Indian Institute of Technology Bombay

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Mathematics of Derivative Pricing

Midterm Report

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Introduction

The study of derivative pricing lies at the heart of modern financial theory and practice. Derivatives are essential instruments that are used to manage risk, improve returns and navigate the complexities of financial markets. This project, *Mathematics of Derivative Pricing*, is a comprehensive journey through the fundamental and advanced concepts that govern derivative pricing and strategic use.

Exploration begins with a fundamental understanding of **financial markets**, where capital flows and assets are exchanged. It delves into the concept of **interest rates**, the backbone of time value and discounting in finance. Based on these principles, the analysis extends to **cash flow streams** and the methods of valuing future payments in the present terms.

A detailed study of **bonds** follows, focusing on their valuation, coupon structures and sensitivity to interest rate movements. The framework then progresses to **portfolio theory**, introducing concepts of diversification, risk-return optimization and the efficient frontier.

The project places significant emphasis on the pricing of **forward contracts** and **futures**, incorporating no-arbitrage pricing models, cost-of-carry relationships and the essential mechanics of these derivative instruments. Based on this, the strategic use of **hedging with futures** is explored, demonstrating how futures contracts can be used to protect portfolios from market fluctuations.

A key part of the project is the introduction to **options**, examining their unique payoff structures, strategic applications and underlying pricing models. The discussion extends to **premium bounds and valuation**, where the upper and lower price limits of options are established.

As the analysis deepens, the project navigates into the realm of **discrete time models** and the foundational ideas of **martingale theory**, which underpin much of modern financial mathematics. The study progresses to **continuous time models**, setting the stage for the derivation and application of the renowned **Black-Scholes pricing model**, a cornerstone in the field of option pricing.

The journey concludes with practical **option trading strategies**, linking theoretical insights to real-world applications. Throughout this project, mathematical rigor is complemented by Python-based simulations, scenario analyses and graphical representations to bridge theory with practice.

This work is both a mathematical exploration and a practical guide to the pricing, management and strategic deployment of financial derivatives.

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1 Financial Markets

1.1 Overview

Financial markets are platforms that facilitate the exchange of financial instruments. They are broadly categorized into:

- **Primary Markets**: Where new securities are issued for the first time (e.g., IPOs).
- **Secondary Markets**: Where previously issued securities are traded (e.g., stock exchanges).

1.2 Types of Markets

- **Capital Markets**: For long-term funding (equity, bonds).
- **Money Markets**: For short-term borrowing/lending (T-bills, commercial paper).
- **Derivatives Markets**: For trading contracts based on underlying assets.
- **Foreign Exchange Markets**: For currency trading.

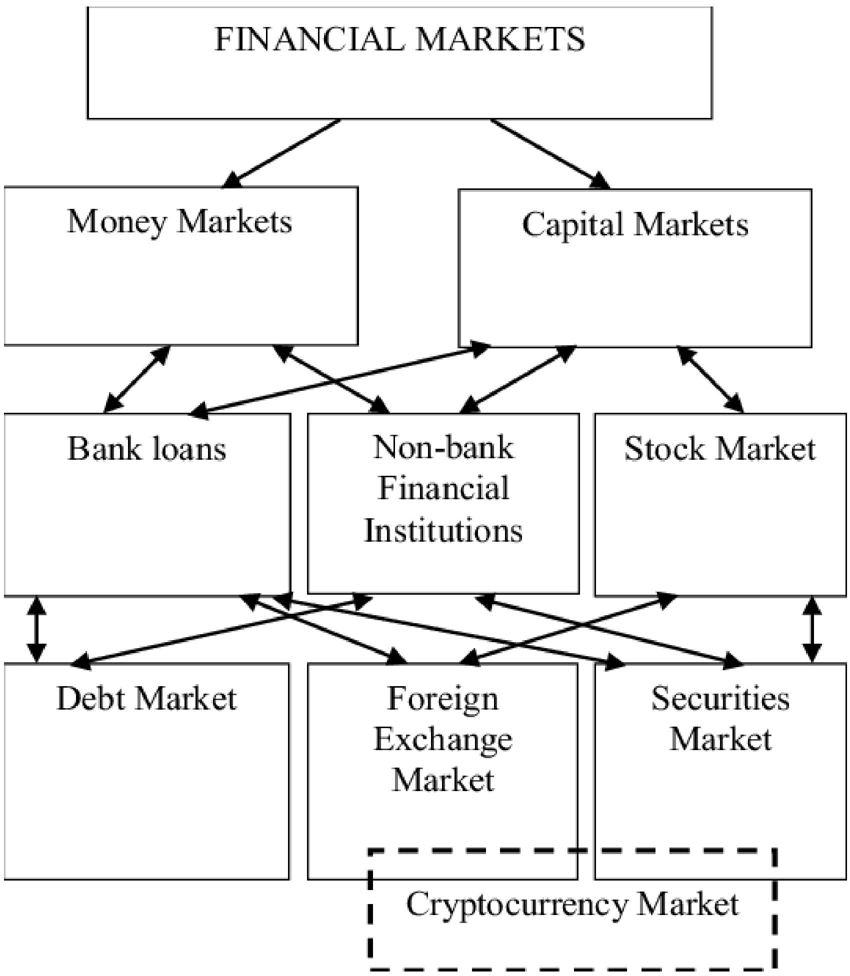

Figure 1: Structure of Financial Markets

1.3 Market Mechanisms

- **Price Discovery**: Determined by supply-demand dynamics.
- **Liquidity**: Ease of trading without price distortion.
- **Efficiency**: How quickly and accurately prices reflect information.

1.4 Relevance to Derivative Pricing

A deep understanding of market microstructure and segmentation is critical before engaging in derivative valuation. Instruments priced in derivative markets ultimately reference the underlying conditions of these primary structures.

> **Insight:** Derivative pricing models assume frictionless, efficient and arbitrage-free markets—assumptions born in the structure of these fundamental markets.

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2 Interest Rates

2.1 Conceptual Foundation

Interest rates represent the time value of money. They quantify the opportunity cost of capital and are used to discount future cash flows to the present.

2.2 Types of Interest Rates

- **Nominal:** Stated rate, not adjusted for inflation.
- **Real:** Adjusted for inflation (approx. Real $\approx$ Nominal $-$ Inflation).
- **Effective Annual Rate (EAR):** Reflects compounding frequency.

2.3 Compounding Mechanisms

Type	**Formula**
Simple	$A = P(1 + rt)$
Compound	$A = P\left(1 + \frac{r}{n}\right)^{nt}$
Continuous	$A = Pe^{rt}$

Table 1: Compounding Mechanisms

2.4 Term Structure of Interest Rates

The term structure, often visualized via the **yield curve**, shows how interest rates change with time to maturity. It forms the basis for:

- Spot rates
- Forward rates
- Zero-coupon curve fitting

2.5 Applications in Derivative Pricing

- Discounting of option payoffs
- Valuing fixed-income derivatives
- Constructing risk-neutral probability measures

> **Insight:** Every modern pricing framework embeds interest rates—often as a stochastic or deterministic process—in its core model structure.

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3 Cash Flow Streams

3.1 Definition

A **cash flow stream** is a series of payments made over time. It is central to pricing nearly all financial instruments.

3.2 Types of Cash Flows

- **Fixed:** Fixed coupons from bonds.
- **Floating:** Interest rates linked to benchmarks (e.g., LIBOR, SOFR).
- **Contingent:** Dependent on future states (e.g., option payoffs).
- **Perpetual:** Payments with infinite time horizons.

3.3 Present Value (PV) of Cash Flows

Given a cash flow CF_t at time t and discount rate r:

$$PV = \sum_{t=1}^{T} \frac{CF_t}{(1+r)^t} \tag{1}$$

For continuous compounding:

$$PV = \sum_{t=1}^{T} CF_t \cdot e^{-rt} \tag{2}$$

3.4 Financial Significance

- All asset pricing models discount future cash flows to their present value.
- Option pricing models compute **expected discounted payoffs**.
- Bond valuation is essentially the present value of a known cash flow stream.

  Insight: Cash flow modeling connects static instruments like bonds to path-dependent instruments like options via time-value principles.

