Filtering in the Frequency Domain (Fundamentals)

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Image and Video Processing

Acknowledgements

- Gonzalez, Rafael C. Digital image processing. Pearson, 4th edition, 2018.
- Jain, Anil K. Fundamentals of digital image processing. Prentice-Hall, Inc., 1989.
- Digital Image Processing course by Brian Mac Namee, Dublin Institute of Technology
- Digital Image Processing course by Christophoros Nikou, University of Ioannina

Filtering in the Frequency Domain

Filter: A device or material for suppressing or minimizing waves or oscillations of certain frequencies.

Frequency: The number of times that a periodic function repeats the same sequence of values during a unit variation of the independent variable.

Webster's New Collegiate Dictionary

Jean Baptiste Joseph Fourier

Source: Google Images

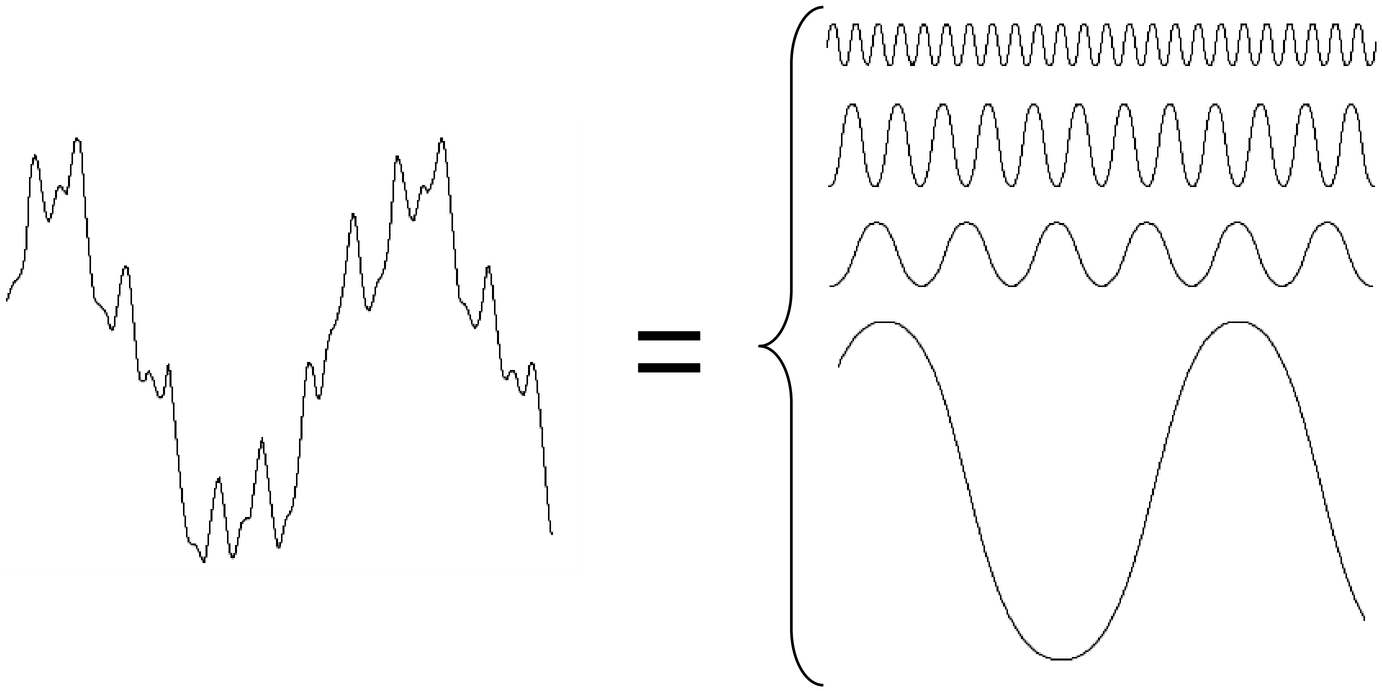
Fourier was born in Auxerre, France in 1768.

- Most famous for his work "*La Théorie Analitique de la Chaleur*" published in 1822.
- Translated into English in 1878: "*The Analytic Theory of Heat*".

Nobody paid much attention when the work was first published.

One of the most important mathematical theories in modern engineering.

The Big Idea



Any function that periodically repeats itself can be expressed as a sum of sines and cosines of different frequencies each multiplied by a different coefficient – a *Fourier series*

Adapted from Gonzalez, Rafael C. Digital image processing. Pearson education India, 2009.

Fourier Series and Fourier Transform:

- Fourier Series

  Any periodic function can be expressed as the sum of sines and /or cosines of different frequencies, each multiplied by a different coefficients

- Fourier Transform

  Any function that is not periodic can be expressed as the integral of sines and /or cosines multiplied by a weighing function

Preliminary Concepts

$j = \sqrt{-1},$ a complex number

$$C = R + jI$$

the conjugate

$$C *= R - jI$$

$|C| = \sqrt{R^2 + I^2}$ and $\theta = \arctan(\, I/R)$

$$C = |C|(\cos\theta + j\sin\theta)$$

Using Euler's formula,

$$C = |C|e^{j\theta}$$

Fourier Series

A function $f(t)$ of a continuous variable t that is periodic with period, T, can be expressed as the sum of sines and cosines multiplied by appropriate coefficients.

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t}$$

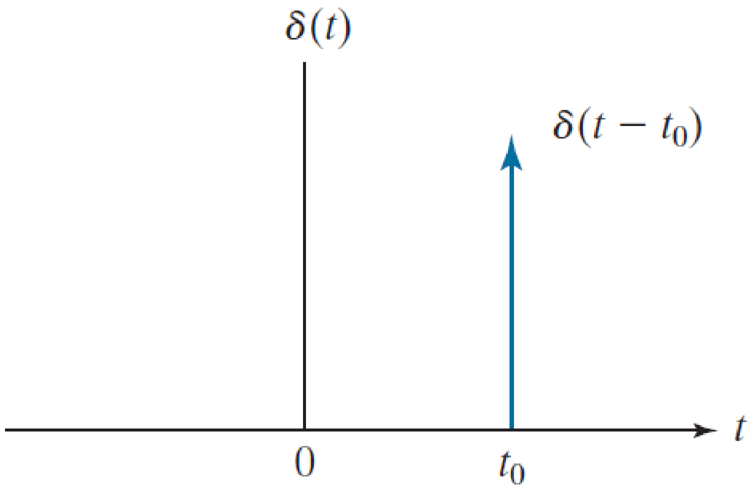
where

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-j\frac{2\pi n}{T}t} dt \qquad \text{for } n = 0, \pm 1, \pm 2, \ldots$$

Impulses and the Sifting Property

A *unit impulse* of a continuous variable t located at t=0, denoted $\delta(t)$, defined as

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$



and is constrained also to satisfy the identity

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

The *sifting property*

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$$

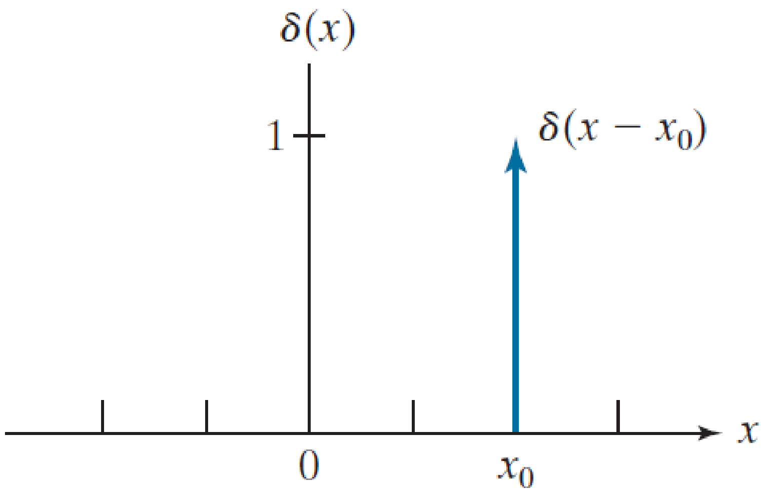
What about $\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt$?

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = f(t_0)$$

Impulses and the Sifting Property (contd.)

A *unit impulse* of a discrete variable *x* located at *x*=0, denoted $\delta(x)$, defined as

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$



and is constrained also to satisfy the identity

$$\sum_{x=-\infty}^{\infty} \delta(x) = 1$$

The *sifting property*

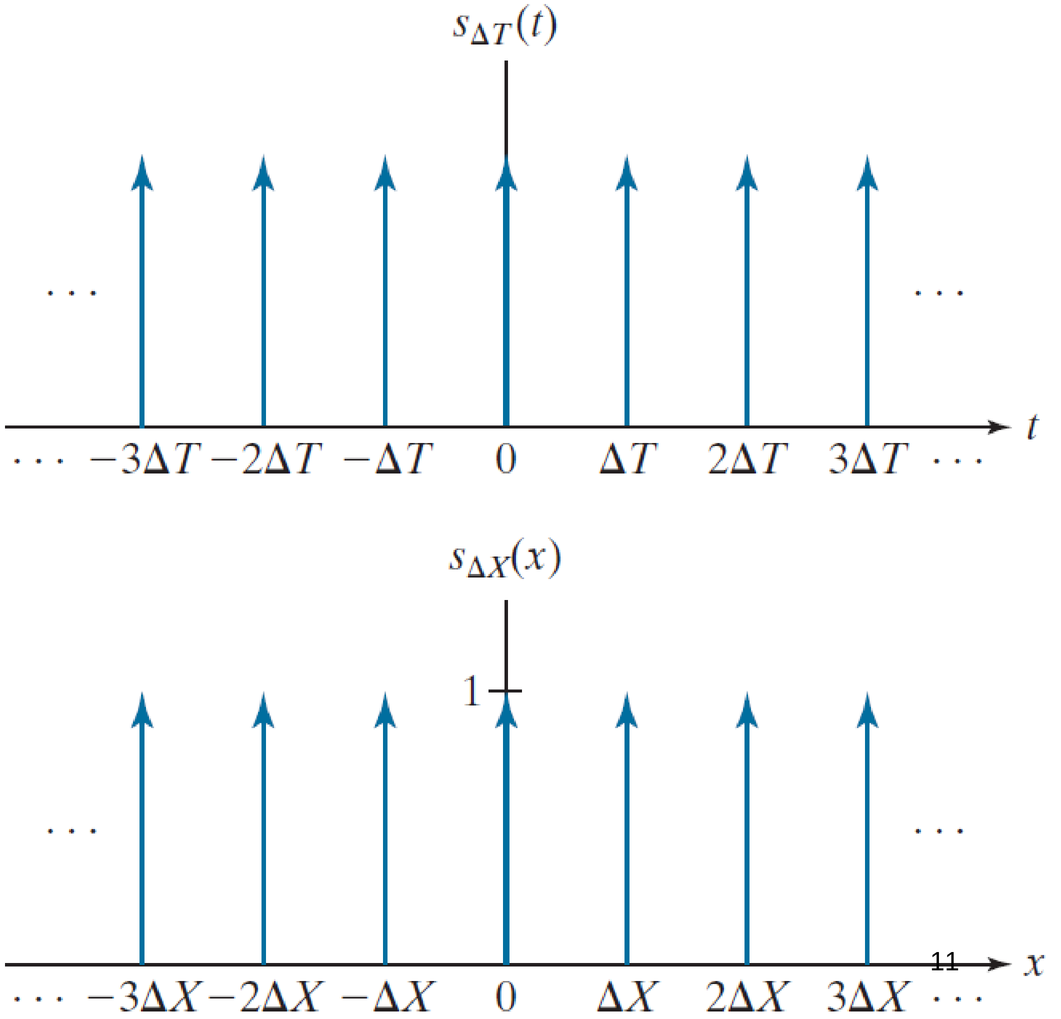
$$\sum_{x=-\infty}^{\infty} f(x)\delta(x) = f(0)$$

$$\sum_{x=-\infty}^{\infty} f(x)\delta(x - x_0) = f(x_0)$$

Impulses and the Sifting Property (contd.)

$$impulse\ train\ s_{\Delta T}(t),$$

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$$



Fourier Transform: One Continuous Variable

The *Fourier Transform* of a continuous function $f(t)$

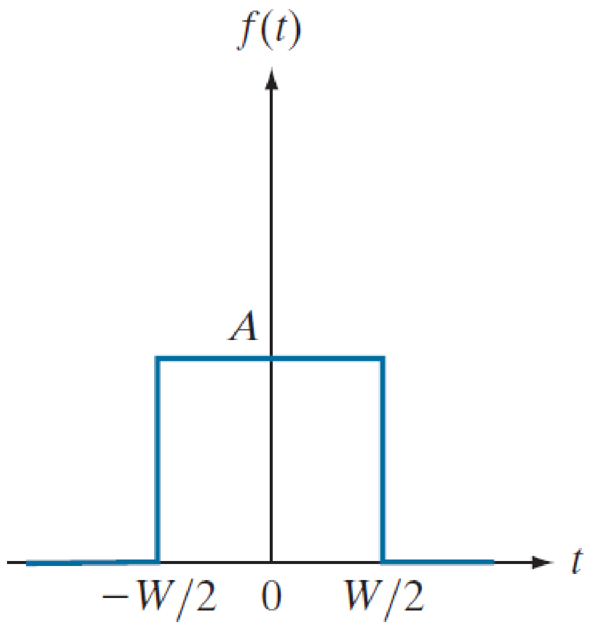
$$F(\mu) = \Im\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-j2\pi\mu t}dt$$

The *Inverse Fourier Transform* of $F(\mu)$

$$f(t) = \Im^{-1}\{F(\mu)\} = \int_{-\infty}^{\infty} F(\mu)e^{j2\pi\mu t}d\mu$$

Fourier Transform: One Continuous Variable (contd.)

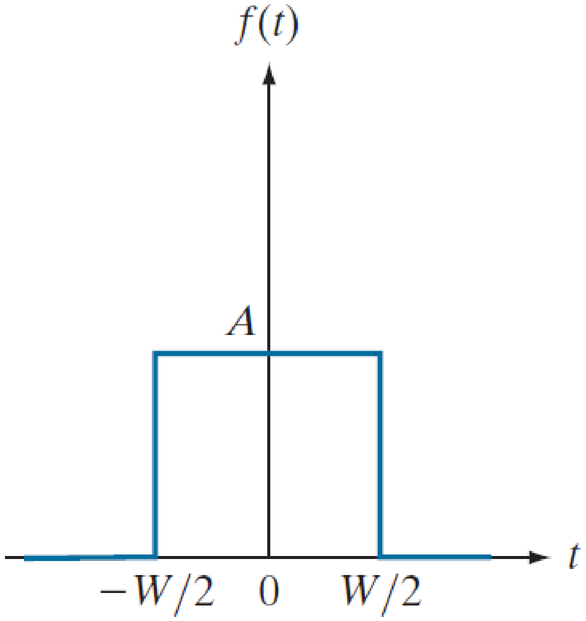
Prove that the Fourier Transform of the box function is



$$AW\frac{\sin(\pi\mu W)}{(\pi\mu W)}$$

Fourier Transform: One Continuous Variable (contd.)

