var(X) = 1}$

13 The 10 randomly selected pages have independent distributions of errors per page. Also $\sum_{i=1}^{m} P(\lambda_i) \sim P\left(\sum_{i=1}^{m} \lambda_i\right)$.

For $i = 1, 2, \ldots, 10$

$$X_i = \begin{cases} 1 & \text{if the error in } i\text{th page} \\ 0 & \text{o.w.} \end{cases}$$

Then X = no. of errors in 10 randomly selected pages where $X_i \sim P(\lambda)$ $\forall\, 1 \le i \le 10$

$= X_1 + X_2 + \cdots + X_{10}$

Thus $X \sim P(10\lambda)$.

Hence $$P(X = 10) = \frac{e^{-10\lambda}(10\lambda)^{10}}{\lfloor 10}$$

13 (b) X :- expected number of rolls

Let $Y = \begin{cases} 1 & \text{A wins on his 1st roll} \\ 2 & \text{B wins on his 1st roll} \\ 3 & \text{otherwise (neither A nor B wins their 1st roll)} \end{cases}$

Let Y be the length of the piece after we break for the first time.

(18) Let X be the length after we break for the second time.

Since the breakpoint is choosen uniformly over a piece of length Y, we have $E[X|Y] = Y/2$.

For a similar reason, we have $E(Y) = l/2$

Thus $E(X) = E(E(X|Y)) = E(Y/2) = \frac{1}{2}E(Y) = l/4$.

(16) The c.d.f of Y_1 is

$$
\begin{aligned}
F_{Y_1}(y) &= P(Y_1 \le y) \\
&= P(\max(X_1, X_2, \ldots, X_n) \le y) \\
&= P(X_1 \le y, X_2 \le y, \ldots, X_n \le y) \quad (\because X_i\text{'s is continuous} \\
&= \prod_{i=1}^{n} P(X_i \le y) = \prod_{i=1}^{n} F_{X_i}(y) \\
&= (F(y))^n
\end{aligned}
$$

The c.d.f. of Y_2 is

$$
\begin{aligned}
F_{Y_2}(y) = P(Y_2 \le y) &= 1 - P(Y_2 > y) \\
&= 1 - P(\min(X_1, X_2, \ldots, X_n) > y) \\
&= 1 - P(X_1 > y, X_2 > y, \ldots, X_n > y) \\
&= 1 - \prod_{i=1}^{n} P(X_i > y) \\
&= 1 - \prod_{i=1}^{n} (1 - P(X_i \le y)) \\
&= 1 - \prod_{i=1}^{n} (1 - F_{X_i}(y))
\end{aligned}
$$

$$\boxed{F_{Y_2}(y) = 1 - (1 - F(y))^n}$$

(19) The m.g.f of ... $\mu_i t + \frac{\ }{2}$

$$\phi_{X_i}(t) = e^{\mu_i t + \frac{\ }{2}}$$

- let $Y = \sum_{i=1}^{n} a_i X_i$.

Then the m.g.f of Y is

$$\phi_Y(t) = E(e^{tY})$$

$$= E\left(e^{t(\sum_{i=1}^{n} a_i X_i)}\right)$$

$$= E\left(\prod_{i=1}^{n} e^{t a_i X_i}\right) = \prod_{i=1}^{n} E\left(e^{t a_i X_i}\right)$$

$$= \prod_{i=1}^{n} \phi_{X_i}(t a_i)$$

$$= \prod_{i=1}^{n} e^{a_i \mu_i t + \frac{a_i^2 \sigma_i^2 t^2}{2}}$$

$$= e^{(\sum_{i=1}^{n} a_i \mu_i) t + \frac{(\sum_{i=1}^{n} a_i^2 \sigma_i^2) t^2}{2}}$$

Hence $Y \sim N\left(\sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} a_i^2 \sigma_i^2\right)$

(20) Same as (19)

