PROBLEM SET 01: THE REAL NUMBER SYSTEM

(1) Let A be a nonempty subset of $\mathbb{R}$ and $M \in \mathbb{R}$. Prove that $M = \sup A$ if and only if
   (a) M is an upper bound of A,
   (b) for any $\varepsilon > 0$, there exists $x \in A$ such that $x > M - \varepsilon$.

(2) Let A, B be nonempty subsets of $\mathbb{R}$ with $A \subset B$. Prove
$$\inf B \leq \inf A \leq \sup A \leq \sup B.$$

(3) Prove that for any $x \in \mathbb{R}$, there exists $m \in \mathbb{N}$ such that $-m < x$.

(4) Let $x, y \in \mathbb{R}$ such that $x < y$. Show that there exist $m, n \in \mathbb{N}$ such that $x < x + \frac{1}{m} < y$ and $x < y - \frac{1}{n} < y$.

(5) (a) Let $x > 0$. Prove that there exists $n \in \mathbb{N}$ such that $x > \frac{1}{n}$.
   (b) Let $x \geq 0$. Prove that $x = 0$ if and only if $x \leq \frac{1}{n}$ for every $n \in \mathbb{N}$.

(6) Let $U_n = (0, \frac{1}{n})$ and $V_n = (\frac{1}{n}, 1)$. Find $\cap_n U_n$ and $\cup_n V_n$.

(7) Find the supremum and infimum of the following sets:
   (a) (a, b), where $a, b \in \mathbb{R}$.
   (b) $\{1 - \frac{1}{n^2} : n \in \mathbb{N}\}$.
   (c) $\{\frac{m+n}{mn} : m,\ n \in \mathbb{N}\}$.
   (d) $\{x \in \mathbb{R} : x^2 - 5x + 6 < 0\}$.
   (e) The set of real numbers in $(0, 1)$ whose decimal expansions contains only 0's and 1's.

(8) Let A be a nonempty subset of $\mathbb{R}$ and $x, M \in \mathbb{R}$. Define the distance between x and A by
$$d(x, A) = \inf\{|x - a| : a \in A\}.$$
If $M = \sup A$, show that $d(M, A) = 0$.

(9) Let $A, B \subset \mathbb{R}$ be nonempty such that $\alpha = \sup A$ and $\beta = \sup B$. Show that $A + B$ is bounded above and $\sup(A + B) = \alpha + \beta$.

(10) **($\mathbb{Q}$ does not have the LUB property)**
   (a) Let $x \in \mathbb{Q}$ and $x > 0$. If $x^2 < 2$, show that there exists $n \in \mathbb{N}$ such that $(x+\frac{1}{n})^2 < 2$. Likewise, if $x^2 > 2$, show that there exists $m \in \mathbb{N}$ such that $(x - \frac{1}{m})^2 > 2$.
   (b) Show that the set $A = \{r \in \mathbb{Q} : r > 0, r^2 < 2\}$ is bounded above in $\mathbb{Q}$ but it does not have the LUB property in $\mathbb{Q}$.
   (c) From part (b), $\mathbb{Q}$ does not have the LUB property.
   (d) Let A be the set defined in part (b) and $M = \sup A$. Show that $M^2 = 2$.

PROBLEM SET 02: SEQUENCES AND THEIR CONVERGENCE

(1) Let $x_n \to \ell$. Show that if we alter a finite number of terms of (x_n), the new sequence still converges to ℓ.

(2) Let $x_n \to \ell$. If $\ell > 0$, then show that except for a finite number of terms, all $x_n > 0$.

(3) Let $x_n \leq y_n$ for all n such that $x_n \to x$ and $y_n \to y$. Show that $x \leq y$.

(4) True or False:

(a) If (x_n) and $(x_n y_n)$ are bounded, then (y_n) is bounded.

(b) If (x_n) and (y_n) are such that $x_n y_n \to 0$, then one of the sequences converges to 0.