Fourier Transform of the box function



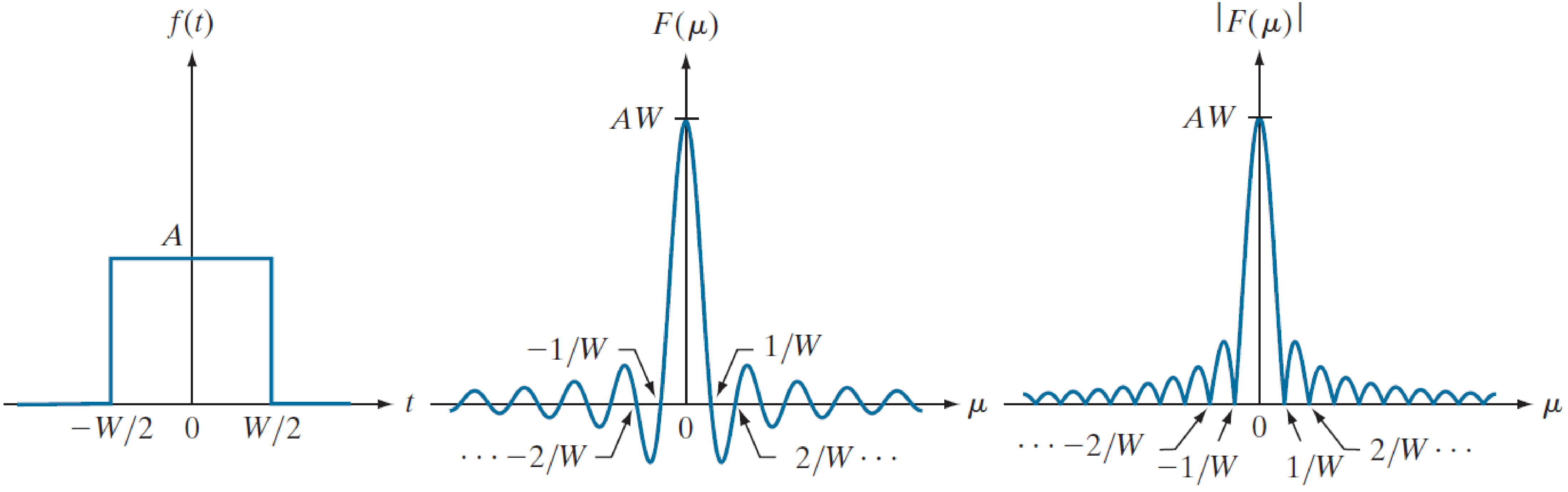
$$F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt = \int_{-W/2}^{W/2} A e^{-j2\pi\mu t} dt$$

$$= \frac{-A}{j2\pi\mu} \left[e^{-j2\pi\mu t} \right]_{-W/2}^{W/2} = \frac{-A}{j2\pi\mu} \left[e^{-j\pi\mu W} - e^{j\pi\mu W} \right]$$

$$= \frac{A}{j2\pi\mu} \left[e^{j\pi\mu W} - e^{-j\pi\mu W} \right]$$

$$= AW \frac{\sin(\pi\mu W)}{(\pi\mu W)}$$

Fourier Transform: One Continuous Variable (contd.)



a b c

FIGURE 4.4 (a) A box function, (b) its Fourier transform, and (c) its spectrum. All functions extend to infinity in both directions. Note the inverse relationship between the width, W, of the function and the zeros of the transform.

Adapted from Gonzalez, Rafael C. Digital image processing. Pearson education India, 2009.

Fourier Transform: One Continuous Variable (contd.)

Fourier Transform of an impulse located at the origin

$$\Im\{\delta(t)\} = F(\mu) = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi\mu t} dt$$

$$= e^{-j2\pi\mu 0}$$

$$=1$$

Fourier Transform: One Continuous Variable (contd.)

Fourier Transform of an impulse located at $t = t_0$

$$\Im\{\delta(t - t_0)\} = F(\mu) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j2\pi\mu t} dt$$

$$= \int_{-\infty}^{\infty} e^{-j2\pi\mu t} \delta(t - t_0) dt = e^{-j2\pi\mu t_0}$$

Fourier Transform: One Continuous Variable (contd.)

The Fourier series expansion of a periodic signal $f(t)$.

$$f(t) = \sum_{n=-\infty}^{+\infty} c_n e^{j\frac{2\pi}{T}nt}$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-j\frac{2\pi}{T}nt}\,dt$$

Fourier Transform: One Continuous Variable (contd.)

The impulse train $s_{\Delta T}$ is periodic with period ΔT can be expressed as a Fourier series

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{\Delta T}t}$$

$$c_n = \frac{1}{\Delta T}\int_{-\Delta T/2}^{\Delta T/2} s_{\Delta T}(t) e^{-j\frac{2\pi n}{\Delta T}t} dt$$

Fourier Transform: One Continuous Variable (contd.)

- Intermediate result
  - The Fourier transform of the impulse train.

$$\sum_{n=-\infty}^{+\infty} \delta(t - n\Delta T) \leftrightarrow \frac{1}{\Delta T} \sum_{n=-\infty}^{+\infty} \delta\left(\mu - \frac{n}{\Delta T}\right)$$

- It is also an impulse train in the frequency domain.
- Impulses are equally spaced every $1/\Delta T$.

Fourier Transform: One Continuous Variable (contd.)

$$c_n = \frac{1}{\Delta T}\int_{-\Delta T/2}^{\Delta T/2} \delta(t) e^{-j\frac{2\pi n}{\Delta T}t} dt = \frac{1}{\Delta T} e^{0} = \frac{1}{\Delta T}$$

$$s_{\Delta T}(t) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}$$

$$\Im\left\{e^{j\frac{2\pi n}{\Delta T}t}\right\} = \delta\left(\mu - \frac{n}{\Delta T}\right)$$

$$S(\mu) = \Im\{s_{\Delta T}(t)\} = \Im\left\{\frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}\right\} = \frac{1}{\Delta T}\Im\left\{\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}\right\} = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty}\delta\left(\mu - \frac{n}{\Delta T}\right)$$

Fourier Transform and Convolution

- Convolution property of the FT.

$$f(t) * h(t) = \int_{-\infty}^{+\infty} f(\tau)h(t-\tau)d\tau$$

$$f(t) * h(t) \leftrightarrow F(\mu)H(\mu)$$

$$f(t)h(t) \leftrightarrow F(\mu) * H(\mu)$$

Fourier Transform and Convolution

$$(f \star h)(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,d\tau$$

$$\begin{aligned}\Im\{(f \star h)(t)\} &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,d\tau\right]e^{-j2\pi\mu t}dt \\ &= \int_{-\infty}^{\infty} f(\tau)\left[\int_{-\infty}^{\infty} h(t-\tau)\,e^{-j2\pi\mu t}dt\right]d\tau\end{aligned}$$

$$\begin{aligned}\Im\{(f \star h)(t)\} &= \int_{-\infty}^{\infty} f(\tau)\left[H(\mu)e^{-j2\pi\mu\tau}\right]d\tau \\ &= H(\mu)\int_{-\infty}^{\infty} f(\tau)e^{-j2\pi\mu\tau}\,d\tau \\ &= H(\mu)F(\mu) \\ &= (H \bullet F)(\mu)\end{aligned}$$

$$(f \star h)(t) \Leftrightarrow (H \bullet F)(\mu)$$

Similarly $\quad (f \bullet h)(t) \Leftrightarrow (H \star F)(\mu)$

Sampling and Fourier Transform of Sampled Functions

- Consider a continuous function, *f(t)*, that we wish to sample at uniform intervals, ΔT, of the independent variable *t.*
- We assume initially that the function extends from $-\infty$ to ∞ with respect to *t*.
- One way to model sampling is to multiply *f(t)* by a sampling function equal to a train of impulses ΔT units apart.

$$\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)$$

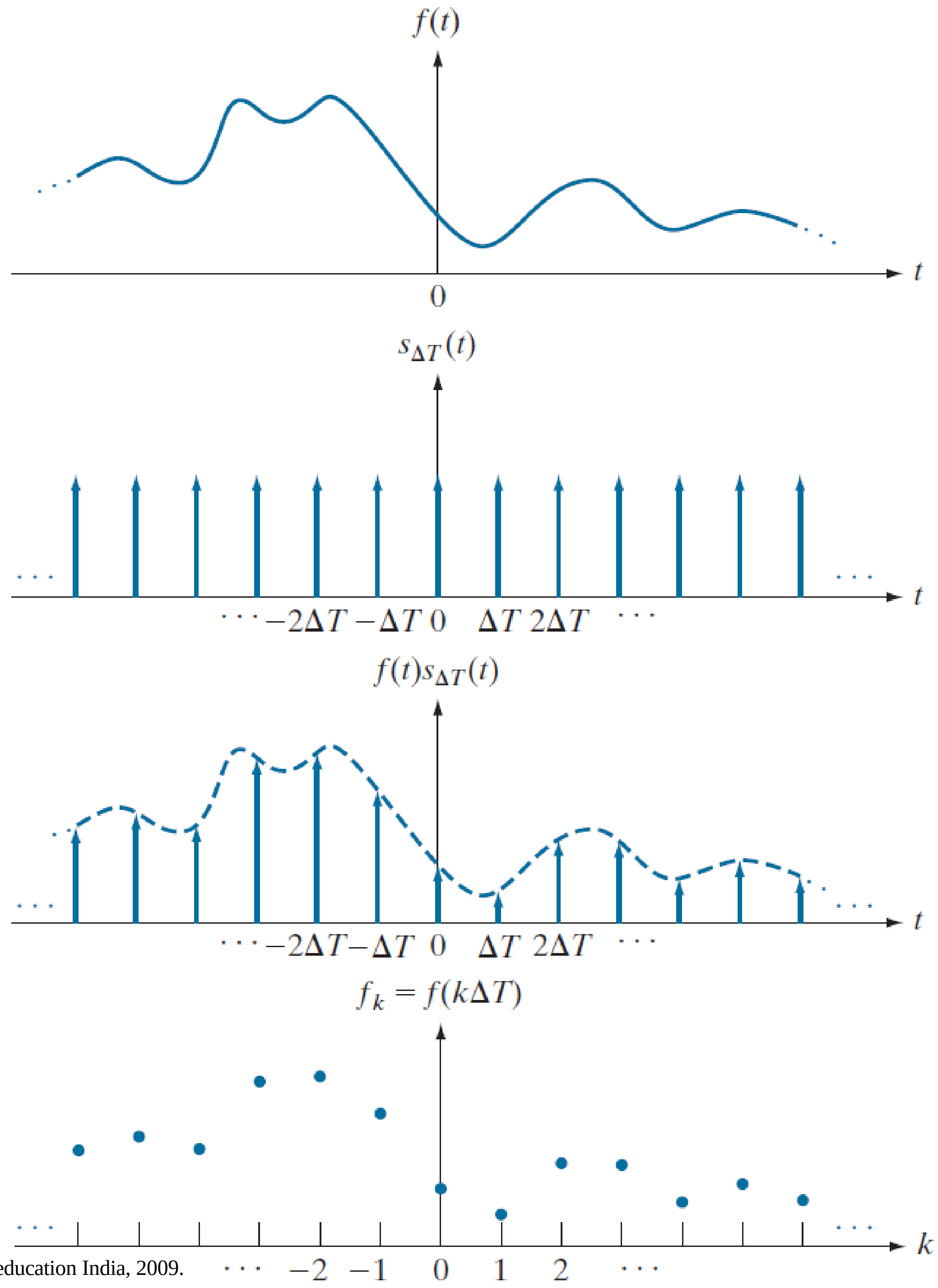


FIGURE 4.5
(a) A continuous function. (b) Train of impulses used to model sampling. (c) Sampled function formed as the product of (a) and (b). (d) Sample values obtained by integration and using the sifting property of impulses. (The dashed line in (c) is shown for reference. It is not part of the data.)

$$f_k = \int_{-\infty}^{\infty} f(t)\delta(t - k\Delta T)\,dt$$
$$= f(k\Delta T)$$

Adapted from Gonzalez, Rafael C. Digital image processing. Pearson education India, 2009.

Sampling and Fourier Transform of Sampled Functions

- Sampling
  - The spectrum of the discrete signal consists of repetitions of the spectrum of the continuous signal every $1/\Delta T$.
  - The Nyquist criterion should be satisfied.

$$\tilde{F}(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{+\infty} F\left(\mu - \frac{n}{\Delta T}\right)$$

Sampling and Fourier Transform of Sampled Functions

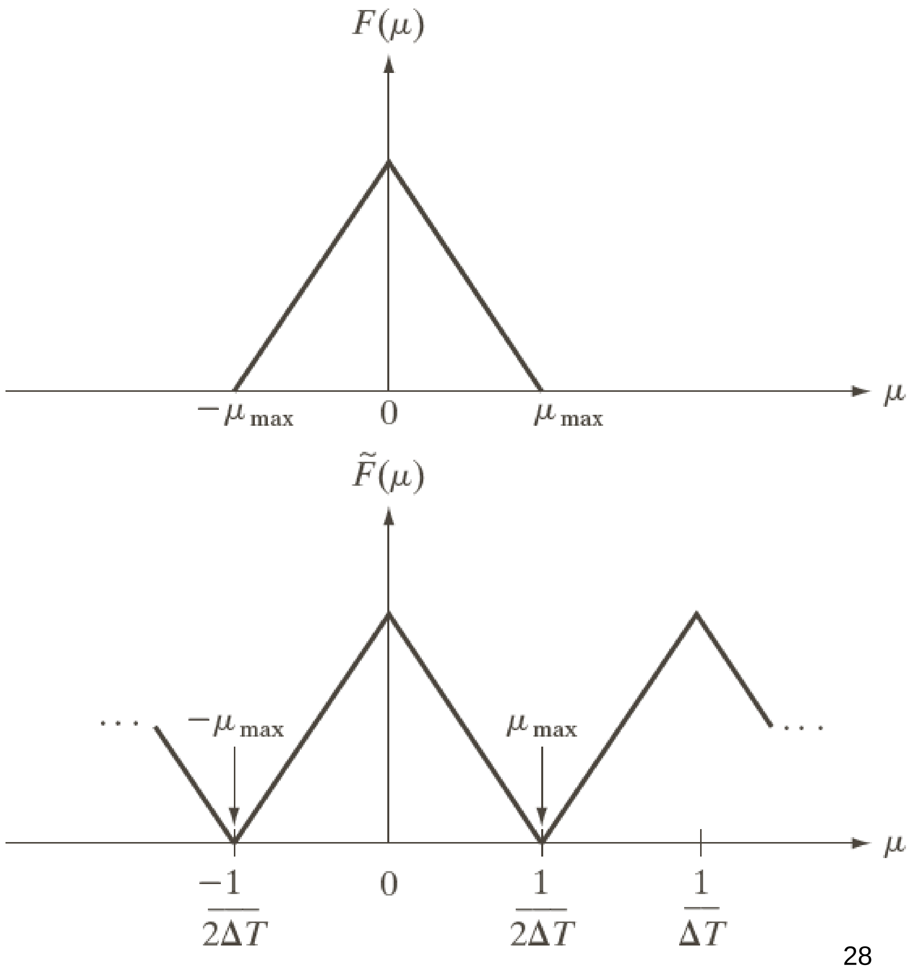
$$\tilde{F}(\mu) = \Im\left\{\tilde{f}(t)\right\} = \Im\left\{f(t)s_{\Delta T}(t)\right\}$$
$$= (F \star S)(\mu)$$

$$S(\mu) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty}\delta\left(\mu - \frac{n}{\Delta T}\right)$$

$$\tilde{F}(\mu) = (F \star S)(\mu) = \int_{-\infty}^{\infty} F(\tau)S(\mu-\tau)\,d\tau$$
$$= \frac{1}{\Delta T}\int_{-\infty}^{\infty} F(\tau)\sum_{n=-\infty}^{\infty}\delta\left(\mu-\tau-\frac{n}{\Delta T}\right)d\tau$$
$$= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} F(\tau)\delta\left(\mu-\tau-\frac{n}{\Delta T}\right)d\tau$$
$$= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} F\left(\mu-\frac{n}{\Delta T}\right)$$

