Problem-set 12

1) Determine the values of $\alpha \in \mathbb{R}$ for which $\sum_{n=1}^{\infty} \left(\frac{\alpha n}{n+1}\right)^n$ converges.

Solⁿ →

Cauchy's root test →

If $|a_n|^{1/n} \leq \alpha$ eventually for some $\alpha < 1$, then $\sum a_n \to$ absolutely convergent.
Divergent for $\alpha > 1$.

$|a_n|^{1/n} = \left|\frac{\alpha n}{n+1}\right| \Rightarrow$ For convergent $\to |\_\_| < 1$.

$\rightarrow \left|\frac{\alpha n}{n+1}\right| \longrightarrow \alpha$ as $n \to \infty$

$\Rightarrow$ Convergant for $|\alpha| < 1$.
DIV. for $|\alpha| > 1$.

2] Consider $\sum a_n$, $a_n > 0 \ \forall n$. Prove/disprove →

(a) If $\frac{a_{n+1}}{a_n} < 1 \ \forall n$, series converges.

Solⁿ → Take $a_n = 1/n$. $\Rightarrow \frac{a_{n+1}}{a_n} = \frac{n}{n+1} < 1 \ \forall n$.

But $a_n \to$ Divergent series.

Hence, false.

(b) If $\frac{a_{n+1}}{a_n} > 1$ $\forall n$, series diverges.

$\underline{\underline{Sol^n}} \rightarrow$ Since $\frac{a_{n+1}}{a_n} > 1 \Rightarrow a_n \not\to 0$.

Hence, Σa_n is divergent.

(3) Show $\frac{1}{1^2} + \frac{1}{2^3} + \frac{1}{3^2} + \frac{1}{4^3} + \frac{1}{5^2} + \frac{1}{6^3} + \cdots$ converges & that the root test & ratio test are not applicable.

$\underline{\underline{Sol^n}} \rightarrow$ Ratio test fails ~~because~~ because we cannot obtain a ratio; it is alternating for diff. terms of the series.

Hence ratio test fails.

Root test fails, since

By Comparison test

4] Consider the rearranged GM: $\frac{1}{2} + 1 + \frac{1}{8} + \frac{1}{4} + \cdots$

Show that the series converges by the root test & that ratio test is not applicable.

Soln → nth term $a_n = \begin{cases} \frac{1}{2^n} & n: \text{odd} \\ \frac{1}{2^{n-2}} & n: \text{even.} \end{cases}$

→ The consecutive ratio is alternating in values b/w $\frac{1}{8}$ & 2, ratio test is not applicable.

However by root test → $a_n^{1/n} \begin{cases} \frac{1}{2}, & n: \text{odd} \\ \frac{1}{2^{1-2/n}} \to \frac{1}{2} & n: \text{even} \end{cases}$

Hence, by root test → $a_n^{1/n} \to \frac{1}{2}$

→ Convergent.

$\boxed{5}$

(a) If $\sum a_n$ & $\sum b_n$ converges absolutely, show that $\sum a_n b_n$ converges absolutely.

Soln → ∵ $\sum b_n$ is conv → $b_n \to 0$. → $|b_n| < \epsilon$

→ $|a_n b_n| \le \epsilon |a_n|$ eventually.

$\underbrace{\qquad}_{\text{convergent.}}$

∴ By Comparison test, $|a_n b_n|$ converges absolutely.

(b) $\sum a_n$ converges absolutely; $(b_n) \to$ Bounded seq^n.

show $|a_n b_n|$ conv.

Sol^n $\to$ $\because$ (b_n) is a bounded seq^n, $|b_n| \leq M$ for some M.

as $|a_n b_n| \leq M|a_n|$.

$\therefore$ By comp. test ________ convergent.

(C) Example of a conv. series $\sum a_n$ & bounded (b_n) s.t. $\sum a_n b_n$ diverges.

Sol^n $\to$ $a_n = \frac{(-1)^n}{n} \to$ converges to 0. , $b_n = (-1)^n \to$ oscillatory seq^n $(-1$ to $1)$.

$a_n b_n = \frac{1}{n} \to$ Div. series

(6) In each of the following cases, discuss the conv/div of $\sum a_n$, where $a_n =$

(a) $\frac{n!}{n^n}$.

Sol^n $\to$ Converges by ratio test.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \times \frac{n^n}{n!} = \frac{n^{n+1}}{(n+1)^{n+1}} = \left(\frac{n}{n+1}\right.$$

(b)

(d) $\dfrac{n^2 2^n}{(2n+1)!}$

Converges by ratio test..

(e) $\left(1-\frac{1}{n}\right)^{n^2}$

Converges by root test $\Rightarrow \left(1-\frac{1}{n}\right)^n = e^{-1}$.

(f) $\dfrac{n^2}{3^n}\left(1+\frac{1}{n}\right)^{n^2}$

Converges by root test $\Rightarrow a_n^{1/n} \longrightarrow \frac{e}{3} < 1$.

(g) $\sin\left(\dfrac{(-1)^n}{n^p}\right)$, $p>0$.

Leibniz test $\Rightarrow$

If a_n be a monotonic seq^n s.t. $a_n \to 0$. Then $(-1)^{n-1} a_n$ $\Rightarrow$ conv.

$\sin\left(\dfrac{(-1)^n}{n^p}\right) = (-1)^n \sin\left(\dfrac{1}{n^p}\right)$

Monotonic, $\to 0$

(h) $\frac{1}{2^n - n}$

(i) $(-1)^n \frac{(\ln n)^3}{n}$ Leibniz test $\Rightarrow$ conv.

If $f(x) = \frac{(\ln x)^3}{x}$, $f'(x) < 0 \quad \forall \ x > e^3$

(j) $(-1)^n (n^{1/n} - 1)^n$

converges absolutely by root test.

k) $\frac{2^n + n^2 - \ln n}{n!}$

Apply LCT, with $\frac{2^n}{n!}$

(l) $\dfrac{\cos(\pi n)\ln n}{n}$

Sol^n $\rightarrow$ conv. by Leibniz test:-

$\cos \pi n = (-1)^n$.

(m) $\left(1+\frac{2}{n}\right)^{n^2-\sqrt{n}}$

Sol^n $u_n \not\rightarrow 0$ as $\left(1+\frac{2}{n}\right) > 1$.

(n) $\dfrac{n^2(2\pi+(-1)^n)^n}{10^n}$

Sol^n $|u_n| \leq \dfrac{n^2(2\pi+1)^n}{10^n}$.

Apply Ratio test.