Surface waves in a Bingham fluid medium

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We study surface waves in a Viscoplastic medium using the Generalized Newtonian Fluid model for treating viscous forces.

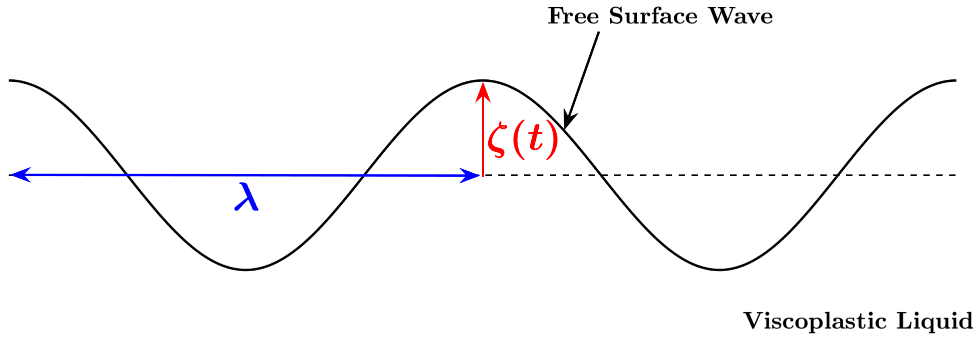

Figure 1: Schematic of the problem: Free Surface Wave in a Viscoplastic Fluid. We track the temporal variation of the amplitude of these oscillations (ζ).

1 Regularization method for Non-Newtonian Fluids

In this section, we briefly describe the regularization method that we have implemented in Basilisk:
We first calculate the second invariant of the deformation tensor ($D_{ij} = 0.5\left(\partial_i U_j + \partial_j U_i\right)$).

$$D_{11} = \frac{\partial u}{\partial x}$$

$$D_{12} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$D_{21} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$D_{22} = \frac{\partial v}{\partial y}$$

The second invariant is $D_2 = \sqrt{D_{ij}D_{ij}}$ (this is the Frobenius norm)

$$D_2^2 = D_{ij}D_{ij} = D_{11}D_{11} + D_{12}D_{21} + D_{21}D_{12} + D_{22}D_{22}$$

The equivalent viscosity is

$$\mu_{eq} = \mu_0 \left(\frac{D_2}{\sqrt{2}}\right)^{N-1} + \frac{\tau_y}{\sqrt{2}D_2}$$

Note: $\|D\| = D_2/\sqrt{2}$.
For Bingham Fluid, N = 1:

$$\mu_{eq} = \mu_0 + \frac{\tau_y}{\sqrt{2}D_2}$$

Finally, μ is the minimum of of μ_{eq} and a large μ_{max}. The fluid flows always, it is not a solid, but a very viscous fluid.