(5) In each of the following sequences, write the first following terms of (x_n), and then investigate its convergence.

(a) $x_n = \frac{n^r}{(1+s)^n}$, where $r, s > 0$.

(b) $x_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} \cdots + \frac{1}{\sqrt{n^2+n}}$.

(c) $x_n = \frac{n^2}{n^3+n+1} + \frac{n^2}{n^3+n+2} + \cdots + \frac{n^2}{n^3+n+n}$.

(d) $x_n = a^n (2n)^b$ where $0 < a < 1$ and $b > 1$.

(e) $x_n = (a^n + b^n)^{1/n}$ where $0 < a < b$.

(f) $x_n = n^\alpha - (n+1)^\alpha$ for some $\alpha \in (0, 1)$.

(6) Show that the sequence (x_n) is bounded and monotone, and find its limit where

(a) $x_1 = 1$ and $x_{n+1} = \sqrt{3x_n}$.

(b) $x_1 = 1$ and $x_{n+1} = \frac{4+3x_n}{3+2x_n}$.

(7) Let $M = \sup A \subset \mathbb{R}$. Show that there exists a sequence (x_n) in A such that $x_n \to M$. Prove the analogous result for infimum as well.

(8) Show that the sequence $x_n = \sum_{k=1}^{n} \frac{1}{k}$ diverges to ∞.

(9) Let $0 < y_1 < x_1$. For $n \geq 2$, define

$$x_{n+1} = \frac{x_n + y_n}{2} \text{ and } y_{n+1} = \sqrt{x_n y_n}.$$

(a) Prove that (y_n) is increasing and bounded above while (x_n) is decreasing and bounded below.

(b) For $n \in \mathbb{N}$, prove that

$$0 < x_{n+1} - y_{n+1} < \frac{1}{2^n}(x_1 - y_1).$$

(c) Prove (x_n) and (y_n) converges to the same limit.

(10) (**Euler's number** e) In general, it may not be possible to explicitly find the limit of a convergent sequence. So, some real numbers are defined as the limit of such sequences.

Let $x_n = (1 + \frac{1}{n})^n$. Prove that (x_n) is increasing and bounded above. Define $e = \lim_{n\to\infty}(1 + \frac{1}{n})^n$. Show that e is irrational.

PROBLEM SET 03: CAUCHY SEQUENCES AND SUBSEQUENCE

(1) Prove that the sum of two Cauchy sequences is Cauchy.

(2) Prove that the product of two Cauchy sequences is Cauchy.

(3) Let (x_n) be a sequence and let $a > 1$. Assume that $|x_{k+1} - x_k| < a^{-k}$ for all $k \in \mathbb{N}$. Show that (x_n) is Cauchy. final GP solution?

(4) Let (x_n) be defined by

$$x_n = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{(-1)^{n+1}}{n!}.$$

Show that (x_n) converges.

(5) Show that the sequences (x_n) defined below are Cauchy.

(a) $x_1 = 1$, $x_{n+1} = \frac{1}{2+x_n^2}$.

(b) $x_1 = 1$, $x_{n+1} = \frac{1}{6}(x_n^2 + 8)$.

(c) $x_1 = \frac{1}{2}$, $x_{n+1} = \frac{1}{7}(x_n^3 + 2)$.

(d) $x_1 = a$, $x_2 = b$, $x_{n+2} = \frac{x_n + x_{n+1}}{2}$, where a and b are two distinct real numbers. and find the limit

(e) Let $0 < a \leq x_1 \leq x_2 \leq b$, $x_{n+2} = \sqrt{x_{n+1}x_n}$.

(6) Show that the following sequences cannot converge.

(a) $x_n = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}$.

(b) $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. x1, x2, x4, x8...

(7) Let (x_n) be a sequence such that $|x_n| \leq \frac{1+n}{1+n+2n^2}$ for all n. Prove that (x_n) is Cauchy.