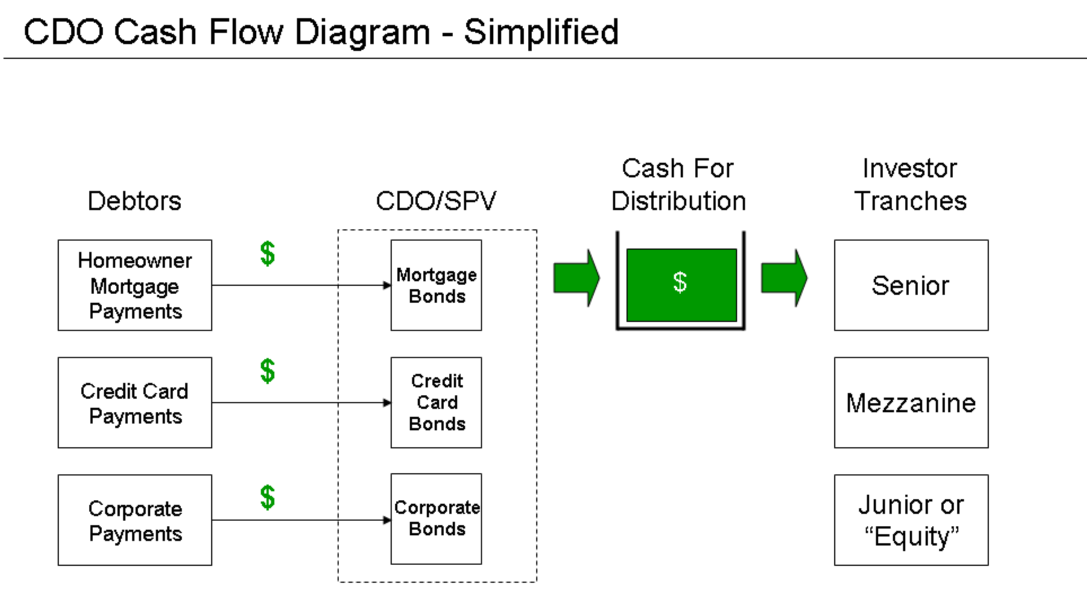

Figure 2: Illustration of a Cash Flow Stream

4 Bond Mathematics

4.1 Bond Pricing

A bond is valued as the present value of future coupon payments and the final principal repayment.

Given:

- Face Value = 1000
- Annual Coupon = 80
- Maturity = 3 years
- Yield to Maturity (YTM) = 6%

Formula:

$$\text{Price} = \frac{80}{1.06} + \frac{80}{(1.06)^2} + \frac{1080}{(1.06)^3}\text{Price} = 1054.30 \tag{3}$$

4.2 Duration Concepts

4.2.1 Macaulay Duration

Weighted average time until cash flows are received:

$$D_{\text{mac}} = \frac{\sum_{t=1}^{n} t \cdot \frac{CF_t}{(1+y)^t}}{\sum_{t=1}^{n} \frac{CF_t}{(1+y)^t}} \tag{4}$$

4.2.2 Modified Duration

Measures price sensitivity to yield changes:

$$D_{\text{mod}} = \frac{D_{\text{Mac}}}{1+y} \tag{5}$$

Interpretation: If Modified Duration = 2.59, then a 1% increase in YTM causes approximately a 2.59% drop in price.

4.3 Practical Use

- Portfolio immunization
- Interest rate risk management
- Building duration-matched portfolios

Insight: Duration is the foundation for convexity, interest rate derivatives, and dynamic hedging.

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5 Yield Curve and Term Structure

5.1 Yield Curve: Concept

A yield curve shows the relationship between interest rates (or bond yields) and different maturities.

Common Shapes:

- **Normal:** Upward sloping → Economic expansion.
- **Inverted:** Downward sloping → Recession signal.
- **Flat:** Uncertainty or policy transitions.

5.2 Example (Schematic)

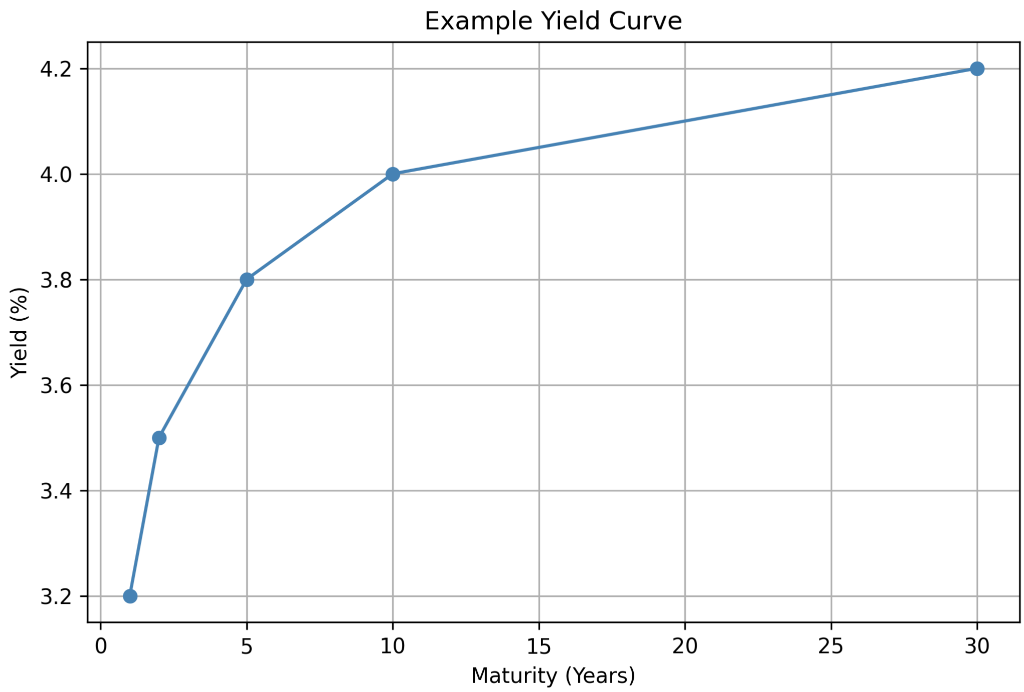

Figure 3: Hypothetical Yield Curve showing maturity vs yield

5.3 Implications

- Spot and forward rate extraction.

- Pricing fixed-income derivatives (e.g., interest rate swaps).
- Economic indicators (via yield spread, e.g., 10Y-2Y).

5.4 Mathematical Linkage

Forward Rate $f_{2,3}$ from spot rates r_2 and r_3:

$$(1 + r_3)^3 = (1 + r_2)^2 \cdot (1 + f_{2,3}) \tag{6}$$

Solving gives forward rates used in bootstrapping, term structure modeling, and swap pricing.

> **Insight:** The yield curve encapsulates market expectations of future rates and inflation, forming the discounting backbone for derivative pricing.

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6 Portfolio Basics

6.1 Portfolio Theory: An Overview

Portfolio theory is a fundamental framework in finance that helps investors understand how to build an optimal combination of assets to balance risk and return. The core idea is that selecting a group of diverse assets can improve the performance of a portfolio by reducing risk without necessarily sacrificing expected returns. This theory was formalized by Harry Markowitz and is widely known as Modern Portfolio Theory (MPT).

- **Diversification:** The principle of diversification suggests that by investing in a variety of assets, an investor can reduce unsystematic risk — the risk specific to individual assets or sectors. Diversification works because not all asset prices move together; gains in some investments can offset losses in others, leading to a more stable overall portfolio.
- **Risk-Return Tradeoff:** Investors constantly face a tradeoff between risk and expected return. Generally, assets that promise higher returns also carry higher risk. Portfolio theory helps investors quantify this relationship and choose the appropriate balance based on their risk tolerance and investment goals.
- **Efficient Frontier:** The efficient frontier represents the set of portfolios that provide the highest expected return for each level of risk. Portfolios lying on the efficient frontier are considered optimal, while those below the frontier are suboptimal because higher returns could be achieved for the same level of risk. Selecting portfolios along this frontier is a key application of MPT.

- **Sharpe Ratio:** The Sharpe ratio is a widely used metric that evaluates the performance of a portfolio relative to its risk. It is defined as the excess return (return above the risk-free rate) per unit of portfolio risk (standard deviation). A higher Sharpe ratio indicates a more attractive risk-adjusted return.

- **Correlation:** The correlation coefficient measures the degree to which two assets move together. Portfolio theory emphasizes combining assets with low or negative correlations to minimize overall portfolio volatility. When assets are less correlated, the combined portfolio tends to experience smaller fluctuations in value.

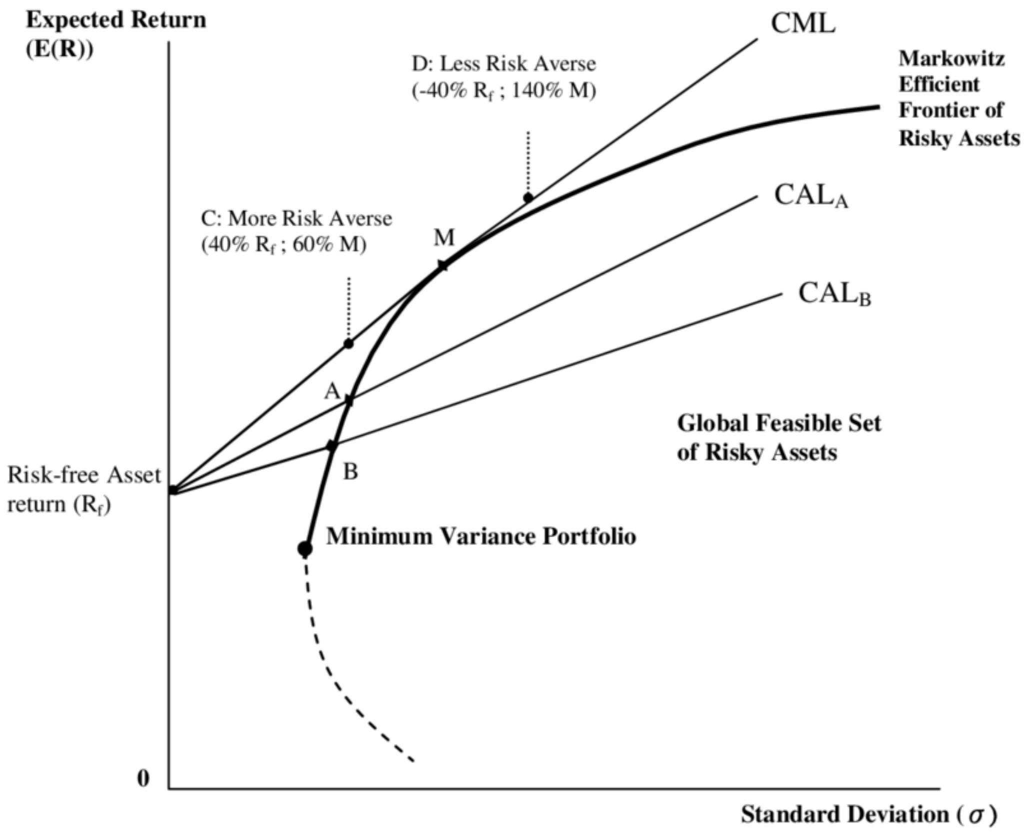

Figure 4: Illustration of Efficient Frontier

Insight: Diversification cannot eliminate all risk, but it is the most effective tool to reduce portfolio volatility. Systematic risk, which is inherent to the market, cannot be diversified away.