$$\mu = min\left(\mu_{eq}, \mu_{max}\right)$$

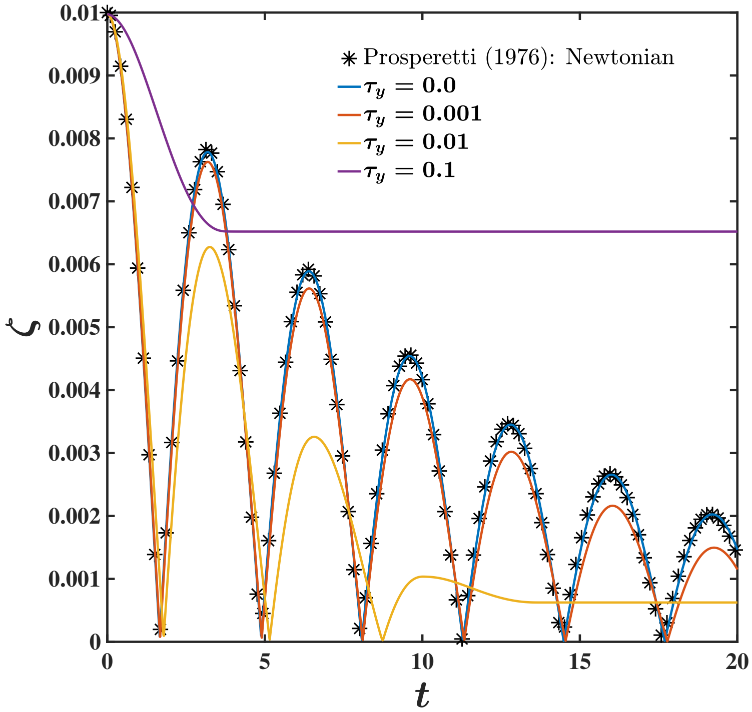

Figure 2: Variation of the amplitude of oscillations with time. Initially, all fluids yield because of the high-stress initial condition. In time, oscillation is arrested as the fluid reaches the plastic limit. This stopping time decreases with increasing yield stress τ_y.

2 Numerical Simulations

We solve the equations above along with the continuity equation ($\partial_i U_i = 0$) and the full Navier-Stokes equation.

$$\hat{\rho}\left(\frac{\partial U_i}{\partial t} + \frac{\partial\left(U_i U_j\right)}{\partial X_j}\right) = -\frac{\partial P}{\partial X_i} + Oh\frac{\partial}{\partial X_j}\left(\hat{\mu}_{eq} D_{ij}\right) + \delta_s \hat{n}_i \tag{1}$$

Symbols have their usual meaning. The Ohnesorge number is defined as $\left(Oh = \mu_0/\sqrt{\sigma\rho\lambda}\right)$. Here, λ is the wavelength of the initialized standing surface wave, such that the amplitude of oscillation for the surface wave is given by:

$$a(x,t) = \zeta(t) cos\left(2\pi\frac{x}{\lambda}\right) \tag{2}$$

We use Volume of Fluid (VOF) and height function (Popinet, 2009) for interface reconstruction. The code used for the present study can be found at Sanjay, 2018. Figure 2 represents the variation of the amplitude of surface wave oscillation over time.

3 Theory

3.1 Newtonian Case

Following Prosperetti, 1976, the equation of motion for the interface can be written as:

$$\frac{\partial^2\zeta}{\partial t^2} + 4\epsilon\frac{\partial\zeta}{\partial t} + \zeta - 4\epsilon^2\int_0^t\left\{\frac{e^{-\epsilon(t-\theta)}}{\sqrt{\pi\epsilon(t-\theta)}} - erfc\left(\sqrt{\epsilon(t-\theta)}\right)\right\}\frac{d}{dt}\left(\zeta(\theta)\right)d\theta = 0 \tag{3}$$

The above equation is a simplified form of the equation presented in Prosperetti, 1981, under the assumption: $\rho_{\text{upper}} \to 0$ & $\rho_{\text{lower}} = \rho, \nu_{\text{lower}} = \nu$ and can be solved using Laplace Transformation. If $\hat{\zeta}(s)$ is the Laplace Transformation of $\zeta(t)$, then the solution of the above equation in the Laplace Space with $\zeta(0) = 1.0$ and

$\dot{\zeta}(0) = 0.0$ is:

$$\hat{\zeta}(s) = \frac{1}{s}\left(1 - \frac{1}{1 + s^2 + 4\epsilon s + 4\epsilon^2 - 4\epsilon\sqrt{(1 + s/\epsilon)}}\right) \tag{4}$$

Here, $\omega^2 = gk + \frac{\sigma}{\rho}k^3$ and $\epsilon = \frac{\nu k^2}{\omega}$. The above equation can be inverted using the MATLAB code by McClure, 2013 which is based on the numerical method described by Abate and Whitt, 2006.

3.2 Non-Newtonian Case

We are still working on this. It would be interesting to study the surface waves in Viscoplastic medium theoretically. Notably, we are looking for a method to get the stopping time of these waves as a function of the Yield Stress. For now, we only have the numerical simulations.

References

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