8.2 Mathematical tools for image formation. Homogeneous transformations

Homogeneous (also called projective) transformations are linear transformations (i.e. matrix multiplications) **between homogeneous coordinates** (vectors). Such coordinates are obtained from Cartesian (inhomogeneous) vectors by **extending them with a non-negative number** (typically 1, for convenience).

Although we are going to explain homogeneous transformations using the 3D space, **this is generalizable to any other number of dimensions**.

A 3D vector (or a 3D point) in **inhomogeneous coordinates** looks like:

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$$

while the same vector in **homogeneous coordinates** has the form (note the tilde in the notation):

$$\tilde{\mathbf{x}} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \equiv \begin{bmatrix} \lambda x \\ \lambda y \\ \lambda z \\ \lambda \end{bmatrix} \in \mathbb{R}^4$$

We can go back by dividing the three first coordinates by the fourth:

$$\tilde{\mathbf{x}} = \begin{bmatrix} \lambda x \\ \lambda y \\ \lambda z \\ \lambda \end{bmatrix} \Rightarrow \mathbf{x} = \begin{bmatrix} x/\lambda \\ y/\lambda \\ z/\lambda \end{bmatrix} \in \mathbb{R}^3$$

This way, the family of homogeneous vectors with $\lambda \neq 0$ represents the same point in $\mathbb{R}^3$, since λ doesn't affect. Another consequence of λ is that **any transformation in homogeneous coordinates holds for any scaled matrix**:

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{bmatrix} \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \\ \lambda \end{bmatrix} = \lambda \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix}$$

Then, the following transformations are equivalent:

$$\begin{bmatrix} 1 & -3 & 2 & 5 \\ 4 & 2 & 1 & 2 \\ 4 & -1 & 0 & 2 \\ -6 & 2 & 1 & 2 \end{bmatrix} \equiv \lambda \begin{bmatrix} 1 & -3 & 2 & 5 \\ 4 & 2 & 1 & 2 \\ 4 & -1 & 0 & 2 \\ -6 & 2 & 1 & 2 \end{bmatrix}$$

This indetermination is typically handled by fixing one entry of the matrix, (e.g. $p_{44} = 1$). Also, these matrices must be non-singular (Rank = 4).

In [1]:
```
import numpy as np
import cv2
import matplotlib.pyplot as plt
import matplotlib
import scipy.stats as stats
from ipywidgets import interact, fixed, widgets
from mpl_toolkits.mplot3d import Axes3D
from math import sin, cos, radians
%matplotlib widget

matplotlib.rcParams['figure.figsize'] = (6.0, 6.0)
images_path = './images/'
```

8.2.1 Why do we want this? Reason one

Now that we know how homogenous coordinates and homogenous transformations works, it's time for understanding **why this is interesting**.

For now, we were performing complete transformations (rotations and translations) by using a rotation matrix and adding a translation vector to the rotated points ($\mathbf{p}' = \mathbf{R}\mathbf{p} + \mathbf{t}$).

The problem of this transformation is that the **concatenation of transformations** when a sequence of transformations has to be done becomes a mess:

$$\mathbf{p}' = \mathbf{R}_1\mathbf{p} + \mathbf{t}_1$$
$$\mathbf{p}'' = \mathbf{R}_2\mathbf{p}' + \mathbf{t}_2 = \mathbf{R}_2(\mathbf{R}_1\mathbf{p} + \mathbf{t}_1) + \mathbf{t}_2 = \mathbf{R}_2\mathbf{R}_1\mathbf{p} + \mathbf{R}_2\mathbf{t}_1 + \mathbf{t}_2$$

In the context of our problem, every time we move the camera we have to concatenate a new transformation. Imagine this in a first-person videogame, where a transformation of the coordinate system is needed in every frame. In just a second more than 60 concatenations should be computed!

What happens if we use homogenous coordinates?

