| 38) ... → ℝ

~~21)~~

20) a) $f: \mathbb{N} \to \mathbb{N}$
$f(x) = x+1$

b) $f: \mathbb{N} \to \mathbb{N}$
$f(x) = \begin{cases} n/2 & \mid n \% 2 = 0 \\ (n-1)/2 & \mid n \% 2 = 1 \end{cases}$

c) $f: \mathbb{N} \to \mathbb{N}$
$f(x) = \begin{cases} x+1 & x \geq 2 \\ 1 & x = 1 \end{cases}$

d) $f: \mathbb{N} \to \mathbb{N}$
$f(x) = \begin{cases} x^2 & x \geq 5 \\ 25 & x < 5 \end{cases}$

22) $f: \mathbb{R} \to \mathbb{R}$

a) $f(x) = -3x+4$

proving one to one - via contradiction

assume $f(x_1) = f(x_2)$
but $x_1 \neq x_2$

$-3x_1 + 4 = -3x_2 + 4$

$-3x_1 = -3x_2$

$x_1 = x_2$

however this contradicts our assumption that $x_1 \neq x_2$ ∴ $x_1 = x_2$ hence this function is one-to-one

proving onto

$y \in \mathbb{R} \to x \in \mathbb{R} \mid f(x) = y$

assume some arbitrary y w/o loss of generality where

$y = f(x)$

$y = -3x+4$

$\frac{y-4}{-3} = x$ ■ since $\frac{y-3}{2}$ is a real number for $y \in \mathbb{R}$ then there for every y there exists x where $f(x) = y$

∴ it is bijective

b) $f(x) = -3x^2 + 7$

prove one-to one

counterexample:

$x_1 = 5$
$x_2 = -5$ $\quad x_1 \neq x_2$

$-3(5^2) + 7 = -3(-5)^2 + 7$

$-75 + 7 = -75 + 7$

$-68 = -68$

$f(x_1) = f(x_2)$ ■

$\exists x_1 x_2$ where $f(x_1) = f(x_2)$ but $x_1 \neq x_2$

∴ not one-to-one ■

Therefore $-3x^2 + 7 = f(x)$ is not a bijection

c) $f(x) = \frac{x+1}{x+2}$

one-to-one proof by contradiction

assume $f(x_1) = f(x_2)$ but $x_2 \neq x_1$

$\frac{x_1+1}{x_1+2} = \frac{x_2+1}{x_2+2}$

$(x_1+1)(x_2+2) = (x_1+2)(x_2+1)$

$\cancel{x_1x_2} + x_2 + 2x_1 + 2 = \cancel{x_1x_2} + x_1 + 2x_2 + 2$

$x_2 + 2x_1 = x_1 + 2x_2$

$-x_2 = -x_1$

$x_2 = x_1$

We have a contradiction where $x_2 = x_1$. This contradicts our assumption ∴ $f(x_1) = f(x_2) \to x_2 = x_1$

∴ it is one-to-one

onto proof?

counterexample:

$f(x) = \frac{x+1}{x+2}$ has a horizontal asymptote @ $y = 1$ therefore

@ $y = 1$ there doesn't exist an x where $y = f(x)$

∴ not onto ■

Therefore the function is NOT A Bijection

d) $f(x) = x^5 + 1$

prove one-to-one

assume $x_1, x_2 \in \mathbb{R}$ and $f(x_1) = f(x_2)$ but $x_1 \neq x_2$

$x_1^5 + 1 = x_2^5 + 1$

$x_1^5 = x_2^5$ - power both sides by $\frac{1}{5}$

$x_1 = x_2$

this contradicts our assumption that $x_1 \neq x_2$ ∴ this function must be one-to-one

prove onto

let $y \in \mathbb{R}$ let $f(x) = y$

$x^5 + 1 = y$

$x = \sqrt[5]{y-1}$

$x \in \mathbb{R}$ for any $y \in \mathbb{R}$ this function is onto

∴ This is a bijection

2.3)

26) a) prove that a strictly increasing function is one-to-one from $\mathbb{R} \to \mathbb{R}$

let $x_1, x_2 \in \mathbb{R}$ such that $x_1 < x_2$ w/o loss of generality

let $f(x_1) < f(x_2)$.

