Tutorial (PS1)

~~ans~~ Random exp
- → All outcomes known in advance
- → Not predictable, a particular outcome of trial.
- → can be repeated,

Sample space ⇒ collection of all outcomes

Event ⇒ subset of sample space.

For uncountable :-

S - uncountable sample

$P(\{x\}) = 0.$

$U \subseteq S$

$U \subseteq V$

not found such that $\boxed{P(V) = 0}$

Proof :-

U-maximal.

$P(U) = 0$

$x \in S - U$

$P(U \cup \{x\}) = 0.$

is contradictory

—x—

Events :- E, F, G

ans. 1)

a) only F occurs.

ans:- $F \cap E^c \cap G^c$

b) both E and F occurs but NOT G.

ans:- $E \cap F \cap G^c$

c) at least one event occurs

ans:- $E \cup F \cup G$

d) at least two event occurs.

ans:- $(E \cap F) \cup (F \cap G) \cup (E \cap G)$

e) all three events occur.

ans:- $E \cap F \cap G$

f) none occurs.

ans:- $E^c \cap F^c \cap G^c$

g) at most one event occurs.

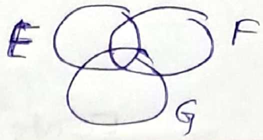
ans:- $(\ )^c$ means → ϕ, 1 occurs

→ (3, 2) atleast 2 event not occurs

$$(E \cap F^c \cap G^c) \cup (F \cap G^c \cap E^c) \cup (G \cap E^c \cap F^c)$$
$$\cup (E^c \cap F^c \cap G^c)$$

(or)

$$(E \cap F)^c \cap (F \cap G)^c \cap (E \cap G)^c$$



h) at most 2 event occur.

ans:- $(E \cap F \cap G)^c$

ans.2) $S = \{0, 1, 2, 3, \ldots\}$ and $E \subseteq S$ (event)

$\hookrightarrow$ countable non-finite set

verify P is probability on S.

a) • It should follow

$\rightarrow P(S) = 1$

$\rightarrow 0 \leq P(E) \leq 1$

$\rightarrow P\left(\bigcup_i E_i\right) = \sum_i P(E_i)$

Probability function :- $P : P(S) \longrightarrow \mathbb{R}$

$\hookrightarrow [0, 1]$

$$P(E) = \sum_{x \in E} \frac{e^{-\lambda} \lambda^x}{x!} \quad \Big\} \text{ convergent series to } 1$$

all terms positive

$\therefore$ sum ≥ 0

$$P(S) = \frac{e^{-\lambda} \cdot \lambda^0}{0!} + \frac{e^{-\lambda} \cdot \lambda^1}{1!} + \frac{e^{-\lambda} \cdot \lambda^2}{2!} + \frac{e^{-\lambda} \cdot \lambda^3}{3!} + \cdots$$

$$= e^{-\lambda} \left[\frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \cdots \right]$$

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$$\therefore \boxed{P(S) = e^{-\lambda} \times e^{\lambda} = 1}$$

—x—

Why converging?

$$\frac{r_{t+1}}{r_t} = \frac{\left(\frac{e^{-\lambda}\cdot\lambda^{t+1}}{(t+1)!}\right)}{\left(\frac{e^{-\lambda}\cdot\lambda^{t}}{t!}\right)} = \frac{\lambda}{(t+1)!} \times t! = \boxed{\frac{\lambda}{t+1}}$$

as $t \to \infty$, $1 > \frac{\lambda}{t+1} > 0$

—x—

$\therefore$ converging
$P(E) \geq 0$
$P(S) = 1$
$P(E) \leq P(S)$
$P(E)$ as "i" increases
$P(E)$ increases
but $P(E)$ bounded
$\therefore$ $0 \leq P(E) \leq 1$

(iii) $P\left(\bigcup_i E_i\right) = \sum_i P(E_i)$

$E_i \Rightarrow$ M.E

Reason:- since $P(S)$ is converging series.

Then $E \subseteq S$, $\therefore P(E)$ will also converging

Also by rearrangement of terms E_i will also Lead to converging series.

ex:- E_1 = even , E_2 = odd.

$$P(E_1 \cup E_2) = \sum_{x \in E_1 \cup E_2} \frac{e^{-\lambda}\cdot\lambda^x}{x!}, \quad \lambda > 0.$$

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$$P\left(\bigcup_i E_i\right) = \sum_{x \in \left(\bigcup_i E_i\right)} \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}, \quad \lambda > 0$$

$$= \sum_i \left(\sum_{x \in E_i} \frac{e^{-\lambda} \cdot \lambda^{x}}{x!} \right)$$

$$= \sum_{x \in E_1} \frac{e^{-\lambda} \cdot \lambda^{x}}{x!} + \sum_{x \in E_2} \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}$$

$$+ \sum_{x \in E_3} \frac{e^{-\lambda} \cdot \lambda^{x}}{x!} + \cdots$$

$$= P(E_1) + P(E_2) + \cdots$$

$$= \sum_i P(E_i)$$

Yes, P.F ✓

(e)

b) $P(E) = \sum_{x \in E} p(1-p)^x, \quad 0 < p < 1$

same as (a)

c) $\because P(E) \geq 0$ and $P(S) = 1$

Hence $0 \leq P(E) \leq 1$

Lets take $E_1 = \text{odd}$ $E_2 = \text{even}$

$\therefore P(E_1 \cup E_2) = P(E_1) + P(E_2)$

$P(S) = 1 + 1$

$\boxed{1 = 2}$ absurd.

also $P(E_1 \cup E_2) \geq 1$, impossible.

$\therefore$ $\boxed{NO}$

—x—

Q. $\boxed{2}$

(a) $S = \mathbb{R}$, $\Sigma = \mathbb{R}$

- $\Sigma =$ ~~all~~ collection of open intervals, unions, (countable) countable intersection, all closed intervals.

Let $A \subseteq \mathbb{R}$

$$P(A) = \frac{1}{\pi} \int_A \frac{1}{1+x^2} \cdot dx.$$

since it is collection of intervals, so breakable at integra

$$P(\mathbb{R}) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{1+x^2}\cdot dx = 1$$

$P(E)$ has all positive terms

$\because$ absolutely convergent, so rearrangement possible.

ans. 3 (b) $\mathbb{R}$ is infinite interval.

$\therefore P(\mathbb{R}) = 0$ instead of $P(\mathbb{R}) = 1$

ans. 3 (c)

$$P(I) = \int_I \frac{1}{2}\cdot dx \quad , \quad I_1 \subseteq [1,\infty)$$

$$= 1 \quad , \text{ if } \quad I_2 \subset (-\infty, 1]$$

exr

$\because \quad (2,6) \in [1,\infty)$

$$P(I_{2-6}) = \boxed{\int_2^6 \frac{1}{2}\cdot dx = 2}$$ impossible.

4) $P\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{i=1}^{n} P(E_i)$

a)

ans:- $n=2$; $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$\therefore \quad P(E_1 \cup E_2) < P(E_1) + P(E_2)$

if $P(E_1 \cap E_2) \neq 0$

$P(E_1 \cup E_2) = P(E_1) + P(E_2)$

if $P(E_1 \cap E_2) = 0$

$n=3$; $P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3)$

$- P(E_1 \cap E_2) + P(E_1 \cap E_2 \cap E_3)$

Tough way :- $- P(E_2 \cap E_3)$

$- P(E_1 \cap E_3)$

Easy way:-

$P(E_1 \cup E_2 \cup E_3) \leq P(E_1 \cup E_2) + P(E_3)$

$\leq P(E_1) + P(E_2) + P(E_3)$

b

b) $P\left(\bigcap_{i=1}^{n} E_i\right) \geq \sum_{i=1}^{n} P(E_i) - (n-1)$

$n=2$ $\quad P(E_1 \cap E_2) = P(E_1) + P(E_2) - P(E_1 \cup E_2)$

$\geq P(E_1) + P(E_2) - (2-1)$

$\because \boxed{P(E_1 \cup E_2) \leq 1}$

$n=k$ Let

$P(E_1 \cap E_2 \cap E_3 \ldots E_k) \geq P(E_1) + P(E_2) + \ldots - (k-1)$

then for $n=k+1$

$P(E_1 \cap E_2 \cap E_3 \ldots E_{k+1}) = P\left((E_1 \cap E_2 \ldots E_k) \cap E_{k+1}\right)$

$= P(E_1 \cap E_2 \ldots E_k) + P(E_{k+1})$

$- P\left(E_{k+1} \cap (E_1 \cap E_2 \ldots E_k)\right)$

$\geq P(E_1) + P(E_2) + \ldots - (k-1) - 1$

$+ P(E_{k+1})$

$\geq P(E_1) + P(E_2) + \ldots + P(E_{k+1})$

$- (k+1-1)$

x——x——x

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(5) Independent Events

→ If E and F are independent

$\because P(E \cap F) = P(E) \times P(F)$ —— (1)

To prove E^c and F are independent

$$P(E^c \cap F) = P(F) - P(E \cap F)$$
$$= P(F) - P(E).P(F) \quad [\text{From (1)}]$$
$$= P(F)[1 - P(E)]$$
$$= P(F) \times P(E^c)$$
$$\therefore \boxed{P(E^c \cap F) = P(F) \times P(E^c)}$$

b) similar as a)

c)
$$P(E^c \cap F^c) = 1 - P[E \cup F]$$
$$= 1 - [P(E) + P(F) - P(E \cap F)]$$
$$= 1 - [P(E) + P(F) - P(E).P(F)]$$
$$= 1 - [P(E)[1 - P(F)] + P(F)]$$
$$= 1 - [P(E).P(F^c) + P(F)]$$
$$= 1 - P(E).P(F^c) - P(F)$$
$$= 1 - P(F) - P(E).P(F^c)$$
$$= P(F^c) - P(E).P(F^c)$$
$$= P(F^c)[1 - P(E)]$$
$$\boxed{P(E^c \cap F^c) = P(F^c).P(E^c)}$$

Tutorial 1)

Q.8] Probability $= \dfrac{\pi (2)^2}{\pi (10)^2} = \boxed{\dfrac{4}{100}}$

Q.9) $\dfrac{366}{7} = 52$ weeks ~~sundays~~ in a Leap year
& 2 days remaining.

2 days $= \{MT, TW, WTh, ThF, FS, S\,Sun, Sun\,M\}$
7 possibilities.

Favour :- $\{S\,Sun, Sun\,M\}$ 2

$\therefore$ Probability $= \boxed{\dfrac{2}{7}}$

Q.10) E :- both are boys

F :- one is boy.

~~$P(E \cap F) = P(boy) P(girl) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$~~

~~$P(F) = \frac{1}{2} \times 1 = \frac{1}{2}$~~

~~$P(E|F) = \dfrac{P(E \cap F)}{P(F)} = \dfrac{1/4}{1/2} = \dfrac{1}{2}$~~

Baye theorem.

$$\text{Prob} = \frac{P(\text{one is boy})}{\text{Total prob}} = \frac{\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\left(\frac{1}{2}\right)}$$

$$= \boxed{\frac{1}{3}}$$

Prob (Both boys) $= \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$.

Prob (1 boy) $= \frac{1}{2}\cdot 1 = \frac{1}{2}$.