We can express a transformation consisting of a rotation + translation using only a matrix multiplication:

$$\tilde{\mathbf{p}}' = \mathbf{T}_1\tilde{\mathbf{p}} \Rightarrow \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} r_{11}x + r_{12}y + r_{13}z + t_x \\ r_{21}x + r_{22}y + r_{23}z + t_y \\ r_{31}x + r_{32}y + r_{33}z + t_z \\ 0x + 0y + 0z + 1 \end{bmatrix} = \begin{bmatrix} r \\ r \\ r \\ \end{bmatrix}$$

Note that the 3×3 left-top submatrix of the $\mathbf{T}_1$ matrix is a rotation matrix while the last column contains the desired translation. This is the main reason for using homogeneous coordinates, look **how concatenation is applied now!**

$$\tilde{\mathbf{p}}' = \mathbf{T}_1\tilde{\mathbf{p}}$$
$$\tilde{\mathbf{p}}'' = \mathbf{T}_2\tilde{\mathbf{p}}' = \mathbf{T}_2\mathbf{T}_1\tilde{\mathbf{p}}$$

Concatenation becomes much easier, being only consecutive matrix multiplications (remember that, nowadays, matrix multiplications are very fast using GPUs).

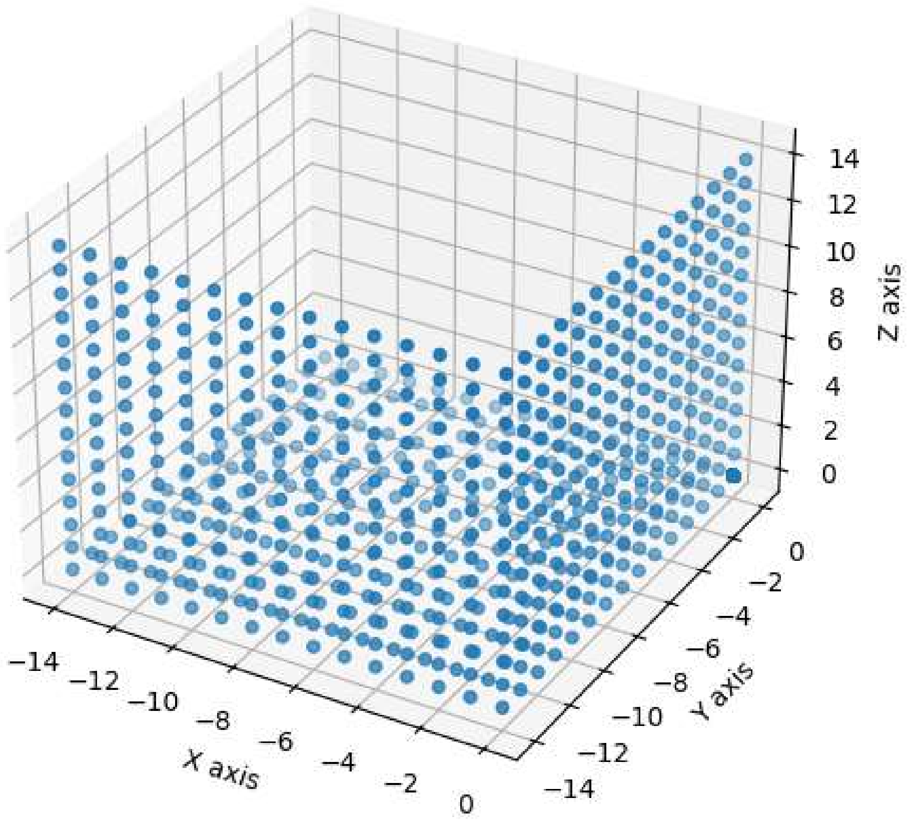
Let's play a bit with homogeneous coordinates. We are going to apply a homogenous transformation to a 3D object (a set of 3D-points, in fact). For this, we have defined `data` $\in \mathbb{R}^4$, a **matrix containing more than 3k points in homogenous coordinates**:

In [2]:

```
# Load data
data = np.load("./data/data.npy")
print('Number of points:', data.size/4)

# Create figure
fig = plt.figure()

# Prepare figure for 3D data
ax = plt.axes(projection='3d')

# Name axes
ax.set_xlabel('X axis')
ax.set_ylabel('Y axis')
ax.set_zlabel('Z axis')

# Plot points
ax.scatter(data[0,:], data[1,:], data[2,:]);
```

```
Number of points: 3375.0
```

Figure



ASSIGNMENT 1: Homogeneous transformations for the win

Now, create a new method called `apply_homogeneous_transformation()` that builds a homogeneous matrix from some *yaw*, *pitch* and *roll* values as well as a translation vector and applies it to the input data matrix `data`. *Note that we are not transforming vectors, but points, so use* `scatter()` *instead of* `quiver()`.

Notice that opposite to the euclidean case, here both rotation and translation are applied just with one matrix multiplication! (`t` in the following code).

Recall the matrices defining the elemental rotations:

$$\mathbf{R}_z(\theta) = \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{R}_y(\theta) = \begin{bmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{bmatrix} \qquad \mathbf{R}_x$$

In [15]:

```
# ASSIGNMENT 7
def apply_homogeneous_transformation(data, yaw, pitch, roll, translation):
    """ Apply a linear transformation to a set of 3D-vectors and plot them

        Args:
            data: Input set of points to transform
            yaw: Degrees to rotate the coordinate system around the 'Z' axis
            pitch: Degrees to rotate the coordinate system around the 'Y' axis
            roll: Degrees to rotate the coordinate system around the 'X' axis
            translation: Column vector containing the translation for each axis
    """
    # Write your code here!

    # Transform to radians
    yaw = radians(yaw)
    pitch = radians(pitch)
    roll = radians(roll)

    # Construct rotation matrices
    Rx = np.array([[1,0,0],[0, cos(roll), -sin(roll)],[0, sin(roll), cos(roll)]]
    Ry = np.array([[cos(pitch), 0, sin(pitch)],[0,1,0],[-sin(pitch), 0, cos(pitc
    Rz = np.array([[cos(yaw), -sin(yaw), 0],[sin(yaw), cos(yaw), 0],[0,0,1]])

    # Combine rotation matrices
    R = np.dot(np.dot(Rz, Ry), Rx)

    # Create homogenous transformation matrix
    t = np.zeros((4,4))
    t[0:3,0:3] = R
    t[0:3,3] = translation
    t[3,3] = 1

    transformed =  t @ data

    # Create figure
    fig = plt.figure()

    # Prepare figure for 3D data
    ax = plt.axes(projection='3d')

    # Name axes
    ax.set_xlabel('X axis')
    ax.set_ylabel('Y axis')
    ax.set_zlabel('Z axis')

    # Plot points
    ax.scatter(transformed[0,:], transformed[1,:], transformed[2,:]);
```