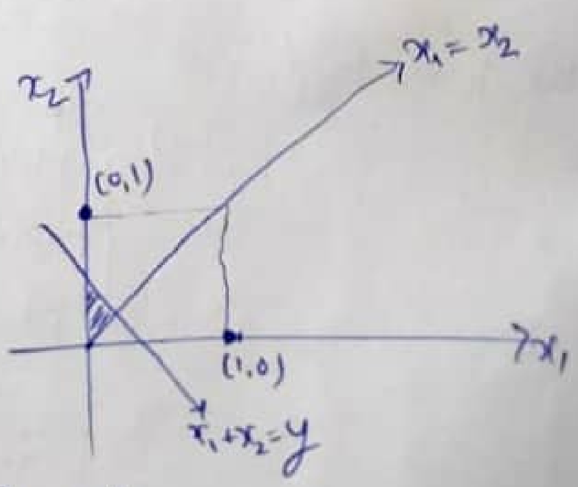
(26)
$$F_Y(y) = P(X_1 + X_2 \le y)$$

$$= \int_0^1 \int_0^1 f(x_1, x_2)\, dx_1\, dx_2$$

$0 \le x_1 \le x_2 \le 1$

$\{(x_1, x_2) : x_1 + x_2 \le y\}$

Clearly for $y < 0$ $F_Y(y) = 0$



For $0 \le y < 1$

$$F_Y(y) = \int_0^{y/2} \left(\int_{x_1}^{y - x_1} 2\, dx_2 \right) dx_1$$

$$= \int_0^{y/2} \left(2(y - 2x_1)\right) dx_1$$

$$= 2\left(y x_1 - x_1^2\right)_0^{y/2} = 2\left(\frac{y^2}{2} - \frac{y^2}{4}\right) = y^2/2$$

(21) Let (X, Y) be random vector with joint p.d.f.

$$f(x,y) = \begin{cases} cx+1 & \text{if } x, y \geq 0 \text{ and } x+y<1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the value of c.

Soln:- To find the value of c, we write

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1$$

$$\Rightarrow \int_{-\infty}^{\infty}\int_{0}^{1-x} (cx+1)\, dy\, dx = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} (cx+1)(1-x)\, dx = 1$$

$$\Rightarrow \int_{0}^{1} (cx+1)(1-x)\, dx = 1$$

$$\Rightarrow \frac{1}{2} + \frac{1}{6}c = 1 \Rightarrow \boxed{c = 3}$$

(b) Find the marginal pdf of X and Y.

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$= \int_{0}^{1-x} (3x+1)\, dy = (3x+1)(1-x) \text{ for } x \in [0,1]$$

Thus, $f_X(x) = \begin{cases} (3x+1)(1-x) & 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$

Similarly, we obtain

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

$$= \int_0^{1-y} (3x+1)\, dx$$

$$= \frac{1}{2}(1-y)(5-3y) \quad : \quad y \in [0,1].$$

Thus, $f_Y(y) = \begin{cases} \frac{1}{2}(1-y)(5-3y) & : 0 \leq y \leq 1 \\ 0 & , \quad \text{o.w.} \end{cases}$

(C) Find $P(Y < 2X^2) = \int_{-\infty}^{\infty} \int_{-\infty}^{2x^2} f(x,y)\, dy\, dx$

$$= \int_0^1 \int_0^{\min(2x^2, 1-x)} (3x+1)\, dy\, dx$$

$$= \int_0^1 (3x+1) \min(2x^2, 1-x)\, dx$$

$$= \int_0^{1/2} (2x^2)(3x+1)\, dx + \int_{1/2}^1 (3x+1)(1-x)\, dx$$

$$= \frac{53}{96}.$$

(22) Let (X, Y) be a random vector with joint pdf

$$f(x,y) = \begin{cases} 6e^{-(2x+3y)} & \text{if } x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Are X and Y are independent? (✓).

Soln:-

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$= \int_0^{\infty} f(x,y)\, dy = \int_0^{\infty} 6e^{(-2x+3y)}\, dy;$$

$$= \int_0^{\infty} 6e^{-2x}e^{-3y}\, dy =$$

$$= 6e^{-2x} \int_0^{\infty} e^{-3y}\, dy = {}^{2}\cancel{6}e^{-2x} \cdot \frac{e^{-3y}}{-\cancel{3}}$$

$$= -2e^{-2x}\left[e^{-3y}\right]_0^{\infty}$$

$$= -2e^{-2x}[0-1]$$

$$= 2e^{-2x}, \quad x \geq 0$$

Similarly, $f_Y(y) = 3e^{-3y}. \quad y \geq 0$

(b) Find $E(Y \mid X>2)$.