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Q.11) non-favour ⇒ 18 {(1 2 3),(2,3,4) + --- (18,19,20)}

Total ⇒ $^{20}C_3 = \frac{20!}{17!\,3!}$

$= \frac{20 \times 19 \times 18}{6}$

$= \boxed{60 \times 19}$

Prob = $1 - \frac{\text{non-favour}}{\text{Total}}$

$= 1 - \frac{18}{60 \times 19}$

$\begin{array}{r} 18 \\ 190 \\ -3 \\ \hline 187 \end{array}$

$= 1 - \frac{3}{190}$

$= \boxed{\frac{187}{190}}$

—x—

Q.13)

a) greater than 5 :- 6, 7, 8

fav $^3C_2 \rightarrow 3$

Total ways :- $^8C_2$

Probability ⇒ $\frac{^3C_2}{^8C_2} = \frac{\frac{3!}{2!\,1!}}{\frac{8!}{2!\,6!}} = \frac{3 \times 2}{8 \times 7}$

$^8C_2 \Rightarrow 28$

$= \boxed{\frac{3}{28}}$

b) $P(\text{Sum} = 5) = \frac{2}{28} = \frac{1}{14}$

ans. 12) $P(A) = 0.4$, $P(B) = 0.7$.

a) Min $P(A \cap B) = 0.1$
Max $P(A \cap B) = 0.4$.

$P(A \cap B) = P(A) + P(B) - P(A \cup B)$

This will be min when $P(A \cup B)$ is max.

$\Rightarrow P(A \cup B)_{max} = 1$

$P(A \cap B)_{min} = 0.4 + 0.7 - 1$
$= \boxed{0.1}$.

Condition:- A & B are exhaustive

b) ~~A∩B~~ similarly $P(A \cap B)$ max when $P(A \cup B)$ min.

$(A \cap B) \subset A$
$(A \cap B) \subset B$.

$P(A \cap B) \leq P(A) = 0.4$

$P(A \cap B) \leq P(B) = 0.7$.

$\therefore P(A \cap B) \leq \min P(A)$ or $P(B)$

$\Rightarrow \boxed{P(A \cap B)_{max} = 0.4}$

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14) ~~ace of spade ⇒ 2×4 = 8~~ ace ⇒ 3 } except spade

even ⇒ 2, 4, 6, 8, 10 ⇒ 5

4 groups ⇒ $5 \times 3 = \boxed{15}$.

spade ⇒ 13.

∴ Total fav ⇒ 3 + 13 + 15 = 31

∴ $\boxed{\text{prob} = \frac{31}{52}}$

15) 2, 3, 4, 5, 6 ⇒ x Total students ⇒ $5x + 2x = 7x$.

1 ⇒ $2x$.

grade 3 ⇒ $\frac{x}{7x} = \boxed{\frac{1}{7}}$.

16) infected + positive ⇒ 99.99 %. ~~infect / neg ⇒ 0.01 %~~

not infect + positive ⇒ 0.02 %. ~~not inf / neg ⇒ 99.98 %~~

nice

generally 1 infected ⇒ $\frac{1}{10{,}000}$ ⇒ 0.0001 %

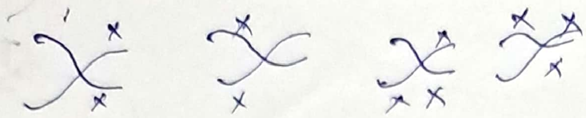
P(Random is positive) = ? ~~P(infect & positive) + P(not infect & positive)~~

~~P(inf & pos) + P(inf & neg) + P(not inf & pos) + P(not inf & neg)~~

P(inf)

~~$= \frac{\frac{99.99}{100} + \frac{0.02}{100}}{\frac{99.99}{100} + \frac{0.01}{100} + \frac{0.02}{100} + \frac{99.98}{100}}$~~

~~$= \frac{100.01}{2} = 50$~~



16) $I =$ people infected
$T =$ test +ve

$$P(T \mid I) = \frac{9999}{10,000}$$

$$P(T \mid I^c) = \frac{2}{10000}$$

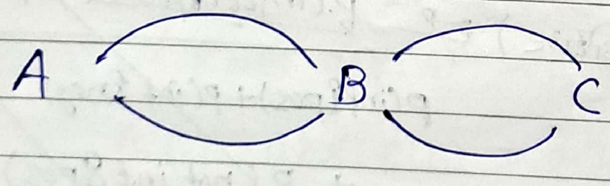
$$P(I) = \frac{1}{10000}$$

$$P(T) = P(T \cap I) + P(T \cap I^c)$$

$$= P(T \mid I) \cdot P(I) + P(T \mid I^c) \cdot P(I^c)$$

H.W solve yourself

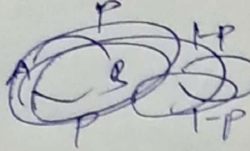
(18)



P(open road A to c) = ??

P (snow - block) = p
p (snow not block) = 1 − p

E = R_i is blocked.
$p(E) = p$
$p(E^c) = 1 - p$.



probability $= P(E_1^c \cup E_2^c) \cdot P(E_3^c \cup E_4^c)$

independent)

$= P(E_1 \cap E_2)^c \cdot P(E_3 \cap E_4)^c$

$= (1 - p \cdot p)(1 - p \cdot p)$ (∵ independent)

$= \boxed{(1-p^2)^2}$

19) Dearrangement of n

$$D_n = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^{n+2} \frac{1}{n!}$$

Given :- E_i = Event that ith letter entered in ith envelope.

E_i^c = ith Letter doesn't enter ith envelope.

$P(E_1^c \cap E_2^c \cap E_3^c \cdots \cap E_n^c)$

$= 1 - P(E_1 \cup E_2 \cup \cdots E_n)$

$$= 1 - \left\{ \sum_{i=1}^{n} P(E_i) - \sum_{i=1}^{^nC_2} P(E_i \cap E_j) + \sum_{i=1}^{^nC_3} P(E_i \cap E_j \cap E_k) \cdots + (-1) \sum_{i=1}^{^nC_k} P(E_i \cap E_j \cdots E_k) \right.$$

$$= 1 - \left\{ n \times \left[\frac{(n-1)!}{n!}\right] - {}^nC_2 \times \left[\frac{(n-2)!}{n!}\right] + {}^nC_3 \left[\frac{(n-3)!}{n!}\right] + \cdots \right.$$

$P(E_i) = P(\text{one goes into its envelope})$

$= \frac{1}{n}.$

$P(E_i \cap E_j) = P(\text{two goes into their envelope})$

$= P(\text{1 goes correctly}) \cdot P(\text{2 goes correctly} \mid \text{1 has gone correctly})$

$= \frac{1}{n} \times \frac{1}{n-1}$

Reason :- $P(E_1 \cap E_2) = P(E_1) \cdot P(E_2 \mid E_1)$

$= \frac{1}{n} \cdot \frac{1}{n-1}$

$\because P(E_1 \cap E_2 \cap E_3) = P(E_1) \cdot P(E_2 \mid E_1) \cdot P(E_3 \mid E_1 \cap E_2)$

$= \frac{1}{n} \times \frac{1}{n-1} \times \frac{1}{n-2}$

$\vdots$

$P(E_1 \cap E_2 \cdots \cap E_n) = \frac{1}{n!}$

$$\Rightarrow 1 - \left\{ n \times \frac{1}{n} - {}^{n}C_2 \times \frac{1}{n} \times \frac{1}{n-1} + \cdots \right\}$$

$$\Rightarrow 1 - \left\{ 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} - \cdots \right\}$$

$$= \left\{ \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} \cdots \cdots (-1)^{n+2} \left(\frac{1}{n!}\right) \right.$$

(20) R, W, B, Y, G

$P(Red) = \frac{1}{5}$

$P(W) = 2/5$

$P(B) = ?$ $P(Y) = ?$ $P(G) = ?$

$$P(G \cup Y \cup B) = 1 - P(R) - P(W)$$

$$= 1 - \frac{1}{5} - \frac{2}{5}$$

$$= \boxed{\frac{2}{5}}$$

(21) $n - (r+2)$

P_1 ___ P_2 ___

1 r 1 $n-(r+2)$

Total seatings P_1 & P_2 pack

P_1 ___ P_2

1 r 1

$\underbrace{P_1 \; \_\_\_ \; P_2}_{r}$ or $\underbrace{P_2 \; \_\_\_ \; P_1}_{r}$

O ___ n^{th}

r

$n-(r+1)$

Total $\Rightarrow$ $n!$

1 ___ 1

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P_1, P_2 interchange   /shifting places   people randomize

$$\text{Prob} \Rightarrow \frac{\text{favour}}{\text{Total}} = \frac{2 \times [n-(r+1)] \times [n-2)]!}{n!}$$

$$\frac{2 \times [n-(r+1))!}{n!}$$

$$= \frac{}{n \times (n-1) \times (n-2) \ldots \times (n-(r+2)) \times (n-(r+3)) \times \ldots 3 \times 2 \times 1}$$

$$= \boxed{\frac{2 \times (n-(r+1))}{n \times (n-1)}}$$

—X—

Charizard

Tutorial 2

Problem set 2

Random variable:-

$X : S \longrightarrow \mathbb{R}$ such that

$\forall B \in B_{\mathbb{R}}$

$X^{-1}(B) \in \Sigma$ = event space

also $X^{-1}\left((-\infty, x]\right) \in \Sigma$

Distribution function:-

$F_X : \mathbb{R} \to \mathbb{R}$ codomain $\mathbb{R}$, Range :- $[0,1]$

such that
- F_X is increasing
- Right continuous
- $F_X(-\infty) = 0$, $F_X(\infty) = 1$

Q.1] a) $F(x) = \begin{cases} 0 & \text{if } x<0 \\ 1-e^{-x} & \text{if } x \geq 0. \end{cases}$

i) increasing :- $x_1 < x_2 \Rightarrow F(x_1) \leq F(x_2)$

if $x_1, x_2 < 0$

$f(x_1) = f(x_2)$.

if $x_1, x_2 \geq 0$

$x_1 < x_2$

$-x_1 > -x_2$

$e^{-x_1} > e^{-x_2}$ (e is increasing)

$-e^{-x_1} < -e^{-x_2}$

$\boxed{1-e^{-x_1} < 1-e^{-x_2}}$

- Function is right continuous.
  (It is continuous everywhere).
  By checking continuity at 0

$\lim_{x \to 0^-} F(x) \Rightarrow 0$ Left continuous ✓

$\lim_{x \to 0^+} F(x) \; (1 - e^{-x}) = 1 - 1 = \boxed{0}$ Right continuous ✓

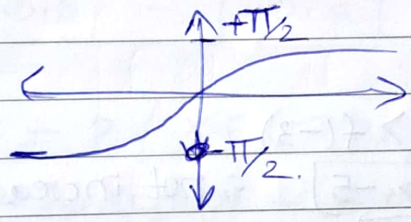
$F_X(-\infty) = \lim_{x \to -\infty} F(x) = 0$

$F_X(+\infty) = \lim_{x \to +\infty} F(x) = 1 - e^{-\infty} = 1 - 0 = \boxed{1}$

—— x ——

b) $F(x) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1} x, \quad -\infty < x < \infty.$

ans- $\tan^{-1} x$



i) Since $\tan^{-1} x$ is increasing function $\left(\because \frac{d(\tan^{-1} x)}{dx} = \frac{1}{1+x^2} > 0, \text{ so increasing.} \right)$

$\therefore$ $F(x)$ is increasing.

ii). $\tan^{-1} x$ is completely continuous
Hence $F(x)$ is also continuous.
($\because \tan^{-1} x$ is differentiable).