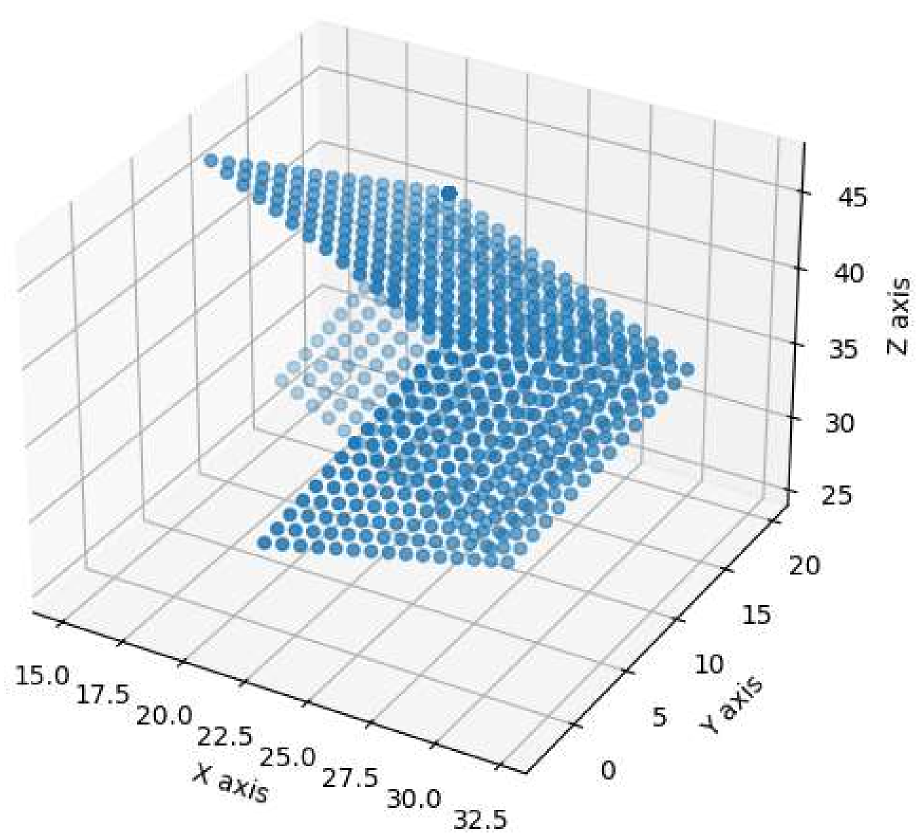
Now apply the following transformation to the object previously loaded:

- **Yaw** rotation: 45 degrees
- **Pitch** rotation: -60 degrees
- **Roll** rotation: 20 degrees
- **Translation**:
  - X-axis: 20 units
  - Y-axis: 20 units
  - Z-axis: 40 units

Remember that this is going to be performed only using a homogeneous transformation (one unique matrix multiplication)!

In [16]:
```
# Load data
data = np.load("./data/data.npy")

# Apply transformation
apply_homogeneous_transformation(data, 45, -60, 20, [20,20,40])
```

Figure



Expected output:

No description has been provided for this image
No description has been provided for this image

By doing these transformations, our graphic engine could represent moving objects in the `WORLD` when the player is still (e.g. flying birds, cars, other players, etc.).

Checking execution time

Just as a curiosity, let's check how much time is required by python to apply a transformation in both euclidean and homogeneous coordinates. *Note: take care with the length, if it is too large it could freeze your computer!*

In [17]:
```
# Set the number of points to transform
length = 6000000

# Prepare data in Euclidean coordinates
Pts = np.random.rand(3,length)
R = np.random.rand(3,3)
T = np.random.rand(3,1)

# Prepare data in Homogeneous coordinates
Ones = np.ones(length)
Pts_H = np.row_stack((Pts,Ones))
T_H = np.random.rand(4,4)

import time

# Apply euclidean transformation
start_time = time.time()
res = R@Pts+T
print("Time spent with Euclidean tranformation: %s seconds ---" % (time.time() -

# Apply Homogeneous transformation
start_time = time.time()
res = T_H@Pts_H
print("Time spent with Homogeneous tranformation: %s seconds ---" % (time.time()
```

```
Time spent with Euclidean tranformation: 0.07600259780883789 seconds ---
Time spent with Homogeneous tranformation: 0.045998334884643555 seconds ---
```

ASSIGNMENT 2: How the player sees the world

One final example, consider the following image, in where our character Joel is looking at a dystopian, post-apocalyptic scenario.

No description has been provided for this image

The `WORLD` reference system is displayed in orange and labeled with $\{\mathbf{W}\}$ while the reference system of the character's view is displayed in red and labeled as $\{\mathbf{C}\}$. We know that:

- the position and orientation of $\{\mathbf{C}\}$ w.r.t. $\{\mathbf{W}\}$ is given by a *yaw* angle of -45°, a *roll* angle of -90 and a translation of [0.0,-4.0,1.2] meters in $[x,y,z]$ axes, respectively, and
- the coordinates of the point $\mathbf{p}^W$ in the world are [30.0,1.0,0.5] meters.