Modern Portfolio Theory is the starting point for many advanced topics in finance, including asset pricing models, risk management and portfolio optimization algorithms. The concepts introduced here provide the mathematical and strategic foundation for the quantitative analyses that follow in this report.

Insight: Diversification does not eliminate all risk but can significantly reduce portfolio volatility by minimizing unsystematic risk.

6.2 Overview

This section covers the essentials of Modern Portfolio Theory (MPT) applied to a seven-stock universe: `AMZN`, `AAPL`, `TSLA`, `NVDA`, `MSFT`, `META`, `GOOGL`.

We analyze historical price data, compute returns, covariances and then optimize for various objectives:

- Sharpe Ratio Maximization
- Maximum Return under Risk Constraints
- Minimum Volatility

6.3 Data Acquisition & Preprocessing

- **Data Source:** Manually downloaded CSVs from Yahoo Finance
- **Ticker List:** AMZN, AAPL, TSLA, NVDA, MSFT, META, GOOGL

For each ticker:

1. Download daily adjusted close prices over a chosen date range.
2. Compute daily log returns:
$$r_t = \ln\left(\frac{P_t}{P_{t-1}}\right) \tag{7}$$
3. Clean missing data by forward-filling or dropping early rows.

6.4 Return & Risk Metrics

Expected Return Vector (μ):

$$\mu_i = \frac{1}{T}\sum_{t=1}^{T} r_{i,t} \tag{8}$$

Covariance Matrix (Σ):

$$\Sigma_{ij} = \frac{1}{T-1}\sum_{t=1}^{T} (r_{i,t} - \bar{r}_i)(r_{j,t} - \bar{r}_j) \tag{9}$$

Portfolio Return (R_p):

$$R_p = w^\top \mu \tag{10}$$

Portfolio Variance (σ_p^2):

$$\sigma_p^2 = w^\top \Sigma w \tag{11}$$

Portfolio Standard Deviation (σ_p):

$$\sigma_p = \sqrt{w^\top \Sigma w} \tag{12}$$

Here, $w = [w_1, w_2, \dots, w_7]^\top$ are portfolio weights with $\sum_{i=1}^{7} w_i = 1$ and $w_i \geq 0$.

6.5 Optimization Objectives

6.5.1 Sharpe Ratio Maximization

Maximize:

$$\text{Sharpe}(w) = \frac{w^\top \mu - r_f}{\sqrt{w^\top \Sigma w}} \tag{13}$$

Subject to:

$$\sum_{i=1}^{7} w_i = 1, \quad w_i \geq 0 \tag{14}$$

Risk-free rate r_f is taken as 2%.

Notebook: `sharpe_maximum_MPT.ipynb`

- Define objective: $-\text{Sharpe}(w)$ (for minimization)
- Use `scipy.optimize.minimize` with constraints
- Plot efficient frontier by varying feasible weight combinations

6.5.2 Return Maximization under Risk Constraint

Maximize:

$$w^\top \mu \tag{15}$$

Subject to:

$$w^\top \Sigma w \leq \sigma_{\max}^2, \quad \sum_{i=1}^{7} w_i = 1, \quad w_i \geq 0 \tag{16}$$

Notebook: `returns_max_MPT.ipynb`

- Specify $\sigma_{\max}$ (e.g., 0.02 daily)
- Use `scipy.optimize.minimize` with variance constraint
- Generate return-risk tradeoff by varying $\sigma_{\max}$

6.5.3 Minimum Volatility Portfolio

Minimize:

$$w^\top \Sigma w \tag{17}$$

Subject to:

$$\sum_{i=1}^{7} w_i = 1, \quad w_i \geq 0 \tag{18}$$

Notebook: `std_dev_min_MPT.ipynb`

- Define objective: $w^\top \Sigma w$
- Apply weight constraints
- Compare with Sharpe-maximizing portfolio