Sampling Theorem

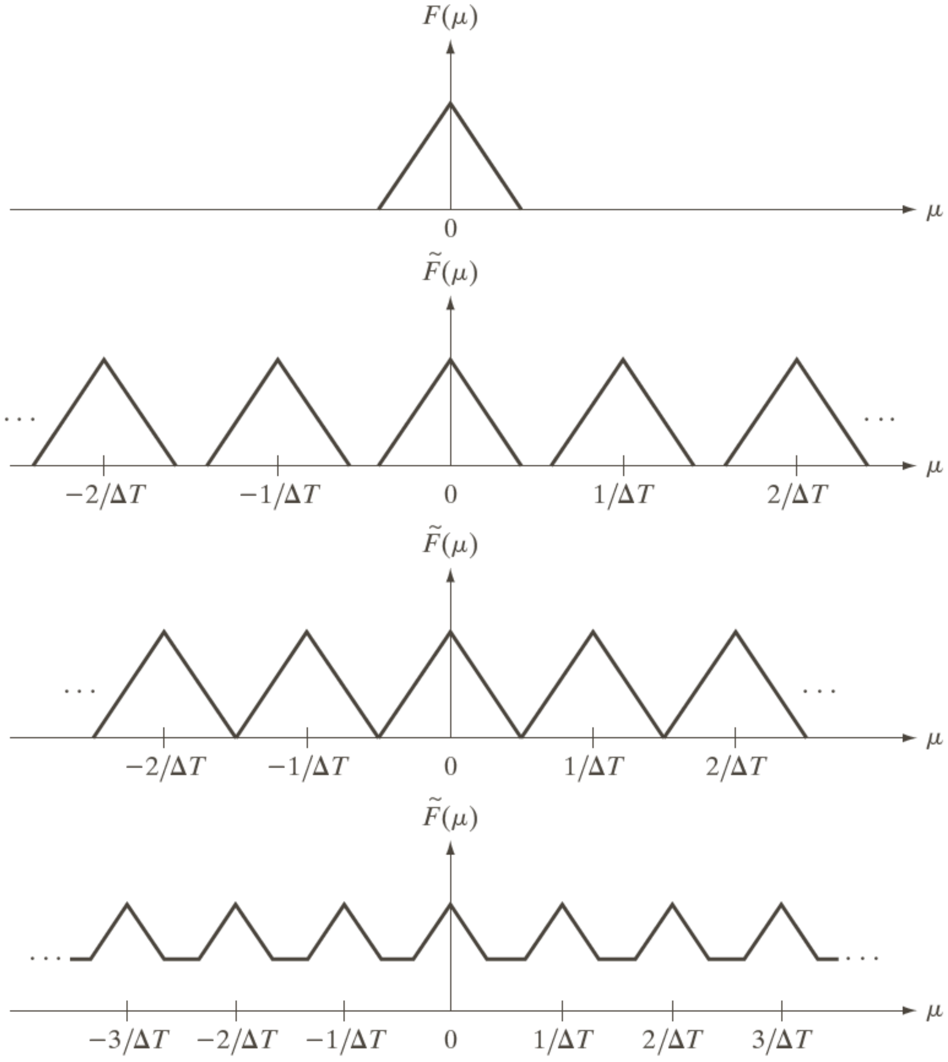
A function *f(t)* whose Fourier transform is zero for values of frequencies outside a finite interval (band) $[-\mu_{\max}, \mu_{\max}]$ about the origin is called a *band-limited* function.



Nyquist theorem

$$\frac{1}{\Delta T} \geq 2\mu_{\max}$$

Sampling Theorem (cont.)



FT of a continuous signal

Oversampling

$$\frac{1}{\Delta T} > 2\mu_{max}$$

Critical sampling with the Nyquist frequency

$$\frac{1}{\Delta T} = 2\mu_{max}$$

Undersampling
Aliasing appears

$$\frac{1}{\Delta T} < 2\mu_{max}$$

Sampling Theorem (cont.)

- **Nyquist-Shannon Sampling Theorem:** A continuous, band-limited function can be recovered completely from a set of its samples if the samples are acquired at a rate exceeding twice the highest frequency content of the function.

Sampling Theorem (cont.)

- Reconstruction (under correct sampling).

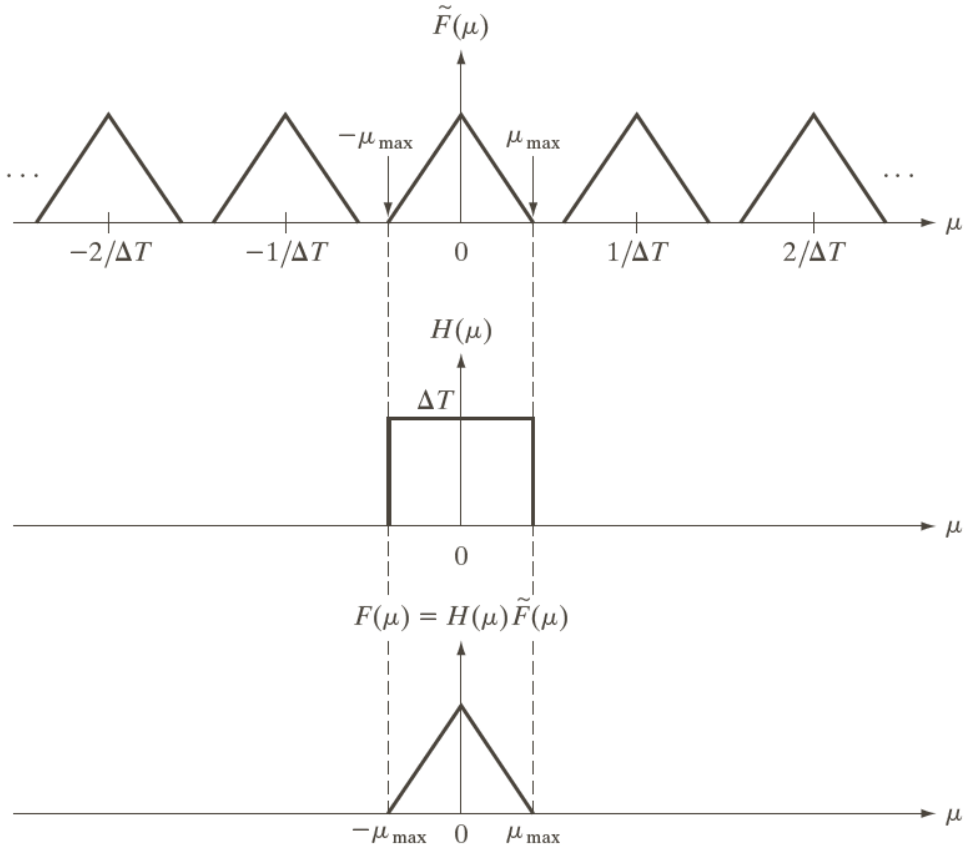
$$H(\mu) = \begin{cases} \Delta T & -\mu_{\max} \le \mu \le \mu_{\max} \\ 0 & \text{otherwise} \end{cases}$$

$$F(\mu) = \tilde{F}(\mu)H(\mu)$$

$$f(t) = \tilde{f}(t) * T\,\mathrm{sin}\,c\left(\frac{t}{\Delta T}\right)$$

$$\mathrm{sinc}(m) = \frac{\sin(\pi m)}{(\pi m)}$$

where sinc (0) = 1 and sinc (*m*) = 0 for all other *integer* values of *m*.



Sampling Theorem (cont.)

- Reconstruction
  - Provided a correct sampling, the continuous signal may be perfectly reconstructed by its samples.

$$f(t) = \sum_{n=-\infty}^{+\infty} f(n\Delta T)\mathrm{sinc}\left[\frac{(t - n\Delta T)}{n\Delta T}\right]$$

Aliasing

- Alias means “a false identity.”
- Aliasing refers to sampling phenomena that cause different signals to become indistinguishable from one another after sampling; or, viewed another way, for one signal to “masquerade” as another.
- The foundation of aliasing phenomena, as it relates to sampling, is that we can describe a digitized function only by the values of its samples.
- This means that it is possible for two (or more) totally different continuous functions to coincide at the values of their respective samples. Still, we would have no way of knowing the characteristics of the functions between those samples.
- Two continuous functions having the characteristics just described are called an aliased pair, and such pairs are indistinguishable after sampling.

a b
c d

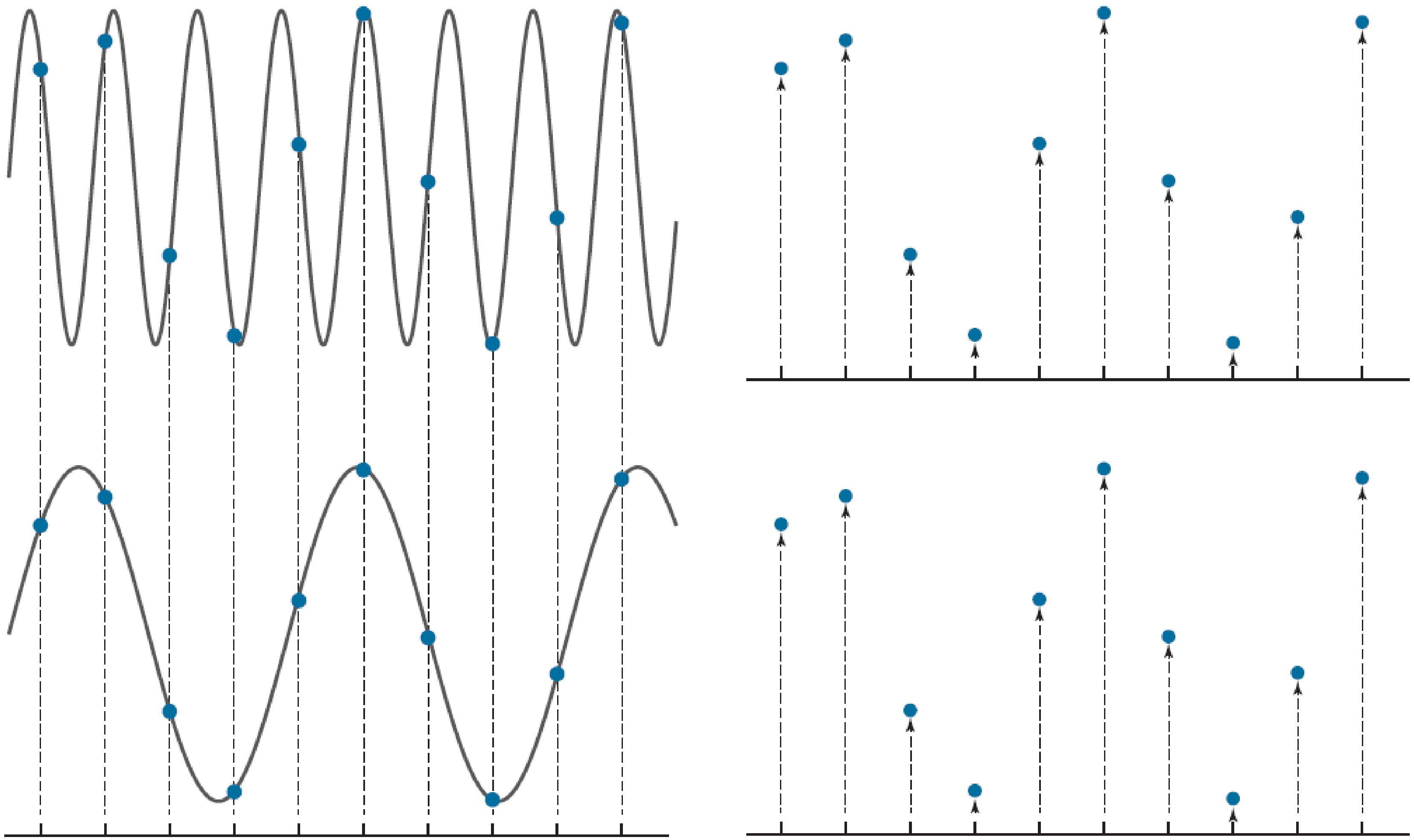


FIGURE 4.9 The functions in (a) and (c) are totally different, but their digitized versions in (b) and (d) are identical. Aliasing occurs when the samples of two or more functions coincide, but the functions are different elsewhere.

- If a band-limited function is sampled at a rate that is less than twice its highest frequency?
- The inverse transform will yield a corrupted function, This effect is known as **frequency aliasing** or simply **aliasing**.

Adapted from Gonzalez, Rafael C. Digital image processing. Pearson education India, 2009.

