CHAPTER

6

LOGARITHMIC FUNCTIONS

Syllabus coverage

TOPIC 4.1: THE LOGARITHMIC FUNCTION

Logarithmic functions

4.1.1 define logarithms as indices: $a^x = b$ is equivalent to $x = \log_a b$ i.e. $a^{\log_a b} = b$
4.1.2 establish and use the algebraic properties of logarithms
4.1.3 examine the inverse relationship between logarithms and exponentials: $y = a^x$ is equivalent to $x = \log_a y$
4.1.4 interpret and use logarithmic scales
4.1.5 solve equations involving indices using logarithms
4.1.6 identify the qualitative features of the graph of $y = \log_a x$ ($a > 1$), including asymptotes, and of its translations $y = \log_a x + b$ and $y = \log_a (x - c)$
4.1.7 solve simple equations involving logarithmic functions algebraically and graphically
4.1.8 identify contexts suitable for modelling by logarithmic functions and use them to solve practical problems

Calculus of the natural logarithmic function

4.1.9 define the natural logarithm $\ln x = \log_e x$
4.1.10 examine and use the inverse relationship of the functions $y = e^x$ and $y = \ln x$

Video playlists (5):

6.1 Logarithms
6.2 Exponential and logarithmic equations
6.3 The logarithmic function $y = \log_a(x)$
6.4 Applications of logarithmic functions
WACE question analysis Logarithmic functions

Worksheets (2):

6.1 Logarithm laws • Logarithms review

Puzzles (2):

6.2 Logarithms – Solving equations 1 • Logarithms – Solving equations 2

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

6.1 Logarithms

Logarithms have many uses in science. pH is logarithmic and is a measure of how acidic or basic a solution is. Other examples include the Richter scale for measuring earthquake strength and decibels (dB), which are used to measure sound intensity.

A **logarithm** (or log) is the power or **exponent** to which a base is raised to yield a certain number.

$3^4 = 81$

This means that when we multiply 3 by itself 4 times, we get 81.

Another way of writing this is by using a **logarithm**, which is abbreviated as **log**.

$\log_3(81) = 4$

This is read 'the logarithm of 81, to base 3, is 4'.

The logarithm of a number to base a is the **power** to which a must be raised to give that number.

For example: $\log_5(625)$ means the power to which 5 is raised to get 625.

The equation $5^4 = 625$, is equivalent to $\log_5(625) = 4$.

Video playlist Logarithms

Worksheets Logarithm laws

Logarithms review

Some special logarithms

Base 10: When the base is omitted from the logarithm, we assume it is a base 10 logarithm.
$\log(100)$ is interpreted as $\log_{10}(100)$ and has a value of 2.

Base e: When a logarithm is to base e, it is called a natural logarithm.
$\log_e(x) = \ln(x)$

Equations in logarithmic form and exponential form

The logarithmic equation $\log_a(b) = x$ can also be written in exponential form as $a^x = b$.

The relations $a^x = b$ and $\log_a(b) = x$ are equivalent as they show the same relationship between a, b and x, but they are written in different forms. The mathematical symbol for equivalence is $\Leftrightarrow$.

Logarithms		
Logarithmic form	$\Leftrightarrow$	Exponential form
$x = \log_a(b)$	$\Leftrightarrow$	$a^x = b$

WORKED EXAMPLE 1	**Converting to logarithmic form**
Write each statement in logarithmic form. **a** $5^2 = 25$	**b** $2^{-3} = \frac{1}{8}$
Steps	**Working**
a Write as $\log_a(b) = x$, where a is the base and x is the power. base = 5, power = 2	$5^2 = 25$ $\log_5(25) = 2$
b base = 2, power = −3	$2^{-3} = \frac{1}{8}$ $\log_2\left(\frac{1}{8}\right) = -3$

WORKED EXAMPLE 2	**Converting to exponential form**
Write each statement in exponential form. **a** $\log_2(64) = 6$	**b** $\log_7\left(\frac{1}{7}\right) = -1$
Steps	**Working**
a Write as $a^x = b$, where a is the base and x is the power. base = 2, power = 6	$\log_2(64) = 6$ $2^6 = 64$
b base = 7, power = −1	$\log_7\left(\frac{1}{7}\right) = -1$ $7^{-1} = \frac{1}{7}$

Exam hack

When you have to find the logarithm of a number to a particular base, you need to think 'What power of this will give the number?'

or

'How many times do I need to multiply the base to get the number?'

WORKED EXAMPLE 3	**Evaluating logarithms**
Evaluate each logarithm. **a** $\log_4(64)$ **b** $\log_6\left(\frac{1}{216}\right)$	**c** $\log_9(1)$
Steps	**Working**
a 1 Think $4^x = 64$. **2** Evaluate the logarithm.	$4^3 = 64$ $\log_4(64) = 3$
b 1 Think $6^x = \frac{1}{216}$. The power must be negative. **2** Evaluate the logarithm.	$6^{-3} = \frac{1}{216}$ $\log_6\left(\frac{1}{216}\right) = -3$
c 1 Think $9^x = 1$. $\log_a(1) = 0$ always, because $a^0 = 1$. **2** Evaluate the logarithm.	$9^0 = 1$ $\log_9(1) = 0$

The algebraic properties of logarithms

6.1

The rules that apply to all logarithms can be established using the algebraic properties of exponentials (index laws).

Logarithm of a product

$$\log_a(mn) = \log_a(m) + \log_a(n)$$

This can be proved using the index law $a^x \times a^y = a^{x+y}$.

Let $\log_a(m) = x$ and $\log_a(n) = y$

$\log_a(m) = x \Leftrightarrow a^x = m$ and $\log_a(n) = y \Leftrightarrow a^y = n$

$$m \times n = a^x \times a^y$$

$$m \times n = a^{x+y} \quad \text{take } \log_a(\ ) \text{ of both sides}$$

$$\log_a(m \times n) = \log_a(a^{x+y})$$

however, $\log_a(a^b) = b$

therefore, $\log_a(m \times n) = x + y$

$$\log_a(mn) = \log_a(m) + \log_a(n)$$

Logarithm of a quotient

$$\log_a\left(\frac{m}{n}\right) = \log_a(m) - \log_a(n)$$

This can be proved using the index law $a^x \div a^y = a^{x-y}$.

Let $\log_a(m) = x$ and $\log_a(n) = y$

$\log_a(m) = x \Leftrightarrow a^x = m$ and $\log_a(n) = y \Leftrightarrow a^y = n$

$$\frac{m}{n} = \frac{a^x}{a^y}$$

$$\frac{m}{n} = a^{x-y} \quad \text{take } \log_a(\ ) \text{ of both sides}$$

$$\log_a\left(\frac{m}{n}\right) = \log_a(a^{x-y})$$

however, $\log_a(a^b) = b$

therefore, $\log_a\left(\frac{m}{n}\right) = x - y$

$$\log_a\left(\frac{m}{n}\right) = \log_a(m) - \log_a(n)$$

Logarithm of a power

$$\log_a(m^k) = k\log_a(m)$$

This can be proved using the index law $(a^x)^k = a^{kx}$.

Let $\log_a(m) = x \Leftrightarrow a^x = m$

$$m^k = (a^x)^k$$

$$m^k = a^{xk}$$

$$\log_a(m^k) = \log_a(a^{kx})$$

however, $\log_a(a^b) = b$

$$\log_a(m^k) = kx$$

therefore, $\log_a(m^k) = k\log_a(m)$

These algebraic properties of logarithms are also called the **laws of logarithms**.

Laws of logarithms

$\log_a(mn) = \log_a(m) + \log_a(n)$

$\log_a\left(\frac{m}{n}\right) = \log_a(m) - \log_a(n)$

$\log_a(m^k) = k\log_a(m)$

$\log_a(1) = 0$

$\log_a(a^b) = b$ and $a^{\log_a(b)} = b$

WORKED EXAMPLE 4 Simplifying and evaluating logarithms using the laws of logarithms

Simplify each expression.

a $\log_4(32) - \log_4(2)$ **b** $\log_6(18) + \log_6(12)$ **c** $\log_2(144) - 2\log_2(3)$

Steps	Working
a **1** Use the logarithm law $\log_a\left(\frac{m}{n}\right) = \log_a(m) - \log_a(n)$.	$\log_4(32) - \log_4(2)$ $= \log_4(32 \div 2)$ $= \log_4(16)$
2 Evaluate the logarithm using $\log_a(a^b) = b$.	$= \log_4(4^2)$ $= 2$
b **1** Use $\log_a(mn) = \log_a(m) + \log_a(n)$.	$\log_6(18) + \log_6(12)$ $= \log_6(18 \times 12)$ $= \log_6(216)$
2 Write 216 as a power of 6 and simplify.	$= \log_6(6^3)$ $= 3$
c Use $\log_a(m^k) = k\log_a(m)$ then use the logarithm of a quotient law.	$\log_2(144) - 2\log_2(3)$ $= \log_2(144) - \log_2(3^2)$ $= \log_2(144 \div 9)$ $= \log_2(16)$ $= \log_2(2^4)$ $= 4$

WORKED EXAMPLE 5 Simplifying logarithms into a single expression

Simplify $3\log_2(x) + \log_2(y) - 4\log_2(x+3)$ to a single logarithm.

Steps	Working
1 Use $k\log_a(m) = \log_a(m^k)$.	$3\log_2(x) + \log_2(y) - 4\log_2(x+3)$ $= \log_2(x^3) + \log_2(y) - \log_2(x+3)^4$
2 Use $\log_a(mn) = \log_a(m) + \log_a(n)$.	$= \log_2(x^3y) - \log_2(x+3)^4$
3 Use $\log_a(m) - \log_a(n) = \log_a\left(\frac{m}{n}\right)$.	$= \log_2\left[\frac{x^3y}{(x+3)^4}\right]$

EXERCISE 6.1 Logarithms

ANSWERS p. 399

6.1

Mastery

1 WORKED EXAMPLE 1 Write the equivalent equation to each of the following, in logarithmic form.

a $7^2 = 49$ **b** $3^3 = 27$ **c** $2^4 = 16$ **d** $5^3 = 125$

e $11^0 = 1$ **f** $(2)^0 = 1$ **g** $5^{-2} = \frac{1}{25}$ **h** $4^{-2} = \frac{1}{16}$

2 WORKED EXAMPLE 2 Write the equivalent equation to each of the following, in exponential form.

a $\log_5(25) = 2$ **b** $\log_4(16) = 2$ **c** $\log_5(125) = 3$ **d** $\log_2(16) = 4$

e $\log_3(3) = 1$ **f** $\log_7(49) = 2$ **g** $\log_2(128) = 7$ **h** $\log_5(1) = 0$

3 WORKED EXAMPLE 3 Evaluate each logarithm.

a $\log_2(64)$ **b** $\log_9(81)$ **c** $\log_3(81)$ **d** $\log_7(343)$

e $\log_6(216)$ **f** $\log_5(1)$ **g** $\log_3(3)$ **h** $\log(100\,000)$

i $\log_3(243)$ **j** $\log_4(1024)$ **k** $\log_{\frac{1}{2}}\left(\frac{1}{16}\right)$ **l** $\log_5\left(\frac{1}{125}\right)$

