

A Stable Lasso

Mahdi Nouraie¹, Houying Zhu¹, and Samuel Muller^{1,2*}

¹School of Mathematical and Physical Sciences, Macquarie University

²School of Mathematics and Statistics, The University of Sydney

Abstract

The Lasso has been widely used as a method for variable selection, valued for its simplicity and empirical performance. However, Lasso's selection stability deteriorates in the presence of correlated predictors. Several approaches have been developed to mitigate this limitation. In this paper, we provide a brief review of existing approaches, highlighting their limitations. We then propose a simple technique to improve the selection stability of Lasso by integrating a weighting scheme into the Lasso penalty function, where the weights are defined as an increasing function of a correlation-adjusted ranking that reflects the predictive power of predictors. Empirical evaluations on both simulated and real-world datasets demonstrate the efficacy of the proposed method. Additional numerical results demonstrate the effectiveness of the proposed approach in stabilizing other regularization-based selection methods, indicating its potential as a general-purpose solution.

Keywords: Bioinformatics, Feature Selection, Lasso, Stability Selection, Variable Selection

1 Introduction

The Least Absolute Shrinkage and Selection Operator (Lasso; Tibshirani, 1996) has been widely used as a regularization-based variable selection method. However, it is well-established that the Lasso becomes unstable in the presence of correlated predictor variables (Wang et al., 2018). Specifically, when an irrelevant variable is highly correlated with relevant variables, the Lasso may be unable to distinguish between them, regardless of the amount of data available or the degree of regularization applied (Leng et al., 2006; Zhao and Yu, 2006). In this paper, we briefly review existing approaches aimed at addressing this issue, highlighting their limitations. We subsequently propose a simple methodology that improves the selection stability of the Lasso without increasing the optimization cost, altering the original data space, or requiring prior knowledge of the underlying grouping structure among predictors, addressing key limitations of existing methods. Our methodology relies on weighting predictors within the Lasso optimization process according to a correlation-adjusted ranking that reflects their predictive power.

*Address for correspondence: samuel.muller@mq.edu.au

We consider a dataset $\mathcal{D} = \{(\mathbf{x}_i^\top, y_i)\}_{i=1}^n$ where each element of \mathcal{D} consists of a univariate response $y_i \in \mathbb{R}$ and a p -dimensional fixed vector of predictor variables $\mathbf{x}_i^\top \in \mathbb{R}^p$. Linear regression is formally defined as $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where \mathbf{Y} denotes the n -vector of the response variable, $\mathbf{X} \in \text{mat}(n, p)$ denotes the fixed design matrix, $\boldsymbol{\beta}$ denotes the p -vector of regression coefficients, and $\boldsymbol{\epsilon}$ denotes the n -vector of the random errors. The intercept term is omitted here because, without loss of generality, we assume that the response vector is centered; that is, $\sum_{i=1}^n y_i = 0$. We also assume that the columns of the design matrix \mathbf{X} are centred and scaled by their means and standard deviations. Following the basic assumptions of the linear regression, it is assumed that $\mathbb{E}(\mathbf{Y}|X)$ is linear in the coefficients, $\mathbb{E}(\boldsymbol{\epsilon}|X) = 0$, and $\text{Var}(\boldsymbol{\epsilon}|X) = \sigma^2 \mathbb{I}_n$ where $\sigma^2 < \infty$ and $\mathbb{I}_n \in \text{mat}(n \times n)$ denotes the identity matrix.

The Lasso performs variable selection by augmenting the least-squares loss with an ℓ_1 -norm penalty on the regression coefficients, thereby encouraging sparsity in the estimated model. The Lasso regression coefficients are estimated through

$$\hat{\boldsymbol{\beta}}(\lambda) = \arg \min_{\boldsymbol{\beta} \in \mathbb{R}^p} \left(\|\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}\|_2^2 + \lambda \|\boldsymbol{\beta}\|_1 \right), \quad (1)$$

where $\lambda \in \mathbb{R}^+$ denotes the Lasso regularization parameter. Numerous extensions of the Lasso have been proposed since its inception (see, e.g., [Zou and Hastie \(2005\)](#), [Tibshirani et al. \(2005\)](#), [Zou \(2006\)](#), [Yuan and Lin \(2006\)](#), [Park and Casella \(2008\)](#), [Lorbert et al. \(2010\)](#), [Grave et al. \(2011\)](#), [Simon et al. \(2013\)](#), [Chen et al. \(2013\)](#), [Kong et al. \(2014\)](#), [Takada et al. \(2018\)](#), [Wang et al. \(2018\)](#), and [Craig et al. \(2024\)](#)), highlighting its popularity and effectiveness.

The stability of a variable selection method pertains to its capacity to consistently identify the same variables across different training sets sampled from the same underlying distribution ([Kalousis et al., 2007](#)). This property serves as an indicator of the reproducibility and generalizability of the selection results and has garnered significant attention in recent years (see, e.g., [Müller and Welsh \(2010\)](#), [Meinshausen and Bühlmann \(2010\)](#), [Wang et al. \(2018\)](#), and [Soloff et al. \(2024\)](#)). The presence of correlated predictor variables in \mathbf{X} is known to adversely affect the stability of variable selection performed by Lasso ([Meinshausen and Bühlmann, 2010](#); [Wang et al., 2018](#); [Faletto and Bien, 2022](#)).

Following [Takada et al. \(2018\)](#), variable selection methods can be broadly categorized into two classes. The first, known as ‘grouping selection’, seeks to retain all correlated variables that collectively provide predictive information for the response. The second, referred to as ‘exclusive selection’, retains only a subset of such correlated variables, based on the premise that including all may lead to redundancy. In this paper, we focus on exclusive selection methods. In this research thread, [Chen et al. \(2013\)](#) introduced the Uncorrelated Lasso, which was later refined by [Takada et al. \(2018\)](#) into the Independently Interpretable Lasso (IILasso). IILasso addresses the challenges posed by correlated variables by incorporating an additional penalty term into the Lasso objective function. This penalty is designed to discourage the simultaneous selection of correlated variables, encouraging sparse solutions within correlated groups. To the best of our knowledge, IILasso represents the current state-of-the-art in exclusive selection. Although effective, its optimization requires the full covariance matrix of the predictor variables, making it computationally expensive in high-dimensional settings, where $p \gg n$. In this paper, we propose an alternative strategy whose optimization reduces to the standard Lasso, enhanced

by variable-specific weights derived from a correlation-adjusted ranking based on the predictive power of predictors.

Certain methods, such as the Sparse Group Lasso (SGL; [Simon et al., 2013](#)), have been developed to encourage sparsity both between and within groups of correlated variables. However, these approaches typically require prior knowledge of the underlying grouping structure among predictors, which distinguishes them from the focus of this paper. In addition, similar to IILasso, these methods are generally more computationally intensive owing to the complexity of their objective functions. On the other hand, Stability Selection was introduced as a general framework to improve the stability of variable selection methods by leveraging random sub-samples of original data ([Meinshausen and Bühlmann, 2010](#); [Shah and Samworth, 2013](#)). Although this approach has shown considerable success, it has been demonstrated that it does not resolve the instability of Lasso in the presence of correlated predictors, due to the vote-splitting effect ([Meinshausen and Bühlmann, 2010](#); [Faletto and Bien, 2022](#)). In this paper, we use Stability Selection to assess the selection stability and to calibrate the regularization parameter λ , following the methodology proposed by [Nouraie and Muller \(2024\)](#), as outlined in Section 2.

[Nouraie et al. \(2025\)](#) proposed an alternative strategy for handling correlated variables by decorrelating them prior to selection. Their method, as a modified version of the classic Gram–Schmidt orthogonalization ([Björck, 1994](#)), applies selection algorithm to an orthonormal surrogate rather than the original data. While effective, this approach alters the original variable space. In this work, we explore a complementary direction that aims to improve selection stability of Lasso while preserving the original design matrix.