(8) Show that if a monotone sequence has a convergent subsequence, then it is convergent.

(9) Let (r_n) be an enumeration of all rational numbers in $[0, 1]$. Show that (r_n) is not convergent.

(10) **(Elaborate version of Bolzano-Weierstrass Theorem)** Let (x_n) be a sequence. If (x_n) is bounded above and does not diverge to $-\infty$, then prove that (x_n) has a convergent subsequence. Likewise, if (x_n) is bounded below and does not diverge to ∞, then prove that (x_n) has a convergent subsequence.

CONTINUITY AND LIMITS

(1) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function such that $f(c) > 0$ for some $c \in \mathbb{R}$. Show that there exists an $\epsilon > 0$ such that $f(x) > 0$ for all $x \in (c - \epsilon, c + \epsilon)$. apply IVP assuming ends are negative

(2) Let $f : (-1, 1) \to \mathbb{R}$ be a continuous function such that in every neighborhood of 0, there exists a point where f takes the value 0. Show that $f(0) = 0$.

(3) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous at 0, show that f is continuous everywhere.

(4) Discuss the continuity/discontinuity for the following functions

(a) $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 3x + 1, & \text{if } x \in \mathbb{Q} \\ x, & \text{otherwise.} \end{cases}$$

discontinuous everywhere except at x = -1/2

(b) $f : [0, \pi] \longrightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ x \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}, & \text{otherwise.} \end{cases}$$

discontinuous at x=0

(5) Let A be a nonempty subset of $\mathbb{R}$ and $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = \inf\{|x - a| : a \in A\}$. Show that f is continuous on $\mathbb{R}$.

(6) Let $J = \{\frac{1}{n} : n \in \mathbb{N}\}$. Show that any function $f : J \longrightarrow \mathbb{R}$ is continuous on J.

(7) A function $f : [a, b] \longrightarrow \mathbb{R}$ is said to be Lipschitz on $[a, b]$ if there exists $L > 0$ such that $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in [a, b]$. Show that any Lipschitz function is continuous.

(8) Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous such that given any two points $x < y$, there exists a point z such that $x < z < y$ and $f(z) = g(z)$. Show that $f(x) = g(x)$ for all x.

(9) Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous. If $f(x) = g(x)$ for $x \in \mathbb{Q}$, then show that $f = g$.

(10) Let $f : [0, \infty) \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1, & \text{if } x = 0, \\ 1/q, & \text{if } x = p/q, \text{ where } p, q \in \mathbb{N} \text{ and } p, q \text{ have no common factors }, \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

Show that f is continuous at every irrational in $[0, \infty)$.

PROPERTIES OF CONTINUOUS FUNCTIONS

(1) Let $f : J \to \mathbb{R}$ be a strictly monotonic function such that $f(J)$ is an interval. Show that f is continuous.

(2) Let $f : [a, b] \to [a, b]$ be continuous. Show that f has a fixed point in $[a, b]$, i.e., $\exists\, c \in [a, b]$ such that $f(c) = c$.

(3) Prove that $x = \cos x$ for some $x \in (0, \pi/2)$.

(4) Prove that $xe^x = 1$ for some $x \in (0, 1)$.

(5) Is there a continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(x) \notin \mathbb{Q}$ for $x \in \mathbb{Q}$ and $f(x) \in \mathbb{Q}$ for $x \notin \mathbb{Q}$.

(6) Show that a polynomial of odd degree has at least one real root.

(7) Show that $x^4 + 5x^3 - 7$ has at least two real roots.

(8) Let $p(X) := a_0 + a_1X + \cdots + a_nX^n$, n is even. If $a_0a_n < 0$, show that p has at least two real roots.

(9) Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous. Show that $f([a, b]) = [c, d]$ for some $c, d \in \mathbb{R}$ with $c \leq d$. Can you identify c, d.

(10) Construct a continuous bijection $f : [a, b] \longrightarrow [c, d]$ such that f^{-1} is continuous.