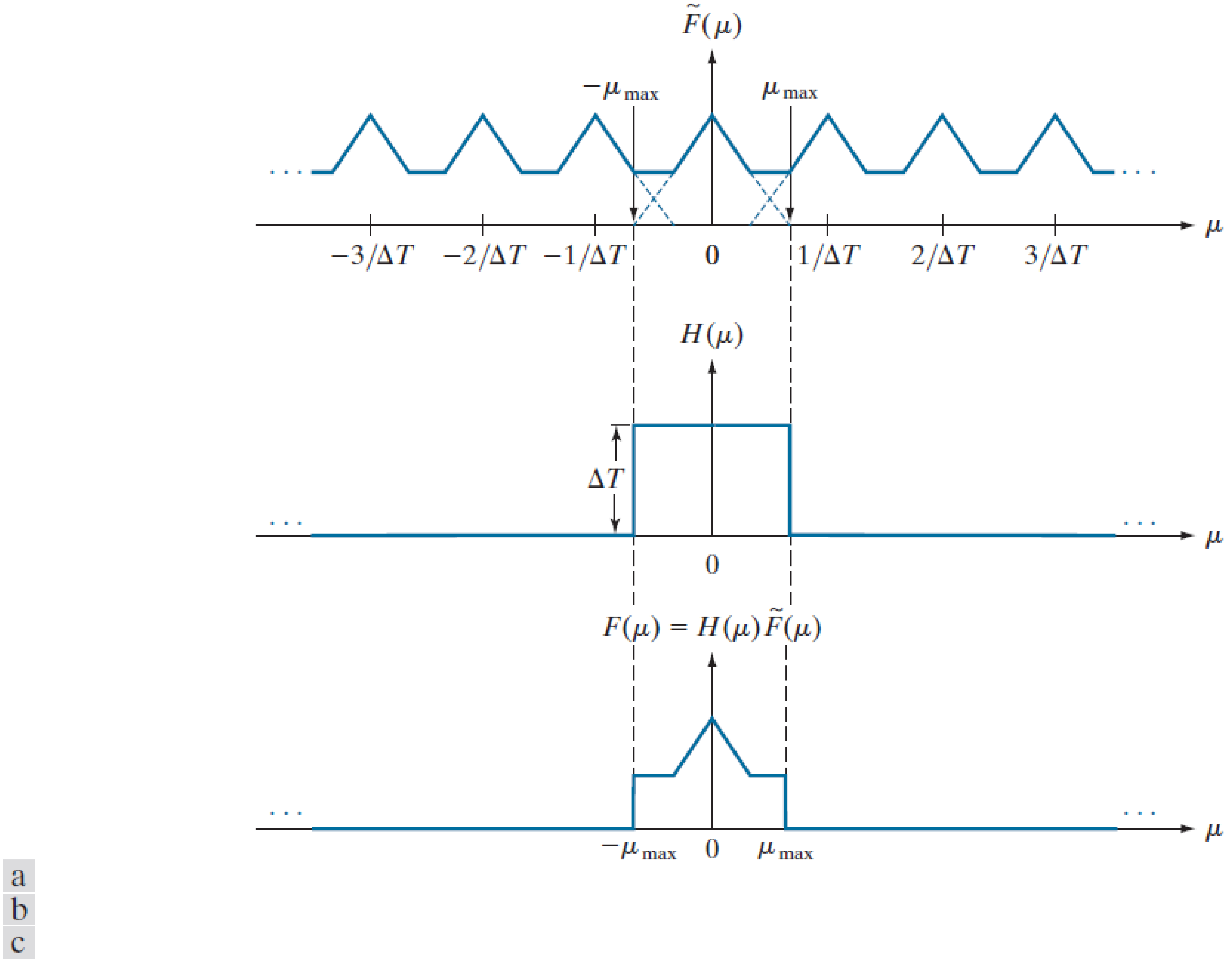


FIGURE 4.10 (a) Fourier transform of an under-sampled, band-limited function. (Interference between adjacent periods is shown dashed). (b) The same ideal lowpass filter used in Fig. 4.8. (c) The product of (a) and (b). The interference from adjacent periods results in aliasing that prevents perfect recovery of $F(\mu)$ and, consequently, of $f(t)$.

Adapted from Gonzalez, Rafael C. Digital image processing. Pearson education India, 2009.

Aliasing (contd.)

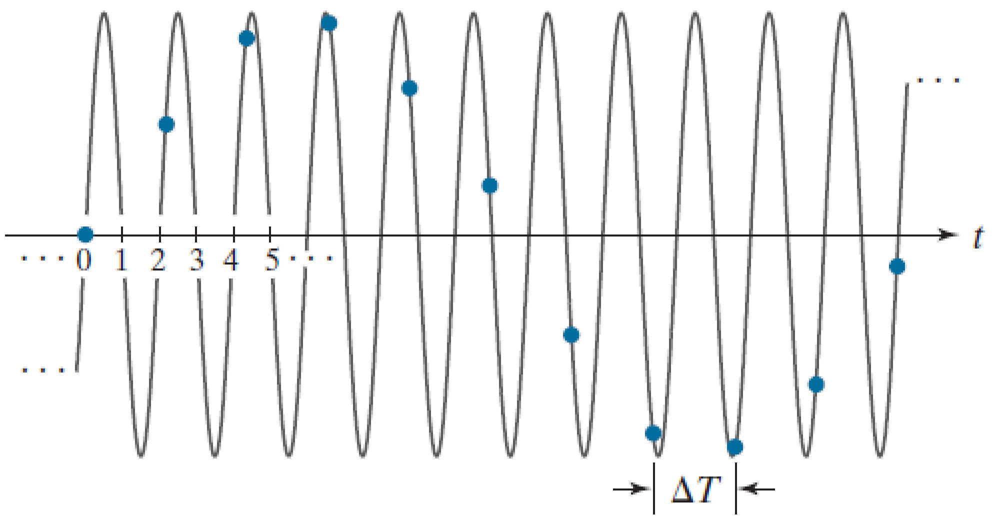


FIGURE 4.11 Illustration of aliasing. The under-sampled function (dots) looks like a sine wave having a frequency much lower than the frequency of the continuous signal. The period of the sine wave is 2 s, so the zero crossings of the horizontal axis occur every second. ΔT is the separation between samples.

Discrete Fourier Transform: One Variable

$$F(\mu) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi\mu x/M}, \qquad \mu = 0, 1, \ldots, M-1$$

$$f(x) = \frac{1}{M} \sum_{\mu=0}^{M-1} F(\mu) e^{j2\pi\mu x/M}, \qquad x = 0, 1, 2, \ldots, M-1$$

Discrete Fourier Transform: One Variable (contd.)

Both the forward and inverse discrete transforms are infinitely periodic, with period *M*.

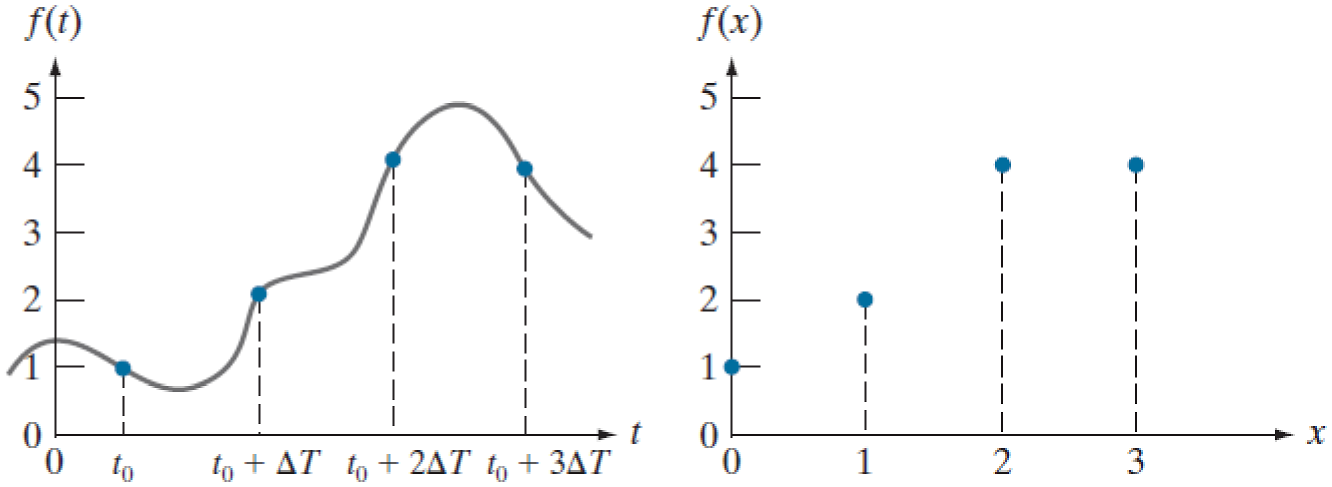
$$F(u) = F(u + kM)$$

$$f(x) = f(x + kM)$$

The discrete equivalent of the 1-D convolution is

$$f(x) \star h(x) = \sum_{m=0}^{M-1} f(m)h(x-m) \quad x = 0, 1, 2, \ldots, M-1$$

Example



a b

FIGURE 4.12 (a) A continuous function sampled ΔT units apart. (b) Samples in the x-domain. Variable t is continuous, while x is discrete.

Compute $F(0)$

$$F(0) = \sum_{x=0}^{3} f(x) = \left[f(0) + f(1) + f(2) + f(3) \right] = 1 + 2 + 4 + 4 = 11$$

Compute $F(1)$

$$F(1) = \sum_{x=0}^{3} f(x) e^{-j2\pi(1)x/4} = 1e^{0} + 2e^{-j\pi/2} + 4e^{-j\pi} + 4e^{-j3\pi/2} = -3 + 2j$$

Compute $F(2)$ and $F(3)$

$$F(2) = -(1 + 0j) \text{ and } F(3) = -(3 + 2j)$$

Compute $f(0)$

$$f(0) = \frac{1}{4} \sum_{u=0}^{3} F(u) e^{j2\pi u(0)} = \frac{1}{4} \sum_{u=0}^{3} F(u) = \frac{1}{4}[11 - 3 + 2j - 1 - 3 - 2j] = \frac{1}{4}[4] = 1$$

Adapted from Gonzalez, Rafael C. Digital image processing. Pearson education India, 2009.

2-D Impulse and Sifting Property: Continuous

The impulse, $\delta(t,z)$, of two continuous variables, *t* and *z*, is defined as before

$$\delta(t,z) = \begin{cases} \infty & \text{if } t = z = 0 \\ 0 & \text{otherwise} \end{cases}$$

and is constrained also to satisfy the identity

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(t,z)\,dt\,dz = 1$$

Sifting property

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)\delta(t,z)\,dt\,dz = f(0,0)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)\delta(t-t_0,z-z_0)\,dt\,dz = f(t_0,z_0)$$

2-D Impulse and Sifting Property: Discrete

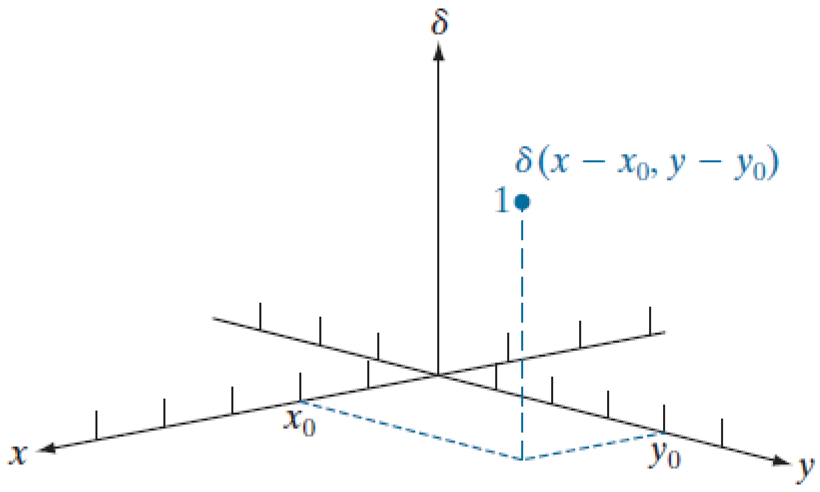
The impulse, δ(x,y), of two discrete variables, *x* and y, is defined as before

$$\delta(x,y)=\begin{cases}1 & \text{if } x=y=0\\ 0 & \text{otherwise}\end{cases}$$

Sifting property

$$\sum_{x=-\infty}^{\infty}\sum_{y=-\infty}^{\infty} f(x,y)\delta(x,y)=f(0,0)$$

$$\sum_{x=-\infty}^{\infty}\sum_{y=-\infty}^{\infty} f(x,y)\delta(x-x_0,y-y_0)=f(x_0,y_0)$$



2-D Fourier Transform: Continuous

$$F(\mu,\nu)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)e^{-j2\pi(\mu t+\nu z)}dtdz$$

and

$$f(t,z)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\mu,\nu)e^{j2\pi(\mu t+\nu z)}d\mu d\nu$$

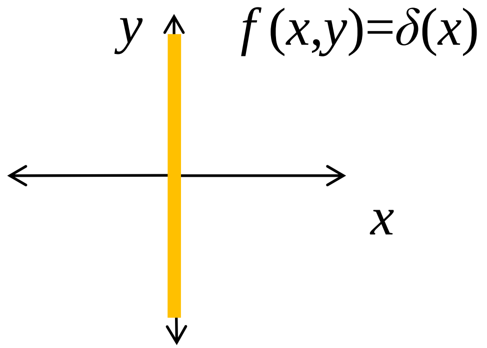
2D continuous signals

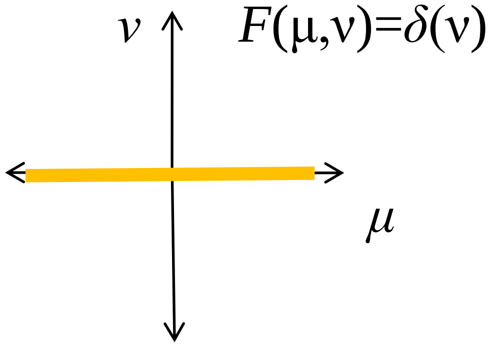
- Example: FT of $f(x,y)=\delta(x)$

$$F(\mu,\nu) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \delta(x) e^{-j2\pi(\mu x+\nu y)} dy dx$$

$$= \int_{-\infty}^{+\infty} \delta(x) e^{-j2\pi\mu x} dx \int_{-\infty}^{+\infty} e^{-j2\pi\nu y} dy$$

$$= \int_{-\infty}^{+\infty} e^{-j2\pi\nu y} dy \qquad = \delta(\nu)$$





2D continuous signals (cont.)