Since, X and Y are independent, we have

$$E(Y \mid X>2) = E(Y).$$

Also, $Y \sim \text{Exp}(3) \Rightarrow E(Y) = \frac{1}{\lambda} = \frac{1}{3}$.

(c) Find $P(X>Y) = \int_0^\infty \int_y^\infty 6e^{-(2x+3y)}\, dx\, dy$

$$= \int_0^\infty 3e^{-5y}\, dy = \frac{3}{5}.$$

23 Let X be a random variable with pdf

$$f_X(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

We know that given $X = x$, the r.v. Y uniformly distributed on $[-x, x]$.

(a) Find the joint pdf of X and Y.

Soln:- Since, $f_{Y|X}(y|x) = \begin{cases} \frac{1}{x-(-x)=2x} & -x \le y \le x \\ 0 & \text{o.w.} \end{cases}$

Thus, we have

$$f_{X,Y}(x,y) = f_{Y|X}(y|x)\, f_X(x)$$

$$= \begin{cases} 1, & 0 \le x \le 1,\ -x \le y \le x \\ 0, & \text{o.w.} \end{cases}$$

$$\Rightarrow f_{XY}(x,y) = \begin{cases} 1, & |y| \le x \le 1 \\ 0, & \text{o.w.} \end{cases}$$

(b) Find pdf. of Y.

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx$$

$$= \int_{|y|}^{1} 1\, dx$$

$$= 1 - |y|, \quad -1 \le y \le 1$$

$$\therefore f_Y(y) = \begin{cases} 1-|y|, & -1 \le y < 1 \\ 0, & \text{o.w.} \end{cases}$$

(c) Find $P(|Y| < X^3)$.

Soln :-

$$A = \{ (x,y) : |y| < x^3 \}$$

$$P(|Y| < X^3) = \iint_A f(x,y)\,dx\,dy$$

$$= \int_0^1 \left(\int_{-x^3}^{x^3} 1\,dy \right) dx$$

$$= \int_0^1 (2x^3)\,dx$$

$$= 2\left(\frac{x^4}{4}\right)_0^1 = 2 \times \frac{1}{4} = \frac{1}{2}$$

(24) Let X be random variable with pdf

$$f_X(x) = \begin{cases} 4x(1-x^2), & 0<x<1 \\ 0, & o.w. \end{cases}$$

For a fixed $x \in (0,1)$, the conditional pdf of Y given $X=x$ is

$$f_{Y/X}(y|x) = \begin{cases} \frac{2y}{1-x^2} & \text{if } x<y<1 \\ 0 & o.w. \end{cases}$$

(a) Find the conditional pdf of X given $Y=y$ for appropriate values of y.

Solⁿ:-
$$f_{X/Y}(x|y) = \frac{f(x,y)}{f_Y(y)}.$$

Now, $f(x,y) = f_{Y/X}(y|x)\, f_X(x)$

$$= \begin{cases} 8xy & \text{if } 0<x<y<1 \\ 0 & o.w. \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

$$= \int_0^y 8xy\, dx \qquad 0<y<1$$

$$= 4y^3, \quad 0<y<1.$$

$\therefore \quad f_{X|Y}(x|y) = \begin{cases} \frac{8xy}{4y^3} & \text{if } 0 < x < y \\ 0 & \text{o.w.} \end{cases}, \quad 0 < y < 1$

(b) Find $E(X|Y=0.5)$ and $\text{var}(X|Y=0.5)$.

