Indian Institute of Information Technology Allahabad
Linear Algebra (LAL)
C2 Review Test Marking Scheme

Program: B.Tech. 1st Semester (IT+ECE)
Duration: **60+10 Minutes** Full Marks: 25
Date: March 19, 2021 Time:: 18:00 IST - 19:00 IST

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1. Let A be a real projection matrix and $f(x)$ be a polynomial over $\mathbb{R}$. Show that $f(A) = aI + bA$, from some $a, b \in \mathbb{R}$. Also, find a and b in terms of the coefficients of f. [2]

   Solution: Let $f(x) = a_0 + a_1x + \cdots + a_nx^n$. Then $f(A) = a_0I + a_1A + \cdots + a_nA^n$. Since A is a projection, $A^2 = A$ so that $f(A) = a_0I + a_1A + a_2A + \cdots + a_nA$. [1]
   Therefore, $a = a_0$ and $b = a_1 + a_2 + \cdots + a_n$. [1]

2. Let $A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 1 & 1 \\ 4 & 5 & 7 & 1 \\ 3 & 0 & -1 & 2 \end{pmatrix}$. Find a basis of the orthogonal complement of the null space of A. [3]

   Solution: By Fundamental Theorem of Linear Algebra, $N(A)^\perp = C(A^T)$=Row space$(A) = L(\{(1,2,3,0),(2,1,1,1),(4,5,7,1),(3,0,-1,2)\})$. [1]

   $$A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 1 & 1 \\ 4 & 5 & 7 & 1 \\ 3 & 0 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 3 & 5 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ [1]

   The non-zero rows of REF are linearly independent, a basis of orthogonal complement of the null space is $\{(1,2,3,0),(0,3,5,-1)\}$. [1]

3. Let $V = \mathbb{P}_3(\mathbb{R})$ with the inner product $\langle p(x), q(x)\rangle = \int_{-1}^{1} p(x)q(x)dx$. Find the orthogonal projection of x^3 on $\mathbb{P}_2(\mathbb{R})$. [8]

   Solution: A basis of $\mathbb{P}_2(\mathbb{R})$ is given by $\{1, x, x^2\}$. By Gram-Schmidt process, an orthonormal basis $\{v_1, v_2, v_3\}$ of V is obtained as follows:

   $u_1 = 1$, $v_1 = \frac{u_1}{\|u_1\|}$, $\|u_1\|^2 = \int_{-1}^{1} 1\,dx = 2 \Rightarrow v_1 = \frac{1}{\sqrt{2}}$. [1]

   $u_2 = x - \langle x, \frac{1}{\sqrt{2}}\rangle \frac{1}{\sqrt{2}} = x$, $v_2 = \frac{u_2}{\|u_2\|}$, $\|u_2\|^2 = \int_{-1}^{1} x^2\,dx = 2/3 \Rightarrow v_2 = \frac{\sqrt{3}}{\sqrt{2}}x$. [1+1]

   $u_3 = x^2 - \langle x^2, \frac{1}{\sqrt{2}}\rangle \frac{1}{\sqrt{2}} - \langle x^2, \frac{\sqrt{3}}{\sqrt{2}}x\rangle \frac{\sqrt{3}}{\sqrt{2}}x = x^2 - \frac{1}{3} - 0 = \frac{3x^2-1}{3}$, [1]

   $v_3 = \frac{u_3}{\|u_3\|}$, $\|u_3\|^2 = \frac{1}{9}\int_{-1}^{1}(3x^2-1)^2\,dx = 8/45 \Rightarrow v_3 = \frac{\sqrt{5}}{\sqrt{8}}(3x^2-1)$. [1]

The orthogonal projection of x^3 on $W = \mathbb{P}_2(\mathbb{R})$ is given by

$P_W(x^3) = \langle x^3, \frac{1}{\sqrt{2}}\rangle \frac{1}{\sqrt{2}} + \langle x^3, \frac{\sqrt{3}}{\sqrt{2}}x\rangle \frac{\sqrt{3}}{\sqrt{2}}x + \langle x^3, \frac{\sqrt{5}}{\sqrt{8}}(3x^2-1)\rangle \frac{\sqrt{5}}{\sqrt{8}}(3x^2-1)$
$= 0 + \frac{3}{5}x + 0 = \frac{3}{5}x$ [1+1+1]

4. Let A be an $n \times n$ real symmetric matrix and $\langle\, , \rangle$ be the usual inner product on $\mathbb{R}^n$. Suppose $\langle x, Ax\rangle = \langle x, x\rangle$ for all $x \in \mathbb{R}^n$. Find the matrix A. [5]

   Solution: Let $A = (a_{ij})$, where $a_{ij} = a_{ji}$ for every $1 \le i, j \le n$.

   Take $x = e_i$, then $\langle e_i, Ae_i\rangle = \langle e_i, e_i\rangle$ [1]

   $\implies a_{ii} = 1$, for all $1 \le i \le n$. [1]

   Now take $x = e_i - e_j$, where $i < j$. [1]

   Then $\langle e_i - e_j, e_i - e_j\rangle = 2$ and $\langle e_i - e_j, A(e_i - e_j)\rangle = \langle e_i, Ae_i\rangle - \langle e_i, Ae_j\rangle - \langle e_j, Ae_i\rangle + \langle e_j, Ae_j\rangle = 1 - a_{ij} - a_{ji} + 1 = 2 - 2a_{ij}$ (since $a_{ij} = a_{ji}$). [1]

   Thus, $a_{ij} = a_{ji} = 0$. [1]

   Therefore $A = I_n$.

5. Find a 3×3 real symmetric matrix A such that $1, 2, 2$ are the eigenvalues of A and $(-1, -1, 1)^t$ is an eigenvector of A corresponding to the eigenvalue 1. [7]

   Solution: Let E_1 and E_2 be the eigenspaces corresponding to eigenvalues 1 and 2 respectively. Since A is symmetric, $\mathbb{R}^3 \cong E_1 \oplus E_2$ and $E_1^{\perp} = E_2$. [1]

   Since $v_2 = (1, 0, 1)^t$ is orthogonal to $(-1, -1, 1)^t$, $v_2 \in E_2$. [1]

   Now, suppose $(x, y, z)^t$ is orthogonal to both $(-1, -1, 1)^t$ and $(1, 0, 1)^t$.

   So we have $-x - y + z = 0, x + z = 0$. Then $v_3 = (-1, 2, 1)^t$ is orthogonal to both $(-1, -1, 1)^t$ and $(1, 0, 1)^t$. [2]

   Now take $P = \begin{pmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}$ [1]

   Hence, $A = P \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} P^t = \begin{pmatrix} \frac{5}{3} & \frac{-1}{3} & \frac{1}{3} \\ \frac{-1}{3} & \frac{5}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{5}{3} \end{pmatrix}$ [1+1]