4 WORKED EXAMPLE 4 Simplify and evaluate each expression.

a $\log_4(10) + \log_4(2) - \log_4(5)$ **b** $\log_5(25) + \log_5(125) - \log_5(625)$

c $\log_8\left(\frac{1}{8}\right) + \log_8(4)$ **d** $\log_2(16) + \log_2(4) + \log_2(8)$

e $\log(400) + \log(10) - \log(4)$ **f** $\log_5(8) - \log_5(4) - \log_5(2)$

g $\log_8(2) - \log_8\left(\frac{1}{4}\right)$ **h** $\log_4(256) - \log_4(32) + \log_4(2)$

5 WORKED EXAMPLE 5 Write each expression as a single logarithm.

a $5\log_4(x) + \log_4(x^2) - \log_4(x^3)$ **b** $3\log_7(x) - 5\log_7(x) + 4\log_7(x)$

c $4\log_6(x) - \log_6(x^2) - \log_6(x^3)$ **d** $\log_2(x+2) - \log_2(x+2)^2$

e $\log_4[(x-1)^3] - \log_4[(x-1)^2]$ **f** $\log_3(x-3) + \log_3(x+3) - \log_3(x^2-9)$

Calculator-free

6 (5 marks) Evaluate each logarithm.

a $\log_2(\sqrt{2})$ (1 mark)

b $\log_9(9\sqrt{9})$ (1 mark)

c $\log_4(\sqrt{64})$ (1 mark)

d $\log_7(\sqrt{343})$ (1 mark)

e $\log_6(\sqrt[3]{36})$ (1 mark)

Video playlist
Exponential and logarithmic equations

Puzzles
Logarithms – Solving equations 1
Logarithms – Solving equations 2

6.2 Exponential and logarithmic equations

Solving exponential equations

Every exponential equation has an equivalent **logarithmic equation**.

It is often necessary when solving an equation in exponential form to express it as an equation in logarithmic form. The two equations are equivalent ways of expressing the same relationship.

WORKED EXAMPLE 6 Solving exponential equations

Solve for x.

a $2^{x-1} = 3$ **b** $e^x = 7$

Steps	Working
a Express the exponential equation as a logarithmic equation and solve for x.	$2^{x-1} = 3$ $x - 1 = \log_2(3)$ $x = \log_2(3) + 1$
b Express the exponential equation as a logarithmic equation. Remember, $\log_e(x) = \ln(x)$.	$e^x = 7$ $x = \log_e(7)$ $x = \ln(7)$

WORKED EXAMPLE 7 Solving exponential equations using the null factor law

Solve $e^x(e^x - 6) = 0$ for x.

Steps	Working
1 Solve using the null factor law. There is only one solution as e^x is always positive.	$e^x - 6 = 0$ $e^x = 6$
2 Express the exponential equation as a logarithmic equation.	$x = \log_e(6)$ $x = \ln(6)$

Solving logarithmic equations

To solve logarithmic equations, we need to use the algebraic properties of logarithms to simplify expressions and be able to change expressions from logarithmic form ($\log_a(x) = b$) to exponential form ($a^b = x$). These laws of logarithms were covered in the previous section.

WORKED EXAMPLE 8 Solving logarithmic equations using the laws of logarithms

Solve $\log_2(x-1) + 2\log_2(5) = 2$ for x.

Steps	Working
1 Use log laws to express the left-hand side of the equation as a single logarithm.	$\log_2(x-1) + 2\log_2(5) = 2$ $\log_2(x-1) + \log_2(5)^2 = 2$ $\log_2(x-1) + \log_2(25) = 2$ $\log_2(25(x-1)) = 2$
2 Change the equation from log form to exponential form.	$25(x-1) = 2^2$
3 Simplify and solve the equation.	$25x - 25 = 4$ $25x = 29$ $x = \frac{29}{25}$

WORKED EXAMPLE 9	Solving equations where every term is a logarithm
Solve $\log_7(x) = \log_7(3) + \log_7(6)$ for x.	
Steps	**Working**
1 Use the laws of logarithms to express the right-hand side of the equation as a single logarithm.	$\log_7(x) = \log_7(3) + \log_7(6)$ $\log_7(x) = \log_7(3 \times 6)$ $\log_7(x) = \log_7(18)$
2 Equate the brackets.	$x = 18$

WORKED EXAMPLE 10	Solving equations using log form to exponential form transformations
a Given that $\log_5(x) = 2$ and $\log_2(y) = 6$, evaluate $2x + y$. **b** Express y in terms of x given that $\log_3(x+y) - 1 = \log_3(x-y)$.	
Steps	**Working**
a 1 Change each equation from logarithmic form to exponential form to solve for x and y.	$\log_5(x) = 2$, $x = 5^2 = 25$ $\log_2(y) = 6$, $y = 2^6 = 64$
2 Find the value of $2x + y$.	$2x + y = 2 \times 25 + 64$ $= 114$
b 1 Transpose the equation and simplify using log laws.	$\log_3(x+y) - 1 = \log_3(x-y)$ $\log_3(x+y) - \log_3(x-y) = 1$ $\log_3\left(\frac{x+y}{x-y}\right) = 1$
2 Change the equation into exponential form and make y the subject.	$\frac{x+y}{x-y} = 3^1$ $x + y = 3x - 3y$ $4y = 2x$ $y = \frac{x}{2}$

WORKED EXAMPLE 11	Solving simultaneous equations involving logarithms
The graph of $y = a\log_2(x-4) + b$ passes through the points $(5, 8)$ and $(12, 17)$. Find the values of a and b.	
Steps	**Working**
1 Substitute the coordinate $(5, 8)$ into the equation and evaluate the logarithms to simplify. Remember, $\log_2(2^b) = b$.	$(5, 8)$ $8 = a\log_2(5-4) + b$ $8 = a\log_2(1) + b$ $8 = a \times 0 + b$ $b = 8$
2 Substitute the coordinate $(12, 17)$ into the equation and evaluate the logarithms to simplify.	$(12, 17)$ $y = a\log_2(x-4) + 8$ $17 = a\log_2(12-4) + 8$ $17 = a\log_2(8) + 8$ $3a + 8 = 17$ $a = 3$

EXERCISE 6.2 Exponential and logarithmic equations

ANSWERS p. 399

Recap

1 Evaluate the following logarithms.

a $\log_2(32)$ **b** $\log_5(125)$ **c** $\log_3\left(\frac{1}{81}\right)$

2 Simplify each expression.

a $\log_2(96) - \log_2(3)$ **b** $\log_5(50) + \log_5(75) - \log_5(6)$

Mastery

3 WORKED EXAMPLE 6 Solve for x.

a $3^{x-5} = 7$ **b** $2^{x+3} - 5 = 7$ **c** $e^{3x} = 9$ **d** $e^{2x+3} = 2$

4 WORKED EXAMPLE 7 Solve for x.

a $e^x(e^x - 8) = 0$ **b** $5^x(5^x - 4) = 0$ **c** $7(2^{3x}) - 6 = 5(2^{3x})$ **d** $(3^x - 1)(3^{2x} - 2) = 0$

5 WORKED EXAMPLE 8 Solve each logarithmic equation for x.

a $\log_3(x + 7) + \log_3(2) = 3$ **b** $2\log_2(3) + \log_2(x + 1) = 4$

c $\log_2(3x - 2) + 2\log_2(4) = 3$ **d** $\log_2(2x - 4) + \log_2(5) = 1$

e $\ln(x - 3) - \ln(4) = 0$ **f** $\log_2(x + 2) - \log_2(3) = 3$

6 WORKED EXAMPLE 9 Solve each logarithmic equation for x.

a $\log_5(x) + \log_5(3) - \log_5(2) = \log_5(6)$

b $\log_2(x) + \log_2(6) = \log_2(3) + \log_2(x + 7)$

c $\log_2(x) - 3\log_2(2) = \log_2(x + 1) - 2\log_2(5)$

7 WORKED EXAMPLE 10

a Given that $\log_7(x) = 2$ and $\log_3(y) = 4$, evaluate $3y - 2x$.

b Express y in terms of x given that $\log_5(x - y) - 2 = \log_5(2y - x)$.

8 WORKED EXAMPLE 11

a The graph of $y = a\log_3(x + 2) + b$ passes through the points $(-1, 10)$ and $(7, 14)$. Find the values of a and b.

b The graph of $y = a\log_2(x - 7) + b$ passes through the points $(9, 26)$ and $(15, 36)$. Find the values of a and b.

Calculator-free

9 © SCSA MM2017 Q3 (4 marks) Solve $4e^{2x} = 81 - 5e^{2x}$ exactly for x.

10 © SCSA MM2016 Q1 (5 marks)

a Given that $\log_8(x) = 2$ and $\log_2(y) = 5$, evaluate $x - y$. (2 marks)

b Express y in terms of x given that $\log_2(x + y) + 2 = \log_2(x - 2y)$. (3 marks)

9780170477536

11 (6 marks) Solve each equation for x.

a $2\log_3(5) - \log_3(2) + \log_3(x) = 2$ (2 marks)

b $\log_e(3x+5) + \log_e(2) = 2$ (2 marks)

c $\log_2(6-x) - \log_2(4-x) = 2$ (2 marks)

6.3

Calculator-assumed

12 (5 marks) The functions f and g are defined as $f(x) = \log_3(x+1) - 3$ and $g(x) = \log_3(2)$. The graphs of the function intersect at the point (a, b).

a Show $\log_3\left(\frac{a+1}{2}\right) = 3$. (2 marks)

b Find the values of a and b. (3 marks)

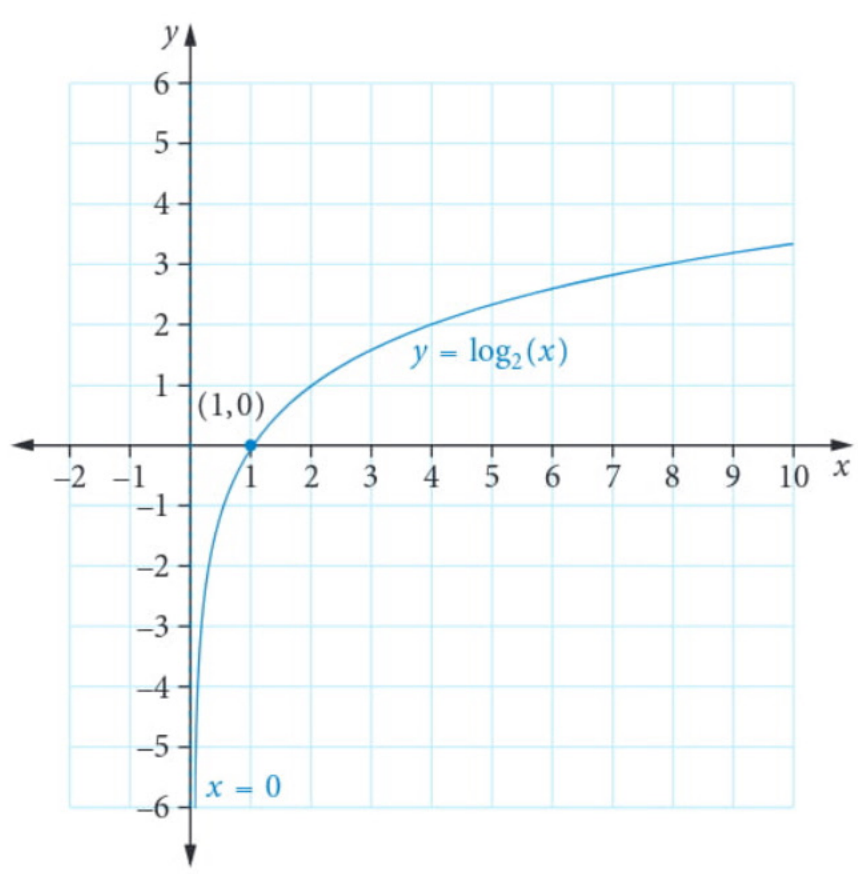
6.3 The logarithmic function $y = \log_a(x)$

Video playlist The logarithmic function $y = \log_a(x)$

Graphing logarithmic functions

We can plot the **logarithmic function** for $y = \log_2(x)$ using a table of values. We substitute powers of 2 for x so that the logarithms are easier to calculate. This table of values and its graphical representation are shown below.