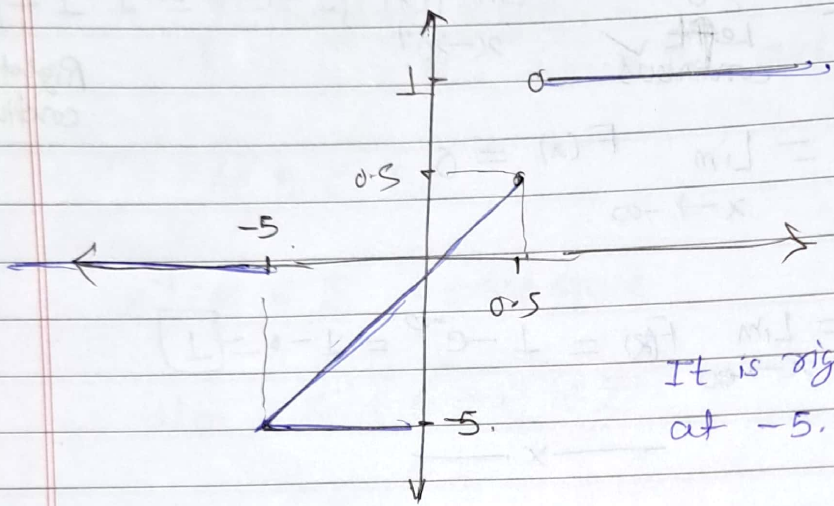
iii) $F_X(+\infty) = \lim_{x \to +\infty} F(x) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1}(+\infty)$

$= \frac{1}{2} + \frac{1}{\pi} \times \frac{\pi}{2}$

$= \frac{1}{2} + \frac{1}{2} = \boxed{1}.$

Similarly $\boxed{F_X(-\infty) = 0}$

c) $F(x) = \begin{cases} 0 & x < -5 \\ x & -5 \le x \le 0.5 \\ 1 & x > 0.5 \end{cases}$



It is right continuous at -5.

disprove :- not right-continuous at 0.5.

$-7 < -3$

but $f(-7) > f(-3)$

$\boxed{0 > -5}$ $\therefore$ not increasing

x ——— x ——— x

ans.2

$$F(x) = \begin{cases} 0 & \text{if } x<0 \\ \frac{2}{3} & 0\leq x<1 \\ \frac{7-6c}{6} & 1\leq x<2 \\ \frac{4c^2-9c+6}{4} & 2\leq x\leq 3. \\ 1 & x>3 \end{cases}$$

Revision

Note:- $\{X\leq x\}$ means $\{\omega\in S : X(\omega)\leq x\}$

$\{X\leq x\} = X^{-1}((-\infty, x])$

$P(a<X\leq b) = P(\{a<X\leq b\})$

$(a,b] = (-\infty,b] - (-\infty,a]$

$P(X\in(a,b]) = P(X\in(-\infty,b]) - P(X\in(-\infty,a])$

$= F_X(b) - F_X(a)$

X — discrete Random variable if $\exists\, E_X \subset \mathbb{R}$

$P(X=x)>0 \quad \forall\, x\in E_X.$

$\&\ P(X\in E_X) = \sum_{x\in E_X} P(X=x) = 1.$

p.m.f $f_X(x) = P(X=x) = F_X(x) - \underbrace{F_X(x^-)}_{\text{c.d.f}}$

$$= \begin{cases} P(X=x & \text{if } x\in E_X \\ 0 & \text{otherwise} \end{cases}$$

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$= P(X \le x)$

$$F_X(x) = \sum f_X(y)$$

$$y \in (-\infty, x] \cap E_X$$

$P(E_X^c) = 0$

$$P(X \in A) = P(X \in A \cap E_X) + P(X \in A \cap E_X^c)$$

$$= P(X \in A \cap E_X)$$

$$= \sum_{x \in A \cap E_x} P(X = x)$$

$$= \sum_{x \in A \cap E_x} f_X(x)$$

Q.2) continue :-

F(x) is right continuous —— (∵ CDF)

checking continuity at 1

$$\frac{2}{3} = \frac{7-6c}{6}$$

$$\frac{12}{3} = 7-6c$$

$$4 = 7-6c$$

$$6c = 3$$

$$\boxed{c = \frac{1}{2}}$$

common mistake

It is not right continuous at 1.

fn is also not increasing at $c = \frac{1}{2}$

since $F(x)$ is right continuous at 3.

$$\lim_{x \to 3^+} F(x) = \lim_{h \to 0} F(3+h) = 1.$$

now, $F(3) = \frac{4c^2 - 9c + 6}{4}$

$$1 = \frac{4c^2 - 9c + 6}{4}$$

$$4 = 4c^2 - 9c + 6$$

$$4c^2 - 9c + 2 = 0$$

$$4c^2 - 8c - c + 2 = 0$$

$$4c(c-2) - 1(c-2) = 0$$

$$(c-2)(4c-1) = 0$$

$$\boxed{c = 2} \quad \boxed{c = \frac{1}{4}}$$

if $c = 2$, f^n will not be creasing

if $\boxed{c = \frac{1}{4}}$, f^n is increasing

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 2/3 & 0 \le x < 1 \\ 11/12 & 1 \le x < 2 \\ 1 & 2 \le x \end{cases}$$

Yes, $f(x)$ is increasing

$\frac{14-3}{2} = \frac{11}{12}$. $\quad 4\left(\frac{1}{16}\right) \quad \frac{1}{4} - \frac{9}{4} + 6 = \frac{-8}{4} + 6 = \frac{-4}{4} = 1$

$7 - \frac{8}{4} \, 3 \, 2$

c.d.f $F(x)$ • is not continuous everywhere
• is Right continuous everywhere
• continuous at E_x^c
• discontinuous at E_x.

$$P(X=0) = F_X(0) - F_X(0^-)$$
$$= \frac{2}{3} - 0$$

$$\boxed{P(X=0) = \frac{2}{3}}$$

similarly $P(X=1) = F_X(1) - F_X(1^-)$
$$= \frac{11}{12} - \frac{2}{3}$$
$$= \frac{3}{12} = \boxed{\frac{1}{4}}.$$

$$P(X=2) = F_X(2) - F_X(2^-)$$
$$= 1 - \frac{11}{12}$$
$$= \boxed{\frac{1}{12}}.$$

$$\sum_{x \in (0,1,2)} P(X=x) = \frac{2}{3} + \frac{1}{4} + \frac{1}{12} = \boxed{1}$$

$\underset{x \in (0,1,2)}{P(X=x)} > 0$ Hence $E_x = \{0\}, \{1\}, \{2\}$

ii) pmf is function from $\mathbb{R} \to \mathbb{R}$

$$f_X(x) = p(x) = \begin{cases} \frac{2}{3} & x=0 \\ \frac{1}{4} & x=1 \\ \frac{1}{12} & x=2 \\ 0 & \text{otherwise} \end{cases}$$

use small p for p.m.f.

or small f

c) $P(1 < X < 2) = P(X<2) - P(X \leq 1)$

$= \cancel{0}\, F_X(2^-) - F_X(1)$

$= \frac{11}{12} - \frac{11}{12}$

$= \boxed{0}.$

$P(2 \leq X < 3) = P(X<3) - P(X<2)$

$= F_X(3^-) - F_X(2^-)$

$= 1 - \frac{11}{12}$

$= \boxed{\frac{1}{12}}.$

$P(0 < X \leq 1) = P(X \leq 1) - P(X \leq 0)$

$= F_X(1) - F_X(0)$

$= \frac{11}{12} - \frac{2}{3}$

$= \frac{3}{12} = \boxed{\frac{1}{4}}.$

$$P(1 \leq X \leq 2) = \boxed{\frac{1}{3}}$$

$$P(X \geq 3) = 1 - 1 = 0.$$

$$P(X = 2.5) = P(X = 2.5) - P(X = 2.5^-)$$
$$= 1 - 1$$
$$= \boxed{0}$$

ans. d) $P(\{X = 1\} \mid \{1 \leq X \leq 2\})$

$$= P(\{A\} \mid \{B\})$$

$$= \frac{P(A \cap B)}{P(B)}$$

$$P(1 \leq X \leq 2)$$
$$= P(X=2) + P(X=1) + \underbrace{P(1 < X < 2)}_{0}$$

$$A \cap B = (X=1) \cap (1 \leq X \leq 2)$$
$$= \boxed{(X=1)}$$

$$B = \{1 \leq X \leq 2\}$$

Homework

$\frac{a}{r}, a, ar^2$

$^nP_r = \frac{n!}{(n-r)!}$ $\quad ^nC_r = \frac{n!}{(n-r)!\,r!}$

$n=3$
$r=1$

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~~Ta~~ $\boxed{PS-1}$

y

$$\frac{2 \times (0)! \times \cancel{(0)}!\ 01! \times {}^{3-2}C_1}{3!} = \frac{1}{3}$$

ans. 22

Exam Question 1–100

$\boxed{\begin{matrix} Q.22 \\ \& \\ Q.28 \end{matrix}}$

(1,2,3), (2,3,4) ——— (48,49,50) ⇒ 48

(1,3,5), (5,7,9) ——— (46,48,50) ⇒ 46

⋮

(1,25,49) (2,26,50) ⇒ 2

$\therefore$ summation ⇒ $2+4+ \cdots + 46+48$

$= 2\,[1+2+ \cdots +24]$

$= 2 \times \frac{24 \times 25}{2}$

$= \boxed{600}$

$\therefore$ Probability $= \frac{600 \text{ A.P ways}}{\text{total ways of selecting } 3} = \boxed{\frac{600}{^{50}C_3}}$

ans 23 If $r>1$ ⇒ Yes G.P

If $r=1$, a,a,a ⇒ (No G.P.) ⇒ hence Rejected.

If $r<1$ ⇒ Reverse order G.P.

Case 1) $r>1$

$a_{min} = 1$

$\therefore ar^2 < 50$

ie $r^2 < 50$

$\therefore \boxed{r \leq 7}$

$\therefore r = \{2,3,4,5,6,7\}$

If $r=2$,

$1 \leq a \leq 12$] $\therefore$ 12 triplets

$ar^2 < 50$

$a < \frac{50}{r^2} = \frac{50}{4} = 12.5$

If $r=3$ $a<\frac{50}{r^2}$, $\frac{50}{9}\Rightarrow 5$ triplets

If $r=4$, $a<\frac{50}{16}\Rightarrow$ 3 triplets

If $r=5$, $a<\frac{50}{25}\Rightarrow 2$

If $r=6$, $a<\frac{50}{36}\Rightarrow 1$

If $r=7\Rightarrow$ $a<\frac{50}{49}\Rightarrow$ 1 triplet

$\therefore$ Total numbers $\Rightarrow 12+5+3+2+1+1=\boxed{24}$

<u>case 2)</u> $r\leq 1$

$r=\frac{m}{n}$, $(m,n)=1$ $\because$ coprime m & n.

$ar^2=\frac{am^2}{n^2}$ is an integer

$\Rightarrow\therefore$ a must be divided by n^2.

ie $n^2|a$

Now, $ar^2\leq 50$

$a\leq\left[\frac{50n^2}{m^2}\right]$.

(m,n)	$\left[1, \left[\frac{50n^2}{m^2}\right]\right)$	
$(3,2)$	$[1,22]$	4, 8, 12, 16, 20.
$(5,2)$	$[1,8]$	4, 8.
$(7,2)$	$[1,4]$	4
$(4,3)$	$[1,28]$	9, 18, 27.
$(5,3)$	$[1,18]$	9, 18.
$(7,3)$	$[1,9]$	9
$(5,4)$	$[1,32]$	16, 32.
$[7,4]$	$[1,16]$	16
$[16,5]$	$[1,34]$	25.
$[7,5]$	$[1,25]$	25.
$[7,6]$	$[1,36]$	1

Total favourable cases

$= 24 + 5 + 2 + 1 + 3 + 2 + 1 + 2 + 1 + 1 + 1 + 1$

$= 44$

$\therefore$ answer $= \boxed{\dfrac{44}{^{50}C_3}}$

ans. 28)

Total favourable cases $\Rightarrow$ Take 3 defective fuses. $\Rightarrow {}^5C_3$.

Total cases $\Rightarrow {}^{20}C_3$

ans:- $\frac{{}^5C_3}{{}^{20}C_3}$

Reason:- Taking balls without replacement = Taking balls together

ans. 30. answer is $\frac{r}{100}$. Let red balls be 'r'.

a) First ball drawn will be red.