6.6 Visualizations

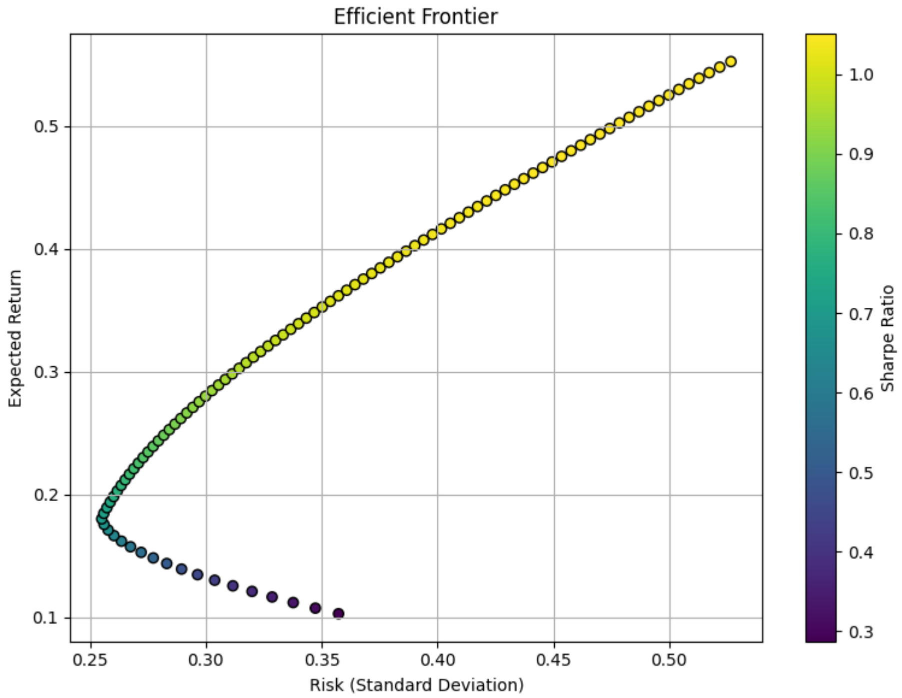

Figure 5: Efficient Frontier Plot

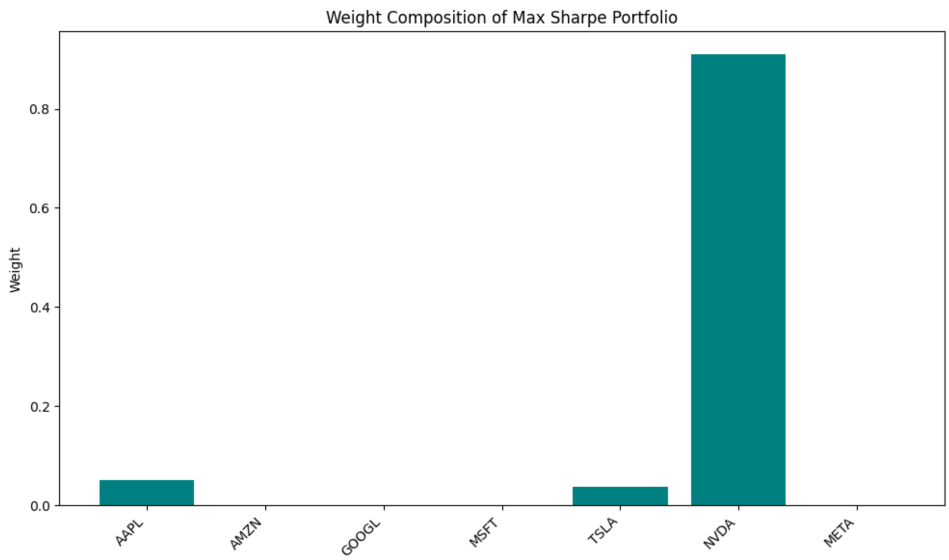

Figure 6: Weight Composition

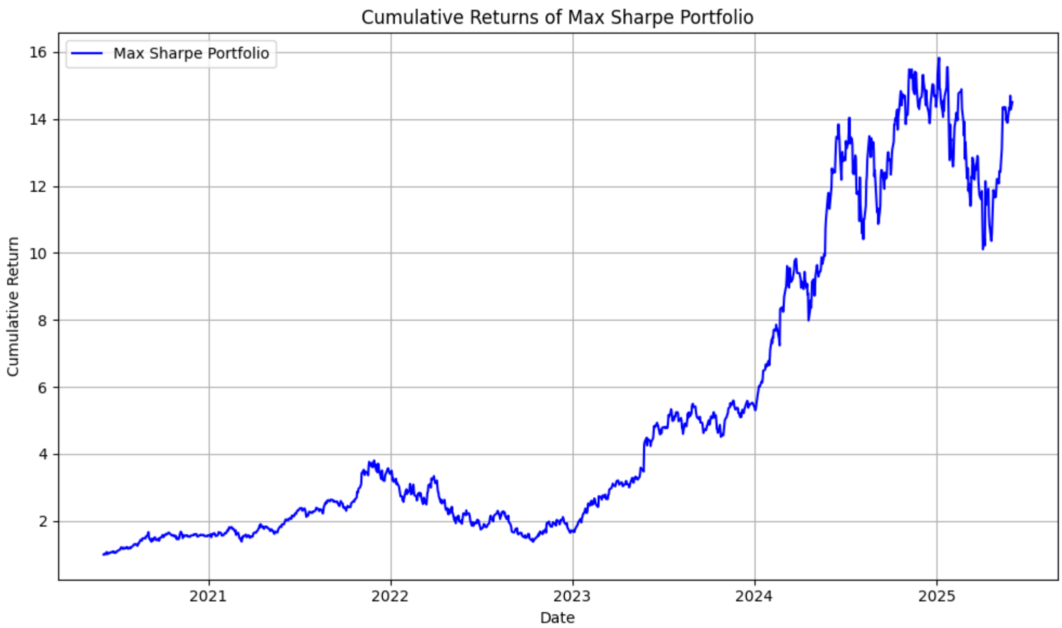

Figure 7: Cumulative Returns

6.7 Summary of Portfolio Results

Portfolio Type	Expected Return (%)	Volatility (%)	Sharpe Ratio
Sharpe-Maximized Portfolio	52.59	50.05	1.0507
Return-Maximized (Risk Capped)	55.25	52.67	1.0490
Minimum Volatility Portfolio	18.08	25.50	0.5911

Table 2: Summary of Portfolio Optimization Results

Note: The numerical values in Table 2 are derived from Python simulations using the historical data of the selected assets.

6.8 Code Implementation

All implementation code resides in the `notebooks/` directory. The following Jupyter notebooks demonstrate the full workflows:

- **sharpe_maximum_MPT.ipynb**
  - Historical data collection
  - Sharpe ratio maximization:

```
# Objective: negative Sharpe ratio
def neg_sharpe(weights, mu, Sigma, rf):
    port_return = weights.dot(mu)
    port_vol = np.sqrt(weights.T @ Sigma @ weights)
    return -(port_return - rf) / port_vol
```

  - Optimization and efficient frontier plotting

- **returns_max_MPT.ipynb**
  - Return maximization under volatility constraint:

```python
# Constraints: portfolio variance <= sigma_max^2
constraints = (
    {'type': 'eq', 'fun': lambda w: np.sum(w) - 1},
    {'type': 'ineq', 'fun': lambda w: sigma_max**2 - w.T
       @ Sigma @ w}
)
```

  - Trade-off curve generation

- **std_dev_min_MPT.ipynb**
  - Minimum variance objective:

```python
# Objective: portfolio variance
def portfolio_var(weights, Sigma):
    return weights.T @ Sigma @ weights
```

  - Weight optimization and comparison with Sharpe-optimal portfolio

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7 Forwards and Pricing Models

7.1 Forwards: A Theoretical Overview

A **forward contract** is a private agreement between two parties to buy or sell an asset at a specified price (the forward price F_0) at a future date. Unlike exchange-traded futures, forwards are over-the-counter (OTC) instruments that can be tailored to the specific needs of the counterparties.

- **Key Features:**
  - Customizable in contract size, maturity, and settlement.
  - Traded directly between parties, exposing them to *counterparty risk*.
  - No initial cash flow at the start of the contract.

- **Payoff at Maturity:**

$$\text{Payoff}(S_T) = S_T - F_0 \tag{19}$$

  where S_T is the spot price at maturity and F_0 is the agreed forward price.

- **Profit in Present Value Terms:**

$$\text{PV Profit} = e^{-rT}(S_T - F_0) \tag{20}$$

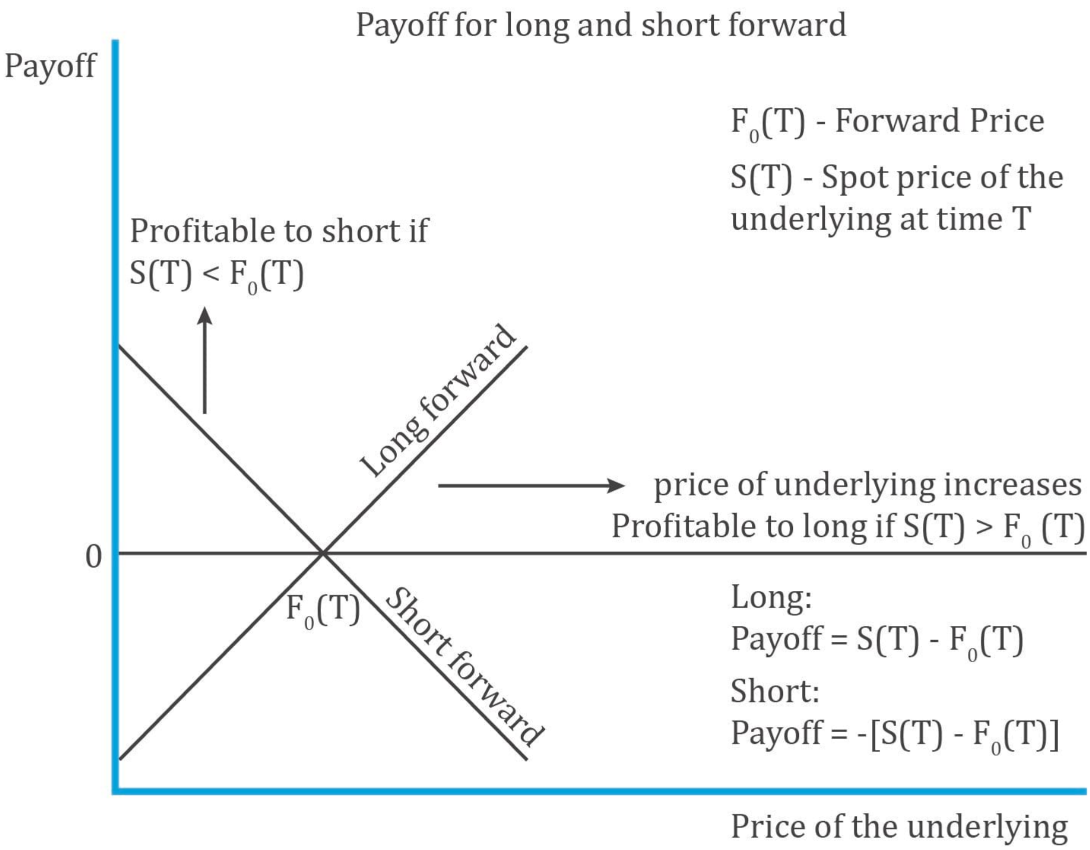

Figure 8: Forward Contract Payoff Diagram

7.2 No-Arbitrage Forward Pricing

The forward price is derived based on the principle of **no-arbitrage**, which ensures that the forward contract price aligns with the cost of carrying the asset to the future settlement date.

7.2.1 Case 1: Continuous Dividend Yield

When the underlying asset pays a continuous dividend yield q, the forward price is:

$$F_0 = S_0 \cdot e^{(r-q)T} \tag{21}$$

7.2.2 Case 2: Discrete Dividends

When the asset pays discrete dividends (known in advance), the forward price adjusts for the present value of these dividends:

$$F_0 = (S_0 - \mathrm{PV}(\mathrm{Dividends})) \cdot e^{rT} \tag{22}$$

7.2.3 Intuition Behind Forward Pricing

- The forward price reflects the **cost of carry** — the net financing cost of holding the asset.

- For dividend-paying assets, expected dividends reduce the forward price.
- Arbitrage ensures that discrepancies between forward prices and theoretical prices are quickly corrected by trading strategies like *cash-and-carry arbitrage* and *reverse cash-and-carry arbitrage*.

Insight: Forward pricing is built on the assumption of frictionless markets, no transaction costs, and no arbitrage opportunities.