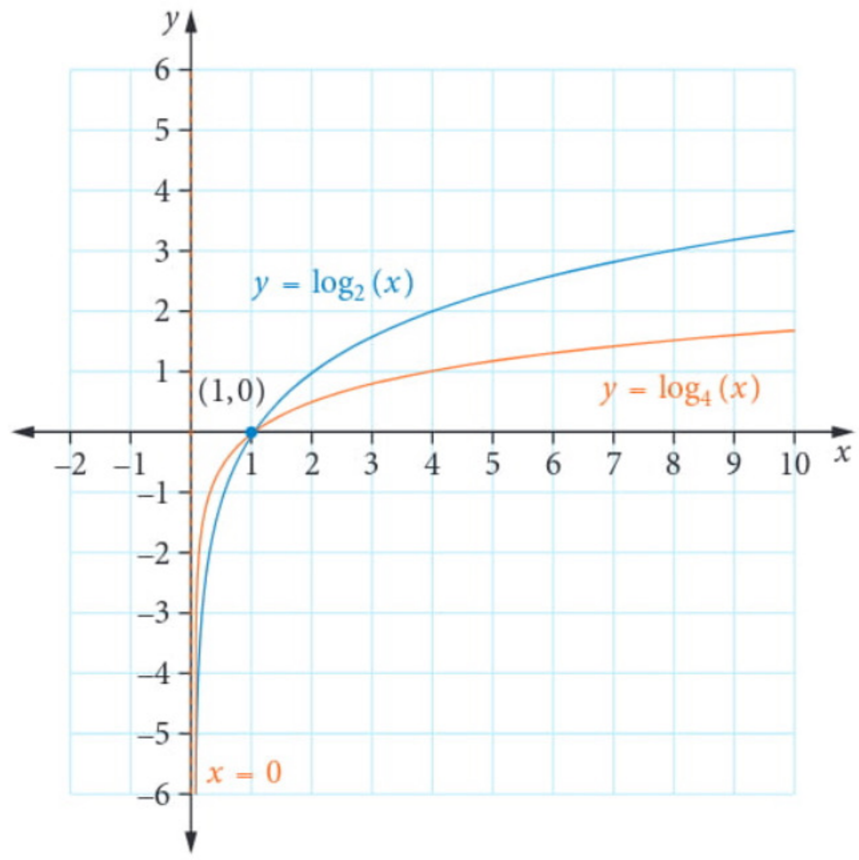
x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$y = \log_2(x)$	$\log_2\left(\frac{1}{4}\right) = -2$	$\log_2\left(\frac{1}{2}\right) = -1$	$\log_2(1) = 0$	$\log_2(2) = 1$	$\log_2(4) = 2$



Properties of the logarithmic function $y = \log_a(x)$

- It is a strictly increasing function, increasing quickly at first, then more slowly.
- The gradient of the graph is always decreasing.
- x-intercept is at $(1, 0)$ as $\log_a(1) = 0$.
- The y-axis $(x = 0)$ is a vertical asymptote.

Changing the base of the logarithm does not alter the basic shape, x-intercept or the asymptote of the graph of the logarithmic function.



Translations of $y = \log_a(x)$

The graph of the function $y = \log_a(x - c) + b$ is a translation of $y = \log_a(x)$, c units right and b units up where b and c are positive real constants.

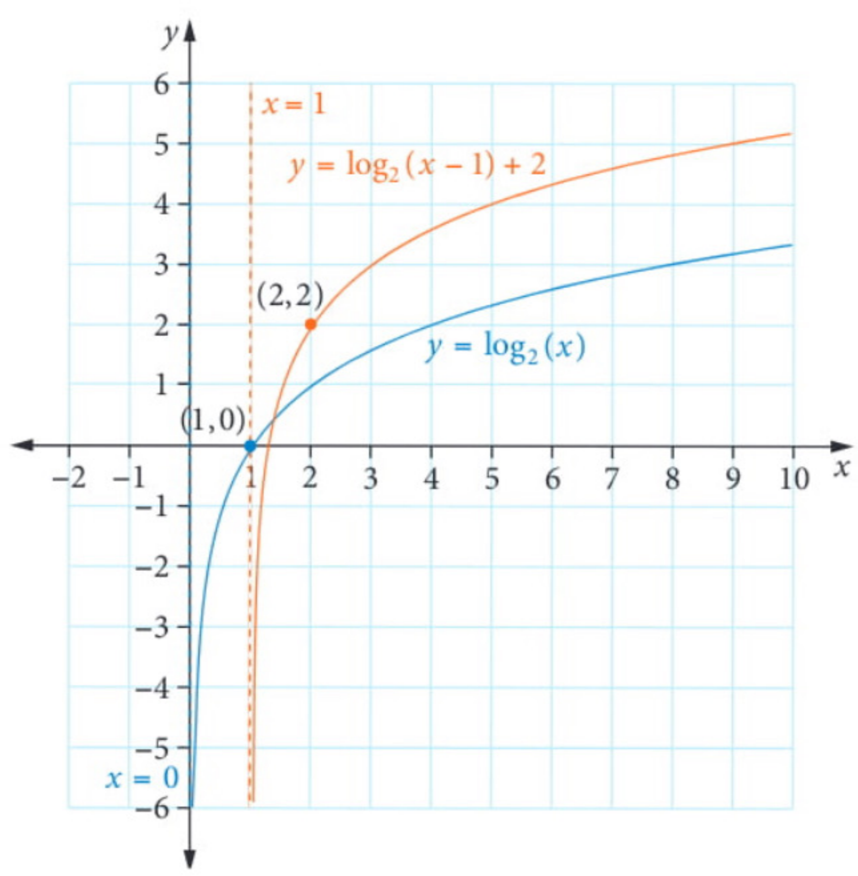
The horizontal translation of c units right produces a vertical asymptote at $x = c$.

The x-intercept $(1, 0)$ of $y = \log_a(x)$ translates to $(1 + c, b)$.

Properties of the logarithmic function $y = \log_a(x - c) + b$, where b and c are positive real constants

- It has the same shape as $y = \log_a(x)$.
- The horizontal translation is c units right and the vertical translation is b units up.
- $x = c$ is the vertical asymptote.
- Include the guiding point $(1 + c, b)$.

The graphs of $y = \log_2(x)$ and $y = \log_2(x - 1) + 2$ are shown below.



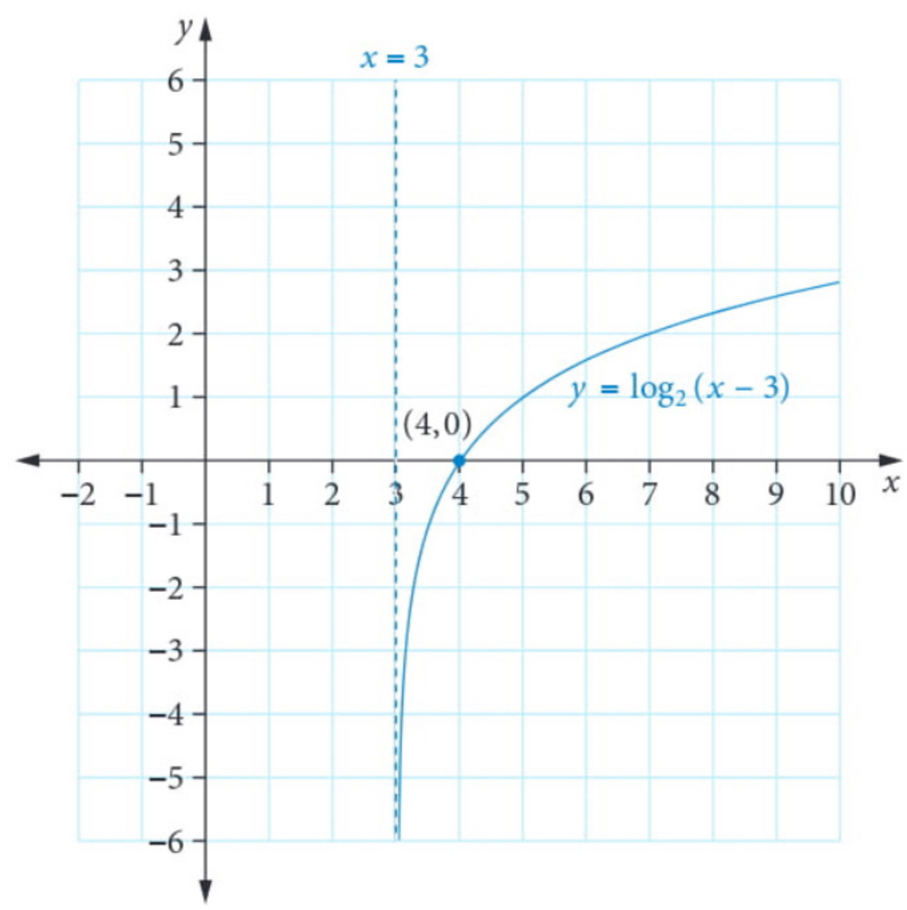
Exam hack

A guiding point is often necessary to improve the accuracy of the sketched graph.

The graph of $y = \log_2(x)$ has been translated horizontally, 1 unit right, and vertically, 2 units up, to produce the graph of $y = \log_2(x - 1) + 2$. Note that the asymptote $x = 0$ and the x-intercept $(1, 0)$ on $y = \log_2(x)$ have translated to $x = 1$ and $(2, 2)$ respectively on $y = \log_2(x - 1) + 2$.

WORKED EXAMPLE 12 Using translations to sketch a logarithmic function

Sketch the graph of $f(x) = \log_2(x-3)$. Label the coordinates of the x-intercept and the asymptote with its equation.

Steps	Working
1 The graph of $y = \log_2(x)$ has a vertical asymptote at $x = 0$ and an x-intercept $(1, 0)$. Translate the function 3 units to the right.	The vertical asymptote is $x = 3$. x-intercept $(y = 0)$: $\log_2(x-3) = 0$ $x - 3 = 2^0$ $x - 3 = 1$ $x = 4$ The x-intercept is $(4, 0)$.
2 Sketch the graph.	