$\rightarrow \frac{r}{100} \times \frac{r-1}{99} + \frac{100-r}{100} \times \frac{r}{99}$

$= \boxed{\frac{r}{100}}$

b). ~~5th~~ 3rd ball will be red.

$\Rightarrow \frac{r}{100} \times \frac{r-1}{99} + \frac{r-2}{98}$

$+ \frac{r}{100} \times \frac{100-r}{99} \times \frac{r-1}{98}$

$+ \frac{100-r}{100} \times \frac{99-r}{100} \times \frac{r}{98}$

$+ \frac{100-r}{100} \times \frac{r}{99} \times \frac{r-1}{98}$

$\Rightarrow \boxed{\frac{r}{100}}$

Hence 50^{th} ball will also be red.

(c) Probability that Last ball is red $= \boxed{\frac{1}{100}}$.

Remember this important result

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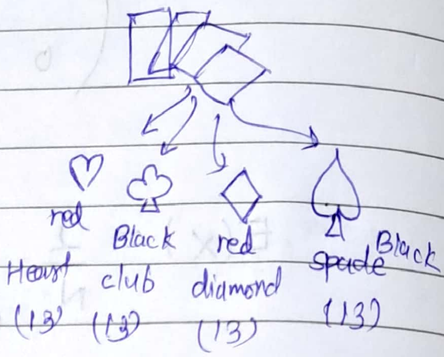
Problem set 2)

ans.3) $52 \longrightarrow 5$ at random without replacement

ways of choosing (sample space) $= {}^{52}C_5 = \binom{52}{5}$



Random variable

$X =$ No. of Hearts

$\therefore X(\omega) = \{0, 1, 2, 3, 4, 5\}$

Ex (support) $= \{0, 1, 2, 3, 4, 5\}$

a) pmf of X.

$$P(X=0) = \frac{\binom{13}{0} \cdot \binom{39}{5}}{\binom{52}{5}}$$

$$P(X=1) = \frac{\binom{13}{1}\binom{39}{4}}{\binom{52}{5}}$$

$$P(X=5) = \frac{\binom{13}{5}\binom{39}{0}}{\binom{52}{5}}$$

$$\therefore \quad P(X=x) = \frac{\binom{13}{x}\binom{39}{5-x}}{\binom{52}{5}}$$

So, pmf will be

$$p(x) = \begin{cases} \dfrac{\binom{13}{x}\binom{39}{5-x}}{\binom{52}{5}} & \text{when } x \in \{0,1,2,3,4,5\} \\ 0 & \text{otherwise} \end{cases}$$

(b) $\quad p(X \leq 1) = p(X=0) + p(X=1)$

[Q.6 is same as Q.3]

Q.4 X – RV with pmf.

$$p(x) = \begin{cases} \dfrac{c}{(2x-1)(2x+1)} & , \text{ if } x \in \{1,2,3,\ldots\} \\ 0 & , \text{ otherwise} \end{cases}$$

c is a constant.

ans:

a) $$\sum_{x \in E_X} p(x) = 1.$$

$$\Rightarrow \sum_{x=1}^{\infty} \frac{c}{(2x-1)(2x+1)} = 1.$$

$$c \sum_{x=1}^{\infty} \frac{1}{(2x-1)(2x+1)} = 1$$

$$\frac{c}{2} \sum_{x=1}^{\infty} \left(\frac{1}{2x-1} - \frac{1}{2x+1} \right) = 1$$

$$\frac{c}{2} \left(\frac{1}{1} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \cdots + \cancel{\frac{1}{2x-1}} - \frac{1}{2x+1} \right) = 1$$

$$\frac{c}{2} \left(1 - \frac{1}{2x+1} \right) = 1$$

$$\boxed{\because \lim_{x \to \infty} \left(\frac{1}{2x+1} \right) = 0}$$

$$\therefore \frac{c}{2}(1-0) = 1$$

$$\boxed{c = 2}$$

b) c.d.f of X.

$$p(x) = \begin{cases} \dfrac{2}{(2x-1)(2x+1)} & x \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

For c.df, find $P(X \leq x)$, $x \in \mathbb{R}$

$$\boxed{F_X(x) = P(X \leq x), \quad x \in \mathbb{R}}$$

$2 \quad \frac{4}{5}$ ———— 13. $\frac{2}{3}+\frac{2}{15}=\frac{12}{15}$

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$$F_X(x) = \begin{cases} 0 & , \text{if } x<1 \\ \frac{2}{1\cdot 3} & , \text{if } x=1 \\ \frac{2}{1\cdot 3} & , 1<x<2 \\ \frac{2}{1\cdot 3}+\frac{2}{3\cdot 5} & , 2\leq x<3 \\ \frac{2}{1\cdot 3}+\frac{2}{3\cdot 5}+\frac{2}{5\cdot 7} & , 3\leq x<4 \end{cases}$$

$$F_X(x) = \begin{cases} 0 & , \text{if } x<1 \\ \frac{2[x]}{2[x]+1} & , \text{if } x\geq 1. \end{cases}$$

———— x ————

$P(X=a) :\rightarrow \{\omega : X(\omega)=a\}$

(c) $m, n \in \mathbb{N}, \quad m < n$

$P(X < m+1) = ?$

ans:-

$P(X < m+1)$

$= P(X \leq m+1) - P(X = m+1)$

$= \underbrace{F_X(m+1)}_{\text{c.df}} - \underbrace{p(m+1)}_{\text{pmf}}$

$= \dfrac{2[m+1]}{2[m+1]+1} - \dfrac{2}{(2(m+1)-1)(2(m+1)+1)}$

$= \dfrac{2}{2(m+1)+1}\left[(m+1) - \dfrac{1}{2(m+1)-1}\right]$

$= \dfrac{2}{2(m+1)+1}\left[\dfrac{(m+1)(2m+1)-1}{2m+1}\right]$

H.W

(d) $P\left(\underbrace{(X>1)}_{A} \mid \underbrace{(1 \leq X < 4)}_{B}\right)$

$= \dfrac{P(A \cap B)}{P(B)}$

$= \dfrac{P(1 < X < 4)}{P(1 \leq X < 4)} = \dfrac{P(X \leq 4) - P(X=4) - P(X \leq 1)}{P(X \leq 4) - P(X=4) - P(X < 1)}$

(5) X — RV, c.d.f

$$F(x) = \begin{cases} 0 & \text{if } x<0 \\ \frac{x^2}{2} & \text{if } 0 \leq x < 1 \\ \frac{x}{2} & \text{if } 1 \leq x < 2 \\ 1 & \text{if } x \geq 2 \end{cases}$$

a) X is of continuous type and then pdf of x??

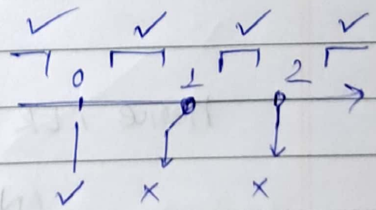
ans:- X is C.T.R.V if cdf is differentiable at critical points. {i.e 0, 1, and 2} → where definition of function changes

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \text{RHL}$$

$$\lim_{h \to 0} \frac{f(a) - f(a-h)}{h} \quad \text{LHL}$$

RHL & LHL should be same.

Observation :- differentiability



Finite no. of non-differentiable points, hence it is continuous Type R.V. (piece-wise continuous)

$$F'(x) = \begin{cases} 0 & , x < 0 \\ x & , 0 \leq x < 1 \\ \frac{1}{2} & , 1 < x < 2 \\ 0 & , x > 2 \end{cases}$$

(Note:- not-differentiable at 1 & 2)

$$= \begin{cases} x & , 0 \leq x < 1 \\ \frac{1}{2} & , 1 < x < 2 \\ 0 & , x \notin \{1, 2\} \\ \text{Any arbitrary constant} & , x \in \{1, 2\} \end{cases}$$

Hence, Ek R.V का pdf unique हो ज़रूरी नहीं
(Not needed that a R.V, should have unique p.d.f)

ans. (b) $P(1 < X < 2)$
$= P(X \leq 2) - P(X = 2) - P(X \leq 1)$
$= P(X < 2) - P(X \leq 1) = F_X(2^-) - F_X(1)$

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Q.6] HW.

Q.7] X - R.V.

$$f(x) = \begin{cases} c\,(4x - 2x^2) & , \text{ if } 0 < x < 2 \\ 0 & \text{otherwise.} \end{cases}$$

a) Find c?

ans:- use properties

$$\int_{-\infty}^{\infty} f(t).dt = 1$$

$$\Rightarrow \int_0^2 c\,(4x - 2x^2).dx = 1.$$

$$c \left[\frac{4x^2}{2} - \frac{2x^3}{3} \right]_0^2 = 1$$

$$c \left[\left(\frac{4 \times 4}{2} - \frac{2 \times 8}{3} \right) - (0) \right] = 1$$

$$c \left(\frac{16}{2} - \frac{16}{3} \right) = 1$$

$$c \left(\frac{3-2}{6} \right) = \frac{1}{16}$$

$$\boxed{c = \frac{6}{16} = \frac{3}{8}}$$

b) cdf of X

$$F_X(x) = P\{X \leq x\}$$

$$= \int_{-\infty}^{x} f(t) \cdot dt$$

$$= \begin{cases} 0 & \text{if } x < 0 \\ \int_0^x \frac{3}{8} \cdot (4t - 2t^2) \cdot dt & \text{if } 0 \leq x < 2 \\ 1 & \text{if } x \geq 2 \end{cases}$$

<u>Question type covered till now:-</u>

$$X \begin{cases} \rightarrow \text{discrete} \longrightarrow \text{cdf} \rightleftarrows \text{pmf} \\ \rightarrow \text{continuous} \longrightarrow \text{cdf} \rightleftarrows \text{pdf} \end{cases}$$

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(8) $P(X=r) = \binom{n}{r} p^r (1-p)^{n-r}$ $\Rightarrow \boxed{P(X\neq r) = 0}$

$r = 0, 1, 2, \ldots n$ $0 \le p \le 1.$

Find pmf

a) $Y = aX + b$ b) $Y = X^2$

continuous continuous

Y are R.V.

ans:- $P(Y=y)$

$P(aX+b = y) = ?$

$Y(\omega) = b \quad \forall \omega$

Note:- $P(X=-2) = 0$
$P(X=-7) = 0$
$\because P(X = \text{negative})$ not defined

case I :- when $a = 0$

$$P(Y=y) = \begin{cases} 1, & y = b \\ 0, & y \neq b \end{cases}$$

case II :- when $a \neq 0$.

$$P(Y=y) = P(aX+b=y)$$
$$= P\left(X = \frac{y-b}{a}\right)$$

We already have $P(X=r)$, use the definition.

$$P\left(X = \frac{y-b}{a}\right) = \begin{cases} \binom{n}{\left(\frac{y-b}{a}\right)} p^{\left(\frac{y-b}{a}\right)} (1-p)^{n-\left(\frac{y-b}{a}\right)} ; \\ \quad \frac{y-b}{a} = 0, 1, 2, \ldots n. \\ \therefore y = b, a+b, 2a+b, \ldots na+b \end{cases}$$

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$$\Big\}\ 0 \quad ; \text{ otherwise}$$

$\therefore$ PMF of $Y = aX + b$.

$$P(Y=y) = P\left(X = \frac{y-b}{a}\right)$$

$$= \begin{cases} \binom{n}{\left(\frac{y-b}{a}\right)} p^{\left(\frac{y-b}{a}\right)} (1-p)^{n-\left(\frac{y-b}{a}\right)} ; & \text{where } y = b,\ a+b,\ 2a+b \cdots na+b \\ 0 ; & \text{otherwise.} \end{cases}$$

b) $P(Y=X^2) = P(X=\sqrt{Y}) = P(X=\sqrt{y})$

$$\therefore \quad = \begin{cases} \binom{n}{\sqrt{y}} p^{(\sqrt{y})} (1-p)^{n-\sqrt{y}} ; & \\ \text{where } \sqrt{y} = 0,1,2 \cdots n & \\ \therefore\ y = 0,1,4,9, \cdots n^2. & \\ 0 ; & \text{otherwise} \end{cases}$$

$$P(Y=X^2) = P(X=\pm\sqrt{Y}) = P(X=\pm\sqrt{y}) = P(X=\sqrt{y}) + \overbrace{P(X=-\sqrt{y})}^{\text{value is zero}}$$

$$= P(X=\sqrt{y}) \text{ only}$$

$$= \begin{cases} \binom{n}{\sqrt{y}} p^{\sqrt{y}} (1-p)^{n-\sqrt{y}} ; & \sqrt{y} = 0,1,2, \cdots n \\ & y = 0,1,4,9, \cdots n^2 \\ 0 ; & \text{otherwise} \end{cases}$$

(9) X has p.d.f $f(x) = \begin{cases} c \cdot e^{-x} ; & 0 < x < \infty \\ 0 ; & \text{elsewhere} \end{cases}$

continuous.