Could you compute what are its coordinates w.r.t our character's point of view in homogeneous coordinates ($\tilde{\mathbf{p}}^C$)? In other words, we have to build the homogeneous transformation $\mathbf{T}$ that produces $\tilde{\mathbf{p}}^C = \mathbf{T}\tilde{\mathbf{p}}^W$. As we will see in future notebooks, knowing such coordinates is vital to get the position of the 3D point **in the image** that Joel would see if this game were in first-person (fortunately it's not!).

In [18]:
```
# ASSIGNMENT 8
# Write your code here!
p = np.array([30,1,0.5,1])
p = np.vstack(p)

# Transform to radians
yaw = radians(-45)
pitch = radians(0)
roll = radians(-90)

# Construct rotation matrices
Rx = np.array([[1,0,0],[0, cos(roll), -sin(roll)],[0, sin(roll), cos(roll)]])
Ry = np.array([[cos(pitch), 0, sin(pitch)],[0,1,0],[-sin(pitch), 0, cos(pitch)]])
Rz = np.array([[cos(yaw), -sin(yaw), 0],[sin(yaw), cos(yaw), 0],[0,0,1]])

# Combine rotation matrices
R = np.dot(np.dot(Rz,Ry), Rx)

# Create homogenous transformation matrix
t = np.zeros((4,4))
t[0:3,0:3] = R
t[0:3,3] = [0.0,-4.0,1.2]
t[3,3] = 1

transformed = np.linalg.inv(t) @ p
print(transformed)
```

```
[[17.67766953]
 [ 0.7       ]
 [24.74873734]
 [ 1.        ]]
```

Expected output (homogeneous):

```
[[17.67766953]
 [ 0.7       ]
 [24.74873734]
 [ 1.        ]]
```

Thinking about it (1)

Now you are in a good position to answer these questions:

- What is the length of a 3D cartesian vector in homogeneous coordinates?

  Va a tener longitud 4 (x, y, z, 1)

- How many operations do you need to transform a point from the world frame to the camera one using euclidean coordinates? and using homogeneous coordinates?

  Coordenadas euclideas: 9 multiplicaciones (punto por la matriz de rotacion) y 6 sumas, y 3 sumas más (por el vector de translacion), lo que en total son 18 operaciones. Coordenadas homogeneas: se hacen 16 multipliaciones y 12 sumas, es decir 28 operaciones

- Explain the difference in the execution time when using the two types of transformations.

La transformacion de coordenadas homogeneas es más rapida cuando concatenamos transformaciones (multiplicaciones de matrices que son muy rapidas en GPUs). Mientras que con la transformacion de las coordenadas euclideas, la complejidad de transformaciones consecutivas aumenta rapido con cada una.

- Why are the rotations applied in that order? Could they have been applied differently?

  Porque la multiplicacion de matrices no cumple la propiedad conmutativa. Si, se podria aplicar de muchas otras formas pero una de las mas utilizadas es Z.Y.X .

8.2.2 Why do we want this? Reason two

There is another reason justifying the utilization of homogeneous coordinates when dealing with transformations: they result in a natural model for the camera, since points in the image plane are projection rays in $\mathbb{R}^3$.

In 1D:

- Cartesian coordinates: $x = x_1 = 3$.
- Homogeneous coordinates: $x = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \equiv \begin{bmatrix} 6 \\ 2 \end{bmatrix} \equiv \begin{bmatrix} 3k \\ k \end{bmatrix}, \ \ k \neq 0$

All the points in homogeneous coordinates represents the same point in cartesians since $x = x_1/x_2 = 3$.