(11) Construct a continuous function from $(0, 1)$ onto $[0, 1]$. Can such a function be one-one.

(12) Construct a continuous one-one function from $(0, 1)$ to $[0, 1]$. Can such a function be onto.

DIFFERENTIABILITY

(1) Discuss the differentiability of the following functions at $x = 0$.

(a) $f(x) = x^{\frac{1}{3}}$

(b) $f(x) = x^2$ for rational x and $f(x) = 0$ for irrational x.

(c) $f(x) = x \sin x \cos \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$.

(d) Let m, n be positive integers. Define

$$f(x) = \begin{cases} x^n, & x \geq 0 \\ x^m, & x < 0. \end{cases}$$

(2) Give an example of a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which is differentiable only at $x = 1$.

(3) Let $n \in \mathbb{N}$. Define $f : \mathbb{R} \longrightarrow \mathbb{R}$ by $f(x) = x^n$ for $x \geq 0$ and $f(x) = 0$ if $x < 0$. For which values of n,

(a) is f continuous at 0?

(b) is f differentiable at 0?

(c) is f' continuous at 0?

(d) is f' differentiable at 0?

(4) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $|f(x)| \leq x^2$ for all $x \in \mathbb{R}$. Discuss the differentiability of f at 0.

(5) Find the number of real solutions of the following equations.

(a) $2x - \cos^2 x + \sqrt{7} = 0$.

(b) $x^{17} - e^{-x} + 5x + \cos x$.

(6) Let $P(X) := \sum_{k=0}^{n} a_k X^k$, $n \geq 2$ be a real polynomial. Assume that all roots of P lie in $\mathbb{R}$. Show that all roots its derivative $P'(X)$ are also real.

(7) Let $a_1, a_2, \ldots, a_n$ be real numbers such that $a_1 + a_2 + \cdots + a_n = 0$. Show that the polynomial $q(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1}$ has at least one real root.

(8) Let $f : [a, b] \longrightarrow \mathbb{R}$ be such that $f'''(x)$ exists for all $x \in [a, b]$. Suppose $f(a) = f(b) = f'(a) = f'(b) = 0$ Show that the equation $f'''(x) = 0$ has a solution.

(9) Let $f : (a, b) \to \mathbb{R}$ be a function. Prove that f is differentiable at x if and only if there exists a (unique) linear map $A : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{h \to 0} \frac{f(x+h) - f(x) - Ah}{|h|} = 0.$$

(10) (**Differentiable Inverse Theorem**). Suppose $f : J \to \mathbb{R}$ is a one-one and continuous function. If f is differentiable at c and $f'(c) \neq 0$, then $f^{-1} : f(J) \to J$ is differentiable at $f(c)$ and $(f^{-1})'(f(c)) = \frac{1}{f'(c)}$.

MEAN VALUE THEOREM

(1) Find values of the constants a, b and c for which the graphs of the two functions $f(x) = x^2 + ax + b$ and $g(x) = x^3 - c$, $x \in \mathbb{R}$ intersect at the point $(1, 2)$ and the have the same tangent there.

(2) Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be differentiable. Assume that $f(0) = g(0)$ and $f'(x) \leq g'(x)$, $\forall\, x \in \mathbb{R}$. Show that $f(x) \leq g(x)$ for $x \geq 0$.

(3) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be differentiable. Assume that $1 \leq f'(x) \leq 2$ for $x \in \mathbb{R}$ and $f(0) = 0$. Prove that $x \leq f(x) \leq 2x$ for $x \geq 0$.

(4) Use MVT to establish the following inequalities

(a) $e^x > 1 + x$, $\forall\, x \in \mathbb{R}$.

(b) $\frac{y-x}{y} < \log \frac{y}{x} < \frac{y-x}{x}$ for $0 < x < y$.

