Tutorial Sheet-1

1. Which of these are propositions?
   (a) Do not pass.
   (b) What time is it?
   (c) $4 + x = 5$.
   (d) The moon is made of green cheese.
   (e) $2^n \geq 100$.

2. Consider the following propositions, $p :=$ Swimming at New Jersey shore is allowed, and $q :=$ Sharks have been spotted near the shore.

   Express each of these compound propositions as an English sentence.
   (a) $\neg q$
   (b) $p \wedge q$
   (c) $\neg p \vee q$
   (d) $p \to \neg q$
   (e) $\neg p \to \neg q$
   (f) $p \leftrightarrow \neg q$
   (g) $\neg p \wedge (p \ \vee \neg q)$

3. Let p, q, and r be propositions, $p :=$ Grizzly bears have been in the area, $q :=$ Hiking is safe on the trail, $r :=$ Berries are ripe along the trail.

   Write these propositions using p, q, and r and logical connectives.
   (a) Berries are ripe along the trail, but grizzly bears have not been seen in the area.
   (b) Grizzly bears have not been seen in the area and hiking on the trail is safe, but berries are ripe along the trail.
   (c) If berries are ripe along the trail, hiking is safe if and only if grizzly bears have not been seen in the area.
   (d) It is not safe to hike on the trail, but grizzly bears have not been seen in the area and the berries along the trail are ripe.
   (e) For hiking on the trail to be safe, it is necessary but not sufficient that berries not be ripe along the trail and for grizzly bears not to have been seen in the area.
   (f) Hiking is not safe on the trail whenever grizzly bears have been seen in the area and berries are ripe along the trail.

4. Determine whether each of these conditional statements is true or false.
   (a) If $1 + 1 = 2$, then $2 + 2 = 5$.
   (b) If $1 + 1 = 3$, then $2 + 2 = 4$.
   (c) If $1 + 1 = 3$, then $2 + 2 = 5$.
   (d) If monkeys can fly, then $1 + 1 = 3$.
   (e) If1+1=2,then dogs can fly.
   (f) If 2+2=4, then $1 + 2 = 3$.
5. State the converse, contrapositive, and inverse of each of these conditional statements.
   (a) If it snows today, I will ski tomorrow.
   (b) I come to class whenever there is going to be a quiz.
   (c) A positive integer is a prime only if it has no divisors other than 1 and itself.
6. Construct a truth table for each of the following compound propositions:
   (a) $p \to \neg p$.
   (b) $p \leftrightarrow \neg p$.
   (c) $p \oplus (p \vee q)$.
   (d) $(p \wedge q) \to (p \vee q)$.
   (e) $(q \to \neg p)$.
   (f) $(p \leftrightarrow q)$.
   (g) $(p \leftrightarrow q) \oplus (p \leftrightarrow \neg q)$.
7. Show that each of these conditional statements is a tautology by using once truth tables and without truth tables.
   (a) $(p \wedge q) \to p$.
   (b) $p \to (p \vee q)$.
   (c) $\neg p \to (p \to q)$.
   (d) $(p \wedge q) \to (p \to q)$.
   (e) $\neg(p \to q) \to p$.
   (f) $\neg(p \to q) \to \neg q$.
8. Show the following.
   (a) $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$.
   (b) $\neg(p \leftrightarrow q) \equiv (p \leftrightarrow \neg q)$.
   (c) $(p \leftrightarrow q) \wedge (p \to r) \equiv p \to (q \wedge r)$.

(d) $(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$.

9. Check the validity of the following argument:

   $p \to \neg q$,

   $r \to q$,

   $\therefore r \to \neg p$.

10. Check the validity of the following argument:

    If you invest in the Gomermatic Corporation, then you get rich,

    You did not invest in the Gomermatic Corporation,

    Therefore you did not get rich.

11. Let $P(x)$ be the statement "x spends more than five hours every weekday in class", where the domain for x consists of all students. Express each of these quantifications in English.

    (a) $\exists x\, P(x)$.

    (b) $\forall x\, P(x)$.

    (c) $\exists x\, \neg P(x)$.

    (d) $\forall\ \neg P(x)$.

12. Suppose that the domain of the propositional function $P(x)$ consists of the integers 0, 1, 2, 3, and 4. Write down each of these propositions using disjunctions, conjunctions, and negations.

    (a) $\exists x\, P(x)$.

    (b) $\forall x\, P(x)$.

    (c) $\exists x\, \neg P(x)$.