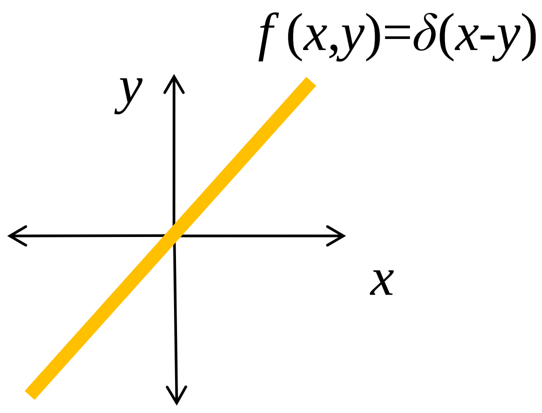
- Example: FT of $f(x,y)=\delta(x-y)$

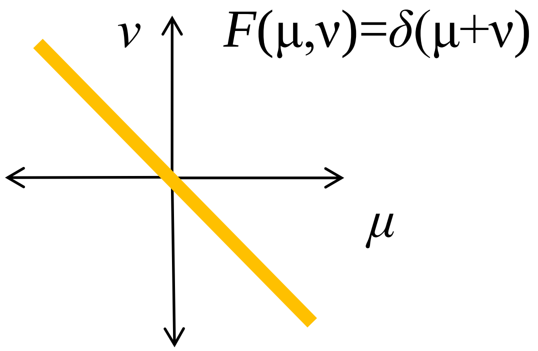
$$F(\mu,\nu)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\delta(x-y)e^{-j2\pi(\mu x+\nu y)}dydx$$

$$=\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}\delta(x-y)e^{-j2\pi\mu x}dx\right]e^{-j2\pi\nu y}dy$$

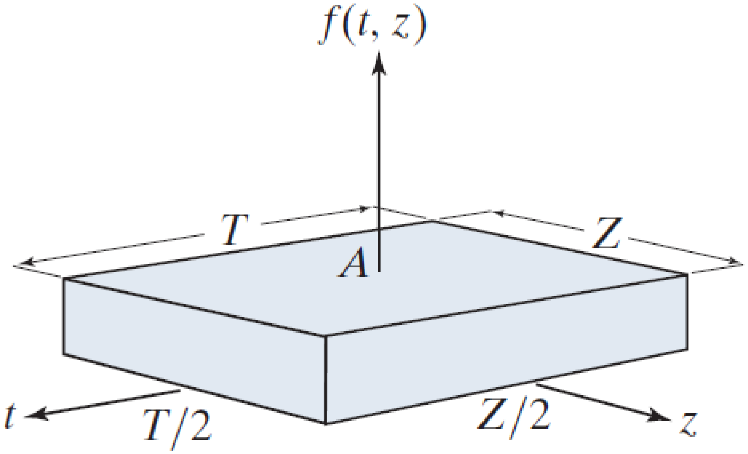
$$=\int_{-\infty}^{+\infty}e^{-j2\pi\mu y}e^{-j2\pi\nu y}dy \qquad =\int_{-\infty}^{+\infty}e^{-j2\pi(\mu+\nu)y}dy$$

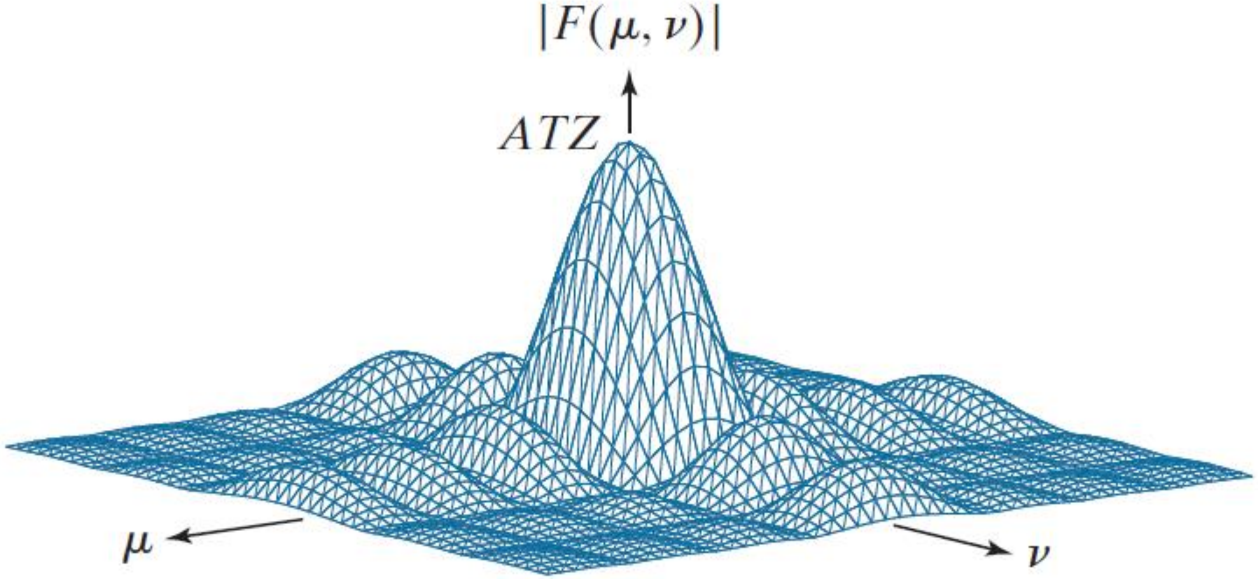
$$=\delta(\mu+\nu)$$





Fourier Transform of a 2-D Box Function



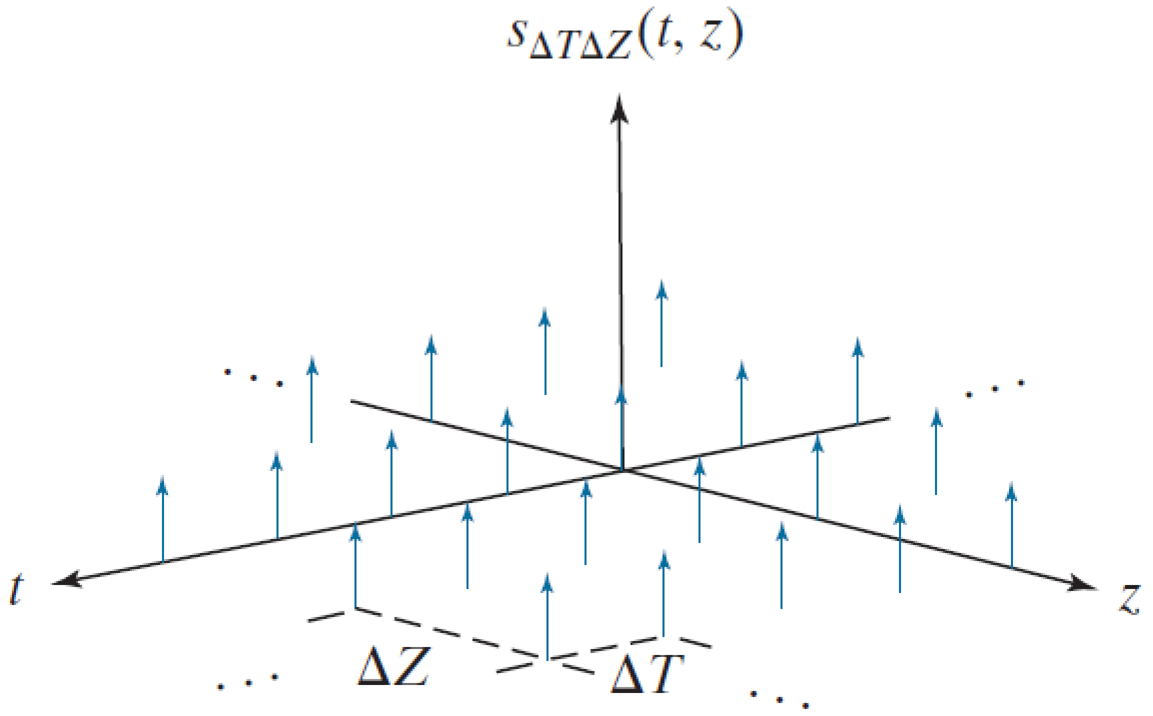


$$
\begin{aligned}
F(\mu,\nu) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)e^{-j2\pi(\mu t+\nu z)}dtdz \\
&= \int_{-T/2}^{T/2}\int_{-Z/2}^{Z/2} Ae^{-j2\pi(\mu t+\nu z)}dtdz \\
&= ATZ\left[\frac{\sin(\pi\mu T)}{\pi\mu T}\right]\left[\frac{\sin(\pi\nu T)}{\pi\nu T}\right]
\end{aligned}
$$

2-D Sampling and 2-D Sampling Theorem

2-D impulse train

$$s_{\Delta T \Delta Z}(t,z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(t - m\Delta T, z - n\Delta Z)$$



2-D Sampling and 2-D Sampling Theorem: (contd.)

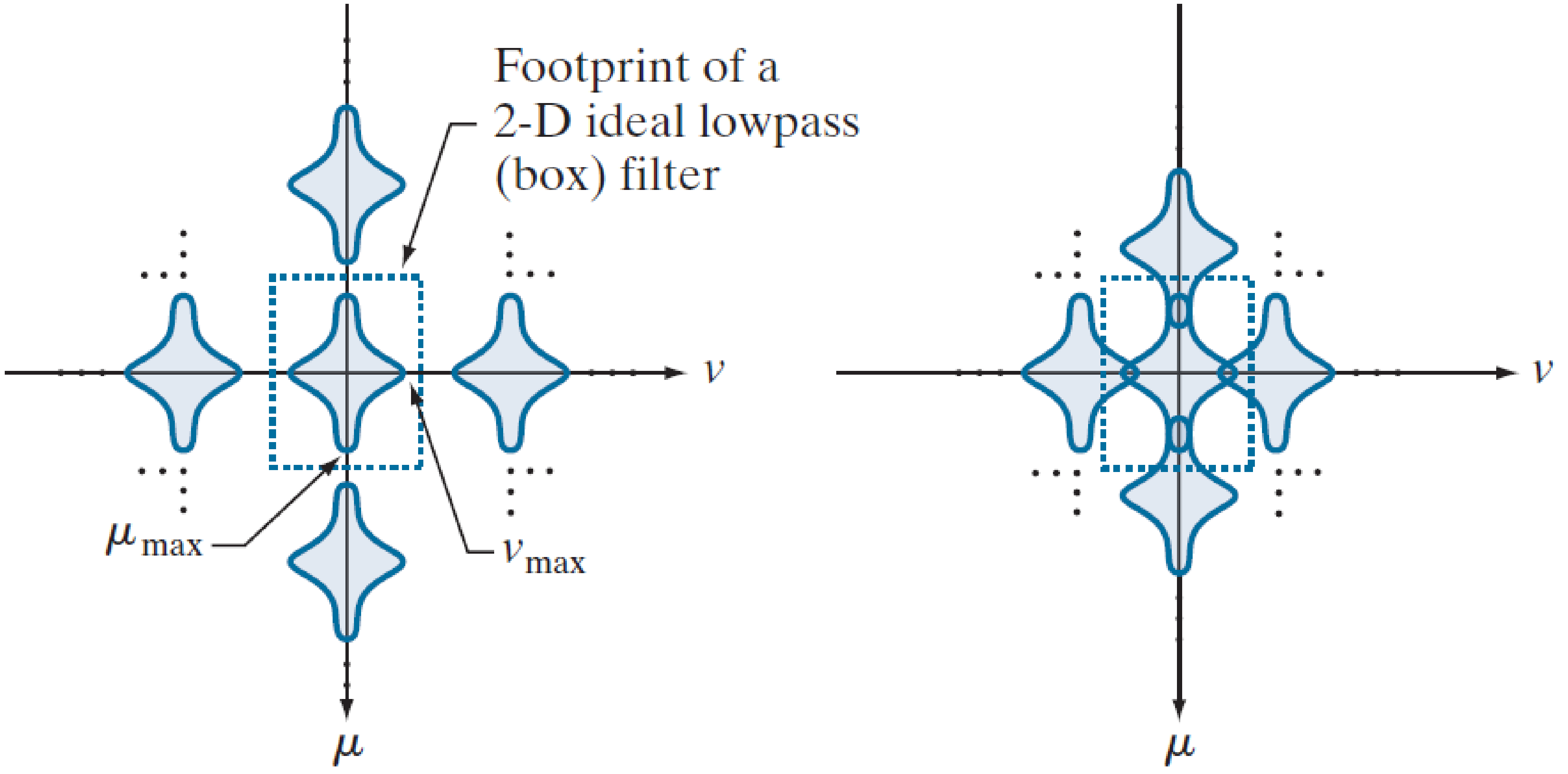
A function *f(t,z)* is said to be *band-limited* if its Fourier transform is zero outside a rectangle established by the intervals $[-\mu_{max}, \mu_{max}]$ and $[-\nu_{max}, \nu_{max}]$, that is

$$F(\mu,\nu)=0 \text{ for } |\mu| \geq \mu_{max} \text{ and } |\nu| \geq \nu_{max}$$

2-D Sampling Theorem: A continuous, band-limited function *f(t,z)* can be recovered with no error from a set of its samples if the sampling intervals are

$$\Delta T < \frac{1}{2\mu_{max}} \text{ and } \Delta Z < \frac{1}{2\nu_{max}}$$

2-D Sampling and 2-D Sampling Theorem: (contd.)



a b

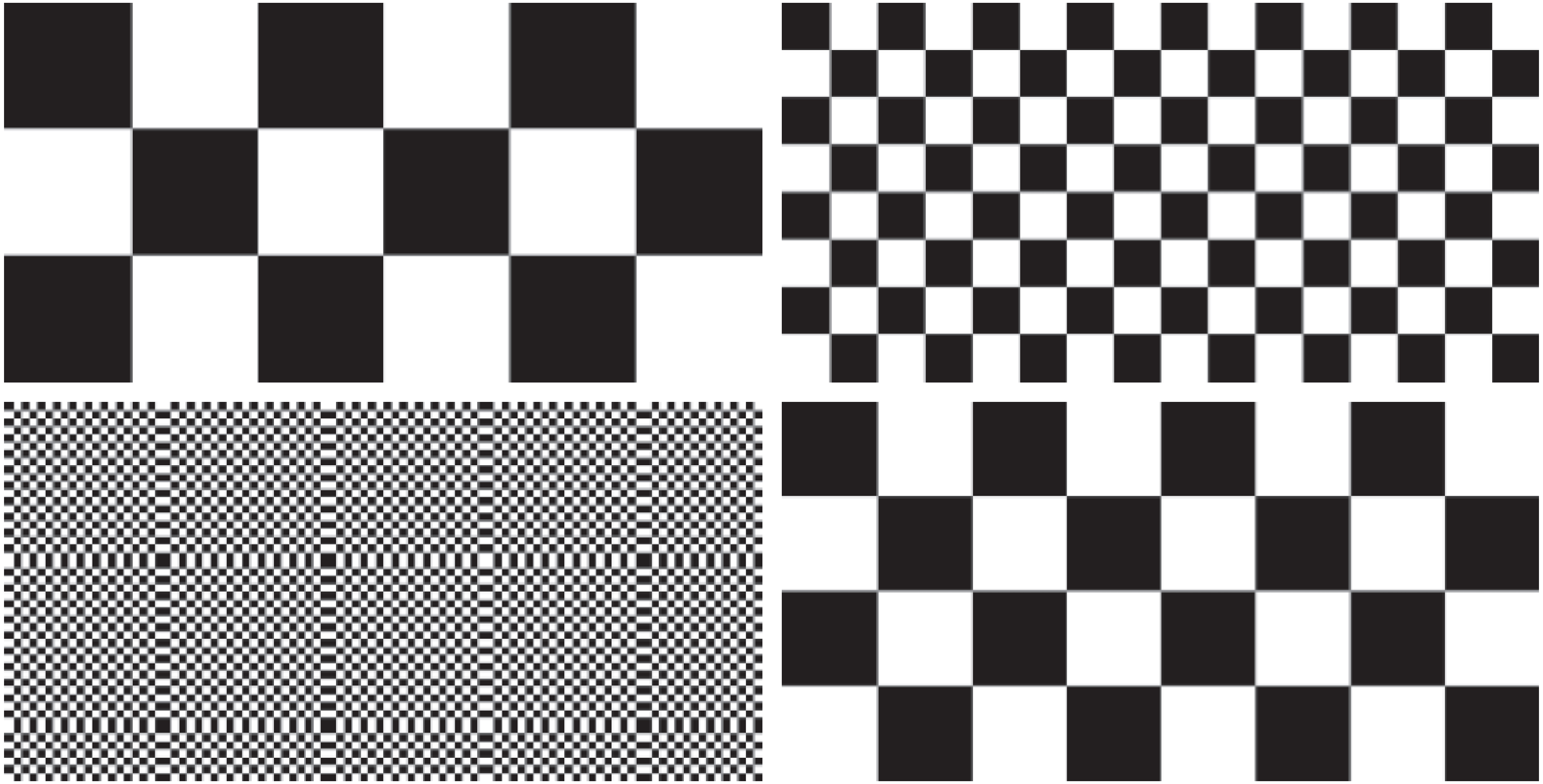
FIGURE 4.16 Two-dimensional Fourier transforms of (a) an over-sampled, and (b) an under-sampled, band-limited function.

