**Indian Institute of Information Technology Allahabad**
**Univariate and Multivariate Calculus**
**C1 Review Test - Tentative Marking Scheme**

Program: B.Tech. $2^{\text{nd}}$ Semester (IT+ECE)
Duration: **40 Minutes** Full Marks: 25
Date: May 22, 2022 Time: 12:00 - 12:40 IST

**Important Instructions:**

1. Attempt all the questions. There is no credit for a solution if the appropriate work is not shown, even if the answer is correct. All the notations are standard and same as used in the lecture notes.
2. Write down your name and enrolment number on a piece of paper. Write the solutions clearly with all the steps in details.
3. Submit the solution in PDF format through Google Classroom. **Name the PDF as Enrolment number-UMC-22.pdf**. We will not accept the solution through emails.
4. Extra 5 minutes is given for submission. Submission after 12:45 PM will attract penalty.

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Attempt all questions.

1. Let $f, g : [0,1] \to [0,\infty)$ be continuous functions satisfying

$$\sup_{0\le x\le 1} f(x) = \sup_{0\le x\le 1} g(x).$$

Prove that there exists $x_0 \in [0,1]$ such that $f(x_0) = g(x_0)$. [5]

**Solution.** Let $M = \sup_{0\le x\le 1} f(x) = \sup_{0\le x\le 1} g(x)$.

Since $f, g$ are continuous functions on $[0,1]$, there exist $c, d \in [0,1]$ such that $f(c) = g(d) = M$. [2]

If $c = d$, we are done. Otherwise, define the function $h(x) = f(x) - g(x)$ on $[0,1]$. Clearly, $h$ is continuous. [1]

Since

$$h(c) = f(c) - g(c) = M - g(c) \ge 0$$

and

$$h(d) = f(d) - g(d) = f(d) - M \le 0,$$

by intermediate value theorem there exists $x_0 \in [c,d]$ such that $h(x_0) = f(x_0) - g(x_0) = 0$. [2]

2. Let $\alpha \neq \beta \in \mathbb{R} \setminus \mathbb{Q}$ and $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \alpha x, & \text{if } x \in \mathbb{Q} \\ \beta - \alpha x, & \text{otherwise }. \end{cases}$$

Find the points of continuity and discontinuity. [4]

**Solution.** Let $x \in \mathbb{R}$. If $x \in \mathbb{Q}$, there exists a sequence $(x_n) \subseteq \mathbb{R} \setminus \mathbb{Q}$ converging to $x$ and if $x \in \mathbb{R} \setminus \mathbb{Q}$, there exists a sequence $(x_n) \subseteq \mathbb{Q}$ converging to $x$. [1]

We know that if $f$ is continuous at $x$ and $x_n \to x$, then $f(x_n) \to f(a)$.

If $x \in \mathbb{Q}$, then $\beta - \alpha x_n \to \alpha x$. But, $\beta - \alpha x_n \to \beta - \alpha x$. [1]

Since limit of a convergent sequence is unique, we have $\beta - \alpha x = \alpha x \Rightarrow x = \frac{\beta}{2\alpha}$. [1]

Similarly, if $x \in \mathbb{R} \setminus \mathbb{Q}$, then $\alpha x_n \to \beta - \alpha x$. Since $\alpha x_n \to \alpha x$, we have $\beta - \alpha x = \alpha x \Rightarrow \frac{\beta}{2\alpha}$.

Hence, $f$ is continuous only at $x = \frac{\beta}{2\alpha}$. [1]

3. Let $f : (0, \infty) \to \mathbb{R}$ be differentiable. Assume that there exist $\xi, \nu \in \mathbb{R}$ such that $\lim_{x\to\infty} f(x) = \xi$ and $\lim_{x\to\infty} f'(x) = \nu$. Using L'Hôpital's Rule, find the value $\nu$. [3]

**Solution.** $\xi = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{e^x f(x)}{e^x}$. [2]

Using L'Hôpital's Rule,

$\xi = \lim_{x\to\infty} \frac{e^x(f(x)+f'(x))}{e^x} = \xi + \nu$. This implies that $\nu = 0$. [1]

4. For $t > 0$, show that

$$t - \frac{t^2}{2} < \log(1 + t) < t - \frac{t^2}{2(1 + t)}. \quad [5]$$

**Solution.** Let $f(t) = \log(1 + t) - t + \frac{t^2}{2}$, for $t \geq 0$.

$\Rightarrow f'(t) = \frac{1}{1+t} - 1 + t = \frac{t^2}{1+t^2} > 0, \forall\, t > 0$. [1]

This implies that $f(t)$ is increasing. [$\frac{1}{2}$]

Since $f(0) = 0$, we have, $f(t) > 0$ for $t > 0 \Rightarrow t - \frac{t^2}{2} < \log(1 + t)$. [1]

Similarly, let $g(t) = t - \frac{t^2}{2(1+t)} - \log(1 + t)$ for $t \geq 0$

$\Rightarrow g'(t) = \frac{t^2}{2(1+t)^2} > 0, \forall\, t > 0$. [1]

This implies that $g(t)$ is increasing. [$\frac{1}{2}$]

Since $f(0) = 0$, we have, $g(t) > 0$ for $t > 0 \Rightarrow \log(1 + t) < t - \frac{t^2}{2(1+t)}$. [1]

5. Let $f(x) = x + \frac{1}{x}$ for $x \in \mathbb{R} \setminus \{0\}$. Find the points of local maxima/minima, points of inflection, domain of convexity/concavity, and horizontal/vertical/oblique asymptotes. Sketch the graph of $f$. [8]

**Solution.** $f(x) = x + \frac{1}{x}$

$f'(x) = 0 \Rightarrow 1 - \frac{1}{x^2} = 0 \Rightarrow x = \pm 1$.

Hence, $x = \pm 1$ are critical points. [1]

and $f"(x) = \frac{2}{x^3} \Rightarrow f"(1) > 0$ and $f"(-1) < 0$. [1]

Thus, $x = 1$ is a point of local minima and $x = -1$ is a point of local maxima. [1]

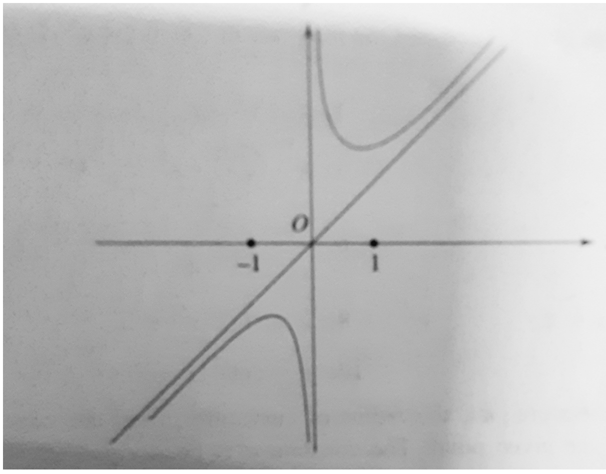
Convexity $\Leftrightarrow f"(x) > 0$. Therefore, $(0, \infty)$ is the domain of convexity and $(-\infty, 0)$ is the domain of concavity. [1]

There is no point of inflection as $f$ is not defined at 0. [1]

$\lim_{x \to 0} f(x) = \infty \Rightarrow x = 0$ is a vertical asymptote. [1]

$\lim_{x \to \infty}(f(x) - x) = 0 \Rightarrow y = x$ is an oblique asymptote. [1]

The sketch of the graph of the $f$ is shown below.




[1]