    (d) $\forall x\, \neg P(x)$.

    (e) $\neg \exists x\, P(x)$.

    (f) $\neg \forall x\, P(x)$.

13. Express the negation of these propositions using quanti- fiers, and then express the negation in English.

    (a) Some drivers do not obey the speed limit.

    (b) All Swedish movies are serious.

    (c) No one can keep a secret.

    (d) There is someone in this class who does not have a good attitude.

14. Express each of these system specifications using predi- cates, quantifiers, and logical connectives.

    (a) Every user has access to an electronic mailbox.

    (b) The system mailbox can be accessed by everyone in the group if the file system is locked.

    (c) The firewall is in a diagnostic state only if the proxy server is in a diagnostic state.

    (d) At least one router is functioning normally if the throughput is between 100 kbps and 500 kbps and the proxy server is not in diagnostic mode.

15. Let A, B, and C be sets. Show that

    (a) $A \cup \varnothing = A$

    (b) $A \cap B \subseteq A$

    (c) $A \cup (B - A) = A \cup B$

    (d) $(A - C) \cap (C - B) = \varnothing$

16. Let A and B be subsets of a Universal set U. Show that $A \subseteq B$ if and only if $B^c \subseteq A^c$.

Tutorial: Relations, Functions, and Equivalent sets

1. Let R be the raltion $R = \{(a,b) \,|\, a < b\}$ on the set of integers $\mathbb{Z}$. Find
   (a) R^{-1} (b) R^c.

2. How many relations are there on a set with n elements that are
   (a) symmetric (b) antisymmetric (c) reflexive.

3. Show that the relation R on a set A is symmetric if and only if $R = R^{-1}$.

4. Let A be the set of non-zero integers and let R be the relation on $A \times A$ defined as follows: $(a,b)\,R\,(c,d)$ whenever $ad = bc$. Prove that R is an equivalence relation.

5. Consider the set of integers $\mathbb{Z}$. Define aRb if $b = a^r$ for some positive integer r. Show that R is a partial on $\mathbb{Z}$.

6. Give an example of relations R on $A = \{1,2,3\}$ having the following property.
   (a) R is both symmetric and antisymmetric (b) R is neither symetric not antisymmetric.
   (c) R is transitive but $R \cup R^{-1}$ is not transitive.

7. Let R be the following equivalence relation on the set $A = \{1,2,\ldots,6\}$:

   $R = \{(1,1),(1,5),(2,2),(2,3),(2,6),(3,2),(3,3),(3,6),(4,4),(5,1),(5,5),(6,2),(6,3),(6,6)\}$. Find the equivalence classes of R, i.e., find the partition of A induced by R.

8. Let $f : A \to B$ be a function and $E, F \subseteq A$ and $G, H \subseteq B$. Then show that
   (a) $f(E \cup F) = f(E) \cup f(F)$ (b) $f(E \cap F) \subseteq f(E) \cap f(F)$
   (c) $f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$ (d) $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$.

9. (a) Show that if $f : A \to B$ is injective and $E \subseteq A$, then $f^{-1}(f(E)) = E$. Give an example that equality need not hold if f is not injective.

   (b) Show that if $f : A \to B$ is surjective and $H \subseteq B$, then $f(f^{-1}(H)) = H$. Give an example that equality need not hold if f is not surjective.

10. Let A and B be sets with $|A| = l$ and $|B| = m$.

    (a) Find the number of injective functions from A to B.

    (b) Find the number of surjective functions from A to B.

    (c) Find the number of bijective functions from A to B.

11. Show that $(0,\infty) \approx (-\infty,\infty) \approx (-\frac{\pi}{2},\frac{\pi}{2})$.

12. $(0,1) \times (0,1) \approx (0,1)$.

13. $\mathbb{R} \approx \mathbb{R} \times \mathbb{R}$.

14. $[0,1] \approx \mathcal{P}(\mathbb{N})$ (Power set of $\mathbb{N}$).

15. Show that $\mathcal{P} = \{p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n \, : a_0, a_1, \ldots, a_n \in \mathbb{Z}\} \approx \mathbb{N}$.

16. A real number r is called algebraic is r is a solution of $p(x) = 0$, where $p(x) \in \mathcal{P}$ (in above). Show that the set A of all algebraic number is equivalent to $\mathbb{N}$.

Tutorial Sheet-3: Proof Techniques

(1) Prove that if n is an integer and n^2 is odd, then n is odd.