(c) $\frac{1}{2\sqrt{n+1}} < \sqrt{n+1} - \sqrt{n} < \frac{1}{2\sqrt{n}}$, $\forall\, n \in \mathbb{N}$.

(d) If $e \leq a < b$, then $a^b > b^a$. (Hint: Use part (b)).

(e) **Bernoullis Inequality:** Let $\alpha > 0$ and $h \geq -1$. Then

$$\begin{aligned} (1+h)^\alpha &\leq 1 + \alpha h, \quad \text{for } 0 < \alpha \leq 1, \\ (1+h)^\alpha &\geq 1 + \alpha h, \quad \text{for } \alpha \geq 1. \end{aligned}$$

(5) Prove that $\frac{\sin x}{x}$ is strictly decreasing on $(0, \pi/2)$.

(6) Let $f : [0, 1] \longrightarrow \mathbb{R}$ be differentiable such that $|f'(x)| < 1$, $\forall\, x \in [0, 1]$. Show that f has at most one fixed point.

(7) Let $f : [0, 1] \longrightarrow \mathbb{R}$ be differentiable and $f(0) = 0$. Suppose that $|f'(x)| \leq |f(x)|$ $\forall\, x \in [0, 1]$. Show that $f = 0$.

(8) Let $f : (0, 1] \longrightarrow \mathbb{R}$ be differentiable with $|f'(x)| < 1$. Define $a_n := f(1/n)$. Show that (a_n) converges.

(9) Let $f : [a, b] \longrightarrow \mathbb{R}$ be differentiable and $a \geq 0$. Using Cauchy mean value theorem, show that there exist $c_1, c_2 \in (a, b)$ such that $\frac{f'(c_1)}{a+b} = \frac{f'(c_2)}{2c_2}$.

LOCAL EXTREMA AND POINTS OF INFLECTION

(1) Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be defined as $h(x) = f(x)g(x)$, where f and g are non-negative functions. Show that h has a local maximum at a if f and g have a local maximum at a.

(2) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = (\sin x - \cos x)^2$. Find the maximum value of f on $\mathbb{R}$.

(3) Let $f : [-2, 0] \longrightarrow \mathbb{R}$ be defined by $f(x) = 2x^3 + 2x^2 - 2x - 1$. Find the maximum and minimum values of f on $[-2, 0]$.

(4) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be such that $f'(x) = 14(x-2)(x-3)^2(x-4)^3(x-5)^4$. Find all the points of local maxima and local minima.

(5) Let $x_1, x_2, \ldots, x_n \in \mathbb{R}$ and $f(x) = \sqrt{(x-x_1)^2 + (x-x_2)^2 + \cdots + (x-x_n)^2}, x \in \mathbb{R}$. Find the point of absolute minimum of the function f.

(6) Find the points of local maxima and local minima of $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = 2x^4 e^{-x^2}$.

(7) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a twice differentiable function with the following properties: $f(-1) = 4$, $f(0) = 2$, $f(1) = 0$, $f'(x) > 0$ for $|x| > 1$, $f'(x) < 0$ for $|x| < 1$, $f'(1) = 0$, $f'(-1) = 0$, $f''(x) < 0$ for $x < 0$ and $''(x) > 0$ for $x > 0$. Sketch the graph of f.

(8) Sketch the graphs of the following functions after finding the intervals of decrease/increase, concavity/convexity, points of local minima/local maxima, points of inflection and asymptotes.

a) $f(x) = \frac{x^2+x-5}{x-1}$ b) $f(x) = \frac{2x^2-1}{x^2-1}$ c) $f(x) = \frac{x^2}{x^2+1}$

d) $f(x) = \frac{2x^3}{x^2-4}$ e) $f(x) = 3x^4 - 8x^3 + 12$.

TAYLOR'S THEOREM

(1) Let $f : [a, b] \longrightarrow \mathbb{R}$ and n be a non-negative integer. Suppose that $f^{(n+1)}$ exists on $[a, b]$. Show that f is a polynomial of degree $\leq n$ if $f^{(n+1)}(x) = 0$ for all $x \in [a, b]$. Observe that the statement for $n = 0$ can be proved by the mean value theorem.

