Indian Institute of Information Technology Allahabad
Probability and Statistics (PAS)
C2 Review Test Tentative Marking Scheme

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1. Let X_1 and X_2 have the joint probability density function [11]

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} 10x_1x_2^2, & 0 < x_1 < x_2 < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the probability density function of $Y = \frac{X_1}{X_2}$ by using transformation of variables technique.

(b) Find $Cov(X_1, X_2)$.

(c) Are Y and X_2 uncorrelated? Justify your answer.

Solution:

(a) Let $Z = X_2$. Then $x_1 = yz$ and $x_2 = z$. [1/2]

Hence

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y} & \frac{\partial x_1}{\partial z} \\ \frac{\partial x_2}{\partial y} & \frac{\partial x_2}{\partial z} \end{vmatrix} = \begin{vmatrix} z & y \\ 0 & 1 \end{vmatrix} = z \quad [1/2]$$

Hence, the joint pdf of Y and Z is

$$\begin{aligned} f_{Y,Z}(y,z) &= f_{X_1,X_2}(yz,z)|J| \\ &= \begin{cases} 10yz^4, & 0 < yz < z < 1 \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} 10yz^4, & 0 < y < 1, 0 < z < 1 \\ 0, & \text{otherwise} \end{cases} \quad [1+1] \end{aligned}$$

Hence the pdf of Y is

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{Y,Z}(y,z)dz \\ &= \begin{cases} \int_0^1 10yz^4 dz, & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} 2y, & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases} \quad [1+1] \end{aligned}$$

(b) $E(X_1X_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2 f_{X_1,X_2}(x_1,x_2)dx_1dx_2 = \int_0^1\int_0^{x_2} 10x_1^2x_2^3 dx_1dx_2 = \frac{10}{21}$ [1]

$E(X_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 f_{X_1,X_2}(x_1,x_2)dx_1dx_2 = \int_0^1\int_0^{x_2} 10x_1^2x_2^2 dx_1dx_2 = \frac{5}{9}$ [1]

$E(X_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_2 f_{X_1,X_2}(x_1,x_2)dx_1dx_2 = \int_0^1\int_0^{x_2} 10x_1x_2^3 dx_1dx_2 = \frac{5}{6}$ [1]

$Cov(X_1,X_2) = E(X_1X_2) - E(X_1)E(X_2) = \frac{10}{21} - \frac{25}{54} = \frac{5}{378}$ [1]

(c) $E(Y) = \int\limits_{-\infty}^{\infty} y f_Y(y) dy = \int\limits_{0}^{1} 2y^2 dy = \frac{2}{3}$ [1]

Now, $Cov(Y, X_2) = E(YX_2) - E(Y)E(X_2) = E(X_1) - E(Y)E(X_2) = \frac{5}{9} - \frac{10}{18} = 0.$
Thus Y and X_2 are uncorrelated. [1]

2. Let $(X_n)_{n\geq 1}$ be a sequence of random variables with corresponding distribution functions given by [7]

$$F_n(x) = \begin{cases} 0, & x < -n \\ \frac{x+n}{2n}, & -n \leq x < n \\ 1, & x \geq n. \end{cases}$$

Does F_n converge to a distribution function. Justify your answer.

Solution: For $x \in \mathbb{R}$ with $x < 0$, we have a $N \in \mathbb{N}$ such that $-N \leq x < -(N-1)$. Then

$$F_n(x) = \begin{cases} 0, n < N \\ \frac{x+n}{2n}, n \geq N \end{cases} \qquad [2]$$

Similarly, for $x \in \mathbb{R}$ with $x > 0$, we have a $M \in \mathbb{N}$ such that $M - 1 \leq x < M$. Then

$$F_n(x) = \begin{cases} 1, n < M \\ \frac{x+n}{2n}, n \geq M \end{cases} \qquad [2]$$

Also $F_n(0) = \frac{1}{2}$, for all $n \in \mathbb{N}$. [1]

Hence $\lim\limits_{n\to\infty} F_n(x) = \frac{1}{2}$, for all $x \in \mathbb{R}$. That is. F_n converges to the function $f(x) = \frac{1}{2}$, for all $x \in \mathbb{R}$, which is not a distrinution function. [1+1]

3. Let $(X_n)_{n\geq 1}$ be a sequence of identically and independently distributed random variables with common probability density function [7]

$$f(x) = \begin{cases} e^{-x+\theta}, & x \geq \theta \\ 0, & x < \theta. \end{cases}$$

Let $\overline{X}_n = \frac{1}{n}\sum\limits_{i=1}^{n} X_i$. Show that $\overline{X}_n$ converges to $1 + \theta$ in probability.

Solution: $E(X_i) = \int\limits_{-\infty}^{\infty} x f(x) dx = \int\limits_{\theta}^{\infty} x e^{-x+\theta} dx = 1 + \theta, \forall\, i \in \mathbb{N}$ [1]

$E(X_i^2) = \int\limits_{-\infty}^{\infty} x^2 f(x) dx = \int\limits_{\theta}^{\infty} x^2 e^{-x+\theta} dx = \theta^2 + 2\theta + 2, \forall\, i \in \mathbb{N}$ [1]

$Var(X_i) = E(X_i^2) - (E(X_i))^2 = 1, \forall\, i \in \mathbb{N}.$ [1]

$E(\overline{X}_n) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = 1 + \theta$ [1]

$Var(\overline{X}_n) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{1}{n}$ [1]

By Chebyshev's Ineuality, for every $\epsilon > 0$, we have $P(\{|\overline{X}_n - (1+\theta)| \geq \epsilon\}) \leq \frac{1}{n\epsilon^2}$ [1]

Since $\lim\limits_{n\to\infty} \frac{1}{n\epsilon} = 0$, $\lim\limits_{n\to\infty} P(\{|\overline{X}_n - (1+\theta)| \geq \epsilon\}) = 0.$ [1]