Find c & $P(1 < X \leq 2) = ?$

ans:-

$\cancel{P(X = S) = 1}$

ie $\cancel{\int_{-\infty}^{+\infty} c \cdot e^{-x} = 1}$

$\int_{-\infty}^{+\infty} f(x) = 1$

$\int_{0}^{\infty} c \cdot e^{-x} = 1$

$-c \left[e^{-x} \right]_{0}^{\infty} = 1$

$-c \left[\frac{1}{e^{\infty}} - \frac{1}{e^{0}} \right] = 1$

$\boxed{c = 1}$

$\cancel{-c \left[e^{-\infty} - e^{0} \right] = 1}$

$P(1 < X \leq 2) = P(X \leq 2) - P(X \leq 1)$

$= F_X(2) - F_X(1)$

$= \int_{-\infty}^{2} f(x) \cdot dx - \int_{-\infty}^{1} f(x) \cdot dx = \int_{1}^{2} f(x) \cdot dx$

$= \boxed{e^{-1} - e^{-2}}$

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ans.10) Let Range $(X) = [0,3]$. $\rightarrow$ support of X. E_x.

pdf $f_X(x) = kx^2$

Let $Y = X^3$.

find

(a) Find k and c.d.f of X??

ans-

$$f(x) = \begin{cases} kx^2 & 0 \leq x \leq 3 \\ 0 & \text{else.} \end{cases}$$

Now $$\int_{-\infty}^{+\infty} f(x) = \int_0^3 kx^2 = 1.$$

$$\boxed{k = \frac{1}{9}}$$

$$F_X(x) = \text{cdf of } X = \begin{cases} P(X \leq x), & x \in E_x \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \int_{-\infty}^{x} f_X(t)\,dt, & x \in E_x \\ 0, & \text{otherwise.} \end{cases}$$

cdf

$$= \begin{cases} P(X < 0) = 0 ; & \underline{x < 0} \\ P(0 \leq X \leq 3) = \int_0^x \frac{x^2}{9} ; \\ P(X > 3) = 1 ; \end{cases}$$

(b) $E(Y) = \int_{-\infty}^{+\infty} y \cdot f_Y(y) \cdot dx.$

expectation.

$E(h(x)) = \int_{-\infty}^{+\infty} h(x) \cdot f_X(x) \cdot dx$

Roadmap:- pdf of X

$\Downarrow$

cdf of ~~pdf~~ $Y = X^3$ $\Rightarrow$ pdf of $Y = X^3$.

X is R.V

$h: R \longrightarrow R$, $h(x) = x^3$.

If h is strictly monotone ✓

+

differentiable ✓.

then

pdf of $h(x)$

$$= \begin{cases} f_X(h^{-1}(y)) \left| \frac{d(h^{-1}(y))}{dy} \right|, & y = h(x). \\ 0, & \text{otherwise} \end{cases}$$

$$E(h(x)) = \int_{-\infty}^{+\infty} h(x) \cdot f_X(x) \cdot dx.$$

$$= \int_0^3 x^3 \cdot f_X(x) \cdot dx.$$

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(c) $Var(Y) = E\left[(y - \mu_y)^2\right]$.

where $\mu_y = E(Y)$

now

$$Var(Y) = E\left[\underbrace{\left(x^3 - \frac{27}{2}\right)^2}_{h(x)}\right]$$

—— x ——

ans. 11) X — RV

$$p(x) = \begin{cases} 1/7 & x \in \{-2, -1, 0, 1\} \\ 3/14 & x \in \{2, 3\} \\ 0 & \text{otherwise} \end{cases}$$

a) Find mgf of x ??

ans. mgf of $x = E(e^{tx})$

$$E(e^{tx}) = \sum_{x \in E_x} e^{tx} \cdot p(x), \quad t \in \mathbb{R}$$

$$= \frac{1}{7}\left(e^{-2t} + e^{-t} + e^0 + e^t\right) + \frac{3}{14}\left(e^{2t} + e^{3t}\right)$$

(b) pmf of $Z = X^2$

$= P_Z(Z)$

$$= \begin{cases} P(Z = z), & z \in E_Z \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} P(X^2 = z), & z \in \{0,1,4,9\} \\ 0, & \text{otherwise} \end{cases}$$

$P(X^2 = 0) = P(X = 0) = \frac{1}{7}$

$P(X^2 = 1) = P(X = 1) + P(X = -1)$

$= \frac{1}{7} + \frac{1}{7}$

$= \boxed{\frac{2}{7}}$

$P(X^2 = 4) = P(X = +2) + P(X = -2)$

$= \frac{3}{14} + \frac{1}{7}$

$= \boxed{\frac{5}{14}}$

$P(X^2 = 9) = P(X = 3) + P(X = -3)$

$= \frac{3}{14} + 0 \hookrightarrow$ ($\because$ not given)

$\therefore$ pmf $Z = X^2 = \begin{cases} P(Z=0) = \frac{1}{7}, \\ P(Z=1) = \frac{2}{7} \\ P(Z=4) = \frac{5}{14} \\ P(Z=9) = \frac{3}{14} \end{cases}$

c.d.f $Z = X^2 = \begin{cases} 0 & Z < 0 \\ \left(\frac{1}{7}\right) & \text{if } 0 \leq Z < 1 \\ \frac{3}{7} & 1 \leq Z < 4 \\ \frac{11}{14} & 4 \leq Z < 9 \\ 1 & Z \geq 9. \end{cases}$

14) X – R.V with pdf

★★★ Exam

$f(x) = \begin{cases} 6x(1-x), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

Find cdf of $Z = X^2(3-2X)$
find its pdf.

ans:- We cannot find inverse of $X^2(3-2X)$

Reason :- $Y = 3X^2 - 2X^3$

$X = f(y)$??

$x(3-2x) \rightarrow -2x^2 - 2x^2$
$+ x^2(-2)$ $(-4x^2)$

Lets go through cdf of Z

$\Downarrow$

$$F_Z(z) = \begin{cases} P(Z \leq z) & , z \in E_Z \\ 0 & , \text{otherwise.} \end{cases}$$

$h(x) \in E_Z$

$h([0,1]) = [0,1]$

cdf of Z

$$\hookrightarrow \begin{cases} 0 & z < 0 \\ \displaystyle\int_{-\infty}^{h^{-1}(z)} h'(x) & 0 < z < 1 \end{cases}$$

$$= \begin{cases} 0 & ; \ z < 0 \\ \displaystyle\int_{-\infty}^{h^{-1}(z)} 6(x) \cdot (1-x) & ; \ 0 \leq h(x) < 1 \\ 1 & h(x) \geq 1 \end{cases}$$

continued on next page.

$Z = X^2 (3-2X) = h(X)$

c.d.f of Z

$$F_Z(z) = \begin{cases} P(Z \leq z), & z \in E_Z \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} P(h(X) \leq z), & \text{—"—} \\ 0, & \text{—"—} \end{cases}$$

$$= \begin{cases} P(X \in \frac{h^{-1}(z)}{x}), & \text{—"—} \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \int_{-\infty}^{h^{-1}(z)} f_X(x)\,dx & z \in E_Z \\ 0, & \text{otherwise} \end{cases}$$

$P(Z \leq z) = P(Z < 1) + P(Z \geq 1)$

$\Rightarrow P(Z < 1)$

$= \boxed{1}$

moment generating function.

ans:- [15] m.g.f $= \dfrac{e^{t} - e^{-2t}}{3t}, \quad t \neq 0.$

$$E(e^{tx}) = \frac{e^{t} - e^{-2t}}{3t}$$

$$\int_{x \in E_x} e^{tx} \cdot f_X(x) \cdot dx = \frac{e^{t} - e^{-2t}}{3t}$$

X ——— X ——— X

~~Replay~~ $M_X(t) = E(e^{xt})$.

continuous

discrete (not our question)

$$M_X(t) = \sum_x e^{xt} \cdot p(x)$$

$$M_X(t) = E(e^{xt})$$

$$= \int_{-\infty}^{+\infty} e^{xt} \cdot f_X(x) \cdot dx$$

ans:-

$$M_X(t) = \int_a^b e^{xt} \cdot f_X(x) \cdot dx = \frac{e^{t} - e^{-2t}}{3t}.$$

∴ It should be :-

$$\int_{-2}^{1} e^{xt} \cdot \underbrace{f_X(x)}_{\left(\frac{1}{3}\right)} \cdot dx = \boxed{\frac{e^{t} - e^{-2t}}{3t}}$$

so, pdf of X.

$$f_X(x) = \begin{cases} 1/3 & , -2 < x < 1 \\ 0 & , \text{otherwise.} \end{cases}$$

17) series is not convergend

18) integration convergent.

19) RV - X.

Mean - μ , Finite second moment.

Show:- $E((X-\mu)^2) \leq E((X-c)^2) \quad \forall c \in \mathbb{R}$

Soln:- $E((X-c)^2) = E(X^2 - 2cX + c^2)$

$= E(X^2) + c^2 - 2c\,E(X)$

let $h(c) = E(X^2) + c^2 - 2c\,E(X)$

To check nature $\quad h'(c) = 2c - 2E(X)$.

$= 2(c - \mu)$.

If $c > \mu$, then $\quad h'(c) > 0$

$c < \mu$, then. $\quad h'(c) < 0$.