7.3 Data Acquisition & Preprocessing

- **Data Source:** Manually exported CSVs from Yahoo Finance
- **Files:** `data/AAPL.csv` (adjusted close), `data/AAPL_dividends.csv`

1. Download price and dividend CSVs for the analysis period.
2. Load into pandas and align on business-day frequency:

```
import pandas as pd
spot = pd.read_csv('data/AAPL.csv', parse_dates=['Date'],
    index_col='Date')['Adj Close']
divs = pd.read_csv('data/AAPL_dividends.csv', parse_dates=['
    Date'], index_col='Date')['Dividends']
df = pd.DataFrame({'Spot': spot, 'Dividends': divs}).asfreq('
    B').ffill()
```

7.4 Implementation Details

7.4.1 Dividend Present Value

```
import numpy as np
r = 0.05            # risk-free rate
end = '2025-05-31'
days = (pd.to_datetime(df.index) - pd.to_datetime(end)).days
pv_divs = (df['Dividends'] * np.exp(-r * days/365)).sum()
print('PV(dividends):', round(pv_divs, 2))
```

7.4.2 Forward Price Computation

```
S0 = df['Spot'].iloc[-1]
T = 0.5             # 6 months in years
q = df['Dividends'].sum() / S0
F_cont = S0 * np.exp((r - q) * T)
F_disc = (S0 - pv_divs) * np.exp(r * T)
print(f'F_cont: {F_cont:.2f}, F_disc: {F_disc:.2f}')
```

7.4.3 Scenario Analysis & Payoff Table

```
scenarios = {'Down 10%': 0.9*S0, 'Base': S0, 'Up 10%': 1.1*S0}
rows = []
for name, ST in scenarios.items():
    payoff = ST - F_cont
    pv_profit = np.exp(-r * T) * payoff
    rows.append({'Scenario': name, 'S_T': ST, 'Payoff': payoff, '
       PV Profit': pv_profit})
payoff_df = pd.DataFrame(rows)
payoff_df.to_csv('Data/forward_payoff_table.csv')
```

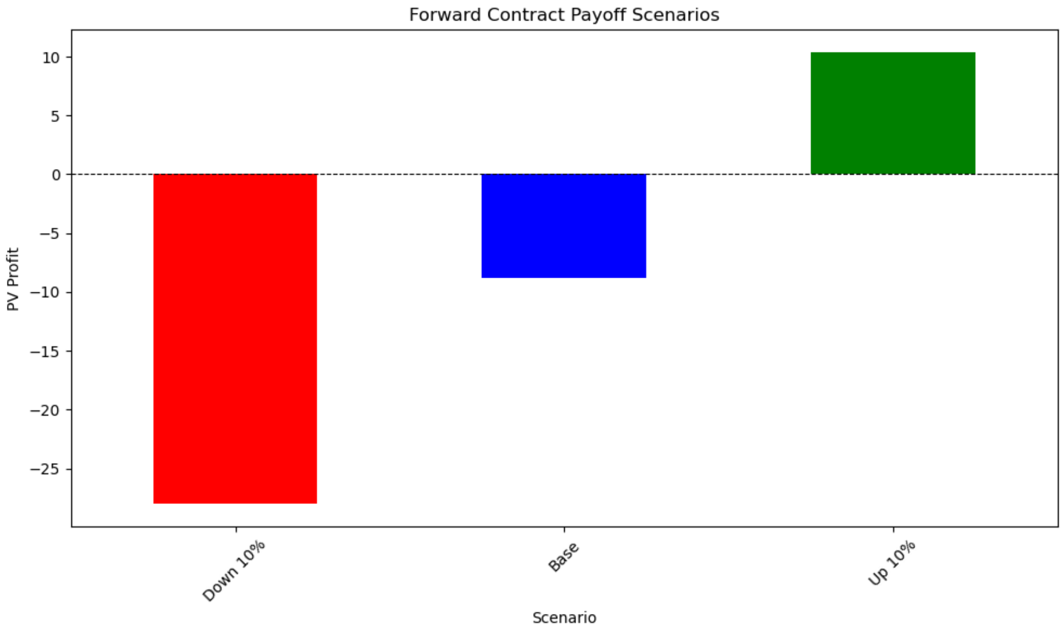

Figure 9: Forward Contract Scenario Payoffs

7.4.4 Arbitrage Strategy

```
F_mkt = 155.00
if F_mkt > F_cont:
    strategy = 'Cash-and-carry: buy spot, finance, short forward'
else:
    strategy = 'Reverse carry: short spot, invest proceeds, long
       forward'
print('Strategy:', strategy)
```

7.4.5 Payoff Visualization

```
import matplotlib.pyplot as plt
ST_vals = np.linspace(0.8*S0, 1.2*S0, 100)
payoffs = ST_vals - F_cont
```

```
plt.plot(ST_vals, payoffs)
plt.axhline(0, color='black', lw=0.5)
plt.title('Forward Payoff at Expiry')
plt.xlabel('S_T')
plt.ylabel('Payoff')
plt.savefig('plots/forward_payoff_plot.png')
```

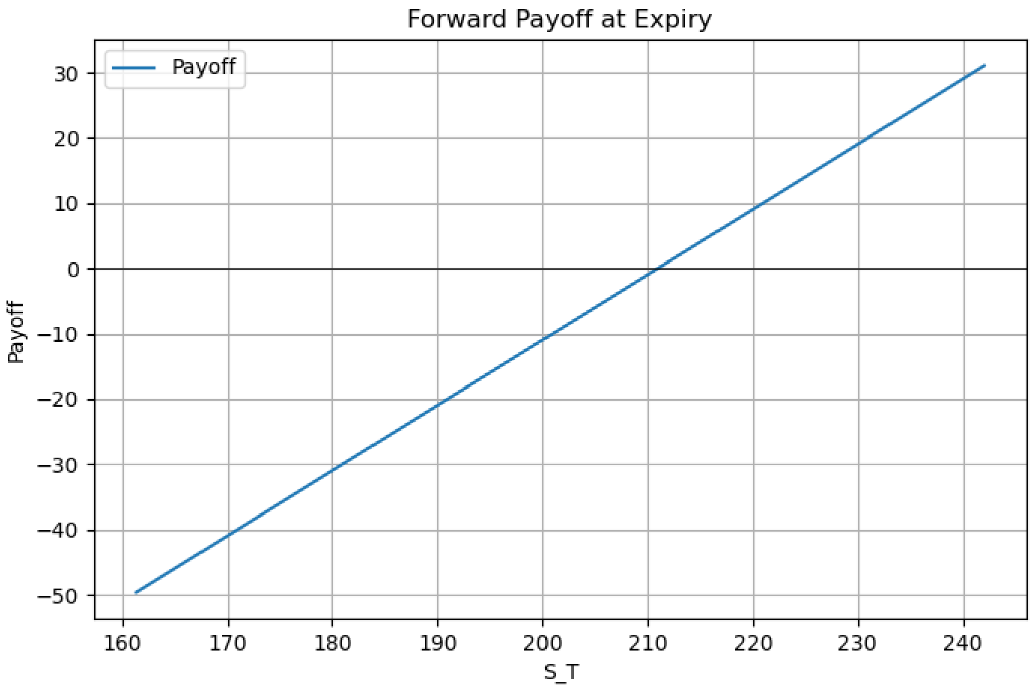

Figure 10: Forward Payoff at Expiry

Notebook: `Forwards_pricingipynb`

7.5 Self-Analysis Summary

- Reviewed the theoretical foundations of forward pricing models under both continuous and discrete dividend assumptions.
- Verified numerical forward prices against Python simulations.
- Conducted sensitivity analysis to evaluate payoff changes across different price scenarios.
- Identified arbitrage conditions and designed cash-and-carry strategies accordingly.

8 Futures and Pricing Models

8.1 Introduction to Futures Contracts

Futures contracts are standardized agreements to buy or sell an asset at a predetermined price at a specified future date. Unlike forward contracts, futures are traded on exchanges, require margin deposits and are settled daily through a process called **mark-to-market (MTM)**.

8.2 Key Features of Futures

- **Standardization:** Futures contracts have fixed specifications such as lot size, expiry date and tick size.
- **Exchange-Traded:** All futures are traded through regulated exchanges, ensuring liquidity and price transparency.
- **Margin Requirements:** Traders must maintain initial and maintenance margins, which are adjusted daily based on MTM gains or losses.
- **Daily Settlement:** Futures are marked to market at the end of each trading day, with profits and losses realized daily.

8.3 Futures Pricing: The Cost-of-Carry Model

The theoretical futures price is derived using the cost-of-carry relationship:

$$F_t = S_t \cdot e^{(r-q)(T-t)} \tag{23}$$

Where:

- F_t = Futures price at time t
- S_t = Spot price at time t
- r = Risk-free interest rate
- q = Continuous dividend yield
- $(T - t)$ = Time remaining until contract expiry

This formula assumes that investors can borrow or lend at the risk-free rate and that there are no arbitrage opportunities.

8.4 Basis and Its Behavior

$$\text{Basis} = F_t - S_t \tag{24}$$

- The basis typically converges to zero as the futures contract approaches maturity.
- **Contango:** When futures prices are above spot prices ($F_t > S_t$).
- **Backwardation:** When futures prices are below spot prices ($F_t < S_t$).

8.5 Mark-to-Market (MTM) and Margining

- Futures positions are settled daily via the MTM process.
- Daily P&L is computed and credited or debited from the trader's margin account.
- The exchange may issue a **margin call** if the account balance falls below the maintenance margin.

8.6 Practical Implications

- Futures contracts allow for effective hedging of price risk in underlying assets.
- MTM ensures that losses are realized and managed promptly.
- Pricing discrepancies between theoretical and market futures can signal arbitrage opportunities.