(2) Prove that $\sqrt{p}$ is irrational, where p is prime number.

(3) Show that the square of an even number is an even number.

(4) Prove or disprove that the product of two irrational numbers is irrational.

(5) Prove that if x is rational and $x \neq 0$, then $\frac{1}{x}$ is rational.

(6) Prove that sum of a rational number and an irrational number is irrational.

(7) Prove that $f(x) = \sin x$ is continuous on $\mathbb{R}$.

(8) Show that and $f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$ is not continuous at $x = 0$.

(9) Prove or disprove $7^n - 4^n$ is divisible by 3, for all $n \in \mathbb{N}$.

(10) Prove or disprove $9(9^n - 1) - 8n$ is divisible by 64, for all $n \in \mathbb{N}$.

(11) Prove that $\arctan \frac{1}{3} + \arctan \frac{1}{7} + \cdots + \arctan \frac{1}{n^2+n+1} = \arctan \frac{n}{n+2}$, for all $n \in \mathbb{N}$.

(12) Prove that if n is an integer, then $n^2 \geq n$.

(13) Show that if a and b are integers and both ab and $a + b$ are even, then both a and b are even.

(14) Prove that $m^2 = n^2$ if and only if $m = n$ or $m = n$.

(15) Prove that if n is a positive integer, then n is even if and only if $7n + 4$ is even.

(16) Prove that if n is a perfect square, then $n + 2$ is not a perfect square.

(17) There is no surjection (onto) from a set X to its power set $P(X)$.

(18) Let $n \in \mathbb{N}$ and suppose we are given real numbers $a_1 \geq a_2 \geq \ldots \geq a_n \geq 0$. Then Arithmetic mean (AM) $= \frac{a_1+a_2+\ldots a_n}{2} \geq (a_1 a_2 \ldots a_n)^{\frac{1}{n}} =$ GM (Geometric mean).

(19) Fix a positive integer n and let A be a set with $|A| = n$. Let $P(A)$ denote the power set of A. Then show that $|P(A)| = 2^n$.

Tutorial Sheet-4: Counting, Generating Functions and Recurrence Relations

(1) How many ways are there to assign three jobs to five employees if each employee can be given more than one job?

(2) How many ways are there to select three unordered elements from a set with five elements when repetition is allowed?

(3) How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 17,$$

where x_1, x_2, x_3, and x_4 are non-negative integers?

(4) How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 21,$$

where $x_i,\ i = 1, 2, 3, 4, 5$, is a non-negative integer such that

(a) $x_1 \geq 1$?

(b) $x_i \geq 2$ for $i = 1, 2, 3, 4, 5$?

(c) $0 \leq x_1 \leq 10$?

(d) $0 \leq x_1 \leq 3,\ 1 \leq x_2 < 4$, and $x_3 \geq 15$? 106

(5) How many solutions are there to the inequality

$$x_1 + x_2 + x_3 \leq 11,$$

where x_1, x_2, and x_3 are non-negative integers?[Hint: Introduce an auxiliary variable x_4 such that $x_1 + x_2 + x_3 + x_4 = 11$.]

(6) Show that if there are 30 students in a class, then at least two have last names that begin with the same letter.

(8) How many numbers must be selected from the set $\{1, 3, 5, 7, 9, 11, 13, 15\}$ to guarantee that at least one pair of these numbers add up to 16?

(9) In how many different orders can five runners finish a race if no ties are allowed?

(10) In how many ways can a set of five letters be selected from the English alphabet?

(11) How many subsets with more than two elements does a set with 100 elements have?

(12) (a) What is the generating function for $\{a_k\}$, where a_k is the number of solutions of

$$x_1 + x_2 + x_3 + x_4 = k,$$

when x_1, x_2, x_3, and x_4 are integers with $x_1 \geq 3,\ 1 \leq x_2 \leq 5,\ 0 \leq x_3 \leq 4$, and $x_4 \geq 1$?

(b) Use your answer to part (a) to find a_7?

(13) Use generating functions to solve the recurrence relation $a_k = 7a_{k-1}$ with the initial condition $a_0 = 5$.

(14) Use generating functions to solve the recurrence relation $a_k = 3a_{k-1} + 2$ with the initial condition $a_0 = 1$.