(2) Show that $1 + \frac{x}{2} - \frac{x^2}{8} \leq \sqrt{1+x} \leq 1 + \frac{x}{2}$ for $x > 0$.

(3) Show that for $x \in \mathbb{R}$ with $|x|^5 < \frac{5!}{10^4}$, we can replace $\sin x$ by $x - \frac{x^3}{6}$ with an error of magnitude less than or equal to 10^{-4}.

(4) Prove the binomial expansion: $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots + x^n,\ x \in \mathbb{R}$.

(5) Using Taylor's theorem compute: $\lim_{x\to 0} \frac{1-\sqrt{1-x^2}\cos x}{x^4}$.

(6) If $x \in [0, 1]$ and $n \in \mathbb{N}$, show that

$$\left| \ln(1+x) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1}\frac{x^n}{n} \right) \right| < \frac{x^{n+1}}{n+1}.$$

(7) (a) Let $f : [a, b] \longrightarrow \mathbb{R}$ be such that $f''(x) \geq 0$ for all $x \in [a, b]$. Suppose $x_0 \in [a, b]$. Show that for any $x \in [a, b]$

$$f(x) \geq f(x_0) + f'(x_0)(x - x_0)$$

i.e., the graph of f lies above the tangent line to the graph at $(x_0, f(x_0))$.

(b) Show that $\cos y - \cos x \geq (x - y)\sin x$ for all $x, y \in [\frac{\pi}{2}, \frac{3\pi}{2}]$.

(8) (a) Let $f : [a, b] \longrightarrow \mathbb{R}$ be such that $f''(x) \geq 0$ for all $x \in [a, b]$. Suppose $x, y \in (a, b), x < y$ and $0 < \lambda < 1$. Show that

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

i.e., the chord joining the two points $(x, f(x))$ and $(y, f(y))$ lies above the portion of the graph $\{(z, f(z)) : z \in (x, y)\}$.

(b) Show that $\lambda \sin x \leq \sin \lambda x$ for all $x \in [0; \pi]$ and $0 < \lambda < 1$.

(9) Let f be a twice differentiable function on $\mathbb{R}$ such that $f''(x) \geq 0$ for all $x \in \mathbb{R}$. Show that if f is bounded then it is a constant function.

(10) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be such that $f'''(x) > 0$ for all $x \in \mathbb{R}$. Suppose that $x_1, x_2 \in \mathbb{R}$ and $x_1 < x_2$. Show that $f(x_2) - f(x_1) > f'\left(\frac{x_1+x_2}{2}\right)(x_2 - x_1)$.

(11) Suppose f is a three times differentiable function on $[-1, 1]$ such that $f(-1) = 0$, $f(1) = 1$ and $f'(0) = 0$. Using Taylor's theorem show that $f'''(c) \geq 3$ for some $c \in (-1, 1)$.

SERIES

(1) Prove that if a series $\sum_{n=1}^{\infty} a_n$ converges, then the sum is unique.

(2) Show that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=k}^{\infty} a_n$ converges for any $k \in \mathbb{N}$.

(3) Let (a_n) be any sequence of real numbers. Show that this sequence converges to a number S if and only if the series

$$a_1 + \sum_{n=2}^{\infty} (a_n - a_{n-1})$$

converges and has sum S. Verify the convergence/divergence of the following series:

(a) $\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)}$.

(b) $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$.

(4) Let $\sum_{n=1}^{\infty} a_n$ converges and $a_n > 0$ for all n. If (a_{n_k}) is a subsequence of (a_n), show that $\sum_{k=1}^{\infty} a_{n_k}$ also converges.