$$E(X|Y=0.5) = \int_{x \in E_X} x \cdot f_{X|Y}(x|y)\, dx$$

$$= \int_0^{y \, \cancel{=0.5}} x \cdot \frac{2x}{y^2}\, dx$$

$$= \frac{2}{3}\left[\frac{x^3}{y^2}\right]_0^y$$

$$= \frac{2}{3}y = \frac{2}{3} \times 0.5 = \frac{1}{3} \quad (\text{for } y = 0.5)$$

$$\text{Var}(X|Y=0.5) = E(X^2|Y=0.5) - E(X|Y=0.5)^2$$

$$= \int_{-\infty}^{\infty} x^2 f_{X|Y}(x|y)\, dx - \frac{1}{9}$$

$$= \int_0^y x^2 \cdot \frac{2x}{y^2}\, dx - \frac{1}{9}$$

$$= \frac{1}{2y^2} y^4 - \frac{1}{9} = \frac{1}{2} \times \left(\frac{1}{2}\right)^2 - \frac{1}{9} = \frac{1}{72}$$

(c) $P(0<Y<\frac{1}{3}) = \int_0^{1/3} f_Y(y)\, dy$

$$= \int_0^{1/3} 4y^3 \, dy$$

$$= \left(\frac{1}{3}\right)^4 = \frac{1}{81}.$$

$$P(\tfrac{1}{3}<Y<\tfrac{2}{3} \mid X=\underset{\frac{1}{2}}{\overset{\shortparallel}{0.5}}) = P\left(\tfrac{1}{2}<Y<\tfrac{2}{3}\right)$$

$$= \int_{1/2}^{2/3} \frac{2y}{1-x^2}\, dy$$

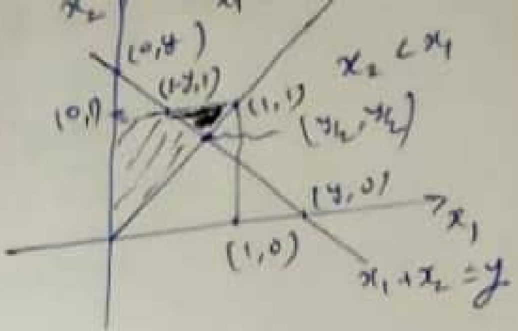
$$= \left[\frac{y^2}{1-x^2}\right]_{1/2}^{2/3}$$

$$= \frac{1}{1-x^2}\left[\frac{4}{9}-\frac{1}{4}\right]$$

$(x=0.5)$ $= \frac{\cancel{4}}{3} \times \frac{7}{9\times\cancel{4}} = \cancel{\frac{21}{}}$

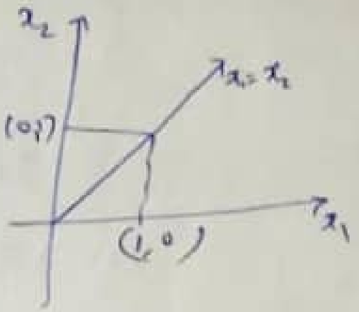
$$= \frac{7}{27}.$$

For $1 \le y < 2$



$$F_Y(y) = 1 - P(X_1+X_2 > y)$$
$$= 1 - 2\cdot\frac{1}{2}(1-\tfrac{y}{2})(2-y)$$
$$= 1 - \frac{1}{2}(2-y)^2$$

For $y \geq 2$



$$F_Y(y) = 2\times\frac{1}{2} = 1$$

Hence the c.d.f of Y is

$$F_Y(y) = \begin{cases} 0 & \text{if } y<0 \\ y^2/2 & \text{if } 0 \le y < 1 \\ 1-\frac{(y-2)^2}{2} & \text{if } 1 \le y < 2 \\ 1 & \text{if } y \geq 2 \end{cases}$$

So the p.d.f of Y is
$$f_Y(y) = \begin{cases} y & \text{if } 0 \le y \le 1 \\ 2-y & 1 \le y \le 2 \\ 0 & \text{o.w.} \end{cases}$$

27) $Y_1 = \frac{X_1}{X_2}$, $Y_2 = X_2$

$y_1 = \frac{x_1}{x_2}$ & $y_2 = x_2 \Rightarrow x_1 = y_1 y_2$ or

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} y_2 & y_1 \\ 0 & 1 \end{vmatrix} = y_2 \neq 0 \text{ on the range of transformation}$$