[Zou \(2006\)](#) proposed the Adaptive Lasso that incorporates variable-specific weights into the Lasso penalty, with the weights derived from initial estimates of the regression coefficients. In high-dimensional settings, obtaining consistent initial estimates is challenging. [Bühlmann and van de Geer \(2011\)](#) suggested using Lasso estimates as a basis for the initial weights; however, given the known instability of Lasso, propagating its estimates into the weighting scheme may reinforce the initial instable results. [Huang et al. \(2008\)](#) showed that when relevant and irrelevant variables are only weakly correlated, marginal univariate regression can provide suitable initial estimates. Nevertheless, this assumption rarely holds in real-world datasets, where strong correlations among predictors are common.

[Meinshausen and Bühlmann \(2010\)](#) introduced the Randomized Lasso, a method conceptually related to the Adaptive Lasso, with the primary distinction that variable weights are assigned randomly. As described by [Meinshausen and Bühlmann \(2010\)](#), the Randomized Lasso modifies the standard Lasso by weighting each predictor in the objective function by $1/w_j$, where $w_j = \alpha$ with probability $p_j \in (0, 1)$, and $w_j = 1$ otherwise for $j = 1, 2, \dots, p$. They demonstrated that this randomization yields asymptotic consistency within the Stability Selection framework, assuming sparse eigenvalues, a condition weaker than the irrepresentable condition ([Zhao and Yu, 2006](#); [Meinshausen and Bühlmann, 2006](#)) which guarantees consistent variable selection for the Lasso. Although the Randomized Lasso achieves asymptotic consistency, we contend that its random weighting scheme may introduce additional variability and potentially undermine the accuracy of the results. A numerical comparison between our method and the Randomized Lasso is provided in Section 3. [Courtois et al. \(2021\)](#) also proposed

two variants of the Adaptive Lasso that aim to improve false discovery rates and sensitivity. While Adaptive Lasso shares a conceptual similarity with our approach, the way we assign the weights is notably different. To the best of our knowledge, no existing variant of the Adaptive Lasso accounts for the correlation structure among predictors and incorporates their predictive relevance rankings to inform variable-specific weights. Other related methods are discussed in Section 4.

[Fan and Lv \(2008\)](#) introduced the Sure Independence Screening (SIS) procedure, which reduces the dimensionality of high-dimensional data by ranking variables based on their marginal correlations with the response variable. This approach asymptotically retains relevant variables with overwhelming probability, a desirable property known as the ‘sure screening property’. According to [Fan and Lv \(2008\)](#), a sufficient condition for SIS to possess this property is that the marginal correlations of relevant predictors must be bounded away from zero. However, this condition is often violated in practice due to the presence of correlation among predictors ([Wang and Leng, 2015](#)). We highlight the similarity between this issue and that discussed above concerning the use of marginal univariate regression coefficients in the Adaptive Lasso. Specifically, when the Adaptive Lasso uses the marginal coefficients to weight predictors, it ranks variables based on their individual predictive strength and applies a penalty that is implicitly an increasing function of their rank, without accounting for the underlying correlation structure among predictors. As mentioned above, this approach is reliable only under conditions of weak correlation.

[Wang and Leng \(2015\)](#) introduced the Ridge High-dimensional Ordinary Least Squares Projection (Ridge-HOLP; [Wang and Leng, 2015](#)), a method that ranks variables by simultaneously accounting for their predictive power and correlation structure. This is achieved by removing the random diagonal matrix D that arises in the singular value decomposition of X ([Wang and Leng, 2015](#)). They demonstrated that when $p > n$, the Moore–Penrose inverse of X can be used to construct a screening-consistent procedure; that is, when selecting a model of the same size as the true model, the method asymptotically identifies the correct set of relevant variables. According to [Wang and Leng \(2015\)](#), a critical condition for the screening consistency of Ridge-HOLP is $\kappa(\Sigma) \leq cn^\tau$, where $c > 0$ and $0 \leq \tau < 1/7.5$, Σ denotes the covariance matrix of X , $\kappa(\Sigma) = \lambda_{\max}(\Sigma)/\lambda_{\min}(\Sigma)$ denotes the condition number of Σ , and $\lambda_{\max}(\Sigma)$ and $\lambda_{\min}(\Sigma)$ denote the largest and smallest eigenvalues of Σ respectively. One instance in which this condition is met is when the groups of correlated variables are mutually independent, each group contains finitely many variables, and the variables within each group exhibit a compound symmetric structure with a correlation value less than one. The assumptions required for the screening consistency of Ridge-HOLP are stated in Theorem 3 of [Wang and Leng \(2015\)](#). Recently, [Joudah et al. \(2025\)](#) introduced Air-HOLP, a data-adaptive extension of Ridge-HOLP, and demonstrated through extensive simulation studies that it performs well in the presence of correlated predictors.

In this paper, we propose to first rank the predictors using a correlation-adjusted ranking method based on [Wang and Leng \(2015\)](#), and then leverage these ranks to weight the predictors in the Lasso objective function, with the aim of enhancing selection stability. Consequently, our method can be viewed as a more stable variant of the Lasso and a generalized form of the Adaptive Lasso. Although our proposed technique is relatively simple, the numerical results

are highly promising compared to existing methods, particularly those with complex objective functions that result in substantially longer running times.

The rest of this paper is organized as follows. Section 2 outlines the proposed methodology. Section 3 describes the real and synthetic datasets used in this paper and presents the corresponding numerical results. Section 4 reviews some related works and highlights some encouraging directions for future research.

2 Methodology

In this section, we propose a methodology to achieve stable variable selection using the Lasso, without increasing the computational cost of the optimization process, altering the original data, or requiring prior knowledge of the underlying grouping structure among predictor variables. The central idea is to assign weights to variables based on a correlation-adjusted ranking that reflects their predictive relevance. By correlation-adjusted, we mean that a relevant variable correlated with irrelevant ones is assigned a higher rank. These ranks are then transformed into variable-specific penalty factors and incorporated into the Lasso objective function. This weighting scheme provides the model with prior information, enabling it to prioritize more promising variables within groups of correlated predictors. As a result, the method yields more stable selection results by mitigating the random selection among correlated variables by the Lasso, as discussed in Meinshausen and Bühlmann (2010), Wang et al. (2018), and Faletto and Bien (2022). However, the overall selection accuracy remains dependent on the output of the initial ranking.

Since this paper focuses on correlated variables and seeks to prioritize the relevant ones among them, it is important to employ a ranking method that assigns higher ranks to relevant variables within correlated groups. As noted by Nouraie et al. (2025), this requirement is weaker than screening consistency, as introduced in Section 1, since it does not demand that all relevant variables to be assigned higher ranks than all irrelevant ones, but only that each relevant variable is ranked higher than the irrelevant variables with which it is correlated. We refer to this property as ‘partial screening consistency’. Accordingly, Ridge-HOLP, which satisfies asymptotic screening consistency, serves as a suitable choice. Ridge-HOLP employs a diagonally dominant projection matrix $X^\top(XX^\top)^{-1}X$, such that multiplying this matrix by β is likely to preserve the rank order of the entries in β (Wang and Leng, 2015).

In this paper, we employ Air-HOLP, a data-adaptive extension of Ridge-HOLP, as the initial ranking method. Assuming $s \asymp n^\nu$ for a $\nu < 1$ while s denotes the size of the true model, we set the screening threshold $d = n/\log(n)$ which is a common choice in the screening literature (Joudah et al., 2025). While alternative choices for the screening threshold exist and have been proposed, a comprehensive comparison of them is beyond the scope of this paper. In addition, although other correlation-adjusted screening and ranking procedures such as the method proposed by Wang et al. (2025) could potentially be integrated into our methodology, a comprehensive comparison of such alternatives is beyond the scope of this paper. The chosen ranking method is expected to order variables such that those with greater predictive relevance to the response are assigned higher ranks.