(5) Let $\sum_{n=1}^{\infty} a_n$ be a convergent series. Show that for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\sum_{n=N+1}^{\infty} a_n < \epsilon$. The series $\sum_{n=N+1}^{\infty} a_n$ is called a tail of the series $\sum_{n=1}^{\infty} a_n$.

(6) Express the infinite repeating decimal

$$0.123451234512345123451234512345\ldots$$

as the sum of a convergent geometric series and compute its sum.

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(7) Show that

$$\frac{1}{r-1} = \frac{1}{r+1} + \frac{2}{r^2+1} + \frac{4}{r^4+1} + \frac{8}{r^8+1} + \cdots$$

for all $r > 1$.

(8) Obtain a formula for the following sums

(a) $2 + \frac{2}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{2\sqrt{2}} + \cdots$.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+2)(k+4)}$.

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{\alpha r + \beta}{k(k+1)(k+2)}$.

(9) Let $\sum_{n=1}^{\infty} a_n$ be a convergent series and $\sum_{n=1}^{\infty} b_n$ is obtained by grouping finite number of terms of $\sum_{n=1}^{\infty} a_n$ such as $(a_1 + a_2 + \cdots + a_{m_1}) + (a_{m_1+1} + a_{m_1+2} + \cdots + a_{m_2}) + \cdots$ for some $m_1, m_2, \ldots$ (Here $b_1 = a_1 + a_2 + \cdots + a_{m_1}, b_2 = a_{m_1+1} + a_{m_1+2} + \cdots + a_{m_2}$ and so on). Show that $\sum_{n=1}^{\infty} b_n$ converges and has the same limit as $\sum_{n=1}^{\infty} a_n$. What happens if $\sum_{n=1}^{\infty} a_n$ diverges?

(10) Let $a_n \geq 0$ for all n such that $\sum_{n=1}^{\infty} a_n$ converges. Suppose $\sum_{n=1}^{\infty} b_n$ is obtained by rearranging the terms of $\sum_{n=1}^{\infty} a_n$ (i.e., the terms of $\sum_{n=1}^{\infty} b_n$ are same as those of $\sum_{n=1}^{\infty} a_n$ but they occur in different order). Show that $\sum_{n=1}^{\infty} b_n$ converges and has the same limit as $\sum_{n=1}^{\infty} a_n$.

(11) Consider the series $\sum_{n=1}^{\infty} a_n$ where $a_n = \frac{(-1)^{n+1}}{n}$. Show that the series

$$(1 - \frac{1}{2}) - \frac{1}{4} + (\frac{1}{3} - \frac{1}{6}) - \frac{1}{8} + (\frac{1}{5} - \frac{1}{10}) - \frac{1}{12} + \cdots$$

which is obtained from $\sum_{n=1}^{\infty} a_n$ by rearranging and grouping, is $\frac{1}{2} \sum_{n=1}^{\infty} a_n$.

CONVERGENCE TESTS I: COMPARISON, LIMIT COMPARISON AND CAUCHY CONDENSATION TESTS

(1) Let $a_n \geq 0$ for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} a_n$ converges, then show that the following series also converge.

(a) $\sum_{n=1}^{\infty} a_n^2$.

(b) $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$.

(c) $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$.

(d) $\sum_{n=1}^{\infty} \frac{a_n+4^n}{a_n+5^n}$.

from 1a, this means that there is a negative term in the sequence

(2) Let $\sum_{n=1}^{\infty} a_n$ be a convergent series. Show that $\sum_{n=1}^{\infty} |a_n|$ diverges if $\sum_{n=1}^{\infty} a_n^2$ diverges.

(3) Let $a_n, b_n \geq 0$ for all $n \in \mathbb{N}$. Show that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converges if and only if $\sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2}$ converges.

(4) Suppose $a_n > 0$ for all n and $\sum_{n=1}^{\infty} a_n$ converges. Show that $\sum_{n=1}^{\infty}(1 - \frac{\sin a_n}{a_n})$ converges.