(15) Solve these recurrence relations together with the initial conditions given

(a) $a_n = a_{n-1} + 6a_{n-2}$ for $n \geq 2,\ a_0 = 3,\ a_1 = 6$

(b) $a_n = 7a_{n-1} - 10a_{n-2}$ for $n \geq 2,\ a_0 = 2,\ a_1 = 1$

(c) $a_n = -6a_{n-1} - 9a_{n-2}$ for $n \geq 2,\ a_0 = 3,\ a_1 = -3$

(d) $a_{n+2} = -4a_{n+1} + 5a_n$ for $n \geq 0,\ a_0 = 2,\ a_1 = 8$.

Tutorial Sheet 5: Graph Theory

(1) Draw these graphs.
$(a)K_7$ $(b)K_{1,8}$ $c)K_{4,4}$ $(d)C_7$ (e) W_7 $(f)Q_4$.

(2) For which values of n are these graphs bipartite?
$(a)K_n$ $(b)C_n$ $(c)W_n$ $(d)Q_n$.

(3) Let n be a positive integer. Show that a subgraph induced by a nonempty subset of the vertex set of K_n is a complete graph.

(4) How many subgraphs with at least one vertex does K_3 have?

(5) For which values of n are these graphs regular?
$a)K_n$ $(b)C_n$ $(c)W_n$ $(d)Q_n$.

(6) Find an adjacency matrix for each of these graphs.
$(a)K_n$ $(b)C_n$ $(c)W_n$ $(d)K_{m,n}$ $(e)Q_n$.

(7) Find a self-complementary simple graph with five vertices.

(8) How many nonisomorphic simple graphs are there with five vertices and three edges?

(9) What is the product of the incidence matrix and its transpose for an undirected graph?

(10) Show that every connected graph with n vertices has at least $n-1$ edges.

(11) Show that if a connected simple graph G is the union of the graphs G_1 and G_2, then G_1 and G_2 have at least one common vertex.

(12) Show that a simple graph G with n vertices is connected if it has more than $(n-1)(n-2)/2$ edges.

(13) For which values of n do these graphs have an Euler circuit?
$(a)K_n$ $(b)C_n$ $(c)W_n$ $(d)Q_n$.

(14) Show that a bipartite graph with an odd number of vertices does not have a Hamilton circuit.

(15) Show that K_5 is nonplanar.

(16) Suppose that a connected bipartite planar simple graph has e edges and v vertices. Show that $e \leq 2v-4$ if $v \geq 3$.

(17) Suppose that a connected planar graph has 30 edges. If a planar representation of this graph divides the plane into 20 regions, how many vertices does this graph have?

Tutorial Sheet 6: Group Theory

(1) Give two reasons why the set of odd integers under addition is not a group.

(2) Show that $\{1, 2, 3\}$ under multiplication modulo 4 is not a group but that $\{1, 2, 3, 4\}$ under multiplication modulo 5 is a group.

(3) Prove that in a group, $(a^{-1})^{-1} = a$ for all a.

(4) Prove that a group G is Abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$ for all a and b in G.

(5) Let p be a prime number and G be a group such that $|G| = p$. The group G is cyclic.

(6) Is D_3 (the set of symmetries of an equilateral triangle) Abelian?

(7) Prove that a group G is Abelian if and only if $(ab)^2 = a^2b^2$ for all a and b in G.

(8) Prove that an Abelian group with two elements of order 2 must have a subgroup of order 4.

(9) Let $\Phi : G \to H$ be a homomorphism. Show that $\Phi(e_G) = e_H$ and $\Phi(a^{-1}) = (\Phi(a))^{-1}$.

(10) Show that A_3 is a normal subgroup of S_3.

(11) Prove or disprove that (a) Every normal subgroup of a group G is cyclic.
(b) Every cyclic subgroup is normal.

(12) Show that the set $Z[x]$ of all polynomials in the variable x with integer coefficients under ordinary addition and multiplication is a commutative ring with unity $f(x) = 1$.

(13) Show that $2\mathbb{Z} \cup 3\mathbb{Z}$ is not a subring of $\mathbb{Z}$.

(14) The ring $\{0, 2, 4, 6, 8\}$ under addition and multiplication modulo 10 has a unity (multiplicative identity). Find it.

(15) Let R be a ring with unity 1. Show that $S = \{n.1 : n \in \mathbb{Z}\}$ is a sub- ring of R.

(16) Suppose that a belongs to a ring and $a^4 = a^2$. Prove that $a^{2n} = a^2$ for all $n \geq 1$.

(17) Let F be a finite field with n elements. Prove that $x^{n-1} = 1$ for all nonzero x in F, 1 denotes the multiplicative identity of F.