WORKED EXAMPLE 13 Using the intercept method to sketch a logarithmic function

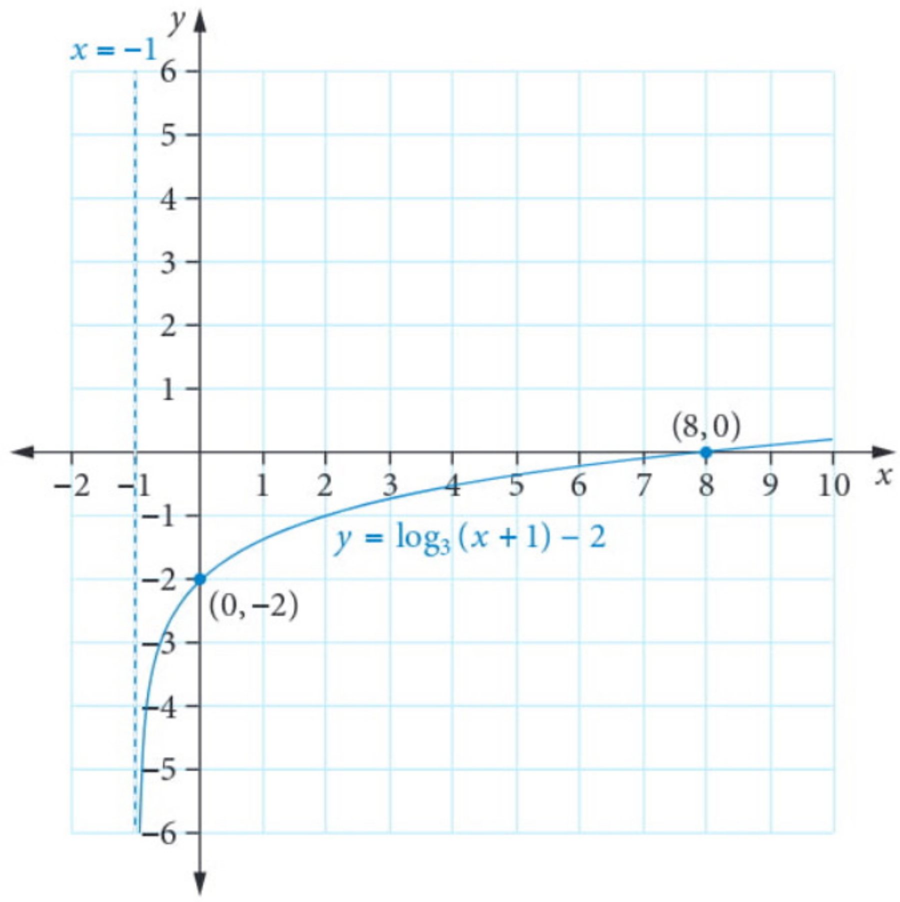
Consider the function $f(x) = \log_3(x+1) - 2$. Describe the translations on $y = \log_3(x)$ and sketch $y = f(x)$. Label axes intercepts and the asymptote with its equation.

Steps	Working
1 State the translation.	The graph of $y = \log_3(x)$ is translated vertically 2 units down, and horizontally 1 unit left. The vertical asymptote is $x = -1$. The point $(1, 0)$ on $y = \log_3(x)$ is translated 1 unit left and 2 units down to $(0, -2)$.
2 Find the x-intercept and y-intercept.	x-intercept $(y = 0)$: $\log_3(x+1) - 2 = 0$ $\log_3(x+1) = 2$ $x = 3^2 - 1 = 8$ The x-intercept is $(8, 0)$. y-intercept $(x = 0)$: $y = \log_3(1) - 2$ $y = -2$ The y-intercept is $(0, -2)$.

3 Sketch the graph.

Exam hack

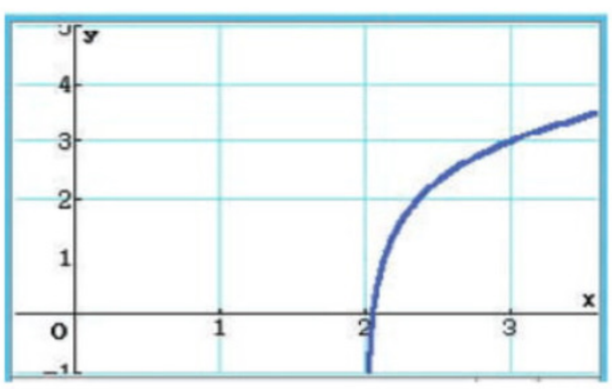
Include the asymptote equation $x = -1$, and the coordinates of any axes intercepts.



USING CAS 1 Graphing logarithmic functions

Graph $y = \ln(x-2) + 3$.

ClassPad

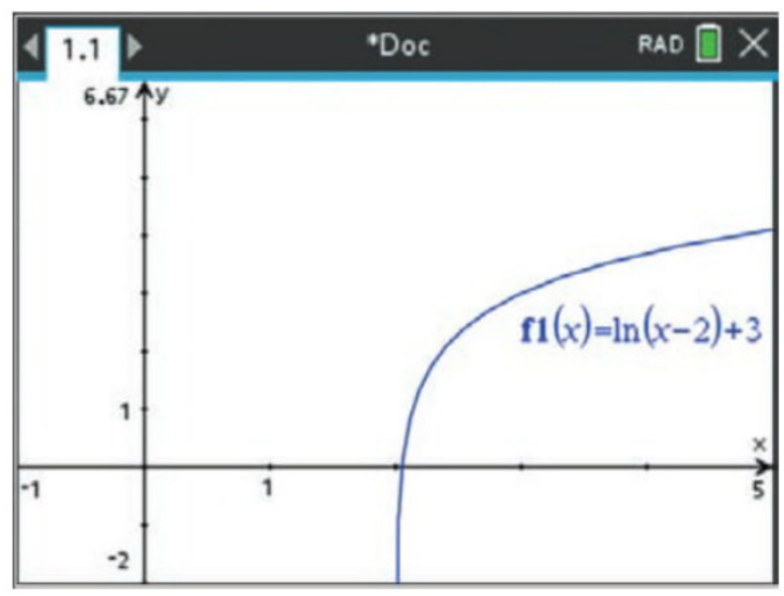


1 In **Main**, enter and highlight the equation using **ln** for $\log_e$ from **Math1**.

2 Tap **Graph** and drag the equation down into the Graph window.

3 Adjust the window settings to suit.

TI-Nspire



1 Add a **Graphs** page and enter the function as shown above. Press **ctrl** + **e^x** for the **ln** function.

2 Adjust the window settings to suit.

Finding equations of logarithmic functions

Finding the equation of logarithmic functions is often a multi-step process, and every problem is different depending on the information and type of graph we are given.

For a logarithmic function $y = \log_a(x - c) + b$, the vertical asymptote is $x = c$.

6.3

WORKED EXAMPLE 14 Finding the rule of a logarithmic function

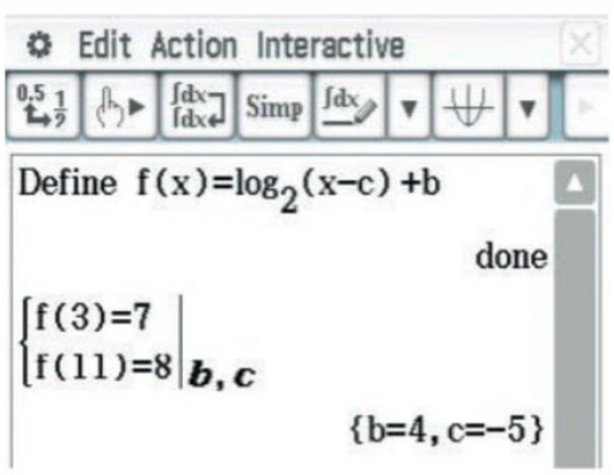
Find the rule for the logarithmic function $y = a\ln(x + b)$ if the vertical asymptote is $x = -3$ and the y-intercept is at the point $(0, 4\ln(9))$.

Steps	Working
1 The function $y = a\ln(x + b)$ has been translated b units to the left and will have an asymptote at $x = -b$.	Vertical asymptote is $x = -3$. $b = 3$ $y = a\ln(x + 3)$
2 Substitute $(0, 4\ln(9))$ into the equation.	$4\ln(9) = a\ln(0 + 3)$
3 Simplify $4\ln(9)$ using the log law $\ln(x^n) = n\ln(x)$.	$4\ln(3^2) = a\ln(3)$ $8\ln(3) = a\ln(3)$ $a = 8$
4 Write the function.	$y = 8\ln(x + 3)$

USING CAS 2 Finding rules for logarithmic functions

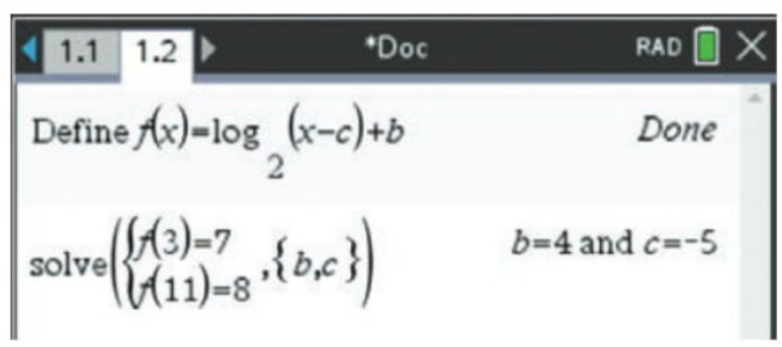
The graph of the logarithmic function $f(x) = \log_2(x - c) + b$ passes through the points $(3, 7)$ and $(11, 8)$. Find the values of b and c.

ClassPad



1 In **Main**, enter and highlight the equation using **log** from **Math1**.
2 Tap **Interactive, Define**.
3 Enter the simultaneous equations as shown and solve for b and c.

TI–Nspire



1 Add a **Calculator** page and define the function $f(x)$.
2 Press **menu** > **algebra** > **Solve system of equations**.
3 Enter the equations as shown.

$b = 4$, $c = -5$.

EXERCISE 6.3 The logarithmic function $y = \log_a(x)$

ANSWERS p. 399

Recap

1 Find the value of $x + y$ if $\log_2(x) = 3$ and $\log_3(y) = 2$.

2 Solve $e^{2x} + 16 = 2e^{2x}$ for x.

Mastery

3 WORKED EXAMPLE 12 Sketch the graph of $f(x) = \log_2(x + 4)$. Label the coordinates of the x- and y-intercepts and the asymptote with its equation.

4 Sketch the graph of $f(x) = \ln(x - 4)$. Label the coordinates of the x-intercept and the asymptote with its equation.

5 WORKED EXAMPLE 13 Consider the function $f(x) = \log_2(x + 2) - 1$. Describe the translations on $y = \log_2(x)$ and sketch $y = f(x)$, labelling axes intercepts and the asymptote.