Applying Air-HOLP to X and \mathbf{Y} and obtaining the ranking of predictors, we define

$$w_j(r_j) := 1 - \frac{1}{r_j}; \quad j = 1, 2, \dots, p, \quad (2)$$

where w_j denotes the weight (penalty factor) assigned to the j th predictor, and r_j denotes the rank assigned to the j th predictor using Air-HOLP. Ranks obtained by Air-HOLP represent the relative predictive relevance of predictors, with higher ranks indicating greater relevance.

By incorporating variable-specific weights from Equation (2), Equation (1) can be extended to

$$\hat{\boldsymbol{\beta}}(\lambda, \mathbf{w}) = \arg \min_{\boldsymbol{\beta} \in \mathbb{R}^p} \left(\|\mathbf{Y} - X\boldsymbol{\beta}\|_2^2 + \lambda \|\mathbf{w} \odot \boldsymbol{\beta}\|_1 \right), \quad (3)$$

where \mathbf{w} denotes the p -vector of w_j values, and \odot denotes the element-wise (Hadamard) product operator. The variable with the highest rank receives no additional penalty beyond the standard least-squares quadratic loss. As the rank increases, the corresponding weight gradually approaches 1, matching the standard Lasso penalty. Hereafter, we refer to the model introduced in Equation (3) as the ‘Stable Lasso’.

Rather than relying solely on the asymptotic screening consistency of the screening method and discarding variables excluded from the screened set, we incorporate a rank-based weighting to guide the Lasso. The slope of the weight function introduced in Equation (2) decreases with increasing rank, implying that a unit increase in rank leads to a more substantial change in the weight for higher-ranked variables than for lower-ranked ones. As a result, the weights become less sensitive to rank changes among lower-ranked variables, effectively applying the standard Lasso to them. This behavior is reminiscent of certain methods, such as those proposed by Fan and Li (2001) and Zhang (2010), which aim to reduce the bias of the Lasso by being more lenient towards variables with larger coefficient magnitudes.

In terms of computational cost, Joudah et al. (2025) noted that in the high-dimensional setting, Ridge-HOLP is more computationally efficient than the standard ridge regression (Hoerl and Kennard, 1970), and showed that while Air-HOLP uses an iterative process to update the initial tuning parameter, it remains considerably faster than the standard ridge regression in the high-dimensional setting.

We embed the Stable Lasso within Stability Selection to assess its stability across random sub-samples and to tune the hyper-parameter λ , as will be explained shortly. In this paper, the binary selection outcomes obtained from Stability Selection with the Lasso for a fixed λ are represented by the matrix $M(\lambda) \in \text{mat}(B \times p)$, where B denotes the number of sub-samples. The stability measure, $\hat{\Phi}(M(\lambda))$, is then computed as defined in Nogueira et al. (2018), corresponding to a scaled average of the column-wise variances of $M(\lambda)$.

Remark 1. Consider Stability Selection with the Lasso for a fixed λ value. Then, employing a fixed non-uniform weighting scheme as penalty factors for the Lasso, applied consistently across all sub-samples and recorded in $M'(\lambda)$, yields greater stability than applying the standard equally weighted Lasso and recording in $M''(\lambda)$, that is, $\hat{\Phi}(M'(\lambda)) \geq \hat{\Phi}(M''(\lambda))$.

The conclusion in Remark 1 follows directly from the influence of weighting on the Lasso estimator. Random selection among correlated predictors is a well-known source of instability

ity in the Lasso. By consistently applying a fixed, non-uniform weighting scheme across all sub-samples in Stability Selection, a systematic preference among predictors is enforced, independent of the value of λ or the specific sub-sample. This consistency aligns selection outcomes across sub-samples, thereby increasing overall stability as measured by $\hat{\Phi}(\mathbf{M}(\lambda))$.

As is evident, Remark 1 is not restricted to the Stable Lasso: applying a non-uniform weighting scheme in the Lasso generally enhances stability. However, some weighted Lasso variants employ nearly uniform weights. For instance, the Randomized Lasso uses only two weights, $1/\alpha$ and 1, while in the Adaptive Lasso, when initial estimates are obtained from the standard Lasso, most regression coefficients are zero. Nevertheless, as will be demonstrated in Section 3, even these methods often exhibit greater stability than the standard Lasso, which employs uniform weights. It should be noted, however, that stability alone is insufficient; selection accuracy must also be taken into account.

In addition, Remark 1 is not restricted to the high-dimensional setting. Therefore, following the findings of Nouraie et al. (2025), we conclude that selection stability is also a concern in low-dimensional contexts. In Section 3, we evaluate the performance of the Stable Lasso in a low-dimensional setting to examine its effectiveness under such conditions.

Remark 2. *Under the assumptions of Theorem 3 in Wang and Leng (2015), for an appropriately chosen λ , the Stable Lasso consistently prioritizes all relevant variables for selection.*

Theorem 3 in Wang and Leng (2015) establishes the asymptotic screening consistency of Ridge-HOLP, showing that, with increasing sample size, all relevant variables are ranked above irrelevant ones with high probability. The Stable Lasso leverages this ranking by applying variable-specific penalty weights, assigning smaller penalties to higher-ranked, potentially relevant variables. Under the sufficient conditions stated in Remark 2, this mechanism promotes the selection of all relevant variables. Notably, the necessary requirement is partial screening consistency, which is substantially weaker than full screening consistency.

For the optimization of Equation (3), we employ a coordinate descent algorithm, as recommended by Friedman et al. (2010), incorporating the weight vector \mathbf{w} as a penalty factor within the Lasso objective function.

Hyper-Parameter Tuning

Equation (3) includes the hyper-parameter $\lambda \in \Lambda$ that governs the overall level of shrinkage applied during estimation. Nouraie and Muller (2024) proposed tuning λ with respect to both selection stability and prediction accuracy, yielding a Pareto optimal solution (Pareto, 1896) on the stability–accuracy Pareto front, as conceptualized by Nogueira et al. (2018), and in line with Sun et al. (2013). They embedded the Lasso in the Stability Selection framework and used the stability measure introduced by Nogueira et al. (2018) to quantify the selection stability. They defined λ_{stable} as the smallest λ achieving a stability value of at least 0.75, and showed its Pareto optimality under two assumptions on the behaviour of the stability and accuracy curves across the regularization grid. If λ_{stable} is not attainable, they proposed selecting the smallest λ whose corresponding stability remains within one standard deviation of

the maximum observed stability, denoted by $\lambda_{\text{stable-1sd}}$. Since the goal of this paper is to obtain stable results, we adopt their approach for tuning λ by jointly considering selection stability and prediction accuracy, rather than relying solely on cross-validation methods that prioritize prediction accuracy. Nonetheless, cross-validation-based methods remain applicable.

3 Numerical Results

We evaluate our method using several synthetic datasets and a real bioinformatics dataset. We used the corresponding GitHub repositories of Joudah et al. (2025), Nogueira et al. (2018), and Nouraie et al. (2025) to execute Air-HOLP, apply the stability measure, and perform variable decorrelation respectively. For regularization tuning, we used the GitHub repository associated with Nouraie and Muller (2024).

To create a set of candidate regularization values Λ , we use a 10-fold cross-validation on the entire dataset \mathcal{D} . The Air-HOLP method takes an input parameter, `Threshold`, which specifies the number of variables to retain during the screening process. In all experiments, this parameter was fixed at $n/\log(n)$ to eliminate its influence on the analysis. In addition, the maximum number of iterations for Air-HOLP was set to 10 across all experiments.