Thus the joint p.d.f of Y_1 & Y_2 is

$$f_Y(y_1,y_2) = f_X(y_1y_2, y_2)\,|J|$$
$$= \begin{cases} 8y_1y_2^3 & 0<y_1<1,\ 0<y_2<1 \\ 0 & \text{o.w} \end{cases}$$

As $0 \le x_1 < x_2 < 1$
$$\Rightarrow 0 < y_1y_2 < y_2 < 1$$
$$\Rightarrow 0 < y_1 < 1 \ \& \ 0 < y_2 < 1$$

Thus the marginal p.d.f of Y_1 is

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_Y(y_1, y_2)\, dy_2$$

$$= \begin{cases} \int_0^1 8 y_1 y_2^3\, dy_2 & \text{if } 0 < y_1 < 1 \\ 0 & \text{o.w.} \end{cases}$$

$$= \begin{cases} 2y_1 & \text{if } 0 < y_1 < 1 \\ 0 & \text{o.w.} \end{cases}$$

(28) (a) $\int_0^{\infty}\left(\int_0^{\infty} \frac{e^{-(y+\frac{x}{y})}}{y}\, dx\right) dy$ ~~Put $\frac{x}{y} = t$~~ dx

$$= \int_0^{\infty}\left(\frac{e^{-y}}{y}(-y)e^{-x/y}\right)_0^{\infty} dy = \int_0^{\infty} e^{-y}\, dy$$

$$= (-e^{-y})_0^{\infty} = 1$$

$\Rightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1.$

(b) $$E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y)\, dx\, dy$$

$$= \int_0^{\infty} e^{-y}\left(\int_0^{\infty} \frac{x}{y} e^{-x/y}\, dx\right) dy$$

$$= \int_0^{\infty} y e^{-y}\, dy = 1$$

$$E(Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f(x,y)\, dx\, dy$$

$$= \int_0^{\infty} e^{-y}\int_0^{\infty} e^{-x/y}\, dx\quad dy$$

$$= \int_0^{\infty} y e^{-y}\, dy = 1$$

$$E(XY) = \int_0^{\infty} y e^{-y} \int_0^{\infty} \frac{x}{y} e^{-x/y}\, dx\, dy$$

$$= \int_0^{\infty} y^2 e^{-y}\, dy = 2$$

Hence $\operatorname{Cov}(X,Y) = E(XY) - E(X)E(Y)$
$= 2 - 1 = 1$

(29) (a)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\Rightarrow \int_0^1 \left(\int_0^1 (x + cy^2)\,dy \right) dx = 1$$

$$\Rightarrow \int_0^1 \left(xy + \frac{cy^3}{3} \right)\Big|_0^1 dx = 1$$

$$\Rightarrow \int_0^1 \left(x + \frac{c}{3}\right) dx = 1 \Rightarrow \left(\frac{x^2}{2} + \frac{cx}{3}\right)\Big|_0^1 = 1$$

$$\Rightarrow \frac{1}{2} + \frac{c}{3} = 1 \Rightarrow \frac{c}{3} = \frac{1}{2}$$

$$\Rightarrow c = 3/2$$

$$F_{(X,Y)}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(x_1,x_2)\,dx_1\,dx_2$$

$$= \cancel{\int_0^x \int_0^y x + cy^2}$$

$$= \begin{cases} 0 & \text{if } x<0 \text{ or } y<0 \\ \int_0^x \int_0^y \left(x_1 + \frac{3}{2}x_2^2\right) dx_2\,dx_1 & \text{if } 0 \le x, y \le 1 \\ \int_0^x \int_0^1 \left(x_1 + \frac{3}{2}x_2^2\right) dx_2\,dx_1 & \text{if } 0 \le x \le 1 \ \& \ y \ge 1 \\ \int_0^1 \int_0^y \left(x_1 + \frac{3}{2}x_2^2\right) dx_2\,dx_1 & \text{if } x \ge 1 \ \& \ 0 \le y \le 1 \\ \int_0^1 \int_0^1 \left(x_1 + \frac{3}{2}x_2^2\right) dx_2\,dx_1 & \text{if } x \ge 1 \ \& \ y \ge 1 \end{cases}$$