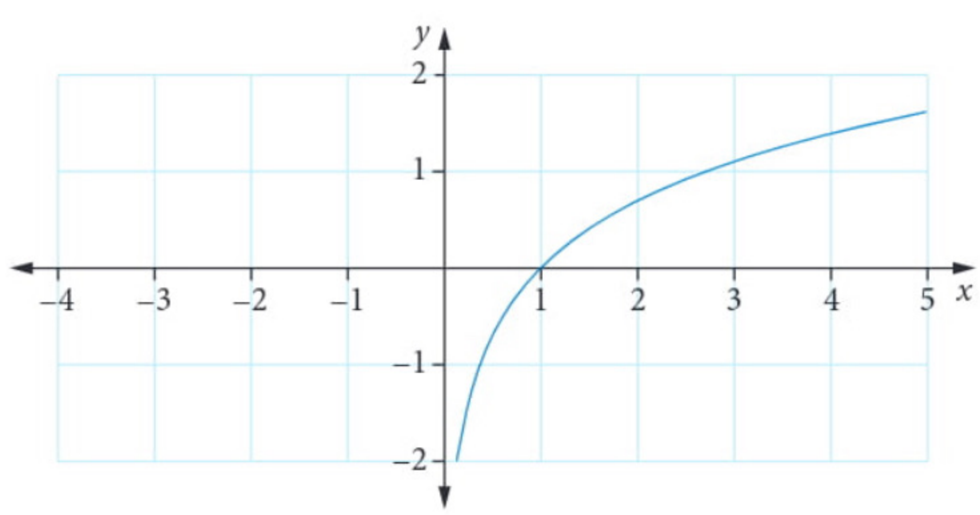
6 Using CAS 1 Sketch the graph of $f(x) = \ln(x + 3) + 5$.

7 WORKED EXAMPLE 14 Find the rule for the logarithmic function $y = a\ln(x + b)$ if the vertical asymptote is $x = -4$ and the y-intercept is at the point $(0, 6\ln(2))$.

8 Using CAS 2 The graph of the logarithmic function $f(x) = \log_2(x - c) + b$ passes through the points $(3, 8)$ and $(33, 12)$ Find the values of b and c.

Calculator-free

9 © SCSA MM2019 Q4b (3 marks) Consider the graph of $y = \ln(x)$ shown below.



Copy the graph and on it sketch the graph of $y = \ln(x - 2) + 1$.

10 (6 marks) Determine the coordinates of the x- and y-intercepts and the equation of the asymptote for each of the logarithmic functions below.

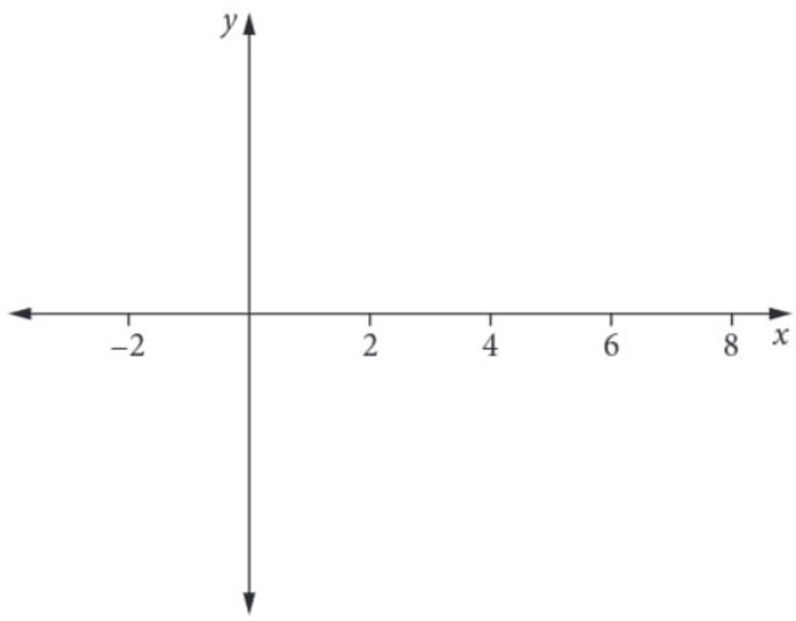
a $f(x) = \log_3(x + 9) - 4$ (3 marks)

b $f(x) = \log_2(x + 8) - 3$ (3 marks)

11 (4 marks) Find the rule for the logarithmic function $y = \log_5(x - c) + b$, if the vertical asymptote is $x = 2$ and the graph passes through the point $(27, 10)$.

12 © SCSA MM2018 Q8 (8 marks) Consider the function $f(x) = \log_a(x-1)$ where $a > 1$.

a Copy the axes below, and on it sketch the graph of $f(x)$, labelling important features. (3 marks)



b Determine the value of m if $f(m) = 1$. (2 marks)

c Determine the coordinates of the x-intercept of $f(x+b)+c$, where b and c are positive real constants. (3 marks)

13 © SCSA MM2021 Q15b (2 marks) The graph of $y = m\log_3(x-p)+q$ has a vertical asymptote at $x = 5$. If this graph passes through the points $(6, 2)$ and $(14, -6)$, determine the values of m, p and q.

6.4 Applications of logarithmic functions

Video playlist
Applications of logarithmic functions

Modelling with logarithmic functions

Logarithms have many applications in the fields of finance and science. In this section we will look at some examples of those applications.

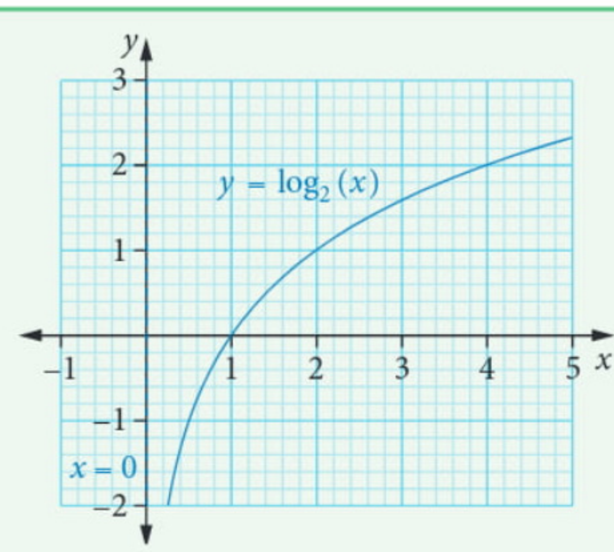
WORKED EXAMPLE 15 Using the graph of a logarithmic function to approximate logarithms and exponentials

Consider the graph of $y = \log_2(x)$ shown.

Use the graph to estimate the value of m in each of the following.

a $0.8 = \log_2(m)$

b $2^{m-2} - 5 = 0$



Steps	Working
a The values of $\log_2(x)$ are on the y-axis. Find the x value on the curve that corresponds to $y = 0.8$.	When $y = 0.8$, $x = 1.8$ $\log_2(1.8) = 0.8$ $m = 1.8$

b **1** Express the exponential equation as a logarithmic equation.

$$2^{m-2} - 5 = 0$$
$$2^{m-2} = 5$$
$$m - 2 = \log_2(5)$$

2 Use the graph to find y when $x = 5$.

Substitute into the equation and solve.

From the graph $\log_2(5) = 2.3$.

$$m - 2 = 2.3$$
$$m = 4.3$$

WORKED EXAMPLE 16 Applying logarithmic functions

The cost of manufacturing bicycle components depends on the number produced each day and this cost influences the profit. A company can manufacture a maximum of 80 components in one day. The daily profit from producing x components each day is given by the function $P(x) = (100 - x)\ln(3x + 1) - x$.

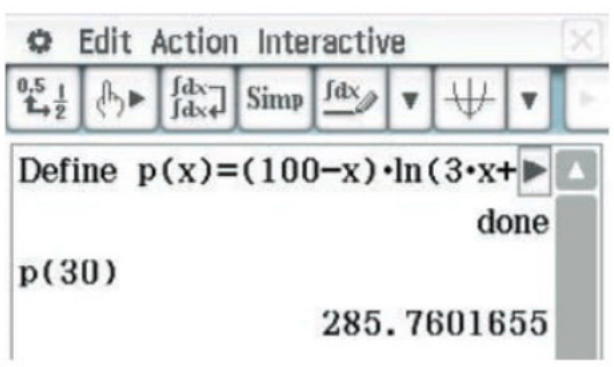
a Find the profit, to the nearest dollar, when 30 components are manufactured in a day.

b Sketch the graph of $P(x)$.

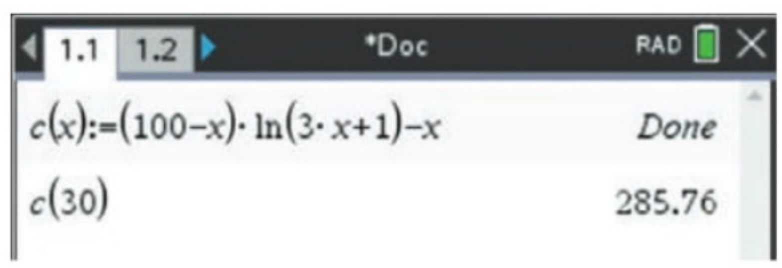
c Determine the number of components that result in the highest profit per day.

Steps	Working
a Define the $P(x)$ function on CAS and find $P(30)$.	$P(30) = 285.76$ \$286 profit is made when 30 components are manufactured.

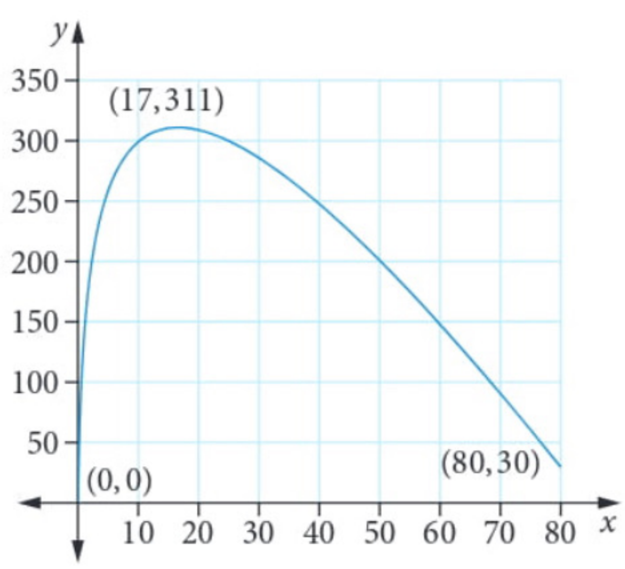
ClassPad

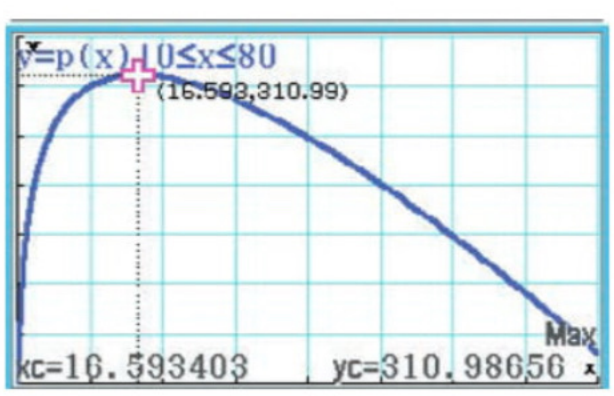


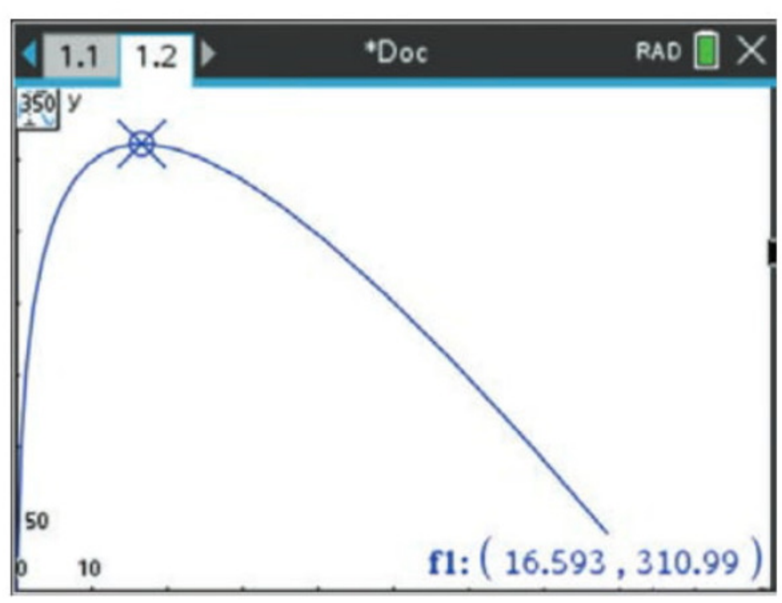
TI-Nspire



b Use CAS to graph the function over the domain $0 \le x \le 80$. Set the window to an appropriate scale and include the coordinates of the stationary point and the endpoints on the graph.