We use the `SGL` R package (Simon et al., 2019) to fit the SGL and the `iilasso` R package¹ to fit the IILasso. For all Adaptive Lasso experiments, we fix the tuning parameter at $\gamma = 1$ to eliminate its influence on the analysis. In addition, to avoid division by zero when computing the adaptive weights, we add a small constant (10^{-6}) to the absolute values of the regression coefficients. In our implementation of Randomized Lasso, we set $\alpha = 0.2$ and $p_j = 0.5$ for all j values. It should be noted that, although Randomized Lasso is intended to assign random weights within Stability Selection (Meinshausen and Bühlmann, 2010), we observed that this implementation leads to poor performance. Consequently, we opted to apply a fixed random weighting consistently across all sub-samples of the data.

We embed selection methods in the Stability Selection framework (Meinshausen and Bühlmann, 2010) to evaluate their selection stability and accuracy. For methods employing variable-specific weights (penalty factors), the weights are obtained from the full dataset \mathcal{D} and subsequently applied consistently across all models within the Stability Selection sub-samples. Simulation R scripts were executed using the `parallel` package to perform parallel computing across 7 CPU cores on a MacBook Air 13.

Synthetic Data

For the synthetic data experiments, we consider a challenging high-dimensional setting characterized by strong correlations among predictors. Specifically, datasets are generated with a sample size of $n = 100$ and $p = 1,000$ predictor variables. For each dataset, the predictor variables \mathbf{x}_i^\top are independently drawn from the Normal distribution $\mathcal{N}(\mathbf{0}, \Sigma)$, where $\Sigma \in \text{mat}(p, p)$

¹<https://github.com/tkdmah/iilasso/tree/master>

has a diagonal of ones, and the off-diagonal elements are defined as follows

$$\sigma_{jk} = \begin{cases} \rho_g, & \text{if } j, k \text{ belong to the same group } G_g, \text{ where } g = 1, \dots, 5, \\ 0, & \text{otherwise;} \end{cases}$$

where each group consists of consecutive indices

$$G_g = \left\{ \frac{(g-1)p}{5} + 1, \dots, \frac{gp}{5} \right\}, \quad g = 1, \dots, 5.$$

Each group G_g is associated with a distinct covariance parameter ρ_g . Specifically, the covariance values are set as $\rho_1 = 0.8$, $\rho_2 = 0.85$, $\rho_3 = 0.9$, $\rho_4 = 0.95$, and $\rho_5 = 0.99$. Within each group, the highest-indexed variable is designated as relevant. The true regression coefficients of relevant variables are defined as $\beta = (3, 2.5, 2, 1.5, 1)^\top$ following the order of the groups and there are 199 irrelevant variables in each group. The error term ϵ is an i.i.d. sample from the standard Normal distribution $\mathcal{N}(0, 1)$.

We generate 100 datasets with varying random seeds, following the setting described above. For each dataset we set the number of Stability Selection sub-samples $B = 100$. In our implementation, we use $\lambda_{\text{stable-1sd}}$ for each dataset, as λ_{stable} may not exist for some datasets. This approach ensures consistent regularization tuning across all datasets.

Figure 1 presents a comparative analysis of Stability Selection using different selection methods, evaluating both selection stability and average F1-score, which is the harmonic mean of precision and recall. Here, the F1-score is computed to assess selection accuracy, treating the selection task as a binary classification problem. Figure 1 compares the performance of Lasso, Stable Lasso, Adaptive Lasso with Lasso initial values (Adaptive Lasso (L)), Adaptive Lasso with univariate regression initial values (Adaptive Lasso (U)), and Randomized Lasso.

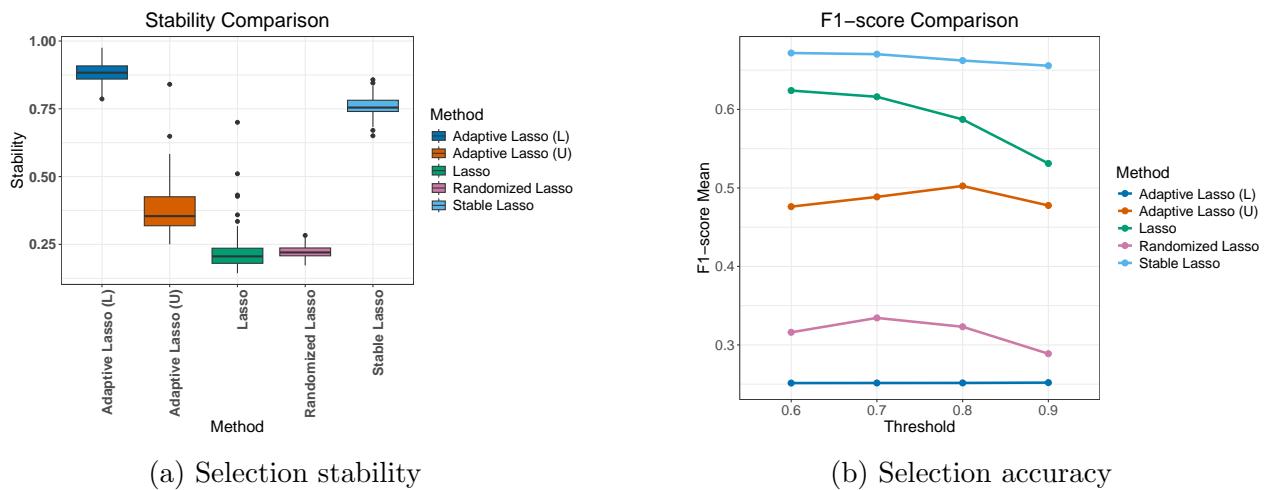


Figure 1: Comparing Stability and F1-score between Stability Selection with Stable Lasso, Lasso, Adaptive Lasso, and Randomized Lasso

Figure 1a demonstrates that the Adaptive Lasso (L) achieves the highest stability among all methods, with Stable Lasso ranking second, attaining a median stability value of 0.75, indicative of excellent selection stability. In contrast, the Lasso exhibits the lowest median stability. Figure 1b presents the selection F1-scores of the methods across different Stability Selection

thresholds π_{thr} . Stable Lasso consistently outperforms all alternatives across all threshold values. Notably, the Lasso ranks second, achieving higher accuracy than the Adaptive Lasso methods. In contrast, the Adaptive Lasso (L) demonstrates poor selection accuracy despite its excellent stability.

Figures 2a and 2b compare Stable Lasso with SGL, which requires prior knowledge of the group structure between variables. In our experiments, this group information was provided to SGL. It is important to note that SGL incurs a higher running time due to its complex optimization procedure. As shown in Figures 2a and 2b, Stable Lasso demonstrates significantly greater stability and accuracy than SGL. This highlights the effectiveness of our simple approach, even when compared to more complex models that leverage grouping structures among predictors.

Figures 2c and 2d compare Stable Lasso with IILasso, which exploits the full covariance structure of the predictors. Due to the high computational cost of IILasso's complex optimization, results are based on 25 simulated datasets, in contrast to the 100 datasets used in the other figures. Notably, even after reducing the number of datasets and employing parallel computing, this experiment still required just over 72 hours to complete. As shown in Figure 2c, Stable Lasso exhibits substantially higher stability than IILasso. Figure 2d shows that the selection accuracy of the two models is comparable, with IILasso performing better at lower thresholds and Stable Lasso better at higher thresholds. Moreover, the selection accuracy of Stable Lasso is more robust to the choice of threshold compared to IILasso. These results highlight the effectiveness of our simple and computationally efficient approach, even when compared to sophisticated methods that exploit the full covariance structure among predictors.