So, h is increasing $c \in (\mu, \infty) \rightarrow \mu < c \Rightarrow h(\mu) \leq h(c)$

h is decreasing $c \in (-\infty, \mu) \rightarrow \mu > c \Rightarrow h(\mu) \leq h(c)$

$$h(\mu) \leq h(c) \quad \forall c \in \mathbb{R}$$

$$E\left((X-\mu)^2\right) \leq E\left((X-c)^2\right) \quad \forall c \in \mathbb{R}$$

Q2(b))

Question is:- $E(X) = \mu$
gives hint $E(X-\mu) = 0$.
ie $E(X) - \mu = 0$

$$E(X-\mu) = \int_{-\infty}^{+\infty} (x-\mu) f_X(x) \cdot dx.$$

$$= \cancel{\int_{-\infty}^{0} (x-\mu) f_X(x) \cdot dx} \quad \mu - x$$

$$+ \cancel{\int_{0}^{\infty} (x-\mu) \cdot f_X(x) \cdot dx} \quad \mu + x.$$

using change of variable

let's first replace x by t

ILLUMINATI

$$= \int_{-\infty}^{0} (t-\mu) f_X(t) \cdot dt + \int_{0}^{\infty} (t-\mu) \cdot f_X(t) \cdot dt$$

Now, Replace t by $(\mu - x)$ Replace t by $(\mu + x)$

$\therefore$ $dt =$

$$= \cancel{\int_{-\infty}^{0} (-x) . f_X(\mu - x) .}$$

$$= \int_{-\infty}^{\mu} (x-\mu)\, f_X(x) . dx + \int_{\mu}^{\infty} (x-\mu)\, f_X(x) . dx$$

$\mu - t \qquad \mu + t$

$$= \int_{+\infty}^{0} (-t) \cdot f_X(\mu - t) . dt + \int t \cdot f_X(\mu + t) . dt$$

— x —

Q.21 (

$$\boxed{\begin{array}{l} \text{Markov's Inequality} \\ X \text{ is R.V, neither discrete nor continuous} \\ \text{where } X \text{ takes non-negative values.} \\ \qquad X(\omega) \in [0, \infty) \\ \qquad P(X \geq 0) = 1 \\ \text{For all } t > 0 \\ \qquad P(X \geq t) \leq \dfrac{E(X)}{t} \end{array}}$$

ans. 21) Given $P(X \leq 0) = 0$, μ is finite

$$P(X \geq 2\mu) \leq 0.5$$

ans:- Replace t by (2μ) in Markov's Inequality.

$$P(X \geq 2\mu) \leq \frac{E(x)}{2\mu}$$

we know, $E(x) = \mu$.

$$\therefore \quad P(X \geq 2\mu) \leq \frac{\mu}{2\mu} = \boxed{0.5}.$$

but all this is true when $2\mu > 0 \Rightarrow \mu > 0$.

Let's prove:-

case I):- X discrete $(\mu > 0)$.

$$\mu = E(X) = \sum x\, p(x)$$

$$= \cancel{\sum_{x \leq 0} x.p(x)} + \underbrace{\sum_{x > 0} x.p(x)}_{\boxed{>0}}. \quad \therefore \mu \text{ finite}$$

case II):- X is continuous R.v

$$\mu = E(X) = \int_{-\infty}^{+\infty} x.f_X(x).dx. \quad \because \mu \text{ finite}$$

$$= \boxed{\int_{-\infty}^{0} x f_X(x).dx} + \underbrace{\int_{0}^{\infty} x.f_X(x).dx}_{\boxed{>0}}.$$

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ans-22) Chebyshev Inequality

If $\varepsilon > 0$

$$P\{|X-c| \geq \varepsilon\} \leq \frac{E((X-c)^2)}{\varepsilon^2}$$

⇓ same as

$$1 - P\{|X-c| < \varepsilon\} \leq \frac{E((X-c)^2)}{\varepsilon^2}$$

$$1 - P\{c-\varepsilon < X < c+\varepsilon\} \leq \frac{E((X-c)^2)}{\varepsilon^2}$$

$\therefore$ $c = 3$
$\varepsilon = 5.$

$E(X) = 3 \quad E(X^2) = 13$

$var(X) = E(X^2) - (E(X))^2 = 4$

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The Bernoulli Distribution

X	0	1
$P(X=x)$	$1/2$	$1/2$

There is single trial.

$P(\text{success}) = p$ $X=1$ is success.

$p(\text{failure}) = 1-p$ $X=0$ is failure

Then X has Bernoulli distribution:

$$P(X=x) = p^x (1-p)^{1-x}$$

Binomial Trials

There are n trials.

Each single Trial $\rightarrow$ failure, success

$P(S) = p$ (success)

$P(F) = 1-p$ (failure)

$$P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$$

X — # of success in n trials.

Q. 60 times Head, in 100 trials.

ans:- $P(X=60) = \binom{100}{60} \cdot \left(\frac{1}{2}\right)^{60} \left(\frac{1}{2}\right)^{40}$

$$= \boxed{\binom{100}{60}\left(\frac{1}{2}\right)^{100}}$$

Q] What is probability in 3 rolls of dice, that 5 comes 2 times?

ans] $\binom{3}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^1 = \cancel{3} \times \frac{1}{\cancel{36}_{12}} \times \frac{5}{6} = \boxed{\frac{5}{72}}$

Problem. set - 3

ans. 3)

$P(X \geq 1, \underbrace{n - X \geq 1}) \geq 0.95$

because atleast a girl also.

$P(1 \leq X \leq n-1) \geq 0.95$

$\Downarrow$ (complement it)

$1 - P(X=0) - P(X=n) \geq 0.95$

$1 - \binom{n}{0}\cdot\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^0 - \binom{n}{n}\cdot\left(\frac{1}{2}\right)^n\cdot\left(\frac{1}{2}\right)^0 \geq 0.95$

$1 - \left(\frac{1}{2}\right)^{n-1} \geq 0.95$

$1 - 0.95 \geq \left(\frac{1}{2}\right)^{n-1}$

$\frac{5}{100}$

$0.05 \geq \left(\frac{1}{2}\right)^{n-1}$

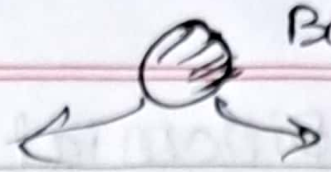
$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$

$\frac{\cancel{5}}{\cancel{100}_{20}} \geq \left(\frac{1}{2}\right)^{n-1}$

$\frac{1}{20} \geq \left(\frac{1}{2}\right)^{n-1}$

$\boxed{\frac{1}{20} \geq \left(\frac{1}{32}\right)}$ $\quad \therefore \left(\frac{1}{2}\right)^5 = \left(\frac{1}{2}\right)^{n-1}$

$\therefore \boxed{n=6}$



(6) 18 balls.

$\underbrace{B_1 \quad B_2 \quad B_3}_{p=\frac{3}{7}} \quad \underbrace{B_4 \quad B_5 \quad B_6 \quad B_7}_{1-p=\left(\frac{4}{7}\right)}$

Trials 18 ball

find probab B_1, B_2, B_3 has 6 balls in total.

$$P(x=6) = \boxed{\left({}^{18}C_6\right) \times \left(\frac{3}{7}\right)^6 \times \left(\frac{4}{7}\right)^{12}}$$

Problem Set–III

Q.1–8 , discrete probability distribution.

Q.1–3 done

Q.4 30 applicants, 20 qualified , 6 selected
find ~~atleast~~ P (atleast 2 qualified)

$P(X \geq 2)$

What is HyperGeometric dist??

ans:-

- Randomly sampling n objects without replacement from a source that contains 'a' success and 'N-a' failures.
- X represents no of success in sample.

$a = 20$, $N-a = 10$, $N = 30$, $n = 6$

$P(X \geq 2) = ?$

$$P(X=x) = \frac{\binom{a}{x}\binom{N-a}{n-x}}{\binom{N}{n}} \qquad x = \max\{0, n-(N-a)\} \text{ ... } \min\{a, n\}$$

We are selecting 6 people. (ie selecting 1-1 without replacement , i.e. not independent events)
Hence, cannot use Binomial dist.

Now, $P(X \geq 2) = 1 - P(X=0) - P(X=1)$

$$= 1 - \frac{\binom{20}{0}\binom{10}{6}}{\binom{30}{6}} - \frac{\binom{20}{1}\binom{10}{5}}{\binom{30}{6}}$$

Confusion Binomial
Hypergeometric.
Negative Binomial

$\frac{9}{N}$

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5) Each shot is independent of other shot
Binomial distribution can be used
but Trials = 2000 shots Large number
Also success probability = 0.002 (very rare event)

X = #(number of) events in a unit of time

$$P(X=x) = \frac{\lambda^x . e^{-\lambda}}{x!} \qquad \lambda = np$$
$$x = 0,1,2 \ldots$$

ans:- $\lambda = np = 2000 \times 0.002 = \boxed{4}$

$$P(X \geq 2) = 1 - P(X<2)$$
$$= 1 - P(X=0) - P(X=1)$$
$$= 1 - \frac{4^0 e^{-4}}{0!} - \frac{4^1 . e^{-4}}{1!}$$
$$= 1 - e^{-4} - 4e^{-4}$$
$$= \boxed{1 - 5e^{-4}}$$

6.7. Homework

ans. 8) A, B play until 1 team wins 5 games.
P(A) = 0.7 P(B) = 0.3.
P(8 games)

X Binomial : # no. of success in n trials
Negative Binomial : # no. of trials to get rth success.

Why Negative : The end match must be success of a team.
(which is sure, so we calculate for r-1 matches.)

A. $8-5=3$

4 match   1 match

End match is win

Negative Binomial

$$\boxed{P(X=x) = \binom{x-1}{r-1} p^{r-1} (1-p)^{(x-1)-(r-1)} \cdot p^{1}}$$

$$= \boxed{\binom{x-1}{r-1} p^{r} (1-p)^{(x-1)-(r-1)}}$$

Here $x=8$, $r=5$

$$P(X=8) = \binom{8-1}{5-1} (0.7)^{5} (0.3)^{(8-1)-(5-1)}$$

$$+ \binom{8-1}{5-1} (0.7)^{(8-1)-(5-1)} (0.3)^{5}$$

$$= \left(\underbrace{\binom{7}{4} (0.7)^{5} (0.3)^{3}}_{A\text{ win}} + \underbrace{\binom{7}{4} (0.7)^{3} (0.3)^{5}}_{B\text{ win}} \right)$$

$\frac{7\times5\times6\times4!}{4!\,3\times2}$

$$= \binom{7}{4} [0.16807)(0.027) + \binom{7}{4} (0.343)(0.00243).$$

$$= 35 [\, 0.00453 + 0.00083349 \,]$$

$$= \cancel{0.1588} \quad \boxed{0.188}$$

---

if someone uses Binomial $\Rightarrow$

which is wrong $\times$ $\quad \binom{8}{5} (0.7)^{5} (0.3)^{3} + \binom{8}{5} (0.7)^{3} (0.3)^{5}$

This number will be wrong

$\boxed{9}$ $X \sim Ge(p)$. $E\left(\frac{1}{2^X}\right)$

ans:-

$$p(X=x) = (1-p)^{x-1} \cdot \underbrace{p^1}_{1 \text{ success.}}$$

$$E\left(\frac{1}{2^x}\right) = \sum_{x=1}^{\infty} \frac{1}{2^x} \cdot (1-p)^{x-1} (p).$$

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ans;(10) The CF of random variable X is defined as:
↳ characteristic function

$$C_x(t) = E\left[e^{itX}\right] \qquad i=\sqrt{-1}$$

$$= \begin{cases} \sum_j e^{itx_j} \cdot p(x_j) & \text{discrete} \\ \int e^{itx} \cdot f(x)dx & \text{continuous} \end{cases}$$

(i) $X \sim Bin(n,p)$

$$\Rightarrow P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$x = 0,1,2,\dots n$

$$C_x(t) = \left\{ \sum_0^n e^{it(x_j)} \cdot p(x_j) \right.$$

$$= e^{it(0)} \cdot p(0) + e^{it(1)} \cdot p(1)$$

$$+ e^{it(2)} \cdot p(2) + \dots + e^{it(n)} \cdot p(n)$$

$$= \sum_{j=0}^n e^{it(x_j)} \cdot P(X = x_j)$$

$\Rightarrow$

$$= \sum_{j=0}^{n} e^{it(x_j)} \cdot \binom{n}{x_j} p^{x_j} \cdot (1-p)^{n-x_j}$$

$$= \sum \underbrace{(e^{it}\cdot p)}_{a}^{x_j} \binom{n}{x_j} \cdot \underbrace{(1-p)}_{b}^{n-x_j}$$

$$= (e^{it}\cdot p + 1 - p)^{n} \qquad \boxed{(a+b)^{n}}$$

(ii) $X \sim$ Poisson (λ).