(5) Let (a_n) be a sequence of positive real numbers such that $a_{n+1} \leq a_n$ for all n and $\sum_{n=1}^{\infty} a_n$ converges. Show that $\sum_{n=1}^{\infty} n(a_n - a_{n+1})$ converges.

(6) Show that $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)(\ln(\ln n))}$ diverges.

(7) In each of the following cases, discuss the convergence/divergence of the series $\sum_{n=2}^{\infty} a_n$ where a_n equals

a) $\frac{1}{(\ln n)^p}$, $(p > 0)$ b) $\frac{\sin \frac{1}{n}}{\sqrt{n}}$ c) $\frac{1}{n^2 - \ln n}$ d) e^{-n^2}

e) $\frac{1}{n^{1+\frac{1}{n}}}$ f) $1 - \cos \frac{\pi}{n}$ g) $(\ln n) \sin \frac{1}{n^2}$ h) $(n+2)(1 - \cos \frac{1}{n})$

i) $\frac{3+\cos n}{e^n}$ j) $\frac{2+\sin^3(n+1)}{2^n+n^2}$ k) $\frac{\sqrt{n+1}-\sqrt{n}}{n}$

CONVERGENCE TESTS II: RATIO, ROOT AND LEIBNIZS TESTS

(1) Determine the values of $\alpha \in \mathbb{R}$ for which $\sum_{n=1}^{\infty}(\frac{\alpha n}{n+1})^n$ converges.

(2) Consider $\sum_{n=1}^{\infty} a_n$, where $a_n > 0$ for all n. Prove or disprove the following statements.

  (a) If $\frac{a_{n+1}}{a_n} < 1$ for all n, then the series converges.

  (b) If $\frac{a_{n+1}}{a_n} > 11$ for all n, then the series diverges.

(3) Show that the series $\frac{1}{1^2} + \frac{1}{2^3} + \frac{1}{3^2} + \frac{1}{4^3} + \frac{1}{5^2} + \frac{1}{6^3} + \cdots$ converges and that the root test and ratio test are not applicable.

(4) Consider the rearranged geometric series $\frac{1}{2} + 1 + \frac{1}{8} + \frac{1}{4} + \frac{1}{32} + \frac{1}{16} + \frac{1}{128} + \frac{1}{64} + \cdots$. Show that the series converges by the root test and that the ratio test is not applicable.

(5) (a) If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converges absolutely, show that $\sum_{n=1}^{\infty} a_n b_n$ converges absolutely.

  (b) If $\sum_{n=1}^{\infty} a_n$ converges absolutely and (b_n) is a bounded sequence, show that $\sum_{n=1}^{\infty} a_n b_n$ converges absolutely.

  (c) Give an example of a convergent series $\sum_{n=1}^{\infty} a_n$ and a bounded sequence (b_n) such that $\sum_{n=1}^{\infty} a_n b_n$ diverges.

(6) n each of the following cases, discuss the convergence/divergence of the series $\sum_{n=1}^{\infty} a_n$ where a_n equals

a) $\frac{n!}{n^n}$ b) $\frac{7^{n+1}}{9^n}$ c) $\frac{n!}{(e)^{n^2}}$ d) $\frac{n^2 2^n}{(2n+1)!}$

e) $(1-\frac{1}{n})^{n^2}$ f) $\frac{n^2}{3^n}(1+\frac{1}{n})^{n^2}$ g) $\sin(\frac{(-1)^n}{n^p})$, $p > 0$ h) $\frac{1}{2^n - n}$

i) $(-1)^n \frac{(\ln n)^3}{n}$ j) $(-1)^n (n^{\frac{1}{n}} - 1)^n$ k) $\frac{2^n + n^2 - \ln n}{n!}$ l) $\frac{\cos(\pi n) \ln n}{n}$

m) $(1+\frac{2}{n})^{n^2 - \sqrt{n}}$ n) $\frac{n^2(2\pi + (-1)^n)^n}{10^n}$