Figures 2e and 2f compare Stable Lasso with the variable decorrelation approach proposed by [Nouraie et al. \(2025\)](#). As shown in Figure 2e, both methods achieve excellent stability, with the variable decorrelation approach showing slightly higher median stability values. This is particularly noteworthy, as their method applies Lasso to an orthonormal surrogate matrix, whereas Stable Lasso attains comparable stability while operating directly on the original, highly correlated data. Figure 2f indicates that their selection accuracy is also comparable, with Stable Lasso outperforming the alternative across all threshold values.

An additional advantage of Stable Lasso over the variable decorrelation method is interpretability. While the selected components in the decorrelation approach serve as surrogates for the original variables, Stable Lasso operates directly in the original data space, selecting the relevant predictors. These results demonstrate that Stable Lasso can deliver stable and accurate variable selection in complex and challenging settings, outperforming several sophisticated alternatives.

Now we extend the initial data generation setting to explore a reduced-dimensional setting, decreasing the number of variables from 1,000 to 80 while preserving the original group structure. This modification allows us to assess the performance of our method in a low-dimensional context. This setting is of particular interest because it falls within the low-dimensional setting, where Ordinary Least Squares (OLS) estimation of regression coefficients is feasible. In the figures, the adaptive Lasso method powered by OLS estimates is labeled Adaptive Lasso (OLS). In addition, we consider an oracle version of the Adaptive Lasso, denoted by Adaptive Lasso (O), which employs the true regression coefficients used to generate the data. The results

for this scenario are presented in Figures 3a and 3b.

Figure 3a shows that Adaptive Lasso (OLS) achieves the highest stability, with Stable Lasso ranking second, slightly above Adaptive Lasso (O). Figure 3b further demonstrates that Adaptive Lasso (O), benefiting from oracle information, attains the highest selection accuracy, followed by Stable Lasso. These results underscore the effectiveness of our method in the low-dimensional setting.

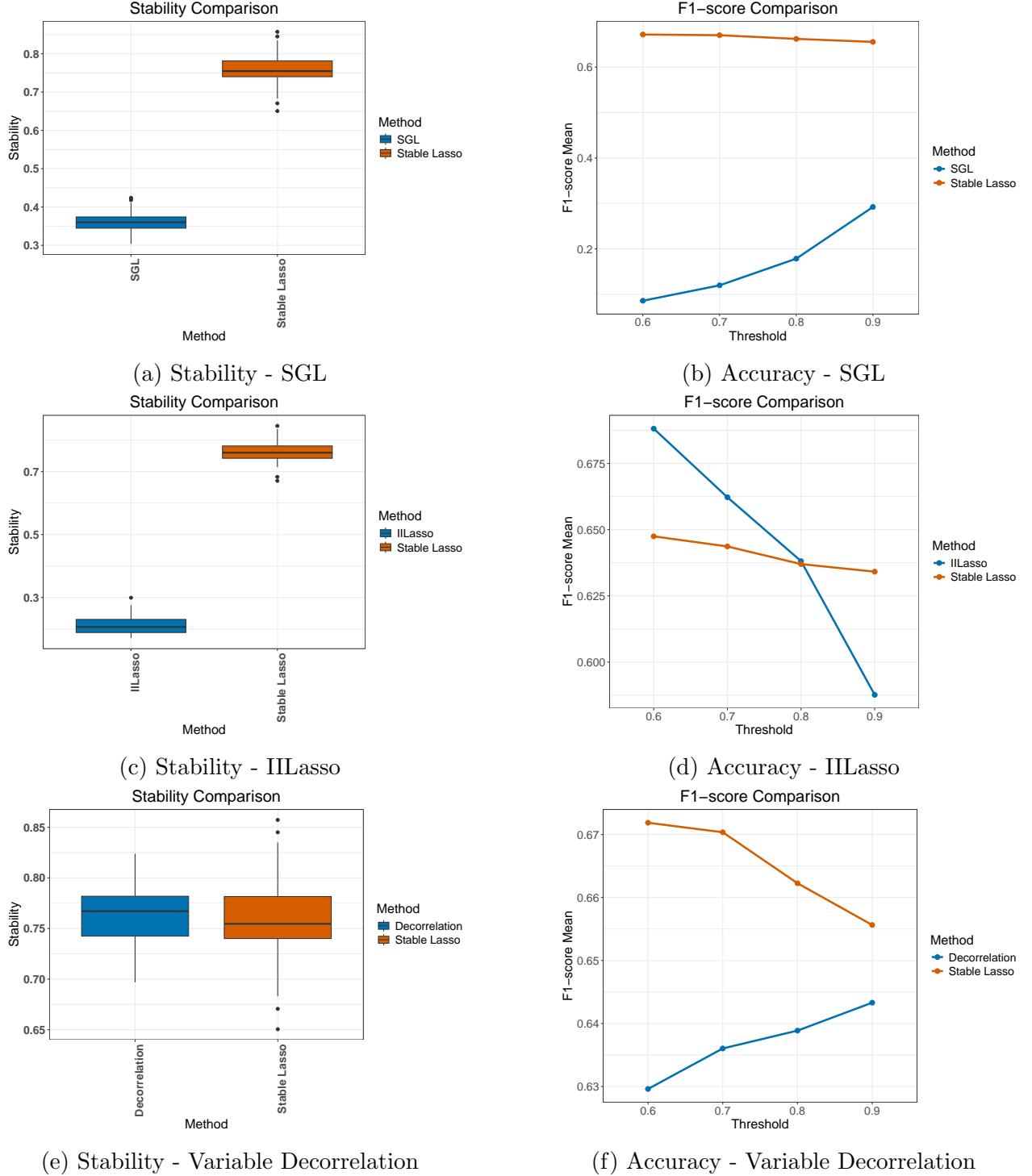


Figure 2: Comparing Stability and F1-score between Stability Selection with Stable Lasso, SGL, IILasso, and variable decorrelation approach