Adapted from Gonzalez, Rafael C. Digital image processing. Pearson education India, 2009.

Aliasing in images

a b
c d

FIGURE 4.18 Aliasing. In (a) and (b) the squares are of sizes 16 and 6 pixels on the side. In (c) and (d) the squares are of sizes 0.95 and 0.48 pixels, respectively. Each small square in (c) is one pixel. Both (c) and (d) are aliased. Note how (d) masquerades as a "normal" image.



Adapted from Gonzalez, Rafael C. Digital image processing. Pearson education India, 2009.

a b c

Adapted from Gonzalez, Rafael C. Digital image processing. Pearson education India, 2009.

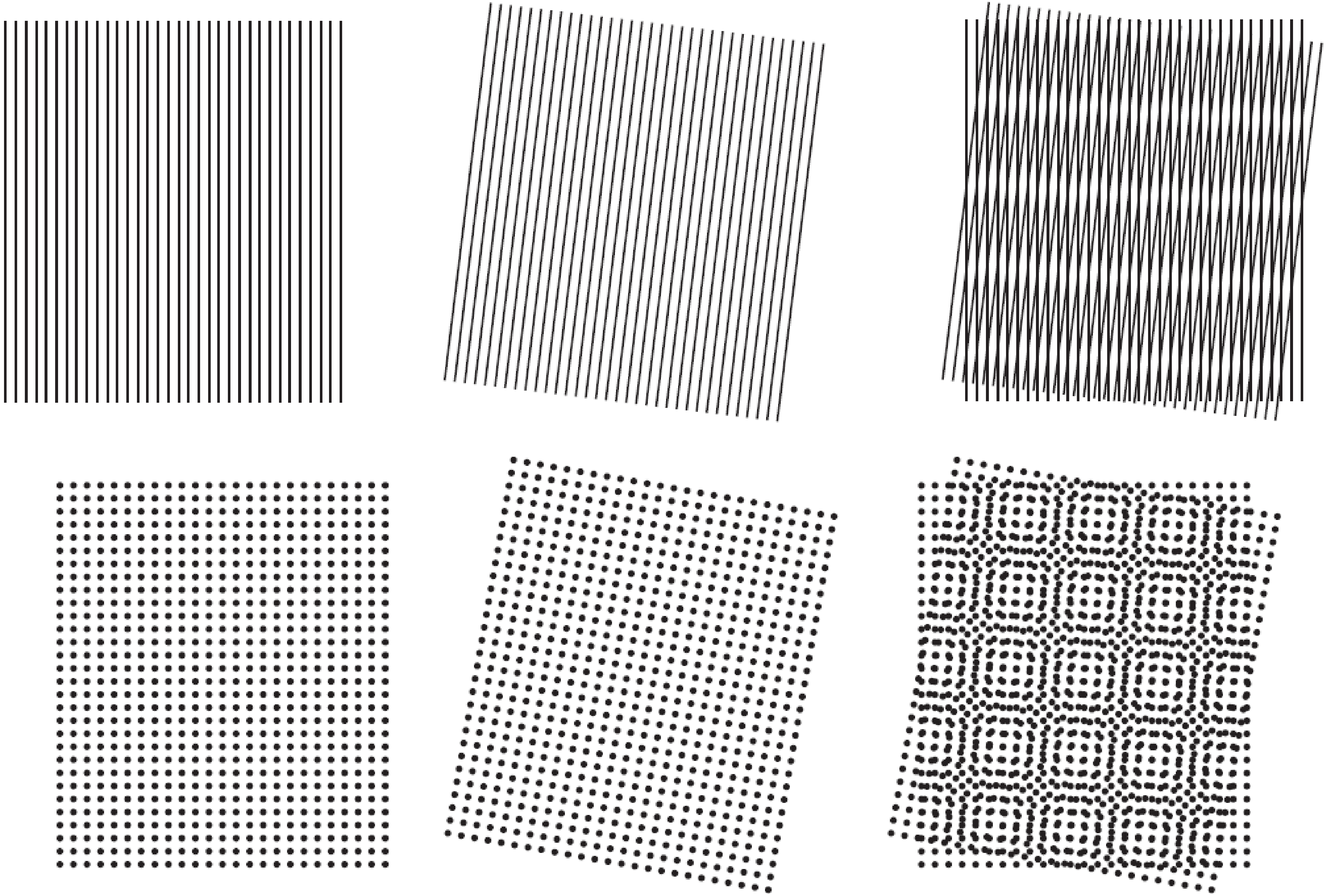
FIGURE 4.19 Illustration of aliasing on resampled natural images. (a) A digital image of size 772×548 pixels with visually negligible aliasing. (b) Result of resizing the image to 33% of its original size by pixel deletion and then restoring it to its original size by pixel replication. Aliasing is clearly visible. (c) Result of blurring the image in (a) with an averaging filter prior to resizing. The image is slightly more blurred than (b), but aliasing is not longer objectionable. (Original image courtesy of the Signal Compression Laboratory, University of California, Santa Barbara.)

Aliasing and Moiré Patterns

- In optics, a moiré pattern is a secondary, visual phenomenon produced, for example, by superimposing two gratings of approximately equal spacing.
- We see them in overlapping insect window screens, on the interference between TV raster lines, striped or highly textured materials in the background, or worn by individuals.
- In digital image processing, moiré-like patterns arise routinely when sampling media print, such as newspapers and magazines, or in images with periodic components whose spacing is comparable to the spacing between samples.
- It is important to note that moiré patterns are more general than sampling artifacts.
- The simple acts of superimposing one pattern on the other creates a pattern with frequencies not present in either of the original patterns.

a b c
d e f

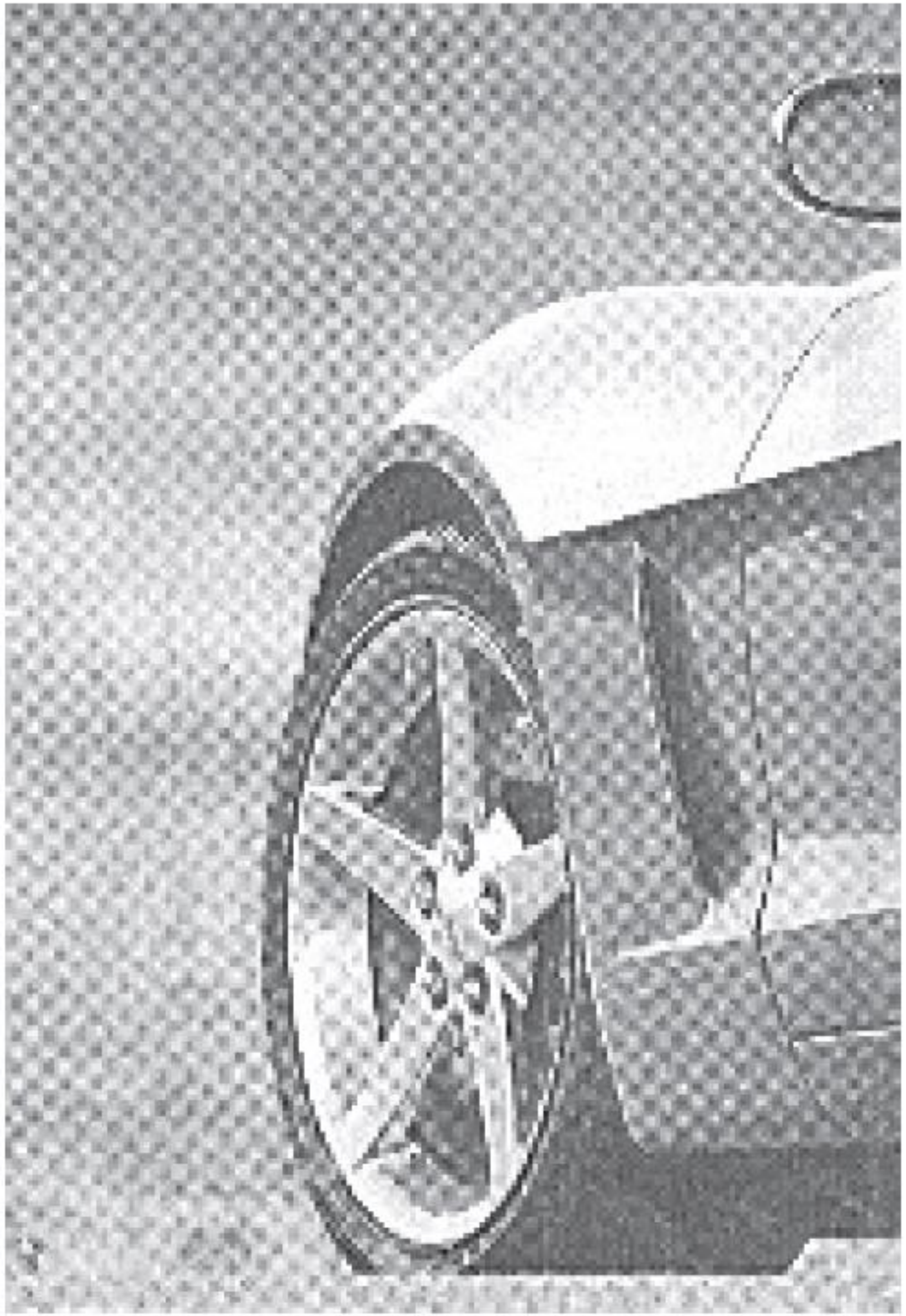
FIGURE 4.20 Examples of the moiré effect. These are vector drawings, not digitized patterns. Superimposing one pattern on the other is analogous to multiplying the patterns.



Adapted from Gonzalez, Rafael C. Digital image processing. Pearson education India, 2009.

FIGURE 4.21

A newspaper image digitized at 75 dpi. Note the moiré-like pattern resulting from the interaction between the ±45° orientation of the half-tone dots and the north-south orientation of the sampling elements used to digitized the image.



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2-D Discrete Fourier Transform (DFT) and its inverse (IDFT)

- 2D DFT pair of image $f(x,y)$ of size $M \times N$.

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

$$f(x,y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) e^{j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

Properties of 2-D DFT and IDFT

Relationship between Spatial and Frequency Intervals

- Let ΔT and ΔZ denote the separations between samples, then the separations between the corresponding discrete, frequency domain variables are given by

$$\Delta u = \frac{1}{M\Delta T}$$

$$\Delta v = \frac{1}{N\Delta Z}$$

Properties of 2-D DFT and IDFT (contd.)

Translation and Rotation

$$f(x,y)e^{j2\pi(u_0x/M + v_0y/N)} \Leftrightarrow F(u-u_0, v-v_0)$$

$$f(x-x_0, y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi(x_0u/M + y_0v/N)}$$

Using the polar coordinates

$$x = r\cos\theta \qquad y = r\sin\theta \qquad u = \omega\cos\varphi \qquad v = \omega\sin\varphi$$

results in the following transform pair:

$$f(r, \theta+\theta_0) \Leftrightarrow F(\omega, \varphi+\theta_0)$$

Properties of 2-D DFT and IDFT (contd.)

Periodicity

$2-D$ Fourier transform and its inverse are infinitely periodic

$$F(u,v)=F(u+k_1M,v)=F(u,v+k_2N)=F(u+k_1M,v+k_2N)$$

$$f(x,y)=f(x+k_1M,y)=f(x,y+k_2N)=f(x+k_1M,y+k_2N)$$

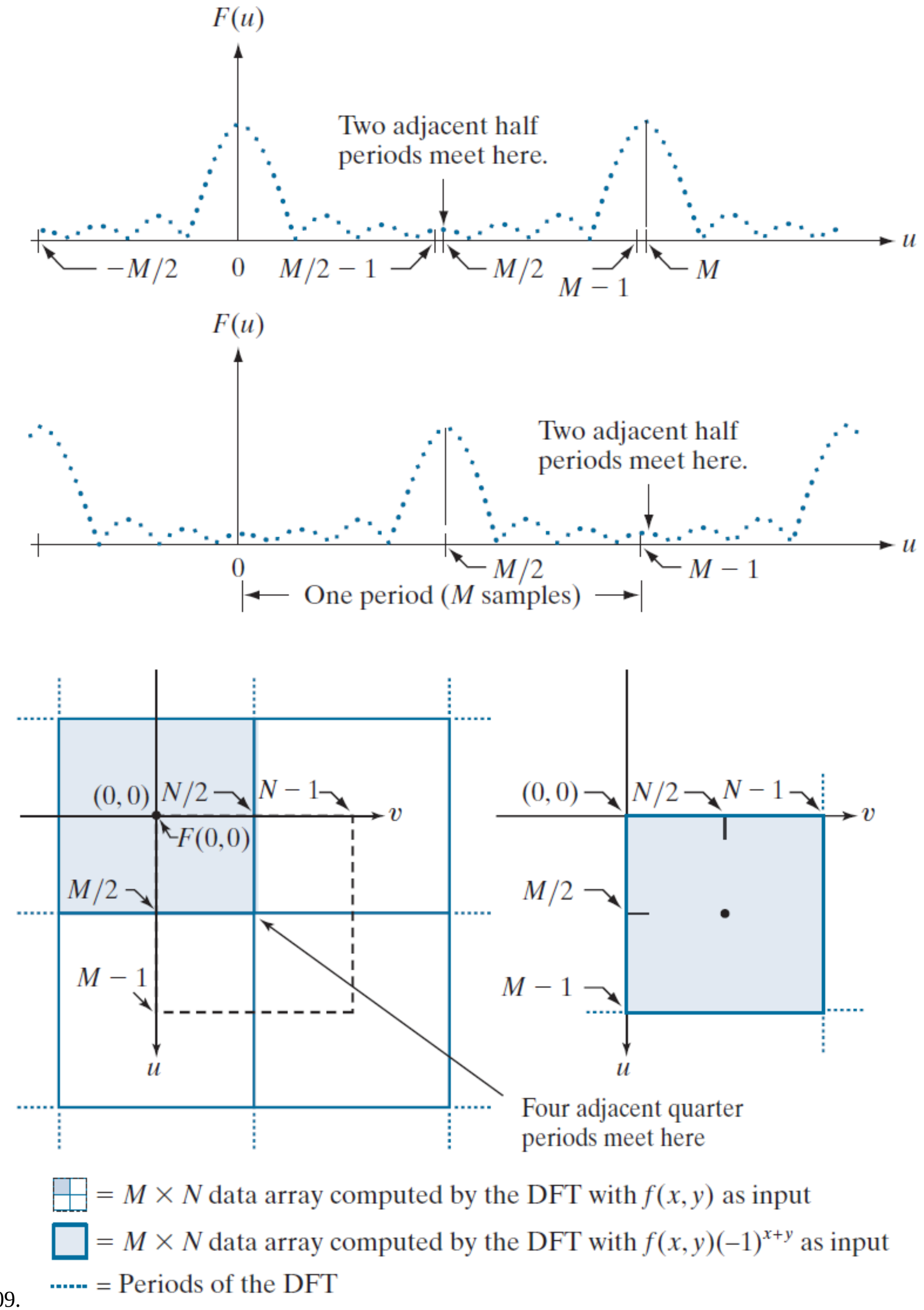
$$f(x)e^{j2\pi(\mu_0 x/M)} \Leftrightarrow F(\mu-\mu_0)$$

$$\mu_0 = M/2, \qquad f(x)(-1)^x \Leftrightarrow F(\mu - M/2)$$

$$f(x,y)(-1)^{x+y} \Leftrightarrow F(\mu - M/2, \nu - N/2)$$

a
b
c d

FIGURE 4.22 Centering the Fourier transform. (a) A 1-D DFT showing an infinite number of periods. (b) Shifted DFT obtained by multiplying $f(x)$ by $(-1)^x$ before computing $F(u)$. (c) A 2-D DFT showing an infinite number of periods. The area within the dashed rectangle is the data array, $F(u,v)$, obtained with Eq. (4-67) with an image $f(x,y)$ as the input. This array consists of four quarter periods. (d) Shifted array obtained by multiplying $f(x,y)$ by $(-1)^{x+y}$ before computing $F(u,v)$. The data now contains one complete, centered period, as in (b).