if $x_1 < x_2$ then $x_1 \neq x_2$ obviously

if $x_1 < x_2$ always results in $f(x_1) < f(x_2)$

$\therefore x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2)$ therefore

this implies that a strictly increasing function is one-to-one

26) b) an increasing function is defined as

$\forall x_1 \forall x_2 ((x_1 < x_2) \rightarrow (f(x_1) \leq f(x_2)))$

$\therefore f(x) = \begin{cases} 25 & x < 5 \\ x^2 & x \geq 5 \end{cases}$

is increasing but not one-to-one as all numbers < 5 map to the image 25

3a) $g: A \to B$ $\quad f \circ g = f(g(x))$

$f: B \to C$ $\quad f \circ g: A \to C$

prove if $f \circ g$ is onto $\to f$ is onto

assume $f \circ g$ is onto and f is onto

let some $c \in C$ and let some $a \in A$

set $f \circ g(a) = c$

$f(g(a)) = c \Rightarrow g(a) = b$ where $b \in B$

$f(b) = c$ this implies $f(x)$ is onto ~~since~~ which contradicts our assumption $\therefore$ $\boxed{\text{if } f \circ g \text{ is onto } f(x) \text{ is onto}}$

see this shows that for each $c \in C$ ~~there~~ exists a $b = g(a) \in B$

~~if for every c there exist a ∈ A where~~

~~f(g(a)) = c then this implies that the~~

~~a that then creates a b ∈ B value using g(x) which then creates~~ C

34) a) suppose $f \circ g$ is onto

this implies ~~[illegible]~~

$\forall c \exists a (c \in C \to (a \in A \wedge f(g(a)) = c))$

by def of $f \circ g$, for every $c \in C$, there exists element $b \in B$ where $b = g(a) \in B$ such that $f(b) = c$ $\therefore$ for every $c \in C$ there is $b \in B$ where $f(b) = c$ $\therefore$ f is onto when $f \circ g$ is onto

b) $f: B \to C$

$g: A \to B$

if $f \circ g$ is one-to-one then

$f \circ g(a_1) = f \circ g(a_2)$ (where $a_1, a_2 \in A$)

imply ~~from~~ $a_1 = a_2$

suppose $g(a_1) = g(a_2)$ for some $a_1, a_2 \in A$

by applying ~~f∘f~~ f to both sides we get

$f(g(a_1)) = f(g(a_2))$ and since $f \circ g$ is one-to-one this implies $a_1 = a_2$ $\therefore$ if $g(a_1) = g(a_2)$ implies $a_1 = a_2$ then one can say g is one-to-one

c) suppose $f \circ g$ is an bijection therefore $f \circ g$ is both one-to-one and onto.

~~[illegible]~~

proof pt 1)

prove that if $f \circ g$ is a bijection and g is onto then f is one-to-one.

$f \circ g$ is one to one $\therefore f \circ g(a_1) = f \circ g(a_2) \to a_1 = a_2$ where $a_1, a_2 \in A$

thus suppose $g(a_1) = b_1$ and $g(a_2) = b_2$ where $b_1, b_2 \in B$ (we can imply this because g is onto)

$\therefore f(b_1) = f(b_2)$ which means that since we know $f \circ g$ is one-to-one this implies

$b_1 = b_2$ $\therefore$ f is one-to-one

Proof pt 2) prove that if $f \circ g$ is a bijection & f is one to one then g is onto

by def. $f \circ g(a) = c$ (for some $c \in C$ there exists $a \in A$)

$f \circ g(a) = c$, for every $c \in C$ we can find $b = g(a) \in B$ where $f(b) = c$, since $f \circ g$ is one-to-one the only way for it cover all elements of C is for $g(a)$ to cover all elements of B $\therefore$ g must be onto ∎

38) $f \circ g: \mathbb{R} \to \mathbb{R}$

$f \circ g = (x+2)^2 + 1 = x^2 + 5 + 2x$

$g \circ f: \mathbb{R} \to \mathbb{R}$

$g \circ f = (x^2+1) + 2 = x^2 + 3$

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$f \circ g = a(cx+d) + b$

$g \circ f = c(ax+b) + d$

$a(cx+d) + b = c(ax+b) + d$

$acx + ad + b = cax + cb + d$

~~ab~~ $ad + b = cb + d$

~~[illegible]~~

$ad - d = cb - b$

$\boxed{d(a-1) = b(c-1)}$

This is the condition for $f \circ g$ to equal $g \circ f$

42) let $f: A \to B$

& $S \subseteq A$ $T \subseteq A$

a) prove $f(S \cup T) = f(S) \cup f(T)$

i) prove $f(S \cup T) \subseteq f(S) \cup f(T)$

$y \in f(S \cup T)$ $\therefore$

$y = f(x)$ where $x \in S \cup T$

case 1)

$x \in S$

$y = f(x) \in f(S)$

case 2)

$x \in T$

$y = f(x) \in f(T)$

$y \in f(S) \cup f(T)$

$\therefore f(S \cup T) \subseteq f(S) \cup f(T)$

ii) prove $f(S) \cup f(T) \subseteq f(S \cup T)$

$y \in f(S) \cup f(T)$ or $(y \in f(S)$ or $y \in f(T))$

let $y = f(x)$ such that $x \in S \vee x \in T$

case 1:

$y = f(x)$ where $x \in S$ $\therefore$ it follows that $x \in S \cup T$ $\therefore y \in f(S \cup T)$

case 2:

$y = f(x)$ where $x \in T$ $\therefore$ it follows that $x \in S \cup T$ $\therefore y \in f(S \cup T)$

$\therefore y \in f(S \cup T)$ $\therefore$

$f(S) \cup f(T) \subseteq f(S \cup T)$ $\therefore$

$f(S \cup T) = f(S) \cup f(T)$

b) prove $f(S \cap T) \subseteq f(S) \cap f(T)$

let $y \in f(S \cap T)$

set $y = f(x)$ where $x \in S \cap T$

$\quad \quad = x \in S \wedge x \in T$

case 1:

$y = f(x)$ $x \in S$ $\therefore y \in f(S)$

Also $\downarrow$ x is also! AND

$y = f(x)$ $x \in T$ $\therefore y \in f(T)$

$\therefore y \in f(S) \wedge f(T)$

$y \in f(S) \cap f(T)$ ■

(2.4)

10) a) $a_n = (-2)a_{n-1}$

$a_0 = -1$

$a_1 = -2(-1) = 2$

$a_2 = -2(2) = -4$

$a_3 = -2(-4) = 8$

$a_4 = -16$

$a_5 = 32$

$a_6 = -64$

b) $a_n = a_{n-1} - a_{n-2}$

$a_0 = 2$

$a_1 = -1$

$a_2 = -1 - 2 = -3$

$a_3 = -3 - -1 = -2$

$a_3 = -2 - -3 = 1$

$a_4 = 1 - -2 = 3$

$a_5 = 3 - 1 = 2$

$a_6 = 2 - 3 = -1$

c) $a_n = 3(a_{n-1})^2$

$\begin{array}{r} 27 \\ \times 27 \end{array}$

$a_0 = 1$

$a_1 = 3$

$a_2 = 27$

$a_3 = 2187$

$a_4 = 14348907$

$a_5 \approx 6.176734 \cdot 10^{14}$

$a_6 \approx 1.1445613 \times 10^{39}$

d) $a_n = n a_{n-1} + a_{n-2}^2$

$a_0 = -1$

$a_1 = 0$

$a_2 = 2(0) + 1 = 1$

$a_3 = 3(1) + 0 = 3$

$a_4 = 4(3) + 1 = 5$

$a_5 = 5(5) + 3^2 = 25 + 9 = 34$

$a_6 = 6(34) + 5^2 = 229$

e) $a_0 = 1$

$a_1 = 1$

$a_2 = 2$

$a_3 = 2 - 1 + 1 = 2$

$a_4 = 2 - 2 + 1 = 1$

$a_5 = 1 - 1 + 2 = 2$

$a_6 = 2 - 2 + 1 = 1$

12) a) ✓✓

$-3(0) + 4(0) = 0$

for all a_n

b) ✓✓

$a_n = 1$ for all n

$-3(1) + 4(1) = 1$ always

c) $(-4)^n = -3(-4)^{n-1} + 4(-4)^{n-2}$

$(-4)^n = 16(-4)^{n-2}$

$(-4)^n = (-4)^n$ ✓✓ valid solution

yup it works!

d) $2(-4)^n + 3 = -3(2(-4)^{n-1} + 3) + 4(2(-4)^{n-2} + 3)$

$= (-6(-4)^{n-1} - 9) + (8(-4)^{n-2} + 12)$

$= -6(-4)^{n-1} + 8(-4)^{n-2} + 3$

$= (-6(-4)) + 8) -4^{n-2} + 3$

$= (32)(-4)^{n-2} + 3$

$= 2(-4)^n + 3$ ✓✓✓

yup it works

16) c)

$a_n = a_{n-1} - n$

$a_0 = 4$

$a_1 = 4 - 1 = 3$

$a_2 = 3 - 2 = 1 = 4 - 1 - 2$

$a_3 = 1 - 3 = -2 = 4 - 1 - 2 - 3$

$a_4 = -2 - 4 = -6 = 4 - 1 - 2 - 3$

$a_5 = -11 = 4 - 1 - 2 - 3 - 4$

$$a_n = 4 - \underbrace{\frac{n(n+1)}{2}}_{\text{Subtracting the sum of integers up to } n}$$

e) $a = (n+1)\ a_{n-1}$

$a_0 = 2$ $\quad a_1 = 2(2) = 4$ $\quad a_2 = 3(4) = 12$

$\cancel{a_2 = 2(2+2) = 4}$

$\cancel{a_2 = 2((2(2))+1) = 10}$

$\cancel{a_3 = 2(10+1) = 22}$

$a_3 = 4(12) = 48$

$a_n = (n+1)(n)(n-1)(n-2)\ldots(2)$

$\boxed{a_n = 2(n+1)!}$