8.7 Python Implementation and Analysis

8.7.1 Data Acquisition and Preprocessing

```python
import pandas as pd

spot = pd.read_csv('data/NIFTY_spot.csv', parse_dates=['Date'], index_col='Date')['Close']
futures = pd.read_csv('data/NIFTY_futures.csv', parse_dates=['Date'], index_col='Date')['Close']

df = pd.DataFrame({'Spot': spot, 'Futures': futures}).dropna()
df.head()
```

Listing 1: Loading Spot and Futures Data

8.7.2 Theoretical Futures Price and Basis

```python
import numpy as np

r = 0.05   # risk-free rate
q = 0.00   # assume zero dividend yield
T = pd.to_datetime('2025-06-30')   # futures expiry

df['Tau'] = (T - df.index).days / 365
df['F_theo'] = df['Spot'] * np.exp((r - q) * df['Tau'])
df['Basis'] = df['Futures'] - df['F_theo']
```

Listing 2: Calculating Theoretical Futures Price and Basis

8.7.3 Spot, Futures and Theoretical Futures Plot

```python
import matplotlib.pyplot as plt

plt.figure(figsize=(10, 6))
plt.plot(df.index, df['Spot'], label='Spot Price')
plt.plot(df.index, df['Futures'], label='Market Futures')
plt.plot(df.index, df['F_theo'], label='Theoretical Futures',
    linestyle='--')
plt.title('Spot vs. Futures vs. Theoretical Futures')
plt.xlabel('Date')
plt.ylabel('Price')
plt.legend()
plt.savefig('assets/futures_price_plot.png')
plt.show()
```

Listing 3: Plotting Spot, Market Futures and Theoretical Futures

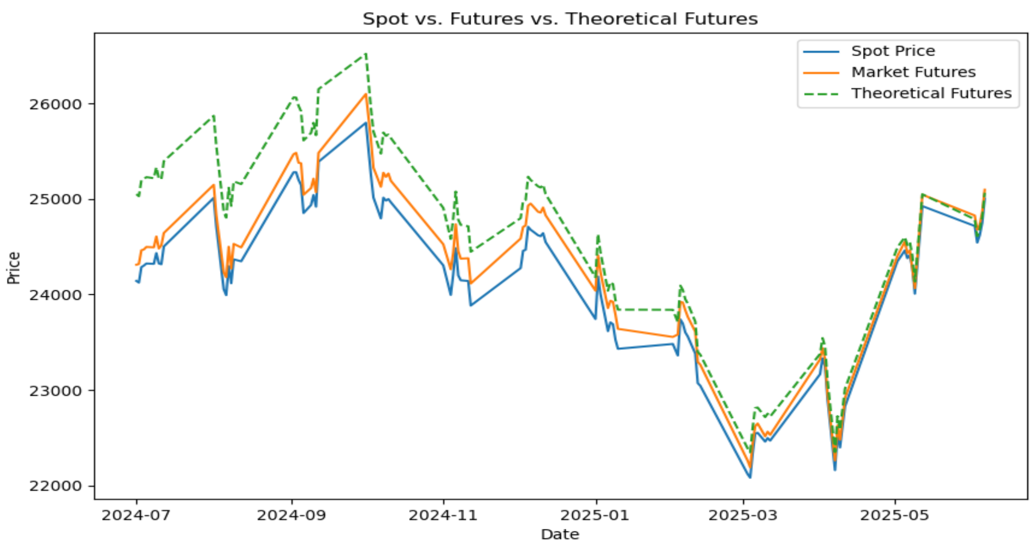

Figure 11: Spot vs. Futures vs. Theoretical Futures

8.7.4 Basis Plot

```
plt.figure(figsize=(8, 4))
plt.plot(df.index, df['Basis'])
plt.axhline(0, color='black', lw=0.5)
plt.title('Basis Over Time')
plt.xlabel('Date')
plt.ylabel('Basis')
plt.savefig('assets/futures_basis_plot.png')
plt.show()
```

Listing 4: Plotting Basis Over Time

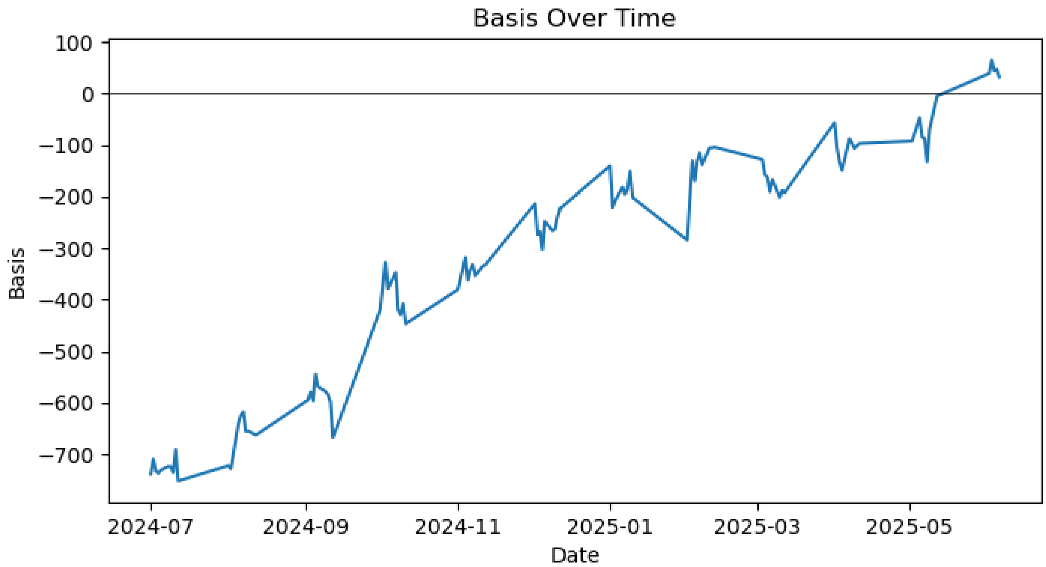

Figure 12: Basis Over Time

8.7.5 Mark-to-Market (MTM) Simulation

```
contract_size = 75  # NIFTY lot size
df['Delta_PnL'] = (df['Futures'].diff()) * contract_size
df['Cumulative_PnL'] = df['Delta_PnL'].cumsum()

plt.figure(figsize=(10, 5))
plt.plot(df.index, df['Cumulative_PnL'], label='MTM P&L')
plt.title('Mark-to-Market P&L Simulation')
plt.xlabel('Date')
plt.ylabel('Cumulative P&L')
plt.legend()
plt.savefig('assets/futures_mtm_pnl.png')
plt.show()
```

Listing 5: Simulating MTM P&L

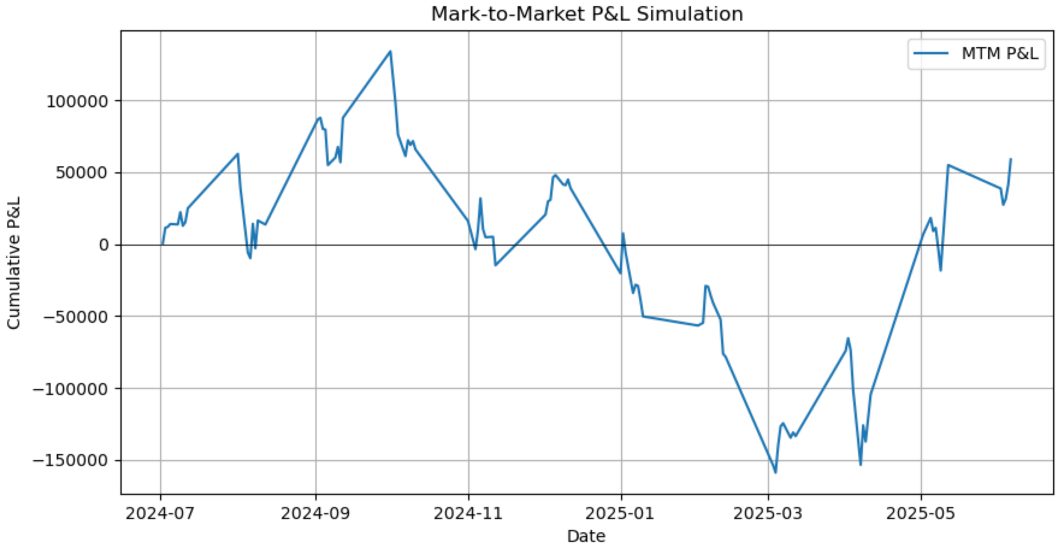

Figure 13: Mark-to-Market P&L Simulation

Notebook: `futures_analysisipynb`

8.8 Self-Analysis Summary

- Compared theoretical futures prices to market futures using the cost-of-carry model.
- Analyzed basis behavior and verified convergence towards zero near expiry.
- Simulated MTM P&L and visualized daily profit and loss accumulation.
- Observed futures dynamics across contango and backwardation market conditions.

---

9 Hedging with Futures

9.1 Introduction to Futures Hedging

Futures hedging is a widely used risk management strategy that involves using standardized, exchange-traded futures contracts to protect against adverse price movements in an underlying asset. It allows portfolio managers, businesses and individual investors to lock in prices and minimize potential losses.

Futures hedging does not require upfront premiums and is highly effective when there is a strong correlation between the futures and the underlying asset. However, it introduces **basis risk**, as futures prices and spot prices may not move perfectly in sync.