Adapted from Gonzalez, Rafael C. Digital image processing. Pearson education India, 2009.

TABLE 4.1
Some symmetry properties of the 2-D DFT and its inverse. $R(u,v)$ and $I(u,v)$ are the real and imaginary parts of $F(u,v)$, respectively. Use of the word *complex* indicates that a function has nonzero real and imaginary parts.

	Spatial Domain†		Frequency Domain†
1)	$f(x,y)$ real	$\Leftrightarrow$	$F^*(u,v) = F(-u,-v)$
2)	$f(x,y)$ imaginary	$\Leftrightarrow$	$F^*(-u,-v) = -F(u,v)$
3)	$f(x,y)$ real	$\Leftrightarrow$	$R(u,v)$ even; $I(u,v)$ odd
4)	$f(x,y)$ imaginary	$\Leftrightarrow$	$R(u,v)$ odd; $I(u,v)$ even
5)	$f(-x,-y)$ real	$\Leftrightarrow$	$F^*(u,v)$ complex
6)	$f(-x,-y)$ complex	$\Leftrightarrow$	$F(-u,-v)$ complex
7)	$f^*(x,y)$ complex	$\Leftrightarrow$	$F^*(-u,-v)$ complex
8)	$f(x,y)$ real and even	$\Leftrightarrow$	$F(u,v)$ real and even
9)	$f(x,y)$ real and odd	$\Leftrightarrow$	$F(u,v)$ imaginary and odd
10)	$f(x,y)$ imaginary and even	$\Leftrightarrow$	$F(u,v)$ imaginary and even
11)	$f(x,y)$ imaginary and odd	$\Leftrightarrow$	$F(u,v)$ real and odd
12)	$f(x,y)$ complex and even	$\Leftrightarrow$	$F(u,v)$ complex and even
13)	$f(x,y)$ complex and odd	$\Leftrightarrow$	$F(u,v)$ complex and odd

†Recall that x, y, u, and v are *discrete* (integer) variables, with x and u in the range $[0, M-1]$, and y and v in the range $[0, N-1]$. To say that a complex function is *even* means that its real *and* imaginary parts are even, and similarly for an *odd* complex function. As before, "$\Leftrightarrow$" indicates a Fourier transform pair.

Symmetry

Adapted from Gonzalez, Rafael C. Digital image processing. Pearson education India, 2009.

Properties of 2-D DFT and IDFT (contd.)

Fourier Spectrum and Phase Angle

2−D DFT in polar form

$$F(u,v) = |F(u,v)|e^{j\phi(u,v)}$$

Fourier spectrum

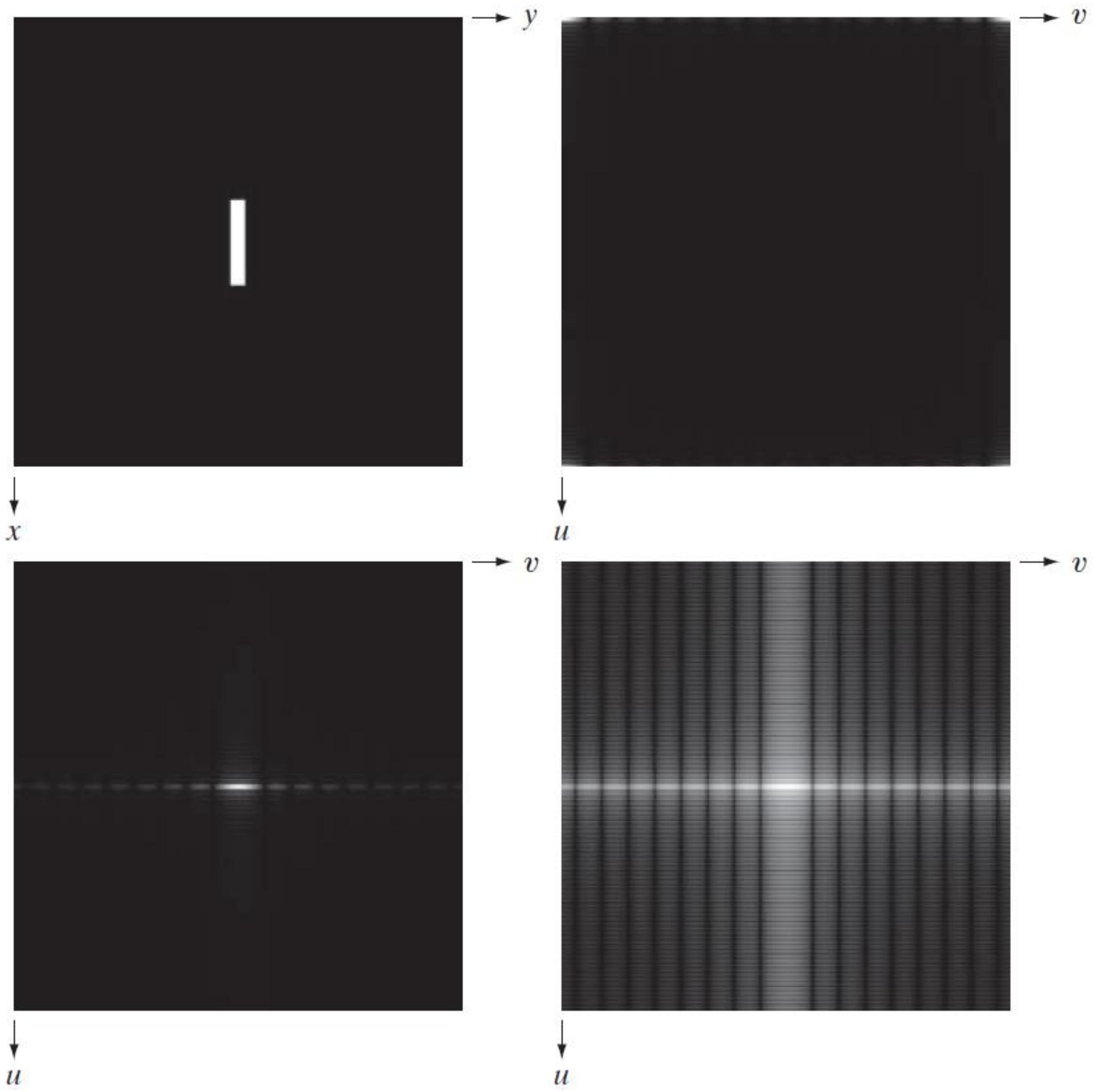
$$|F(u,v)| = [R^2(u,v) + I^2(u,v)]^{1/2}$$

Power spectrum

$$P(u,v) = |F(u,v)|^2 = R^2(u,v) + I^2(u,v)$$

a	b
c	d

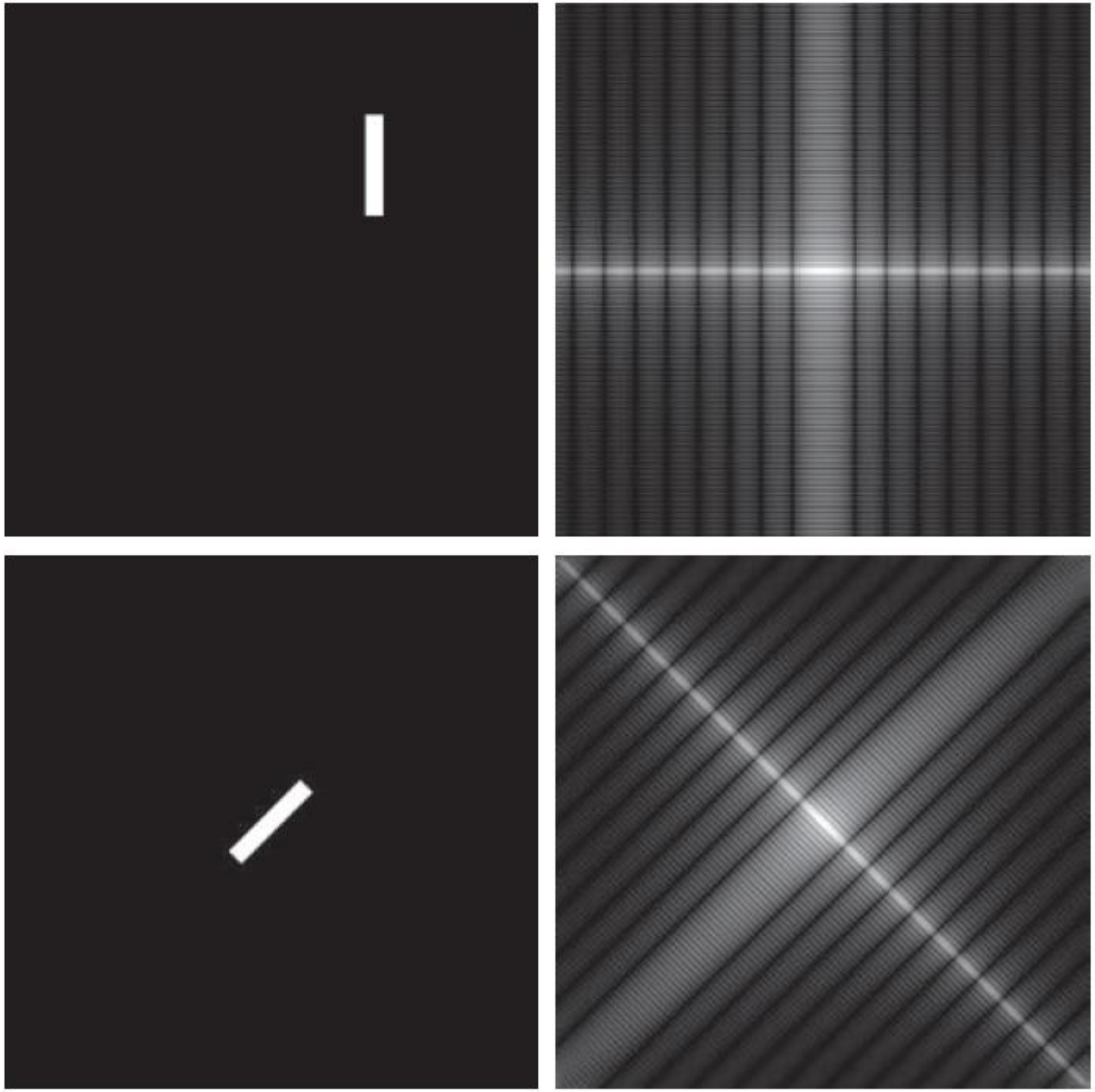
FIGURE 4.23
(a) Image.
(b) Spectrum, showing small, bright areas in the four corners (you have to look carefully to see them).
(c) Centered spectrum.
(d) Result after a log transformation. The zero crossings of the spectrum are closer in the vertical direction because the rectangle in (a) is longer in that direction. The right-handed coordinate convention used in the book places the origin of the spatial and frequency domains at the top left (see Fig. 2.19).



Adapted from Gonzalez, Rafael C. Digital image processing. Pearson education India, 2009.

a b
c d

FIGURE 4.24
(a) The rectangle in Fig. 4.23(a) translated.
(b) Corresponding spectrum.
(c) Rotated rectangle.
(d) Corresponding spectrum.
The spectrum of the translated rectangle is identical to the spectrum of the original image in Fig. 4.23(a).



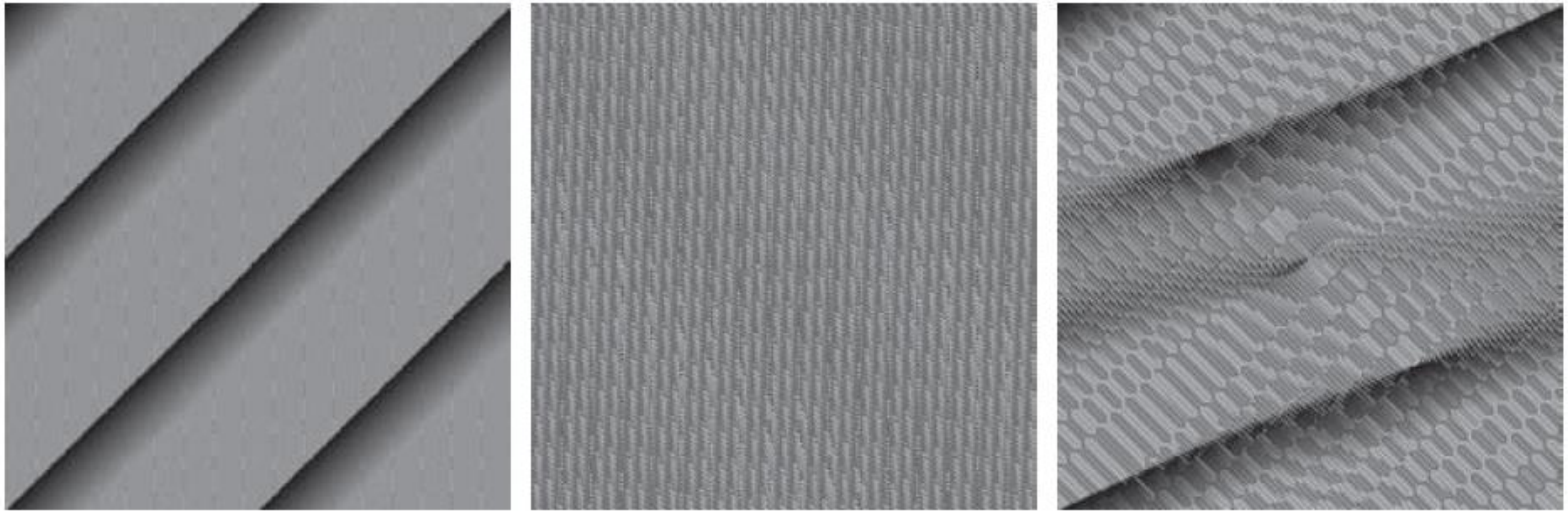
Adapted from Gonzalez, Rafael C. Digital image processing. Pearson education India, 2009.

Properties of 2-D DFT and IDFT (contd.)