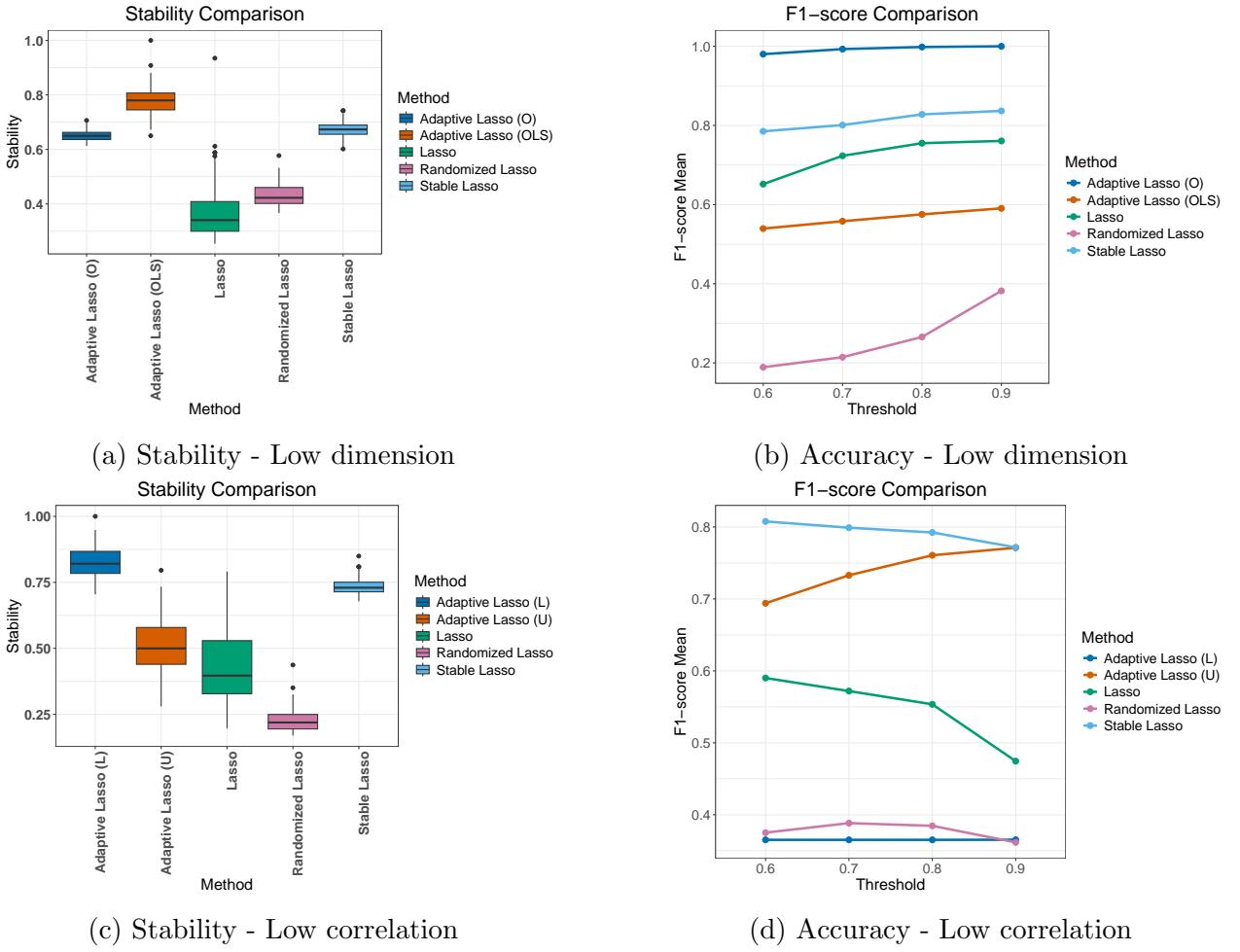


Figure 3: Comparing Stability and F1-score between Stability Selection with Stable Lasso, Lasso, Adaptive Lasso, and Randomized Lasso in the low-dimension, and low-correlation scenarios

In addition, we consider a scenario with attenuated covariance values to examine our method's performance under less extreme correlation structures. This scenario retains the original high-dimensional setting with $n = 100$ and $p = 1000$, but reduces the correlation levels among the predictors. This analysis aims to determine whether the strong results reported earlier are specific to extremely correlated data, or if they remain robust under more moderate correlation structures. Specifically, we set the correlation parameters to $\rho_1 = 0.4$, $\rho_2 = 0.5$, $\rho_3 = 0.6$, $\rho_4 = 0.7$, and $\rho_5 = 0.8$. The results for this scenario are presented in Figures 3c and 3d.

As shown in Figure 3c, Adaptive Lasso (L) exhibits the highest stability, followed by Stable Lasso. Figure 3d shows that Stable Lasso achieves the highest average F1-score across all decision thresholds, followed by Adaptive Lasso (U). These results indicate that the performance advantages of Stable Lasso persist even under moderate correlation structures.

Mice PCR Data

As a real-world example, we employ a Polymerase Chain Reaction (PCR) dataset, which serves to validate and verify the effectiveness of our proposed approach. Lan et al. (2006) conducted an experiment which examines the genetics of two inbred mouse populations (C57BL/6J and

BTBR). A total of $n = 60$ second generation (F2) samples, with 31 female and 29 male mice, were used to monitor the expression levels of $p = 22,575$ genes. Some physiological phenotypes, including numbers of phosphoenopyruvate carboxykinase, glycerol-3-phosphate acyltransferase and stearoyl-CoA desaturase 1 were measured by quantitative realtime PCR. In this example, we study the relationship between phosphoenopyruvate carboxykinase as the response variable and the gene expression levels. The gene expression data is standardized and the response vector is centered before the analysis. [Song and Liang \(2014\)](#) approached this problem with a split-and-merge Bayesian approach and identified 1429089_s_at, 1430779_at, 1432745_at, 1437871_at, 1440699_at, and 1459563_x_at as relevant genes. Later [Staerk et al. \(2024\)](#) identified 1438937_x_at and 1438936_s_at as relevant genes.

We set the number of Stability Selection sub-samples $B = 100$. Applying Lasso within Stability Selection on this dataset produced a maximum stability value of 0.054 (rounded to three decimals), which is very low. This indicates that using Stability Selection alone, even with careful regularization tuning, does not yield stable results, highlighting the need for a more advanced selection approach. Here, with $\lambda_{\text{stable-1sd}} = 0.394$ and $\pi_{\text{thr}} = 0.2$, the gene 1438937_x_at is identified as relevant, with a selection frequency of 0.35 under $\lambda_{\text{stable-1sd}}$. This gene was also identified by [Staerk et al. \(2024\)](#).

Applying Stable Lasso resulted in a maximum selection stability of 0.85 (rounded to three decimals), substantially higher than the previous approach. In this case, λ_{stable} exists and is 0.433. Using this λ_{stable} , the genes 1437871_at and 1438937_x_at were identified with selection frequencies of 1 and 0.88, respectively. The first gene was reported by [Song and Liang \(2014\)](#), while the latter was identified by [Staerk et al. \(2024\)](#).

4 Discussion and Conclusion

In this section, we briefly discuss some related methods which, while not explicitly framed within the context of stability, fall within the broader scope of this paper. Our approach shares certain similarities with these methods but differs fundamentally in how stability is defined and quantified, thereby underscoring the unique contribution of our framework.

Sorted L-One Penalized Estimation (SLOPE; [Bogdan et al., 2015](#)) introduces a ranking-based penalty on the regression coefficients, encouraging variable selection by ordering predictors according to their estimated effect magnitudes during the optimization process. While conceptually related to our approach, [Zeng and Figueiredo \(2014\)](#) demonstrated that SLOPE generalizes the Octagonal Shrinkage and Clustering Algorithm for Regression (OSCAR; [Bondell and Reich, 2008](#)), which is designed to perform grouping selection. This contrasts with the objective of the present work, which emphasizes exclusive selection. [Feser and Evangelou \(2023\)](#) introduced Sparse Group SLOPE, which performs bi-level selection in a manner similar to the SGL. However, as with our discussion of SGL, we emphasize that our work differs from theirs.

[Chatterjee et al. \(2025\)](#) recently introduced Univariate-Guided Sparse Regression, a two-step procedure that preserves the signs of univariate regression coefficients throughout the optimization process and incorporates their magnitudes. Although there are some conceptual

similarities between their approach and ours, the two methods are fundamentally different and address distinct objectives. While their method is designed to enhance sign stability of the Lasso, our focus is on improving its selection stability.

Furthermore, our approach, which enhances the selection stability of the Lasso by incorporating prior information derived from a variable ranking method, bears conceptual resemblance to Bayesian variable selection methods, such as those proposed by Park and Casella (2008) and Nouraie et al. (2024).

Finally, we present numerical results obtained by applying our proposed weighting method to the Smoothly Clipped Absolute Deviation (SCAD; Fan and Li 2001) and Minimax Concave Penalty (MCP; Zhang 2010) methods. We adopt the primary simulation scenario with $n = 100$ and $p = 1000$. The outcomes of this experiment are summarized in Figure 4. Analogous to our implementation of the method as ‘Stable Lasso’, we refer to SCAD and MCP when augmented with the weights defined in Equation (2) as ‘Stable SCAD’ and ‘Stable MCP’, respectively. Computations for SCAD and MCP were performed using the `ncvreg` R package (Breheny and Huang, 2011). As illustrated in Figures 4a and 4b, SCAD equipped with our proposed weighting scheme exhibits substantially improved stability and accuracy compared with the standard SCAD. Similarly, Figures 4c and 4d demonstrate the same enhancement for MCP. These results underscore the potential of our approach as a general-purpose solution for stable variable selection, motivating further investigation and development.