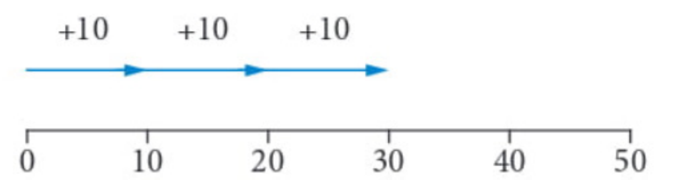


Steps	Working
c Find the maximum function value using CAS and state the x-coordinate to the nearest integer.	$f(16) = 310.91$ and $f(17) = 310.95$ The maximum profit is made when 17 components are made each day.

9780170477536

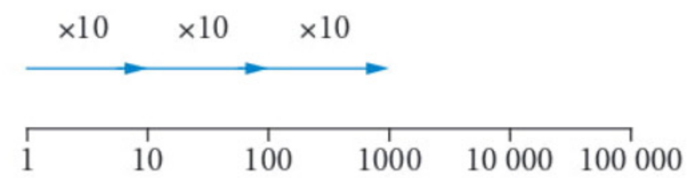
Logarithmic scales

If a graph has a linear scale, we *add* the same number to move from one scale mark to the next. In the example on the right, the linear scale involves adding 10 each time.



In situations where the values we need to plot cover a very large range, it is better to use a **logarithmic scale** (or **log scale**).

On a log scale, we *multiply* by the same number to move from one scale mark to the next. In the example on the right, the log scale involves multiplying by 10 each time.



This is called a 'log base 10' or '$\log_{10}$' scale.

Measurements of acidity (pH), earthquake strength (Richter magnitude) and sound intensity (decibels – dB) are examples of logarithms.

WORKED EXAMPLE 17 Comparing values measured on a logarithm scale

© SCSA MM2016 Q12 MODIFIED

The Richter magnitude, M, of an earthquake is determined from the logarithm of the amplitude, A, of waves recorded by seismographs.

$M = \log_{10} \frac{A}{A_0}$, where A_0 is a reference value.

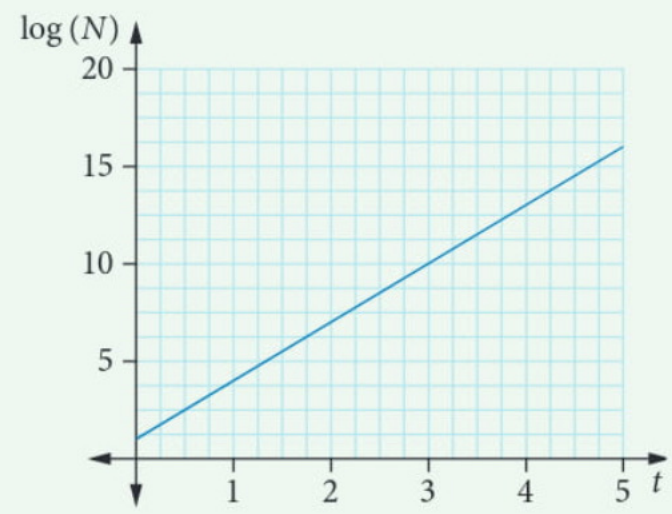
An earthquake in Double Spring Flat (NV, USA) in 1994 was estimated at 6.0 on the Richter scale, whereas the Java earthquake (Indonesia) in 2009 measured 7.0 on the same scale. How many times larger was the amplitude of the waves in Java compared to those of Double Spring Flat?

Steps	Working
1 Change the equation from logarithmic form to exponential form.	$M = \log_{10} \frac{A}{A_0}$ $\frac{A}{A_0} = 10^M$ $A = A_0 \times 10^M$
2 Substitute USA: $A = A_1$, $M_1 = 6$ and Indonesia: $A = A_2$, $M_2 = 7$ into: $A = A_0 \times 10^M$	USA: $A_1 = A_0 \times 10^6$ Indonesia: $A_2 = A_0 \times 10^7$
3 Calculate $\frac{A_2}{A_1}$.	$\frac{A_2}{A_1} = \frac{A_0 \times 10^7}{A_0 \times 10^6} = 10$ The amplitude of the waves in Java was 10 times larger than the waves in Double Spring Flat.

WORKED EXAMPLE 18 Problems involving log scales

© SCSA MM2019 Q17a–bi MODIFIED

A scientist is studying the growth of a certain type of bacteria under controlled laboratory conditions. A population of bacteria is incubated at a temperature of 30°C and the size of the population is measured at hourly intervals for five hours. The logarithm of the population size (N) appears to lie on a straight line when plotted against time (measured in hours) and the line of best fit, as shown on the axes below.



a On the basis of the graph above, what is the size of the bacteria population

i after one hour?

ii after two hours?

b The equation of the line can be written in the form $\log_{10}(N) = At + B$. Use the graph to determine the values of A and B.

c Use the equation to predict the number of bacteria after 20 minutes.

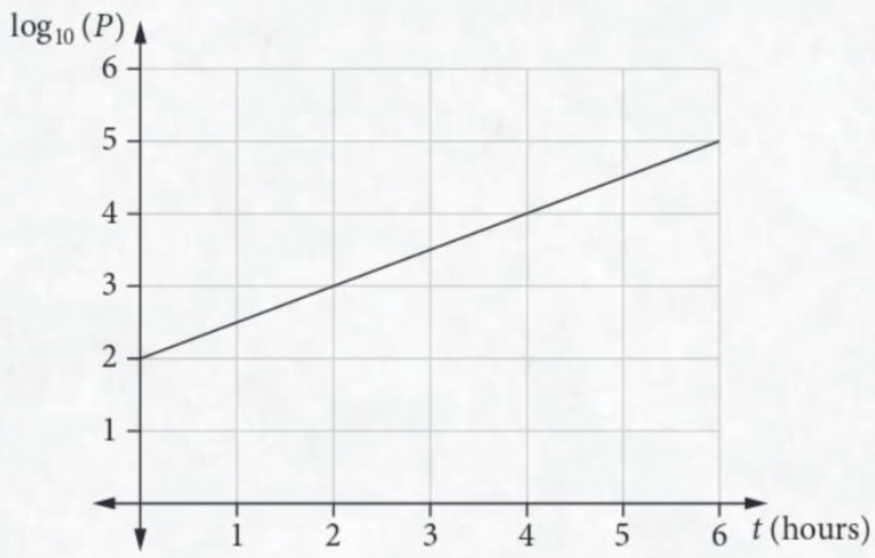
d Express the above equation in the form $N = a(10)^{bt}$.

Steps	Working
a **i** Find the value of $\log(N)$ when $t = 1$ and solve to find N.	$t = 1$, $\log_{10}(N) = 4$ Change to exponential form: $N = 10^4 = 10\,000$ bacteria
ii Repeat for $t = 2$. $\log(N) = \log_{10}(N)$	$t = 2$, $\log_{10}(N) = 7$ $N = 10^7 = 10\,000\,000$ bacteria
b Find the gradient and y-intercept from the graph. Use the linear form $y = mx + c$ to write the equation.	Use the points $(1, 4)$ and $(2, 7)$. $A = \frac{7-4}{2-1} = 3$ Vertical axis intercept, $B = 1$. $\log_{10}(N) = 3t + 1$
c Change 20 minutes into hours then substitute the value of t into $\log_{10}(N) = 3t + 1$.	$t = \frac{20}{60} = \frac{1}{3}$ $\log_{10}(N) = 3\left(\frac{1}{3}\right) + 1$ $\log_{10}(N) = 2$ $N = 10^2 = 100$
d Change the equation into exponential form and simplify using index laws.	$\log_{10}(N) = 3t + 1$ $N = 10^{3t+1}$ $N = 10^{3t} \times 10^1$ $N = 10(10)^{3t}$

WACE QUESTION ANALYSIS

© SCSA MM2019 Q17 Calculator-assumed (15 marks)

A microbiologist is studying the effect of temperature on the growth of a certain type of bacteria under controlled laboratory conditions. A population of bacteria is incubated at a temperature of 30°C and the size of the population measured at hourly intervals for six hours. The logarithm of the population size appears to lie on a straight line when plotted against time (measured in hours) and the line of best fit shown on the axes below.



a **i** On the basis of the graph above, what is the size of the bacteria population after two hours? (2 marks)

ii The equation of the line can be written in the form $\log_{10}(P) = At + B$. Use the graph to determine the values of A and B. (2 marks)

Another population of the same bacteria is cultured at 40°C. The size of the population, P, after t hours satisfies the equation

$$\log_{10}(P) = \frac{1}{3}t + 2.$$

b **i** Express the above equation in the form $P = A(10)^{Bt}$. (3 marks)

ii Determine the size of the population after exactly four hours to the nearest whole number. (1 mark)

iii Express the above equation in the form $t = C\log_{10}\left(\frac{P}{D}\right)$. (3 marks)

iv How many minutes does it take for the population to reach a size of 5000? Give your answer to the nearest minute. (2 marks)

c With reference to parts **a** and **b**, describe the effect of temperature on the population growth of this type of bacteria. (2 marks)

Reading the question

- The graph of the population is a log scale, where the log to base 10 of the population is on the vertical axis.
- Highlight the type, accuracy and units of the answer required in each part.
- Take note of the number of marks allocated to each part of the question. This will give an indication of the amount of working required.

Thinking about the question

- This question requires a knowledge of graphs using logarithmic scales.
- You will also need to be able to transform an equation from logarithmic form to exponential form.
- You will need to be able to find the equation of a straight line using a gradient and y-intercept. Remember, in this case, the subject of the linear equation is $\log_{10}(P)$.