In 2D:

- Cartesian coordinates: $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$.
- Homogeneous coordinates: $x = k \begin{bmatrix} x_2 \\ x_1 \\ 1 \end{bmatrix}, \ \ k \neq 0$

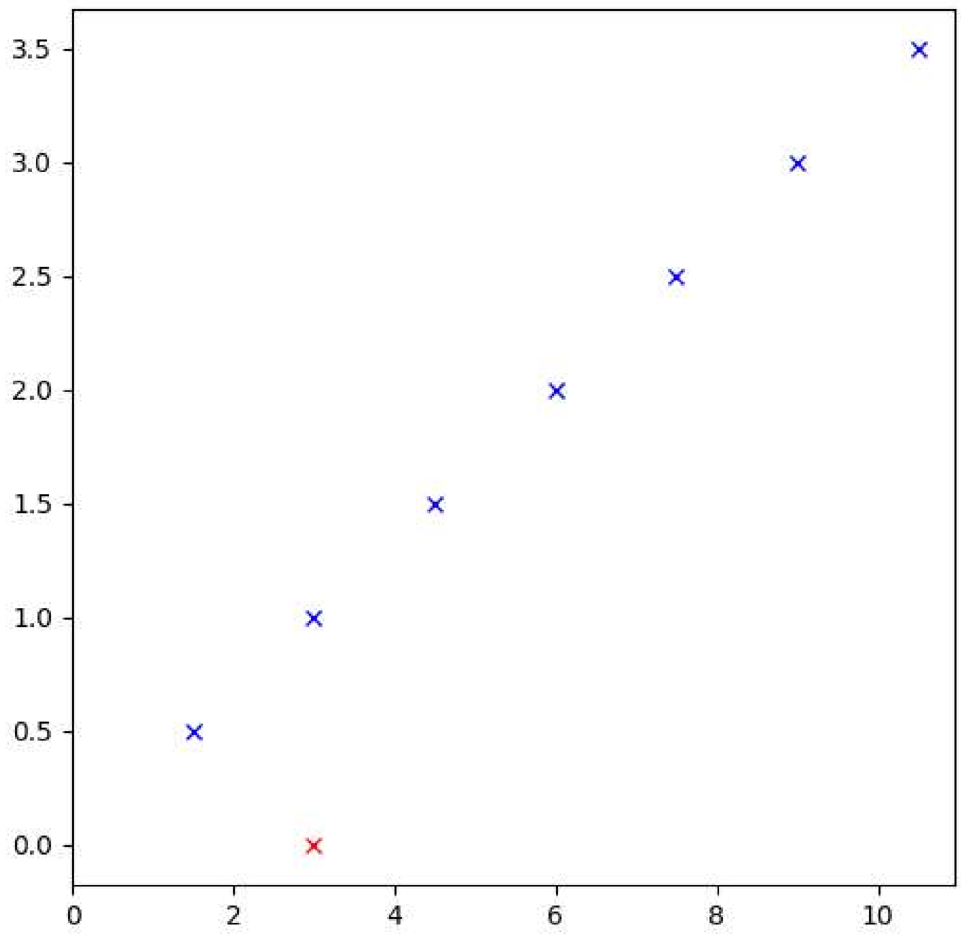
So in homogeneous coordinates, a point in the plane $\mathbb{R}^2$ transforms to a line passing through the origin in a reference frame parallel to the image plane (perpendicular to x_3).

The following code illustrates this for a 1D point.

In [19]:
```
# 1D point
x1 = 3
plt.figure()
plt.plot(x1,0,'rx')

# Point in homogeneous
x = np.array([3,1])

# Equivalent points by multiplying by lambda
for lambda_ in np.arange(0.5,4,0.5):
    plt.plot(x[0]*lambda_,x[1]*lambda_,'bx')
```

```
plt.xlim(0)
plt.show()
```

Figure



Conclusion

Awesome!

In his notebook we have learned:

- The principles of homogeneous coordinates.
- How to rotate and translate points or vectors using homogeneous coordinates.

Let's keep learning!