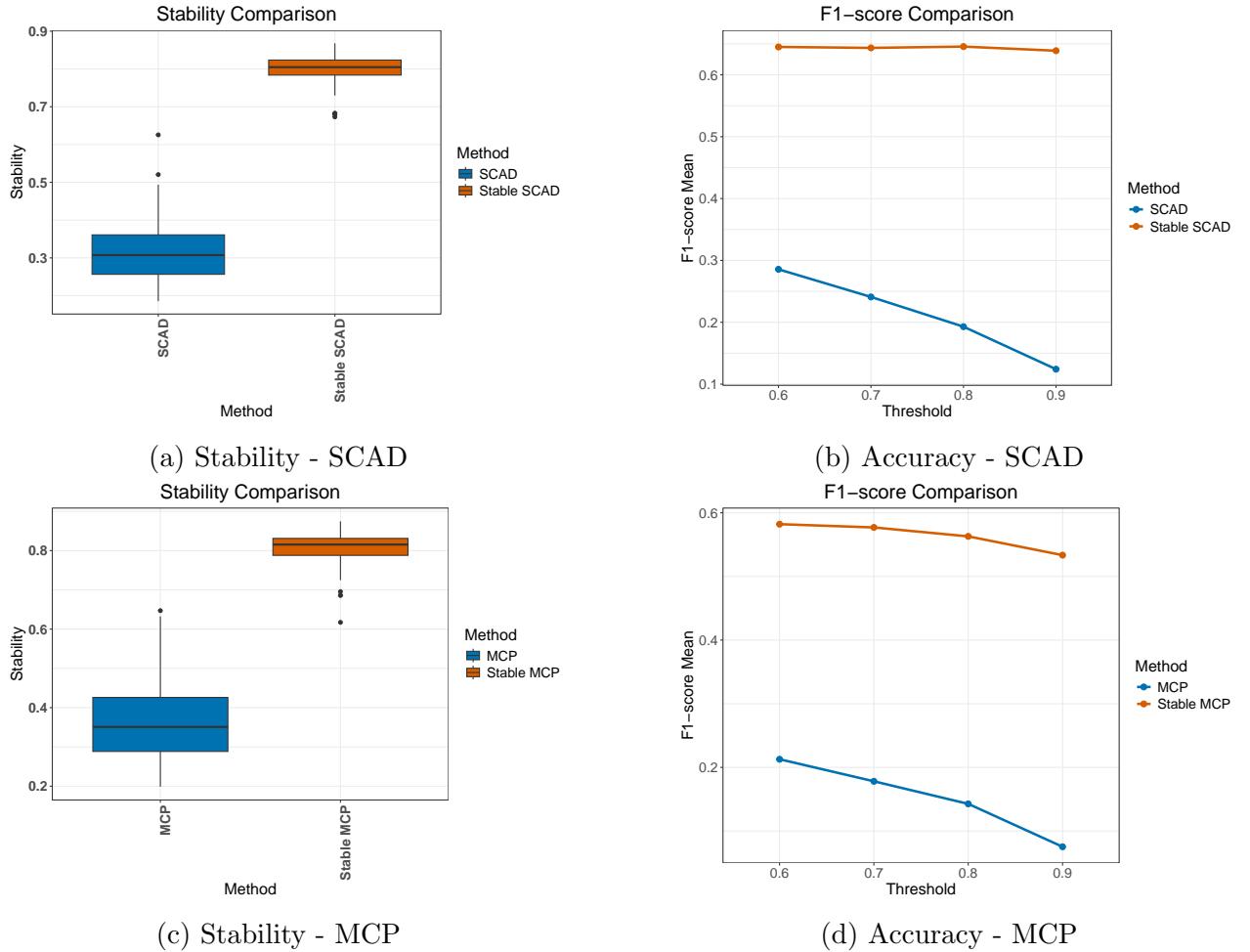


Figure 4: Comparing Stability and F1-score between Stability Selection with Stable SCAD, SCAD, Stable MCP, and MCP

Competing interests

The authors declare that they have no conflict of interest.

Author contributions statement

Mahdi Nouraie was responsible for drafting the manuscript, the development of the research methodology and for writing the computer code used throughout. Samuel Muller and Houying Zhu provided critical feedback on the content of the manuscript, refining the clarity and scope of the manuscript and the computer code.

Data and Code Availability

The mice PCR dataset is accessible JRSS(B) Datasets Vol.77(5) at <https://fslive.oup.com/www.oup.com/academic/content/rss-datasets/jrsssb/77-5-2015-B.zip>.

The source code used for the paper is accessible through the following GitHub repository: <https://github.com/MahdiNouraie/Stable-Lasso>.

Acknowledgments

Mahdi Nouraie was supported by the Macquarie University Research Excellence Scholarship (20213605) and a top-up scholarship from the Statistical Society of Australia. Samuel Muller was supported by the Australian Research Council Discovery Project Grant (DP230101908).

References

- Å. Björck. Numerics of Gram-Schmidt Orthogonalization. *Linear Algebra and its Applications*, 197-198:297–316, 1994.
- Małgorzata Bogdan, Ewout Van Den Berg, Chiara Sabatti, Weijie Su, and Emmanuel J Candès. SLOPE—adaptive variable selection via convex optimization. *The Annals of Applied Statistics*, 9(3):1103–1140, 2015.
- Howard D. Bondell and Brian J. Reich. Simultaneous Regression Shrinkage, Variable Selection, and Supervised Clustering of Predictors with OSCAR. *Biometrics*, 64(1):115–123, 2008.
- Patrick Breheny and Jian Huang. Coordinate descent algorithms for nonconvex penalized regression, with applications to biological feature selection. *Annals of Applied Statistics*, 5 (1):232–253, 2011.
- Peter Bühlmann and Sara van de Geer. *Statistics for High-Dimensional Data: Methods, Theory and Applications*. Springer Berlin Heidelberg, 2011.
- Sourav Chatterjee, Trevor Hastie, and Robert Tibshirani. Univariate-guided sparse regression. *arXiv preprint arXiv:2501.18360*, 2025.

Si-Bao Chen, Chris Ding, Bin Luo, and Ying Xie. Uncorrelated Lasso. *Proceedings of the AAAI Conference on Artificial Intelligence*, 27(1):166–172, 2013.

Émeline Courtois, Pascale Tubert-Bitter, and Ismaïl Ahmed. New adaptive lasso approaches for variable selection in automated pharmacovigilance signal detection. *BMC Medical Research Methodology*, 21(271):1–17, 2021.

Erin Craig, Mert Pilancı, Thomas Le Menestrel, Balasubramanian Narasimhan, Manuel A. Rivas, Stein-Erik Gullaksen, Roozbeh Dehghannasiri, Julia Salzman, Jonathan Taylor, and Robert Tibshirani. Pretraining and the Lasso. *arXiv preprint arXiv:2401.12911*, 2024.

Gregory Faletto and Jacob Bien. Cluster Stability Selection. *arXiv preprint arXiv:2201.00494*, 2022.

Jianqing Fan and Runze Li. Variable selection via nonconcave penalized likelihood and its oracle properties. *Journal of the American Statistical Association*, 96(456):1348–1360, 2001.

Jianqing Fan and Jinchi Lv. Sure independence screening for ultrahigh dimensional feature space. *Journal of the Royal Statistical Society Series B: Statistical Methodology*, 70(5):849–911, 2008.

Fabio Feser and Marina Evangelou. Sparse-group SLOPE: adaptive bi-level selection with FDR-control. *arXiv preprint arXiv:2305.09467*, 2023.

Jerome Friedman, Trevor Hastie, and Rob Tibshirani. Regularization Paths for Generalized Linear Models via Coordinate Descent. *Journal of Statistical Software*, 33(1):1–22, 2010.