6.4

Video WACE question analysis: Logarithmic functions

Worked solution (✓ = 1 mark)

a **i** $\log_{10}(P) = 3$ ✓

$P = 10^3 = 1000$ ✓

ii gradient of $\frac{1}{2}$ and vertical axis intercept of 2

$A = \frac{1}{2}$ ✓ $B = 2$ ✓

$\log_{10}(P) = \frac{1}{2}t + 2$

b **i** $\log_{10}(P) = \frac{1}{3}t + 2$

$P = 10^{\frac{1}{3}t+2}$ ✓

$P = 10^2 \times 10^{\frac{1}{3}t}$ ✓

$P = 100(10)^{\frac{1}{3}t}$ ✓

ii $P = 100 \cdot 10^{\frac{4}{3}}$

$P = 2154$ ✓

iii

$$\begin{aligned}\log_{10}(P) &= \frac{1}{3}t + 2\\ \log_{10}(P) - 2 &= \frac{1}{3}t\\ \log_{10}(P) - \log_{10}(100) &= \frac{1}{3}t\\ \log_{10}\left(\frac{P}{100}\right) &= \frac{1}{3}t\\ t &= 3\log_{10}\left(\frac{P}{100}\right)\end{aligned}$$

expresses 2 in terms of a log of base 10 ✓

applies appropriate log law to arrive at single log expression (second last line) ✓

determines correct expression ✓

iv $t = 3\log_{10}\left(\frac{5000}{100}\right)$

$t = 5.0969$ **hours** ✓

$t = 306$ **minutes** ✓

c The equation at 30°C has a greater slope than that of the 40°C equation, which indicates a greater growth rate. Parts **a** and **b** would seem to indicate that the lower temperature incubation results in a higher growth rate.

identifies features of the equations in parts a and b that relate to growth ✓

states lower temperature has higher growth ✓

EXERCISE 6.4 Applications of logarithmic functions

ANSWERS p. 400

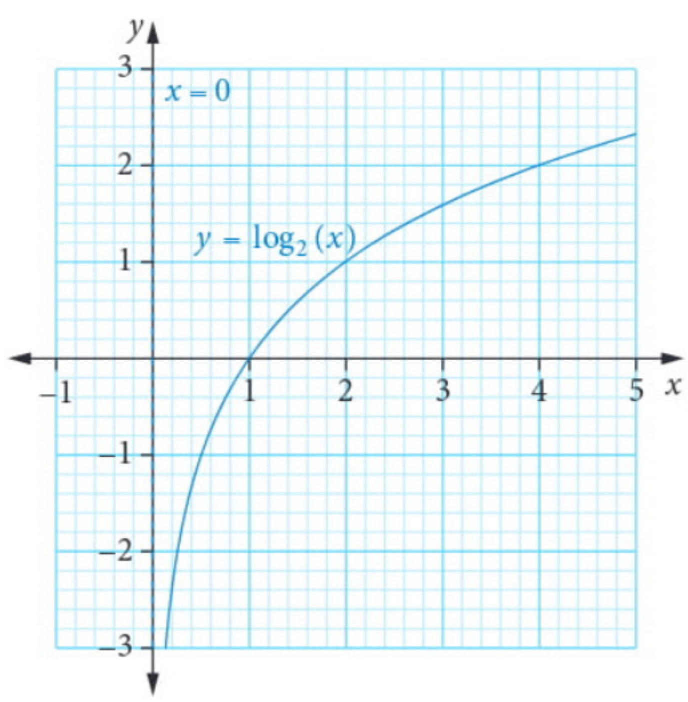
Recap

1 The graph of the logarithmic function $f(x) = \log_2(x - c) + b$ has a vertical asymptote at $x = 10$ and passes through the point $(12, 21)$. Find the values of b and c.

2 The graph of the logarithmic function $f(x) = \log_3(x - c) + b$ passes through the points $(17, 4)$ and $(11, 3)$. Find the values of b and c.

Mastery

3 WORKED EXAMPLE 15 Consider the graph of $y = \log_2(x)$ shown below.



Use the graph to estimate the value of n in each of the following.

a $\log_2(n) = 1.9$

b $2^{n-3} - 0.7 = 0$

4 WORKED EXAMPLE 16 A farmer finds that the cost of raising sheep for wool, meat and breeding stock varies depending on the number of sheep. The farm can support a maximum of 400 sheep. The monthly cost $C(x)$ of raising x sheep is given by the function

$$C(x) = (x - 200)\ln(0.5x + 1) - x + 1000.$$

a Find the monthly cost, to the nearest dollar, when the farmer raises 100 sheep.

b Determine the number of sheep the farmer must raise to produce the least cost.

c How many sheep should the farmer raise to keep costs below \$600 per month?

5 A small colony of black peppered moths live on a small isolated island. In summer, the population begins to increase. If t is the number of days after 12 midnight on 1 January, the equation that best models the number of moths in the colony at any given time is

$$N(t) = 500\ln(21t + 3).$$

a What is the population of the species on 1 January?

b What is the population of moths after 30 days?

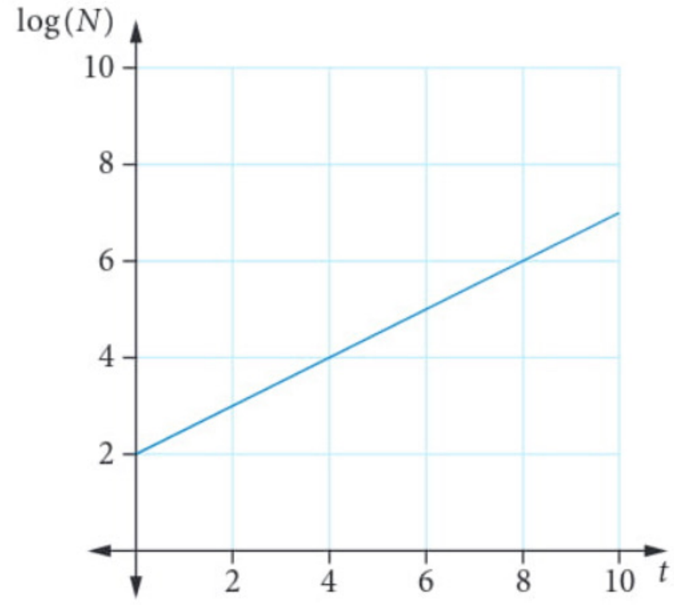
c On which day is the population first greater than 2000?

6 WORKED EXAMPLE 17 The Richter magnitude, M, of an earthquake is determined from the logarithm of the amplitude, A, of waves recorded by seismographs.

$M = \log_{10} \frac{A}{A_0}$, where A_0 is a reference value.

An earthquake in México City (Mexico) in 1985 was estimated at 8.0 on the Richter scale, while the San Francisco Bay Area earthquake (CA, USA) in 1989 measured 6.9 on the same scale. How many times larger was the amplitude of the waves in México City compared to the waves in San Francisco Bay Area?

7 WORKED EXAMPLE 18 The number of cases (N) of a virus is recorded daily. The logarithm of the number of cases appears to lie on a straight line when plotted against time (measured in days) and the line of best fit, as shown on the axes below.



a On the basis of the graph above, how many virus cases are recorded

i at the start of the outbreak

ii after two days.

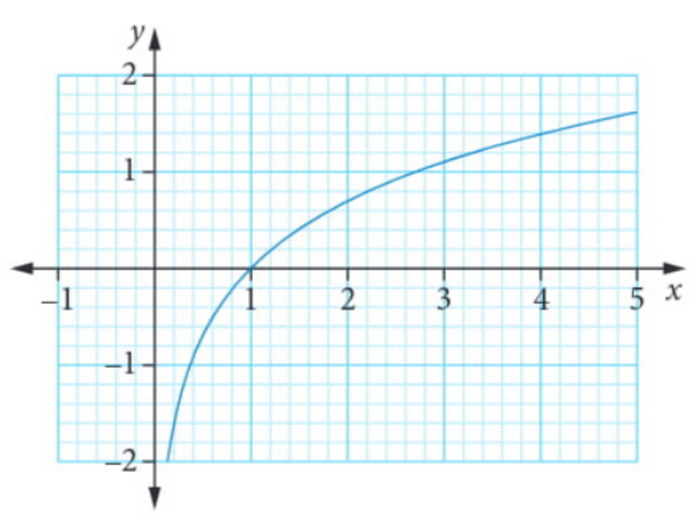
b The equation of the line can be written in the form $\log_{10}(N) = At + B$. Use the graph to determine the values of A and B.

c Use the equation to predict the number of cases after six days.

d Express the above equation in the form $N = a(10)^{bt}$.

Calculator-free

8 © SCSA MM2019 Q4a (3 marks) Consider the graph of $y = \ln(x)$ shown below.



Use the graph to estimate the value of p in each of the following.

a $1.4 = \ln(p)$ (1 mark)

b $e^{p+1} - 3 = 0$ (2 marks)

Calculator-assumed

9 © SCSA MM2019 Q12 (6 marks) Part of Josie's workout at her gym involves a 10 minute run on a treadmill. The treadmill's program makes her run at a constant 12.3 km/h for the first 2 minutes and then her speed, $s(t)$, is determined by the equation below, where t is the time in minutes after she began running.

$$s(t) = 10 - \frac{\ln(t - 1.99)}{t} \text{ km/h}$$

a Sketch the graph of her speed during this run versus time. (3 marks)

b At what time(s) is Josie's speed 10 km/h? (1 mark)

c At what time(s) during her run is Josie's acceleration zero? (2 marks)

10 © SCSA MM2016 Q12 (3 marks) The Richter magnitude, M, of an earthquake is determined from the logarithm of the amplitude, A, of waves recorded by seismographs.

$M = \log_{10} \dfrac{A}{A_0}$, where A_0 is a reference value.

An earthquake in a town in New Zealand in November 2015 was estimated at 5.5 on the Richter scale, while the earthquake just north of Hayman Island measured 3.4 on the same scale. How many times larger was the amplitude of the waves in New Zealand compared to those at Hayman Island?

11 © SCSA MM2018 Q18ab (4 marks) The ear has the remarkable ability to handle an enormous range of sound levels. In order to express levels of sound meaningfully in numbers that are more manageable, a logarithmic scale is used, rather than a linear scale. This scale is the decibel (dB) scale.

The sound intensity level, L, is given by the formula below:

$$L = 10\log\left(\frac{I}{I_0}\right) \text{dB}$$

where I is the sound intensity and I_0 is the reference sound intensity.

I and I_0 are measured in watt/m^2.

a Listening to a sound intensity of 5 billion times that of the reference intensity ($I = 5 \times 10^9 I_0$) for more than 30 minutes is considered unsafe. To what sound intensity level does this correspond? (2 marks)

b The reference sound intensity, I_0, has a sound intensity level of 0 dB. If a household vacuum cleaner has a sound intensity $I = 1 \times 10^{-5}$ watt/m^2 and this corresponds to a sound intensity level $L = 70$ dB, determine I_0. (2 marks)

6 Chapter summary

Logarithms

A **logarithm** (or log) is the power or **exponent** to which a base is raised to yield a certain number.

When a logarithm is to base e it is called a natural logarithm.

$\log_e(x) = \ln(x)$

Solving logarithmic and exponential equations

It is often necessary when solving an equation in exponential form to express it as an equation in logarithmic form.