$$= \begin{cases} 0 & \text{if } x<0 \text{ or } y < 0 \\ \frac{1}{2}x^2 y + \frac{1}{2}xy^3 & \text{if } 0 \le x, y \le 1 \\ \frac{1}{2}x^2 + \frac{1}{2}x & \text{if } 0 \le x \le 1 \ \& \ y \ge 1 \\ \frac{1}{2}y + \frac{1}{2}y^3 & \text{if } 0 \le y \le 1 \ \& \ x \ge 1 \\ 1 & \text{if } x \ge 1 \ \& \ y \ge 1 \end{cases}$$

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(c) $P(0 \le X \le \frac{1}{2},\ 0 \le Y \le \frac{1}{2})$

$$= \int_0^{1/2}\int_0^{1/2} (x + cy^2)\,dx\,dy$$

$$= \int_0^{1/2} \left(\frac{x^2}{2} + \frac{3}{2}xy^2\right)_0^{1/2} dy$$

$$= \int_0^{1/2} \left(\frac{1}{8} + \frac{3}{4}y^2\right) dy$$

$$= \left(\frac{1}{8}y + \frac{3}{4}\frac{y^3}{3}\right)_0^{1/2} = \frac{1}{16} + \frac{1}{32} = \frac{3}{32}$$

(d) The marginal p.d.f of X & Y is

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \qquad \forall x \in \mathbb{R}$$

$$= \begin{cases} \int_0^1 \left(x + \frac{3}{2}y^2\right) dy & 0 \le x \le 1 \\ 0 & \text{o.w.} \end{cases}$$

$$= \begin{cases} \left(xy + \frac{y^3}{2}\right)_0^1 & 0 \le x \le 1 \\ 0 & \text{o.w.} \end{cases} = \begin{cases} x + \frac{1}{2} \\ 0 \end{cases}$$

&

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx \qquad \forall y \in \mathbb{R}$$

$$= \begin{cases} \int_0^1 \left(x + \frac{3}{2}y^2\right) dx & 0 \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$$

$$= \begin{cases} \left(\frac{x^2}{2} + \frac{3}{2}y^2x\right)_0^1 & 0 \le y \le 1 \\ 0 & \text{o.w.} \end{cases} = \begin{cases} \frac{3}{2}y^2 + \frac{1}{2} \\ 0 \end{cases}$$

Then $E(Y) = 1700 \times 0.6 \ldots$

(30) $y_1 = x_1 + x_2$ & $y_2 = \frac{x_2}{x_1+x_2}$

$\Rightarrow x_2 = y_1 y_2$ & $x_1 = y_1(1-y_2)$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} 1-y_2 & -y_1 \\ y_2 & y_1 \end{vmatrix} = y_1 - \cancel{y_1 y_2} + \cancel{y_1 y_2} = y_1 \neq 0$$

in the range of transformation

Hence the joint p.d.f of Y_1 & Y_2

$$f_{(Y_1,Y_2)}(y_1,y_2) = f_{\underline{X}}(y_1(1-y_2), y_1 y_2)\,|J|$$

$$= \begin{cases} \frac{1}{2} y_1 e^{-y_1(1-y_2)} & \text{if } 0 < y_1 y_2 < y_1(1-y_2) < \infty \\ \frac{1}{2} y_1 e^{-y_1 y_2} & \text{if } 0 < y_1(1-y_2) < y_1 y_2 < \infty \\ 0 & \text{o.w.} \end{cases}$$

The p.d.f of X_i is

$$f_{X_i}(x_i) = \begin{cases} e^{-x_i} & \text{if } x_i > 0 \\ 0 & \text{o.w.} \end{cases}$$