9.2 Hedge Strategies

- **Short Hedge:** Used by holders of the underlying asset (long position) to protect against price declines by selling futures.
- **Long Hedge:** Used by entities with future purchasing needs (short position) to protect against price increases by buying futures.

9.3 The Optimal Hedge Ratio

The **optimal hedge ratio** determines the size of the futures position required to minimize the variance of the combined spot and futures position. It is calculated as:

$$h^* = \frac{\sigma_S}{\sigma_F}\rho_{SF} \tag{25}$$

Where:

- σ_S = Standard deviation of spot price changes
- σ_F = Standard deviation of futures price changes
- ρ_{SF} = Correlation between spot and futures returns

A higher correlation between the spot and futures improves hedge effectiveness.

9.4 Practical Setup

For this analysis:

- Spot and futures data were manually downloaded from the NSE India website.
- Hedging was performed using NIFTY futures contracts.

9.5 Data Acquisition and Preprocessing

```python
import pandas as pd

spot = pd.read_csv('data/NIFTY_spot.csv', parse_dates=['Date'],
    index_col='Date')['Close']
futures = pd.read_csv('data/NIFTY_futures.csv', parse_dates=['
    Date'], index_col='Date')['Close']

df = pd.DataFrame({'Spot': spot, 'Futures': futures}).dropna()
df['Spot_Returns'] = df['Spot'].pct_change()
df['Futures_Returns'] = df['Futures'].pct_change()
df = df.dropna()
```

Listing 6: Data Loading and Preprocessing

9.6 Hedge Ratio Calculation

```
import numpy as np

sigma_S = df['Spot_Returns'].std()
sigma_F = df['Futures_Returns'].std()
rho = df['Spot_Returns'].corr(df['Futures_Returns'])

h_star = (sigma_S / sigma_F) * rho
print('Optimal Hedge Ratio:', round(h_star, 3))
```

Listing 7: Hedge Ratio Calculation

9.7 Hedge Simulation

```
# Assume exposure size
exposure = 1_000_000  # INR

# Futures contract size (example)
contract_size = 75

# Number of contracts
futures_position = h_star * exposure / (df['Futures'].iloc[-1] *
    contract_size)
print('Futures Contracts:', round(futures_position))

# Hedged P&L Calculation
df['Hedged_PnL'] = exposure * df['Spot_Returns'] -
    futures_position * contract_size * df['Futures_Returns']
df['Cumulative_Hedged_PnL'] = df['Hedged_PnL'].cumsum()
```

Listing 8: Hedge Simulation

9.8 Hedging Performance Visualization

```
import matplotlib.pyplot as plt

plt.figure(figsize=(10,5))
plt.plot(df.index, df['Cumulative_Hedged_PnL'], label='Cumulative
    Hedged P&L')
plt.title('Hedging with Futures: Performance')
plt.xlabel('Date')
plt.ylabel('Cumulative P&L (INR)')
plt.legend()
plt.savefig('assets/futures_hedge_performance.png')
plt.show()
```

Listing 9: Plotting Hedged P&L

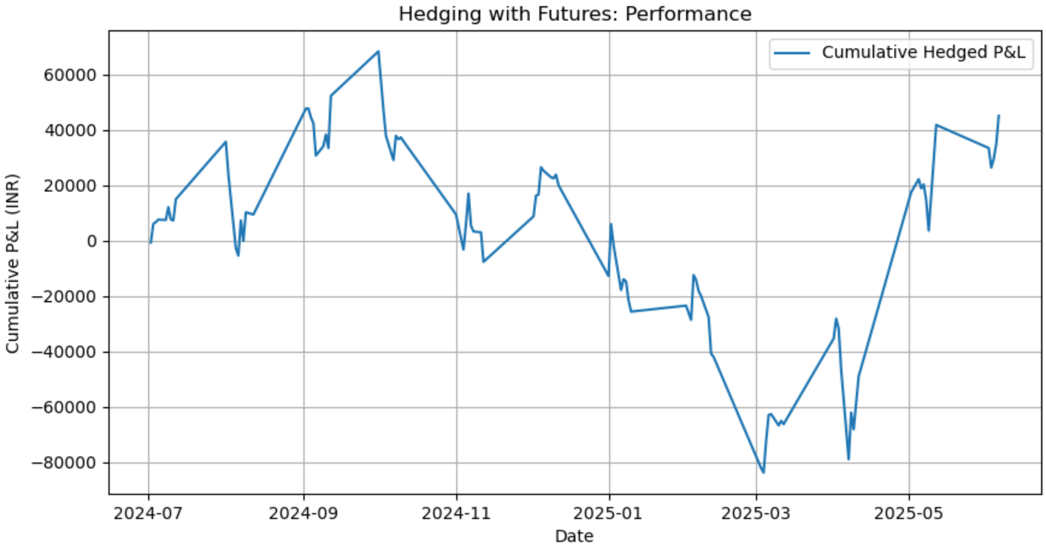

Figure 14: Futures Hedging Performance: Cumulative P&L

Notebook: `futures_hedgingipynb`

9.9 Self-Analysis Summary

- Calculated the optimal hedge ratio using real historical spot and futures data.
- Simulated a dynamic futures-based hedge to track profit and loss.
- Visualized hedge effectiveness and residual risk using cumulative P&L plots.
- Observed that futures hedging is powerful but not perfect due to basis risk and slippage.

---

10 Options: Basics, Payoffs, and Hedging Strategies

10.1 Introduction to Options

Options are versatile financial derivatives that grant the holder the **right, but not the obligation** to buy or sell an underlying asset at a pre-specified price (strike price) on or before a predetermined expiration date.

Options provide flexible tools for risk management, speculation and strategic portfolio positioning. Unlike futures contracts, the maximum loss for option buyers is limited to the premium paid.

10.2 Types of Options

- **Call Option:** Grants the right to *buy* the underlying asset.
- **Put Option:** Grants the right to *sell* the underlying asset.

10.3 Option Payoff Structures

- **Call Option Payoff:**
$$\text{Payoff}_{\text{Call}} = \max(S_T - K, 0) \tag{26}$$
- **Put Option Payoff:**
$$\text{Payoff}_{\text{Put}} = \max(K - S_T, 0) \tag{27}$$

  Where:

  - S_T = Spot price at expiry
  - K = Strike price

10.4 Moneyness of Options

Term	Condition
In-the-Money (ITM)	Call: $S_T > K$, Put: $S_T < K$
At-the-Money (ATM)	Call/Put: $S_T = K$
Out-of-the-Money (OTM)	Call: $S_T < K$, Put: $S_T > K$

Table 3: Option Moneyness Classification

10.5 Protective Put and Covered Call Strategies

Options are commonly used in combination with existing asset positions to create hedging strategies that balance risk and return.

10.5.1 Protective Put

A protective put involves **buying a put option while holding the underlying asset**. This strategy limits downside risk while preserving upside potential.

- Long Asset + Long Put
- Acts like insurance: maximum loss is capped at the difference between the asset price and the strike price, minus the premium.

10.5.2 Covered Call

A covered call involves **selling a call option while holding the underlying asset**. This strategy generates additional income from the option premium but limits the upside profit.

- Long Asset + Short Call
- Premium collected offers downside cushion, but profits are capped above the strike price.

10.6 Practical Code Simulations

10.6.1 Scenario-Based Payoff Simulation: Call Option Example

```python
import numpy as np
import pandas as pd

S0 = 22000
K_call = 22500
premium_call = 150

ST = np.array([19800, 22000, 24200])
call_payoff = np.maximum(ST - K_call, 0) - premium_call

df_call = pd.DataFrame({'Scenario': ['Down', 'Flat', 'Up'],
                        'S_T': ST,
                        'Call Payoff': call_payoff})
df_call.to_csv('assets/call_option_payoff_table.csv')
df_call
```

Listing 10: Call Option Payoff Calculation

10.6.2 Scenario-Based Payoff Simulation: Put Option Example

```python
K_put = 21500
premium_put = 200
put_payoff = np.maximum(K_put - ST, 0) - premium_put

df_put = pd.DataFrame({'Scenario': ['Down', 'Flat', 'Up'],
                       'S_T': ST,
                       'Put Payoff': put_payoff})
df_put.to_csv('assets/put_option_payoff_table.csv')
df_put
```

Listing 11: Put Option Payoff Calculation

10.6.3 Option Payoff Visualization

```
import matplotlib.pyplot as plt

plt.figure(figsize=(8,5))
plt.plot(ST, call_payoff, label='Call Option Payoff')
plt.plot(ST, put_payoff, label='Put Option Payoff')
plt.axhline(0, color='black', lw=0.5)
plt.title('Option Payoff Diagrams')
plt.xlabel('S_T (Spot Price at Expiry)')
plt.ylabel('Payoff (INR)')
plt.legend()
plt.savefig('assets/option_payoff_diagrams.png')
plt.show()
```