Logarithmic form	$\Leftrightarrow$	Exponential form
$x = \log_a(b)$	$\Leftrightarrow$	$a^x = b$

Laws of Logarithms

$\log_a(mn) = \log_a(m) + \log_a(n)$

$\log_a\left(\frac{m}{n}\right) = \log_a(m) - \log_a(n)$

$\log_a(m^k) = k\log_a(m)$

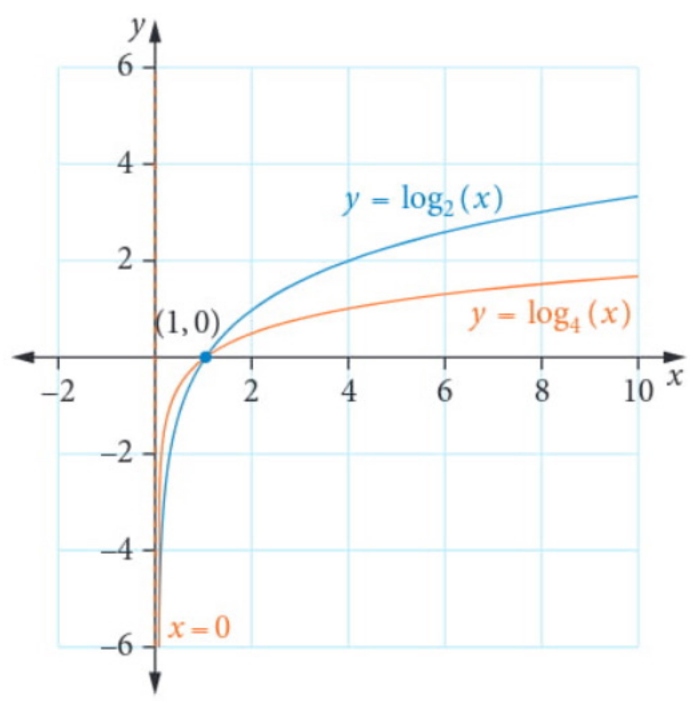
$\log_a 1 = 0$

$\log_a(a^b) = b$ and $a^{\log_a(b)} = b$

Properties of the logarithmic function $y = \log_a(x)$

- It is a strictly increasing function, increasing quickly at first, then more slowly.
- The gradient of the graph is always decreasing.
- The x-intercept is at $(1, 0)$ as $\log_a(1) = 0$.
- The y-axis $(x = 0)$ is a vertical asymptote.

Changing the base of the logarithm does not alter the basic shape, x-intercept or the asymptote of the graph of the logarithmic function.



Properties of the logarithmic function $y = \log_a(x - c) + b$, where b and c are positive real constants

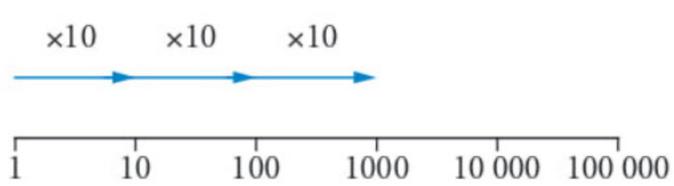
- It has the same shape as $y = \log_a(x)$.
- The horizontal translation is c units right and the vertical translation is b units up.
- $x = c$ is the vertical asymptote.
- Include the guiding point $(1 + c, b)$.

$f(x) = \ln(x)$ is called the **natural logarithmic function**, and is also written as $f(x) = \log_e(x)$.

Logarithmic scales

On a log scale, we *multiply* by the same number to move from one scale mark to the next. In the example on the right, the log scale involves multiplying by 10 each time.

This is called a 'log base 10' or '$\log_{10}$' scale.

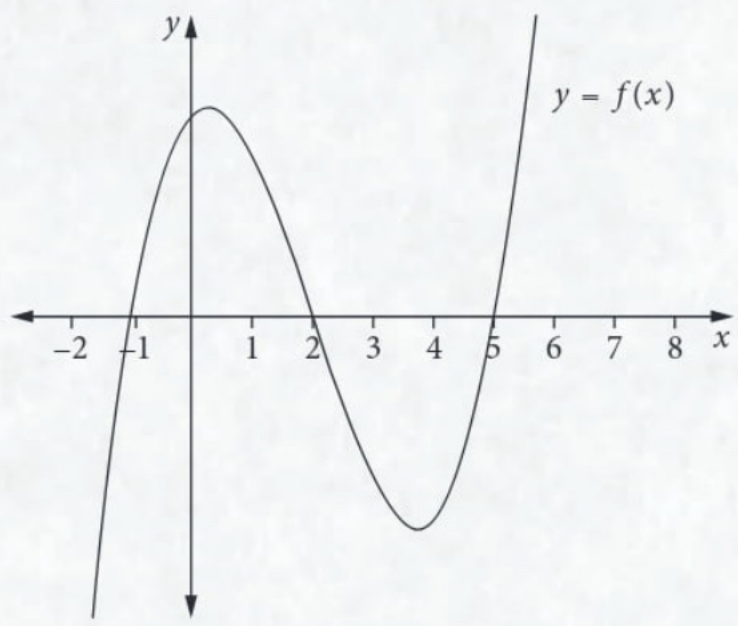


Cumulative examination: Calculator-free

Total number of marks: 22 Reading time: 3 minutes Working time: 22 minutes

1 (5 marks) Given $y = x + \sqrt{x^2 - 4}$, show that $(x^2 - 4)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = 0$.

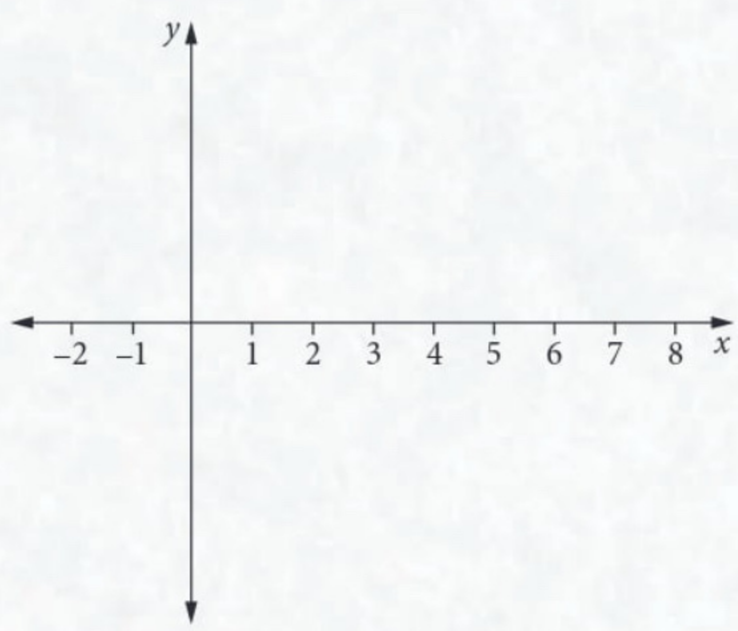
2 © SCSA MM2016 Q5 (6 marks) Consider the graph of $y = f(x)$ which is drawn below.



Let $A(x)$ be defined by the integral $A(x) = \int_{-1}^{x} f(t)\,dt$ for $-1 \le x \le 6$.

It is known that $A(2) = 15$, $A(5) = 0$ and $A(6) = 8$.

Copy the axes below and on them sketch the function of $A(x)$ for $-1 \le x \le 6$, labelling clearly key features such as x-intercepts, turning points and inflection points if any.



3 (8 marks) Solve each equation for x.

a $e^{2x+3} = 11$ (2 marks)

b $5e^{2x} = 27 + 2e^{2x}$ (2 marks)

c $\log_2(3x - 2) = 4$ (2 marks)

d $\ln(8x + 4) - \ln(2) = 3$ (2 marks)

4 (3 marks) Find the values of a and b if the logarithmic function $f(x) = \ln(x - a) + b$ has a vertical asymptote with equation $x = -6$ and passes through the point $(-5, 4)$.

Cumulative examination: Calculator-assumed

Total number of marks: 30 Reading time: 3 minutes Working time: 30 minutes

1 © SCSA MM2020 Q11 (9 marks) The line $y = x + c$ is tangent to the graph of $f(x) = e^x$.

a Obtain the coordinates of the point of intersection of the tangent with the graph of $f(x)$. (2 marks)

b What is the value of c? (1 mark)

c Sketch the graph of $f(x)$ and the tangent on the same axes. (1 mark)

d Evaluate the exact area between the graph of $f(x)$, the tangent line, and the line $x = \ln 2$. (3 marks)

e Given that $g(x)$ is the inverse function of $f(x)$, write a definite integral that could be used to determine the area between the graph of $g(x)$, the x-axis, and the line $x = \ln 2$. (2 marks)

2 (5 marks) A roulette wheel has thirty-seven slots, numbered 0 to 36. Slot 0 is green. Eighteen of the remaining slots are black, and the other eighteen are red. In a single game, a person spins the wheel and at the same time rolls a ball around the wheel in the opposite direction. As the wheel slows, the ball falls into one of the slots. The wheel is carefully balanced so that the ball is equally likely to fall into any of the slots.

a In a single game, what is the probability, correct to three decimal places, that the ball falls into a black slot? (1 mark)

Several games are played one after the other. Assume that the result of each game is independent of the result of any other game.

b What is the probability, correct to three decimal places, that the first time that the ball falls into a black slot is in the sixth game? (2 marks)

c What is the probability, correct to three decimal places, that the ball falls into a black slot three times in the first six games? (2 marks)

3 © SCSA MM2020 Q13 (7 marks) A company manufactures small machine components. They can manufacture up to 200 of a particular component in one day. The total cost, in hundreds of dollars, incurred in manufacturing the components is given by

$$C(x) = \frac{x \ln(2x + 1)}{3} - 2x + 120$$

where x is the number of components that will be produced on that day.

a Determine the total cost of manufacturing 20 components in one day. (1 mark)

b Sketch the graph of $C(x)$. (3 marks)

c With reference to your graph in part **b**, explain how many components the company should manufacture per day if the total cost is to be as low as possible. (3 marks)

4 © SCSA MM2021 Q16 MODIFIED (9 marks) An analyst was hired by a large company at the beginning of 2021 to develop a model to predict profit. At that time, the company's profit was $4 million. The model developed by the analyst was:

$$P(x) = \frac{20 \ln (x + a)}{x + 5}$$

where $P(x)$ is the profit in millions of dollars after x weeks and a is a constant.

a Show that $a = e$. (2 marks)

b What does the model predict the profit will be after five weeks? (1 mark)

c What is this maximum profit and during which week will it occur? (2 marks)

d According to the model, during which week will the company's profit fall below its value at the beginning of 2021? (1 mark)

The model proved accurate and after 10 weeks the company implemented some changes. From this time the analyst used a new model to predict the profit:

$$N(y) = 2e^{b(10+y)}$$

where $N(y)$ is the profit in millions of dollars y weeks from this point in time and b is a constant.

e The company is projecting its profit to exceed $5 million. During which week does the new model suggest this will happen? (3 marks)

CUMULATIVE EXAMINATION: CALCULATOR-ASSUMED