The joint p.d.f of X_1, X_2, X_3 is

$$f_{\underline{X}}(x_1,x_2,x_3) = f_{X_1}(x_1)\, f_{X_2}(x_2)\, f_{X_3}(x_3) \quad \forall (x_1,x_2,x_3) \in \mathbb{R}^3$$

$$= \begin{cases} e^{-(x_1+x_2+x_3)} & \text{if } x_1 > 0, x_2 > 0, x_3 > 0 \\ 0 & \text{o.w.} \end{cases}$$

Let $y_1 = x_1 + x_2 + x_3$, $y_2 = \frac{x_1+x_2}{x_1+x_2+x_3}$ & $y_3 = \frac{x_1}{x_1+x_2}$

$\Rightarrow y_1 y_2 = x_1 + x_2$ & $y_3 = \frac{x_1}{x_1+x_2}$ $\Rightarrow$ $x_1 = y_1 y_2 y_3$

$x_2 = y_1 y_2 (1-y_3)$

$x_3 = y_1(1-y_2)$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \frac{\partial x_1}{\partial y_3} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \frac{\partial x_2}{\partial y_3} \\ \frac{\partial x_3}{\partial y_1} & \frac{\partial x_3}{\partial y_2} & \frac{\partial x_3}{\partial y_3} \end{vmatrix} = \begin{vmatrix} y_2 y_3 & y_1 y_3 & y_1 y_2 \\ y_2(1-y_3) & y_1(1-y_3) & -y_1 y_2 \\ 1-y_2 & -y_1 & 0 \end{vmatrix} = -y_1^2 y_2$$

as $x_i > 0$

Since $y_1 = x_1 + x_2 + x_3$, $0 < y_1 < \infty$

$y_1 y_2 (1-y_3) > 0$ & $y_1(1-y_2) > 0$

Since $y_1 > 0$, $1-y_2 > 0 \Rightarrow y_2 < 1$

Clearly $0 < y_2$. Hence $0 < y_2 < 1$

As $y_1 y_2 > 0 \Rightarrow 1 - y_3 > 0 \Rightarrow y_3 < 1$

$\Rightarrow 0 < y_3 < 1$.

The joint p.d.f of Y_1, Y_2, Y_3 is

$$f_{(Y_1,Y_2,Y_3)}(y_1,y_2,y_3) = f_{\underline{X}}\big(y_1y_2y_3,\ y_1y_2(1-y_3),\ y_1(1-y_2)\big)|J|$$

$$= \begin{cases} y_1^2 y_2 e^{-y_1} & \text{if } 0 < y_1 < \infty,\ 0 < y_2, y_3 < 1 \\ 0 & \text{o.w.} \end{cases}$$

The marginal p.d.f of Y_1 is

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{(Y_1,Y_2,Y_3)}(y_1,y_2,y_3)\, dy_2\, dy_3$$

$$= \begin{cases} \int_0^1 \int_0^1 y_1^2 y_2 e^{-y_1}\, dy_2\, dy_3 & \text{if } 0 < y_1 < \infty \\ 0 & \text{o.w.} \end{cases}$$

$$= \begin{cases} \frac{y_1^2}{2} e^{-y_1} & \text{if } 0 < y_1 < \infty \\ 0 & \text{o.w} \end{cases}$$

The marginal p.d.f of Y_2 is

$$f_{Y_2}(y_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{(Y_1,Y_2,Y_3)}(y_1,y_2,y_3)\, dy_3\, dy_1$$

$$= \begin{cases} \int_0^{\infty} \int_0^1 y_1^2 y_2 e^{-y_1}\, dy_3\, dy_1, & 0 < y_2 < 1 \\ 0 & \text{o.w.} \end{cases}$$

$$= \begin{cases} 2y_2 & \text{if } 0 < y_2 < 1 \\ 0 & \text{o.w.} \end{cases}$$

The marginal p.d.f of Y_3 is

$$f_{Y_3}(y_3) = \begin{cases} \int_0^{\infty} \int_0^1 y_1^2 y_2 e^{-y_1}\, dy_2\, dy_1, & 0 < y_3 < 1 \\ 0 & \text{o.w.} \end{cases}$$

$$= \begin{cases} 1 & \text{if } 0 < y_3 < 1 \\ & \text{o.w.} \end{cases}$$