$C_X(t) = ?$, $x = 0, 1, 2, \ldots$

$C_X(t) = /$

11)

$$K_x(t) = \ln\left(\phi_x(t)\right)$$

where ϕ_x is m.g.f of X.

ans:- show $K_x'(0) = E[x]$

$$K_x''(0) = Var[x]$$

$$\phi_x(t) = E[e^{tx}]$$

$$= E\left[1 + tx + \frac{t^2x^2}{2!} + \cdots\right]$$

$$= E(1) + t.E(x) + \frac{t^2}{2!}E(x^2) + \cdots\cdots$$

$$= 1 + \underbrace{\sum_{r=1}^{\infty} \frac{t^r}{r!} E(x^r)}_{\text{Let } \alpha}$$

$$K_x(t) = \ln\left(\phi_x(t)\right)$$

$$K_x(t) = \ln(1+\alpha).$$

$$= \alpha \underbrace{- \frac{\alpha^2}{2} + \frac{\alpha^3}{3} \cdots\cdots}_{\text{these terms can be ignored}}$$

$$K_X'(t)\Big|_{t=0} = E(X)$$

$$K_X''(t)\Big|_{t=0} = E(X^2) - E(X)^2 = var(X)$$

Q.12 Refer to IIT Kanpur solutions.

Q.13) $X \sim U(0,1)$
$Y = -\ln(1-X)$
Find CDF and PDF of X.

Soln:- CDF of Y:-

$$F_Y(y) = P(Y \leq y)$$
$$= P(-\ln(1-X) \leq y)$$
$$= P(\ln(1-X) \geq -y)$$
$$= P(1-X \geq e^{-y})$$
$$= P(-X \geq e^{-y}-1)$$
$$= P(X \leq 1-e^{-y})$$
$$= \int_{-\infty}^{1-e^{-y}} \underbrace{f_X}_{1} \, dx$$
$$= \int_{0}^{1-e^{-y}} 1 \cdot dx$$

$$\boxed{F_Y(y) = \begin{cases} (1-e^{-y}) & ; 0 \leq y < \infty \\ 0 & ; \text{otherwise} \end{cases}}$$

$$f_Y(y) = \begin{cases} e^{-y} & ; \quad 0 < y < \infty \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Q.14) ~~[illegible]~~ 15–20 min in sunny day prob = 2/3
Alvin 20–25 min rainy day prob = 1/3
All times being equally likes.
What is probability

ans :-

In short, this means pdf is constant in each interval. [15, 20] & [20, 25].
(pdf will be different due to different events but they will be constant.)

Let pdf [15, 20] → c_1
pdf [20, 25] → c_2

$$\text{p.d.f of } X = f_X(x) = \begin{cases} c_1 & ; \quad 15 < x < 20 \\ c_2 & ; \quad 20 < x < 25 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

CDF of X :- $P(X \leq x)$

$$F_X(x) = P(\text{sunny day})$$

$$= \int_{15}^{20} c_1 \cdot dx = \frac{2}{3}$$

$$c_1 \cdot 5 = \frac{2}{3}$$

$$\boxed{c_1 = \frac{2}{15}}$$

$$\int_{20}^{25} C_2 \cdot dx = \frac{1}{3} \Rightarrow \boxed{C_2 = \frac{1}{15}}$$

H.W 15-20

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Q.12 Let $X \sim U(0,k)$ where k is +ve integer.

and $Y = X - [x]$

show :- $Y \sim U(0,1)$.

ans :-

$$f_X(x) = \begin{cases} \frac{1}{k-0} & ; \quad x \in (0,k) \\ 0 & ; \quad \text{otherwise} \end{cases}$$

$$F_Y(y) = P(Y \leq y)$$

$$= P(x - [x] \leq y)$$

$$= \sum_{i=0}^{k-1} P(x - i \leq y) \quad x \in [i, i+1)$$

$$= \sum_{i=0}^{1} P(X \leq y + i)$$

(or)

$$= \sum_{j=1}^{k} P(X - [x] \leq y ; j-1 \leq X < j)$$

$$= \sum_{j=1}^{k} P(x - (j-1) \leq y , j-1 \leq X < j)$$

$$= \sum P(X \leq (j-1) , j-1 \leq X < j)$$

$$= \sum_{j=1}^{k} P(j-1 \leq X < \min\{j-1+y , j\})$$

$$= \begin{cases} \sum_{j=1}^{k} P(j-1 \leq X < (j-1+y)) \end{cases}$$

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CDF

$$= \begin{cases} 0 & y \leq 0 \\ \sum_{j=1}^{k} P(j-1 \leq X < (j-1+y) & 1 < y < 0 \\ 1 & \text{otherwise} \end{cases}$$

PDF of Y_1 $= \begin{cases} 1 & 0 < y < 1 \\ 0 & \text{interval} \end{cases}$

PS | Tutorial-3

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Saathi

Q16 $X \sim N(\mu, \sigma^2)$, $\mu \in \mathbb{R}$, $\sigma > 0$

$Y = aX + b$, $a \in \mathbb{R} - \{0\}$
$b \in \mathbb{R}$

Show $Y \sim N(a\mu + b, a^2\sigma^2)$.

ans:-

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \times e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}} \quad ; -\infty < x < \infty.$$

$$h(x) = ax + b.$$

$$h^{-1}(y) = \frac{Y-b}{a}, \quad a \neq 0$$

$$f_Y(y) = f(h^{-1}(Y)) \cdot \left| \frac{d}{dY}(h^{-1}(y)) \right|$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \times e^{-\frac{1}{2}\frac{\left(\left(\frac{Y-b}{a}\right)-\mu\right)^2}{\sigma^2}} \times \left[\frac{1}{a}\right]$$

$$= \frac{e^{-\frac{1}{2}\frac{\left(\left(\frac{Y-b}{a}\right)-\mu\right)^2}{\sigma^2}}}{a\sigma\sqrt{2\pi}}$$

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Q 18 Diameter of certain car component.

$$X \sim N\left(\underset{\mu}{10}, \underset{\sigma^2}{32}\right)$$

Standard Normal
$Y \sim N(0,1)$

$P(X > 13.4) = ?$

ans:- $\Phi = 1.13$ ——— ?

If $X \sim N(\mu, \sigma^2)$

If $\frac{X-\mu}{\sigma} \sim N(0,1)$

$\sigma > 0.$

$$P(X > 13.4) = 1 - P(X \leq 13.4)$$
$$= 1 - P\left(\frac{X-\mu}{\sigma} < \frac{13.4-\mu}{\sigma}\right)$$
$$= 1 - P\left(\frac{X-10}{3} \leq \frac{3.4}{3}\right)$$

Q.19 $X \sim Exp(\lambda)$.

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & , x > 0, \\ 0 & ; x \leq 0. \end{cases}$$

ans:- $P(X > 5000) = 1 - F_X(5000) = 1 - \left(1 - e^{\frac{-1}{10,000} \times 5000}\right)$

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & ; x > 0 \\ 0 & ; \text{else.} \end{cases}$$

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Ex:- Let $X = (X, Y, Z)$ be a random vector with joint p.m.f

$$f(x,y,z) = \begin{cases} \dfrac{xyz}{72} & , \text{ if } (x,y,z) \in \{1,2\} \times \{1,2,3\} \times \{1,3\} \\ 0 & , \text{ otherwise.} \end{cases}$$

(i) Find the conditional p.m.f of X given $(Y,Z) = (2,1)$.

(ii) Find the conditional p.m.f of (X,Z) given $Y = 3$.

ans:-

(1) $$f_{X|(Y,Z)}(x|(2,1)) = \frac{f(x,2,1)}{f_{Y,Z}(2,1)}$$

$$R_{(2,1)} = \{x \in \mathbb{R},\ (x,2,1) \in E_X\}$$

$$= \{1,2\}$$

$$f_{Y,Z}(2,1) = \sum_{x \in R_{(2,1)}} f(x,2,1).$$

$$= f(1,2,1) + f(2,2,1)$$

$$= \frac{2}{72} + \frac{4}{72} = \frac{6}{72} = \boxed{\frac{1}{12}}.$$

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$$f_X(2,1) = \begin{cases} \frac{x}{3} & \text{if } x \in \{1,2\} \\ 0 & \text{otherwise.} \end{cases}$$

(ii) $$f_{(X,Z)|Y}((x,z)|3) = \frac{f(x,z,3)}{f_Y(3)}$$

$$R_3 = \{(x,z) \in \mathbb{R}^2 ; (x,3,z) \in E_X\}$$

$$= \{1,2\} \times \{1,3\}$$

$$f_Y(z) = \sum_{(x,z) \in R_3} f(x,3,z) = f(1,3,1) + f(1,3,3) + f(2,3,1) + f(2,3,3)$$

$$= \frac{1}{2}.$$

$$\therefore f_{(X,Z)|Y}((x,z)|3) = \begin{cases} \frac{\frac{3xz}{72}}{1/2} = \frac{xz}{12} & \text{if } (x,z) \in \{1,2\} \times \{1,3\} \\ 0 & \text{, otherwise.} \end{cases}$$

13 october

Tutorial

("Problem set - IV")

Refer Lec-13

Thm 10

Thm 11

Saathi

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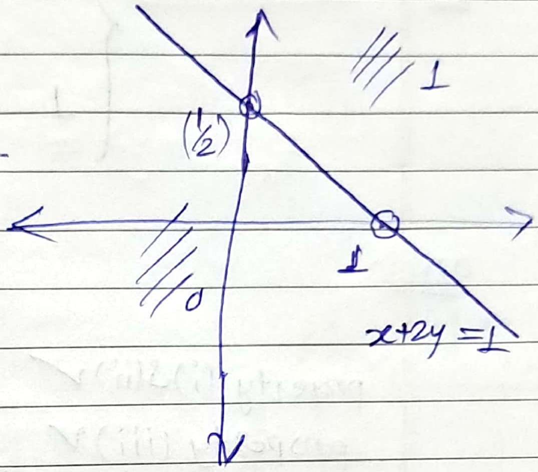
1)

a) $F(x,y) = \begin{cases} 1 & \text{if } x+2y \geq 1 \\ 0 & \text{if } x+2y < 1 \end{cases}$



Properties :- $F(x,y) = 1$

i) $\lim x \to \infty$
$y \to \infty$

ii) $\lim_{x \to -\infty} F(x,y) = 0$ (y is some constant now)

$\lim_{y \to -\infty} F(x,y) = 0.$

iii) $F_X(x,y)$ is right continuous & non-decreasing.

fix any 1 cordinate

$F_X(x,k)$ & $F_X(k,y)$ are right continuous & non-decreasing

because value 0 to 1, increasing

iv) For each $\overset{X}{(a_1,b_1]} \times \overset{Y}{(a_2,b_2]}$ in $\mathbb{R}^2$.

$\Delta = F(b_1,b_2) - F(a_1,b_2) - F(b_1,a_2) + F(a_1,a_2).$
≥ 0

take $(0,1] \times (0,1]$

$\Delta = 1 - 1 - 1 - 0 = -1 < 0$

$\therefore$ property (iv) disproved.