Listing 12: Option Payoff Diagram Plot

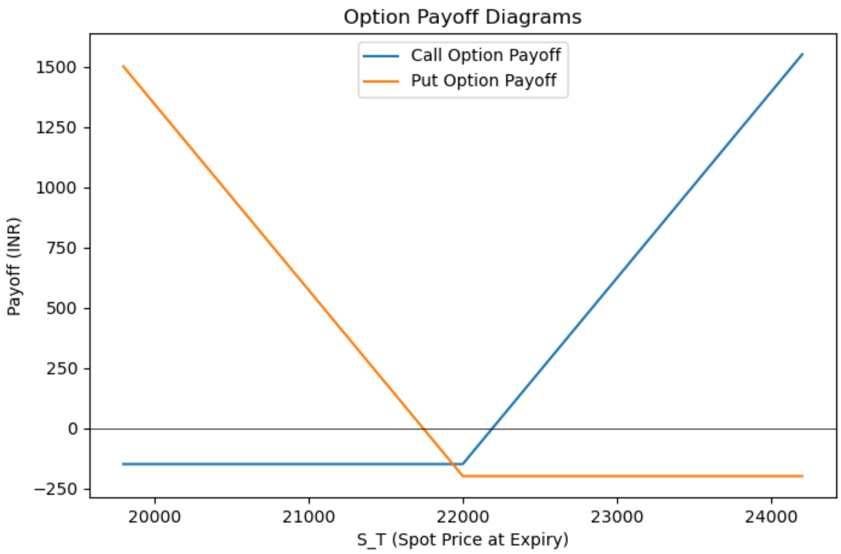

Figure 15: Option Payoff Diagrams for Call and Put Options

10.7 Hedging Strategy Simulations

10.7.1 Protective Put Simulation

```
spot_change = ST - S0
protective_put_total = spot_change + np.maximum(K_put - ST, 0) -
    premium_put
df_protective = pd.DataFrame({'Scenario': ['Down', 'Flat', 'Up'],
                              'S_T': ST,
                              'Total Payoff':
                                  protective_put_total})
df_protective
```

Listing 13: Protective Put Payoff Simulation

10.7.2 Covered Call Simulation

```
covered_call_total = spot_change - np.maximum(ST - K_call, 0) +
    premium_call
df_covered = pd.DataFrame({'Scenario': ['Down', 'Flat', 'Up'],
                           'S_T': ST,
                           'Total Payoff': covered_call_total})
df_covered
```

Listing 14: Covered Call Payoff Simulation

10.7.3 Combined Payoff Visualization

```
plt.figure(figsize=(8,5))
plt.plot(ST, spot_change, label='Unhedged Spot')
plt.plot(ST, protective_put_total, label='Protective Put')
plt.plot(ST, covered_call_total, label='Covered Call')
plt.axhline(0, color='black', lw=0.5)
plt.title('Options Hedging Strategies Payoff')
plt.xlabel('S_T (Spot Price at Expiry)')
plt.ylabel('Total Payoff (INR)')
plt.legend()
plt.savefig('assets/options_hedging_payoff.png')
plt.show()
```

Listing 15: Combined Hedging Payoff Plot

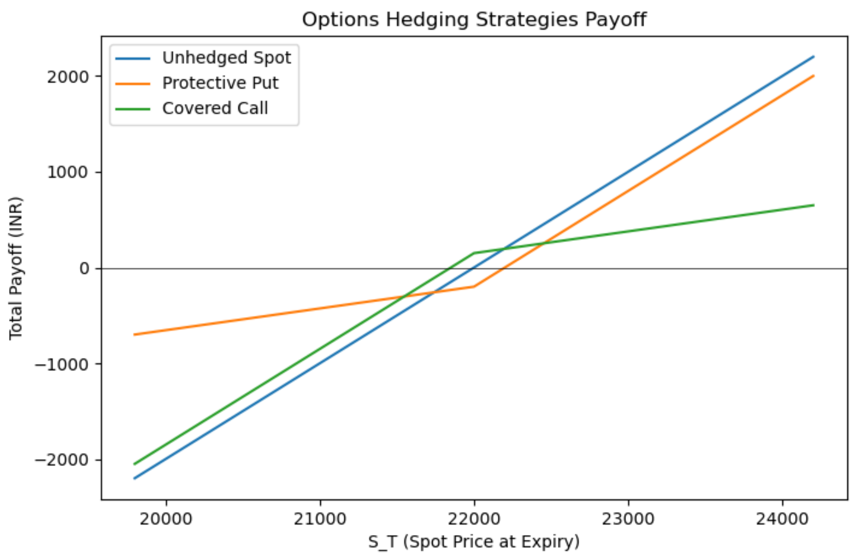

Figure 16: Combined Payoff: Protective Put, Covered Call, and Unhedged Spot

Notebook: `Optionsipynb`

10.8 Key Learnings

- **Protective Put:** Offers downside protection with unlimited upside but requires premium payment.
- **Covered Call:** Generates income but limits profit potential if the price rises significantly.
- Option-based hedging allows customization of risk profiles.
- Scenario-based simulations provide intuitive understanding of option strategies.

---

11 Conclusion

11.1 Summary of Work

This report presents a detailed exploration of the fundamental and practical aspects of derivative pricing. Throughout the project, the journey traversed from the basics of financial markets and interest rate structures to more advanced derivative instruments such as forwards, futures and options. Each topic was supported by hands-on Python simulations, data-driven analysis and real-world scenarios to solidify both theoretical and practical understanding. Key themes covered include:

- Core principles of financial markets, cash flow streams, bond pricing and yield curves.
- Portfolio optimization using Modern Portfolio Theory, focusing on Sharpe ratio maximization, return maximization under risk constraints and minimum volatility portfolios.
- Forward and futures pricing models, including cost-of-carry relationships and basis behavior.
- Hedging strategies using futures, focusing on optimal hedge ratio determination and futures-based risk management.
- Options fundamentals, payoff structures and hedging strategies using protective puts and covered calls.

Each module included extensive simulations using manually collected historical data to bridge the gap between theoretical models and practical financial decision-making.

11.2 Learning Outcomes

The most valuable learnings from this project include:

- Developed a clear conceptual understanding of derivatives, their pricing models and their strategic applications.
- Gained proficiency in using Python libraries (`pandas`, `numpy`, `matplotlib`, `scipy`) for financial data analysis and simulation.
- Built confidence in implementing portfolio optimization and hedging strategies programmatically.
- Cultivated the ability to critically assess the assumptions behind derivative pricing models, such as the reliance on arbitrage-free markets and perfect liquidity.
- Learned how to structure financial analyses and communicate results effectively through data visualizations and scenario tables.

11.3 Future Endeavors

Moving forward, I aim to:

- Deepen my understanding of complex derivative models, particularly **Discrete Time Models**, **Martingale Theory** and **Continuous Time Models**.
- Study the **Black-Scholes Pricing Model** in detail and apply it to real option pricing scenarios.
- Explore advanced option trading strategies, including spreads, straddles and dynamic hedging techniques.
- Gain exposure to **stochastic processes** and **risk-neutral pricing** to build a stronger quantitative foundation.

This midterm project has laid a solid groundwork and I am excited to continue building upon this knowledge in the upcoming phases of the Mathematics of Derivative Pricing journey.

References

[1] Zvi Bodie, Alex Kane, and Alan J. Marcus. *Investments*. McGraw-Hill Education, 10th edition, 2014. Introduces core concepts in portfolio management, asset pricing, market efficiency and derivatives in investment management.

[2] Nasdaq Exchange. Nasdaq historical market data, 2025. Manually downloaded historical stock price and dividend data used for forward pricing analysis.

[3] John C. Hull. *Options, Futures and Other Derivatives.* Pearson Education Limited, 10th edition, 2017. A comprehensive textbook covering the theoretical foundations and practical applications of derivatives, including forwards, futures, options and exotic derivatives.

[4] CFA Institute. Cfa level i derivatives full playlist, 2023. Comprehensive YouTube playlist covering derivative pricing, hedging strategies, portfolio management, options and futures contracts.

[5] CFA Institute. Hedging with futures — cfa level i, 2023. Explains futures contracts, hedging mechanisms and real-world applications of futures in portfolio management.

[6] CFA Institute. Option hedging basics — cfa level i, 2023. Provides an introduction to options hedging strategies, focusing on protective puts, covered calls and risk management.

[7] CFA Institute. Pricing and valuation of forward commitments — cfa level ii, 2023. Covers forward pricing models, including continuous and discrete dividend cases, with practical valuation methods.

[8] CFA Institute. Pricing and valuation of futures contracts — cfa level i, 2023. Explains futures pricing using the cost-of-carry model, introduces basis risk and demonstrates mark-to-market procedures.

[9] CFA Institute. Protective puts and covered calls — cfa level i, 2023. A step-by-step guide to understanding option strategies that limit losses or generate additional income.

[10] Robert L. McDonald. *Derivatives Markets.* Pearson Education Limited, 3rd edition, 2013. Detailed explanation of derivative instruments with extensive focus on pricing models, hedging strategies and arbitrage arguments.

[11] National Stock Exchange of India. Historical market data – nse india, 2025. Manually downloaded historical spot and futures price data used in futures and hedging analysis.

[12] Overleaf. Overleaf online latex editor, 2024. The collaborative online LaTeX editor used for compiling and submitting this midterm report.

[13] FinQuiz Pro. Pricing and valuation of forward contracts, 2023. Detailed CFA Level II explanation of forward contracts, practical pricing scenarios and arbitrage conditions.

[14] Vatsh Van. Mathematics of derivative pricing – github repository, 2025. Complete project repository including Jupyter notebooks, code implementations, scenario simulations and plotted visualizations.