Édouard Grave, Guillaume Obozinski, and Francis Bach. Trace Lasso: a trace norm regularization for correlated designs. In *Proceedings of the 24th International Conference on Neural Information Processing Systems*, NIPS’11, page 2187–2195. Curran Associates Inc., 2011.

Arthur E Hoerl and Robert W Kennard. Ridge regression: Biased estimation for nonorthogonal problems. *Technometrics*, 12(1):55–67, 1970.

Jian Huang, Shuangge Ma, and Cun-Hui Zhang. Adaptive Lasso for Sparse High-Dimensional Regression Models. *Statistica Sinica*, 18(4):1603–1618, 2008.

Ibrahim Joudah, Samuel Muller, and Houying Zhu. Air-HOLP: adaptive regularized feature screening for high dimensional correlated data. *Statistics and Computing*, 35(63), 2025.

Alexandros Kalousis, Julien Prados, and Melanie Hilario. Stability of feature selection algorithms: a study on high-dimensional spaces. *Knowledge and Information Systems*, 12(1):95–116, 2007.

Deguang Kong, Ryohei Fujimaki, Ji Liu, Feiping Nie, and Chris Ding. Exclusive Feature Learning on Arbitrary Structures via $l_{1,2}$ -norm. In *Advances in Neural Information Processing Systems*, volume 27. Curran Associates, Inc., 2014.

Hong Lan, Meng Chen, Jessica B Flowers, Brian S Yandell, Donnie S Stapleton, Christine M Mata, Eric Ton-Keen Mui, Matthew T Flowers, Kathryn L Schueler, Kenneth F Manly, Robert W Williams, Christina Kendziorski, and Alan D Attie. Combined Expression Trait Correlations and Expression Quantitative Trait Locus Mapping. *PLOS Genetics*, 2(1):0051–0061, 2006.

Chenlei Leng, Yi Lin, and Grace Wahba. A note on the Lasso and related procedures in model selection. *Statistica Sinica*, 16(4):1273–1284, 2006.

Alexander Lorbert, David Eis, Victoria Kostina, David M. Blei, and Peter J. Ramadge. Exploiting Covariate Similarity in Sparse Regression via the Pairwise Elastic Net. In *Proceedings of the Thirteenth International Conference on Artificial Intelligence and Statistics*, volume 9 of *Proceedings of Machine Learning Research*, pages 477–484. PMLR, 2010.

Nicolai Meinshausen and Peter Bühlmann. High-dimensional graphs and variable selection with the Lasso. *The Annals of Statistics*, 34(3):1436–1462, 2006.

Nicolai Meinshausen and Peter Bühlmann. Stability selection. *Journal of the Royal Statistical Society Series B: Statistical Methodology*, 72(4):417–473, 2010.

Samuel Müller and Alan H. Welsh. On model selection curves. *International Statistical Review*, 78(2):240–256, 2010.

Sarah Nogueira, Konstantinos Sechidis, and Gavin Brown. On the Stability of Feature Selection Algorithms. *Journal of Machine Learning Research*, 18(174):1–54, 2018.

Mahdi Nouraie and Samuel Muller. On the Selection Stability of Stability Selection and Its Applications. *arXiv preprint arXiv:2411.09097*, 2024.

Mahdi Nouraie, Connor Smith, and Samuel Muller. Bayesian Stability Selection and Inference on Selection Probabilities. *arXiv preprint arXiv:2410.21914*, 2024.

Mahdi Nouraie, Connor Smith, and Samuel Muller. Stability Selection via Variable Decorrelation. *arXiv preprint arXiv:2505.20864*, 2025.

Vilfredo Pareto. *Cours D'Économie Politique*. Rouge, Lausanne, 1896.

Trevor Park and George Casella. The Bayesian Lasso. *Journal of the American Statistical Association*, 103(482):681–686, 2008.

Rajen D Shah and Richard J Samworth. Variable selection with error control: another look at stability selection. *Journal of the Royal Statistical Society Series B: Statistical Methodology*, 75(1):55–80, 2013.

Noah Simon, Jerome Friedman, Trevor Hastie, and Robert Tibshirani. A Sparse-Group Lasso. *Journal of Computational and Graphical Statistics*, 22(2):231–245, 2013.

Noah Simon, Jerome Friedman, Trevor Hastie, and Rob Tibshirani. *SGL: Fit a GLM (or Cox Model) with a Combination of Lasso and Group Lasso Regularization*, 2019. R package version 1.3.

Jake A. Soloff, Rina Foygel Barber, and Rebecca Willett. Bagging provides assumption-free stability. *Journal of Machine Learning Research*, 25(131):1–35, 2024.

Qifan Song and Faming Liang. A Split-and-Merge Bayesian Variable Selection Approach for Ultrahigh Dimensional Regression. *Journal of the Royal Statistical Society Series B: Statistical Methodology*, 77(5):947–972, 2014.

Christian Staerk, Maria Kateri, and Ioannis Ntzoufras. A Metropolized Adaptive Subspace Algorithm for High-Dimensional Bayesian Variable Selection. *Bayesian Analysis*, 19(1):261 – 291, 2024.

Wei Sun, Junhui Wang, and Yixin Fang. Consistent selection of tuning parameters via variable selection stability. *Journal of Machine Learning Research*, 14(1):3419–3440, 2013.

Masaaki Takada, Taiji Suzuki, and Hironori Fujisawa. Independently Interpretable Lasso: A New Regularizer for Sparse Regression with Uncorrelated Variables. In *Proceedings of the Twenty-First International Conference on Artificial Intelligence and Statistics*, volume 84 of *Proceedings of Machine Learning Research*, pages 454–463. PMLR, 2018.

Robert Tibshirani. Regression Shrinkage and Selection via the Lasso. *Journal of the Royal Statistical Society Series B: Statistical Methodology*, 58(1):267–288, 1996.

Robert Tibshirani, Michael Saunders, Saharon Rosset, Ji Zhu, and Keith Knight. Sparsity and smoothness via the fused Lasso. *Journal of the Royal Statistical Society Series B: Statistical Methodology*, 67(1):91–108, 2005.

Haohan Wang, Benjamin J Lengerich, Bryon Aragam, and Eric P Xing. Precision Lasso: accounting for correlations and linear dependencies in high-dimensional genomic data. *Bioinformatics*, 35(7):1181–1187, 2018.

Run Wang, An Nguyen, Somak Dutta, and Vivekananda Roy. Ridge partial correlation screening for ultrahigh-dimensional data. *arXiv preprint arXiv:2504.19393*, 2025.

Xiangyu Wang and Chenlei Leng. High dimensional ordinary least squares projection for screening variables. *Journal of the Royal Statistical Society Series B: Statistical Methodology*, 78(3):589–611, 2015.

Ming Yuan and Yi Lin. Model selection and estimation in regression with grouped variables. *Journal of the Royal Statistical Society Series B: Statistical Methodology*, 68(1):49–67, 2006.

Xiangrong Zeng and Mário A. T. Figueiredo. Decreasing weighted sorted ℓ_1 regularization. *IEEE Signal Processing Letters*, 21(10):1240–1244, 2014.

Cun-Hui Zhang. Nearly unbiased variable selection under minimax concave penalty. *The Annals of Statistics*, 38(2):894–942, 2010.

Peng Zhao and Bin Yu. On Model Selection Consistency of Lasso. *Journal of Machine Learning Research*, 7:2541–2563, 2006.

Hui Zou. The Adaptive Lasso and Its Oracle Properties. *Journal of the American Statistical Association*, 101(476):1418–1429, 2006.

Hui Zou and Trevor Hastie. Regularization and variable selection via the elastic net. *Journal of the Royal Statistical Society Series B: Statistical Methodology*, 67(2):301–320, 2005.