k2-worksheet

The focal length f of an optical instrument is needed. This is determined by using the thin lens formula,

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{f},$$

where r_1 is the distance from the lens to the object and r_2 is the distance from the lens to the real image of the object. The distance r_1 is independently measured 36 times and r_2 is independently measured 40 times. The mean of the measurements is the actual distances, 10 centimeters and 18 centimeters, respectively. The standard deviation of the measurement is 0.1 centimeters for r_1 and 0.5 centimeter for r_2.

a. Let $\bar{R}_1$ be the sample mean of the 36 measurements to the object. Find $\mu_{\overline{R}_1}$ and $\sigma_{\overline{R}_1}$.

The sample mean in this case is the same as the given mean which is 10 cm. The standard deviation is $\frac{0.1}{\sqrt{36}} = 0.0167$. We found this using the law of large numbers

b. Let $\bar{R}_2$ be the sample mean of the 40 measurements to the image. Estimate, using the central limit theorem, $P\{\bar{R}_2 < 17.9\text{cm}\}$.

The sample mean here is 18 and the standard deviation is $\frac{0.5}{\sqrt{40}} = 0.0791$. Therefore, the z score is $\frac{17.9-18}{0.0791} = -1.26502$ Now $P\{\bar{R}_2 < 17.9\text{cm}\} = 0.1038$ (Using the table)

c. For measurements $r_{1,1}, \ldots, r_{1,36}$ and $r_{2,1}, \ldots, r_{2,40}$, estimate the focal length using

$$\frac{1}{\bar{r}_1} + \frac{1}{\bar{r}_2} = \frac{1}{\hat{f}}.$$

Use the delta method to give an estimate of the mean and standard deviation of $\hat{f}$. How do your results from the delta method hand calculation compare to the mean and sd of the simulations below?

Here is a simulation of this protocol 10000 times using the `rnorm` command to simulate sample means for r_1 and r_2.

We see that the values we got for the mean and the standard deviation for $\hat{f}$ are quite similar to the simulated vaues.

```
r1_bars <- rnorm(10000,10,0.1/sqrt(36))
r2_bars <- rnorm(10000,18,0.5/sqrt(40))

f_ests <- (r1_bars*r2_bars)/(r1_bars+r2_bars)

mean(f_ests)
```

```
## [1] 6.428386
```

```
sd(f_ests)
```

```
## [1] 0.01212693
```

$$\frac{1}{\hat f} = \frac{1}{\bar r_1} + \frac{1}{r_2} = \frac{r_2 + r_1}{\bar r_1 \bar r_2} \quad \Bigg\| \quad \hat f = \frac{\bar r_1 \bar r_2}{\bar r_1 + \bar r_2}$$

$$\hat f_{r_1} = \frac{\bar r_2(\bar r_1 + \bar r_2) - \bar r_1 \bar r_2 (1)}{(\bar r_1 + \bar r_2)^2} = \frac{\cancel{\bar r_1 \bar r_2} + \bar r_2^{\,2} - \cancel{\bar r_1 \bar r_2}}{(\bar r_1 + \bar r_2)^2}$$

$$= \frac{\bar r_2^{\,2}}{(\bar r_1 + \bar r_2)^2}$$

Similarly $\hat f_{r_2} = \dfrac{\bar r_1^{\,2}}{(\bar r_1 + \bar r_2)^2}$

$$\mu_{\hat f} = \frac{\mu_{r_1} \mu_{r_2}}{\mu_{r_1} + \mu_{r_2}} = 6.428$$

$$\sigma_{\hat f} = \sqrt{\left|\frac{\partial \hat f}{\partial \bar r_1}(\mu_{r_1}, \mu_{r_2})\right|^2 \frac{\sigma_{r_1}^{\,2}}{\sqrt{n_{r_1}}} + \left|\frac{\partial \hat f}{\partial \bar r_2}(\mu_{r_1}, \mu_{r_2})\right|^2 \frac{\sigma_{r_2}^{\,2}}{\sqrt{n_{r_2}}}}$$

$$= \sqrt{\left|\frac{18^2}{28^2}\right|^2 \left(\frac{0.1}{\sqrt{36}}\right)^2 + \left|\frac{10^2}{28^2}\right|^2 \times \left(\frac{0.5}{\sqrt{40}}\right)^2}$$

$$= \cancel{0.0147} \; 0.0122$$

Figure 1: image