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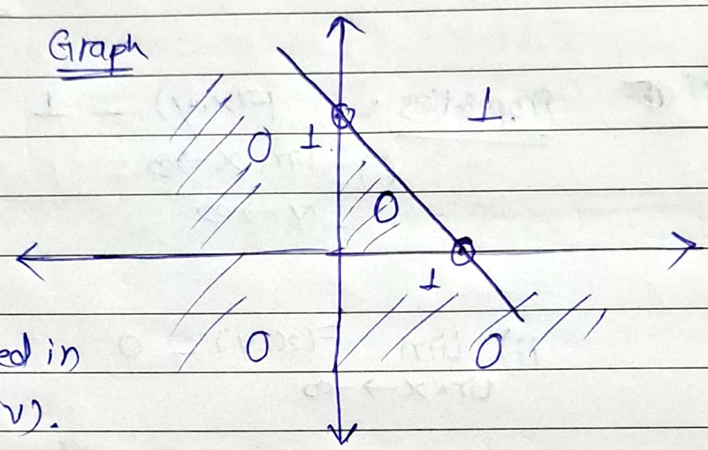
b) $F(x,y) = \begin{cases} 0, & \text{if } x<0 \text{ or } x+y<1 \text{ or } y<0 \\ 1, & \text{otherwise.} \end{cases}$

ans:-

Graph



property (i) & (ii) ✓
property (iii) ✓

It will be trapped in property (iv).
similar as (a)

Hence, not Random vector

2) ans:- X : Heads
Y : | Heads − tails |

$\underline{X} = (X, Y)$ → discrete R. vector ??

$$X(\omega) = \begin{cases} 3, & \omega \in \{HHH\} \\ 2, & \omega \in \{HHT, HTH, THH\} \\ 1, & \omega \in \{HTT, THT, TTH\} \\ 0, & \omega \in \{TTT\} \end{cases}$$

$$Y(\omega) = \begin{cases} 1 & \omega \in \{HTH, THH, HHT, TTH, THT, HTT\} \\ 3 & \omega \in \{HHH, TTT\} \end{cases}$$

~~X \\ Y~~	~~1~~	~~3~~

f_X
Marginal X.

X \\ Y	1	3	Total.
0	$P(X=0, Y=1)$ 0	$P(X=0, Y=3)$ $1/8$	$P(X=0)$ $\frac{1}{8}$
1	$P(X=1, Y=1)$ $\frac{3}{8}$	$P(X=1, Y=3)$ 0	$P(X=1)$ $\frac{3}{8}$
2	$P(X=2, Y=1)$ $\frac{3}{8}$	$P(X=2, Y=3)$ 0	$P(X=2)$ $\frac{3}{8}$
3	$P(X=3, Y=1)$ 0	$P(X=3, Y=3)$ $1/8$	$P(X=3)$. $1/8$
Marginal Y Total.	$P(Y=1)$ ~~How to find~~. $\frac{3}{4}$	$P(Y=3)$ $\frac{1}{4}$	1.

f_Y

How to find $P(X=0, Y=1)$.

Total sample space = 8.

$P(X=0) \Rightarrow$ TTT

$P(Y=1) \Rightarrow$ HTH, THH, HHT, TTH, THT, HTT

) intersection $= \phi$

$\therefore P(X=0, Y=1) = \boxed{\frac{0}{8}}$

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Joint p.m.f

$$f_{XY}(x,y) = \begin{cases} 0 & , (x,y) \in \{(0,1),(1,3)\\ & (2,3),(3,1)\} \\ \frac{1}{8} & , (x,y) \in \{(0,3)\ (3,3)\} \\ \frac{3}{8} & , (x,y) \in \{(1,1)\ (2,1)\}. \end{cases}$$

$$E_X = \{(0,3)(1,1)(2,1)(3,3)\}$$

$p > 0$

$$\& \sum p(x,y) = 1 = \left(\tfrac{1}{8}\right)\times 2 + \left(\tfrac{3}{8}\right)\times 2 + (0)\times 4$$

Since, support ~~are~~ is ~~a~~ finite set of points, the sum of probabilities at that point $= 1$. (sum $= 1$, verifies that E_X is support & finite points have $p > 0$).

or

A Random vector ~~vector~~ is discrete if its Random variables are discrete.

Hence, will need to prove E_X & E_Y are finit

$$\& \sum_{x \in E_X} P(x) = 1 \quad \& \sum_{y \in E_y} P(y) = 1.$$

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Theory.

ans-3)

Independence of Events	Independence
•) X & Y are events	•) X & Y are R.Vs
•) X & Y are independent iff	•) X & Y are independent iff
$P(X \cap Y) = P(X) \cdot P(Y)$	Joint c.d.f = product of marginal c.d.f
	$F_{XY}(x,y) = F_X(x) \cdot F_Y(x)$
	or
	Joint p.m.f = product of marginal p.m.f
	$f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$.

Let A & B are two events

X & Y are R.V.

$$X(\omega) = \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{otherwise} \end{cases}$$

$$Y(\omega) = \begin{cases} 1, & \text{if } \omega \in B \\ 0, & \text{otherwise} \end{cases}$$

Prove :- X & Y are independent

$\Updownarrow$

A & B are independent

$\boxed{\text{Revision: } f_X(x) = P(X=x)}$

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$f_{XY}(x,y) = P(X=x, Y=y)$

(means probability of intersection of $X=x$ & $Y=y$)

$= P(X=0, Y=0) = P(A^c \cap B^c)$ when $(x,y)=(0,0)$

$P(X=0, Y=1) = P(A^c \cap B)$ when $(x,y) = (0,1)$

$P(X=1, Y=0) = P(A \cap B^c)$ when $(x,y) = (1,0)$

$P(X=1, Y=1) = P(A \cap B)$ when $(x,y) = (1,1)$

$f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$

$$f_X(x) = \begin{cases} P(A^c) & \text{when } x=0 \\ P(A) & \text{when } x=1 \end{cases} \qquad f_Y(y) = \begin{cases} P(B^c) & \text{when } y=0 \\ P(B) & \text{when } y=1 \end{cases}$$

We want $P(A \cap B) = P(A) \cdot P(B)$.

Take $P(X=1, Y=1) = P(A \cap B)$

$\therefore$

$f_{XY}(1,1) = f_X(1) \cdot f_Y(1)$

$P(A \cap B) = P(A) \cdot P(B)$.

By theorem if A & B are independent then A^c & B, A & B^c, A^c & B^c, A & B are independent

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Now, $P(A^c \cap B) = P(A^c) \cdot P(B)$

$f_{XY}(0,1) = f_X(0) \cdot f_Y(1)$

similarly all 4 points can be covered.

Hence A & B independent

$\Updownarrow$

X & Y independent

— X —

Note:- $\lim_{x \to \infty} F(x,y) = F(\infty, y)$

ans. 5) Joint pdf

$$f(x,y) = \begin{cases} \dfrac{1+xy}{4}, & \text{if } |x|<1, |y|<1 \\ 0, & \text{otherwise} \end{cases}$$

Prove:- X and Y are not independent

$\Downarrow$

X^2 & Y^2 are independent

Homework, Please verify.

X, Y independent $\xRightarrow{\text{True}}$ X^2 & Y^2 independent

but converse of this is not true.

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To show X and Y are not independent,

$$f_{XY}(x,y) \neq f_X(x) \cdot f_Y(y).$$

$$F_X(x) = \lim_{y \to \infty} f(x,y)$$

$$= \int_{-\infty}^{+\infty} f(x,y) \cdot dy \qquad (\text{since } |y|<1)$$

$$= \begin{cases} \int_{-1}^{1} \frac{1+xy}{4} \cdot dy = \boxed{\frac{1}{2}} & |x|<1 \quad -1<x<1 \\ 0 & \text{otherwise.} \end{cases}$$

$f_Y(y)$ will be exact same as $f_X(x)$

$$f_Y(y) = \begin{cases} \frac{1}{2} & -1<y<1 \\ 0 & \text{otherwise} \end{cases}$$

take $|y|<1$ & $|x|<1$.

$$f_{XY}(x,y) = \frac{1+xy}{4}$$

$$f_X(x) \cdot f_Y(y) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

not equal at $|y|<1$ & $|x|<1$

$\therefore$ not independent

Homework do questions that are given in PS4 solution pdf.

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Date __ / __ / __

To show X^2 and Y^2 are independent

$$f_{X^2Y^2}(x,y) = f_{X^2}(x) \cdot f_{Y^2}(y).$$
is hard,

$$F_{X^2Y^2}(x,y) = F_{X^2}(x) \cdot F_{Y^2}(y)$$ is easy.

$$F_{X^2Y^2}(x,y) = P(X^2 \le x, Y^2 \le y)$$

20 October

Problem set 4

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12) $E(X) = E\big(\overbrace{E[X \mid Y]}^{g(y)}\big)$

$$= \sum_{y} E[X \mid Y = y]\, P(Y = y).$$

~~X~~ We know,

$E(h(X))$ exists.

then

$$E(h(X)) = E(E(h(X) \mid Y))$$

- X denote the time until the miner reaches safety.
- Y denote the door he opens initially.

$E(X) = ?$

its equally likely $\therefore$ P(he chooses a door) $= \frac{1}{3}$.

Now, P(he reaches safety)

$$= E[X \mid Y=1] \cdot \underbrace{P(Y=1)}_{1/3}$$

$$+ E[X \mid Y=2] \cdot \underbrace{P(Y=2)}_{1/3}$$

$$+ E[X \mid Y=3] \cdot \underbrace{P(Y=3)}_{1/3}$$

$$E(x) = \frac{1}{3}\left[2 + (3 + E(X)) + (5 + E(X))\right]$$

$\therefore \boxed{E(X) = 10}$

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$X = \#$ of men who got own hat.

$i = 1, 2, \ldots N$

$$X_i = \begin{cases} 1, & i\text{th man gets own hat.} \\ 0, & \text{otherwise.} \end{cases}$$

$$X = X_1 + X_2 + X_3 + \ldots\ldots + X_n.$$

$$E(X) = E(\Sigma X_i)$$
$$= \Sigma E(X_i).$$

Now

$$E(X_1) = \sum_{x=0}^{1} x \cdot P(X_1 = x)$$
$$= \frac{1}{N}$$
$$= E(X_1^2).$$

Now

$\forall i$, $$E(X_i) = \sum_{x=0}^{1} x \cdot P(X = x)$$
$$= \frac{1}{N}$$
$$= E(X_i^2)$$

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$$Var(X) = E(X^2) - E(X)^2$$

$$\downarrow$$

$$E((X_1 + X_2 + \dots X_n)^2)$$

$$= E\left((X_1^2) + \dots + (X_n^2)\right) + 2(E(X_1 X_2)) \times {}^N C_2$$

$$= N \times \frac{1}{N} + 2(E(X_i X_{i+1})) \times {}^N C_2$$

Note: $E(X_1 X_2) = \sum xy \cdot P(X_1 = x, X_2 = y)$

$$= P(X_1 = 1, X_2 = 1) = \frac{1}{N} \times \frac{1}{N-1}$$

[if $X_1 = 0$, whole term is zero]
or $X_2 = 0$

$$\therefore Var[X] = E(X^2) - [E(X)^2] \quad E(X)^2 \rightarrow 1$$

$$E(X^2) = 1 + 2 \times \frac{1}{N(N-1)} \times \frac{N(N-1)}{2}$$

$$= \boxed{2}$$

$$\therefore Var(X) = 2 - 1 = \boxed{1}$$

C3

Last tut

PS-5

ans. 1) Theoritical
ask Sumit Sir
/other teachers

Saathi

Date __/__/__

~~CLT~~ Poisson $\rightarrow$ discrete $\rightarrow$ continuous correction

ans. 2) By CLT.

$$\frac{(X_1+X_2+\cdots X_{10})-10\times 1}{\sqrt{10}\times\sqrt{1}} \sim N(0,1)$$

$$P(X_1+X_2+\cdots X_{10} \geq 15).$$

$\boxed{\text{continuous correction.}}$

$$P(X_1+\cdots X_{10} > 14.5).$$

$$\approx P\left(\frac{(X_1+\cdots X_{10})-10}{\sqrt{10}} > \frac{14.5-10}{\sqrt{10}}\right)$$

$$\approx P\left(Z > \frac{4.5}{\sqrt{10}}\right)$$

$$\approx 1-P\left(Z \leq \frac{4.5}{\sqrt{10}}\right).$$

$$= 1-\Phi\left(\frac{4.5}{\sqrt{10}}\right).$$

ans:- 3) same as 2.

OVER

REFER THE OTHER PS-4,5 PDF

for complete (nearly)

solution.