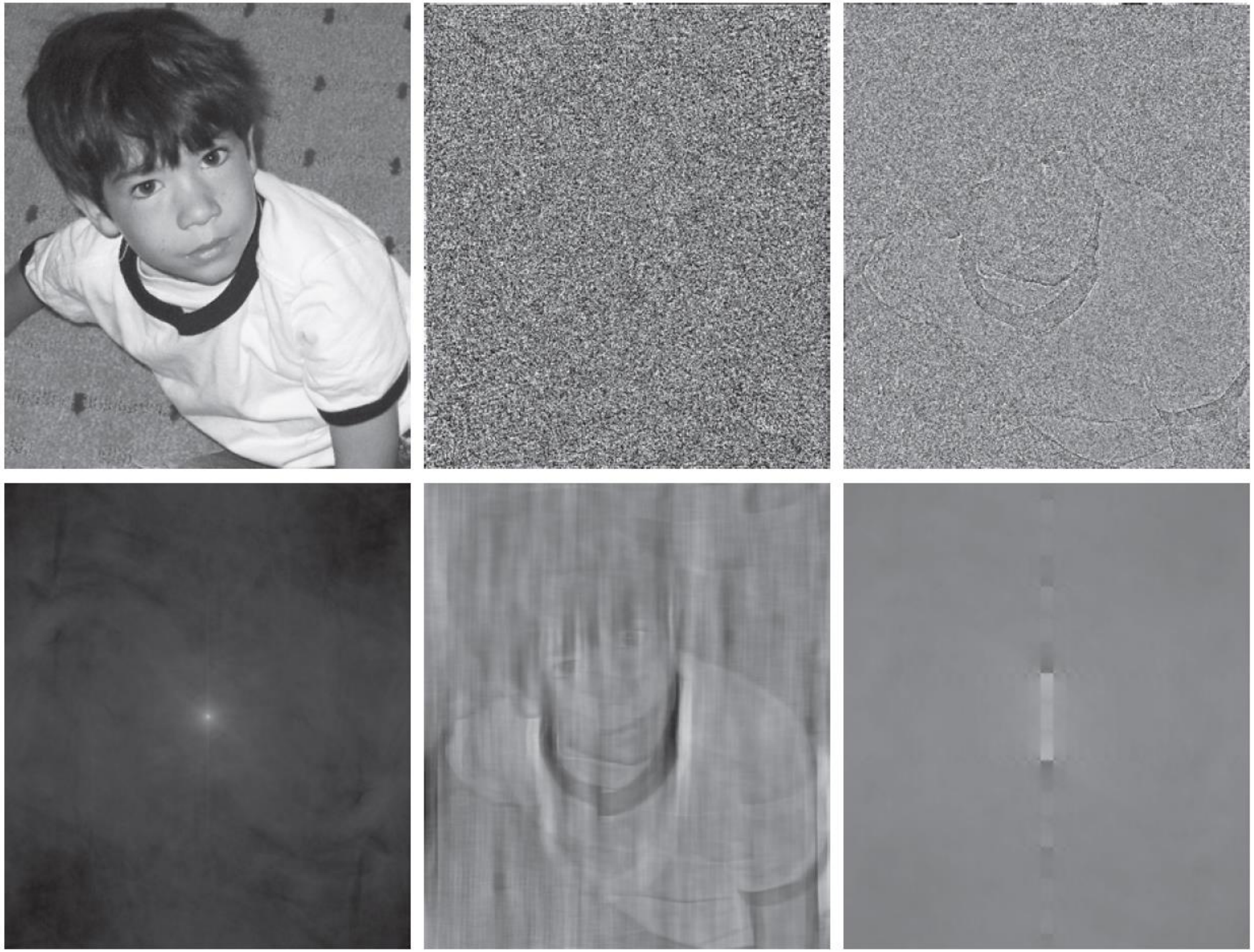
Fourier Spectrum and Phase Angle

a b c

FIGURE 4.25 Phase angle images of (a) centered, (b) translated, and (c) rotated rectangles.



Adapted from Gonzalez, Rafael C. Digital image processing. Pearson education India, 2009.



a	b	c
d	e	f

FIGURE 4.26 (a) Boy image. (b) Phase angle. (c) Boy image reconstructed using only its phase angle (all shape features are there, but the intensity information is missing because the spectrum was not used in the reconstruction). (d) Boy image reconstructed using only its spectrum. (e) Boy image reconstructed using its phase angle and the spectrum of the rectangle in Fig. 4.23(a). (f) Rectangle image reconstructed using its phase and the spectrum of the boy's image.

Adapted from Gonzalez, Rafael C. Digital image pr

2-D Convolution Theorem

1–D convolution

$$f(x)\star h(x) = \sum^{M-1} f(m)h(x-m)$$

2-D convolution

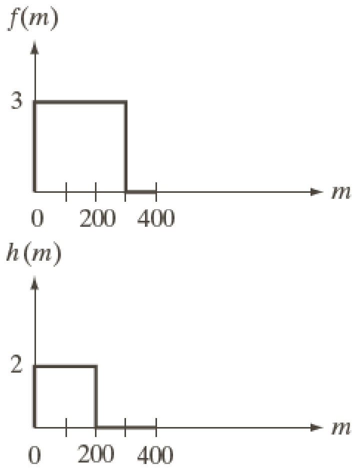
$$(f\star h)(x,y) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)h(x-m,y-n)$$

$$x = 0,1,2,\ldots,M-1; y = 0,1,2,\ldots,N-1.$$

$$f(x,y)\star h(x,y) \Leftrightarrow F(u,v)H(u,v)$$

$$f(x,y)h(x,y) \Leftrightarrow F(u,v)\star H(u,v)$$

An Example of Convolution



Mirroring h about the origin

Translating the mirrored function by x

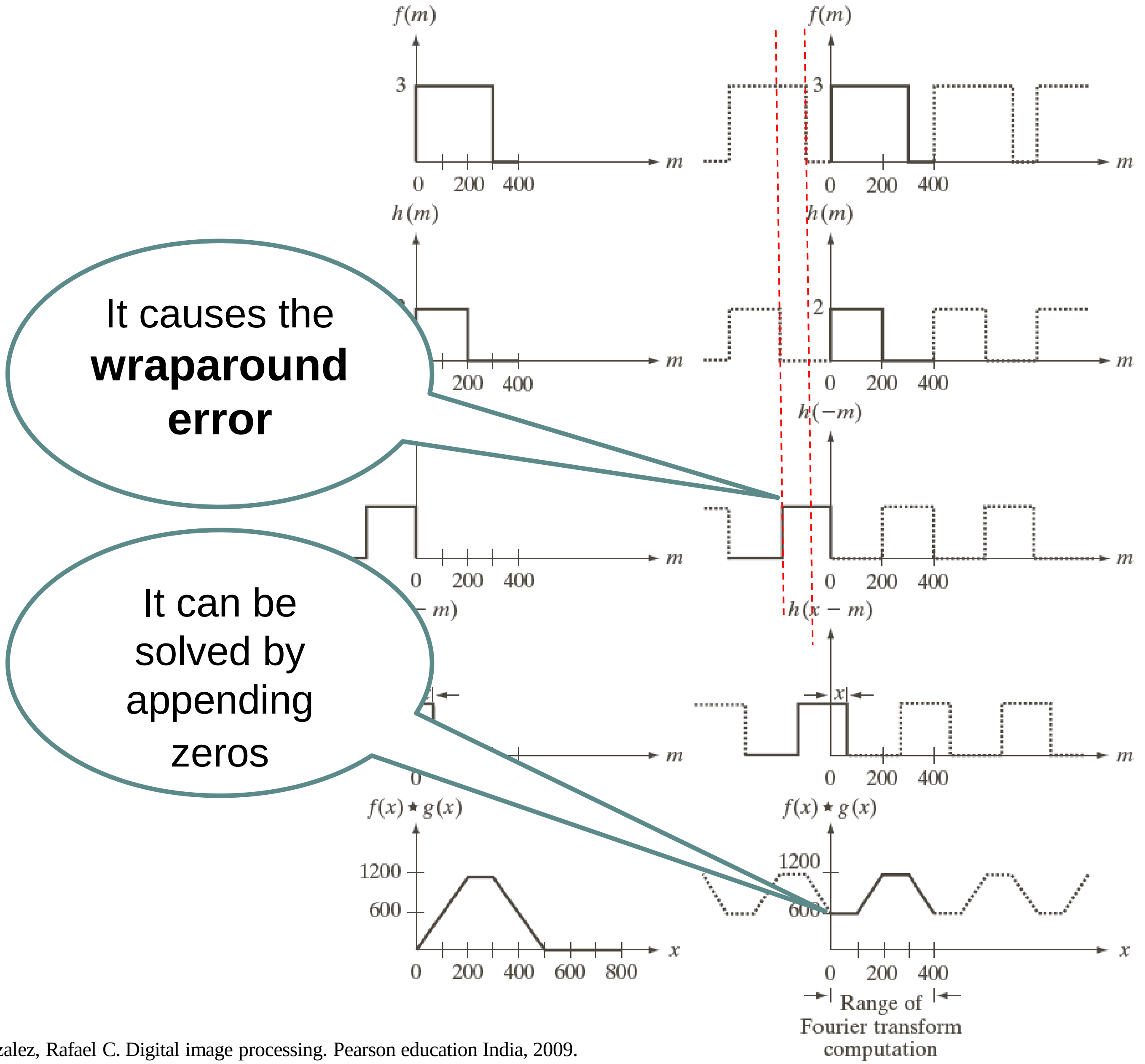
Computing the sum for each x

a f
b g
c h
d i
e j

FIGURE 4.28 Left column: convolution of two discrete functions obtained using the approach discussed in Section 3.4.2. The result in (e) is correct. Right column: Convolution of the same functions, but taking into account the periodicity implied by the DFT. Note in (j) how data from adjacent periods produce wraparound error, yielding an incorrect convolution result. To obtain the correct result, function padding must be used.

Adapted from Gonzalez, Rafael C. Digital image processing. Pearson education India, 2009.

An Example of Convolution



a	f
b	g
c	h
d	i
e	j

FIGURE 4.28 Left column: convolution of two discrete functions obtained using the approach discussed in Section 3.4.2. The result in (e) is correct. Right column: Convolution of the same functions, but taking into account the periodicity implied by the DFT. Note in (j) how data from adjacent periods produce wraparound error, yielding an incorrect convolution result. To obtain the correct result, function padding must be used.

Adapted from Gonzalez, Rafael C. Digital image processing. Pearson education India, 2009.

Zero Padding

- Consider two functions f(x) and h(x) composed of A and B samples, respectively

- Append zeros to both functions so that they have the same length, denoted by P, then wraparound is avoided by choosing

$$P \geq A+B-1$$

Zero Padding

- Let f(x,y) and h(x,y) be two image arrays of sizes $A \times B$ and $C \times D$ pixels, respectively. Wraparound error in their convolution can be avoided by padding these functions with zeros

$$f_p(x,y) = \begin{cases} f(x,y) & 0 \le x \le A-1 \; and \; 0 \le y \le B-1 \\ 0 & A \le x \le P \; or \; B \le y \le Q \end{cases}$$

$$h_p(x,y) = \begin{cases} h(x,y) & 0 \le x \le C-1 \text{ and } 0 \le y \le D-1 \\ 0 & C \le x \le P \text{ or } D \le y \le Q \end{cases}$$

Here $P \ge A + C - 1; Q \ge B + D - 1$

TABLE 4.4
Summmary of DFT pairs. The closed-form expressions in 12 and 13 are valid only for continuous variables. They can be used with discrete variables by sampling the continuous expressions.

Name	DFT Pairs
1) Symmetry properties	See Table 4.1
2) Linearity	$af_1(x,y) + bf_2(x,y) \Leftrightarrow aF_1(u,v) + bF_2(u,v)$
3) Translation (general)	$f(x,y)e^{j2\pi(u_0x/M + v_0y/N)} \Leftrightarrow F(u-u_0, v-v_0)$ $f(x-x_0, y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi(ux_0/M + vy_0/N)}$
4) Translation to center of the frequency rectangle, $(M/2, N/2)$	$f(x,y)(-1)^{x+y} \Leftrightarrow F(u-M/2, v-N/2)$ $f(x-M/2, y-N/2) \Leftrightarrow F(u,v)(-1)^{u+v}$
5) Rotation	$f(r,\theta+\theta_0) \Leftrightarrow F(\omega,\varphi+\theta_0)$ $r = \sqrt{x^2+y^2} \quad \theta = \tan^{-1}(y/x) \quad \omega = \sqrt{u^2+v^2} \quad \varphi = \tan^{-1}(v/u)$
6) Convolution theorem†	$f \star h)(x,y) \Leftrightarrow (F \bullet H)(u,v)$ $(f \bullet h)(x,y) \Leftrightarrow (1/MN)[(F \star H)(u,v)]$
7) Correlation theorem†	$(f \,☆\, h)(x,y) \Leftrightarrow (F^* \bullet H)(u,v)$ $(f^* \bullet h)(x,y) \Leftrightarrow (1/MN)[(F \,☆\, H)(u,v)]$
8) Discrete unit impulse	$\delta(x,y) \Leftrightarrow 1$ $1 \Leftrightarrow MN\delta(u,v)$
9) Rectangle	$\text{rec}[a,b] \Leftrightarrow ab\dfrac{\sin(\pi ua)}{(\pi ua)}\dfrac{\sin(\pi vb)}{(\pi vb)}e^{-j\pi(ua+vb)}$
10) Sine	$\sin(2\pi u_0x/M + 2\pi v_0y/N) \Leftrightarrow \dfrac{jMN}{2}[\delta(u+u_0, v+v_0) - \delta(u-u_0, v-v_0)]$
11) Cosine	$\cos(2\pi u_0x/M + 2\pi v_0y/N) \Leftrightarrow \dfrac{1}{2}[\delta(u+u_0, v+v_0) + \delta(u-u_0, v-v_0)]$
The following Fourier transform pairs are derivable only for continuous variables, denoted as before by t and z for spatial variables and by μ and ν for frequency variables. These results can be used for DFT work by sampling the continuous forms.	
12) Differentiation (the expressions on the right assume that $f(\pm\infty, \pm\infty) = 0$.	$\left(\dfrac{\partial}{\partial t}\right)^m\left(\dfrac{\partial}{\partial z}\right)^n f(t,z) \Leftrightarrow (j2\pi\mu)^m(j2\pi\nu)^n F(\mu,\nu)$ $\dfrac{\partial^m f(t,z)}{\partial t^m} \Leftrightarrow (j2\pi\mu)^m F(\mu,\nu); \quad \dfrac{\partial^n f(t,z)}{\partial z^m} \Leftrightarrow (j2\pi\nu)^n F(\mu,\nu)$
13) Gaussian	$A2\pi\sigma^2 e^{-2\pi^2\sigma^2(t^2+z^2)} \Leftrightarrow Ae^{-(\mu^2+\nu^2)/2\sigma^2} \quad (A \text{ is a constant})$

†Assumes that $f(x,y)$ and $h(x,y)$ have been properly padded. Convolution is associative, commutative, and distributive. Correlation is distributive (see Table 3.5). The products are elementwise products (see Section 2.6).

Adapted from Gonzalez, Rafael C. Digital image processing. Pearson education India, 2009.